



HAESE MATHEMATICS

Mathematics

**Analysis and
Approaches SL**

2



**Michael Haese
Mark Humphries
Chris Sangwin
Ngoc Vo**

for use with

IB Diploma Programme

**Bradley Steventon
Charlotte Frost**

**Joseph Small
Michael Mampusti**

WORKED SOLUTIONS

MATHEMATICS: ANALYSIS AND APPROACHES SL WORKED SOLUTIONS

Bradley Steventon	B.Ma.Sc.
Charlotte Frost	B.Sc.
Joseph Small	B.Ma.Sc.
Michael Mampusti	B.Ma.Adv.(Hons.), Ph.D.

Haese Mathematics
152 Richmond Road, Marleston, SA 5033, AUSTRALIA
Telephone: +61 8 8210 4666, Fax: +61 8 8354 1238
Email: info@haesemathematics.com
Web: www.haesemathematics.com

National Library of Australia Card Number & ISBN 978-1-925489-83-5

© Haese & Harris Publications 2020

Published by Haese Mathematics.
152 Richmond Road, Marleston, SA 5033, AUSTRALIA

First Edition 2020

Artwork by Brian Houston.

Typeset in Australia by Deanne Gallasch. Typeset in Times Roman 10.

This book is available on Snowflake only.

This book has been developed independently of the International Baccalaureate Organization (IBO). This book is in no way connected with, or endorsed by, the IBO.

This book is copyright. Except as permitted by the Copyright Act (any fair dealing for the purposes of private study, research, criticism or review), no part of this publication may be reproduced, stored in a retrieval system, or transmitted in any form or by any means, electronic, mechanical, photocopying, recording or otherwise, without the prior permission of the publisher. Enquiries to be made to Haese Mathematics.

Copying for educational purposes: Where copies of part or the whole of the book are made under Part VB of the Copyright Act, the law requires that the educational institution or the body that administers it has given a remuneration notice to Copyright Agency Limited (CAL). For information, contact the Copyright Agency Limited.

Acknowledgements: While every attempt has been made to trace and acknowledge copyright, the authors and publishers apologise for any accidental infringement where copyright has proved untraceable. They would be pleased to come to a suitable agreement with the rightful owner.

Disclaimer: All the internet addresses (URLs) given in this book were valid at the time of publication. While the authors and publisher regret any inconvenience that changes of address may cause readers, no responsibility for any such changes can be accepted by either the authors or the publisher.

FOREWORD

This book gives you fully worked solutions for every question in Exercises, Review Sets, Activities, and Investigations (which do not involve student experimentation) in each chapter of our textbook *Mathematics: Analysis and Approaches SL*.

Correct answers can sometimes be obtained by different methods. In this book, where applicable, each worked solution is modelled on the worked example in the textbook.

Be aware of the limitations of calculators and computer modelling packages. Understand that when your calculator gives an answer that is different from the answer you find in the book, you have not necessarily made a mistake, but the book may not be wrong either.

We have a list of errata for our books on our website. Please contact us if you notice any errors in this book.

BS CF JS MM

e-mail: info@haesemathematics.com

web: www.haesemathematics.com

TABLE OF CONTENTS

Chapter 1	THE BINOMIAL THEOREM	5
Chapter 2	QUADRATIC FUNCTIONS	25
Chapter 3	FUNCTIONS	98
Chapter 4	TRANSFORMATIONS OF FUNCTIONS	152
Chapter 5	EXPONENTIAL FUNCTIONS	187
Chapter 6	LOGARITHMS	226
Chapter 7	THE UNIT CIRCLE AND RADIAN MEASURE	274
Chapter 8	TRIGONOMETRIC FUNCTIONS	313
Chapter 9	TRIGONOMETRIC EQUATIONS AND IDENTITIES	356
Chapter 10	REASONING AND PROOF	401
Chapter 11	INTRODUCTION TO DIFFERENTIAL CALCULUS	417
Chapter 12	RULES OF DIFFERENTIATION	449
Chapter 13	PROPERTIES OF CURVES	520
Chapter 14	APPLICATIONS OF DIFFERENTIATION	617
Chapter 15	INTRODUCTION TO INTEGRATION	655
Chapter 16	TECHNIQUES FOR INTEGRATION	673
Chapter 17	DEFINITE INTEGRALS	709
Chapter 18	KINEMATICS	770
Chapter 19	BIVARIATE STATISTICS	806
Chapter 20	DISCRETE RANDOM VARIABLES	848
Chapter 21	THE NORMAL DISTRIBUTION	881

Chapter 1

THE BINOMIAL THEOREM

EXERCISE 1A

1 a $2! = 2 \times 1$
 $= 2$

c $4! = 4 \times 3 \times 2 \times 1$
 $= 24$

e $6! = 6 \times 5 \times 4 \times 3 \times 2 \times 1$
 $= 720$

b $3! = 3 \times 2 \times 1$
 $= 6$

d $5! = 5 \times 4 \times 3 \times 2 \times 1$
 $= 120$

f $10! = 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1$
 $= 3\,628\,800$

2 a $4 \times 3 \times 2 \times 1 = 4!$

c $6 \times 5 = \frac{6 \times 5 \times \cancel{4} \times \cancel{3} \times \cancel{2} \times \cancel{1}}{\cancel{4} \times \cancel{3} \times \cancel{2} \times \cancel{1}}$
 $= \frac{6!}{4!}$

b $7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 7!$

d $8 \times 7 \times 6 = \frac{8 \times 7 \times 6 \times \cancel{5} \times \cancel{4} \times \cancel{3} \times \cancel{2} \times \cancel{1}}{\cancel{5} \times \cancel{4} \times \cancel{3} \times \cancel{2} \times \cancel{1}}$
 $= \frac{8!}{5!}$

e $10 \times 9 \times 8 \times 7 = \frac{10 \times 9 \times 8 \times 7 \times \cancel{6} \times \cancel{5} \times \cancel{4} \times \cancel{3} \times \cancel{2} \times \cancel{1}}{\cancel{6} \times \cancel{5} \times \cancel{4} \times \cancel{3} \times \cancel{2} \times \cancel{1}}$
 $= \frac{10!}{6!}$

f $15 \times 14 \times 13 \times 12 = \frac{15 \times 14 \times 13 \times 12 \times \cancel{11} \times \cancel{10} \times \cancel{9} \times \cancel{8} \times \cancel{7} \times \cancel{6} \times \cancel{5} \times \cancel{4} \times \cancel{3} \times \cancel{2} \times \cancel{1}}{\cancel{11} \times \cancel{10} \times \cancel{9} \times \cancel{8} \times \cancel{7} \times \cancel{6} \times \cancel{5} \times \cancel{4} \times \cancel{3} \times \cancel{2} \times \cancel{1}}$
 $= \frac{15!}{11!}$

g $\frac{9 \times 8 \times 7}{3 \times 2 \times 1} = \frac{9 \times 8 \times 7 \times \cancel{6} \times \cancel{5} \times \cancel{4} \times \cancel{3} \times \cancel{2} \times \cancel{1}}{3 \times 2 \times 1 \times \cancel{6} \times \cancel{5} \times \cancel{4} \times \cancel{3} \times \cancel{2} \times \cancel{1}}$
 $= \frac{9!}{3!6!}$

h $\frac{13 \times 12 \times 11 \times 10}{4 \times 3 \times 2 \times 1} = \frac{13 \times 12 \times 11 \times 10 \times \cancel{9} \times \cancel{8} \times \cancel{7} \times \cancel{6} \times \cancel{5} \times \cancel{4} \times \cancel{3} \times \cancel{2} \times \cancel{1}}{4 \times 3 \times 2 \times 1 \times \cancel{9} \times \cancel{8} \times \cancel{7} \times \cancel{6} \times \cancel{5} \times \cancel{4} \times \cancel{3} \times \cancel{2} \times \cancel{1}}$
 $= \frac{13!}{4!9!}$

i $\frac{15 \times 14 \times 13 \times 12 \times 11}{5 \times 4 \times 3 \times 2 \times 1} = \frac{15 \times 14 \times 13 \times 12 \times 11 \times \cancel{10} \times \cancel{9} \times \cancel{8} \times \cancel{7} \times \cancel{6} \times \cancel{5} \times \cancel{4} \times \cancel{3} \times \cancel{2} \times \cancel{1}}{5 \times 4 \times 3 \times 2 \times 1 \times \cancel{10} \times \cancel{9} \times \cancel{8} \times \cancel{7} \times \cancel{6} \times \cancel{5} \times \cancel{4} \times \cancel{3} \times \cancel{2} \times \cancel{1}}$
 $= \frac{15!}{5!10!}$

3 a $\frac{7!}{6!} = \frac{7 \times \cancel{6!}}{\cancel{6!}}$
 $= 7$

b $\frac{8!}{6!} = \frac{8 \times 7 \times \cancel{6!}}{\cancel{6!}}$
 $= 56$

c $\frac{12!}{10!} = \frac{12 \times 11 \times \cancel{10!}}{\cancel{10!}}$
 $= 132$

d $\frac{120!}{119!} = \frac{120 \times \cancel{119!}}{\cancel{119!}}$
 $= 120$

e $\frac{10!}{8! \times 2!} = \frac{10 \times 9 \times \cancel{8!}}{\cancel{8!} \times 2 \times 1}$
 $= \frac{90}{2}$
 $= 45$

f $\frac{100!}{98! \times 2!} = \frac{100 \times 99 \times \cancel{98!}}{\cancel{98!} \times 2 \times 1}$
 $= \frac{9900}{2}$
 $= 4950$

$$\begin{aligned} \text{4 a } \frac{n!}{(n-1)!} &= \frac{n \times \cancel{(n-1)!}}{\cancel{(n-1)!}} \\ &= n, \quad n \geq 1 \end{aligned}$$

$$\begin{aligned} \text{b } \frac{(n+2)!}{n!} &= \frac{(n+2) \times (n+1) \times \cancel{n!}}{\cancel{n!}} \\ &= (n+2)(n+1), \quad n \geq 0 \end{aligned}$$

$$\begin{aligned} \text{c } \frac{(n+1)!}{(n-1)!} &= \frac{(n+1) \times n \times \cancel{(n-1)!}}{\cancel{(n-1)!}} \\ &= (n+1)n, \quad n \geq 1 \end{aligned}$$

INVESTIGATION 1

THE BINOMIAL EXPANSION

$$\begin{aligned} \text{1 } (a+b)^4 &= (a+b)(a+b)^3 \\ &= (a+b)(a^3 + 3a^2b + 3ab^2 + b^3) \\ &= a^4 + 3a^3b + 3a^2b^2 + ab^3 + a^3b + 3a^2b^2 + 3ab^3 + b^4 \\ &= a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4 \end{aligned}$$

$$\begin{aligned} \text{2 } (a+b)^5 &= (a+b)(a+b)^4 \\ &= (a+b)(a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4) \\ &= a^5 + 4a^4b + 6a^3b^2 + 4a^2b^3 + ab^4 + a^4b + 4a^3b^2 + 6a^2b^3 + 4ab^4 + b^5 \\ &= a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5 \end{aligned}$$

3 a When the terms are written in this order, the powers of b increase.

b Yes, as the powers of a decrease, the powers of b increase.

$$\begin{array}{ccccccc} \text{c } n=1 & & 1 & & 1 & & \\ n=2 & & 1 & 2 & 1 & & \\ n=3 & & 1 & 3 & 3 & 1 & \\ n=4 & & 1 & 4 & 6 & 4 & 1 \\ n=5 & & 1 & 5 & 10 & 10 & 5 & 1 \end{array}$$

4 a The values on the end of each row are always 1, and each of the remaining values is found by adding the two values diagonally above it.

$$\begin{array}{ccccccc} \text{b } & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & & \text{row 5} \\ & \swarrow & \searrow & \swarrow & \searrow & \swarrow & \searrow & \swarrow & \searrow & \swarrow & \searrow & \swarrow & \searrow & \\ 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 & & \text{row 6} \end{array}$$

$$\text{c } (a+b)^6 = a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6$$

$$\begin{aligned} \text{d } (a+b)^6 &= (a+b)(a+b)^5 \\ &= (a+b)(a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5) \\ &= a^6 + 5a^5b + 10a^4b^2 + 10a^3b^3 + 5a^2b^4 + ab^5 \\ &\quad + a^5b + 5a^4b^2 + 10a^3b^3 + 10a^2b^4 + 5ab^5 + b^6 \\ &= a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6 \quad \checkmark \end{aligned}$$

EXERCISE 1B

$$1 \quad (a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$

$$\begin{aligned} \mathbf{a} \quad (p + q)^3 \\ = p^3 + 3p^2q + 3pq^2 + q^3 \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad (x + 1)^3 \\ = x^3 + 3x^2(1)^1 + 3x(1)^2 + (1)^3 \\ = x^3 + 3x^2 + 3x + 1 \end{aligned}$$

$$\begin{aligned} \mathbf{c} \quad (x - 3)^3 \\ = x^3 + 3x^2(-3) + 3x(-3)^2 + (-3)^3 \\ = x^3 - 9x^2 + 27x - 27 \end{aligned}$$

$$\begin{aligned} \mathbf{d} \quad (2 + x)^3 \\ = 2^3 + 3(2)^2x + 3(2)x^2 + x^3 \\ = 8 + 12x + 6x^2 + x^3 \end{aligned}$$

$$\begin{aligned} \mathbf{e} \quad (3x - 1)^3 = (3x)^3 + 3(3x)^2(-1) + 3(3x)(-1)^2 + (-1)^3 \\ = 27x^3 - 27x^2 + 9x - 1 \end{aligned}$$

$$\begin{aligned} \mathbf{f} \quad (2x + 5)^3 = (2x)^3 + 3(2x)^2(5) + 3(2x)(5)^2 + (5)^3 \\ = 8x^3 + 60x^2 + 150x + 125 \end{aligned}$$

$$\begin{aligned} \mathbf{g} \quad (2a - b)^3 = (2a)^3 + 3(2a)^2(-b) + 3(2a)(-b)^2 + (-b)^3 \\ = 8a^3 - 12a^2b + 6ab^2 - b^3 \end{aligned}$$

$$\begin{aligned} \mathbf{h} \quad (3x - \frac{1}{3})^3 = (3x)^3 + 3(3x)^2(-\frac{1}{3}) + 3(3x)(-\frac{1}{3})^2 + (-\frac{1}{3})^3 \\ = 27x^3 - 9x^2 + x - \frac{1}{27} \end{aligned}$$

$$\begin{aligned} \mathbf{i} \quad \left(2x + \frac{1}{x}\right)^3 = (2x)^3 + 3(2x)^2\left(\frac{1}{x}\right) + 3(2x)\left(\frac{1}{x}\right)^2 + \left(\frac{1}{x}\right)^3 \\ = 8x^3 + 12x + \frac{6}{x} + \frac{1}{x^3} \end{aligned}$$

$$\begin{aligned} \mathbf{j} \quad (\sqrt{x} - 1)^3 = (\sqrt{x})^3 + 3(\sqrt{x})^2(-1) + 3(\sqrt{x})(-1)^2 + (-1)^3 \\ = x\sqrt{x} - 3x + 3\sqrt{x} - 1 \end{aligned}$$

$$\begin{aligned} \mathbf{k} \quad (x^2 + 2)^3 = (x^2)^3 + 3(x^2)^2(2) + 3(x^2)(2)^2 + (2)^3 \\ = x^6 + 6x^4 + 12x^2 + 8 \end{aligned}$$

$$\begin{aligned} \mathbf{l} \quad \left(x^2 - \frac{1}{x^2}\right)^3 = (x^2)^3 + 3(x^2)^2\left(-\frac{1}{x^2}\right) + 3(x^2)\left(-\frac{1}{x^2}\right)^2 + \left(-\frac{1}{x^2}\right)^3 \\ = x^6 - 3x^2 + \frac{3}{x^2} - \frac{1}{x^6} \end{aligned}$$

$$2 \quad (a + b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$$

$$\begin{aligned} \mathbf{a} \quad (1 + x)^4 = 1^4 + 4(1)^3x + 6(1)^2x^2 + 4(1)x^3 + x^4 \\ = 1 + 4x + 6x^2 + 4x^3 + x^4 \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad (p - q)^4 = p^4 + 4p^3(-q) + 6p^2(-q)^2 + 4p(-q)^3 + (-q)^4 \\ = p^4 - 4p^3q + 6p^2q^2 - 4pq^3 + q^4 \end{aligned}$$

$$\begin{aligned} \mathbf{c} \quad (x - 2)^4 = x^4 + 4x^3(-2) + 6x^2(-2)^2 + 4x(-2)^3 + (-2)^4 \\ = x^4 - 8x^3 + 24x^2 - 32x + 16 \end{aligned}$$

$$\begin{aligned} \text{d } (3-x)^4 &= (3)^4 + 4(3)^3(-x) + 6(3)^2(-x)^2 + 4(3)(-x)^3 + (-x)^4 \\ &= 81 - 108x + 54x^2 - 12x^3 + x^4 \end{aligned}$$

$$\begin{aligned} \text{e } (1+2x)^4 &= (1)^4 + 4(1)^3(2x) + 6(1)^2(2x)^2 + 4(1)(2x)^3 + (2x)^4 \\ &= 1 + 8x + 24x^2 + 32x^3 + 16x^4 \end{aligned}$$

$$\begin{aligned} \text{f } (2x-3)^4 &= (2x)^4 + 4(2x)^3(-3) + 6(2x)^2(-3)^2 + 4(2x)(-3)^3 + (-3)^4 \\ &= 16x^4 - 12 \times 8x^3 + 54 \times 4x^2 - 108 \times 2x + 81 \\ &= 16x^4 - 96x^3 + 216x^2 - 216x + 81 \end{aligned}$$

$$\begin{aligned} \text{g } (2x+b)^4 &= (2x)^4 + 4(2x)^3b + 6(2x)^2b^2 + 4(2x)b^3 + b^4 \\ &= 16x^4 + 32x^3b + 24x^2b^2 + 8xb^3 + b^4 \end{aligned}$$

$$\begin{aligned} \text{h } \left(x + \frac{1}{x}\right)^4 &= x^4 + 4x^3\left(\frac{1}{x}\right) + 6x^2\left(\frac{1}{x}\right)^2 + 4x\left(\frac{1}{x}\right)^3 + \left(\frac{1}{x}\right)^4 \\ &= x^4 + 4x^2 + 6 + \frac{4}{x^2} + \frac{1}{x^4} \end{aligned}$$

$$\begin{aligned} \text{i } \left(2x - \frac{1}{x}\right)^4 &= (2x)^4 + 4(2x)^3\left(-\frac{1}{x}\right) + 6(2x)^2\left(-\frac{1}{x}\right)^2 + 4(2x)\left(-\frac{1}{x}\right)^3 + \left(-\frac{1}{x}\right)^4 \\ &= 16x^4 - 32x^2 + 24 - \frac{8}{x^2} + \frac{1}{x^4} \end{aligned}$$

$$\begin{aligned} \text{3 a i } (a-b)^3 &= a^3 + 3a^2(-b) + 3a(-b)^2 + (-b)^3 \\ &= a^3 - 3a^2b + 3ab^2 - b^3 \end{aligned}$$

$$\begin{aligned} \text{ii } (a-b)^4 &= a^4 + 4a^3(-b) + 6a^2(-b)^2 + 4a(-b)^3 + (-b)^4 \\ &= a^4 - 4a^3b + 6a^2b^2 - 4ab^3 + b^4 \end{aligned}$$

b The terms are the same, except for their signs. The signs in the expansions of $(a+b)^3$ and $(a+b)^4$ are all positive, whereas the signs in the expansions of $(a-b)^3$ and $(a-b)^4$ start with a positive term and then alternate ($a > 0$, $b > 0$).

$$\begin{array}{ccccccccc} \text{4 a} & & 1 & & 4 & & 6 & & 4 & & 1 & \leftarrow \text{the 4th row} \\ & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & \leftarrow \text{the 5th row} \end{array}$$

$$\text{b } (a+b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$$

$$\begin{aligned} \text{c i } (x+2)^5 &= x^5 + 5x^4(2) + 10x^3(2)^2 + 10x^2(2)^3 + 5x(2)^4 + (2)^5 \\ &= x^5 + 10x^4 + 40x^3 + 80x^2 + 80x + 32 \end{aligned}$$

$$\begin{aligned} \text{ii } (1-x)^5 &= (1)^5 + 5(1)^4(-x) + 10(1)^3(-x)^2 + 10(1)^2(-x)^3 + 5(1)(-x)^4 + (-x)^5 \\ &= 1 - 5x + 10x^2 - 10x^3 + 5x^4 - x^5 \end{aligned}$$

$$\begin{aligned} \text{iii } (1+2x)^5 &= (1)^5 + 5(1)^4(2x) + 10(1)^3(2x)^2 + 10(1)^2(2x)^3 + 5(1)(2x)^4 + (2x)^5 \\ &= 1 + 10x + 40x^2 + 80x^3 + 80x^4 + 32x^5 \end{aligned}$$

$$\begin{aligned} \text{iv } (x-2y)^5 &= x^5 + 5x^4(-2y) + 10x^3(-2y)^2 + 10x^2(-2y)^3 + 5x(-2y)^4 + (-2y)^5 \\ &= x^5 - 10x^4y + 40x^3y^2 - 80x^2y^3 + 80xy^4 - 32y^5 \end{aligned}$$

$$\begin{aligned} \text{v } (x^2+1)^5 &= (x^2)^5 + 5(x^2)^4(1) + 10(x^2)^3(1)^2 + 10(x^2)^2(1)^3 + 5(x^2)(1)^4 + (1)^5 \\ &= x^{10} + 5x^8 + 10x^6 + 10x^4 + 5x^2 + 1 \end{aligned}$$

$$\begin{aligned}
 \text{vi } \left(x - \frac{1}{x}\right)^5 &= x^5 + 5x^4\left(-\frac{1}{x}\right) + 10x^3\left(-\frac{1}{x}\right)^2 + 10x^2\left(-\frac{1}{x}\right)^3 + 5x\left(-\frac{1}{x}\right)^4 + \left(-\frac{1}{x}\right)^5 \\
 &= x^5 - 5x^3 + 10x - \frac{10}{x} + \frac{5}{x^3} - \frac{1}{x^5}
 \end{aligned}$$

5 a

1	5	10	10	5	1	← the 5th row
1	6	15	20	15	6	1 ← the 6th row

b $(a + b)^6 = a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6$

c i $(x + 2)^6 = x^6 + 6x^5(2) + 15x^4(2)^2 + 20x^3(2)^3 + 15x^2(2)^4 + 6x(2)^5 + (2)^6$
 $= x^6 + 12x^5 + 60x^4 + 160x^3 + 240x^2 + 192x + 64$

ii $(2x - 1)^6 = (2x)^6 + 6(2x)^5(-1) + 15(2x)^4(-1)^2 + 20(2x)^3(-1)^3 + 15(2x)^2(-1)^4$
 $+ 6(2x)(-1)^5 + (-1)^6$
 $= 64x^6 - 6 \times 32x^5 + 15 \times 16x^4 - 20 \times 8x^3 + 15 \times 4x^2 - 6 \times 2x + 1$
 $= 64x^6 - 192x^5 + 240x^4 - 160x^3 + 60x^2 - 12x + 1$

iii $\left(x + \frac{1}{x}\right)^6$
 $= x^6 + 6x^5\left(\frac{1}{x}\right) + 15x^4\left(\frac{1}{x}\right)^2 + 20x^3\left(\frac{1}{x}\right)^3 + 15x^2\left(\frac{1}{x}\right)^4 + 6x\left(\frac{1}{x}\right)^5 + \left(\frac{1}{x}\right)^6$
 $= x^6 + 6x^4 + 15x^2 + 20 + \frac{15}{x^2} + \frac{6}{x^4} + \frac{1}{x^6}$

6 a $(1 + \sqrt{2})^3 = (1)^3 + 3(1)^2(\sqrt{2}) + 3(1)(\sqrt{2})^2 + (\sqrt{2})^3$
 $= 1 + 3\sqrt{2} + 3 \times 2 + 2 \times \sqrt{2}$
 $= 1 + 3\sqrt{2} + 6 + 2\sqrt{2}$
 $= 7 + 5\sqrt{2}$

b $(\sqrt{5} + 2)^4 = (\sqrt{5})^4 + 4(\sqrt{5})^3(2) + 6(\sqrt{5})^2(2)^2 + 4(\sqrt{5})(2)^3 + (2)^4$
 $= 25 + 8 \times 5\sqrt{5} + 24 \times 5 + 32\sqrt{5} + 16$
 $= 25 + 40\sqrt{5} + 120 + 32\sqrt{5} + 16$
 $= 161 + 72\sqrt{5}$

c $(2 - \sqrt{2})^5$
 $= (2)^5 + 5(2)^4(-\sqrt{2}) + 10(2)^3(-\sqrt{2})^2 + 10(2)^2(-\sqrt{2})^3 + 5(2)^1(-\sqrt{2})^4 + (-\sqrt{2})^5$
 $= 32 - 80\sqrt{2} + 160 - 80\sqrt{2} + 40 - 4\sqrt{2}$
 $= 232 - 164\sqrt{2}$

7 a $(2 + x)^6 = (2)^6 + 6(2)^5x + 15(2)^4x^2 + 20(2)^3x^3 + 15(2)^2x^4 + 6(2)x^5 + x^6$
 $= 64 + 192x + 240x^2 + 160x^3 + 60x^4 + 12x^5 + x^6$

b $(2.01)^6$ is obtained by letting $x = 0.01$

$$\begin{aligned}\therefore (2.01)^6 &= 64 + 192 \times (0.01) + 240 \times (0.01)^2 + 160 \times (0.01)^3 \\ &\quad + 60 \times (0.01)^4 + 12 \times (0.01)^5 + (0.01)^6 \\ &= 65.944\,160\,601\,201\end{aligned}$$

$$\begin{array}{r} 64 \\ 1.92 \\ 0.024 \\ 0.000\,16 \\ 0.000\,000\,6 \\ 0.000\,000\,001\,2 \\ + 0.000\,000\,000\,001 \\ \hline 65.944\,160\,601\,201 \end{array}$$

8 a $(2x + 3)(x + 1)^4$

$$\begin{aligned}&= (2x + 3)(x^4 + 4x^3 + 6x^2 + 4x + 1) \\ &= 2x^5 + 8x^4 + 12x^3 + 8x^2 + 2x + 3x^4 + 12x^3 + 18x^2 + 12x + 3 \\ &= 2x^5 + 11x^4 + 24x^3 + 26x^2 + 14x + 3\end{aligned}$$

b $(x - 1)(2x + 1)^3$

$$\begin{aligned}&= (x - 1)[(2x)^3 + 3(2x)^2(1) + 3(2x)(1)^2 + (1)^3] \\ &= (x - 1)(8x^3 + 12x^2 + 6x + 1) \\ &= 8x^4 + 12x^3 + 6x^2 + x - 8x^3 - 12x^2 - 6x - 1 \\ &= 8x^4 + 4x^3 - 6x^2 - 5x - 1\end{aligned}$$

9 a $(3a + b)^5 = (3a)^5 + 5(3a)^4b + 10(3a)^3b^2 + \dots$

\therefore the coefficient of a^3b^2 is $10 \times 3^3 = 270$

b $(2a + 3b)^6 = (2a)^6 + 6(2a)^5(3b) + 15(2a)^4(3b)^2 + 20(2a)^3(3b)^3 + \dots$

\therefore the coefficient of a^3b^3 is $20 \times 2^3 \times 3^3 = 4320$

ACTIVITY

1 Diagonal 1: 1

Diagonal 2: $1 + 1 = 2$

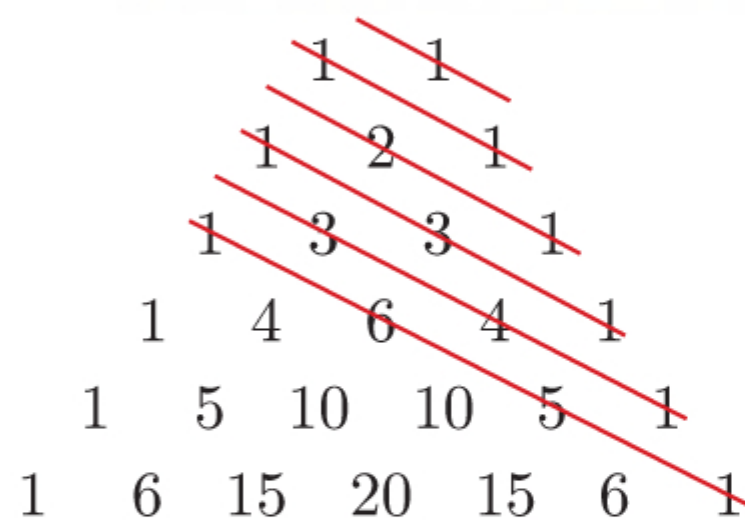
Diagonal 3: $2 + 1 = 3$

Diagonal 4: $1 + 3 + 1 = 5$

Diagonal 5: $3 + 4 + 1 = 8$

Diagonal 6: $1 + 6 + 5 + 1 = 13$

\vdots



2 The sequence of numbers formed by the answer to **1** is the Fibonacci sequence. The sum of the terms in the n th shallow diagonal ($n \geq 3$) is the sum of the terms in the $(n - 2)$ th and $(n - 1)$ th shallow diagonals.

For example, consider the term 6 in diagonal 6. It is obtained by adding together the 3 in diagonal 4 and the 3 in diagonal 5. We could repeat this process for the other terms in diagonal 6.

INVESTIGATION 2**THE BINOMIAL COEFFICIENT****PART 1: COUNTING**

- 1** **a** **i** There are 8 options for who can be listed third.
 ii There are 7 options for who can be listed fourth.
 iii There are 6 options for who can be listed fifth.
- b** The total number of orders in which the members can be listed is:
 options for first \times options for second \times ... \times options for tenth
 $= 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1$
 $= 10!$
- 2** **a** The total number of ways in which the first four members can be listed is:
 options for first \times options for second \times options for third \times options for fourth
 $= 10 \times 9 \times 8 \times 7$
- b** $10 \times 9 \times 8 \times 7 = \frac{10 \times 9 \times 8 \times 7 \times \cancel{6 \times 5 \times 4 \times 3 \times 2 \times 1}}{\cancel{6 \times 5 \times 4 \times 3 \times 2 \times 1}}$
 $= \frac{10!}{6!}$
- c** The 6 people *not* in the team can be ordered in $6 \times 5 \times 4 \times 3 \times 2 \times 1 = 6!$ ways.
- d** Since the order of the 6 people left out of the team is not important, we divided the total number of orders in which the members can be listed by $6!$
- 3** Of the 4 members who *are* in the team:
 options for first \times options for second \times options for third \times options for fourth
 $= 4 \times 3 \times 2 \times 1$
 $= 4!$
 There are $4! = 24$ ways in which the 4 members who *are* in the team can be ordered.
- 4** The total number of ways in which the first four members can be listed is
 $10 \times 9 \times 8 \times 7 = \frac{10!}{6!}$. {from **2**}
- Since the order of the four members in the team is not important, we divide by the number of possible orderings $4!$ {from **3**}
- \therefore the number of ways in which the team of four can be chosen is $\frac{10!}{4! \times 6!}$.
- 5** The total number of ways in which r members of a team of n can be listed is
 $\frac{n!}{(n-r)!}$. {similar to **4**}
- Since the order of the r members in the team is not important, we divide by the number of possible orderings $r!$ {similar to **4**}
- \therefore the number of ways in which the team of r can be chosen is $\frac{n!}{r! \times (n-r)!}$.

d For $\left(2x^2 - \frac{1}{x}\right)^{21}$, $a = (2x^2)$, $b = \left(-\frac{1}{x}\right)$, and $n = 21$

Now $T_{r+1} = \binom{n}{r} a^{n-r} b^r$ and letting $r = 8$ gives $T_9 = \binom{21}{8} (2x^2)^{13} \left(-\frac{1}{x}\right)^8$.

$$\begin{aligned} \mathbf{3} \quad (a-b)^n &= a^n + \binom{n}{1} a^{n-1}(-b) + \binom{n}{2} a^{n-2}(-b)^2 + \dots + \binom{n}{r} a^{n-r}(-b)^r + \dots + (-b)^n \\ &= \binom{n}{0} (-1)^0 a^n b^0 + \binom{n}{1} (-1)^1 a^{n-1} b^1 + \binom{n}{2} (-1)^2 a^{n-2} b^2 + \dots \\ &\quad + \binom{n}{r} (-1)^{n-r} a^n b^{n-r} + \dots + \binom{n}{n} (-1)^n a^0 b^n \\ &= \sum_{r=0}^n \binom{n}{r} (-1)^r a^{n-r} b^r \end{aligned}$$

4 a For $(x+2)^8$, $a = x$, $b = 2$, and $n = 8$

Now $T_{r+1} = \binom{n}{r} a^{n-r} b^r$

$$\therefore T_{r+1} = \binom{8}{r} x^{8-r} 2^r$$

b If $8 - r = 5$

then $r = 3$

$$\therefore T_4 = \binom{8}{3} x^5 2^3$$

\therefore the coefficient of x^5 is $\binom{8}{3} 2^3 = 448$.

5 a For $(x+b)^7$, $a = x$, $b = b$, and $n = 7$

\therefore the general term $T_{r+1} = \binom{7}{r} x^{7-r} b^r$

b If $x^{7-r} = x^4$ then $7 - r = 4$

$$\therefore r = 3$$

$$\text{Now } T_4 = \binom{7}{3} x^4 b^3$$

\therefore the coefficient of x^4 is $\binom{7}{3} b^3 = 35b^3$

But the coefficient of x^4 is -280

$$\text{So, } 35b^3 = -280$$

$$\therefore b^3 = -8$$

$$\therefore b = \sqrt[3]{-8}$$

$$\therefore b = -2$$

6 a For $\left(x + \frac{2}{x^2}\right)^{15}$, $a = x$, $b = \left(\frac{2}{x^2}\right)$, and $n = 15$

Now $T_{r+1} = \binom{n}{r} a^{n-r} b^r$

$$= \binom{15}{r} x^{15-r} \left(\frac{2}{x^2}\right)^r$$

$$= \binom{15}{r} x^{15-r} \frac{2^r}{x^{2r}}$$

$$= \binom{15}{r} 2^r x^{15-3r}$$

The constant term does not contain x .

$$\therefore 15 - 3r = 0$$

$$\therefore r = 5$$

$$\text{so } T_6 = \binom{15}{5} 2^5 x^0$$

$$\therefore \text{ the constant term is } \binom{15}{5} 2^5 = 96\,096.$$

b For $\left(x - \frac{3}{x^2}\right)^9$, $a = x$, $b = \left(-\frac{3}{x^2}\right)$, and $n = 9$

$$\begin{aligned} \text{Now } T_{r+1} &= \binom{n}{r} a^{n-r} b^r \\ &= \binom{9}{r} x^{9-r} \left(-\frac{3}{x^2}\right)^r \\ &= \binom{9}{r} x^{9-r} \frac{(-3)^r}{x^{2r}} \\ &= \binom{9}{r} (-3)^r x^{9-3r} \end{aligned}$$

The constant term does not contain x .

$$\therefore 9 - 3r = 0$$

$$\therefore r = 3$$

$$\text{so } T_4 = \binom{9}{3} (-3)^3 x^0$$

$$\therefore \text{ the constant term is } \binom{9}{3} (-3)^3 = -2268.$$

7 a In $(3 + 2x^2)^{10}$, $a = 3$, $b = (2x^2)$, and $n = 10$

$$\begin{aligned} \text{Now } T_{r+1} &= \binom{n}{r} a^{n-r} b^r \\ &= \binom{10}{r} 3^{10-r} (2x^2)^r \\ &= \binom{10}{r} 3^{10-r} 2^r x^{2r} \end{aligned}$$

We now let $2r = 10$

$$\therefore r = 5$$

$$\text{So, } T_6 = \binom{10}{5} 3^5 2^5 x^{10}$$

$$\therefore \text{ the coefficient of } x^{10} \text{ is } \binom{10}{5} 3^5 2^5 = 1\,959\,552.$$

b In $\left(2x^2 - \frac{3}{x}\right)^6$, $a = (2x^2)$, $b = \left(-\frac{3}{x}\right)$, and $n = 6$

$$\begin{aligned} \text{Now } T_{r+1} &= \binom{n}{r} a^{n-r} b^r \\ &= \binom{6}{r} (2x^2)^{6-r} \left(-\frac{3}{x}\right)^r \\ &= \binom{6}{r} 2^{6-r} x^{12-2r} \frac{(-3)^r}{x^r} \\ &= \binom{6}{r} 2^{6-r} (-3)^r x^{12-3r} \end{aligned}$$

We now let $12 - 3r = 3$

$$\therefore 3r = 9$$

$$\therefore r = 3$$

$$\text{So, } T_4 = \binom{6}{3} 2^3 (-3)^3 x^3$$

$$\therefore \text{ the coefficient of } x^3 \text{ is } \binom{6}{3} 2^3 (-3)^3 = -4320.$$

c In $(2x^2 - 3y)^6$, $a = (2x^2)$, $b = (-3y)$, and $n = 6$

$$\begin{aligned}\text{Now } T_{r+1} &= \binom{n}{r} a^{n-r} b^r \\ &= \binom{6}{r} (2x^2)^{6-r} (-3y)^r \\ &= \binom{6}{r} 2^{6-r} x^{12-2r} (-3)^r y^r \\ &= \binom{6}{r} 2^{6-r} (-3)^r x^{12-2r} y^r\end{aligned}$$

We find r such that $12 - 2r = 6$ and $r = 3$
 $\therefore r = 3$ is the solution

So, $T_4 = \binom{6}{3} 2^3 (-3)^3 x^6 y^3$
 \therefore the coefficient of $x^6 y^3$ is $\binom{6}{3} 2^3 (-3)^3 = -4320$.

d In $\left(2x^2 - \frac{1}{x}\right)^{12}$, $a = (2x^2)$, $b = \left(-\frac{1}{x}\right)$, and $n = 12$

$$\begin{aligned}\text{Now } T_{r+1} &= \binom{n}{r} a^{n-r} b^r \\ &= \binom{12}{r} (2x^2)^{12-r} \left(-\frac{1}{x}\right)^r \\ &= \binom{12}{r} 2^{12-r} x^{24-2r} \frac{(-1)^r}{x^r} \\ &= \binom{12}{r} 2^{12-r} (-1)^r x^{24-3r}\end{aligned}$$

We now let $24 - 3r = 12$
 $\therefore 3r = 12$
 $\therefore r = 4$

So, $T_5 = \binom{12}{4} 2^8 (-1)^4 x^{12}$
 \therefore the coefficient of x^{12} is $\binom{12}{4} 2^8 = 126\,720$.

8 In $(x^2y - 2y^2)^6$, $a = (x^2y)$, $b = (-2y^2)$, and $n = 6$.

$$\begin{aligned}\text{Now } T_{r+1} &= \binom{n}{r} a^{n-r} b^r \\ &= \binom{6}{r} (x^2y)^{6-r} (-2y^2)^r \\ &= \binom{6}{r} x^{12-2r} y^{6-r} (-2)^r y^{2r} \\ &= \binom{6}{r} (-2)^r x^{12-2r} y^{6+r}\end{aligned}$$

Since x and y are raised to the same power,

$$\begin{aligned}12 - 2r &= 6 + r \\ \therefore 3r &= 6 \\ \therefore r &= 2 \\ T_3 &= \binom{6}{2} (-2)^2 x^8 y^8 \\ &= 60x^8 y^8\end{aligned}$$

9 $(1+x)^n$ has $T_3 = \binom{n}{2} 1^{n-2} x^2 = \binom{n}{2} x^2$ and $n \geq 2$

But this term is $36x^2 \therefore \binom{n}{2} = 36$

$$\therefore \frac{n(n-1)}{2} = 36$$

$$\therefore n(n-1) = 72$$

$$\therefore n^2 - n - 72 = 0$$

$$\therefore (n-9)(n+8) = 0$$

$$\therefore n = 9 \text{ or } -8$$

But $n \geq 2$, so $n = 9$

and $T_4 = \binom{n}{3} 1^{n-3} x^3$

$$\therefore T_4 = \binom{9}{3} x^3$$

$$= 84x^3$$

10 $T_{r+1} = \binom{n}{r} a^{n-r} b^r$ where $n = 10$, $a = (x^2)$, $b = \left(\frac{1}{ax}\right)$

$$= \binom{10}{r} (x^2)^{10-r} \left(\frac{1}{ax}\right)^r$$

$$= \binom{10}{r} x^{20-2r} \times \frac{1}{a^r x^r}$$

$$= \binom{10}{r} x^{20-3r} \times \frac{1}{a^r}$$

We let $20 - 3r = 11$ and $T_4 = \binom{10}{3} x^{11} \times \frac{1}{a^3}$

$$\therefore 3r = 9$$

$$\therefore r = 3$$

$$= \frac{\binom{10}{3}}{a^3} x^{11}$$

So, $\frac{\binom{10}{3}}{a^3} = 15$

$$\therefore \frac{120}{a^3} = 15$$

$$\therefore a^3 = 8$$

$$\therefore a = 2$$

11 a $(x+4)(x-3)^6$

$$= (x+4)[x^6 + \binom{6}{1} x^5(-3) + \binom{6}{2} x^4(-3)^2 + \binom{6}{3} x^3(-3)^3 + \dots]$$

$$= (x+4)(x^6 - 3\binom{6}{1} x^5 + \binom{6}{2}(-3)^2 x^4 + \binom{6}{3}(-3)^3 x^3 + \dots)$$

So, the terms containing x^4 are $\binom{6}{3}(-3)^3 x^4$ from (1)

and $4\binom{6}{2}(-3)^2 x^4$ from (2)

\therefore the coefficient of x^4 is $\binom{6}{3}(-3)^3 + 4\binom{6}{2}(-3)^2 = 0$

$$\begin{aligned}
 \text{b} \quad & (x+2)(x^2+1)^8 \\
 &= (x+2)[(x^2)^8 + \binom{8}{1}(x^2)^7(1) + \binom{8}{2}(x^2)^6(1)^2 + \dots + \binom{8}{6}(x^2)^2(1)^6 \\
 &\quad + \binom{8}{7}(x^2)^1(1)^7 + \binom{8}{8}(1)^8] \\
 &= (x+2)(x^{16} + \binom{8}{1}x^{14} + \binom{8}{2}x^{12} + \dots + \binom{8}{6}x^4 + \binom{8}{7}x^2 + \binom{8}{8})
 \end{aligned}$$

So, the only term containing x^5 is $\binom{8}{6}x^5$.

\therefore the coefficient of x^5 is $\binom{8}{6} = 28$.

$$\begin{aligned}
 \text{c} \quad & (2-x)(3x+1)^9 \\
 &= (2-x)[(3x)^9 + \binom{9}{1}(3x)^8(1) + \binom{9}{2}(3x)^7(1)^2 + \binom{9}{3}(3x)^6(1)^3 + \binom{9}{4}(3x)^5(1)^4 + \dots] \\
 &= (2-x)(3^9x^9 + \binom{9}{1}3^8x^8 + \binom{9}{2}3^7x^7 + \binom{9}{3}3^6x^6 + \binom{9}{4}3^5x^5 + \dots)
 \end{aligned}$$

So, the terms containing x^6 are $2\binom{9}{3}3^6x^6$ from (1)

and $-\binom{9}{4}3^5x^6$ from (2)

\therefore the term containing x^6 is $2\binom{9}{3}3^6x^6 - \binom{9}{4}3^5x^6 = 91\,854x^6$.

$$\begin{aligned}
 \text{12} \quad & (1+kx)^n = 1^n + \binom{n}{1}1^{n-1}(kx) + \binom{n}{2}1^{n-2}(kx)^2 + \dots \\
 &= 1 + \binom{n}{1}kx + \binom{n}{2}k^2x^2 + \dots
 \end{aligned}$$

$$\therefore \binom{n}{1}k = -12 \quad \text{and} \quad \binom{n}{2}k^2 = 60$$

$$\therefore nk = -12 \quad \text{and} \quad \frac{n(n-1)}{2}k^2 = 60$$

$$\therefore n(n-1)k^2 = 120$$

$$\text{But } k = -\frac{12}{n} \quad \therefore n(n-1)\frac{144}{n^2} = 120$$

$$\therefore 144(n-1) = 120n \quad \{n \geq 2\}$$

$$\therefore 144n - 120n = 144$$

$$\therefore 24n = 144$$

$$\therefore n = 6 \quad \text{and so } k = -2$$

$$\begin{array}{rcl}
 \text{13} \quad \text{a} & \begin{array}{cccc} 1 & 1 & & \end{array} & \leftarrow \text{row 1} \\
 & \begin{array}{cccc} 1 & 2 & 1 & \end{array} & \leftarrow \text{row 2} \\
 & \begin{array}{cccc} 1 & 3 & 3 & 1 \end{array} & \leftarrow \text{row 3} \\
 & \begin{array}{ccccc} 1 & 4 & 6 & 4 & 1 \end{array} & \leftarrow \text{row 4} \\
 & \begin{array}{ccccc} 1 & 5 & 10 & 10 & 5 & 1 \end{array} & \leftarrow \text{row 5}
 \end{array}$$

$$\begin{array}{rcl}
 \text{b} \quad \text{i} & \text{sum} = 1 + 1 & = 2 = 2^1 \\
 \text{ii} & \text{sum} = 1 + 2 + 1 & = 4 = 2^2 \\
 \text{iii} & \text{sum} = 1 + 3 + 3 + 1 & = 8 = 2^3 \\
 \text{iv} & \text{sum} = 1 + 4 + 6 + 4 + 1 & = 16 = 2^4 \\
 \text{v} & \text{sum} = 1 + 5 + 10 + 10 + 5 + 1 & = 32 = 2^5
 \end{array}$$

c The sum of the numbers in row n of Pascal's triangle is 2^n .

$$\begin{aligned}
 \text{d} \quad & (1+x)^n \\
 &= \binom{n}{0}1^n + \binom{n}{1}1^{n-1}x + \binom{n}{2}1^{n-2}x^2 + \dots + \binom{n}{n-1}1^1x^{n-1} + \binom{n}{n}x^n \\
 &= \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \dots + \binom{n}{n-1}x^{n-1} + \binom{n}{n}x^n \\
 &\quad \{\text{as all powers of 1 are 1}\} \quad \checkmark
 \end{aligned}$$

e i Letting $x = 1$ in **d**, $\text{LHS} = (1 + 1)^n = 2^n$

$$\text{and } \text{RHS} = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \binom{n}{3} + \dots + \binom{n}{n-1} + \binom{n}{n}$$

$$\therefore \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \dots + \binom{n}{n-1} + \binom{n}{n} = 2^n \quad \checkmark$$

ii Letting $x = -1$ gives $\text{LHS} = (1 + (-1))^n = 0$

$$\begin{aligned} \text{and } \text{RHS} &= \binom{n}{0} + \binom{n}{1}(-1) + \binom{n}{2}(-1)^2 + \binom{n}{3}(-1)^3 + \dots \\ &\quad + \binom{n}{n-1}(-1)^{n-1} + \binom{n}{n}(-1)^n \\ &= \binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \dots + (-1)^n \binom{n}{n} \end{aligned}$$

$$\therefore \binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \dots + (-1)^n \binom{n}{n} = 0 \quad \checkmark$$

f
$$\sum_{r=0}^n 2^r \binom{n}{r} = 2^0 \binom{n}{0} + 2^1 \binom{n}{1} + 2^2 \binom{n}{2} + \dots + 2^{n-1} \binom{n}{n-1} + 2^n \binom{n}{n}$$

Using **d**, $(1 + x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \dots + \binom{n}{n-1}x^{n-1} + \binom{n}{n}x^n$

Letting $x = 2$, $(1 + 2)^n = \binom{n}{0} + \binom{n}{1}2 + \binom{n}{2}2^2 + \dots + \binom{n}{n-1}2^{n-1} + \binom{n}{n}2^n$

$$\therefore 3^n = 2^0 \binom{n}{0} + 2^1 \binom{n}{1} + 2^2 \binom{n}{2} + \dots + 2^{n-1} \binom{n}{n-1} + 2^n \binom{n}{n}$$

$$\therefore \sum_{r=0}^n 2^r \binom{n}{r} = 3^n$$

14 a $(3 + x)^n = 3^n + \binom{n}{1}3^{n-1}x + \binom{n}{2}3^{n-2}x^2 + \binom{n}{3}3^{n-3}x^3 + \dots + \binom{n}{n-1}3x^{n-1} + x^n$

b Letting $x = 1$ in **a**,

$$\text{LHS} = (3 + 1)^n = 4^n$$

$$\text{and } \text{RHS} = 3^n + \binom{n}{1}3^{n-1} + \binom{n}{2}3^{n-2} + \binom{n}{3}3^{n-3} + \dots + 3n + 1$$

$$\therefore 3^n + \binom{n}{1}3^{n-1} + \binom{n}{2}3^{n-2} + \binom{n}{3}3^{n-3} + \dots + 3n + 1 = 4^n$$

REVIEW SET 1A

1 a $8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 8!$

b
$$10 \times 9 \times 8 = \frac{10 \times 9 \times 8 \times \cancel{7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}}{\cancel{7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}}$$

$$= \frac{10!}{7!}$$

2 a
$$\frac{n!}{(n-2)!} = \frac{n \times (n-1) \times \cancel{(n-2)!}}{\cancel{(n-2)!}}$$

$$= n(n-1), \quad n \geq 2$$

b
$$\frac{n! + (n+1)!}{n!} = \frac{n! + (n+1)n!}{n!}$$

$$= \frac{\cancel{n!}(1 + (n+1))}{\cancel{n!}}$$

$$= 1 + n + 1$$

$$= n + 2$$

3 a
$$(x + 3)^3 = x^3 + 3x^2(3) + 3x(3)^2 + 3^3$$

$$= x^3 + 9x^2 + 27x + 27$$

$$\begin{aligned}\text{b } (x-2)^5 &= x^5 + 5x^4(-2) + 10x^3(-2)^2 + 10x^2(-2)^3 + 5x(-2)^4 + (-2)^5 \\ &= x^5 - 10x^4 + 40x^3 - 80x^2 + 80x - 32\end{aligned}$$

$$4 \quad \text{a } \text{For } (2x+3)^9, \quad a = (2x), \quad b = 3, \quad \text{and } n = 9$$

$$\text{Now } T_{r+1} = \binom{n}{r} a^{n-r} b^r \quad \text{and letting } r = 4 \quad \text{gives } T_5 = \binom{9}{4} (2x)^5 3^4.$$

$$\text{b } \text{For } \left(3x - \frac{1}{x}\right)^{12}, \quad a = (3x), \quad b = \left(-\frac{1}{x}\right), \quad \text{and } n = 12$$

$$\text{Now } T_{r+1} = \binom{n}{r} a^{n-r} b^r \quad \text{and letting } r = 7 \quad \text{gives } T_8 = \binom{12}{7} (3x)^5 \left(-\frac{1}{x}\right)^7.$$

$$\begin{aligned}5 \quad \text{a } (5 + \sqrt{3})^3 &= 5^3 + 3(5)^2(\sqrt{3}) + 3(5)(\sqrt{3})^2 + (\sqrt{3})^3 \\ &= 125 + 75\sqrt{3} + 15 \times 3 + 3\sqrt{3} \\ &= 125 + 75\sqrt{3} + 45 + 3\sqrt{3} \\ &= 170 + 78\sqrt{3}\end{aligned}$$

$$\begin{aligned}\text{b } (x+3)(x-1)^4 &= (x+3)(x^4 - 4x^3 + 6x^2 - 4x + 1) \\ &= x^5 - 4x^4 + 6x^3 - 4x^2 + x + 3x^4 - 12x^3 + 18x^2 - 12x + 3 \\ &= x^5 - x^4 - 6x^3 + 14x^2 - 11x + 3\end{aligned}$$

$$6 \quad (4+x)^3 = 4^3 + 3(4)^2x + 3(4)x^2 + x^3 \\ = 64 + 48x + 12x^2 + x^3$$

$$(4.02)^3 \text{ is obtained by letting } x = 0.02$$

$$\begin{aligned}\therefore (4.02)^3 &= 64 + 48 \times 0.02 + 12 \times (0.02)^2 + (0.02)^3 \\ &= 64.964808\end{aligned}$$

$$\begin{array}{r} 64 \\ 0.96 \\ 0.0048 \\ + 0.000008 \\ \hline 64.964808 \end{array}$$

$$7 \quad \text{The sixth row of Pascal's triangle is } \quad 1 \quad 6 \quad 15 \quad 20 \quad 15 \quad 6 \quad 1$$

$$\therefore (a+b)^6 = a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6$$

$$\begin{aligned}\text{a } (x-3)^6 &= x^6 + 6x^5(-3) + 15x^4(-3)^2 + 20x^3(-3)^3 + 15x^2(-3)^4 + 6x(-3)^5 + (-3)^6 \\ &= x^6 - 18x^5 + 135x^4 - 540x^3 + 1215x^2 - 1458x + 729\end{aligned}$$

$$\begin{aligned}\text{b } \left(2 + \frac{1}{x}\right)^6 \\ &= (2)^6 + 6(2)^5 \left(\frac{1}{x}\right) + 15(2)^4 \left(\frac{1}{x}\right)^2 + 20(2)^3 \left(\frac{1}{x}\right)^3 + 15(2)^2 \left(\frac{1}{x}\right)^4 + 6(2) \left(\frac{1}{x}\right)^5 + \left(\frac{1}{x}\right)^6 \\ &= 64 + \frac{192}{x} + \frac{240}{x^2} + \frac{160}{x^3} + \frac{60}{x^4} + \frac{12}{x^5} + \frac{1}{x^6}\end{aligned}$$

8 In $\left(2x - \frac{3}{x^2}\right)^{12}$, $a = (2x)$, $b = \left(-\frac{3}{x^2}\right)$, $n = 12$

$$\begin{aligned}\text{Now } T_{r+1} &= \binom{n}{r} a^{n-r} b^r \\ &= \binom{12}{r} (2x)^{12-r} \left(-\frac{3}{x^2}\right)^r \\ &= \binom{12}{r} 2^{12-r} x^{12-r} \frac{(-3)^r}{x^{2r}} \\ &= \binom{12}{r} 2^{12-r} (-3)^r x^{12-3r}\end{aligned}$$

For the coefficient of x^{-6} we let $12 - 3r = -6$

$$\therefore 3r = 18$$

$$\therefore r = 6$$

$$\text{So, } T_7 = \binom{12}{6} 2^6 (-3)^6 x^{-6}$$

$$\therefore \text{ the coefficient of } x^{-6} \text{ is } \binom{12}{6} 2^6 (-3)^6 = 43\,110\,144.$$

9 $(2x + 3)(x - 2)^6$

$$= (2x + 3) \left[x^6 + \binom{6}{1} x^5 (-2) + \binom{6}{2} x^4 (-2)^2 + \dots \right]$$

So, the terms containing x^5 are $2 \binom{6}{2} (-2)^2 x^5$ from (1)

and $3 \binom{6}{1} (-2) x^5$ from (2)

$$\therefore \text{ the coefficient of } x^5 \text{ is } 8 \binom{6}{2} - 6 \binom{6}{1} = 84$$

10 $(1 + cx)(1 + x)^4 = (1 + cx) \left(1^4 + \binom{4}{1} 1^3 x + \binom{4}{2} 1^2 x^2 + \binom{4}{3} 1 x^3 + x^4 \right)$

So, the terms containing x^3 are $\binom{4}{3} x^3$ from (1)

and $c \binom{4}{2} x^3$ from (2)

$$\therefore \text{ the coefficient of } x^3 \text{ is } 4 + 6c$$

But the coefficient of x^3 is 22, so $4 + 6c = 22$

$$\therefore 6c = 18$$

$$\therefore c = 3$$

11 a $(2 + x)^n = 2^n + \binom{n}{1} 2^{n-1} x + \binom{n}{2} 2^{n-2} x^2 + \binom{n}{3} 2^{n-3} x^3 + \dots + \binom{n}{n-1} 2x^{n-1} + x^n$

b Letting $x = 1$ in **a**,

$$\text{LHS} = (2 + 1)^n = 3^n$$

$$\text{and RHS} = 2^n + \binom{n}{1} 2^{n-1} + \binom{n}{2} 2^{n-2} + \binom{n}{3} 2^{n-3} + \dots + 2n + 1$$

$$\therefore 2^n + \binom{n}{1} 2^{n-1} + \binom{n}{2} 2^{n-2} + \binom{n}{3} 2^{n-3} + \dots + 2n + 1 = 3^n$$

12 In $\left(2x + \frac{1}{ax^2}\right)^9$, $a = (2x)$, $b = \left(\frac{1}{ax^2}\right)$, and $n = 9$.

$$\begin{aligned}\text{Now } T_{r+1} &= \binom{n}{r} a^{n-r} b^r \\ &= \binom{9}{r} (2x)^{9-r} \left(\frac{1}{ax^2}\right)^r \\ &= \binom{9}{r} 2^{9-r} a^{-r} x^{9-3r}\end{aligned}$$

Letting $r = 2$, $T_3 = \binom{9}{2} 2^7 a^{-2} x^3$

But the coefficient of x^3 is 288

$$\therefore \frac{\binom{9}{2} 2^7}{a^2} = 288$$

$$\therefore \frac{4608}{a^2} = 288$$

$$\therefore a^2 = \frac{4608}{288}$$

$$\therefore a^2 = 16$$

$$\therefore a = \pm 4$$

REVIEW SET 1B

1 a $\frac{9!}{7!} = \frac{9 \times 8 \times \cancel{7!}}{\cancel{7!}}$
 $= 72$

b $\frac{8!}{3!5!} = \frac{8 \times 7 \times 6 \times \cancel{5!}}{3 \times 2 \times 1 \times \cancel{5!}}$
 $= \frac{336}{6}$
 $= 56$

2 a $7 \times 6 \times 5 \times 4 = \frac{7 \times 6 \times 5 \times 4 \times \cancel{3} \times \cancel{2} \times \cancel{1}}{\cancel{3} \times \cancel{2} \times \cancel{1}}$
 $= \frac{7!}{3!}$

b $\frac{11 \times 10 \times 9}{3 \times 2 \times 1} = \frac{11 \times 10 \times 9 \times \cancel{8} \times \cancel{7} \times \cancel{6} \times \cancel{5} \times \cancel{4} \times \cancel{3} \times \cancel{2} \times \cancel{1}}{\cancel{3!} \times \cancel{8} \times \cancel{7} \times \cancel{6} \times \cancel{5} \times \cancel{4} \times \cancel{3} \times \cancel{2} \times \cancel{1}}$
 $= \frac{11!}{3!8!}$

3 a $(x - 2y)^3 = x^3 + 3x^2(-2y) + 3x(-2y)^2 + (-2y)^3$
 $= x^3 - 6x^2y + 12xy^2 - 8y^3$

b $(3x + 2)^4 = (3x)^4 + 4(3x)^3(2) + 6(3x)^2(2)^2 + 4(3x)(2)^3 + (2)^4$
 $= 81x^4 + 216x^3 + 216x^2 + 96x + 16$

4 In $(2x + 5)^6$, $a = (2x)$, $b = 5$, and $n = 6$

$$\begin{aligned}\text{Now } T_{r+1} &= \binom{n}{r} a^{n-r} b^r \\ &= \binom{6}{r} (2x)^{6-r} 5^r \\ &= \binom{6}{r} 2^{6-r} x^{6-r} 5^r\end{aligned}$$

If $6 - r = 3$

then $r = 3$

$$\therefore T_4 = \binom{6}{3} 2^3 5^3 x^3$$

$$\therefore \text{the coefficient of } x^3 \text{ is } \binom{6}{3} 2^3 5^3 = 20\,000.$$

5 In $\left(2x^2 - \frac{1}{x}\right)^6$, $a = (2x^2)$, $b = \left(-\frac{1}{x}\right)$, and $n = 6$

Now $T_{r+1} = \binom{n}{r} a^{n-r} b^r$ For the constant term we let $12 - 3r = 0$
 $= \binom{6}{r} (2x^2)^{6-r} \left(-\frac{1}{x}\right)^r$ $\therefore r = 4$
 $= \binom{6}{r} 2^{6-r} x^{12-2r} (-1)^r x^{-r}$ $\therefore T_5 = \binom{6}{4} 2^2 (-1)^4 x^0$
 $= \binom{6}{r} 2^{6-r} (-1)^r x^{12-3r}$ \therefore the constant term is $\binom{6}{4} 2^2 (-1)^4 = 60$.

6 a $(2 - \sqrt{2})^6 = 2^6 + 6(2)^5(-\sqrt{2}) + 15(2)^4(-\sqrt{2})^2 + 20(2)^3(-\sqrt{2})^3$
 $+ 15(2)^2(-\sqrt{2})^4 + 6(2)(-\sqrt{2})^5 + (-\sqrt{2})^6$
 $= 64 - 192\sqrt{2} + 480 - 320\sqrt{2} + 240 - 48\sqrt{2} + 8$
 $= 792 - 560\sqrt{2}$

b $(x + 3)(2x + 1)^3 = (x + 3)[(2x)^3 + 3(2x)^2(1) + 3(2x)(1)^2 + (1)^3]$
 $= (x + 3)(8x^3 + 12x^2 + 6x + 1)$
 $= 8x^4 + 12x^3 + 6x^2 + x$
 $+ 24x^3 + 36x^2 + 18x + 3$
 $= 8x^4 + 36x^3 + 42x^2 + 19x + 3$

7 a $(2x - 7)^{10}$
 $= (2x)^{10} + \binom{10}{1}(2x)^9(-7) + \binom{10}{2}(2x)^8(-7)^2 + \dots + \binom{10}{9}(2x)(-7)^9 + (-7)^{10}$

b $\left(3x + \frac{4}{x}\right)^{13}$
 $= (3x)^{13} + \binom{13}{1}(3x)^{12}\left(\frac{4}{x}\right) + \binom{13}{2}(3x)^{11}\left(\frac{4}{x}\right)^2 + \dots + \binom{13}{12}(3x)\left(\frac{4}{x}\right)^{12} + \left(\frac{4}{x}\right)^{13}$

8 In $\left(\frac{3}{x^2} - 4x\right)^{10}$, $a = \left(\frac{3}{x^2}\right)$, $b = (-4x)$, and $n = 10$

Now $T_{r+1} = \binom{n}{r} a^{n-r} b^r$
 $= \binom{10}{r} \left(\frac{3}{x^2}\right)^{10-r} (-4x)^r$
 $= \binom{10}{r} \frac{3^{10-r}}{x^{20-2r}} (-4)^r x^r$
 $= \binom{10}{r} 3^{10-r} (-4)^r x^{3r-20}$

We now let $3r - 20 = 10$
 $\therefore 3r = 30$
 $\therefore r = 10$

So, $T_{11} = \binom{10}{10} 3^0 (-4)^{10} x^{10}$
 $= (-4)^{10} x^{10}$

\therefore the coefficient of x^{10} is $(-4)^{10} = 1\,048\,576$.

- 9** For $\left(3x^2 + \frac{1}{x}\right)^9$, $a = (3x^2)$, $b = \left(\frac{1}{x}\right)$, and $n = 9$

$$\begin{aligned} T_{r+1} &= \binom{n}{r} a^{n-r} b^r \\ &= \binom{9}{r} (3x^2)^{9-r} \left(\frac{1}{x}\right)^r \\ &= \binom{9}{r} 3^{9-r} x^{18-2r} x^{-r} \\ &= \binom{9}{r} 3^{9-r} x^{18-3r} \end{aligned}$$

- a** If $18 - 3r = 12$

$$\text{then } 3r = 6$$

$$\therefore r = 2$$

$$\therefore T_3 = \binom{9}{2} 3^7 x^{12}$$

\therefore the coefficient of x^{12} is

$$\binom{9}{2} 3^7 = 78\,732.$$

- b** If $18 - 3r = 0$

$$\text{then } 3r = 18$$

$$\therefore r = 6$$

$$\therefore T_7 = \binom{9}{6} 3^3 x^0$$

\therefore the constant term is

$$\binom{9}{6} 3^3 = 2268.$$

10 $(1 + kx)^n = 1^n + \binom{n}{1} 1^{n-1} (kx)^1 + \binom{n}{2} 1^{n-2} (kx)^2 + \dots$
 $= 1 + nkx + \binom{n}{2} k^2 x^2 + \dots$

$$\therefore nk = -4 \quad \text{and} \quad \binom{n}{2} k^2 = \frac{15}{2}$$

$$\therefore k = -\frac{4}{n} \quad \dots (*) \quad \therefore \frac{n(n-1)}{2} k^2 = \frac{15}{2}$$

$$\therefore n(n-1)k^2 = 15$$

$$\therefore n(n-1) \left(-\frac{4}{n}\right)^2 = 15 \quad \{\text{using } (*)\}$$

$$\therefore n(n-1) \left(\frac{16}{n^2}\right) = 15$$

$$\therefore 16(n-1) = 15n \quad \{n \geq 2\}$$

$$\therefore 16n - 16 = 15n$$

$$\therefore n = 16 \quad \text{and so } k = -\frac{4}{16} = -\frac{1}{4}$$

11 $(m - 2n)^{10} = m^{10} + \binom{10}{1} m^9 (-2n) + \binom{10}{2} m^8 (-2n)^2 + \dots + (-2n)^{10}$
 $= m^{10} - 20m^9 n + 45m^8 (4n^2) - \dots + 1024n^{10}$
 $= m^{10} - 20m^9 n + 180m^8 n^2 - \dots + 1024n^{10}$

$$\therefore k = 180$$

$$\begin{aligned} \mathbf{12} \quad \left(x^3 + \frac{q}{x^3}\right)^8 \quad \text{has} \quad T_{r+1} &= \binom{8}{r} (x^3)^{8-r} \left(\frac{q}{x^3}\right)^r \\ &= \binom{8}{r} x^{24-3r} \frac{q^r}{x^{3r}} \\ &= \binom{8}{r} x^{24-6r} q^r \end{aligned}$$

which has constant term $\binom{8}{4} q^4$ $\{24 - 6r = 0 \text{ when } r = 4\}$

$$\begin{aligned} \left(x^3 + \frac{q}{x^3}\right)^4 \quad \text{has} \quad T_{r+1} &= \binom{4}{r} (x^3)^{4-r} \left(\frac{q}{x^3}\right)^r \\ &= \binom{4}{r} x^{12-3r} q^r x^{-3r} \\ &= \binom{4}{r} x^{12-6r} q^r \end{aligned}$$

which has constant term $\binom{4}{2} q^2$ $\{12 - 6r = 0 \text{ when } r = 2\}$

$$\begin{aligned} \therefore \binom{8}{4} q^4 &= \binom{4}{2} q^2 \\ \therefore 70q^4 - 6q^2 &= 0 \\ \therefore q^2(70q^2 - 6) &= 0 \\ \therefore 70q^2 - 6 &= 0 \quad \{q = 0 \text{ gives a trivial solution}\} \\ \therefore q^2 &= \frac{6}{70} \\ &= \frac{3}{35} \\ \therefore q &= \pm \sqrt{\frac{3}{35}} \end{aligned}$$

Chapter 2

QUADRATIC FUNCTIONS

EXERCISE 2A

- 1 a** $y = 2x^2 - 4x + 10$ is a relationship between two variables x and y which is in the form $y = ax^2 + bx + c$ where a, b, c are constants, $a \neq 0$.
 $\therefore y = 2x^2 - 4x + 10$ is a quadratic function.
- b** $y = 8x + 3$ cannot be written in the form $y = ax^2 + bx + c$, $a \neq 0$.
 $\therefore y = 8x + 3$ is not a quadratic function.
- c** $y = -2x^2$ is a relationship between two variables x and y which is in the form $y = ax^2 + bx + c$ where a, b, c are constants, $a \neq 0$.
 $\therefore y = -2x^2$ is a quadratic function.
- d** $y = \frac{1}{3}x + 6 - x^2$ can be written as $y = -x^2 + \frac{1}{3}x + 6$.
 $\therefore y = \frac{1}{3}x + 6 - x^2$ is a relationship between two variables x and y which can be written in the form $y = ax^2 + bx + c$ where a, b, c are constants, $a \neq 0$.
 $\therefore y = \frac{1}{3}x + 6 - x^2$ is a quadratic function.
- e** $2y + x - 3 = 0$ cannot be written in the form $y = ax^2 + bx + c$, $a \neq 0$.
 $\therefore 2y + x - 3 = 0$ is not a quadratic function.
- f** $y - 2x^2 = 3x - 1$ can be written as $y = 2x^2 + 3x - 1$.
 $\therefore y - 2x^2 = 3x - 1$ is a relationship between two variables x and y which can be written in the form $y = ax^2 + bx + c$ where a, b, c are constants, $a \neq 0$.
 $\therefore y - 2x^2 = 3x - 1$ is a quadratic function.

2 a When $x = 1$,
 $y = 1^2 + 3(1) - 7$
 $= 1 + 3 - 7$
 $= -3$

c When $x = 3$,
 $y = 3(3)^2 - 2(3) - 5$
 $= 27 - 6 - 5$
 $= 16$

b When $x = -2$,
 $y = -2(-2)^2 + 5(-2) + 2$
 $= -8 - 10 + 2$
 $= -16$

d When $x = -1$,
 $y = -3(-1)^2 + 7(-1) - 2$
 $= -3 - 7 - 2$
 $= -12$

3 a $y = x^2 - 3x + 1$

x	-2	-1	0	1	2
y	11	5	1	-1	-1

c $y = 2x^2 - x + 3$

x	-4	-2	0	2	4
y	39	13	3	9	31

b $y = x^2 + 2x - 5$

x	-2	-1	0	1	2
y	-5	-6	-5	-2	3

d $y = -3x^2 + 2x + 4$

x	-4	-2	0	2	4
y	-52	-12	4	-4	-36

4 a When $x = 0$,

$$y = 2(0)^2 + 5 \\ = 5$$

$\therefore (0, 4)$ does not satisfy the function
 $y = 2x^2 + 5$.

c When $x = -1$,

$$y = -(-1)^2 + 2(-1) - 5 \\ = -1 - 2 - 5 \\ = -8$$

$\therefore (-1, -8)$ satisfies the function
 $y = -x^2 + 2x - 5$.

e When $x = 2$,

$$y = 3(2)^2 - 4(2) + 10 \\ = 12 - 8 + 10 \\ = 14$$

$\therefore (2, 10)$ does not satisfy the function
 $y = 3x^2 - 4x + 10$.

b When $x = 2$,

$$y = (2)^2 - 3(2) + 2 \\ = 4 - 6 + 2 \\ = 0$$

$\therefore (2, 0)$ satisfies the function
 $y = x^2 - 3x + 2$.

d When $x = 3$,

$$y = -2(3)^2 - 3 + 6 \\ = -18 - 3 + 6 \\ = -15$$

$\therefore (3, -15)$ satisfies the function
 $y = -2x^2 - x + 6$.

f When $x = 2$,

$$y = -\frac{1}{2}(2)^2 + 4(2) - 1 \\ = -2 + 8 - 1 \\ = 5$$

$\therefore (2, 5)$ satisfies the function
 $y = -\frac{1}{2}x^2 + 4x - 1$.

5 a If $y = 4$ then

$$x^2 + 3x + 6 = 4 \\ \therefore x^2 + 3x + 2 = 0 \\ \therefore (x + 1)(x + 2) = 0 \\ \therefore x = -1 \text{ or } -2$$

c If $y = -4$ then

$$x^2 - 6x + 1 = -4 \\ \therefore x^2 - 6x + 5 = 0 \\ \therefore (x - 1)(x - 5) = 0 \\ \therefore x = 1 \text{ or } 5$$

e If $y = 1$ then

$$\frac{1}{2}x^2 + \frac{5}{2}x - 2 = 1 \\ \therefore \frac{1}{2}x^2 + \frac{5}{2}x - 3 = 0 \\ \therefore x^2 + 5x - 6 = 0 \\ \therefore (x + 6)(x - 1) = 0 \\ \therefore x = -6 \text{ or } 1$$

b If $y = 3$ then

$$x^2 - 4x + 7 = 3 \\ \therefore x^2 - 4x + 4 = 0 \\ \therefore (x - 2)^2 = 0 \\ \therefore x = 2$$

d If $y = 4$ then

$$2x^2 + 5x + 1 = 4 \\ \therefore 2x^2 + 5x - 3 = 0 \\ \therefore (2x - 1)(x + 3) = 0 \\ \therefore x = \frac{1}{2} \text{ or } -3$$

f If $y = 2$ then

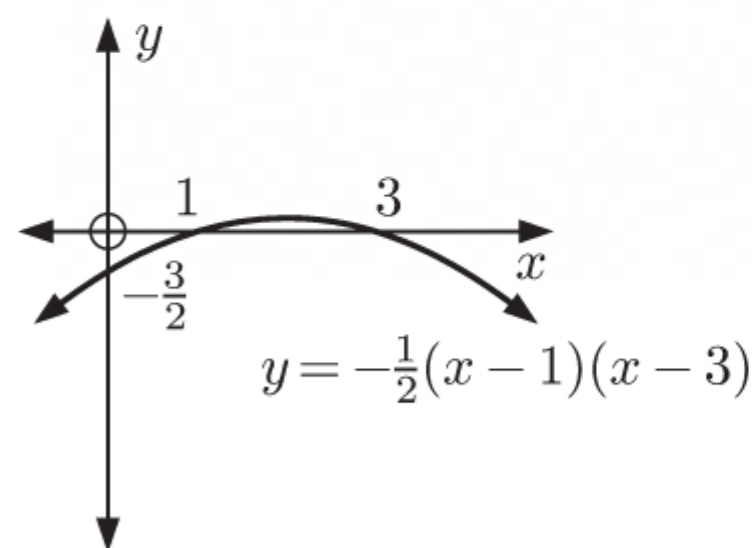
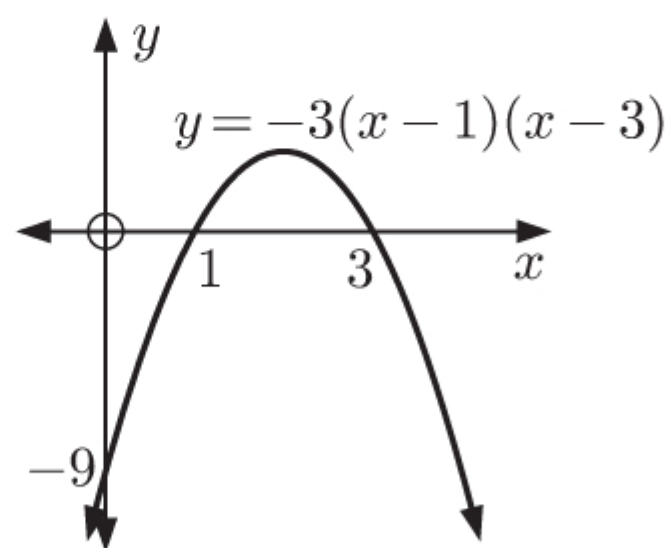
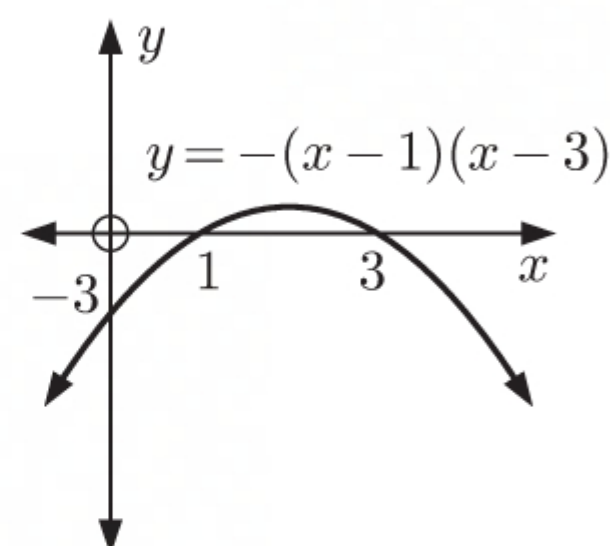
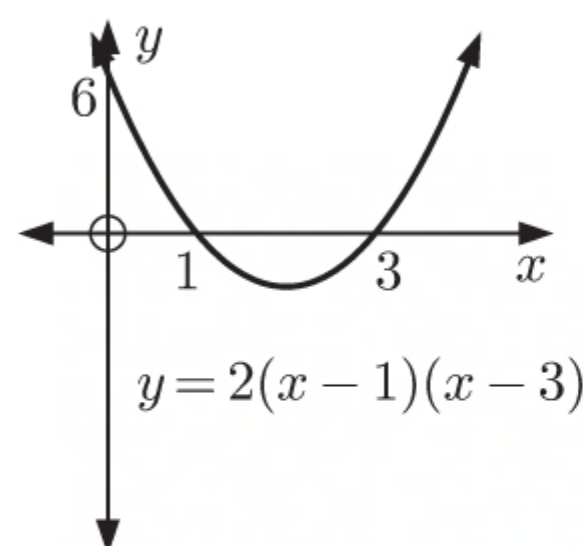
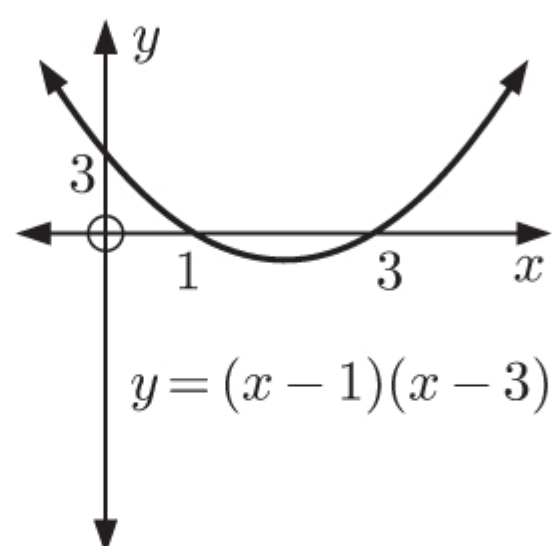
$$-\frac{1}{2}x^2 + 2x - 1 = 2 \\ \therefore -\frac{1}{2}x^2 + 2x - 3 = 0 \\ \therefore x^2 - 4x + 6 = 0 \\ \text{which has } \Delta = (-4)^2 - 4(1)(6) \\ = -8 < 0$$

\therefore the equation has no real solutions.

INVESTIGATION 1


GRAPHING $y = a(x - p)(x - q)$

1 a



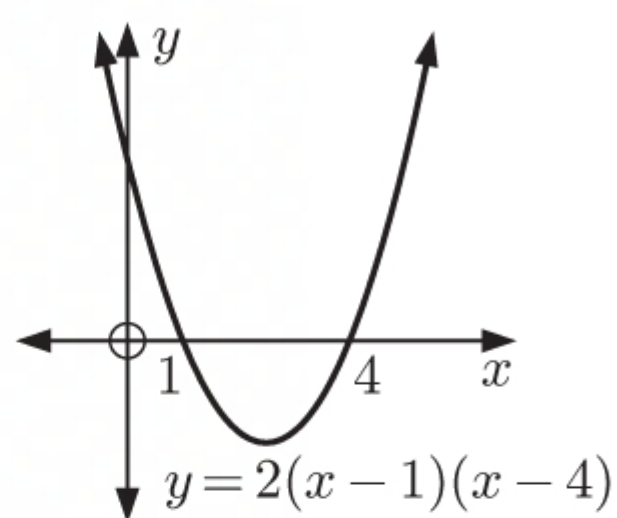
b The x -intercepts for each function in a are 1 and 3.

c For $y = a(x - 1)(x - 3)$: When $a > 0$ the shape is .

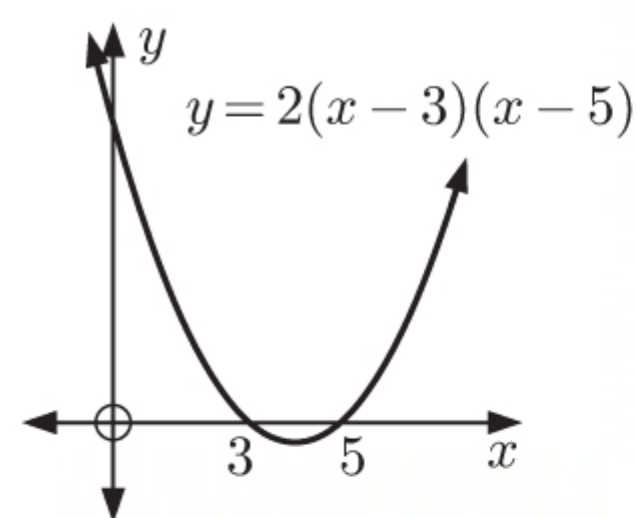
When $a < 0$ the shape is .

As $|a|$ increases, the graph becomes narrower.

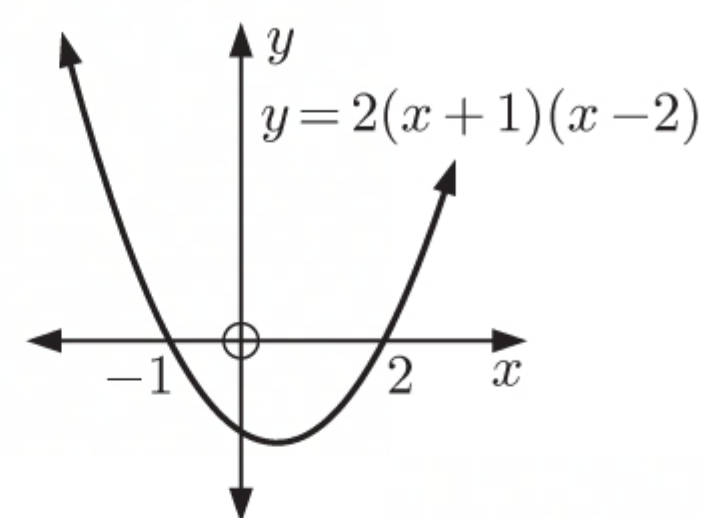
2 a, b



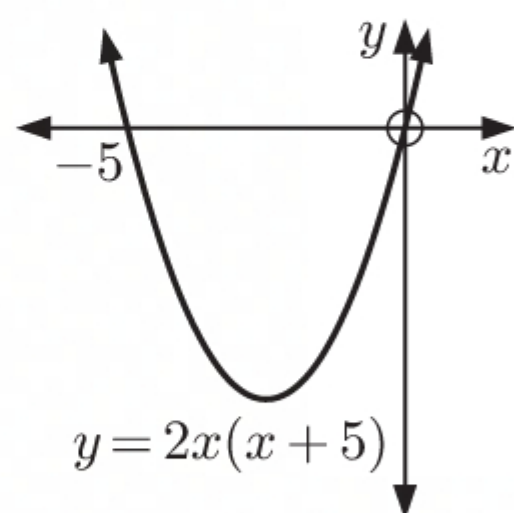
x -intercepts 1 and 4



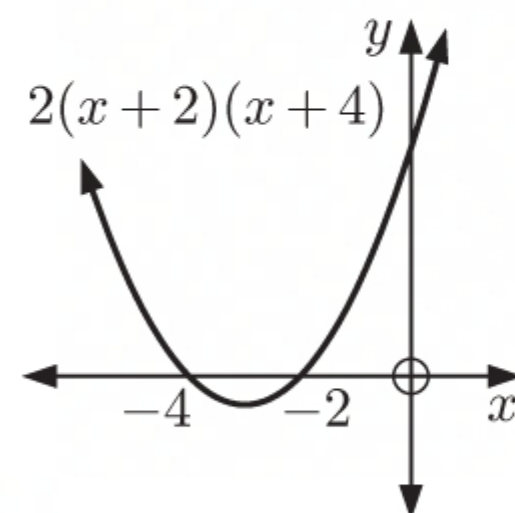
x -intercepts 3 and 5



x -intercepts -1 and 2

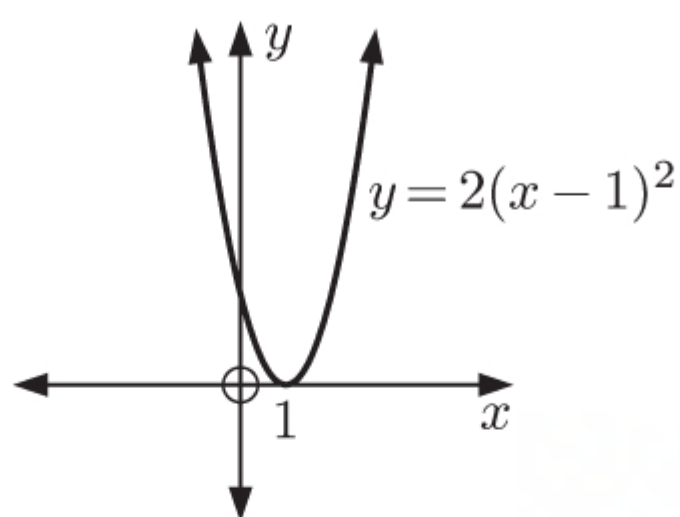
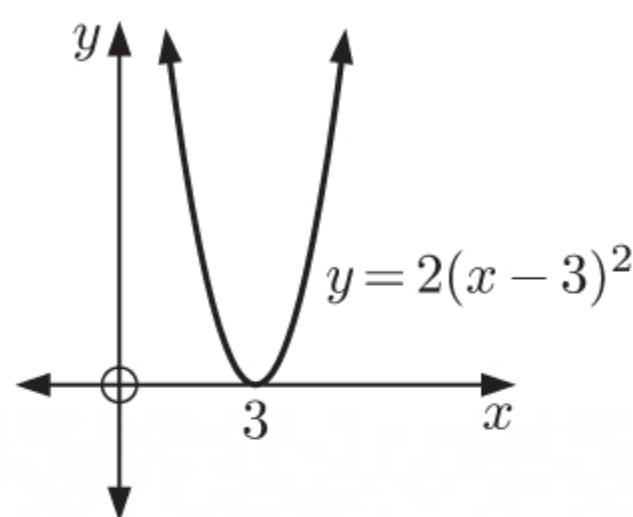
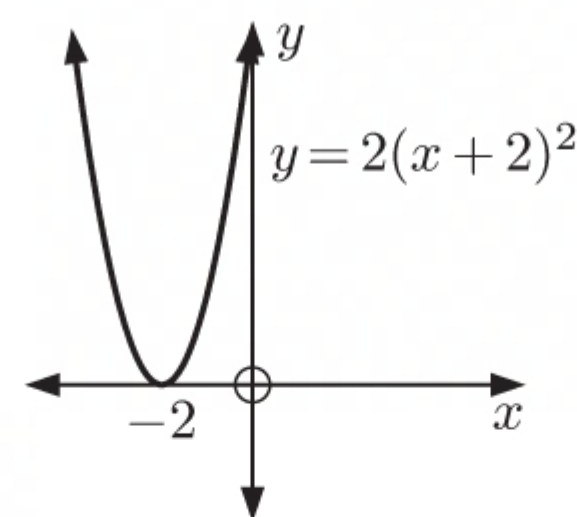
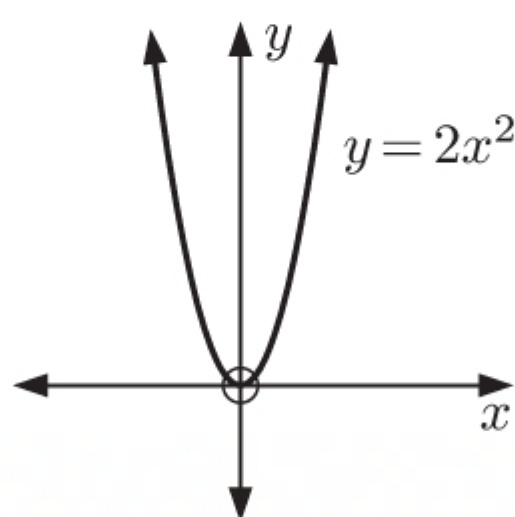


x -intercepts 0 and -5



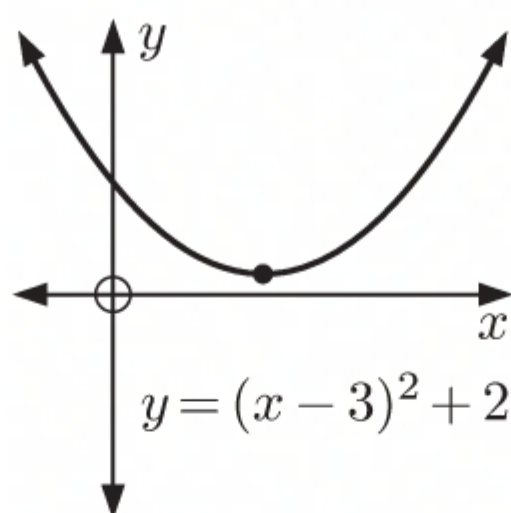
x -intercepts -2 and -4

c For $y = 2(x - p)(x - q)$, $p \neq q$, the graph has x -intercepts p and q , where it cuts the x -axis.

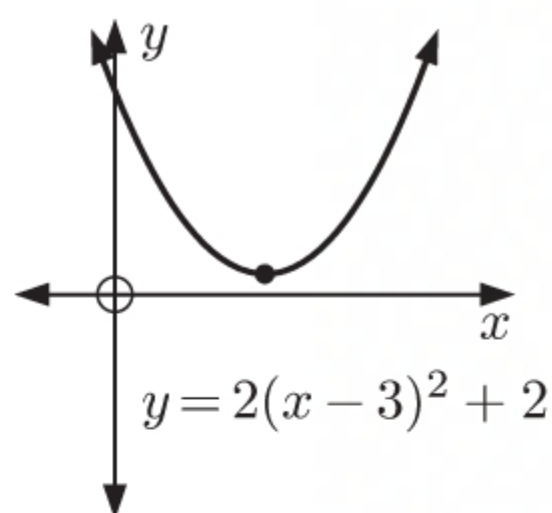
3 a, b x -intercept 1 x -intercept 3 x -intercept -2 x -intercept 0

c For $y = 2(x - p)^2$, the graph has x -intercept p , where it *touches* the x -axis.

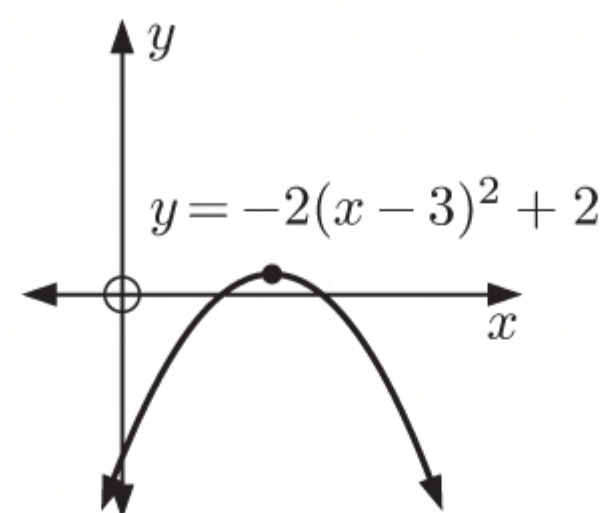
- 4**
- If a quadratic has the form $y = a(x - p)(x - q)$ then it cuts the x -axis at p and q .
 - If a quadratic has the form $y = a(x - p)^2$ then it touches the x -axis at p .

INVESTIGATION 2**GRAPHING $y = a(x - h)^2 + k$** **1 a, b**

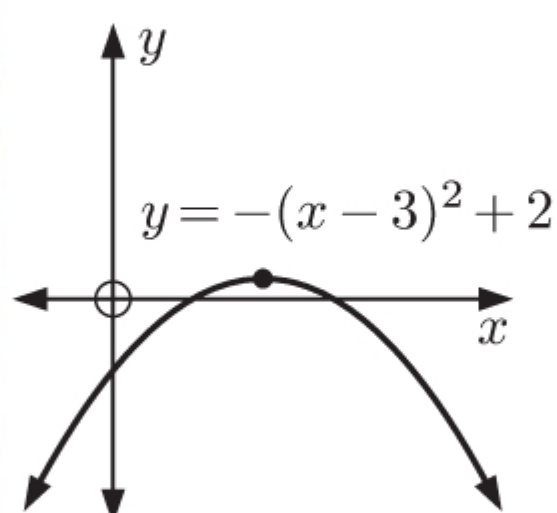
vertex is (3, 2)



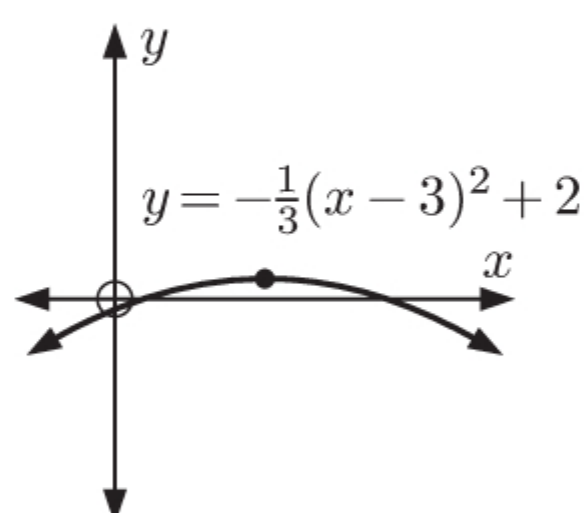
vertex is (3, 2)



vertex is (3, 2)




vertex is (3, 2)

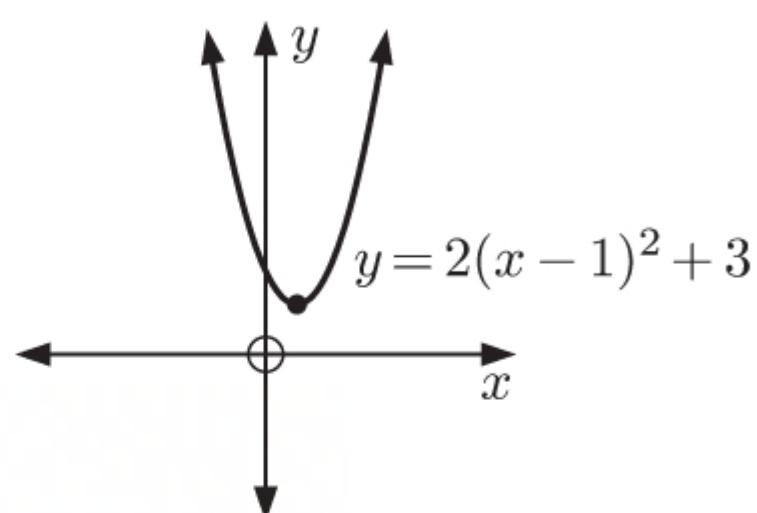
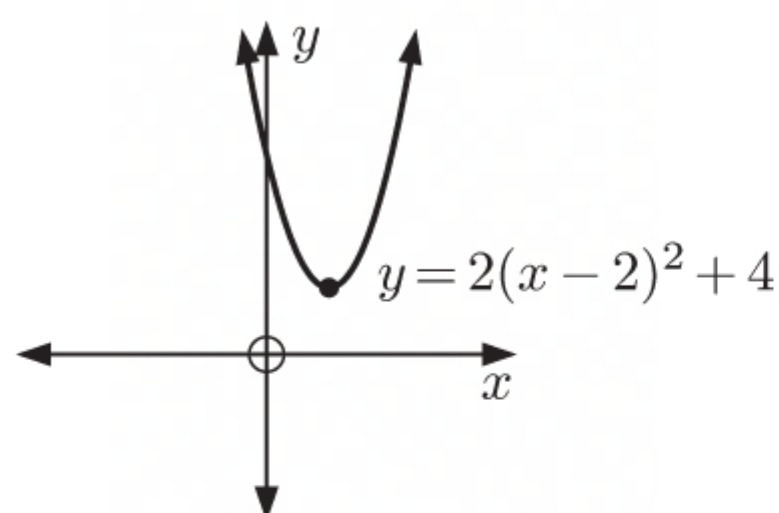
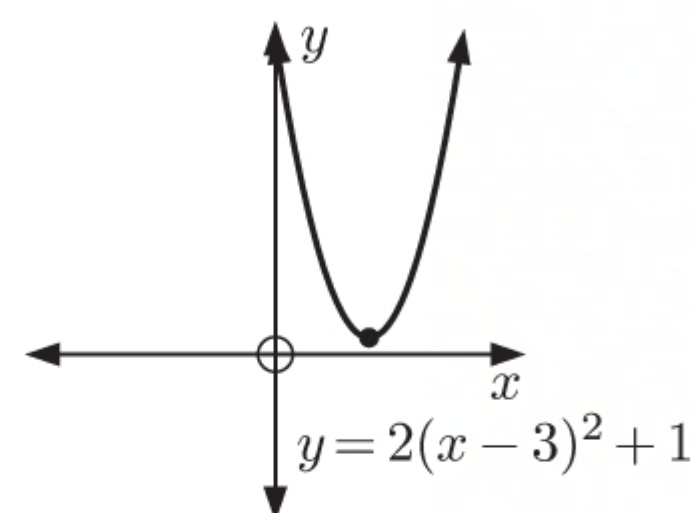
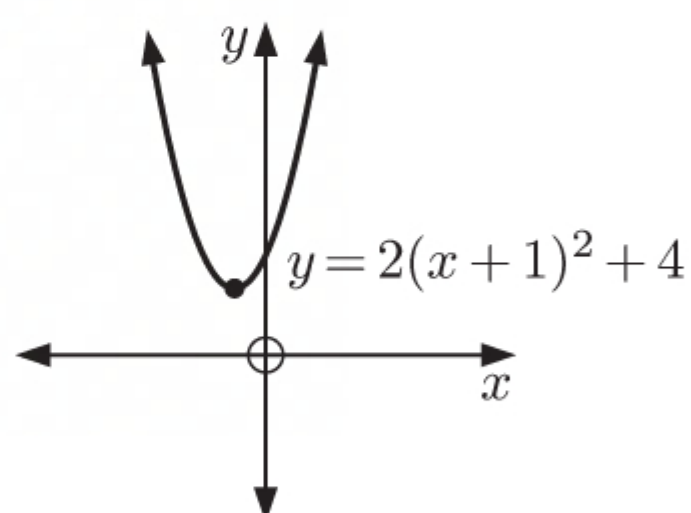
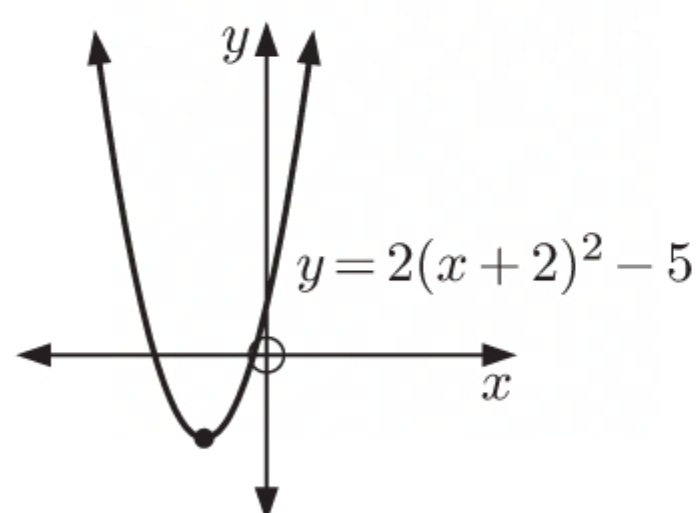
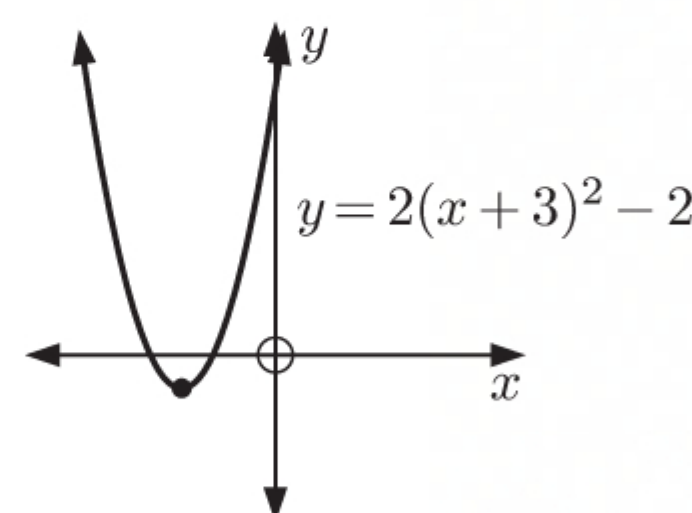


vertex is (3, 2)

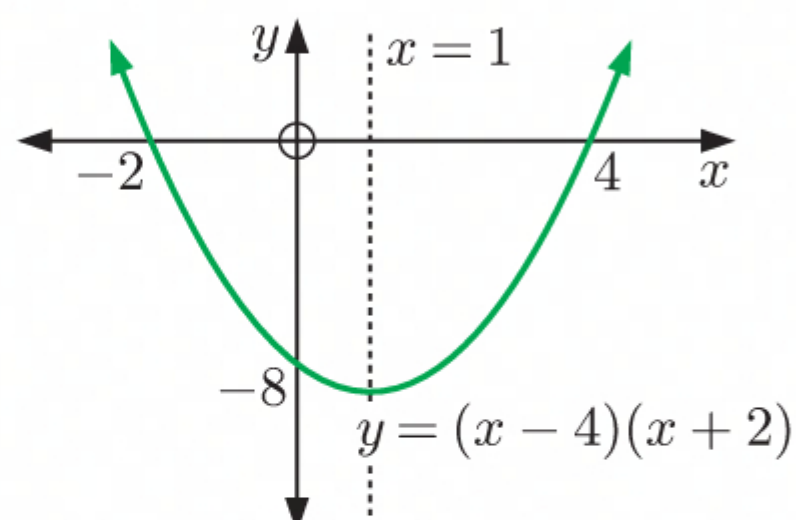
c For $y = a(x - 3)^2 + 2$: When $a > 0$ the shape is .

When $a < 0$ the shape is .

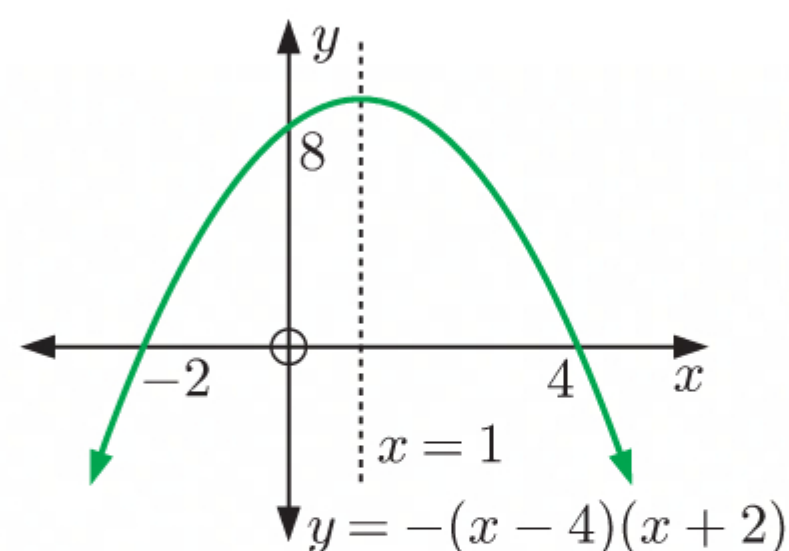
As $|a|$ increases, the graph becomes narrower.

2 a, bvertex is $(1, 3)$ vertex is $(2, 4)$ vertex is $(3, 1)$ vertex is $(-1, 4)$ vertex is $(-2, -5)$ vertex is $(-3, -2)$ **c** For $y = 2(x - h)^2 + k$, the vertex is at (h, k) .**3** If a quadratic has the form $y = a(x - h)^2 + k$ then its vertex has coordinates (h, k) .**EXERCISE 2B.1**

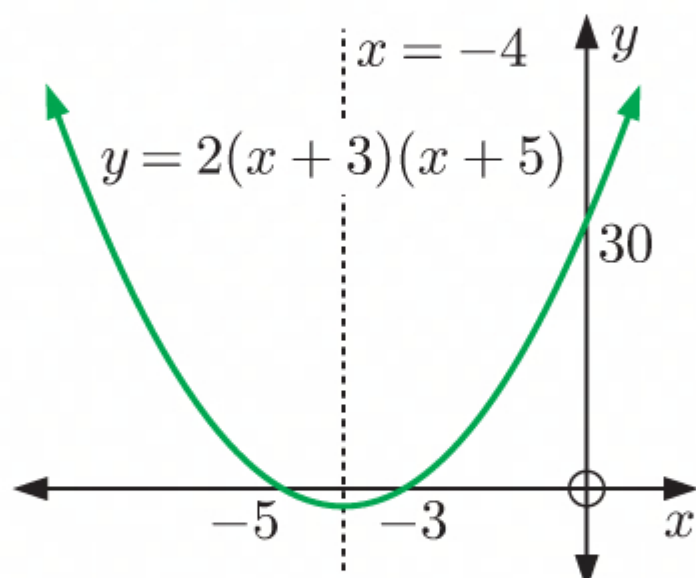
1 a $y = (x - 4)(x + 2)$
 has x -intercepts 4 and -2
 When $x = 0$, $y = (-4)(2)$
 $= -8$
 \therefore the y -intercept is -8 .



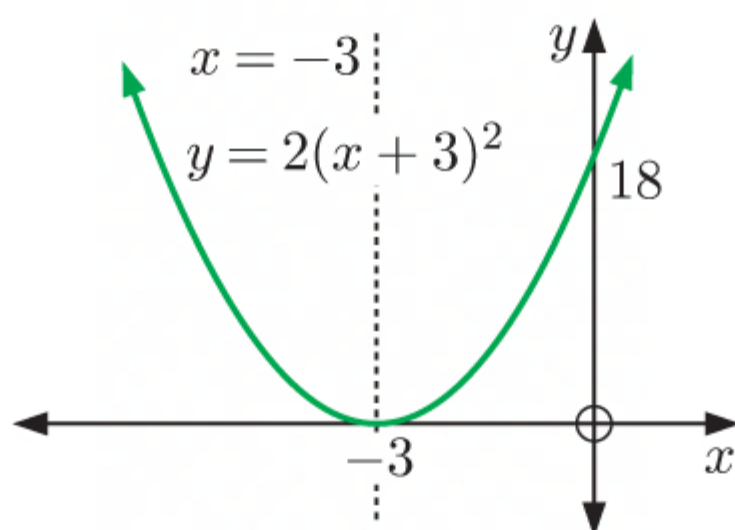
b $y = -(x - 4)(x + 2)$
 has x -intercepts 4 and -2
 When $x = 0$, $y = -(-4)(2)$
 $= 8$
 \therefore the y -intercept is 8.



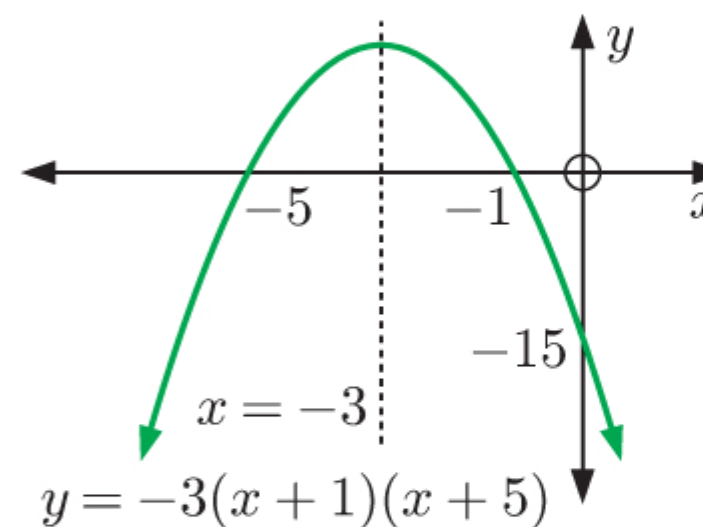
- c** $y = 2(x + 3)(x + 5)$
 has x -intercepts -3 and -5
 When $x = 0$, $y = 2(3)(5)$
 $= 30$
 \therefore the y -intercept is 30 .



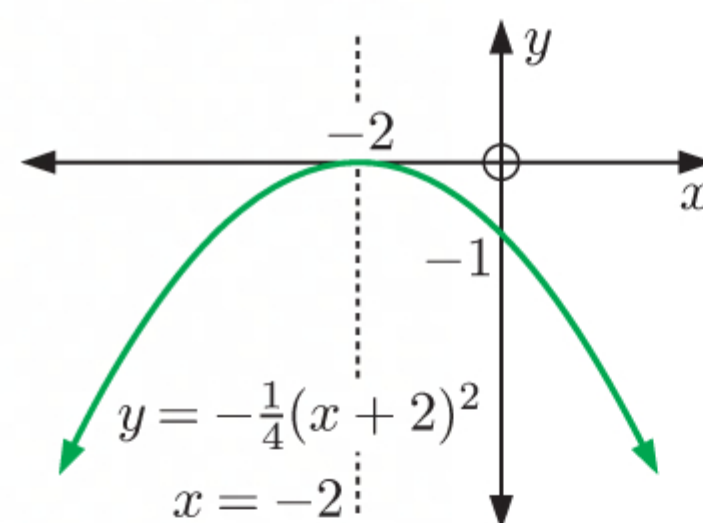
- e** $y = 2(x + 3)^2$
 touches the x -axis at -3
 When $x = 0$, $y = 2(3)^2$
 $= 18$
 \therefore the y -intercept is 18 .



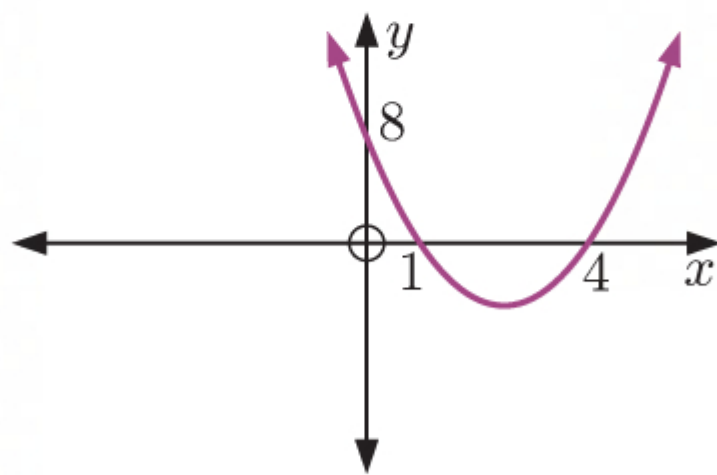
- d** $y = -3(x + 1)(x + 5)$
 has x -intercepts -1 and -5
 When $x = 0$, $y = -3(1)(5)$
 $= -15$
 \therefore the y -intercept is -15 .



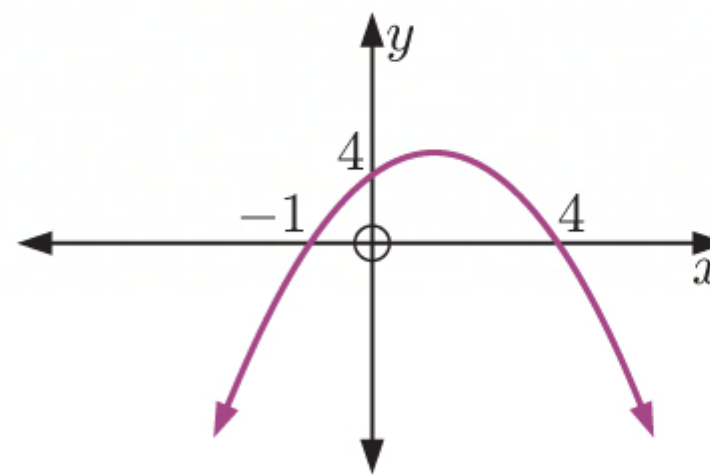
- f** $y = -\frac{1}{4}(x + 2)^2$
 touches the x -axis at -2
 When $x = 0$, $y = -\frac{1}{4}(2)^2$
 $= -1$
 \therefore the y -intercept is -1 .



- 2 a** $y = 2(x - 1)(x - 4)$
 has x -intercepts 1 and 4
 When $x = 0$, $y = 2(-1)(-4)$
 $= 8$
 \therefore the y -intercept is 8 .
 The only graph with these x and y -intercepts is **C**.



- b** $y = -(x + 1)(x - 4)$
 has x -intercepts -1 and 4
 When $x = 0$, $y = -(1)(-4)$
 $= 4$
 \therefore the y -intercept is 4 .
 The only graph with these x and y -intercepts is **E**.



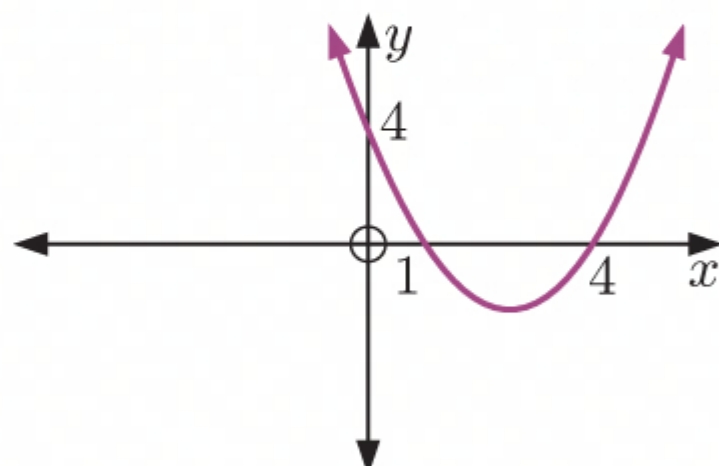
c $y = (x - 1)(x - 4)$

has x -intercepts 1 and 4

$$\text{When } x = 0, \quad y = (-1)(-4) = 4$$

\therefore the y -intercept is 4.

The only graph with these x and y -intercepts is **B**.



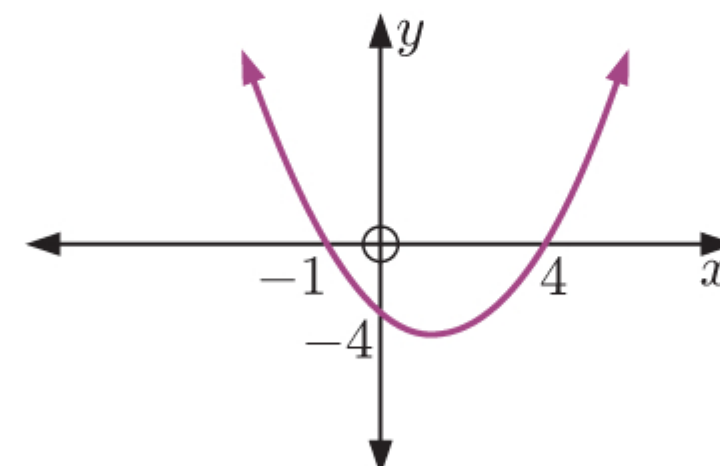
d $y = (x + 1)(x - 4)$

has x -intercepts -1 and 4

$$\text{When } x = 0, \quad y = (1)(-4) = -4$$

\therefore the y -intercept is -4.

The only graph with these x and y -intercepts is **F**.



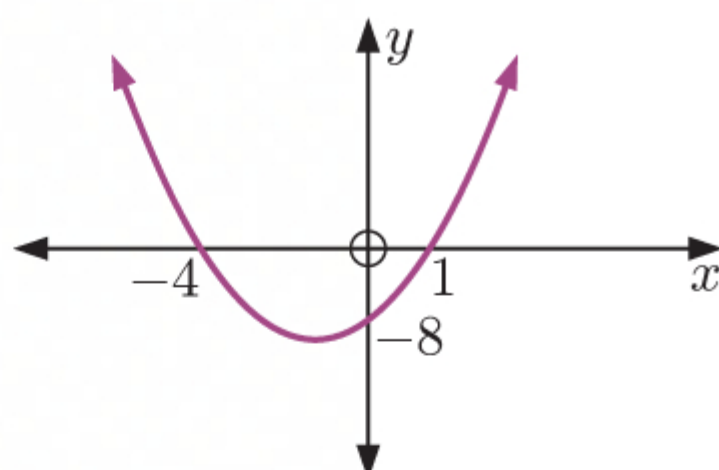
e $y = 2(x + 4)(x - 1)$

has x -intercepts -4 and 1

$$\text{When } x = 0, \quad y = 2(4)(-1) = -8$$

\therefore the y -intercept is -8.

The only graph with these x and y -intercepts is **G**.



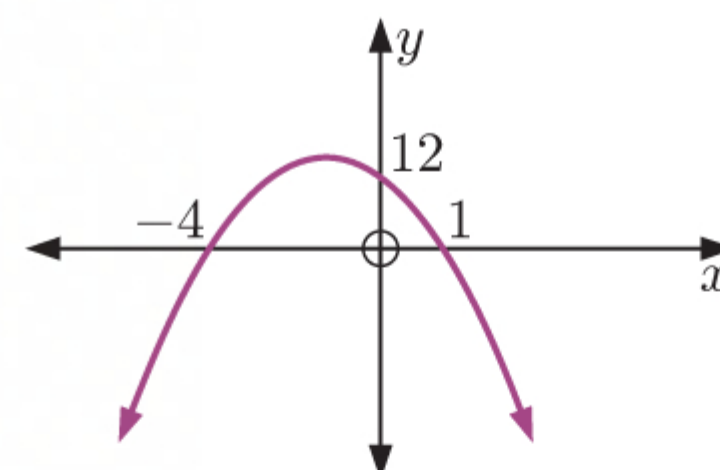
f $y = -3(x + 4)(x - 1)$

has x -intercepts -4 and 1

$$\text{When } x = 0, \quad y = -3(4)(-1) = 12$$

\therefore the y -intercept is 12.

The only graph with these x and y -intercepts is **H**.



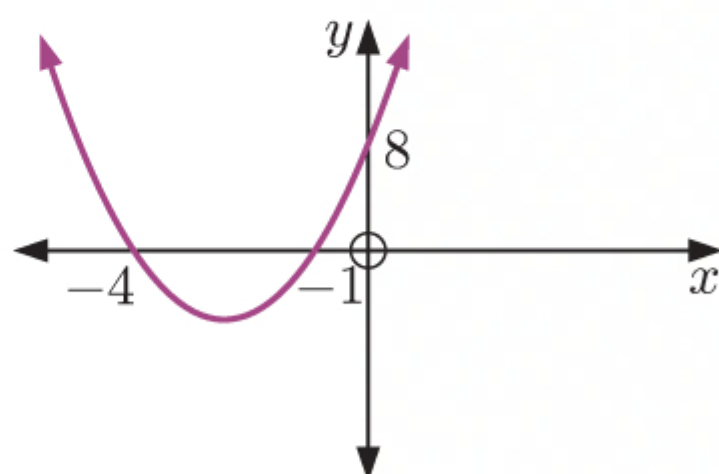
g $y = 2(x + 1)(x + 4)$

has x -intercepts -1 and -4

$$\text{When } x = 0, \quad y = 2(1)(4) = 8$$

\therefore the y -intercept is 8.

The only graph with these x and y -intercepts is **I**.



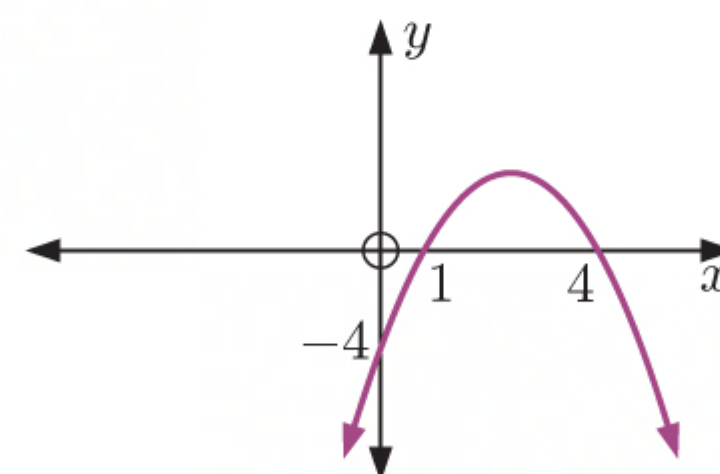
h $y = -(x - 1)(x - 4)$

has x -intercepts 1 and 4

$$\text{When } x = 0, \quad y = -(-1)(-4) = -4$$

\therefore the y -intercept is -4.

The only graph with these x and y -intercepts is **A**.



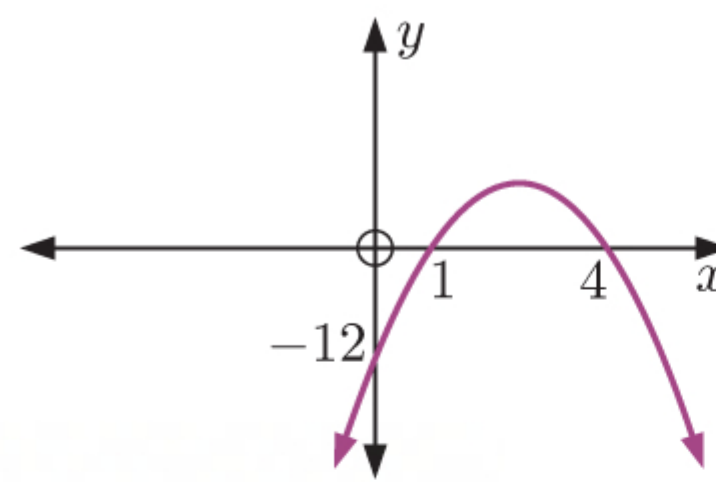
i $y = -3(x - 1)(x - 4)$

has x -intercepts 1 and 4

When $x = 0$, $y = -3(-1)(-4) = -12$

\therefore the y -intercept is -12 .


The only graph with these x and y -intercepts is **D**.

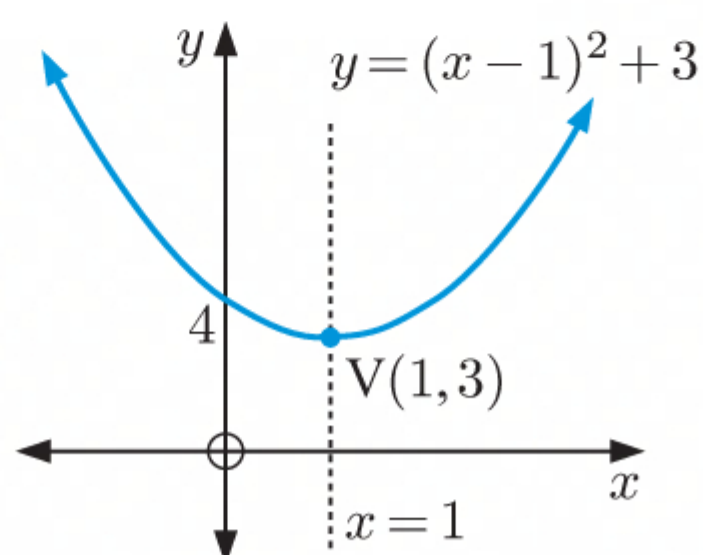


3 a $y = (x - 1)^2 + 3$ has vertex $(1, 3)$.

The axis of symmetry is $x = 1$.

When $x = 0$, $y = (-1)^2 + 3 = 4$


$a > 0$ so the shape is 

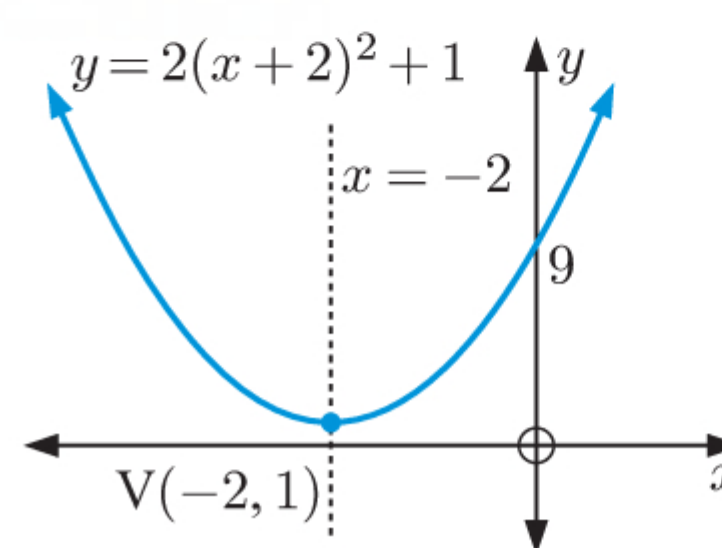


b $y = 2(x + 2)^2 + 1$ has vertex $(-2, 1)$.

The axis of symmetry is $x = -2$.

When $x = 0$, $y = 2(2)^2 + 1 = 9$


$a > 0$ so the shape is 

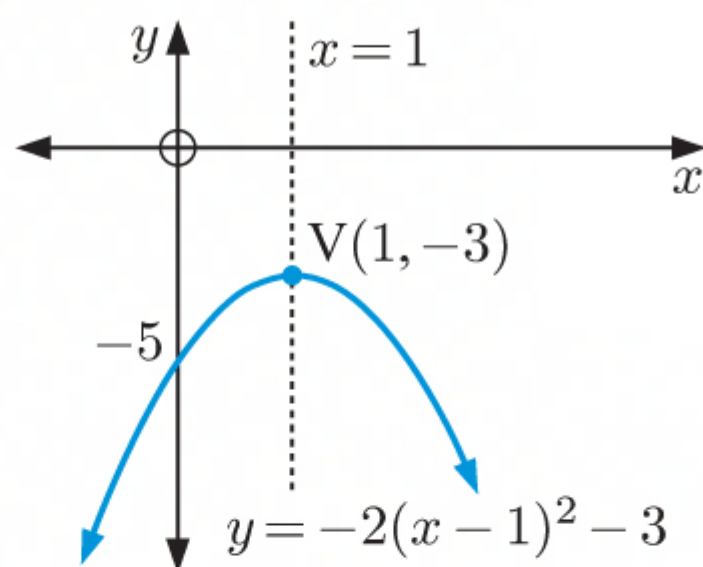


c $y = -2(x - 1)^2 - 3$ has vertex $(1, -3)$.

The axis of symmetry is $x = 1$.

When $x = 0$, $y = -2(-1)^2 - 3 = -5$


$a < 0$ so the shape is 

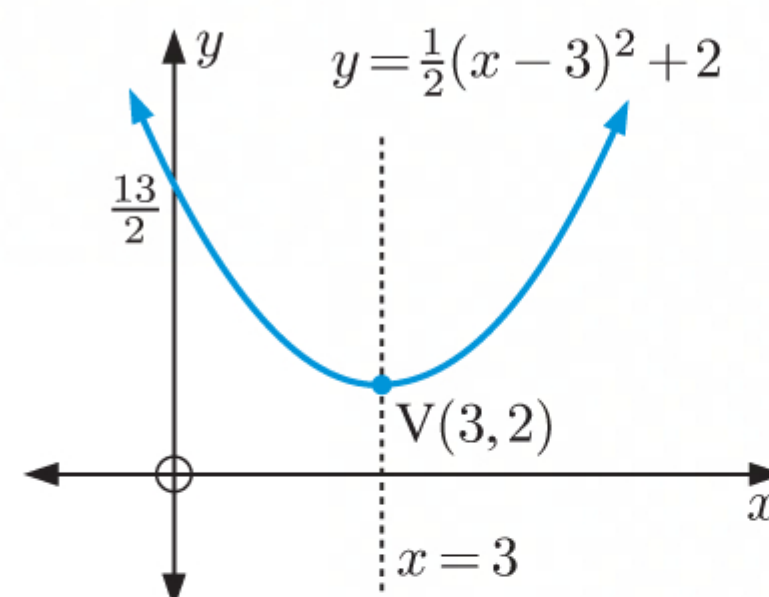


d $y = \frac{1}{2}(x - 3)^2 + 2$ has vertex $(3, 2)$.

The axis of symmetry is $x = 3$.

When $x = 0$, $y = \frac{1}{2}(-3)^2 + 2 = \frac{13}{2}$


$a > 0$ so the shape is 

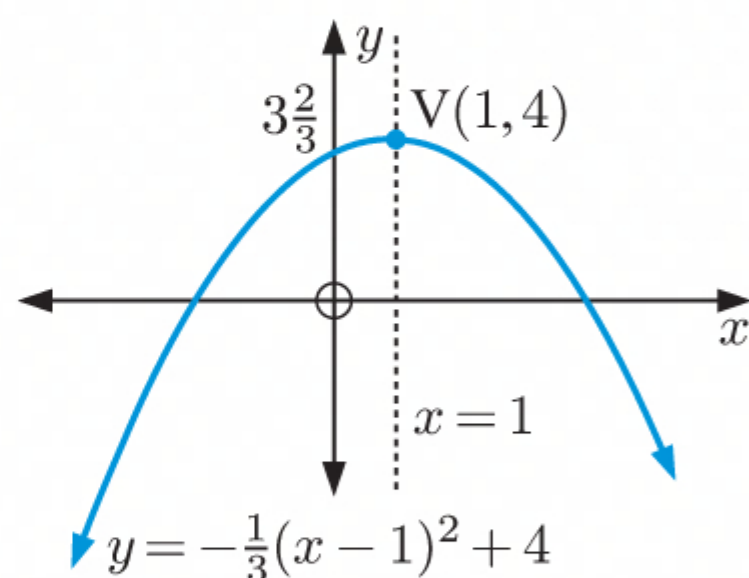


e $y = -\frac{1}{3}(x-1)^2 + 4$ has vertex $(1, 4)$.

The axis of symmetry is $x = 1$.

$$\begin{aligned}\text{When } x = 0, \quad y &= -\frac{1}{3}(-1)^2 + 4 \\ &= 3\frac{2}{3}\end{aligned}$$


$a < 0$ so the shape is 

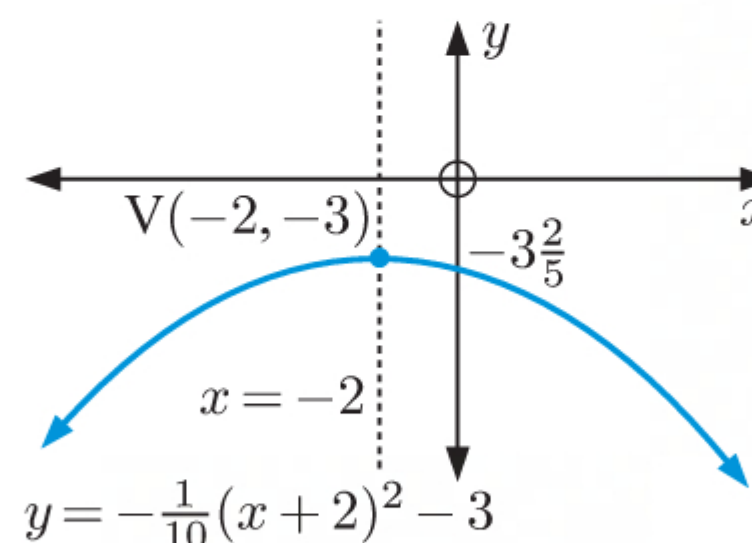


f $y = -\frac{1}{10}(x+2)^2 - 3$ has vertex $(-2, -3)$.

The axis of symmetry is $x = -2$.

$$\begin{aligned}\text{When } x = 0, \quad y &= -\frac{1}{10}(2)^2 - 3 \\ &= -3\frac{2}{5}\end{aligned}$$

$a < 0$ so the shape is 




4 a $y = -(x+1)^2 + 3$

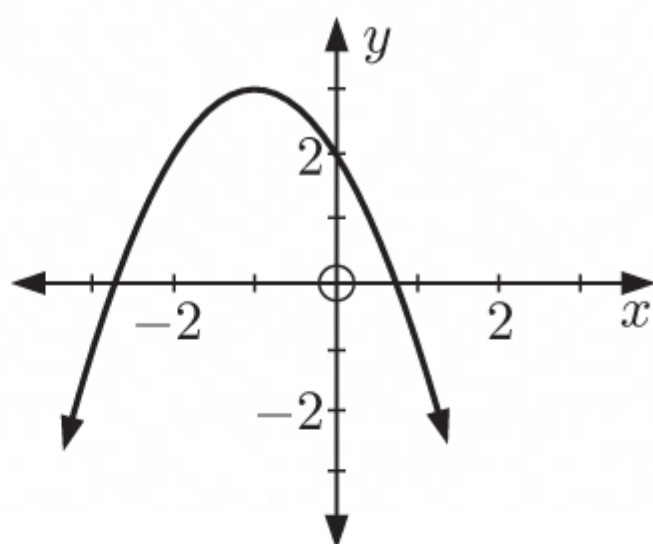
has vertex $(-1, 3)$.

The axis of symmetry is $x = -1$.

$$\begin{aligned}\text{When } x = 0, \quad y &= -(1)^2 + 3 \\ &= 2\end{aligned}$$

$a < 0$ so the shape is 

The only graph with all of these properties is **G**.




b $y = -2(x-3)^2 + 2$

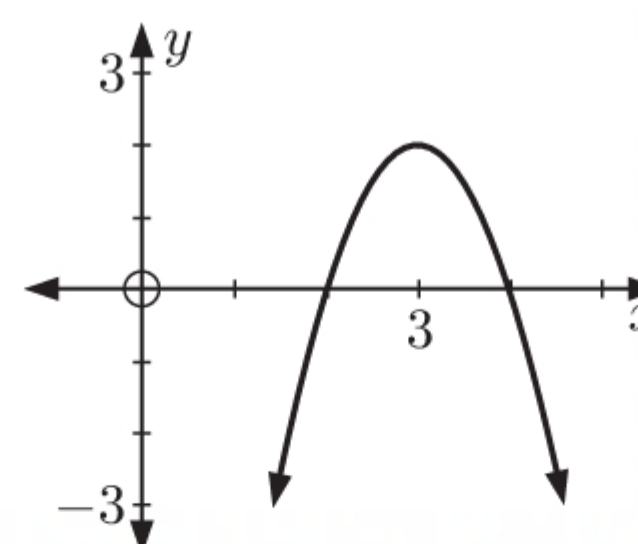
has vertex $(3, 2)$.

The axis of symmetry is $x = 3$.

$$\begin{aligned}\text{When } x = 0, \quad y &= -2(-3)^2 + 2 \\ &= -16\end{aligned}$$

$a < 0$ so the shape is 


The only graph with all of these properties is **A**.



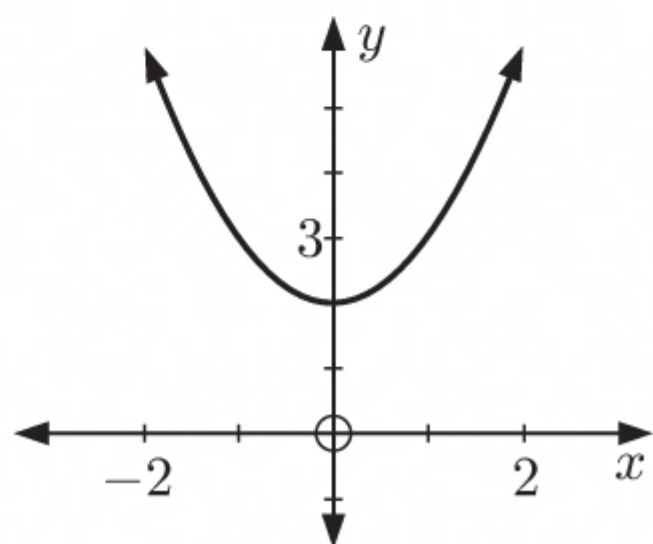
c $y = x^2 + 2$ has vertex $(0, 2)$.

The axis of symmetry is $x = 0$.

$$\text{When } x = 0, \quad y = 0^2 + 2 = 2$$

$a > 0$ so the shape is 


The only graph with all of these properties is **E**.



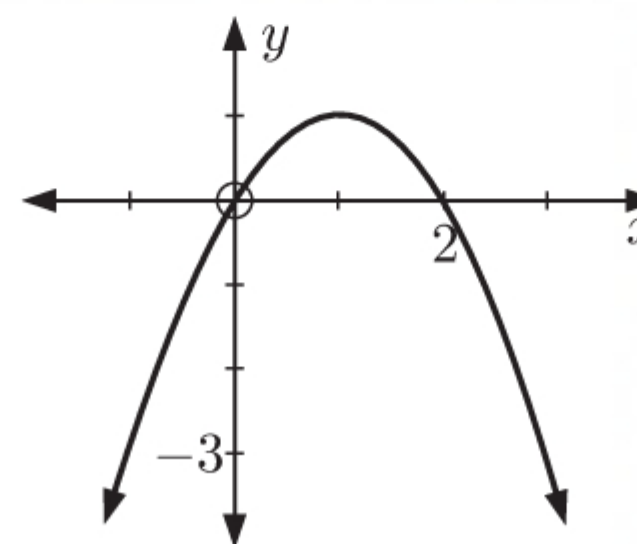
d $y = -(x-1)^2 + 1$ has vertex $(1, 1)$.

The axis of symmetry is $x = 1$.

$$\text{When } x = 0, \quad y = -(-1)^2 + 1 = 0$$

$a < 0$ so the shape is 


The only graph with all of these properties is **B**.



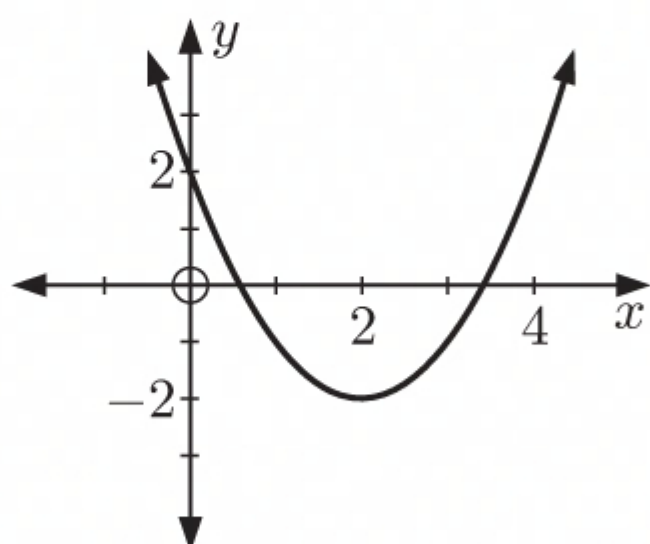
e $y = (x - 2)^2 - 2$ has vertex $(2, -2)$.

The axis of symmetry is $x = 2$.

When $x = 0$, $y = (-2)^2 - 2$
 $= 2$

$a > 0$ so the shape is 

The only graph with all of these properties is **I**.




f $y = \frac{1}{3}(x + 3)^2 - 3$

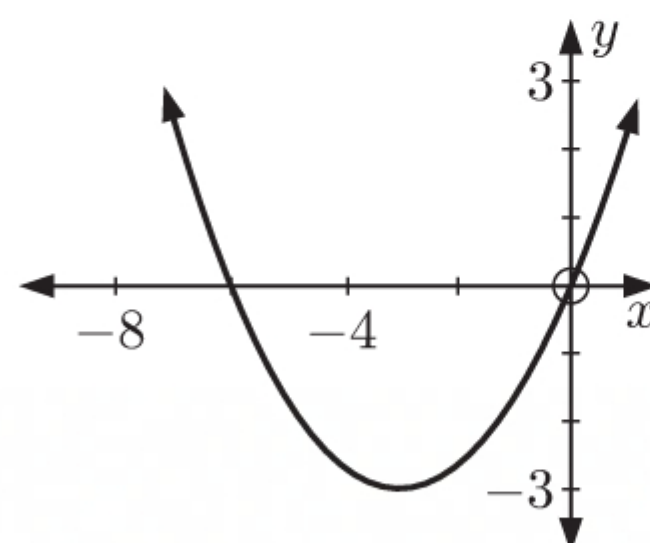
has vertex $(-3, -3)$.

The axis of symmetry is $x = -3$.

When $x = 0$, $y = \frac{1}{3}(3)^2 - 3$
 $= 0$

$a > 0$ so the shape is 


The only graph with all of these properties is **C**.



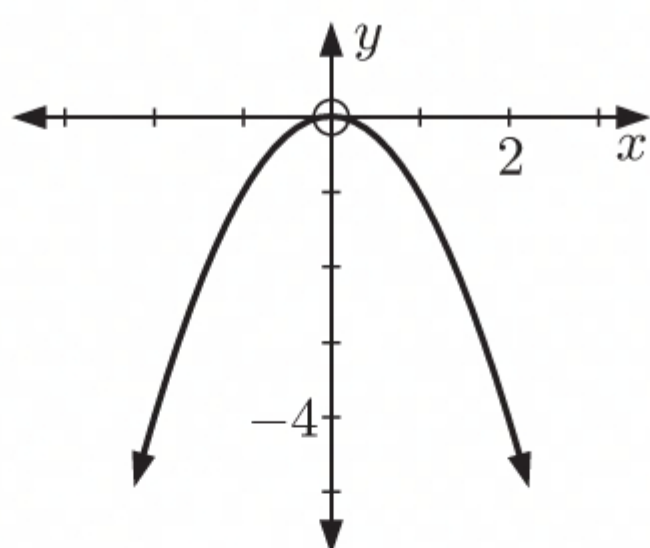
g $y = -x^2$ has vertex $(0, 0)$.

The axis of symmetry is $x = 0$.

When $x = 0$, $y = 0$

$a < 0$ so the shape is 


The only graph with all of these properties is **D**.



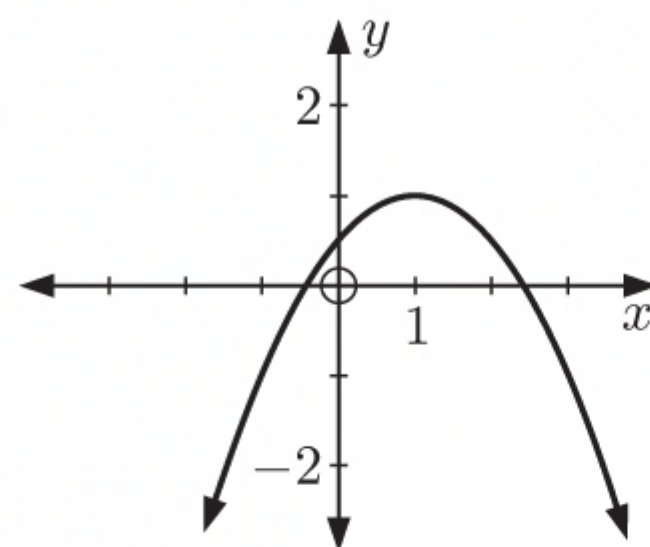
h $y = -\frac{1}{2}(x - 1)^2 + 1$ has vertex $(1, 1)$.

The axis of symmetry is $x = 1$.

When $x = 0$, $y = -\frac{1}{2}(-1)^2 + 1$
 $= \frac{1}{2}$

$a < 0$ so the shape is 


The only graph with all of these properties is **F**.



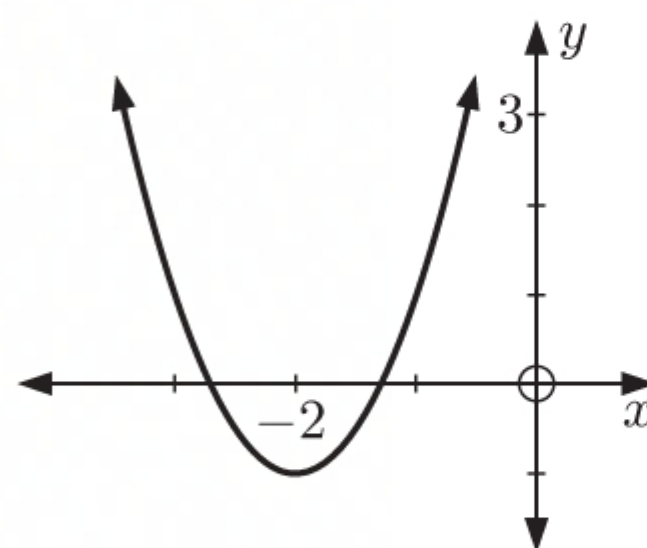
i $y = 2(x + 2)^2 - 1$ has vertex $(-2, -1)$.

The axis of symmetry is $x = -2$.

When $x = 0$, $y = 2(2)^2 - 1$
 $= 7$

$a > 0$ so the shape is 

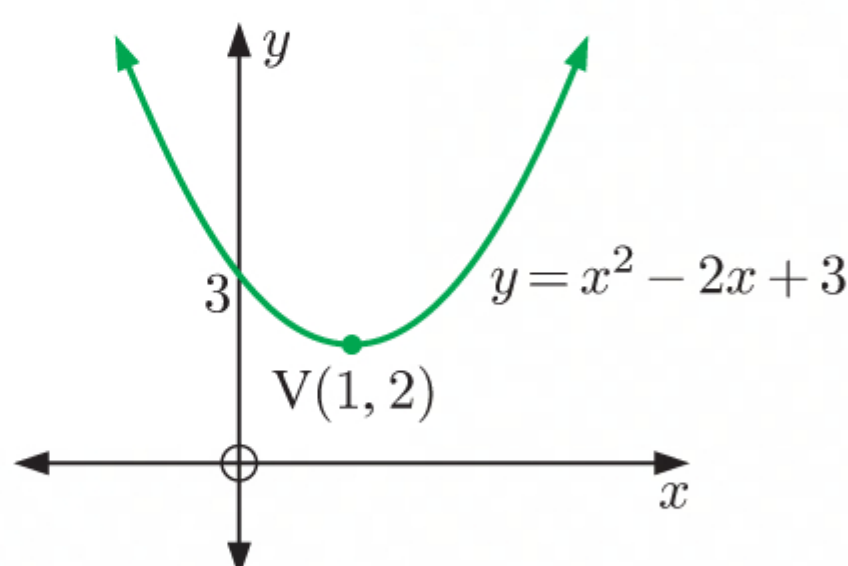
The only graph with all of these properties is **H**.



EXERCISE 2B.2

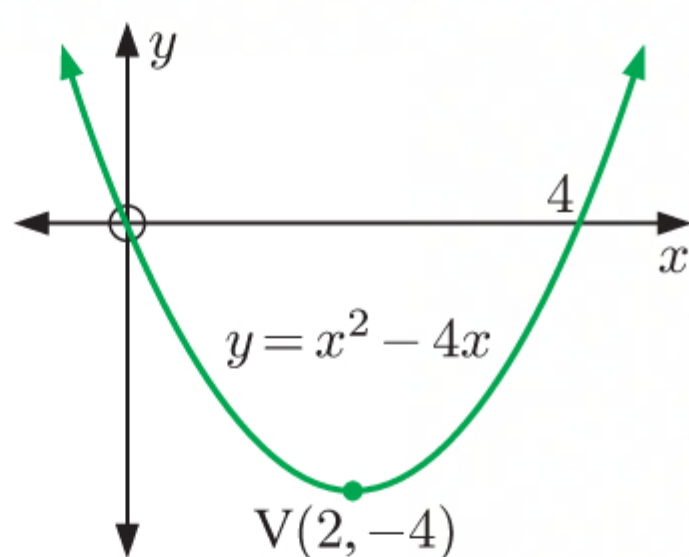
1 a $y = x^2 - 2x + 3$
 $\therefore y = x^2 - 2x + (-1)^2 + 3 - (-1)^2$
 $\therefore y = (x - 1)^2 + 2$

The vertex is $(1, 2)$, and the y -intercept is 3.



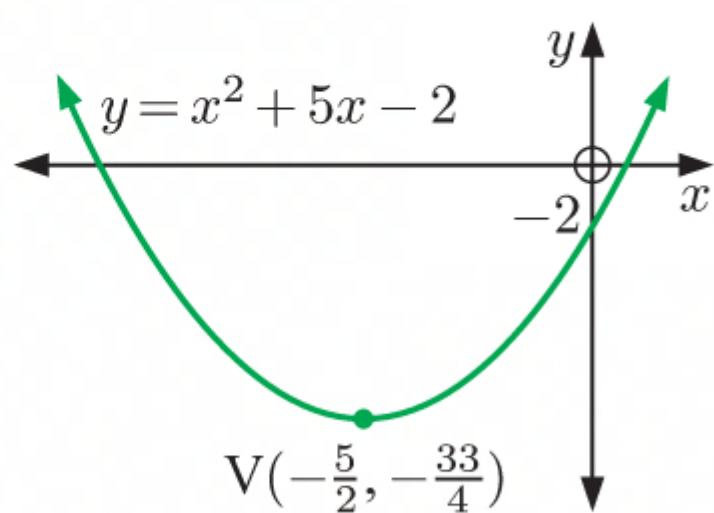
c $y = x^2 - 4x$
 $\therefore y = x^2 - 4x + (-2)^2 - (-2)^2$
 $\therefore y = (x - 2)^2 - 4$

The vertex is $(2, -4)$, and the y -intercept is 0.



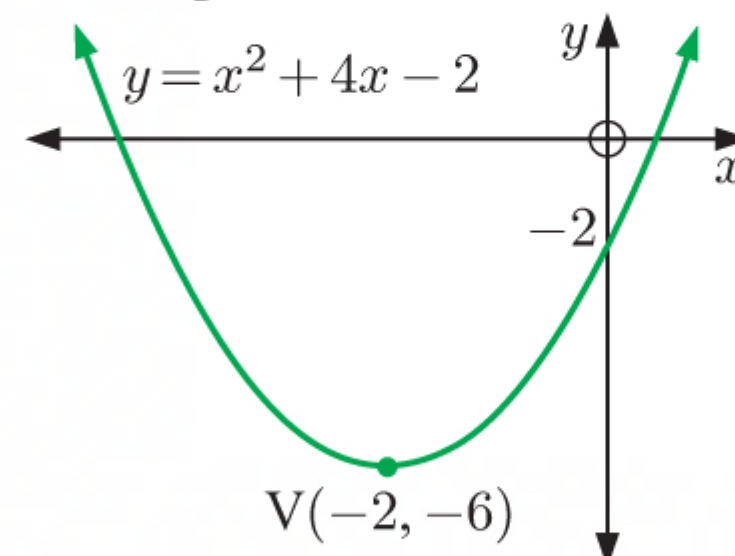
e $y = x^2 + 5x - 2$
 $\therefore y = x^2 + 5x + (\frac{5}{2})^2 - 2 - (\frac{5}{2})^2$
 $\therefore y = (x + \frac{5}{2})^2 - \frac{33}{4}$

The vertex is $(-\frac{5}{2}, -\frac{33}{4})$, and the y -intercept is -2 .



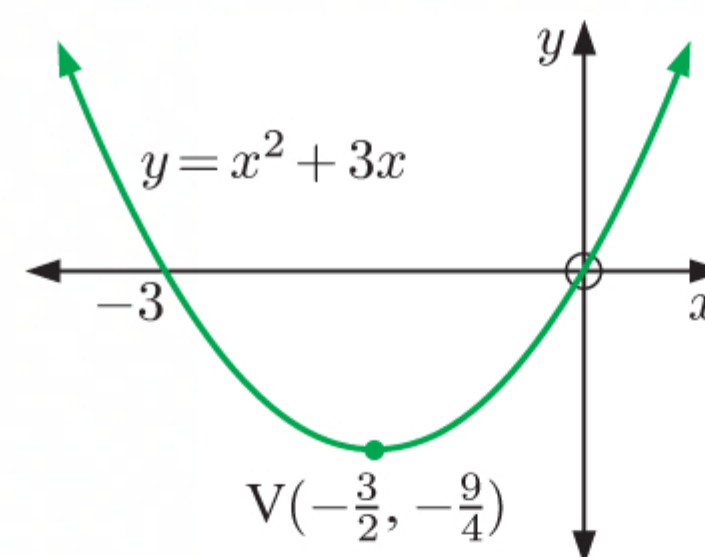
b $y = x^2 + 4x - 2$
 $\therefore y = x^2 + 4x + 2^2 - 2 - 2^2$
 $\therefore y = (x + 2)^2 - 6$

The vertex is $(-2, -6)$, and the y -intercept is -2 .



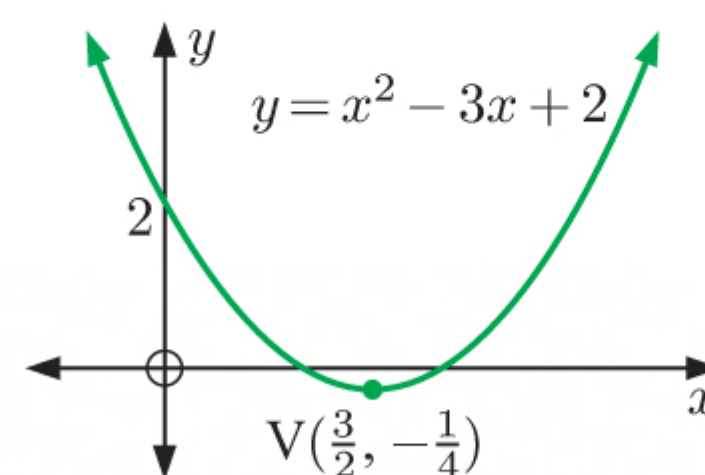
d $y = x^2 + 3x$
 $\therefore y = x^2 + 3x + (\frac{3}{2})^2 - (\frac{3}{2})^2$
 $\therefore y = (x + \frac{3}{2})^2 - \frac{9}{4}$

The vertex is $(-\frac{3}{2}, -\frac{9}{4})$, and the y -intercept is 0.



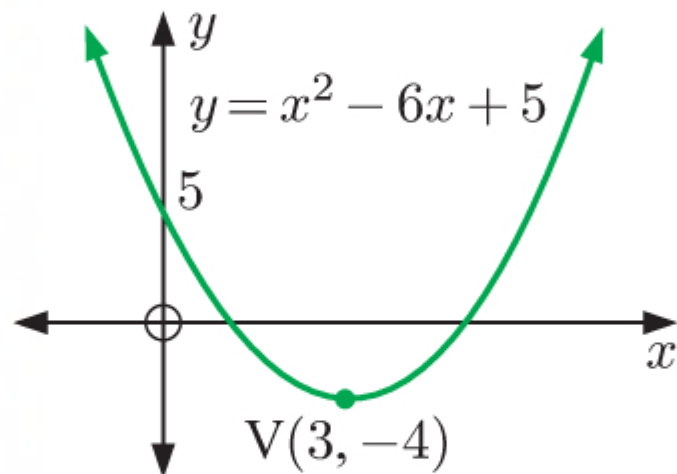
f $y = x^2 - 3x + 2$
 $\therefore y = x^2 - 3x + (-\frac{3}{2})^2 + 2 - (-\frac{3}{2})^2$
 $\therefore y = (x - \frac{3}{2})^2 - \frac{1}{4}$

The vertex is $(\frac{3}{2}, -\frac{1}{4})$, and the y -intercept is 2.



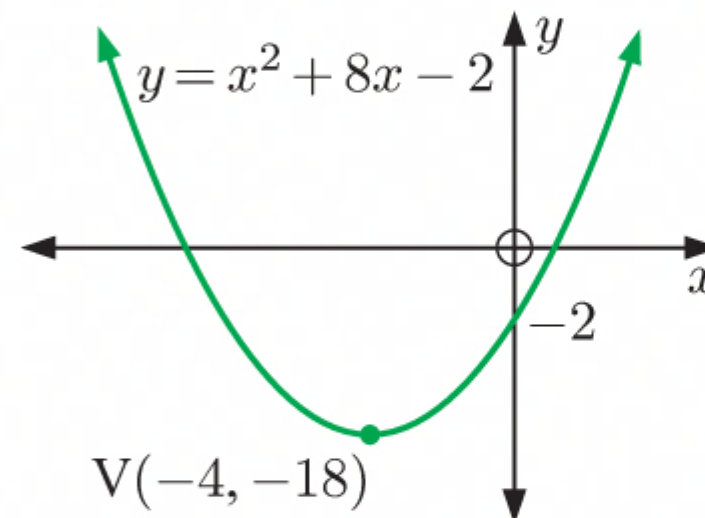
g $y = x^2 - 6x + 5$
 $\therefore y = x^2 - 6x + (-3)^2 + 5 - (-3)^2$
 $\therefore y = (x - 3)^2 - 4$

The vertex is $(3, -4)$, and the y -intercept is 5.



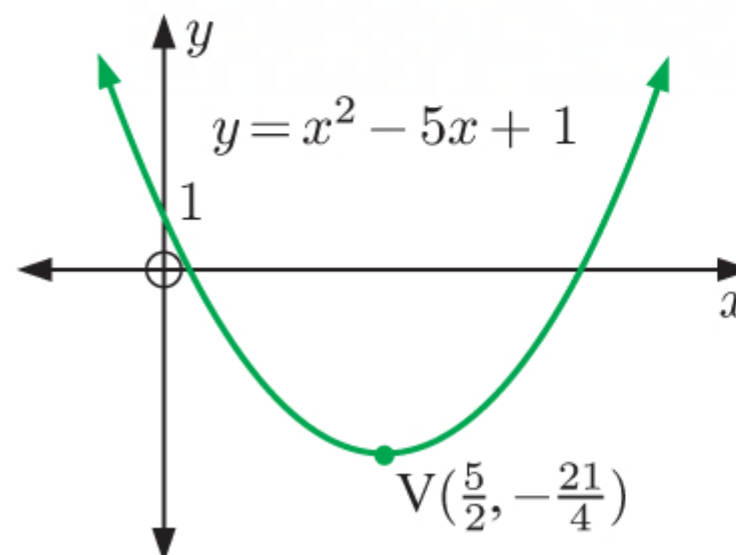
h $y = x^2 + 8x - 2$
 $\therefore y = x^2 + 8x + 4^2 - 2 - 4^2$
 $\therefore y = (x + 4)^2 - 18$

The vertex is $(-4, -18)$, and the y -intercept is -2 .



i $y = x^2 - 5x + 1$
 $\therefore y = x^2 - 5x + (-\frac{5}{2})^2 + 1 - (-\frac{5}{2})^2$
 $\therefore y = (x - \frac{5}{2})^2 - \frac{21}{4}$

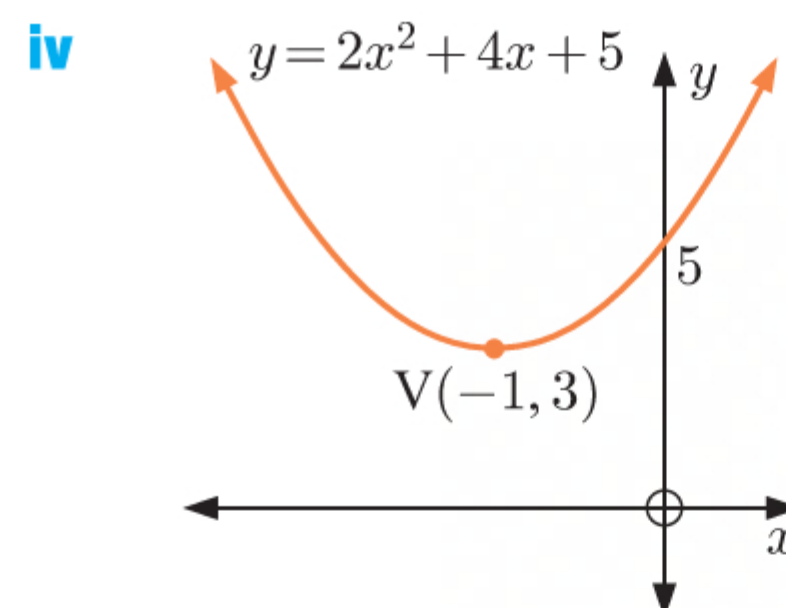
The vertex is $(\frac{5}{2}, -\frac{21}{4})$, and the y -intercept is 1.



2 a i $y = 2x^2 + 4x + 5$
 $= 2[x^2 + 2x + \frac{5}{2}]$
 $= 2[x^2 + 2x + 1^2 + \frac{5}{2} - 1^2]$
 $= 2[(x + 1)^2 + \frac{3}{2}]$
 $\therefore y = 2(x + 1)^2 + 3$

ii The vertex is $(-1, 3)$.

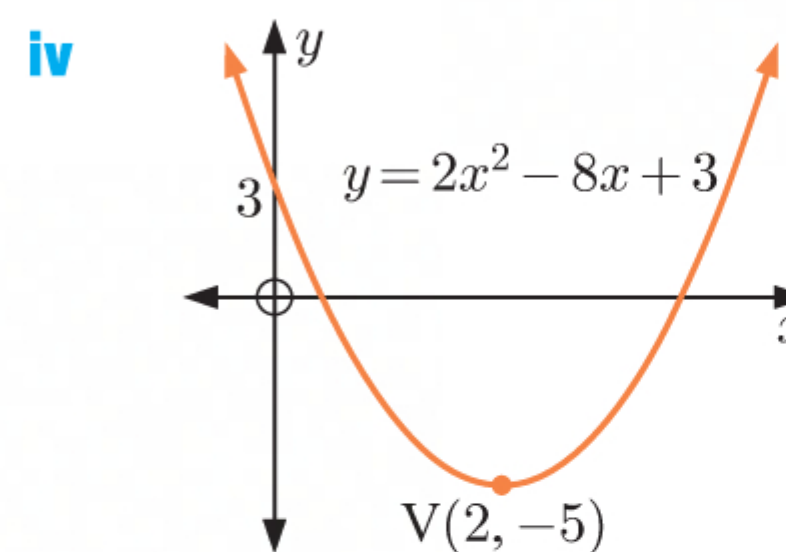
iii The y -intercept is 5.



b i $y = 2x^2 - 8x + 3$
 $= 2[x^2 - 4x + \frac{3}{2}]$
 $= 2[x^2 - 4x + (-2)^2 + \frac{3}{2} - (-2)^2]$
 $= 2[(x - 2)^2 - \frac{5}{2}]$
 $\therefore y = 2(x - 2)^2 - 5$

ii The vertex is $(2, -5)$.

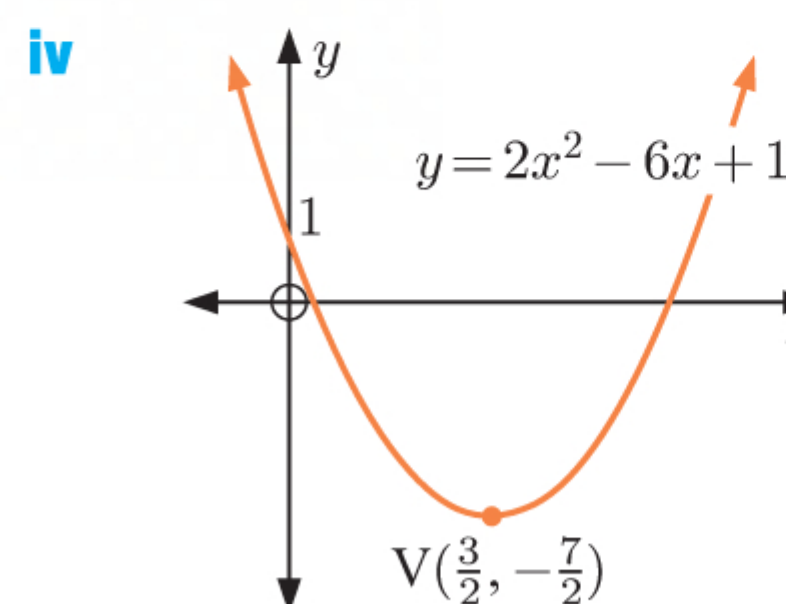
iii The y -intercept is 3.



c i $y = 2x^2 - 6x + 1$
 $= 2[x^2 - 3x + \frac{1}{2}]$
 $= 2[x^2 - 3x + (-\frac{3}{2})^2 + \frac{1}{2} - (-\frac{3}{2})^2]$
 $= 2[(x - \frac{3}{2})^2 - \frac{7}{4}]$
 $\therefore y = 2(x - \frac{3}{2})^2 - \frac{7}{2}$

ii The vertex is $(\frac{3}{2}, -\frac{7}{2})$.

iii The y -intercept is 1.



d i $y = 3x^2 - 6x + 5$
 $= 3[x^2 - 2x + \frac{5}{3}]$
 $= 3[x^2 - 2x + (-1)^2 + \frac{5}{3} - (-1)^2]$
 $= 3[(x-1)^2 + \frac{2}{3}]$
 $\therefore y = 3(x-1)^2 + 2$

ii The vertex is (1, 2).

iii The y -intercept is 5.

e i $y = -x^2 + 4x + 2$
 $= -[x^2 - 4x - 2]$
 $= -[x^2 - 4x + (-2)^2 - 2 - (-2)^2]$
 $= -[(x-2)^2 - 6]$
 $\therefore y = -(x-2)^2 + 6$

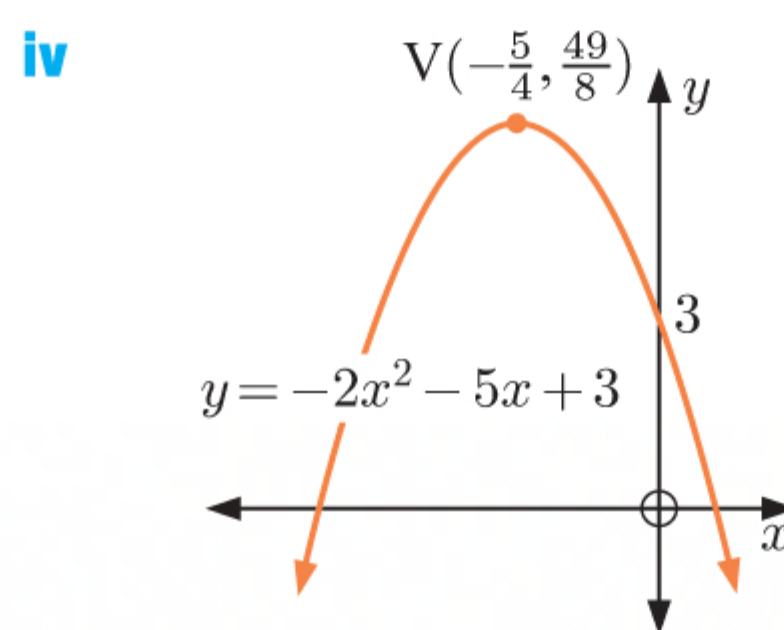
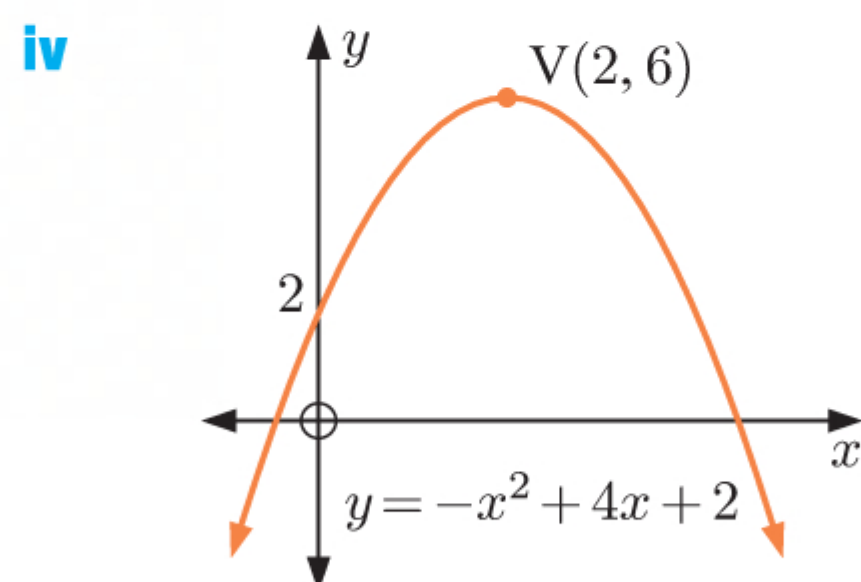
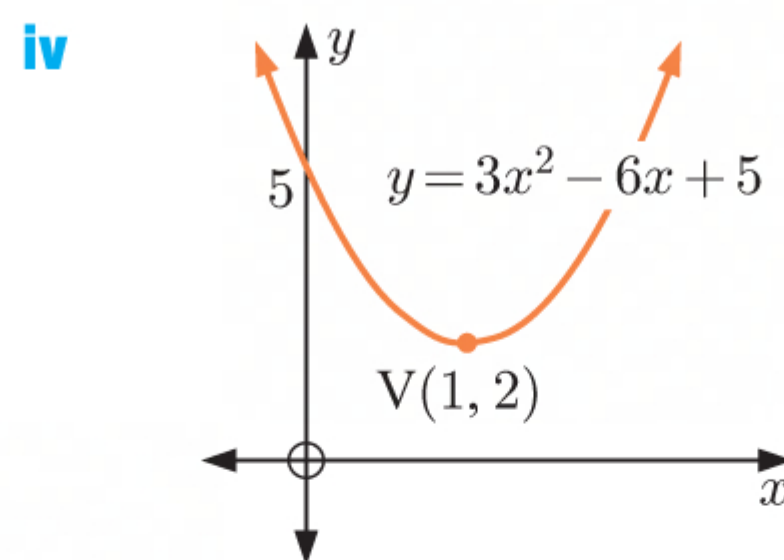
ii The vertex is (2, 6).

iii The y -intercept is 2.

f i $y = -2x^2 - 5x + 3$
 $= -2[x^2 + \frac{5}{2}x - \frac{3}{2}]$
 $= -2[x^2 + \frac{5}{2}x + (\frac{5}{4})^2 - \frac{3}{2} - (\frac{5}{4})^2]$
 $= -2[(x + \frac{5}{4})^2 - \frac{24}{16} - \frac{25}{16}]$
 $= -2[(x + \frac{5}{4})^2 - \frac{49}{16}]$
 $\therefore y = -2(x + \frac{5}{4})^2 + \frac{49}{8}$


ii The vertex is $(-\frac{5}{4}, \frac{49}{8})$.

iii The y -intercept is 3.




EXERCISE 2B.3

1 a i $y = x^2 - 4x + 2$
has $a = 1$, $b = -4$, $c = 2$
 $\frac{-b}{2a} = \frac{-(-4)}{2(1)} = 2$
The axis of symmetry is $x = 2$.
When $x = 2$,
 $y = 2^2 - 4(2) + 2 = -2$
 \therefore the vertex is (2, -2).

ii $a > 0$, so the shape is 
 \therefore the vertex (2, -2) is a minimum turning point.

b i $y = x^2 + 2x - 3$
has $a = 1$, $b = 2$, $c = -3$
 $\frac{-b}{2a} = \frac{-2}{2(1)} = -1$
The axis of symmetry is $x = -1$.
When $x = -1$,
 $y = (-1)^2 + 2(-1) - 3 = -4$
 \therefore the vertex is (-1, -4).

ii $a > 0$, so the shape is 
 \therefore the vertex (-1, -4) is a minimum turning point.

c i $y = 2x^2 + 4$


has $a = 2$, $b = 0$, $c = 4$

$$\frac{-b}{2a} = \frac{-0}{2(2)} = 0$$

The axis of symmetry is $x = 0$.

When $x = 0$, $y = 4$

\therefore the vertex is $(0, 4)$.

- ii** $a > 0$, so the shape is 
 \therefore the vertex $(0, 4)$ is a minimum turning point.

e i $y = 2x^2 + 8x - 7$

has $a = 2$, $b = 8$, $c = -7$


$$\frac{-b}{2a} = \frac{-8}{2(2)} = -2$$

The axis of symmetry is $x = -2$.

When $x = -2$,

$$y = 2(-2)^2 + 8(-2) - 7 = -15$$

\therefore the vertex is $(-2, -15)$.

- ii** $a > 0$, so the shape is 
 \therefore the vertex $(-2, -15)$ is a minimum turning point.

g i $y = 2x^2 + 6x - 1$

has $a = 2$, $b = 6$, $c = -1$


$$\frac{-b}{2a} = \frac{-6}{2(2)} = -\frac{3}{2}$$

The axis of symmetry is $x = -\frac{3}{2}$.

When $x = -\frac{3}{2}$,

$$y = 2\left(-\frac{3}{2}\right)^2 + 6\left(-\frac{3}{2}\right) - 1 = \frac{9}{2} - 9 - 1 = -\frac{11}{2}$$

\therefore the vertex is $\left(-\frac{3}{2}, -\frac{11}{2}\right)$.

- ii** $a > 0$, so the shape is 
 \therefore the vertex $\left(-\frac{3}{2}, -\frac{11}{2}\right)$ is a minimum turning point.

d i $y = -3x^2 + 1$


has $a = -3$, $b = 0$, $c = 1$

$$\frac{-b}{2a} = \frac{-0}{2(-3)} = 0$$

The axis of symmetry is $x = 0$.

When $x = 0$, $y = 1$

\therefore the vertex is $(0, 1)$.

- ii** $a < 0$, so the shape is 
 \therefore the vertex $(0, 1)$ is a maximum turning point.

f i $y = -x^2 - 4x - 9$

has $a = -1$, $b = -4$, $c = -9$


$$\frac{-b}{2a} = \frac{-(-4)}{2(-1)} = -2$$

The axis of symmetry is $x = -2$.

When $x = -2$,

$$y = -(-2)^2 - 4(-2) - 9 = -4 + 8 - 9 = -5$$

\therefore the vertex is $(-2, -5)$.

- ii** $a < 0$, so the shape is 
 \therefore the vertex $(-2, -5)$ is a maximum turning point.

h i $y = 2x^2 - 10x + 3$

has $a = 2$, $b = -10$, $c = 3$


$$\frac{-b}{2a} = \frac{-(-10)}{2(2)} = \frac{5}{2}$$

The axis of symmetry is $x = \frac{5}{2}$.

When $x = \frac{5}{2}$,

$$y = 2\left(\frac{5}{2}\right)^2 - 10\left(\frac{5}{2}\right) + 3 = \frac{25}{2} - \frac{50}{2} + 3 = -\frac{19}{2}$$

\therefore the vertex is $\left(\frac{5}{2}, -\frac{19}{2}\right)$.

- ii** $a > 0$, so the shape is 
 \therefore the vertex $\left(\frac{5}{2}, -\frac{19}{2}\right)$ is a minimum turning point.


i **i** $y = -\frac{1}{2}x^2 + x - 5$
 has $a = -\frac{1}{2}$, $b = 1$, $c = -5$
 $\frac{-b}{2a} = \frac{-1}{2(-\frac{1}{2})} = 1$

The axis of symmetry is $x = 1$.

When $x = 1$,

$$y = -\frac{1}{2}(1)^2 + 1 - 5 \\ = -\frac{9}{2}$$

\therefore the vertex is $(1, -\frac{9}{2})$.

ii $a < 0$, so the shape is 

\therefore the vertex $(1, -\frac{9}{2})$ is a maximum turning point.


j **i** $y = \frac{1}{4}x^2 - 7x + 6$
 has $a = \frac{1}{4}$, $b = -7$, $c = 6$
 $\frac{-b}{2a} = \frac{-(-7)}{2(\frac{1}{4})} = 14$

The axis of symmetry is $x = 14$.


When $x = 14$,

$$y = \frac{1}{4}(14)^2 - 7(14) + 6 \\ = -43$$

\therefore the vertex is $(14, -43)$.

ii $a > 0$, so the shape is 

\therefore the vertex $(14, -43)$ is a minimum turning point.

2 a $y = x^2 - 8x + 7$ has $a = 1$, $b = -8$, $c = 7$. Since $a > 0$, the shape is 

i $\frac{-b}{2a} = \frac{-(-8)}{2(1)} = 4$

The axis of symmetry is $x = 4$.

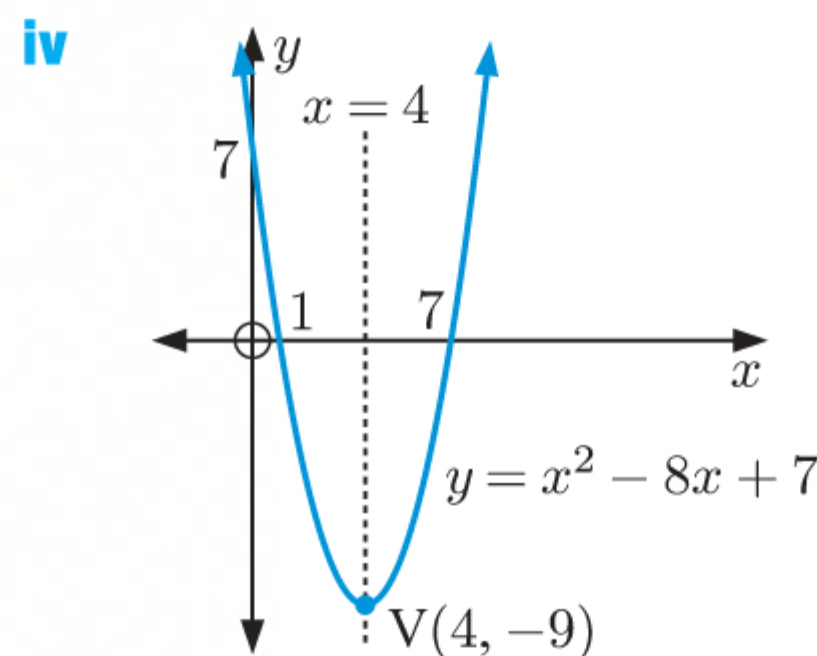
ii When $x = 4$, $y = 4^2 - 8(4) + 7 \\ = -9$

\therefore the vertex is $(4, -9)$.

iii The y -intercept is 7.

When $y = 0$, $x^2 - 8x + 7 = 0$
 $\therefore (x - 1)(x - 7) = 0$
 $\therefore x = 1$ or 7

\therefore the x -intercepts are 1 and 7.



b $y = -x^2 - 6x - 8$ has $a = -1$, $b = -6$, $c = -8$.

Since $a < 0$, the shape is 

i $\frac{-b}{2a} = \frac{-(-6)}{2(-1)} = -3$

The axis of symmetry is $x = -3$.

ii When $x = -3$,

$$y = -(-3)^2 - 6(-3) - 8$$

$$= 1$$

\therefore the vertex is $(-3, 1)$.

iii The y -intercept is -8 .

When $y = 0$,

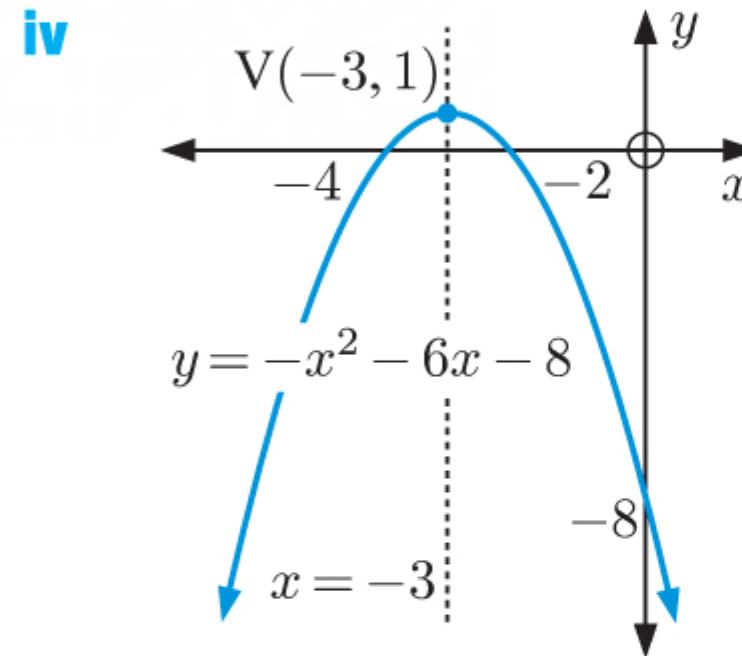
$$-x^2 - 6x - 8 = 0$$


$$\therefore -(x^2 + 6x + 8) = 0$$

$$\therefore -(x + 2)(x + 4) = 0$$

$$\therefore x = -2 \text{ or } -4$$

$$\therefore \text{the } x\text{-intercepts are } -2 \text{ and } -4.$$



c $y = 6x - x^2$ has $a = -1$, $b = 6$, $c = 0$. Since $a < 0$, the shape is 

i $\frac{-b}{2a} = \frac{-6}{2(-1)} = 3$

The axis of symmetry is $x = 3$.

ii When $x = 3$, $y = 6(3) - 3^2$

$$= 9$$

$$\therefore \text{the vertex is } (3, 9).$$

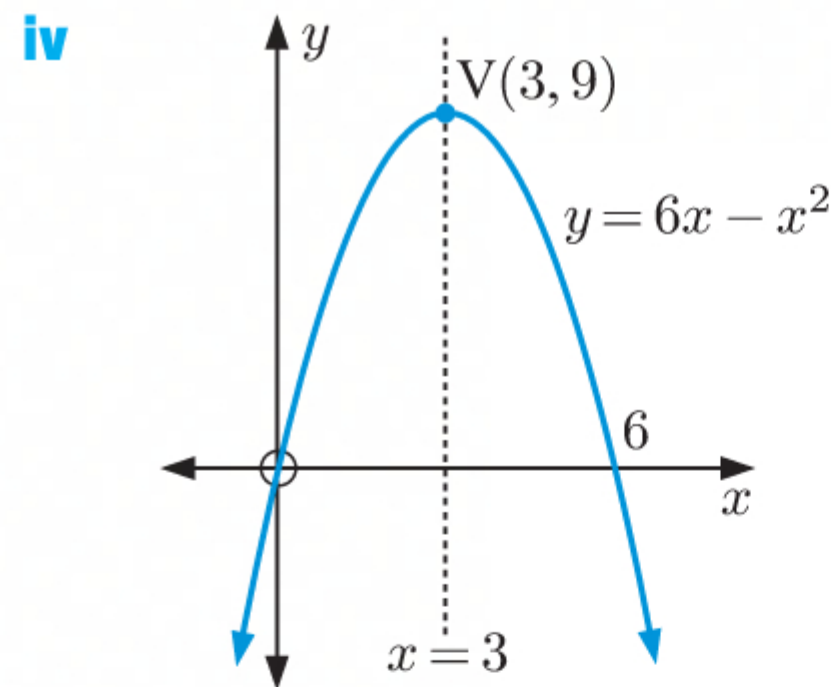
iii The y -intercept is 0.


When $y = 0$, $6x - x^2 = 0$

$$\therefore x(6 - x) = 0$$

$$\therefore x = 0 \text{ or } 6$$

$$\therefore \text{the } x\text{-intercepts are } 0 \text{ and } 6.$$



d $y = -x^2 + 3x - 2$ has $a = -1$, $b = 3$, $c = -2$. Since $a < 0$, the shape is 

i $\frac{-b}{2a} = \frac{-3}{2(-1)} = \frac{3}{2}$

The axis of symmetry is $x = \frac{3}{2}$.

ii When $x = \frac{3}{2}$,

$$\begin{aligned} y &= -\left(\frac{3}{2}\right)^2 + 3\left(\frac{3}{2}\right) - 2 \\ &= -\frac{9}{4} + \frac{18}{4} - 2 \\ &= \frac{1}{4} \end{aligned}$$

\therefore the vertex is $\left(\frac{3}{2}, \frac{1}{4}\right)$.

iii The y -intercept is -2 .

When $y = 0$,

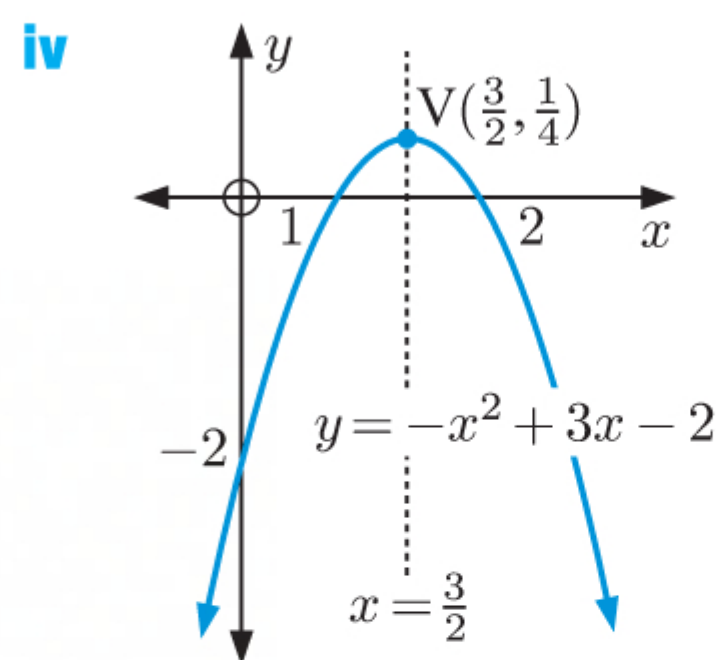
$$-x^2 + 3x - 2 = 0$$


$$\therefore -(x^2 - 3x + 2) = 0$$

$$\therefore -(x - 1)(x - 2) = 0$$

$$\therefore x = 1 \text{ or } 2$$

\therefore the x -intercepts are 1 and 2.



e $y = 2x^2 + 4x - 24$ has $a = 2$, $b = 4$, $c = -24$. Since $a > 0$, the shape is 

i $\frac{-b}{2a} = \frac{-4}{2(2)} = -1$

The axis of symmetry is $x = -1$.

ii When $x = -1$,

$$\begin{aligned} y &= 2(-1)^2 + 4(-1) - 24 \\ &= -26 \end{aligned}$$

\therefore the vertex is $(-1, -26)$.

iii The y -intercept is -24 .

When $y = 0$,

$$2x^2 + 4x - 24 = 0$$

$$\therefore x^2 + 2x - 12 = 0$$

$$\therefore x^2 + 2x = 12$$

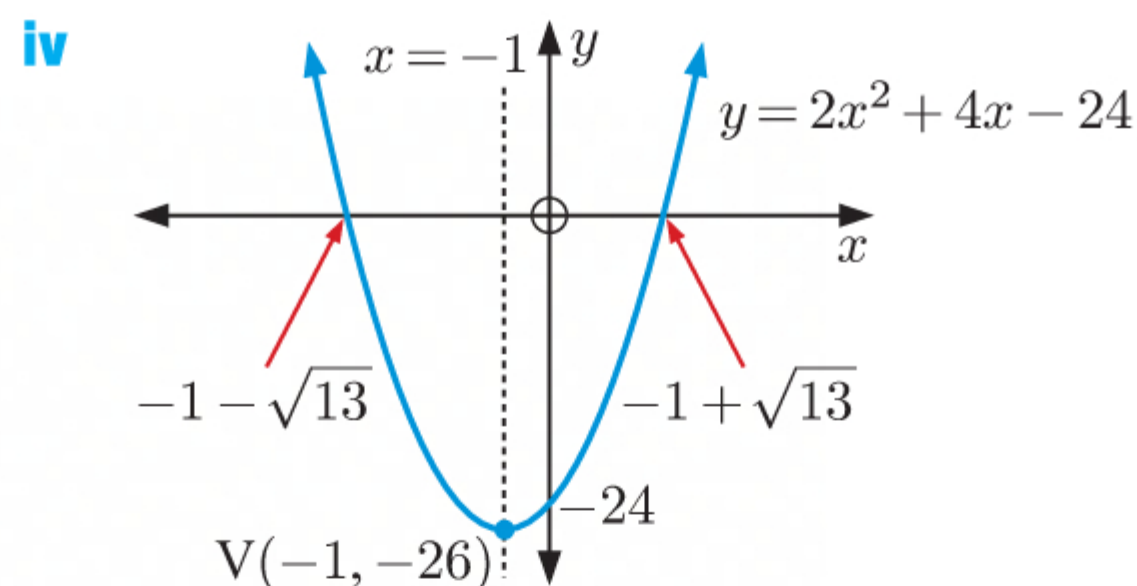
$$\therefore x^2 + 2x + 1^2 = 12 + 1^2$$

$$\therefore (x + 1)^2 = 13$$

$$\therefore x + 1 = \pm\sqrt{13}$$

$$\therefore x = -1 \pm \sqrt{13}$$

\therefore the x -intercepts are $-1 \pm \sqrt{13}$.



f $y = -3x^2 + 4x - 1$ has $a = -3$, $b = 4$, $c = -1$.

Since $a < 0$, the shape is 

i $\frac{-b}{2a} = \frac{-4}{2(-3)} = \frac{2}{3}$

The axis of symmetry is $x = \frac{2}{3}$.

ii When $x = \frac{2}{3}$,

$$\begin{aligned} y &= -3\left(\frac{2}{3}\right)^2 + 4\left(\frac{2}{3}\right) - 1 \\ &= -\frac{4}{3} + \frac{8}{3} - 1 \\ &= \frac{1}{3} \end{aligned}$$

\therefore the vertex is $\left(\frac{2}{3}, \frac{1}{3}\right)$.

iii The y -intercept is -1 .

When $y = 0$,

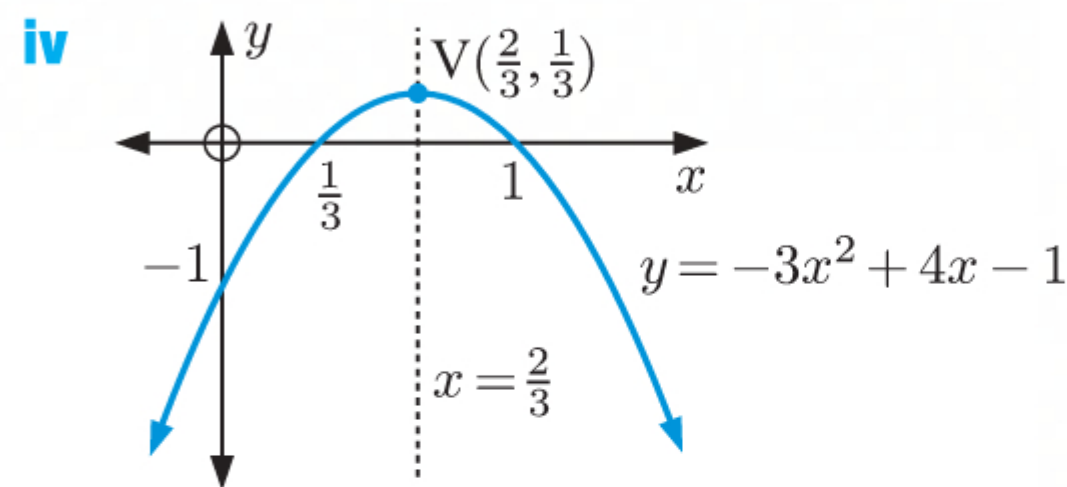
$$-3x^2 + 4x - 1 = 0$$


$$\therefore -(3x^2 - 4x + 1) = 0$$

$$\therefore -(3x - 1)(x - 1) = 0$$

$$\therefore x = \frac{1}{3} \text{ or } 1$$

\therefore the x -intercepts are $\frac{1}{3}$ and 1 .



g $y = 2x^2 - 5x + 2$ has $a = 2$, $b = -5$, $c = 2$. Since $a > 0$, the shape is 

i $\frac{-b}{2a} = \frac{-(-5)}{2(2)} = \frac{5}{4}$

The axis of symmetry is $x = \frac{5}{4}$.

ii When $x = \frac{5}{4}$,

$$\begin{aligned} y &= 2\left(\frac{5}{4}\right)^2 - 5\left(\frac{5}{4}\right) + 2 \\ &= -\frac{9}{8} \end{aligned}$$

\therefore the vertex is $\left(\frac{5}{4}, -\frac{9}{8}\right)$.

iii The y -intercept is 2 .

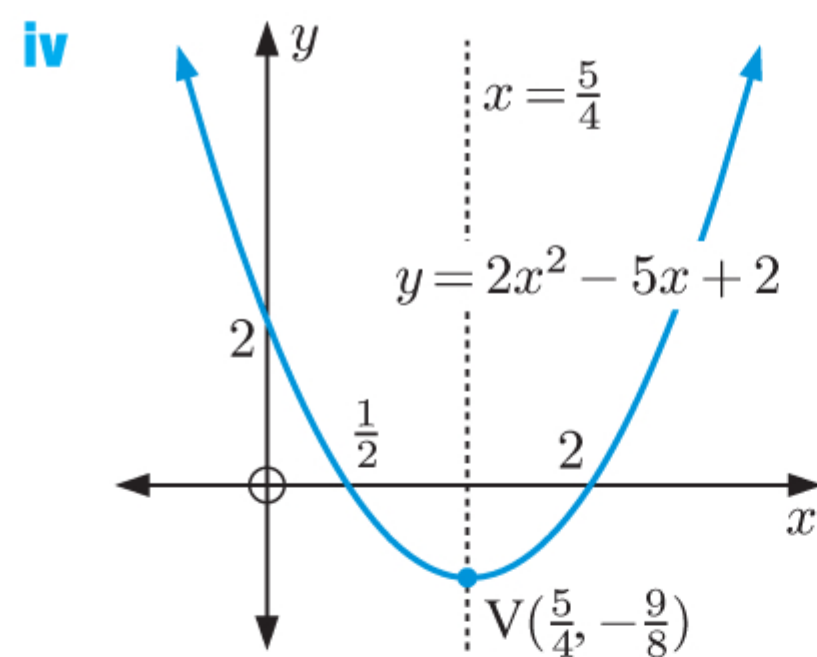
When $y = 0$,


$$2x^2 - 5x + 2 = 0$$

$$\therefore (2x - 1)(x - 2) = 0$$

$$\therefore x = \frac{1}{2} \text{ or } 2$$

\therefore the x -intercepts are $\frac{1}{2}$ and 2 .



h $y = 4x^2 - 8x - 5$ has $a = 4$, $b = -8$, $c = -5$. Since $a > 0$, the shape is 

i $\frac{-b}{2a} = \frac{-(-8)}{2(4)} = 1$

The axis of symmetry is $x = 1$.

ii When $x = 1$, $y = 4(1)^2 - 8(1) - 5$
 $= -9$

\therefore the vertex is $(1, -9)$.

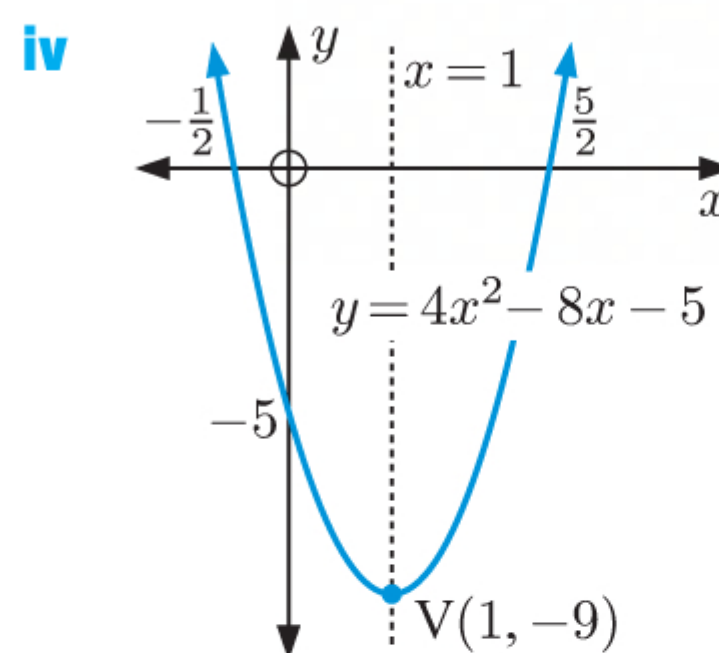
iii The y -intercept is -5 .

When $y = 0$, $4x^2 - 8x - 5 = 0$

$\therefore (2x + 1)(2x - 5) = 0$

$\therefore x = -\frac{1}{2} \text{ or } \frac{5}{2}$

\therefore the x -intercepts are $-\frac{1}{2}$ and $\frac{5}{2}$.



i $y = -\frac{1}{4}x^2 + 2x - 3$ has $a = -\frac{1}{4}$, $b = 2$, $c = -3$.

Since $a < 0$, the shape is 

i $\frac{-b}{2a} = \frac{-2}{2(-\frac{1}{4})} = 4$

The axis of symmetry is $x = 4$.

ii When $x = 4$, $y = -\frac{1}{4}(4)^2 + 2(4) - 3$
 $= 1$

\therefore the vertex is $(4, 1)$.

iii The y -intercept is -3 .

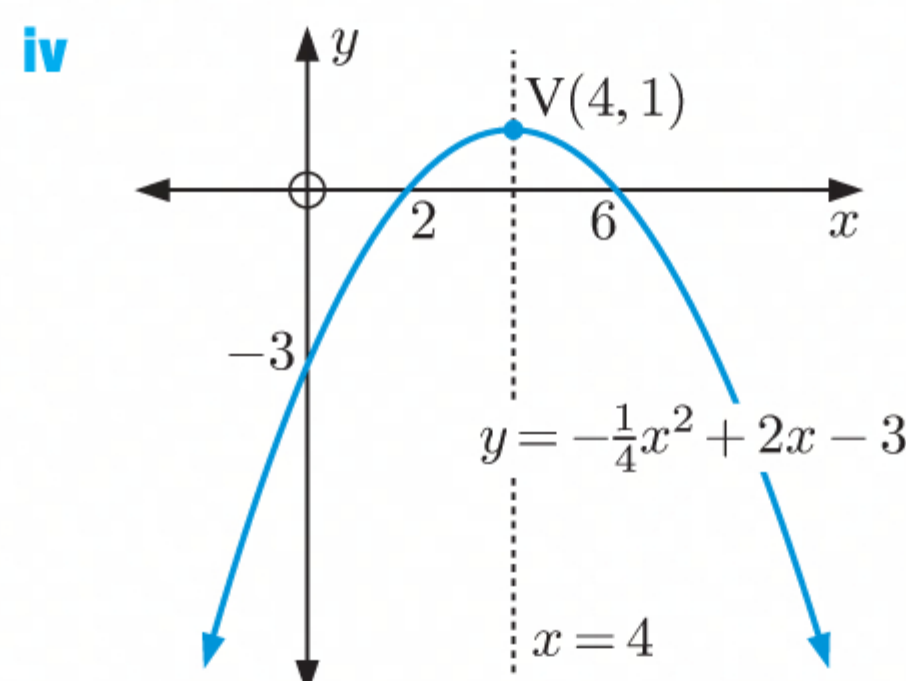
When $y = 0$, $-\frac{1}{4}x^2 + 2x - 3 = 0$

$\therefore x^2 - 8x + 12 = 0$

$\therefore (x - 2)(x - 6) = 0$

$\therefore x = 2 \text{ or } 6$

\therefore the x -intercepts are 2 and 6.



EXERCISE 2C

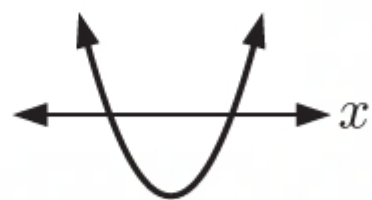
1 a $y = x^2 + x - 2$

has $a = 1$, $b = 1$, $c = -2$

$$\begin{aligned}\therefore \Delta &= b^2 - 4ac \\ &= 1^2 - 4(1)(-2) \\ &= 9\end{aligned}$$

Since $\Delta > 0$, the graph cuts the x -axis twice.

Since $a > 0$, the graph is concave up.



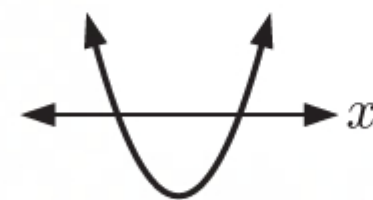
b $y = x^2 - 4x + 1$

has $a = 1$, $b = -4$, $c = 1$

$$\begin{aligned}\therefore \Delta &= b^2 - 4ac \\ &= (-4)^2 - 4(1)(1) \\ &= 12\end{aligned}$$

Since $\Delta > 0$, the graph cuts the x -axis twice.

Since $a > 0$, the graph is concave up.



c $y = -x^2 - 3$

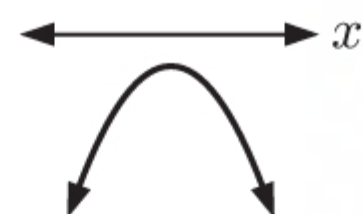
has $a = -1$, $b = 0$, $c = -3$

$$\begin{aligned}\therefore \Delta &= b^2 - 4ac \\ &= 0^2 - 4(-1)(-3) \\ &= -12\end{aligned}$$

Since $\Delta < 0$, the graph does not cut the x -axis.

Since $a < 0$, the graph is concave down.

The graph is negative definite.



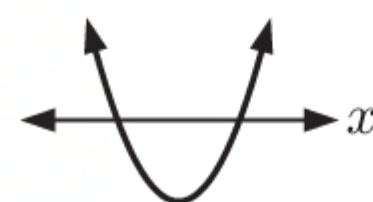
d $y = x^2 + 7x - 2$

has $a = 1$, $b = 7$, $c = -2$

$$\begin{aligned}\therefore \Delta &= b^2 - 4ac \\ &= 7^2 - 4(1)(-2) \\ &= 57\end{aligned}$$

Since $\Delta > 0$, the graph cuts the x -axis twice.

Since $a > 0$, the graph is concave up.



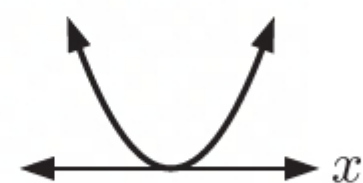
e $y = x^2 + 8x + 16$

has $a = 1$, $b = 8$, $c = 16$

$$\begin{aligned}\therefore \Delta &= b^2 - 4ac \\ &= 8^2 - 4(1)(16) \\ &= 0\end{aligned}$$

Since $\Delta = 0$, the graph touches the x -axis.

Since $a > 0$, the graph is concave up.



f $y = -2x^2 + 3x + 1$

has $a = -2$, $b = 3$, $c = 1$

$$\begin{aligned}\therefore \Delta &= b^2 - 4ac \\ &= 3^2 - 4(-2)(1) \\ &= 17\end{aligned}$$

Since $\Delta > 0$, the graph cuts the x -axis twice.

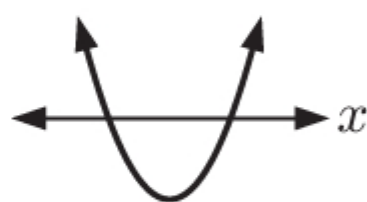
Since $a < 0$, the graph is concave down.



g $y = 6x^2 + 5x - 4$
 has $a = 6$, $b = 5$, $c = -4$
 $\therefore \Delta = b^2 - 4ac$
 $= 5^2 - 4(6)(-4)$
 $= 121$

Since $\Delta > 0$, the graph cuts the x -axis twice.

Since $a > 0$, the graph is concave up.



h $y = -x^2 + x + 6$
 has $a = -1$, $b = 1$, $c = 6$
 $\therefore \Delta = b^2 - 4ac$
 $= 1^2 - 4(-1)(6)$
 $= 25$

Since $\Delta > 0$, the graph cuts the x -axis twice.

Since $a < 0$, the graph is concave down.



i $y = 9x^2 + 6x + 1$
 has $a = 9$, $b = 6$, $c = 1$
 $\therefore \Delta = b^2 - 4ac$
 $= 6^2 - 4(9)(1)$
 $= 0$

Since $\Delta = 0$, the graph touches the x -axis.

Since $a > 0$, the graph is concave up.



2 $y = 2x^2 - 5x + 1$ has $a = 2$, $b = -5$, $c = 1$

a Since $a > 0$, the graph is concave up.

b $\Delta = b^2 - 4ac$
 $= (-5)^2 - 4(2)(1)$
 $= 17$

Since $\Delta > 0$, the graph cuts the x -axis twice.



c When $y = 0$, $2x^2 - 5x + 1 = 0$

$$x = \frac{-b \pm \sqrt{\Delta}}{2a}$$

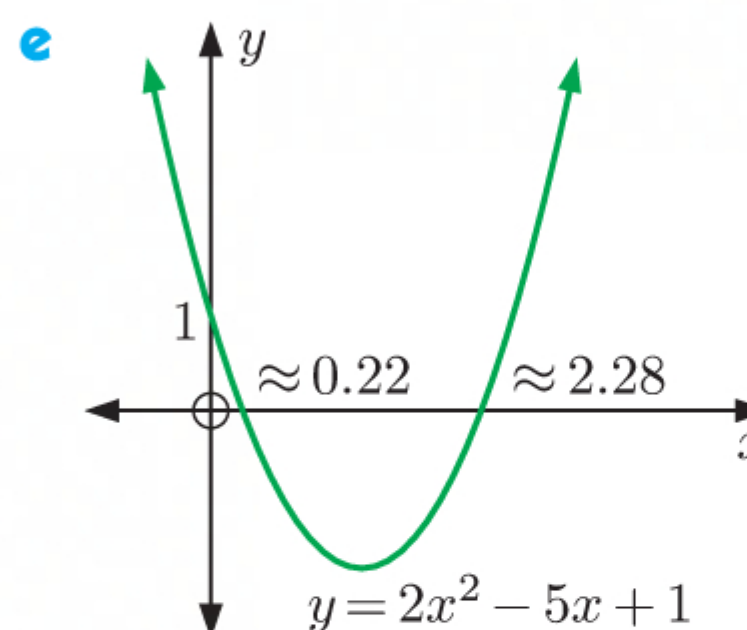
$$\therefore x = \frac{-(-5) \pm \sqrt{17}}{2(2)}$$

$$\therefore x = \frac{5}{4} \pm \frac{\sqrt{17}}{4}$$

$$\therefore x \approx 2.28 \text{ or } 0.22$$

\therefore the x -intercepts are ≈ 2.28 and ≈ 0.22 .

d The y -intercept is 1.



3 a $y = -x^2 + 4x - 7$ has $a = -1$, $b = 4$, $c = -7$

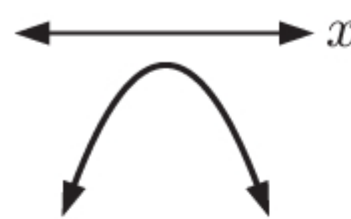
$$\therefore \Delta = b^2 - 4ac$$

$$= 4^2 - 4(-1)(-7)$$

$$= -12$$

Since $\Delta < 0$, the graph does not cut the x -axis.

- b** The graph is negative definite, since $a < 0$ and $\Delta < 0$. This means that it lies entirely below the x -axis.



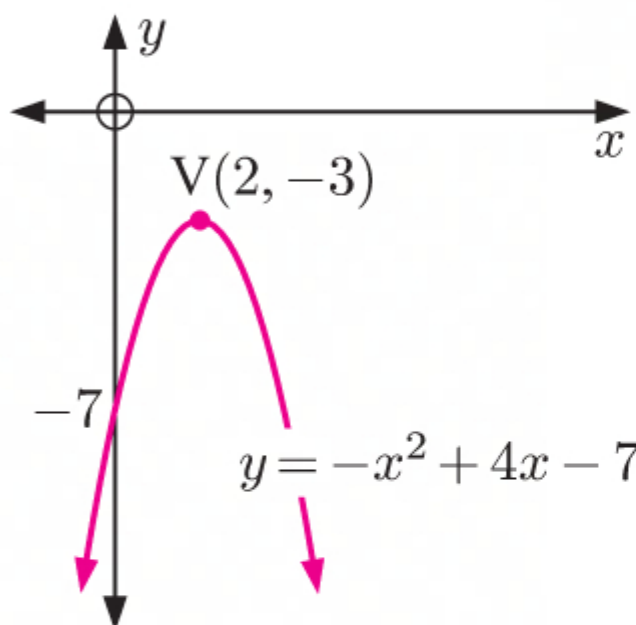
c $x = \frac{-b}{2a} = \frac{-4}{2(-1)} = 2$

When $x = 2$, $y = -2^2 + 4(2) - 7$
 $= -3$

\therefore the vertex is $(2, -3)$.

The y -intercept is -7 .

d



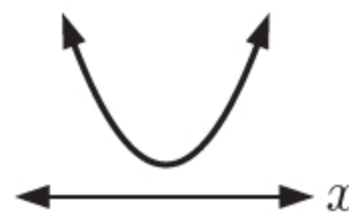
4 a $2x^2 - 4x + 7$ has $a = 2$, $b = -4$, $c = 7$

$\therefore \Delta = b^2 - 4ac$
 $= (-4)^2 - 4(2)(7)$
 $= -40$

Since $\Delta < 0$, the graph does not cut the x -axis.

Since $a > 0$, the graph is concave up.

The graph is positive definite, which means that it lies entirely above the x -axis.



b $-2x^2 + 3x - 4$ has $a = -2$, $b = 3$, $c = -4$

$\therefore \Delta = b^2 - 4ac$
 $= 3^2 - 4(-2)(-4)$
 $= -23$

Since $\Delta < 0$, the graph does not cut the x -axis.

Since $a < 0$, the graph is concave down.

The graph is negative definite, which means that it lies entirely below the x -axis.



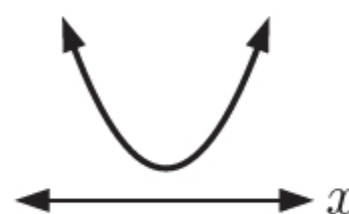
c $x^2 - 3x + 6$ has $a = 1$, $b = -3$, $c = 6$

$\therefore \Delta = b^2 - 4ac$
 $= (-3)^2 - 4(1)(6)$
 $= -15$

Since $\Delta < 0$, the graph does not cut the x -axis.

Since $a > 0$, the graph is concave up.

The graph is positive definite, which means that it lies entirely above the x -axis.



$\therefore x^2 - 3x + 6 > 0$ for all x .

d $4x - x^2 - 6$ has $a = -1$, $b = 4$, $c = -6$

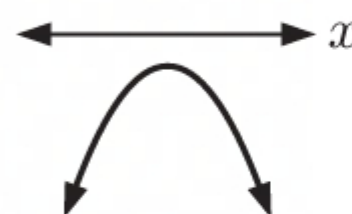
$$\begin{aligned}\therefore \Delta &= b^2 - 4ac \\ &= 4^2 - 4(-1)(-6) \\ &= -8\end{aligned}$$

Since $\Delta < 0$, the graph does not cut the x -axis.

Since $a < 0$, the graph is concave down.

The graph is negative definite, which means that it lies entirely below the x -axis.

$$\therefore 4x - x^2 - 6 < 0 \quad \text{for all } x.$$



5 Consider $y = ax^2 + bx + c$.

The graph is concave up, so $a > 0$.

The y -intercept is positive, so $c > 0$.

The axis of symmetry is to the right of the

y -axis, so $\frac{-b}{2a} > 0$

$$\therefore b < 0 \quad \{a > 0\}$$

The graph does not cut the x -axis, so $\Delta_1 < 0$.

Consider $y = dx^2 + ex + f$.

The graph is concave down, so $d < 0$.

The y -intercept is 0, so $f = 0$.

The axis of symmetry is to the right of the

y -axis, so $\frac{-e}{2d} > 0$

$$\therefore e > 0 \quad \{d < 0\}$$

The graph cuts the x -axis twice, so $\Delta_2 > 0$.

Constant	a	b	c	d	e	f	Δ_1	Δ_2
Sign	+	-	+	-	+	0	-	+

6 a $a = 1$, $b = 3$, $c = k$

$$\begin{aligned}\therefore \Delta &= b^2 - 4ac \\ &= (3)^2 - 4(1)(k) \\ &= 9 - 4k\end{aligned}$$

i The graph cuts the x -axis twice if $\Delta > 0$.

$$\therefore 9 - 4k > 0$$

$$\therefore 4k < 9$$

$$\therefore k < \frac{9}{4}$$

ii The graph touches the x -axis twice if $\Delta = 0$.

$$\therefore 9 - 4k = 0$$

$$\therefore 4k = 9$$

$$\therefore k = \frac{9}{4}$$

iii The graph does not cut the x -axis if $\Delta < 0$.

$$\therefore 9 - 4k < 0$$

$$\therefore 4k > 9$$

$$\therefore k > \frac{9}{4}$$

b $a = k$, $b = -4$, $c = 1$

$$\begin{aligned}\therefore \Delta &= b^2 - 4ac \\ &= (-4)^2 - 4(k)(1) \\ &= 16 - 4k\end{aligned}$$

i The graph cuts the x -axis twice if $\Delta > 0$.

$$\therefore 16 - 4k > 0$$

$$\therefore 4k < 16$$

$$\therefore k < 4, \quad k \neq 0$$

Also, if $k = 0$ then the function is the line $y = -4x + 1$, in which case it cuts the x -axis only *once* at $x = \frac{1}{4}$.

ii The graph touches the x -axis if $\Delta = 0$.

$$\therefore 16 - 4k = 0$$

$$\therefore 4k = 16$$

$$\therefore k = 4$$

iii The graph does not cut the x -axis if $\Delta < 0$.

$$\therefore 16 - 4k < 0$$

$$\therefore 4k > 16$$

$$\therefore k > 4$$

$$\begin{aligned}
 \text{c } a &= k+1, \quad b = -2k, \quad c = k-4 \\
 \therefore \Delta &= b^2 - 4ac \\
 &= (-2k)^2 - 4(k+1)(k-4) \\
 &= 4k^2 - 4(k^2 - 3k - 4) \\
 &= 4k^2 - 4k^2 + 12k + 16 \\
 &= 12k + 16
 \end{aligned}$$

Also, if $k = -1$ then the function is the line $y = 2x - 5$, in which case it cuts the x -axis only *once* at $x = \frac{5}{2}$.

i The graph cuts the x -axis twice if $\Delta > 0$.

$$\begin{aligned}
 \therefore 12k + 16 &> 0 \\
 \therefore 12k &> -16 \\
 \therefore k &> -\frac{4}{3}, \quad k \neq -1
 \end{aligned}$$

ii The graph touches the x -axis if $\Delta = 0$.

$$\begin{aligned}
 \therefore 12k + 16 &= 0 \\
 \therefore 12k &= -16 \\
 \therefore k &= -\frac{4}{3}
 \end{aligned}$$

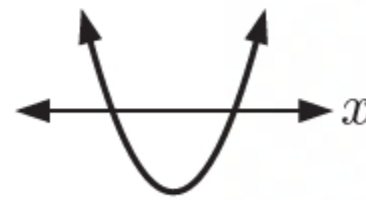
iii The graph does not cut the x -axis if $\Delta < 0$.

$$\begin{aligned}
 \therefore 12k + 16 &< 0 \\
 \therefore 12k &< -16 \\
 \therefore k &< -\frac{4}{3}
 \end{aligned}$$

7 $3x^2 + kx - 1$ has $a = 3, b = k, c = -1$

Since $a > 0$, the graph is concave up.

$$\begin{aligned}
 \Delta &= b^2 - 4ac \\
 &= k^2 - 4(3)(-1) \\
 &= k^2 + 12
 \end{aligned}$$



Now, $k^2 + 12 > 0$ for all k as $k^2 \geq 0$ for all real values of k .

\therefore the graph cuts the x -axis twice for all k .

$\therefore 3x^2 + kx - 1$ is never positive definite for any value of k .

8 $y = \frac{1}{2}x^2 + (k-2)x + k^2 + 4$ has $a = \frac{1}{2}, b = k-2, c = k^2 + 4$

$$\begin{aligned}
 \therefore \Delta &= b^2 - 4ac \\
 &= (k-2)^2 - 4\left(\frac{1}{2}\right)(k^2 + 4) \\
 &= k^2 - 4k + 4 - 2k^2 - 8 \\
 &= -k^2 - 4k - 4 \\
 &= -(k^2 + 4k + 4) \\
 &= -(k+2)^2
 \end{aligned}$$

So, $a > 0$, and $\Delta < 0$ for all $k \neq -2$

\therefore the graph is positive definite for all $k \neq -2$

\therefore the graph is *not* positive definite if $k = -2$, the graph touches the x -axis in this case.

EXERCISE 2D

1 a Since the x -intercepts are 1 and 2, $y = a(x-1)(x-2)$.

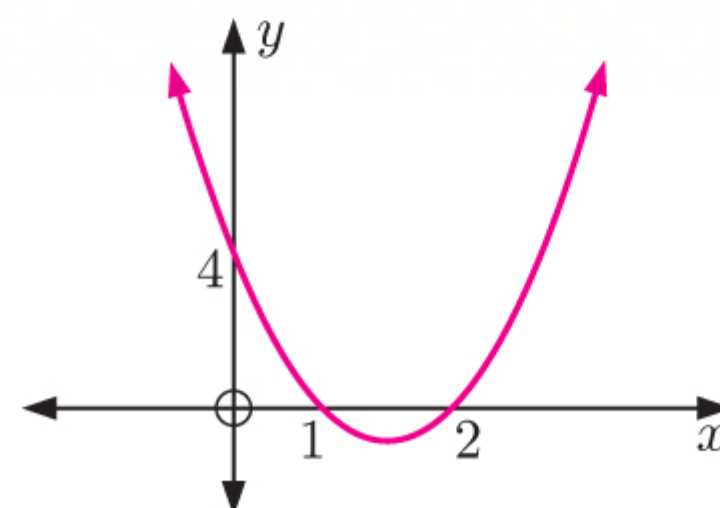
The graph is concave up, so $a > 0$.

When $x = 0$, $y = 4$

$$\therefore 4 = a(-1)(-2)$$

$$\therefore a = 2$$

The quadratic is $y = 2(x-1)(x-2)$.



- b** The graph touches the x -axis at $x = 2$, so $y = a(x - 2)^2$.

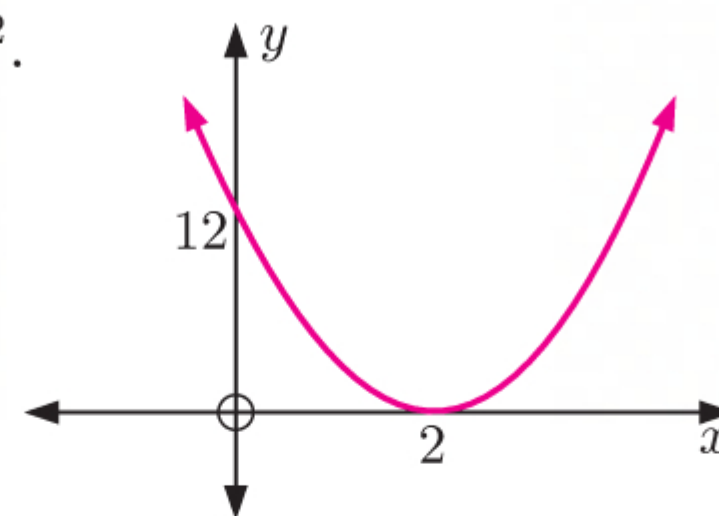
The graph is concave up, so $a > 0$.

When $x = 0$, $y = 12$

$$\therefore 12 = a(-2)^2$$

$$\therefore a = 3$$

The quadratic is $y = 3(x - 2)^2$.



- c** Since the x -intercepts are 1 and 3, $y = a(x - 1)(x - 3)$.

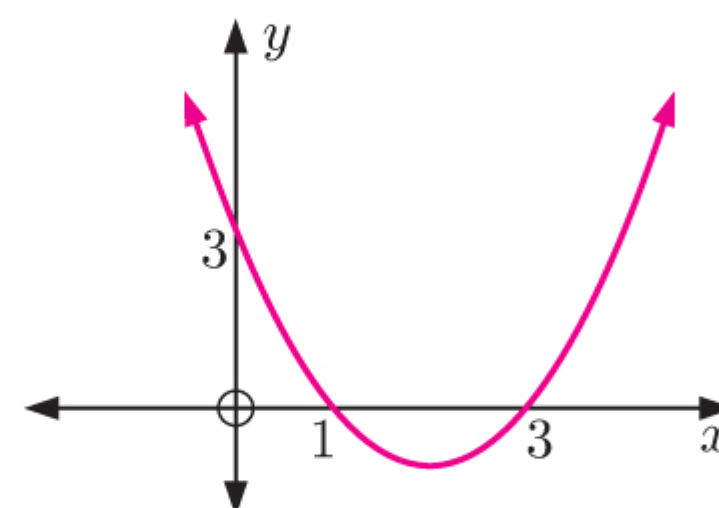
The graph is concave up, so $a > 0$.

When $x = 0$, $y = 3$

$$\therefore 3 = a(-1)(-3)$$

$$\therefore a = 1$$

The quadratic is $y = (x - 1)(x - 3)$.



- d** Since the x -intercepts are -1 and 3 , $y = a(x + 1)(x - 3)$.

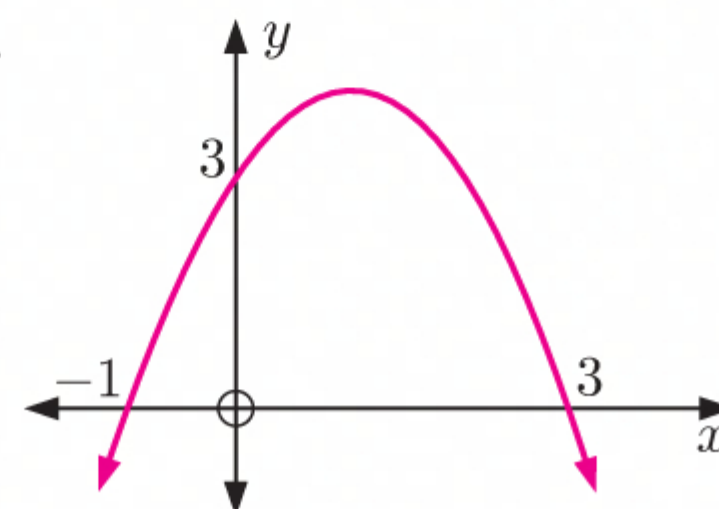
The graph is concave down, so $a < 0$.

When $x = 0$, $y = 3$

$$\therefore 3 = a(1)(-3)$$

$$\therefore a = -1$$

The quadratic is $y = -(x + 1)(x - 3)$.



- e** The graph touches the x -axis at $x = 1$, so $y = a(x - 1)^2$.

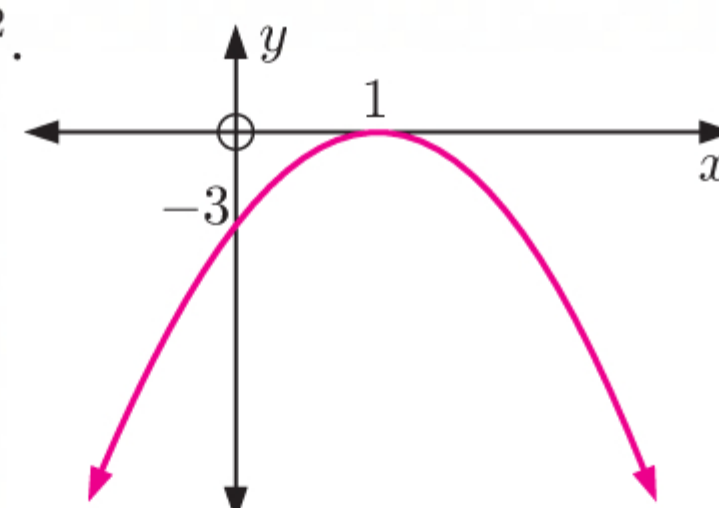
The graph is concave down, so $a < 0$.

When $x = 0$, $y = -3$

$$\therefore -3 = a(-1)^2$$

$$\therefore a = -3$$

The quadratic is $y = -3(x - 1)^2$.



- f** Since the x -intercepts are -2 and 3 , $y = a(x + 2)(x - 3)$.

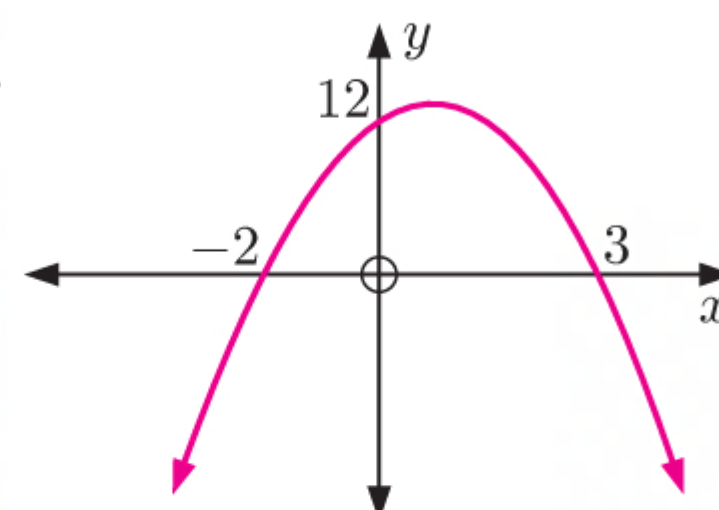
The graph is concave down, so $a < 0$.

When $x = 0$, $y = 12$

$$\therefore 12 = a(2)(-3)$$

$$\therefore a = -2$$

The quadratic is $y = -2(x + 2)(x - 3)$.



- 2 a** The axis of symmetry $x = 3$ lies midway between the x -intercepts.

\therefore the other x -intercept is 4.

\therefore the quadratic has the form

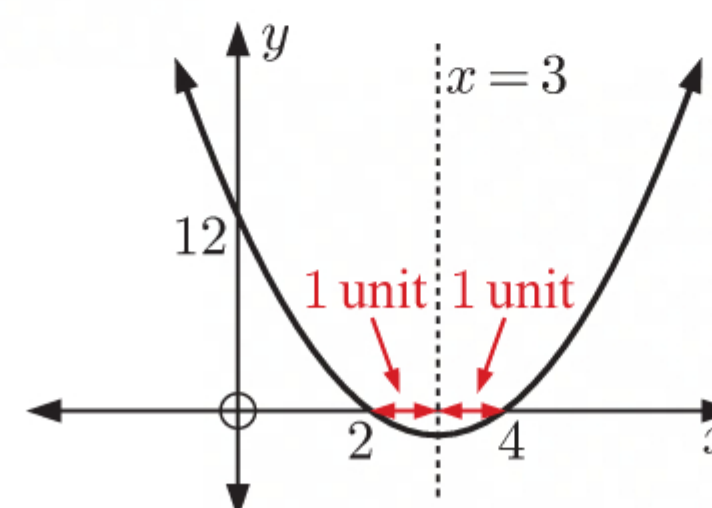
$$y = a(x - 2)(x - 4) \quad \text{where } a > 0$$

But when $x = 0$, $y = 12$

$$\therefore 12 = a(-2)(-4)$$

$$\therefore a = \frac{3}{2}$$

The quadratic is $y = \frac{3}{2}(x - 2)(x - 4)$.



- b** The axis of symmetry $x = -1$ lies midway between the x -intercepts.

\therefore the other x -intercept is 2.

\therefore the quadratic has the form

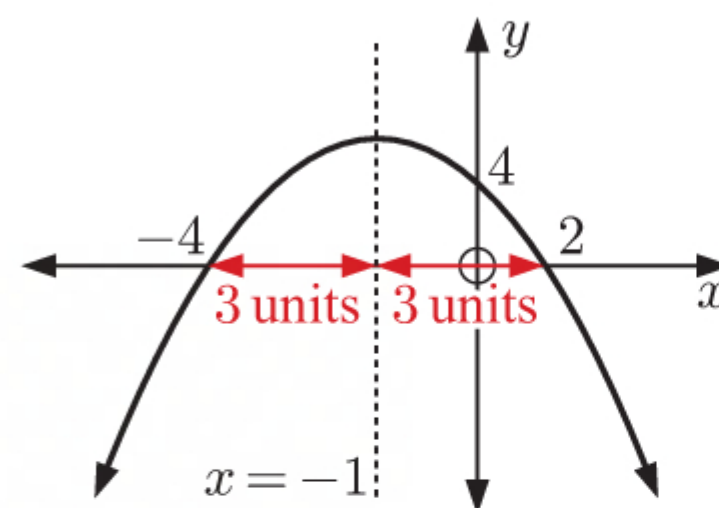
$$y = a(x + 4)(x - 2) \quad \text{where } a < 0$$

But when $x = 0$, $y = 4$

$$\therefore 4 = a(4)(-2)$$

$$\therefore a = -\frac{1}{2}$$

The quadratic is $y = -\frac{1}{2}(x + 4)(x - 2)$.



- c** The graph touches the x -axis at $x = -3$.

\therefore the quadratic has the form

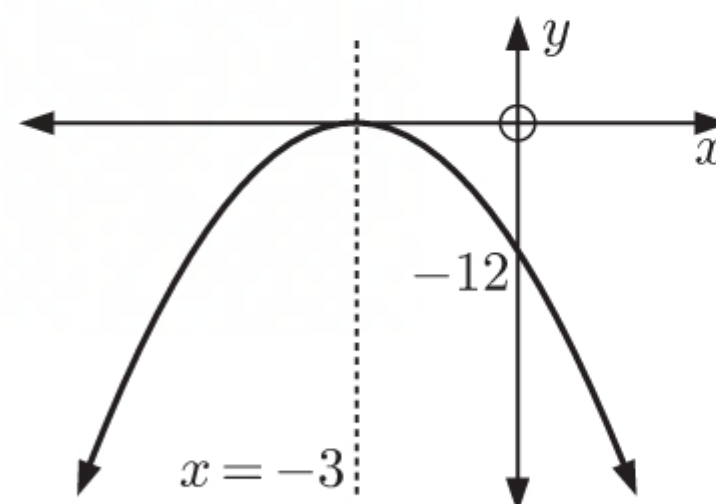
$$y = a(x + 3)^2 \quad \text{where } a < 0$$

But when $x = 0$, $y = -12$

$$\therefore -12 = a(3)^2$$

$$\therefore a = -\frac{4}{3}$$

The quadratic is $y = -\frac{4}{3}(x + 3)^2$.



- 3 a** Since the x -intercepts are 5 and 1, the quadratic has the form

$$y = a(x - 5)(x - 1), \quad a \neq 0.$$

When $x = 2$, $y = -9$

$$\therefore -9 = a(2 - 5)(2 - 1)$$

$$\therefore -9 = a(-3)(1)$$

$$\therefore a = 3$$

The quadratic is $y = 3(x - 5)(x - 1)$

$$= 3(x^2 - 6x + 5)$$

$$\therefore y = 3x^2 - 18x + 15$$

- b** Since the x -intercepts are 2 and $-\frac{1}{2}$, the quadratic has the form

$$y = a(x - 2)(2x + 1), \quad a \neq 0.$$

When $x = 3$, $y = -14$

$$\therefore -14 = a(3 - 2)(2(3) + 1)$$

$$\therefore -14 = a(1)(7)$$

$$\therefore a = -2$$

The quadratic is $y = -2(x - 2)(2x + 1)$

$$= -2(2x^2 - 3x - 2)$$

$$\therefore y = -4x^2 + 6x + 4$$

- c Since the graph touches the x -axis at 3, the quadratic has the form

$$y = a(x - 3)^2, \quad a \neq 0.$$

When $x = -2$, $y = -25$

$$\therefore -25 = a(-2 - 3)^2$$

$$\therefore -25 = a(-5)^2$$

$$\therefore a = -1$$

The quadratic is $y = -(x - 3)^2$
 $= -(x^2 - 6x + 9)$
 $\therefore y = -x^2 + 6x - 9$

- d Since the graph touches the x -axis at -2 , the quadratic has the form

$$y = a(x + 2)^2, \quad a \neq 0.$$

When $x = -1$, $y = 4$

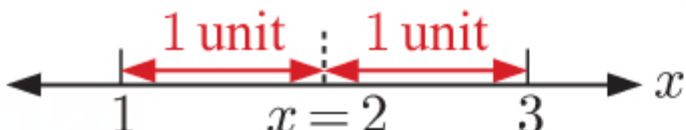
$$\therefore 4 = a(-1 + 2)^2$$

$$\therefore 4 = a(1)^2$$

$$\therefore a = 4$$

The quadratic is $y = 4(x + 2)^2$
 $= 4(x^2 + 4x + 4)$
 $\therefore y = 4x^2 + 16x + 16$

- e The axis of symmetry $x = 2$ lies midway between the x -intercepts.

\therefore the other x -intercept is 1. 

Since the x -intercepts are 3 and 1, the quadratic has the form

$$y = a(x - 3)(x - 1), \quad a \neq 0.$$

When $x = 5$, $y = 12$

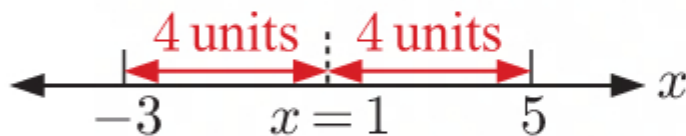
$$\therefore 12 = a(5 - 3)(5 - 1)$$

$$\therefore 12 = a(2)(4)$$

$$\therefore a = \frac{3}{2}$$

The quadratic is $y = \frac{3}{2}(x - 3)(x - 1)$
 $= \frac{3}{2}(x^2 - 4x + 3)$
 $\therefore y = \frac{3}{2}x^2 - 6x + \frac{9}{2}$

- f The axis of symmetry $x = 1$ lies midway between the x -intercepts.

\therefore the other x -intercept is -3 . 

Since the x -intercepts are 5 and -3 , the quadratic has the form

$$y = a(x - 5)(x + 3), \quad a \neq 0.$$

When $x = 2$, $y = 5$

$$\therefore 5 = a(2 - 5)(2 + 3)$$

$$\therefore 5 = a(-3)(5)$$

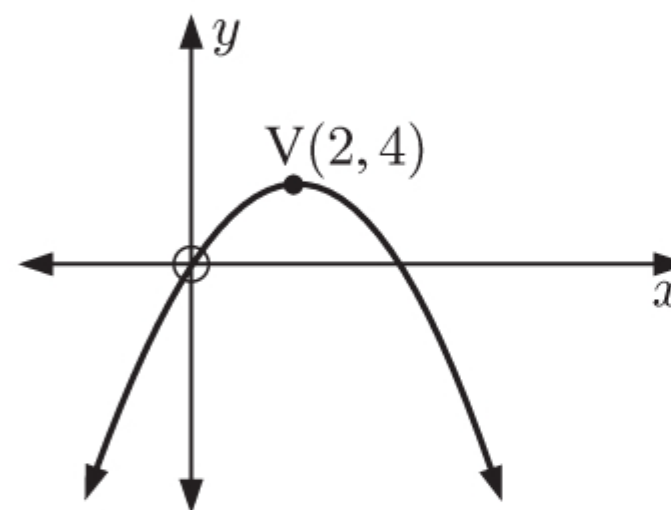
$$\therefore a = -\frac{1}{3}$$

The quadratic is $y = -\frac{1}{3}(x-5)(x+3)$
 $= -\frac{1}{3}(x^2 - 2x - 15)$
 $\therefore y = -\frac{1}{3}x^2 + \frac{2}{3}x + 5$

- 4 a** Since the vertex is $(2, 4)$, the quadratic has the form $y = a(x-2)^2 + 4$, where $a < 0$.

When $x = 0$, $y = 0$
 $\therefore 0 = a(-2)^2 + 4$
 $\therefore 0 = 4a + 4$
 $\therefore -4 = 4a$
 $\therefore a = -1$

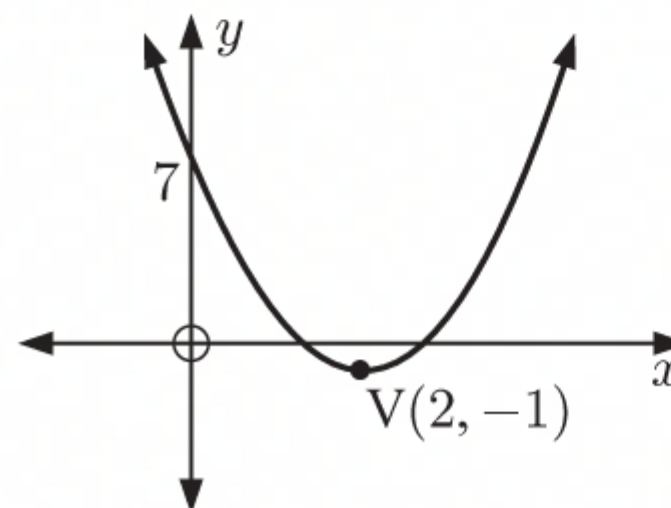
The quadratic is $y = -(x-2)^2 + 4$.



- b** Since the vertex is $(2, -1)$, the quadratic has the form $y = a(x-2)^2 - 1$, where $a > 0$.

When $x = 0$, $y = 7$
 $\therefore 7 = a(-2)^2 - 1$
 $\therefore 7 = 4a - 1$
 $\therefore 8 = 4a$
 $\therefore a = 2$

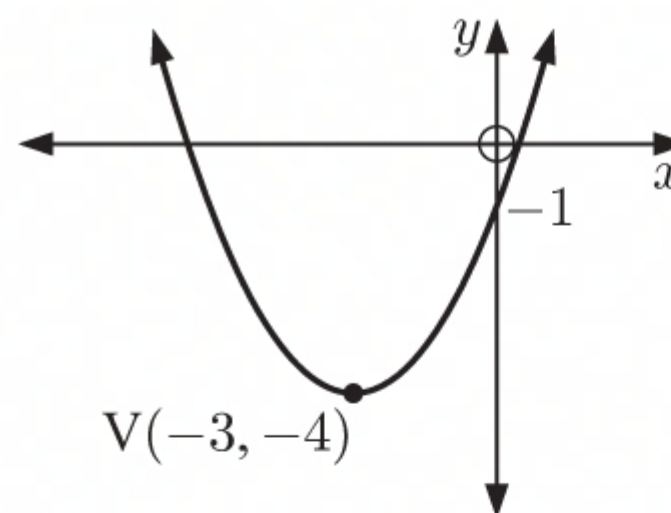
The quadratic is $y = 2(x-2)^2 - 1$.



- c** Since the vertex is $(-3, -4)$, the quadratic has the form $y = a(x+3)^2 - 4$, where $a > 0$.

When $x = 0$, $y = -1$
 $\therefore -1 = a(0+3)^2 - 4$
 $\therefore -1 = a(3)^2 - 4$
 $\therefore -1 = 9a - 4$
 $\therefore 3 = 9a$
 $\therefore a = \frac{1}{3}$

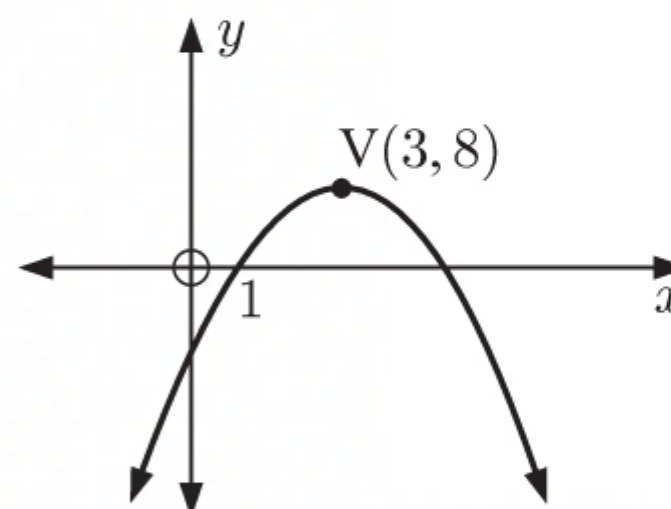
The quadratic is $y = \frac{1}{3}(x+3)^2 - 4$.



- d** Since the vertex is $(3, 8)$, the quadratic has the form $y = a(x-3)^2 + 8$, where $a < 0$.

When $x = 1$, $y = 0$
 $\therefore 0 = a(1-3)^2 + 8$
 $\therefore 0 = a(-2)^2 + 8$
 $\therefore 0 = 4a + 8$
 $\therefore -8 = 4a$
 $\therefore a = -2$

The quadratic is $y = -2(x-3)^2 + 8$.



- e** Since the vertex is $(4, -6)$, the quadratic has the form $y = a(x - 4)^2 - 6$, where $a > 0$.

When $x = 7$, $y = 0$

$$\therefore 0 = a(7 - 4)^2 - 6$$

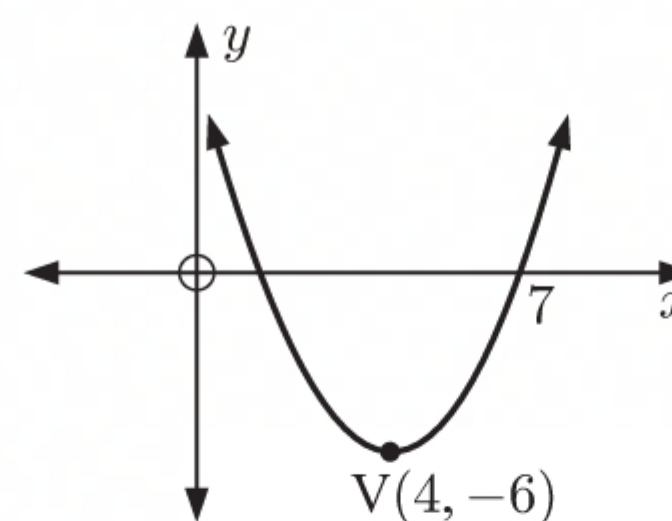
$$\therefore 0 = a(3)^2 - 6$$

$$\therefore 0 = 9a - 6$$

$$\therefore 6 = 9a$$

$$\therefore a = \frac{2}{3}$$

The quadratic is $y = \frac{2}{3}(x - 4)^2 - 6$.



- f** Since the vertex is $(-2, 5)$, the quadratic has the form $y = a(x + 2)^2 + 5$, where $a < 0$.

When $x = -5$, $y = 0$

$$\therefore 0 = a(-5 + 2)^2 + 5$$

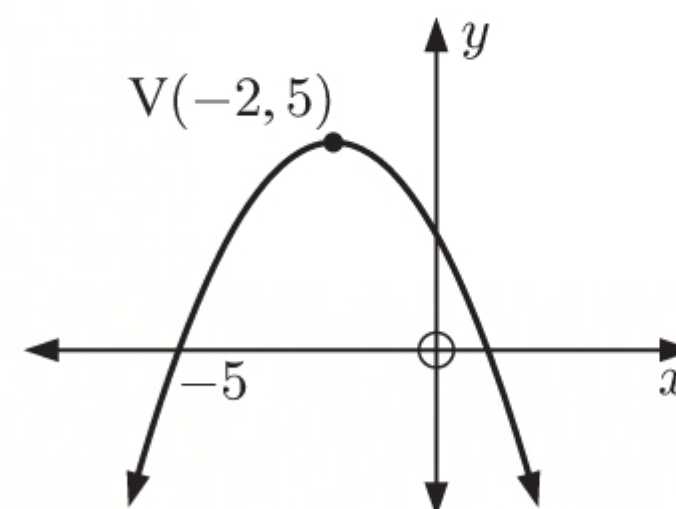
$$\therefore 0 = a(-3)^2 + 5$$

$$\therefore 0 = 9a + 5$$

$$\therefore 9a = -5$$

$$\therefore a = -\frac{5}{9}$$

The quadratic is $y = -\frac{5}{9}(x + 2)^2 + 5$.



- g** Since the vertex is $(2, 3)$, the quadratic has the form $y = a(x - 2)^2 + 3$, where $a < 0$.

When $x = 3$, $y = 1$

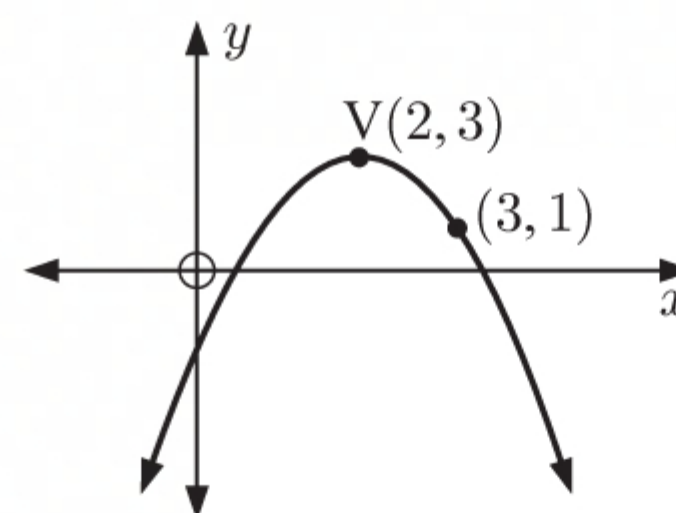
$$\therefore 1 = a(3 - 2)^2 + 3$$

$$\therefore 1 = a(1)^2 + 3$$

$$\therefore 1 = a + 3$$

$$\therefore a = -2$$

The quadratic is $y = -2(x - 2)^2 + 3$.



- h** Since the vertex is $(-4, 3)$, the quadratic has the form $y = a(x + 4)^2 + 3$, where $a > 0$.

When $x = -6$, $y = 9$

$$\therefore 9 = a(-6 + 4)^2 + 3$$

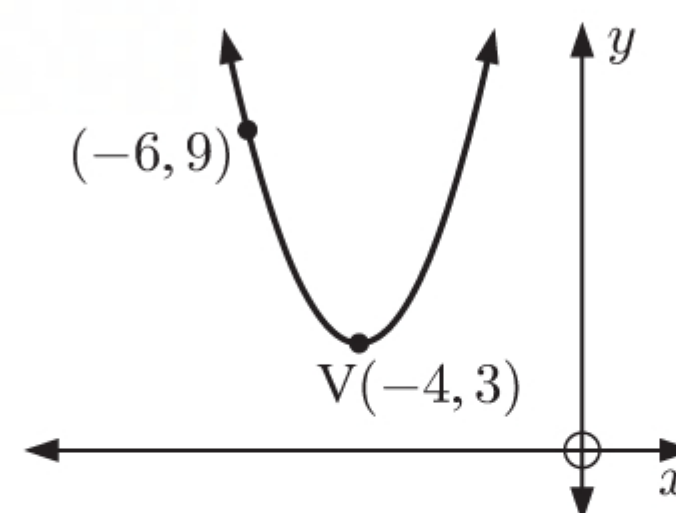
$$\therefore 9 = a(-2)^2 + 3$$

$$\therefore 9 = 4a + 3$$

$$\therefore 6 = 4a$$

$$\therefore a = \frac{3}{2}$$

The quadratic is $y = \frac{3}{2}(x + 4)^2 + 3$.



- i Since the vertex is $(\frac{1}{2}, -\frac{3}{2})$, the quadratic has the form $y = a(x - \frac{1}{2})^2 - \frac{3}{2}$, where $a > 0$.

When $x = \frac{3}{2}$, $y = \frac{1}{2}$

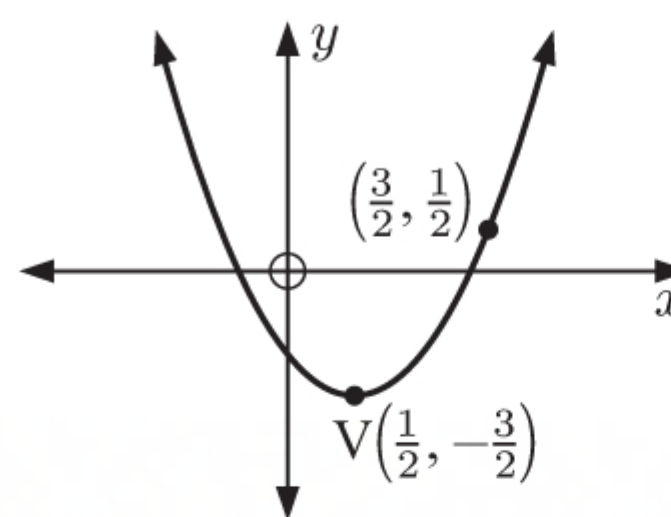
$$\therefore \frac{1}{2} = a(\frac{3}{2} - \frac{1}{2})^2 - \frac{3}{2}$$

$$\therefore \frac{1}{2} = a(1)^2 - \frac{3}{2}$$

$$\therefore \frac{1}{2} = a - \frac{3}{2}$$

$$\therefore a = 2$$

The quadratic is $y = 2(x - \frac{1}{2})^2 - \frac{3}{2}$.



- 5 Since the vertex is $(2, -5)$, the quadratic has the form $y = a(x - 2)^2 - 5$, $a \neq 0$.

When $x = -1$, $y = 13$

$$\therefore 13 = a(-1 - 2)^2 - 5$$

$$\therefore 13 = a(-3)^2 - 5$$

$$\therefore 18 = 9a$$

$$\therefore a = 2$$

The quadratic is $y = 2(x - 2)^2 - 5$.

When $x = 4$, $y = 2(4 - 2)^2 - 5$

$$= 2(2)^2 - 5$$

$$\therefore y = 3$$

INVESTIGATION 3

FINDING QUADRATICS

1 a $y = x^2 + 4x + 3$

x	0	1	2	3	4	5
y	3	8	15	24	35	48
Δ_1	5	7	9	11	13	
Δ_2	2	2	2	2		

b $y = 3x^2 - 4x$

x	0	1	2	3	4	5
y	0	-1	4	15	32	55
Δ_1	-1	5	11	17	23	
Δ_2	6	6	6	6		

c $y = 5x - x^2$

x	0	1	2	3	4	5
y	0	4	6	6	4	0
Δ_1	4	2	0	-2	-4	
Δ_2	-2	-2	-2	-2		

d $y = 4x^2 - 5x + 2$

x	0	1	2	3	4	5
y	2	1	8	23	46	77
Δ_1	-1	7	15	23	31	
Δ_2	8	8	8	8		

- 2 For each quadratic in 1, every value in the Δ_2 row is a single constant.

3 a $y = ax^2 + bx + c$

x	0	1	2	3	4	5
y	c	$a + b + c$	$4a + 2b + c$	$9a + 3b + c$	$16a + 4b + c$	$25a + 5b + c$
Δ_1	$a + b$	$3a + b$	$5a + b$	$7a + b$	$9a + b$	
Δ_2	$2a$	$2a$	$2a$	$2a$	$2a$	

- b** Every value in the Δ_2 row is the constant $2a$.
- c** We can use the circled numbers to find the constants a , b , and c in $y = ax^2 + bx + c$ if we are given a quadratic as a table of values.

4 a

x	0	1	2	3	4
y	⑥	5	8	15	26
Δ_1	①	3	7	11	
Δ_2		④	4	4	

$$c = 6, \quad 2a = 4 \quad a + b = -1$$

$$\therefore a = 2 \quad \therefore 2 + b = -1$$

$$\therefore b = -3$$

\therefore the quadratic with this table of values is $y = 2x^2 - 3x + 6$.

b

x	0	1	2	3	4
y	⑧	10	18	32	52
Δ_1		②	8	14	20
Δ_2			⑥	6	6

$$c = 8, \quad 2a = 6 \quad a + b = 2$$

$$\therefore a = 3 \quad \therefore 3 + b = 2$$

$$\therefore b = -1$$

\therefore the quadratic with this table of values is $y = 3x^2 - x + 8$.

c

x	0	1	2	3	4
y	①	2	-1	-8	-19
Δ_1		①	-3	-7	-11
Δ_2			①	-4	-4

$$c = 1, \quad 2a = -4 \quad a + b = -1$$

$$\therefore a = -2 \quad \therefore -2 + b = 1$$

$$\therefore b = 3$$

\therefore the quadratic with this table of values is $y = -2x^2 + 3x + 1$.

d

x	0	1	2	3	4
y	⑤	3	-1	-7	-15
Δ_1		①	-4	-6	-8
Δ_2			①	-2	-2

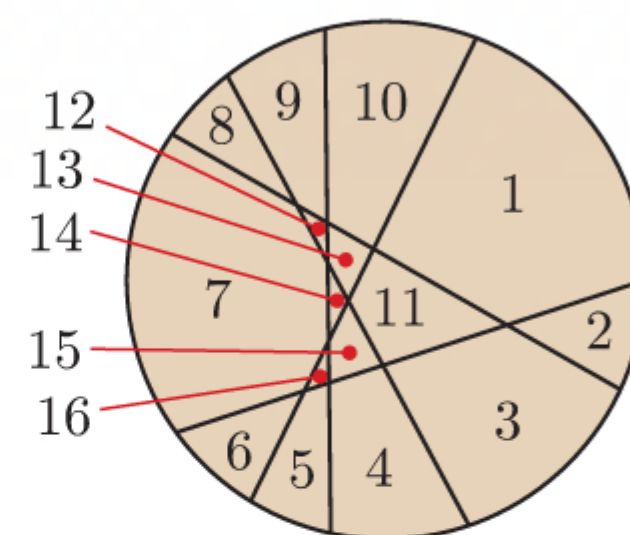
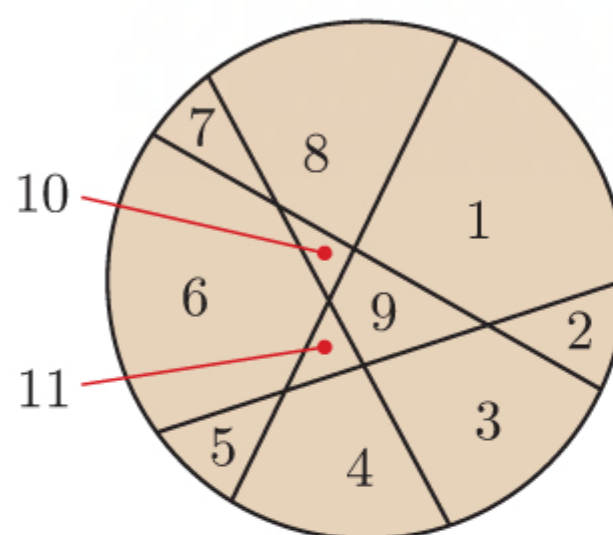
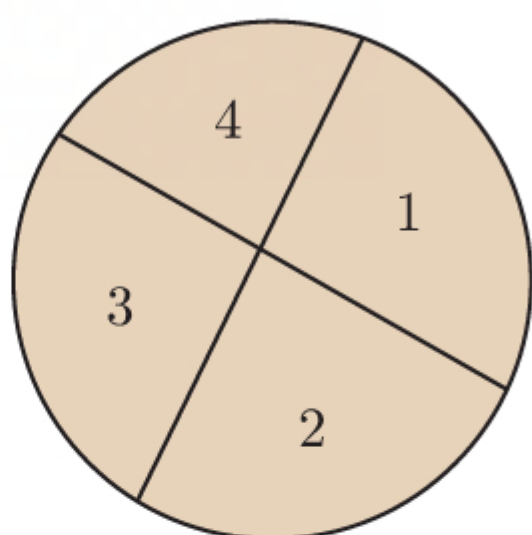
$$c = 5, \quad 2a = -2 \quad a + b = -2$$

$$\therefore a = -1 \quad \therefore -1 + b = -2$$

$$\therefore b = -1$$

\therefore the quadratic with this table of values is $y = -x^2 - x + 5$.

- 5**
- for $n = 2$ we can make 4 pieces
 - for $n = 4$ we can make 11 pieces.
 - for $n = 5$ we can make 16 pieces.



a

Number of cuts, n	0	1	2	3	4	5
Maximum number of pieces, P_n	①	2	4	7	11	16

b

Δ_1	①	2	3	4	5	
Δ_2		①	1	1	1	

$$c = 1, \quad 2a = 1 \quad a + b = 1$$

$$\therefore a = \frac{1}{2} \quad \therefore \frac{1}{2} + b = 1$$

$$\therefore b = \frac{1}{2}$$

$$\therefore P_n = \frac{1}{2}n^2 + \frac{1}{2}n + 1$$

$$\begin{aligned}
 \text{c If } n = 12, \quad P_n &= \frac{1}{2}(12)^2 + \frac{1}{2}(12) + 1 \\
 &= 72 + 6 + 1 \\
 &= 79
 \end{aligned}$$

\therefore the maximum number of pieces for a pizza with 12 cuts is 79.

EXERCISE 2E

1 a $y = x^2 - 2x + 8$ meets $y = x + 6$ where

$$\begin{aligned}
 x^2 - 2x + 8 &= x + 6 \\
 \therefore x^2 - 3x + 2 &= 0 \\
 \therefore (x - 1)(x - 2) &= 0 \\
 \therefore x &= 1 \text{ or } 2
 \end{aligned}$$

Substituting into $y = x + 6$, when $x = 1$, $y = 7$ and when $x = 2$, $y = 8$.
 \therefore the graphs meet at $(1, 7)$ and $(2, 8)$.

b $y = -x^2 + 3x + 9$ meets $y = 2x - 3$ where

$$\begin{aligned}
 -x^2 + 3x + 9 &= 2x - 3 \\
 \therefore x^2 - x - 12 &= 0 \\
 \therefore (x - 4)(x + 3) &= 0 \\
 \therefore x &= 4 \text{ or } -3
 \end{aligned}$$

Substituting into $y = 2x - 3$, when $x = 4$, $y = 2(4) - 3 = 5$ and when $x = -3$, $y = 2(-3) - 3 = -9$.
 \therefore the graphs meet at $(4, 5)$ and $(-3, -9)$.

c $y = x^2 - 4x + 3$ meets $y = 2x - 6$ where

$$\begin{aligned}
 x^2 - 4x + 3 &= 2x - 6 \\
 \therefore x^2 - 6x + 9 &= 0 \\
 \therefore (x - 3)^2 &= 0 \\
 \therefore x &= 3
 \end{aligned}$$

Substituting into $y = 2x - 6$, when $x = 3$, $y = 0$.
 \therefore the graphs touch at $(3, 0)$.

d $y = -x^2 + 4x - 7$ meets $y = 5x - 4$ where

$$\begin{aligned}
 -x^2 + 4x - 7 &= 5x - 4 \\
 \therefore x^2 + x + 3 &= 0
 \end{aligned}$$

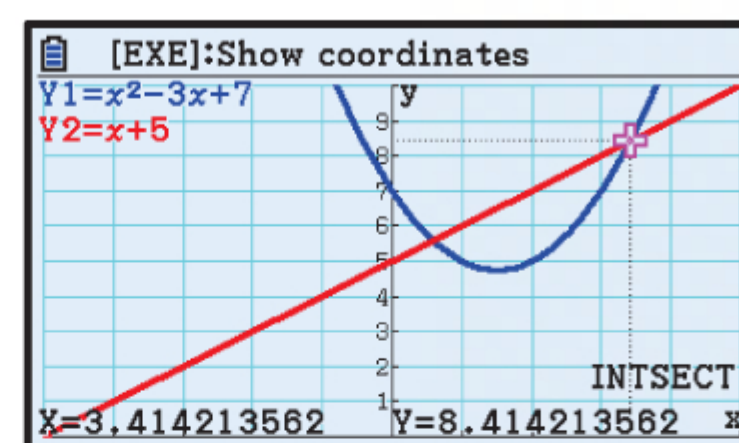
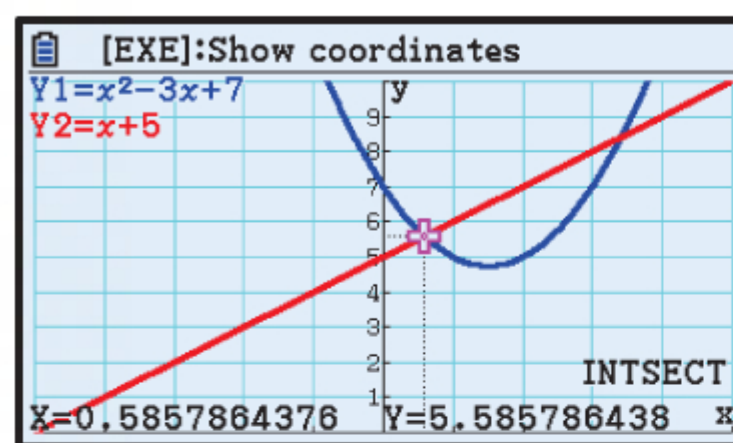
which has $a = 1$, $b = 1$, $c = 3$

$$\therefore x = \frac{-1 \pm \sqrt{1^2 - 4(1)(3)}}{2(1)}$$

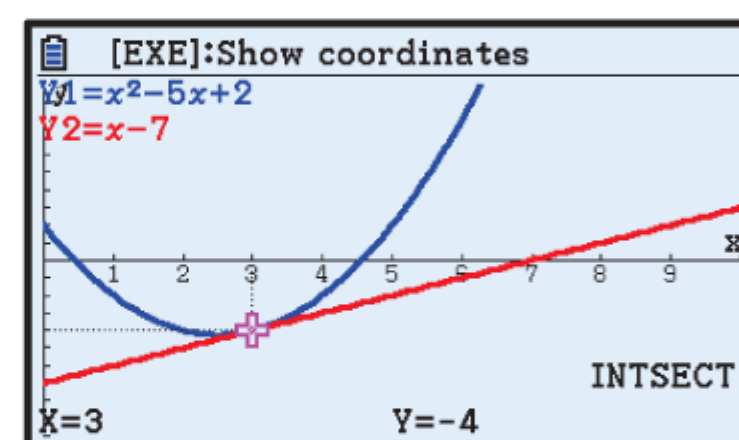
$$\therefore x = \frac{-1 \pm \sqrt{-11}}{2}$$

\therefore there are no real solutions
 \therefore the graphs do not meet.

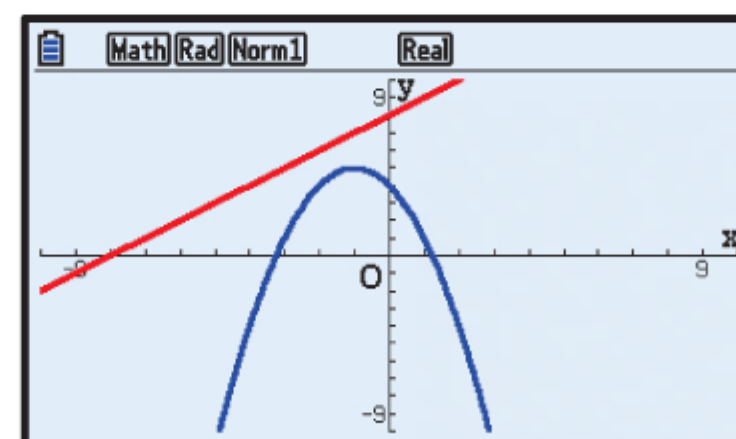
- 2 a $y = x^2 - 3x + 7$ and $y = x + 5$ intersect at $(0.586, 5.59)$ and $(3.41, 8.41)$.



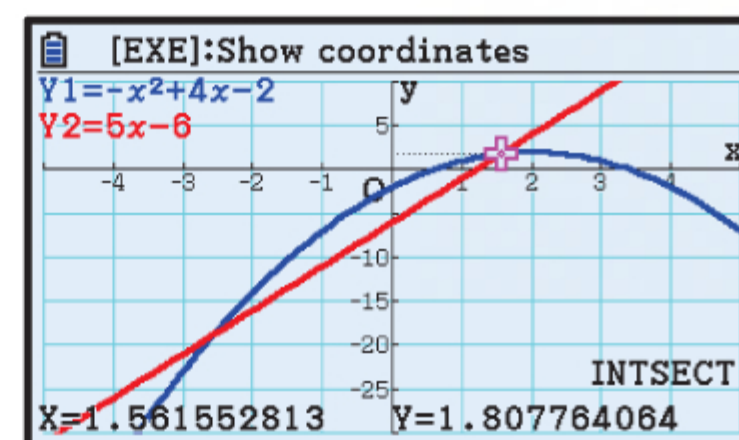
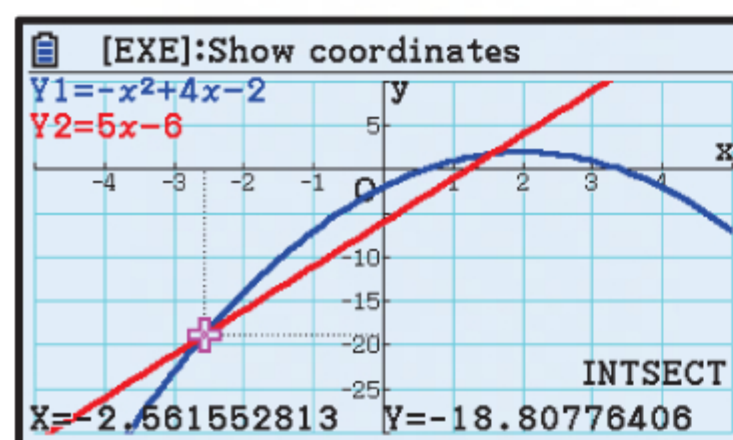
- b $y = x^2 - 5x + 2$ and $y = x - 7$ intersect at $(3, -4)$ (touching).



- c $y = -x^2 - 2x + 4$ and $y = x + 8$ do not intersect.



- d $y = -x^2 + 4x - 2$ and $y = 5x - 6$ intersect at $(-2.56, -18.8)$ and $(1.56, 1.81)$.



- 3 a $y = x^2$ meets $y = x + 2$ where

$$x^2 = x + 2$$

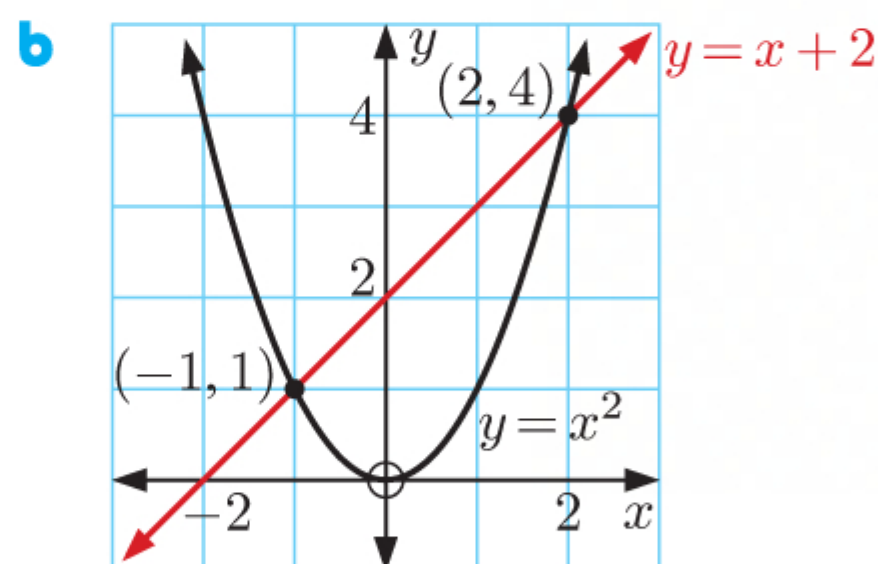
$$\therefore x^2 - x - 2 = 0$$

$$\therefore (x + 1)(x - 2) = 0$$

$$\therefore x = -1 \text{ or } 2$$

Substituting into $y = x + 2$, when $x = -1$, $y = 1$ and when $x = 2$, $y = 4$.

\therefore the graphs meet at $(-1, 1)$ and $(2, 4)$.



- c If $x^2 > x + 2$, the graph of $y = x^2$ is above the graph of $y = x + 2$. This occurs when $x < -1$ or $x > 2$.

- 4 a** $y = x^2 + 2x - 3$ meets $y = x - 1$ where

$$x^2 + 2x - 3 = x - 1$$

$$\therefore x^2 + x - 2 = 0$$

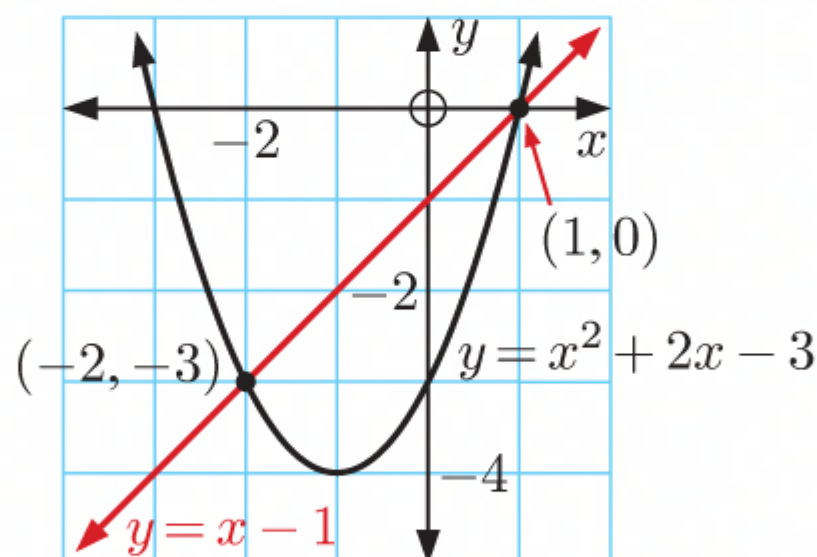
$$\therefore (x + 2)(x - 1) = 0$$

$$\therefore x = -2 \text{ or } 1$$

Substituting into $y = x - 1$, when $x = -2$, $y = -3$ and when $x = 1$, $y = 0$.

\therefore the graphs meet at $(-2, -3)$ and $(1, 0)$.

b



- c** If $x^2 + 2x - 3 > x - 1$, the graph of $y = x^2 + 2x - 3$ is above the graph of $y = x - 1$. This occurs when $x < -2$ or $x > 1$.

- 5 a** $y = 2x^2 - x + 3$ meets $y = 2 + x + x^2$ where

$$2x^2 - x + 3 = 2 + x + x^2$$

$$\therefore x^2 - 2x + 1 = 0$$

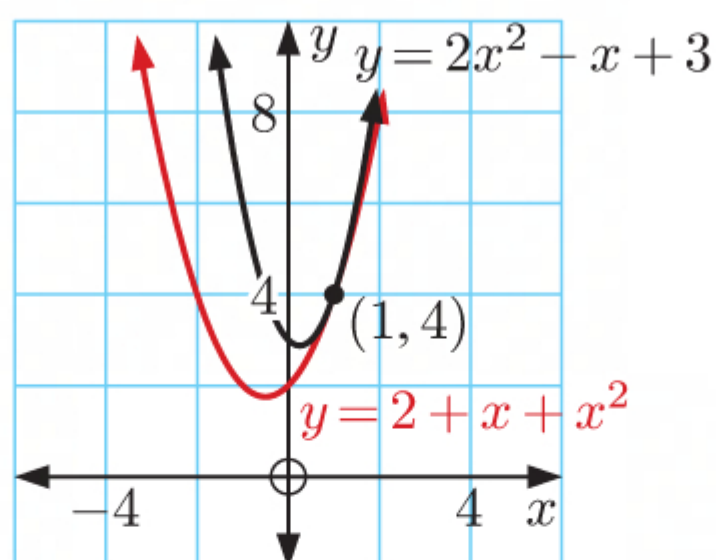
$$\therefore (x - 1)^2 = 0$$

$$\therefore x = 1$$

Substituting into $y = 2 + x + x^2$, when $x = 1$, $y = 4$.

\therefore the graphs meet at $(1, 4)$.

b

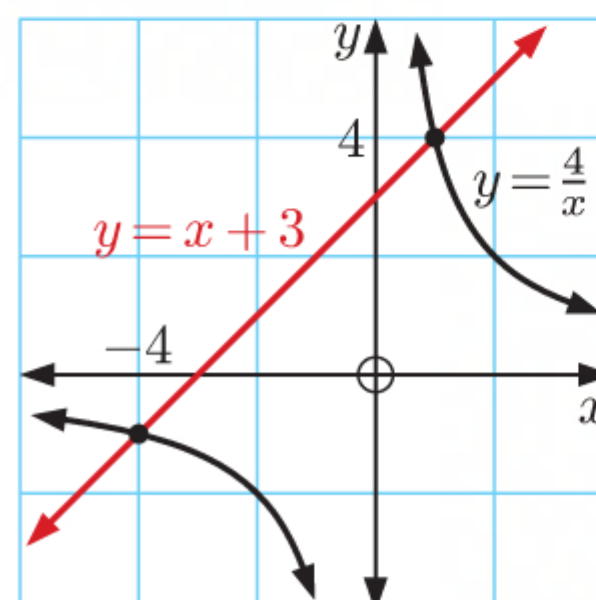


- c** If $2x^2 - x + 3 > 2 + x + x^2$, the graph of $y = 2x^2 - x + 3$ is above the graph of $y = 2 + x + x^2$.

This occurs for all $x \in \mathbb{R}$, $x \neq 1$.

- 6 a** $\frac{4}{x} = x + 3$
 $\therefore 4 = x^2 + 3x$
 $\therefore x^2 + 3x - 4 = 0$
 $\therefore (x + 4)(x - 1) = 0$
 $\therefore x = -4 \text{ or } 1$

b



- If $\frac{4}{x} > x + 3$, the graph of $y = \frac{4}{x}$ is above the graph of $y = x + 3$.

This occurs when $x < -4$ or $0 < x < 1$.

- 7 $y = 3x + c$ is a tangent to $y = x^2 - 5x + 7$ if they meet at exactly one point (touch).

$y = x^2 - 5x + 7$ meets $y = 3x + c$ where $x^2 - 5x + 7 = 3x + c$

$$\therefore x^2 - 8x + (7 - c) = 0$$

The graphs meet exactly once when this equation has a repeated root $\therefore \Delta = 0$

$$\therefore (-8)^2 - 4(1)(7 - c) = 0$$

$$\therefore 64 - 28 + 4c = 0$$

$$\therefore 4c = -36$$

$$\therefore c = -9$$

- 8 $y = mx - 2$ is a tangent to $y = x^2 - 4x + 2$ if they meet at exactly one point (touch).

$y = x^2 - 4x + 2$ meets $y = mx - 2$ where $x^2 - 4x + 2 = mx - 2$

$$\therefore x^2 - (m + 4)x + 4 = 0$$

The graphs meet exactly once when this equation has a repeated root $\therefore \Delta = 0$

$$\therefore (-(m + 4))^2 - 4(1)(4) = 0$$

$$\therefore m^2 + 8m + 16 - 16 = 0$$

$$\therefore m(m + 8) = 0$$

$$\therefore m = 0 \text{ or } -8$$

- 9 Lines with y -intercept 1 have the form $y = mx + 1$.

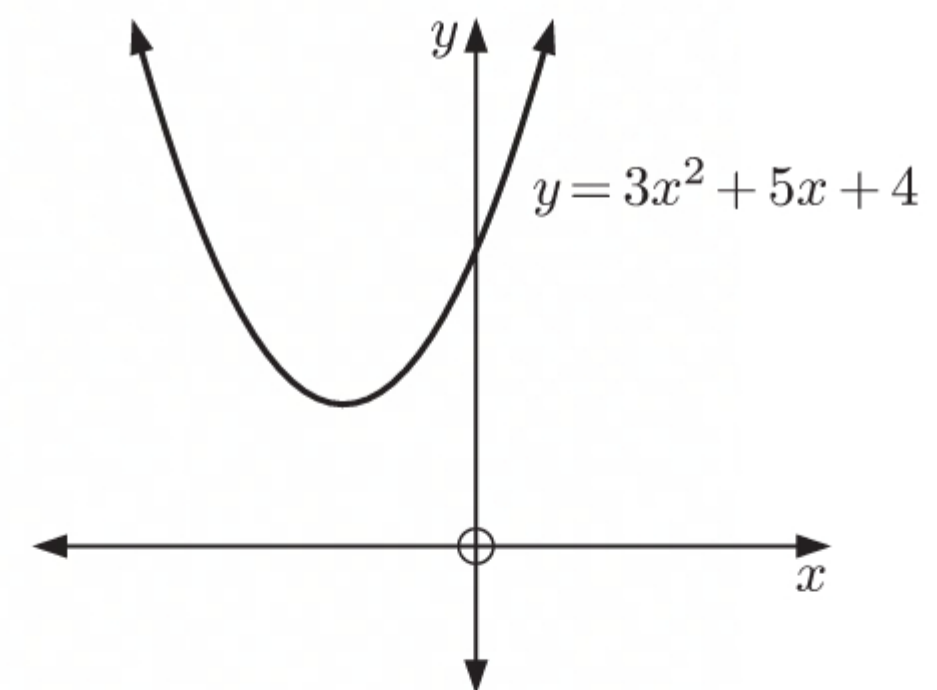
$y = mx + 1$ is a tangent to $y = 3x^2 + 5x + 4$

if they meet at exactly one point (touch).

$y = 3x^2 + 5x + 4$ meets $y = mx + 1$

where $3x^2 + 5x + 4 = mx + 1$

$$\therefore 3x^2 + (5 - m)x + 3 = 0$$



The graphs meet exactly once when this equation has a repeated root $\therefore \Delta = 0$

$$\therefore (5 - m)^2 - 4(3)(3) = 0$$

$$\therefore 25 - 10m + m^2 - 36 = 0$$

$$\therefore m^2 - 10m - 11 = 0$$

$$\therefore (m + 1)(m - 11) = 0$$

$$\therefore m = -1 \text{ or } 11$$

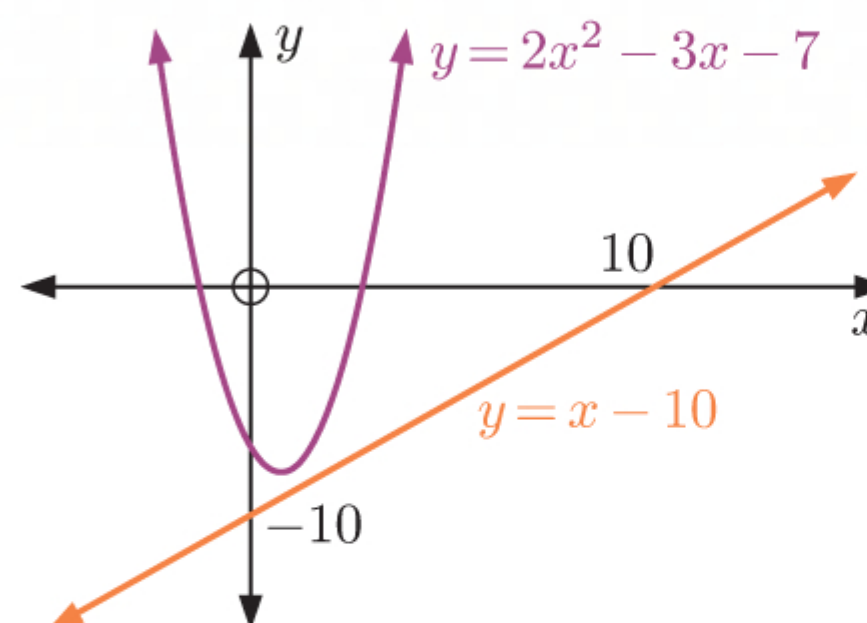
\therefore the required lines have gradient -1 or 11 .

- 10 a** $y = x + c$ meets $y = 2x^2 - 3x - 7$
 where $2x^2 - 3x - 7 = x + c$
 $\therefore 2x^2 - 4x - (7 + c) = 0$

The graphs will never meet if this equation has no real roots.

$$\begin{aligned}\therefore \Delta &< 0 \\ \therefore (-4)^2 - 4(2)(-(7 + c)) &< 0 \\ \therefore 16 + 56 + 8c &< 0 \\ \therefore 8c &< -72 \\ \therefore c &< -9\end{aligned}$$

- b Note:** Other solutions are possible.
 Choose c such that $c < -9$,
 for example $c = -10$:



- 11** Let the two quadratics be $y = a_1x^2 + b_1x + c_1$ and $y = a_2x^2 + b_2x + c_2$.

The graphs meet where $a_1x^2 + b_1x + c_1 = a_2x^2 + b_2x + c_2$

$$\therefore (a_1 - a_2)x^2 + (b_1 - b_2)x + (c_1 - c_2) = 0$$

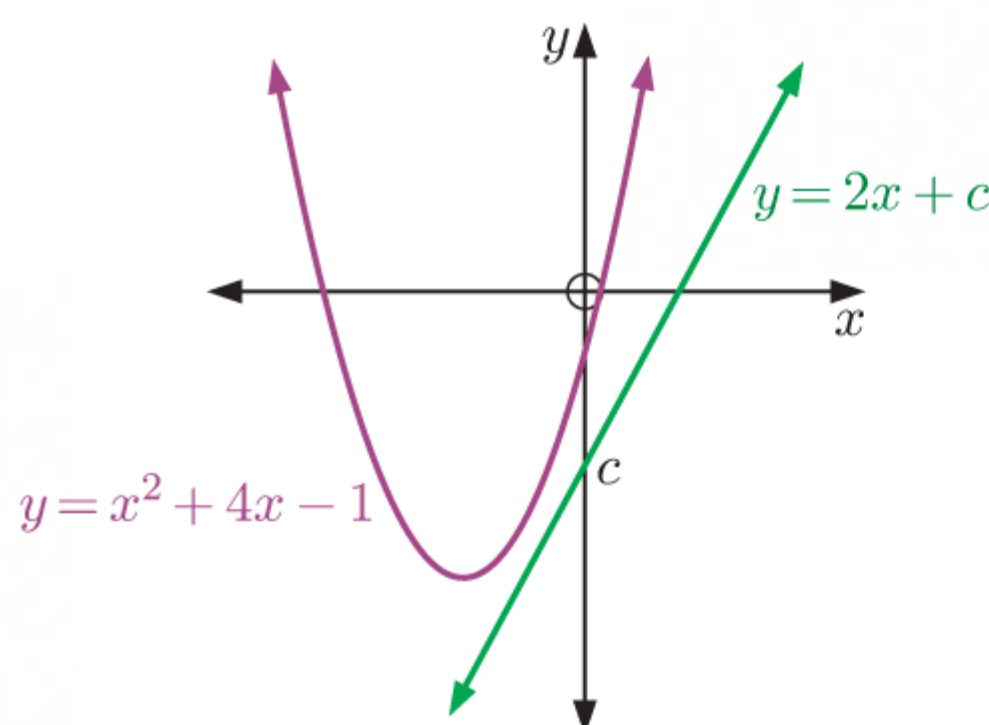
which is a quadratic with at most two solutions (when $\Delta > 0$).

\therefore two quadratic functions can intersect at most twice.

- 12** $y = 2x + c$ meets $y = x^2 + 4x - 1$ where

$$\begin{aligned}x^2 + 4x - 1 &= 2x + c \\ \therefore x^2 + 2x - (1 + c) &= 0\end{aligned}$$

$$\begin{aligned}\text{Now } \Delta &= b^2 - 4ac \\ &= 2^2 - 4(1)(-(1 + c)) \\ &= 4 + 4(1 + c) \\ &= 4 + 4 + 4c \\ &= 4c + 8\end{aligned}$$



- a** The graphs meet twice if $\Delta > 0$
 $\therefore 4c + 8 > 0$
 $\therefore 4c > -8$
 $\therefore c > -2$

- b** The graphs touch if $\Delta = 0$
 $\therefore 4c + 8 = 0$
 $\therefore 4c = -8$
 $\therefore c = -2$

- c** The graphs do not meet if $\Delta < 0$
 $\therefore 4c + 8 < 0$
 $\therefore 4c < -8$
 $\therefore c < -2$

- 13** Let the linear function have equation $y = mx + c$.

Since the y -intercept is 3, then $y = mx + 3$.

$y = mx + 3$ meets $y = 2x^2 - x - 2$ where

$$2x^2 - x - 2 = mx + 3$$

$$\therefore 2x^2 - (1 + m)x - 5 = 0$$

Now $\Delta = b^2 - 4ac$

$$= (-(1 + m))^2 - 4(2)(-5)$$

$$= 1 + 2m + m^2 + 40$$

$$= m^2 + 2m + 41$$

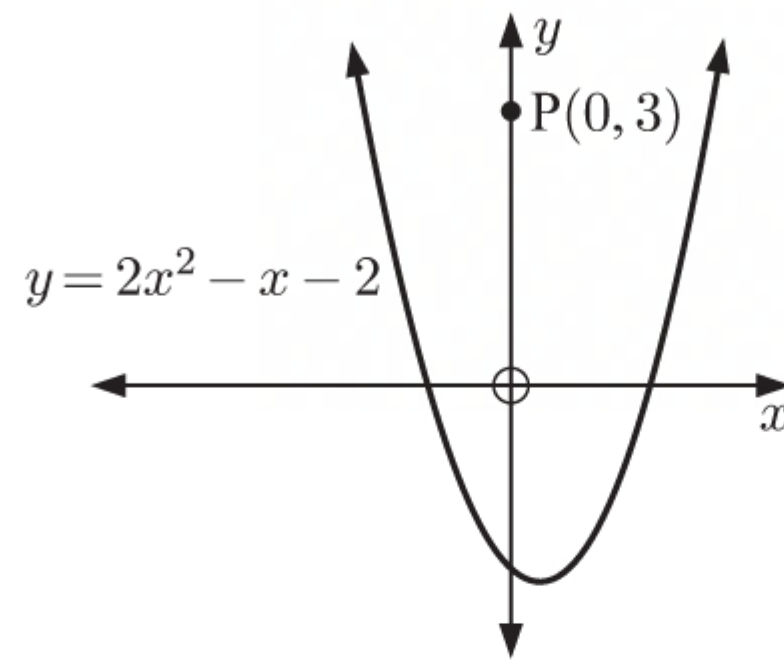
$$= m^2 + 2m + 1^2 + 41 - 1^2$$

$$= (m + 1)^2 + 40$$

which is always > 0 for any value of m

$$\therefore \Delta > 0 \text{ for any value of } m$$

\therefore the linear function $y = mx + 3$ will always meet the curve $y = 2x^2 - x - 2$ twice.



- 14** $y = (x - 2)^2$ and $y = -x^2 + bx + c$ touch when $x = 3$

$$\therefore (3 - 2)^2 = -(3)^2 + b(3) + c$$

$$\therefore 1 = -9 + 3b + c$$

$$\therefore 3b + c = 10$$

$$\therefore c = 10 - 3b \quad \dots (1)$$

Now, consider $(x - 2)^2 = -x^2 + bx + c$

$$\therefore x^2 - 4x + 4 = -x^2 + bx + c$$

$$\therefore -2x^2 + (b + 4)x + c - 4 = 0$$

This quadratic has $\Delta = 0$ since the graphs *touch*.

$$\therefore (b + 4)^2 - 4(-2)(c - 4) = 0$$

$$\therefore b^2 + 8b + 16 + 8c - 32 = 0$$

$$\therefore b^2 + 8b - 16 + 8(10 - 3b) = 0 \quad \{\text{using (1)}\}$$

$$\therefore b^2 + 8b - 16 + 80 - 24b = 0$$

$$\therefore b^2 - 16b + 64 = 0$$

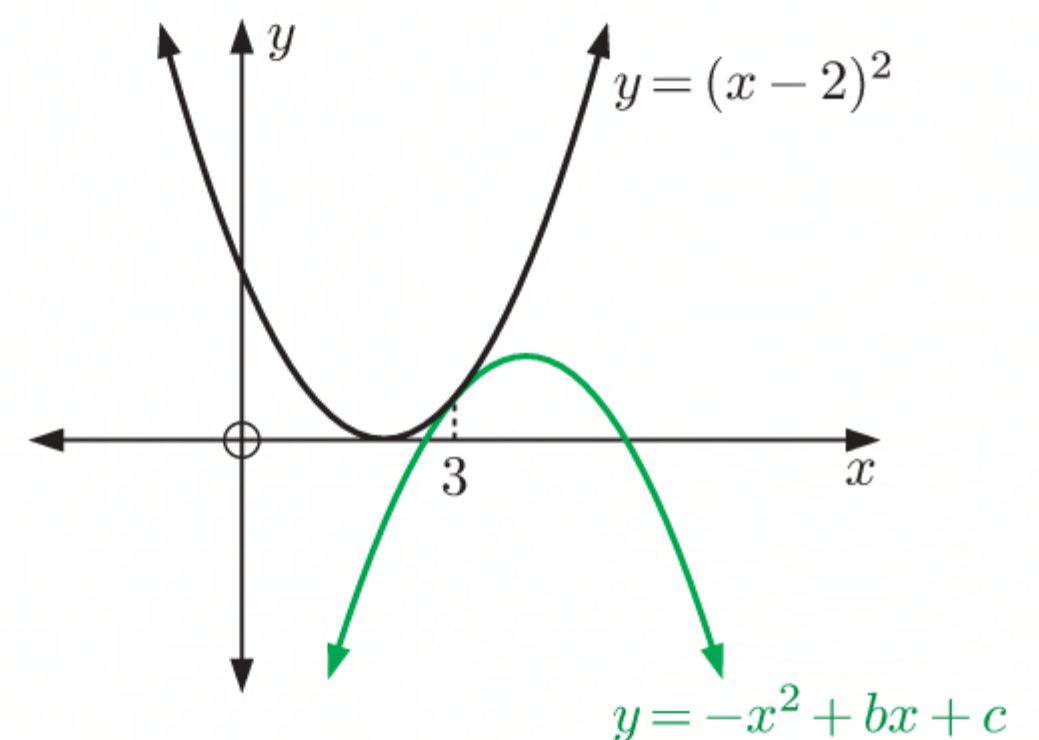
$$\therefore (b - 8)^2 = 0$$

$$\therefore b = 8$$

Substituting into (1) gives $c = 10 - 3(8)$

$$= -14$$

$$\therefore b = 8, \quad c = -14$$



EXERCISE 2F

- 1 Let the smaller of the integers be x . The other integer is $(x + 12)$.

$$\therefore \text{the sum of their squares is } x^2 + (x + 12)^2 = 74$$

$$\therefore x^2 + x^2 + 24x + 144 = 74$$

$$\therefore 2x^2 + 24x + 70 = 0$$

$$\therefore x^2 + 12x + 35 = 0$$

$$\therefore (x + 7)(x + 5) = 0$$

$$\therefore x = -7 \text{ or } -5$$

So, the integers are 7 and -5 , or -7 and 5.

- 2 Let the number be x , so its reciprocal is $\frac{1}{x}$.

$$\text{They have sum } x + \frac{1}{x} = \frac{26}{5}$$

$$\therefore x^2 + 1 = \frac{26}{5}x$$

$$\therefore x^2 - \frac{26}{5}x + 1 = 0$$

$$\therefore 5x^2 - 26x + 5 = 0$$

$$\therefore (5x - 1)(x - 5) = 0$$

$$\therefore x = \frac{1}{5} \text{ or } 5$$

So, the number is either 5 or $\frac{1}{5}$.

- 3 Let the number be x .

The sum of the number and its square is 210.

$$\therefore x + x^2 = 210$$

$$\therefore x^2 + x - 210 = 0$$

$$\therefore (x + 15)(x - 14) = 0$$

$$\therefore x = -15 \text{ or } 14$$

However, the number is a natural number.

\therefore the number is 14.

- 4 Suppose the numbers are x and $(x + 2)$.

$$\text{Then } x(x + 2) = 360$$

$$\therefore x^2 + 2x - 360 = 0$$

$$\therefore (x + 20)(x - 18) = 0$$

$$\therefore x = -20 \text{ or } 18$$

\therefore the numbers are 18 and 20, or -20 and -18 .

- 5 Suppose the numbers are x and $(x + 2)$.

$$\text{Then } x(x + 2) = 255$$

$$\therefore x^2 + 2x - 255 = 0$$

$$\therefore (x - 15)(x + 17) = 0$$

$$\therefore x = 15 \text{ or } -17$$

\therefore the numbers are 15 and 17, or -17 and -15 .

6 If the polygon has 90 diagonals, then $\frac{n}{2}(n-3) = 90$

$$\therefore \frac{1}{2}n^2 - \frac{3}{2}n = 90$$

$$\therefore n^2 - 3n - 180 = 0$$

$$\therefore (n-15)(n+12) = 0$$

$$\therefore n = -12 \text{ or } 15$$

We reject the negative solution, as a polygon must have a positive number of sides.

\therefore the polygon has 15 sides.

7 If the width of the rectangle is w cm, then its length is $(w+4)$ cm.

$$\therefore \text{the area is } w(w+4) = 26$$

$$\therefore w^2 + 4w - 26 = 0$$

which has $a = 1$, $b = 4$, $c = -26$

$$\therefore w = \frac{-4 \pm \sqrt{4^2 - 4(1)(-26)}}{2(1)}$$

$$\therefore w = \frac{-4 \pm \sqrt{120}}{2}$$

$$\therefore w = -2 \pm \sqrt{30}$$

$$\text{But } w > 0, \text{ so } w = -2 + \sqrt{30} \\ \approx 3.477$$

So, the width is approximately 3.48 cm.

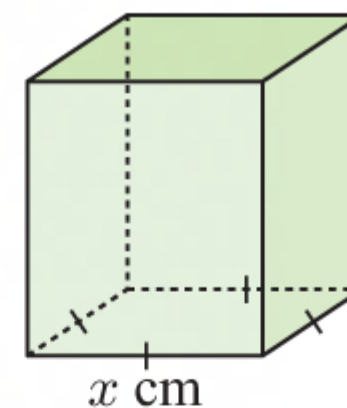
8 a The base has sides of length x cm, so the areas of the top and bottom surfaces are both x^2 cm².

The box has height $(x+1)$ cm, so the area of each of the side faces is $x(x+1)$ cm².

$$\therefore \text{the total surface area is } A = 2x^2 + 4x(x+1)$$

$$= 2x^2 + 4x^2 + 4x$$

$$\therefore A = 6x^2 + 4x \text{ cm}^2$$



b $6x^2 + 4x = 240$

$$\therefore 3x^2 + 2x - 120 = 0$$

$$\therefore (3x+20)(x-6) = 0$$

$$\therefore x = -\frac{20}{3} \text{ or } 6$$

$$\text{but } x > 0, \text{ so } x = 6$$

\therefore the box is 6 cm by 6 cm by 7 cm.

- 9 Suppose the tinplate was $x \text{ cm} \times x \text{ cm}$.

When $3 \text{ cm} \times 3 \text{ cm}$ squares are cut from the corners, the base of the open box formed is $(x-6) \text{ cm} \times (x-6) \text{ cm}$.

The open box has height 3 cm , so its volume is

$$3 \times (x-6) \times (x-6) = 80$$

$$\therefore 3(x^2 - 12x + 36) = 80$$

$$\therefore 3x^2 - 36x + 108 = 80$$

$$\therefore 3x^2 - 36x + 28 = 0$$

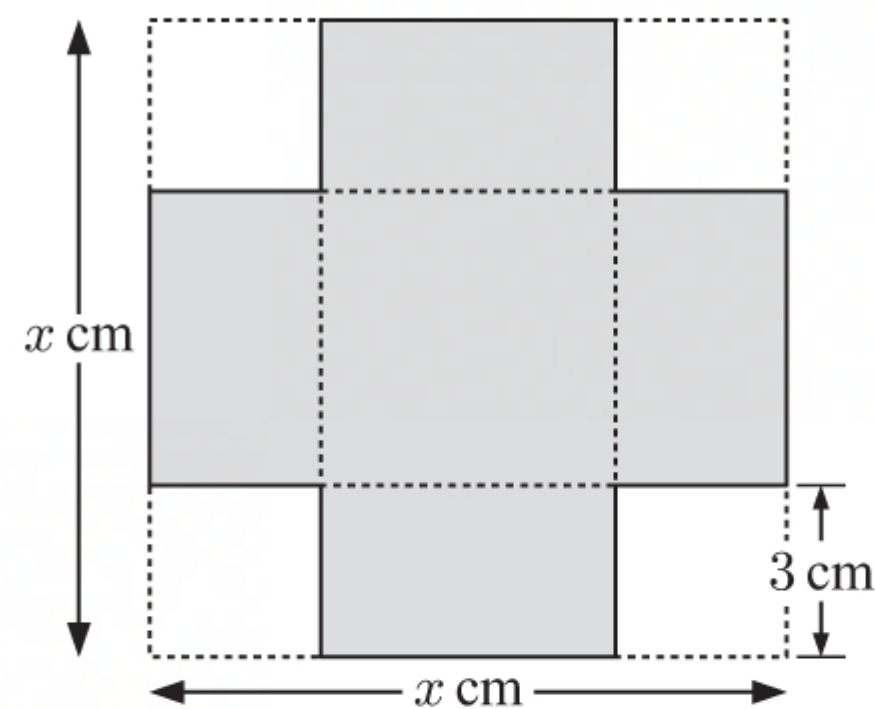
which has $a = 3$, $b = -36$, $c = 28$

$$\therefore x = \frac{-(-36) \pm \sqrt{(-36)^2 - 4(3)(28)}}{2(3)}$$

$$= \frac{36 \pm \sqrt{960}}{6} \quad \text{and since } x > 6,$$

$$x = 6 + \frac{\sqrt{960}}{6} \approx 11.16$$

\therefore the original piece of tinplate was about 11.2 cm square.

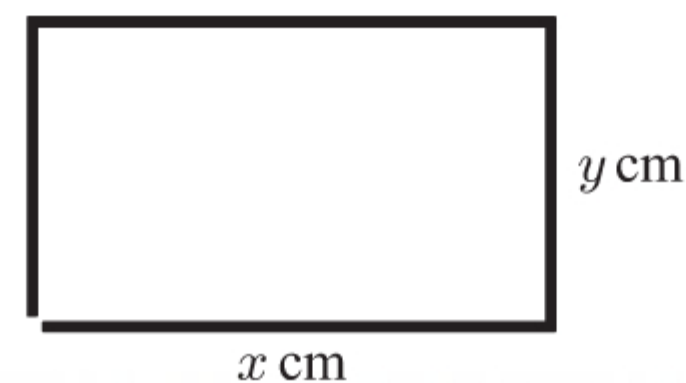


- 10 Suppose one side of the rectangle has length $x \text{ cm}$ and the other has length $y \text{ cm}$.

The perimeter is $(2x + 2y) \text{ cm}$, so $2x + 2y = 20$

$$\therefore 2y = 20 - 2x$$

$$\therefore y = 10 - x$$



The area $A = x(10 - x) \text{ cm}^2$.

If the area is 30 cm^2 , then $x(10 - x) = 30$

$$\therefore 10x - x^2 = 30$$

$$\therefore x^2 - 10x + 30 = 0$$

Now $\Delta = (-10)^2 - 4(1)(30)$

$$= 100 - 120$$

$$= -20 \quad \text{which is } < 0$$

There are no real solutions, indicating that this situation is **impossible**.

- 11 The smaller rectangle is similar to the original rectangle.

$$\therefore \frac{AB}{AD} = \frac{BC}{BY}$$

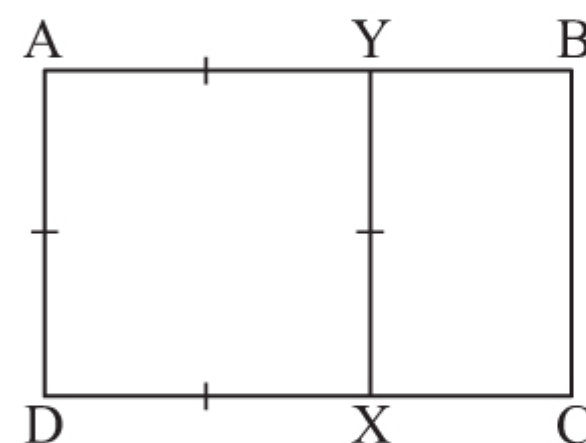
Suppose $AB = x$ units, and $AD = BC = 1$ unit

$$\therefore \frac{x}{1} = \frac{1}{x-1}$$

$$\therefore x(x-1) = 1$$

$$\therefore x^2 - x - 1 = 0$$

which has $a = 1$, $b = -1$, $c = -1$



$$\therefore x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(-1)}}{2(1)}$$

$$= \frac{1 \pm \sqrt{1+4}}{2}$$

$$= \frac{1 \pm \sqrt{5}}{2}$$

$$\therefore x = \frac{1 + \sqrt{5}}{2} \quad \{\text{since } x > 0\}$$

But $x = \frac{AB}{AD}$, which is the golden ratio

\therefore the golden ratio is $\frac{1 + \sqrt{5}}{2}$.

12 Let the speed of the normal train be $x \text{ km h}^{-1}$.

\therefore the speed of the express train is $(x + 10) \text{ km h}^{-1}$.

Time = $\frac{\text{distance}}{\text{speed}}$, so the normal train takes $\frac{160}{x}$ hours, and the express train takes $\frac{160}{x + 10}$ hours

to travel 160 km.

The express train takes half an hour less time than the normal train to travel 160 km.

$$\therefore \frac{160}{x} - \frac{160}{x + 10} = \frac{1}{2}$$

$$\therefore 160(x + 10) - 160x = \frac{1}{2}x(x + 10)$$

$$\therefore 160x + 1600 - 160x = \frac{1}{2}x^2 + 5x$$

$$\therefore \frac{1}{2}x^2 + 5x - 1600 = 0$$

$$\therefore x \approx 51.8 \text{ or } -61.8 \quad \{\text{using technology}\}$$

But $x > 0$, so $x \approx 51.8$

\therefore the speed of the express train is $x + 10 \approx 51.8 + 10 \approx 61.8 \text{ km h}^{-1}$.

13 Let the number of elderly citizens in the original group be x .

The amount paid by each elderly citizen was originally $\frac{160}{x}$ dollars.

Now, 8 elderly citizens fell ill, so the amount paid by each elderly citizen is $\frac{160}{x - 8}$ dollars.

These elderly citizens had to pay \$1 more than before.

$$\therefore \frac{160}{x - 8} = \frac{160}{x} + 1$$

$$\therefore 160x = 160(x - 8) + x(x - 8)$$

$$\therefore 160x = 160x - 1280 + x^2 - 8x$$

$$\therefore x^2 - 8x - 1280 = 0$$

$$\therefore (x + 32)(x - 40) = 0$$

$$\therefore x = -32 \text{ or } 40$$

But $x > 0$, so $x = 40$

$\therefore x - 8 = 40 - 8 = 32$ elderly citizens went on the trip.

- 14 a** The parabola has vertex $(0, 8)$, so it has equation

$$y = a(x - 0)^2 + 8$$

$$\therefore y = ax^2 + 8$$

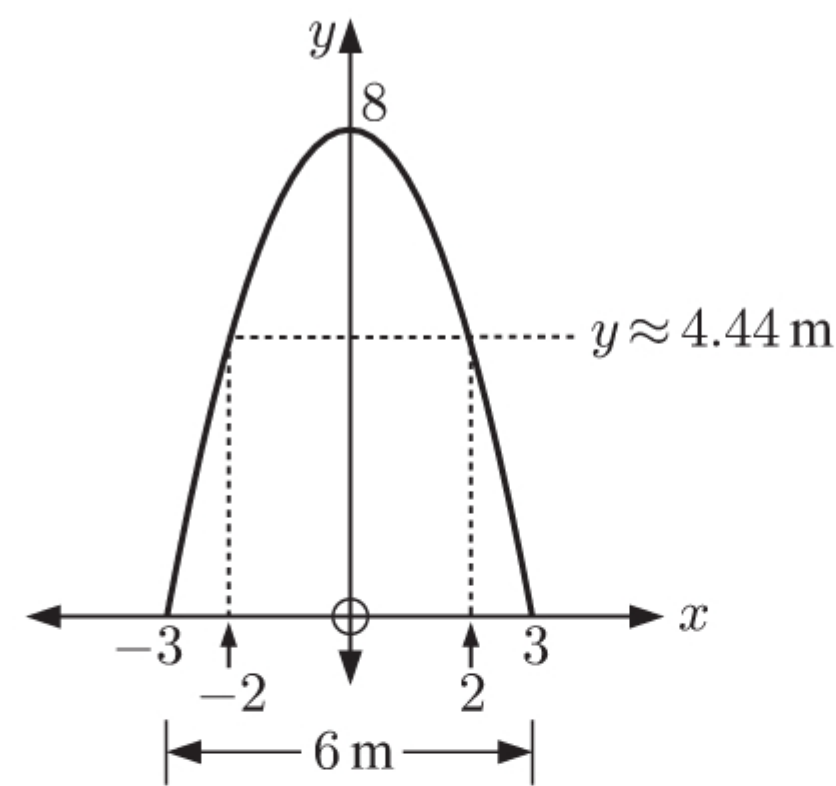
When $x = 3$, $y = 0$, so

$$0 = a(3^2) + 8$$

$$\therefore 9a = -8$$

$$\therefore a = -\frac{8}{9}$$

\therefore the equation of the parabola is $y = -\frac{8}{9}x^2 + 8$.



- b** The truck is 4 m wide, so we use the equation in **a** to find the height of the tunnel when it is 4 m wide.

$$\begin{aligned} \text{When } x = \pm 2, \quad y &= -\frac{8}{9}(2)^2 + 8 \\ &= -\frac{32}{9} + 8 \\ &= \frac{40}{9} \approx 4.44 \text{ m} \end{aligned}$$

For heights greater than 4.44 m, the tunnel is less than 4 m wide. But the truck is 5 m high.
 \therefore the truck will not fit through the tunnel.

- 15 a** The position of the stone above sea level is plotted on the graph as shown, where the maximum height reached is 80 m, when $t = 2$ s.

So, the vertex is $(2, 80)$, and the h -intercept is 60, as the height is 60 m when $t = 0$.

The quadratic has the form

$$h = a(t - 2)^2 + 80, \text{ where } a < 0$$

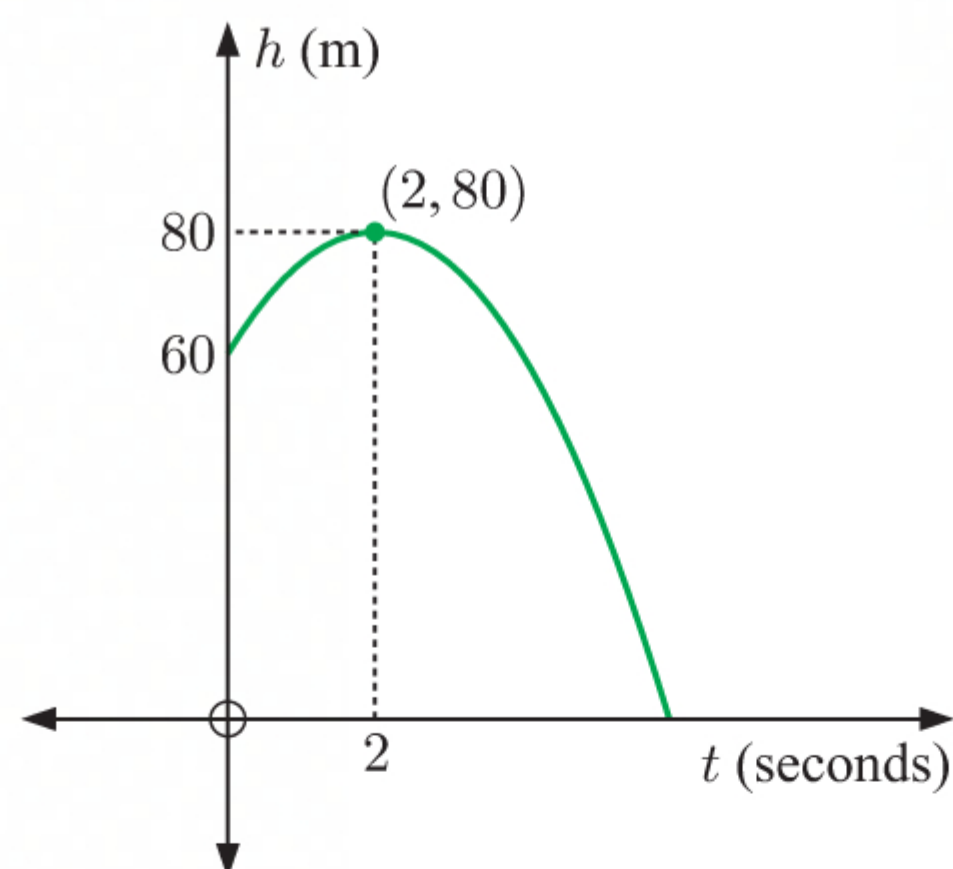
When $t = 0$, $h = 60$

$$\therefore 60 = a(-2)^2 + 80$$

$$\therefore 4a = -20$$

$$\therefore a = -5$$

The quadratic is $h = -5(t - 2)^2 + 80$, which gives the stone's height above sea level for any $t \geq 0$.



- b** When $t = 3$, $h = -5(3 - 2)^2 + 80$
 $= -5(1) + 80$
 $= 75$

The stone is 75 m above sea level after 3 seconds.

- c** The stone will hit the water when $h = 0$.

$$\therefore -5(t - 2)^2 + 80 = 0$$

$$\therefore 5(t - 2)^2 = 80$$

$$\therefore (t - 2)^2 = 16$$


$$\therefore t - 2 = \pm\sqrt{16}$$

$$\therefore t = 2 \pm 4$$

$$\text{but } t \geq 0, \therefore t = 2 + 4 = 6$$


It will take 6 seconds for the stone to hit the water.

EXERCISE 2G

- 1 a** $y = x^2 - 2x$
 has $a = 1$, $b = -2$, $c = 0$.
 Since $a > 0$, the shape is 
 The minimum value occurs when


$$x = \frac{-b}{2a} = \frac{2}{2(1)} = 1$$

 and $y = 1^2 - 2(1) = -1$
 So, the minimum value of $y = x^2 - 2x$
 is -1 , occurring when $x = 1$.

- c** $y = 8 + 2x - 3x^2$
 has $a = -3$, $b = 2$, $c = 8$.
 Since $a < 0$, the shape is 
 The maximum value occurs when

$$x = \frac{-b}{2a} = \frac{-2}{2(-3)} = \frac{1}{3}$$

 and $y = 8 + 2(\frac{1}{3}) - 3(\frac{1}{3})^2 = 8\frac{1}{3}$
 So, the maximum value of
 $y = 8 + 2x - 3x^2$ is $8\frac{1}{3}$, occurring
 when $x = \frac{1}{3}$.

- e** $y = 4x^2 - x + 5$
 has $a = 4$, $b = -1$, $c = 5$.
 Since $a > 0$, the shape is 
 The minimum value occurs when


$$x = \frac{-b}{2a} = \frac{-(-1)}{2(4)} = \frac{1}{8}$$

 and $y = 4(\frac{1}{8})^2 - \frac{1}{8} + 5$

$$= \frac{1}{16} - \frac{1}{8} + 5$$


$$= 4\frac{15}{16}$$

 So, the minimum value of
 $y = 4x^2 - x + 5$ is $4\frac{15}{16}$, occurring
 when $x = \frac{1}{8}$.

- b** $y = 7 - 2x - x^2$
 has $a = -1$, $b = -2$, $c = 7$.
 Since $a < 0$, the shape is 
 The maximum value occurs when

$$x = \frac{-b}{2a} = \frac{-(-2)}{2(-1)} = -1$$

 and $y = 7 - 2(-1) - (-1)^2 = 8$
 So, the maximum value of
 $y = 7 - 2x - x^2$ is 8 , occurring when
 $x = -1$.


- d** $y = 2x^2 + x - 1$
 has $a = 2$, $b = 1$, $c = -1$.
 Since $a > 0$, the shape is 
 The minimum value occurs when

$$x = \frac{-b}{2a} = \frac{-1}{2(2)} = -\frac{1}{4}$$

 and $y = 2(-\frac{1}{4})^2 + (-\frac{1}{4}) - 1$

$$= \frac{1}{8} - \frac{1}{4} - 1 = -1\frac{1}{8}$$

 So, the minimum value of
 $y = 2x^2 + x - 1$ is $-1\frac{1}{8}$, occurring
 when $x = -\frac{1}{4}$.

- f** $y = 7x - 2x^2$
 has $a = -2$, $b = 7$, $c = 0$.
 Since $a < 0$, the shape is 
 The maximum value occurs when

$$x = \frac{-b}{2a} = \frac{-7}{2(-2)} = \frac{7}{4}$$

 and $y = 7(\frac{7}{4}) - 2(\frac{7}{4})^2$

$$= \frac{49}{4} - \frac{49}{8}$$

$$= \frac{49}{8} \text{ or } 6\frac{1}{8}$$

 So, the maximum value of
 $y = 7x - 2x^2$ is $6\frac{1}{8}$, occurring
 when $x = \frac{7}{4}$.

2 a $P = -3x^2 + 240x - 800$ has $a = -3$, $b = 240$, $c = -800$.

Since $a < 0$, the shape is 

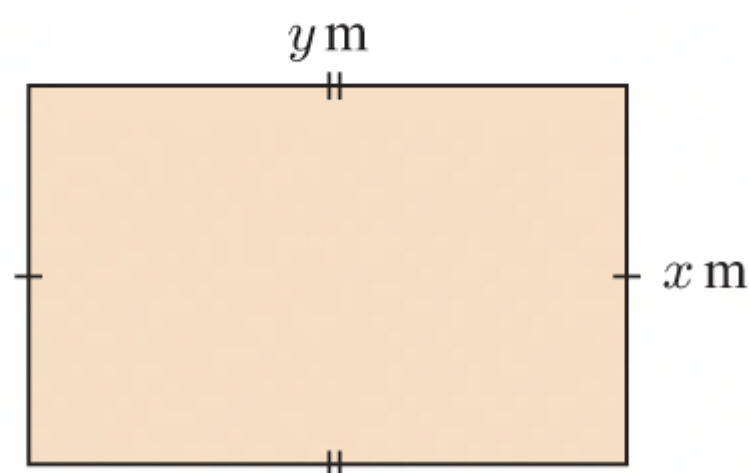
The maximum profit occurs when $x = \frac{-b}{2a} = \frac{-240}{2(-3)} = 40$

So, 40 refrigerators should be made each day to maximise the total profit.

b When $x = 40$, $P = -3(40)^2 + 240(40) - 800$
 $= 4000$

So, the maximum profit is €4000.

3 a Let the other side be y m long.



The perimeter is 200 m.

$$\therefore 2x + 2y = 200$$

$$\therefore x + y = 100$$

$$\therefore y = 100 - x$$

The area $A = xy$

$$\therefore A = x(100 - x)$$

$$\therefore A = 100x - x^2$$

b $A = 100x - x^2$ is a quadratic function with $a = -1$, $b = 100$, $c = 0$.

Since $a < 0$, the shape is 

The area is maximised when

$$x = \frac{-b}{2a} = \frac{-100}{2(-1)} = 50$$

and $y = 100 - 50 = 50$

So, the area of the rectangle is maximised when $x = y = 50$, which is when the rectangle is a square.

4 Let the dimensions of the paddock be x m \times y m.

If 1000 m of fence is available, then


$$2x + y = 1000 \quad \{\text{perimeter}\}$$

$$\therefore y = 1000 - 2x \quad \dots (1)$$

The area of the enclosure $A = xy$

$$\text{Since } y = 1000 - 2x, \quad A = x(1000 - 2x) \\ = 1000x - 2x^2$$

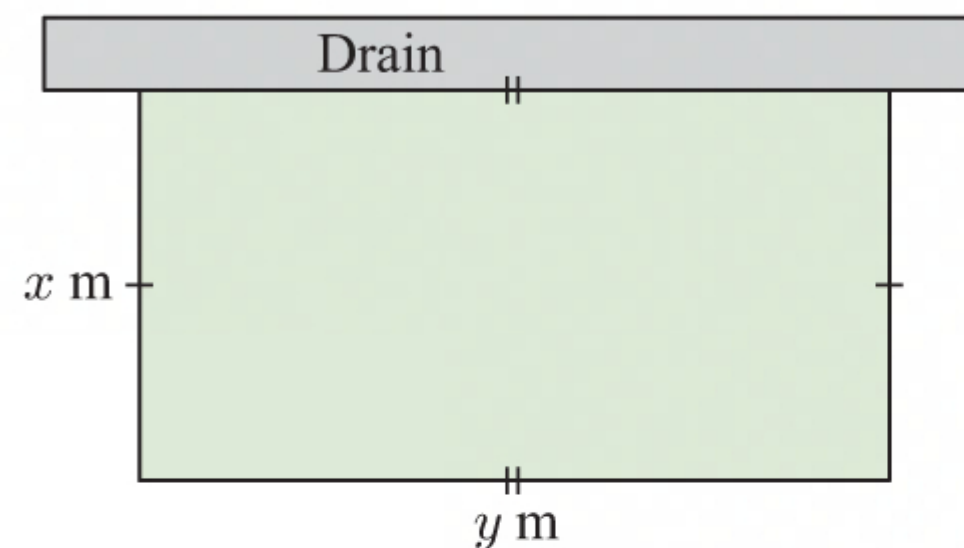
$$\therefore A = -2x^2 + 1000x$$

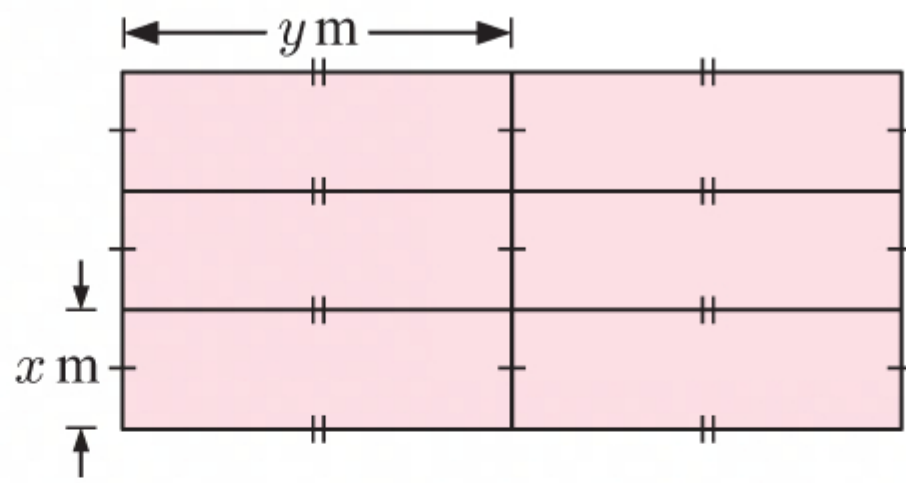
A is a quadratic and $a < 0$, so its shape is 

So, the area is maximised when $x = \frac{-b}{2a} = \frac{-1000}{2(-2)} = 250$

and when $x = 250$, $y = 1000 - 2(250) = 500$

So, the paddock has a maximum area when the dimensions are 250 m \times 500 m.



5 a

The length of fence required for this enclosure is $9x + 8y$. If 1800 m is available for this enclosure, then $9x + 8y = 1800$.

b If $9x + 8y = 1800$, then $y = \frac{1800 - 9x}{8}$.


The area of each pen is $A = xy$.

Substituting $y = \frac{1800 - 9x}{8}$ into A we get

$$A = x \left(\frac{1800 - 9x}{8} \right)$$

$$\therefore A = \frac{1800x}{8} - \frac{9x^2}{8}$$

$$\therefore A = -\frac{9}{8}x^2 + 225x \text{ m}^2$$

c The area A is a quadratic with $a < 0$, so its shape is . So, the area is maximised when $x = \frac{-b}{2a} = \frac{-225}{2(-\frac{9}{8})} = 100$

$$\text{and when } x = 100, y = \frac{1800 - 9(100)}{8} = 112.5$$

Hence, the area is maximised when the dimensions are $100 \text{ m} \times 112.5 \text{ m}$.

6 a Let $x \text{ m} \times y \text{ m}$ be the dimensions of a single pen as shown below.

So, the total length of fencing required is $6x + 6y$.

If there is 500 m of fencing available, then

$$6x + 6y = 500$$


$$\therefore x + y = 83\frac{1}{3}$$

$$\therefore y = 83\frac{1}{3} - x \quad \dots (1)$$

The area of each pen will be $A = xy$ and substituting equation (1), we have

$$A = x(83\frac{1}{3} - x)$$

$$\therefore A = -x^2 + 83\frac{1}{3}x$$

which is a quadratic with $a < 0$, so its shape is .

So, at $x = \frac{-b}{2a}$ we have a maximum value of A .

$$\therefore x = \frac{-83\frac{1}{3}}{2(-1)} = 41\frac{2}{3} \quad \text{and so} \quad y = 83\frac{1}{3} - 41\frac{2}{3} = 41\frac{2}{3}$$

So, the dimensions that maximise the area are $41\frac{2}{3} \text{ m} \times 41\frac{2}{3} \text{ m}$.

b Let $x \text{ m} \times y \text{ m}$ be the dimensions of a single pen as shown below.

So, the total length of fencing required is $5x + 8y$.

If there is 500 m of fencing available, then

$$5x + 8y = 500$$

$$\therefore 8y = 500 - 5x$$

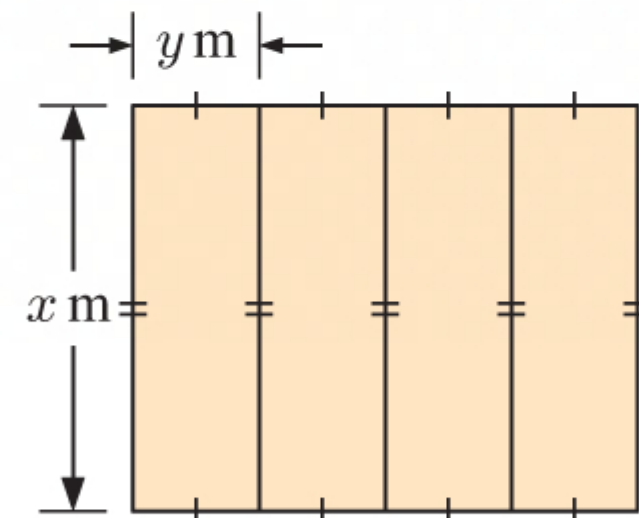
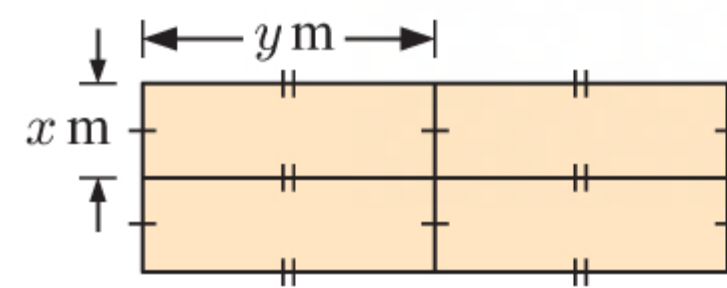
$$\therefore y = \frac{500 - 5x}{8}$$

$$\therefore y = 62\frac{1}{2} - \frac{5}{8}x \quad \dots (1)$$

The area of each pen will be $A = xy$ and substituting equation (1), we have

$$A = x(62\frac{1}{2} - \frac{5}{8}x)$$

$\therefore A = -\frac{5}{8}x^2 + 62\frac{1}{2}x$ which is a quadratic with $a < 0$, so its shape is .



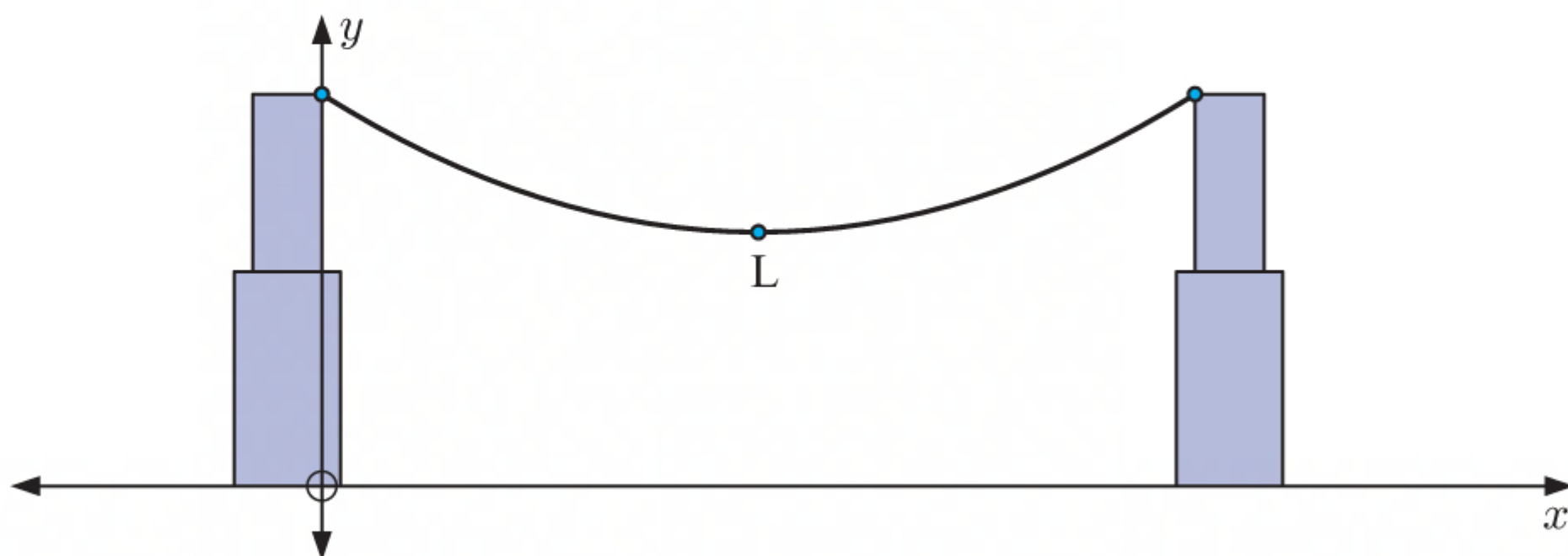
So, when $x = \frac{-b}{2a}$ we have a maximum value of A .

$$\therefore x = \frac{-62\frac{1}{2}}{2(-\frac{5}{8})} = 50, \text{ and substituting } x = 50 \text{ into } y = 62\frac{1}{2} - \frac{5}{8}x, \text{ we have}$$

$$y = 31\frac{1}{4}$$

So, the dimensions that maximise the area are $50 \text{ m} \times 31\frac{1}{4} \text{ m}$.

7



a $y = 0.008x^2 - 0.8x + 50$ has y -intercept when $x = 0$
 $\therefore y = 0.008(0)^2 - 0.8(0) + 50$
 $= 50$

So, the height of the platforms is 50 m.

b In order to find the coordinates of L, we need to find the distance between the two platforms.

When $y = 50$, $50 = 0.008x^2 - 0.8x + 50$

$$\therefore 0.008x^2 - 0.8x = 0$$

$$\therefore x(0.008x - 0.8) = 0$$

$$\therefore x = 0 \text{ or } 0.008x - 0.8 = 0$$

$$\therefore x = 0 \text{ or } x = 100$$

So, the distance between the two platforms is 100 m.

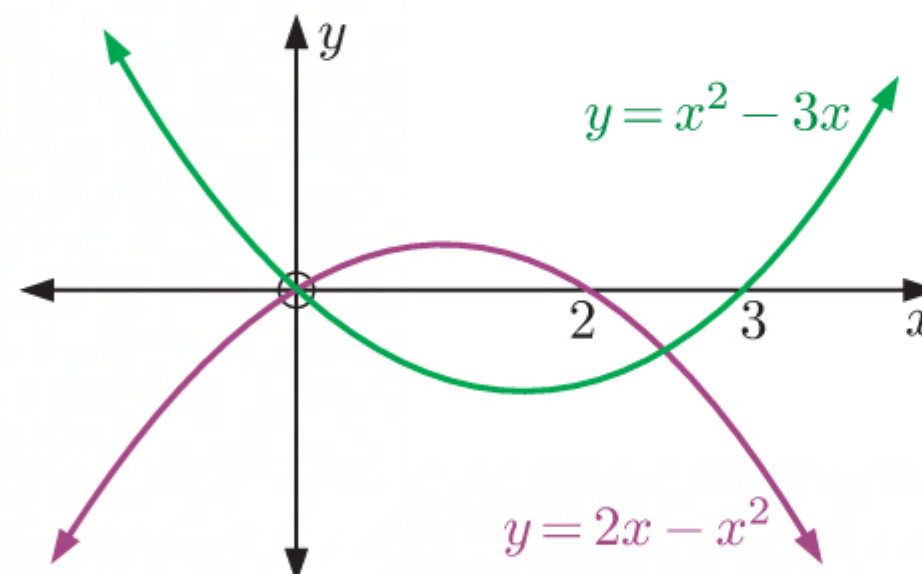
L lies halfway between the two platforms.

\therefore L is 50 m from each platform.

When $x = 50$, $y = 0.008(50)^2 - 0.8(50) + 50$
 $= 30$

So, L is 30 m above ground level.

8 a The graphs of $y = x^2 - 3x$ and $y = 2x - x^2$
meet where $x^2 - 3x = 2x - x^2$
 $\therefore 2x^2 - 5x = 0$
 $\therefore x(2x - 5) = 0$
 $\therefore x = 0 \text{ or } 2\frac{1}{2}$




b The vertical separation between the curves is given by

$$S = (2x - x^2) - (x^2 - 3x) \quad \{y = 2x - x^2 \text{ is above } y = x^2 - 3x \text{ for } 0 \leq x \leq 2\frac{1}{2}\}$$

$$\therefore S = 2x - x^2 - x^2 + 3x$$

$$\therefore S = -2x^2 + 5x$$

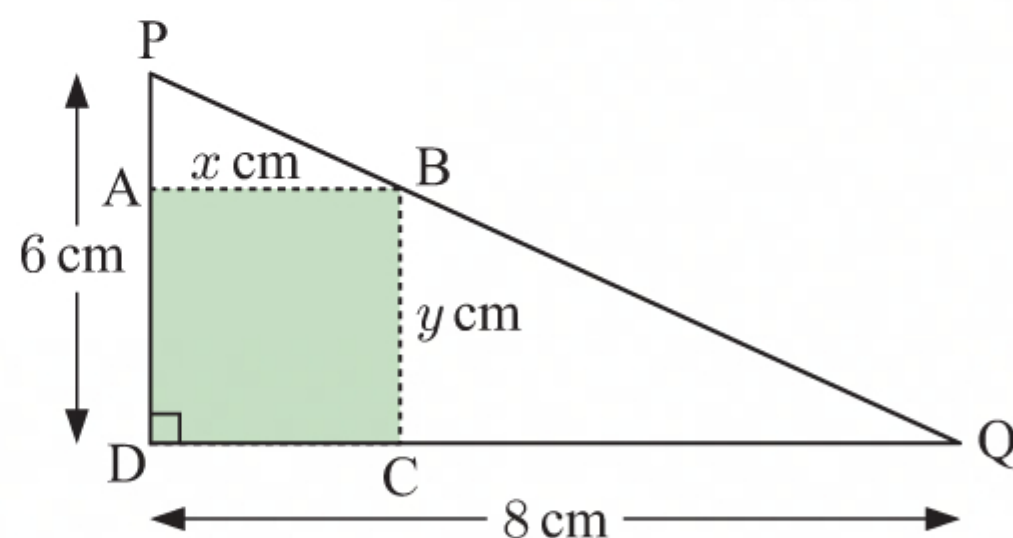
Thus S is a quadratic with $a < 0$, so its shape is 

The maximum separation occurs when $x = \frac{-b}{2a} = \frac{-5}{2(-2)} = \frac{5}{4}$

$$\begin{aligned}\text{and } S &= -2\left(\frac{5}{4}\right)^2 + 5\left(\frac{5}{4}\right) \\ &= -\frac{25}{8} + \frac{25}{4} \\ &= \frac{25}{8} \text{ or } 3\frac{1}{8}\end{aligned}$$

So, the maximum vertical separation between the curves for $0 \leq x \leq 2\frac{1}{2}$ is $3\frac{1}{8}$ units.

9 a



Triangles PAB and PDQ are similar

{ \widehat{APB} is common,

$\widehat{ABP} = \widehat{DQP}$ as $[AB] \parallel [DQ]$ }


$$\therefore \frac{PA}{PD} = \frac{AB}{DQ}$$

$$\therefore \frac{6-y}{6} = \frac{x}{8}$$

$$\therefore 6-y = \frac{3}{4}x$$

$$\therefore y = 6 - \frac{3}{4}x$$

b Rectangle ABCD has area $A = xy$
 $= x\left(6 - \frac{3}{4}x\right)$
 $= -\frac{3}{4}x^2 + 6x$

which is a quadratic with $a < 0$, so its shape is 

The area is maximised when $x = \frac{-b}{2a} = \frac{-6}{2(-\frac{3}{4})} = 4$

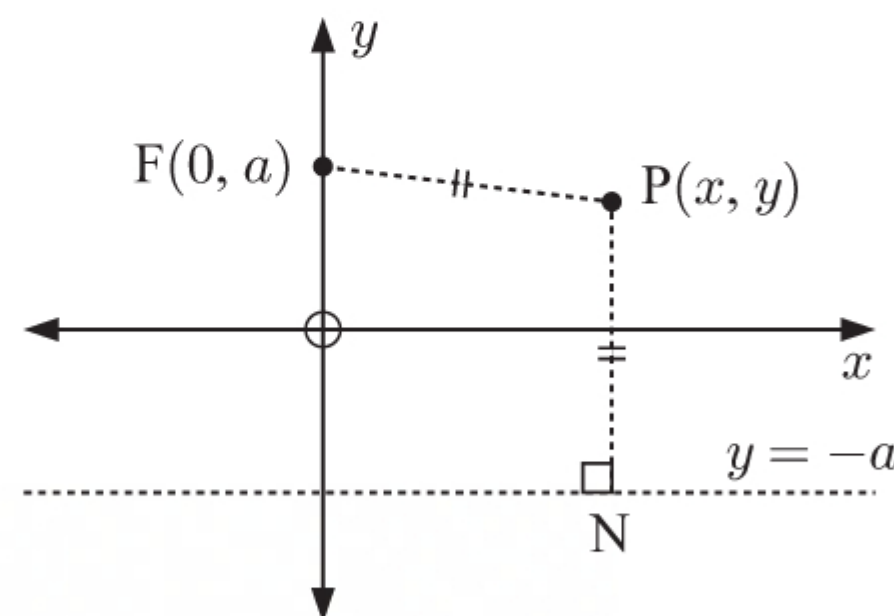
$$\text{and when } x = 4, \quad y = 6 - \frac{3}{4}(4) = 3$$

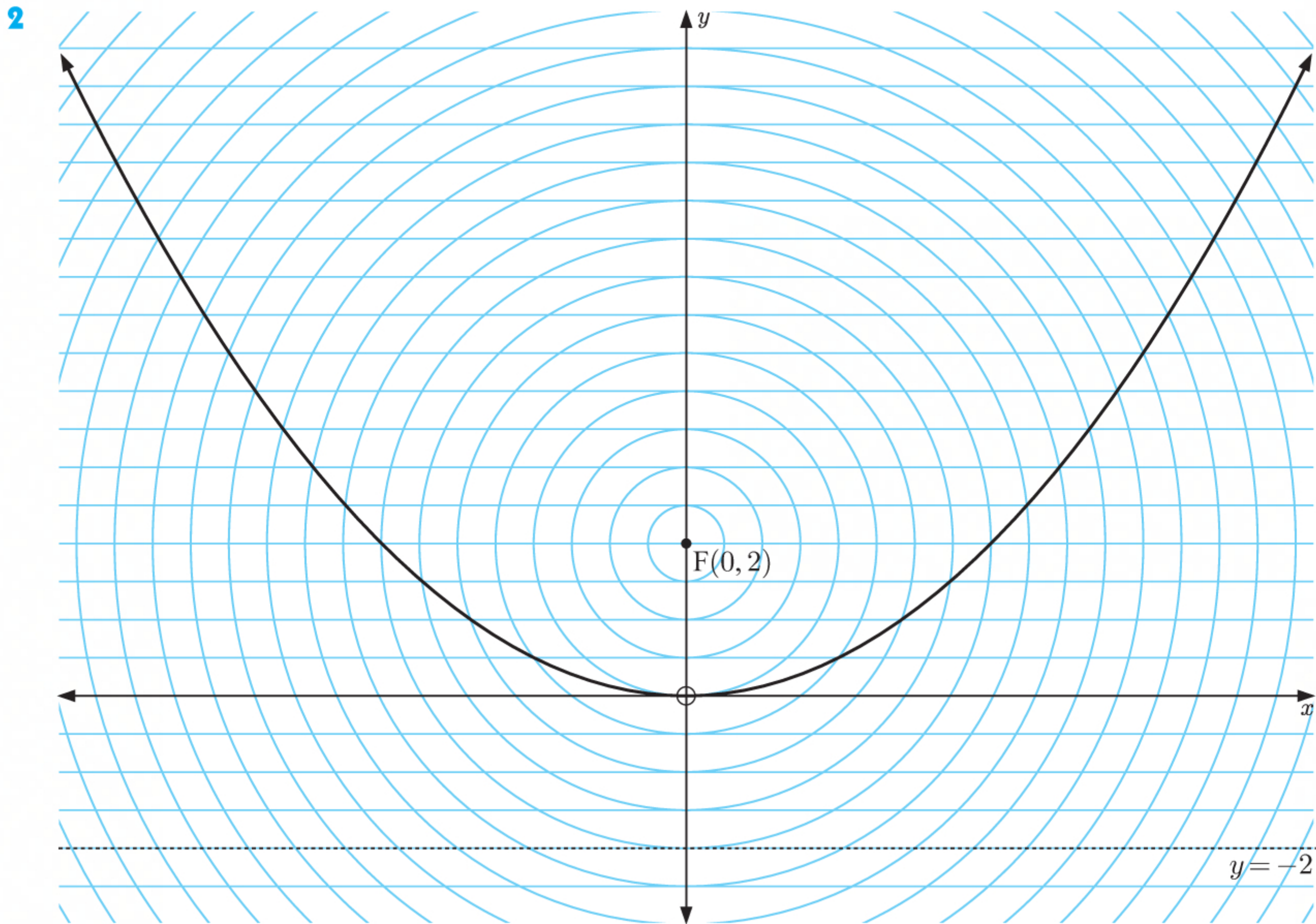
So, the dimensions of rectangle ABCD of maximum area are 4 cm \times 3 cm.

INVESTIGATION 4

THE GEOMETRIC DEFINITION OF A PARABOLA

- 1 If we let the focus be at $(0, a)$ and the directrix be $y = -a$, then the origin $(0, 0)$ is always the vertex of the parabola.





3 a N is $(x, -a)$

b $FP = \sqrt{(x-0)^2 + (y-a)^2}$ $NP = y + a$
 $= \sqrt{x^2 + y^2 - 2ay + a^2}$

c The parabola is the set of all points P such that $FP = NP$

$$\begin{aligned} \therefore \sqrt{x^2 + y^2 - 2ay + a^2} &= y + a \\ \therefore x^2 + y^2 - 2ay + a^2 &= (y + a)^2 \\ \therefore x^2 + y^2 - 2ay + a^2 &= y^2 + 2ay + a^2 \\ \therefore x^2 &= 4ay \\ \therefore y &= \frac{x^2}{4a} \end{aligned}$$

- 4 a The midpoint M has coordinates $\left(\frac{0+X}{2}, \frac{a+(-a)}{2}\right)$ which is $\left(\frac{X}{2}, 0\right)$.

b [MP] has gradient $\frac{\frac{X^2}{4a} - 0}{X - \frac{X}{2}} = \frac{\frac{X^2}{4a}}{\frac{X}{2}} = \frac{X}{2a}$

[MP] passes through $\left(\frac{X}{2}, 0\right)$, so it has

$$\text{equation } y = \frac{X}{2a} \left(x - \frac{X}{2}\right) + 0$$

$$\therefore y = \frac{X}{2a} \left(x - \frac{X}{2}\right)$$

- c The parabola and (MP) meet where $\frac{x^2}{4a} = \frac{X}{2a} \left(x - \frac{X}{2}\right)$

$$\therefore \frac{1}{4a}x^2 = \frac{X}{2a}x - \frac{X^2}{4a}$$

$$\therefore \frac{1}{4a}x^2 - \frac{X}{2a}x + \frac{X^2}{4a} = 0$$

which is a quadratic in x with $\Delta = \left(-\frac{X}{2a}\right)^2 - 4\left(\frac{1}{4a}\right)\left(\frac{X^2}{4a}\right)$

$$= \frac{X^2}{4a^2} - \frac{X^2}{4a^2}$$

$$= 0$$

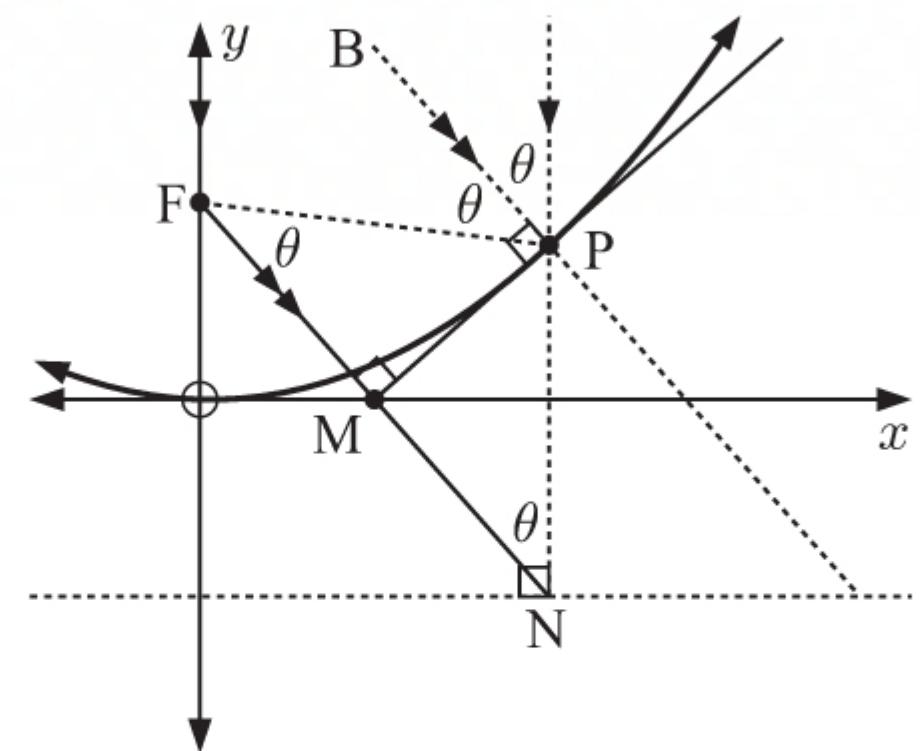
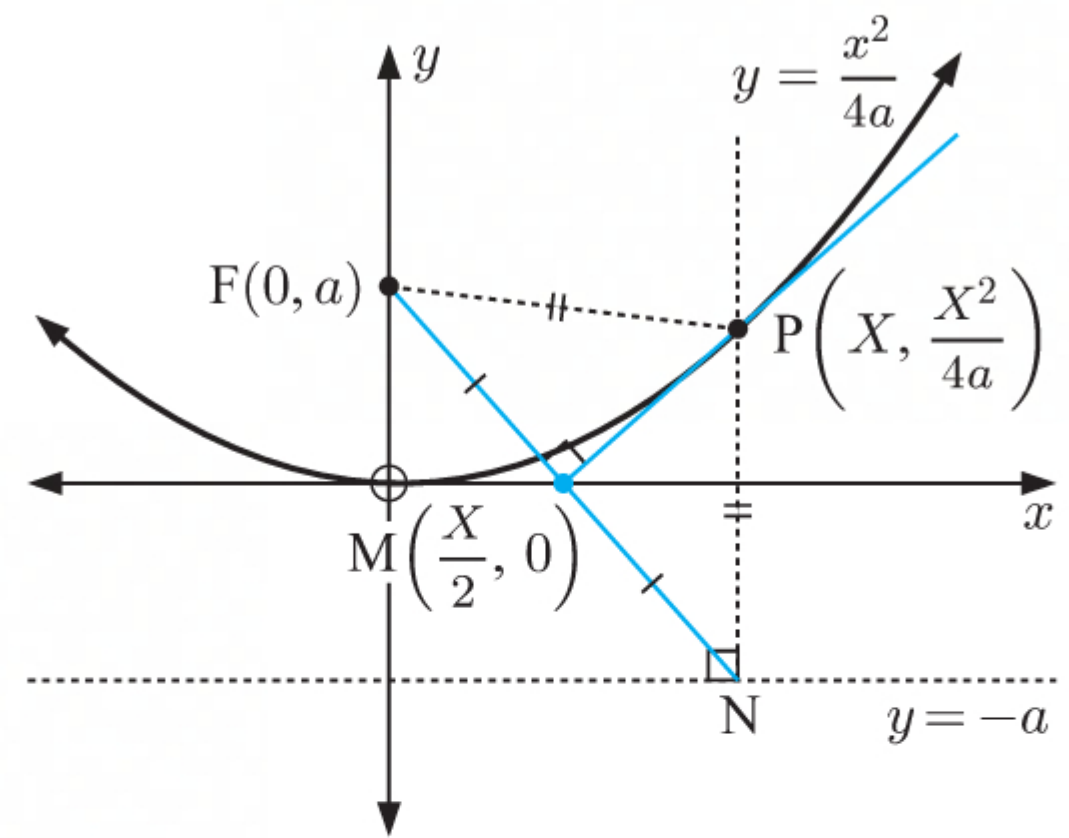
So, the graphs *touch* at one point.

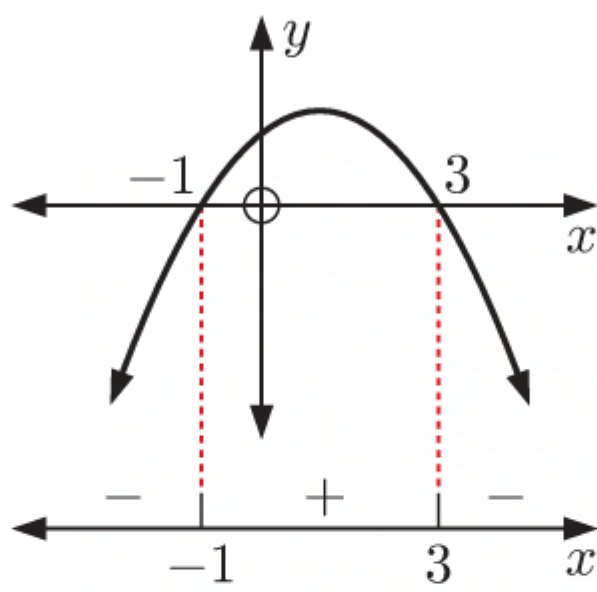
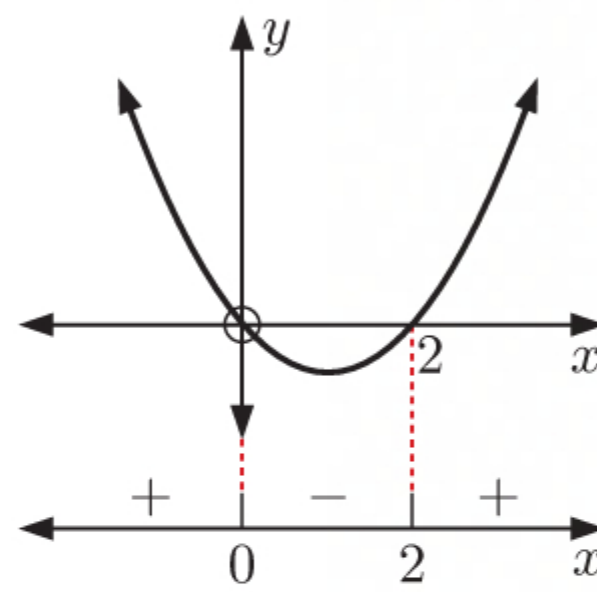
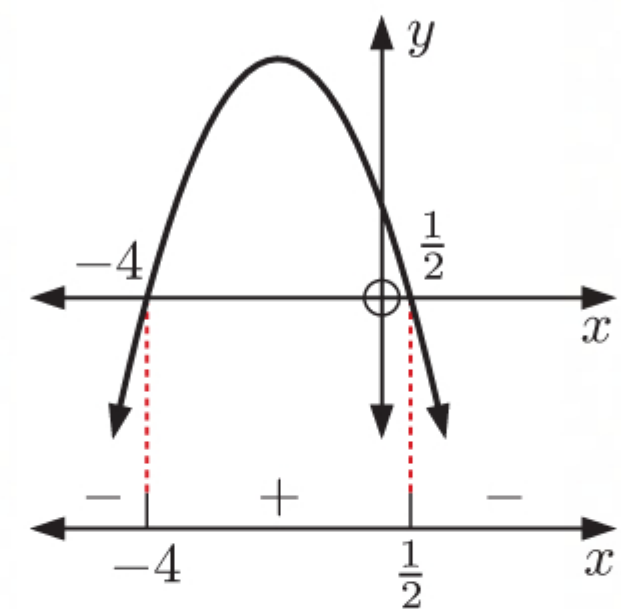
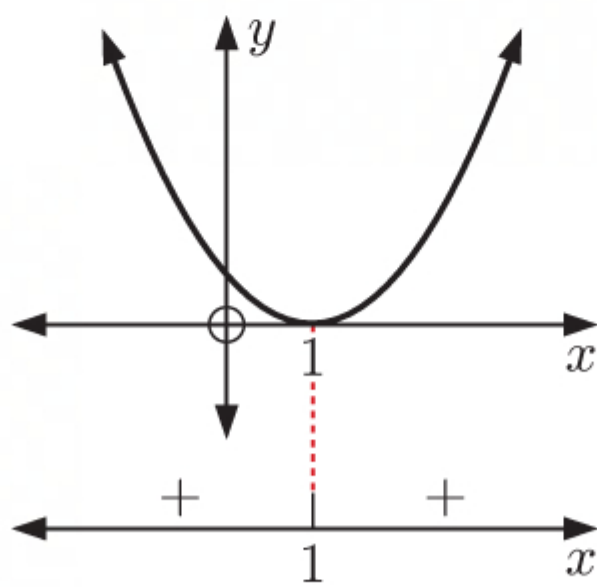
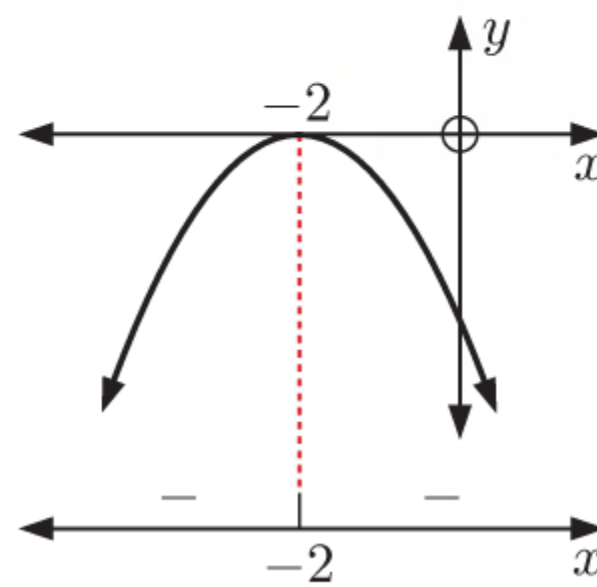
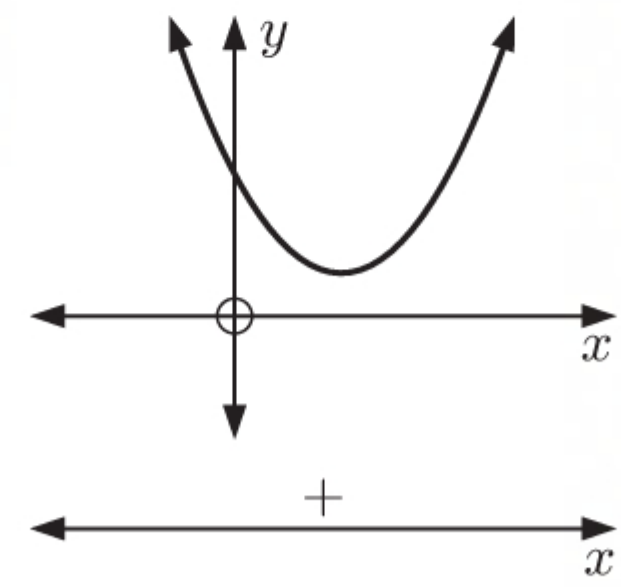
\therefore (MP) is a tangent to the parabola.

- d i $\widehat{MNP} = \theta$ {corresponding angles}
- ii Triangle FPM is isosceles
 $\therefore \widehat{MFP} = \widehat{MNP}$
 {base angles of an isosceles \triangle }
 $= \theta$
- iii $\widehat{FPB} = \widehat{MFP}$ {alternate angles}
 $= \theta$
- iv angle of incidence = angle of reflection
 {from the **Opening Problem**}
 \therefore angle of reflection $= \theta$

But the angle between the normal and the focus $\widehat{FPB} = \theta$, so a vertical ray of light is always reflected to the focus.

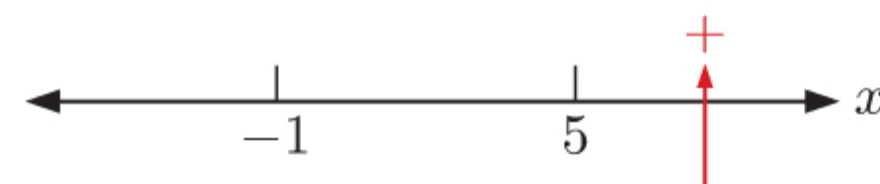
- v Misha needs to use a *parabolic* mirror, and place his cup at the *focus* of the parabola.



EXERCISE 2H.1**1 a****b****c****d****e****f****2 a** $(x + 4)(x - 2)$ has zeros -4 and 2 .

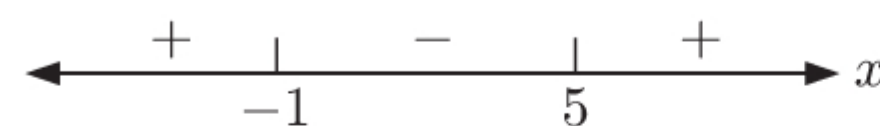
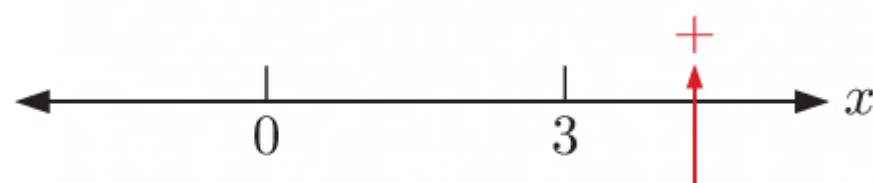
When $x = 3$ we have $(7)(1) > 0$,
so we put a $+$ sign here.

As the factors are single, the signs
alternate.

**b** $(x + 1)(x - 5)$ has zeros -1 and 5 .

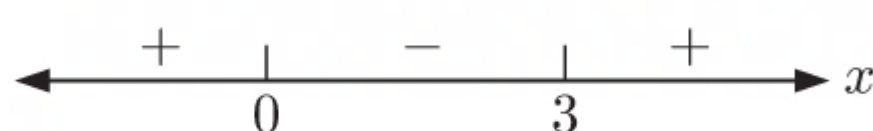
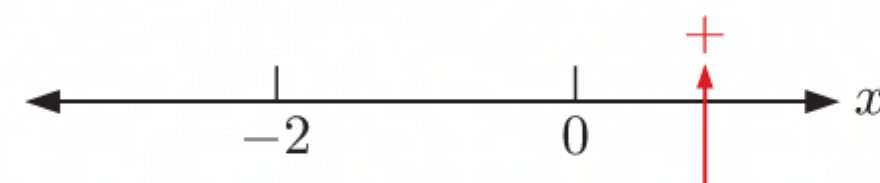
When $x = 6$ we have $(7)(1) > 0$,
so we put a $+$ sign here.

As the factors are single, the signs
alternate.

**c** $x(x - 3)$ has zeros 0 and 3 .

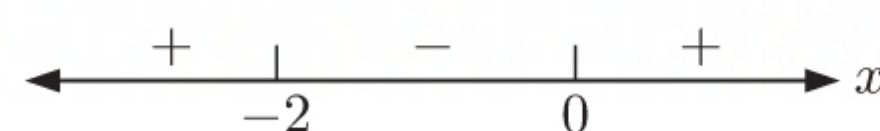
When $x = 4$ we have $(4)(1) > 0$,
so we put a $+$ sign here.

As the factors are single, the signs
alternate.

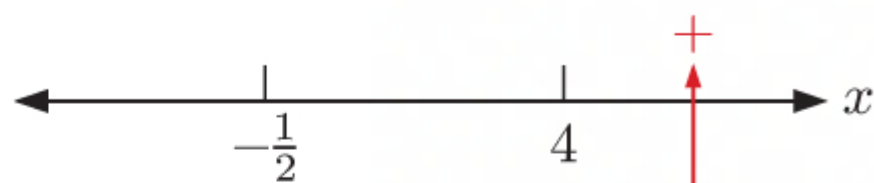
**d** $x(x + 2)$ has zeros 0 and -2 .

When $x = 1$ we have $(1)(3) > 0$,
so we put a $+$ sign here.

As the factors are single, the signs
alternate.



e $(2x + 1)(x - 4)$ has zeros $-\frac{1}{2}$ and 4.

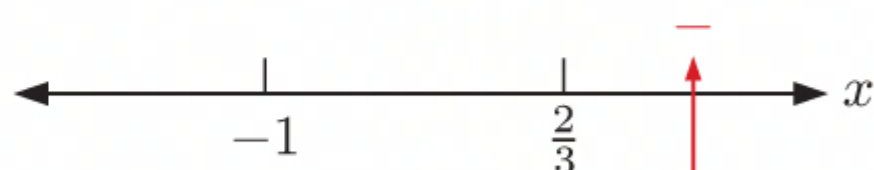


When $x = 5$ we have $(11)(1) > 0$,
so we put a + sign here.

As the factors are single, the signs alternate.

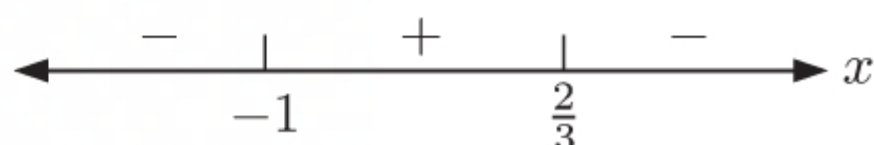


g $-(3x - 2)(x + 1)$ has zeros $\frac{2}{3}$ and -1 .



When $x = 1$ we have $-(1)(2) < 0$,
so we put a - sign here.

As the factors are single, the signs alternate.

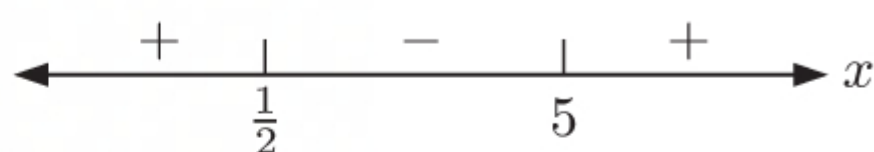


i $(5 - x)(1 - 2x)$ has zeros 5 and $\frac{1}{2}$.

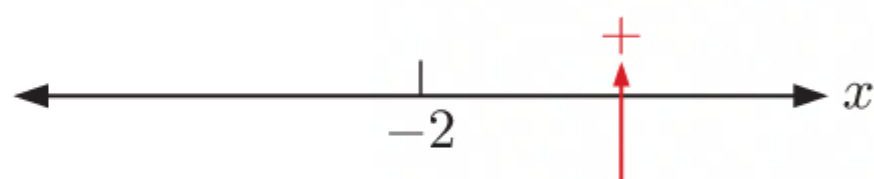


When $x = 6$ we have $(-1)(-11) > 0$,
so we put a + sign here.

As the factors are single, the signs alternate.

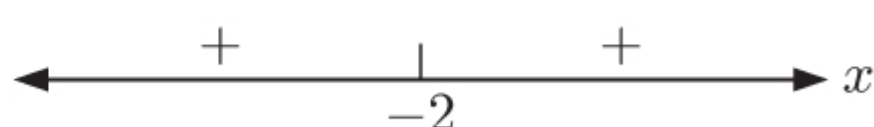


3 a $(x + 2)^2$ has zero -2 .

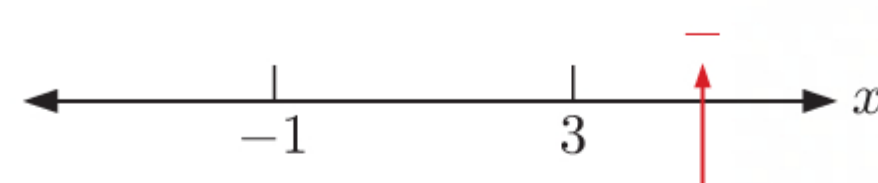


When $x = 0$ we have $(2)^2 > 0$,
so we put a + sign here.

As the factor is squared, the signs do not change.

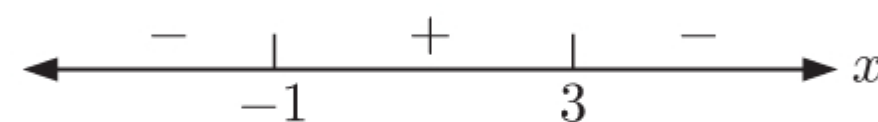


f $-(x + 1)(x - 3)$ has zeros -1 and 3.

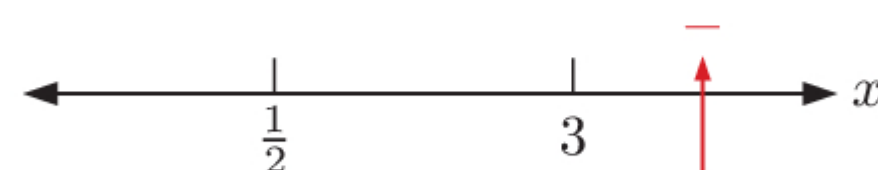


When $x = 4$ we have $-(5)(1) < 0$,
so we put a - sign here.

As the factors are single, the signs alternate.

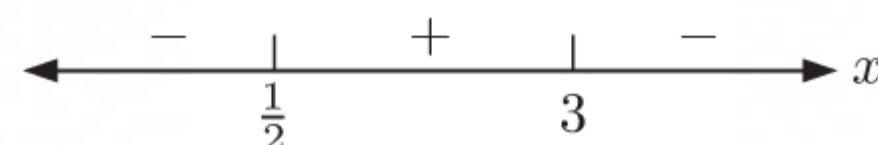


h $(2x - 1)(3 - x)$ has zeros $\frac{1}{2}$ and 3.

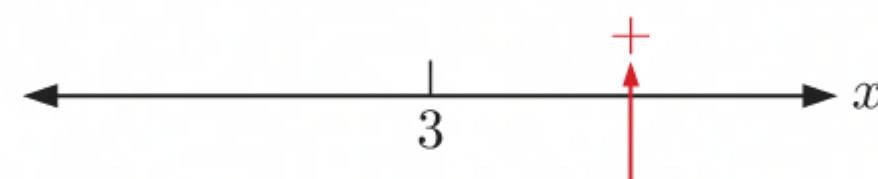


When $x = 4$ we have $(7)(-1) < 0$,
so we put a - sign here.

As the factors are single, the signs alternate.

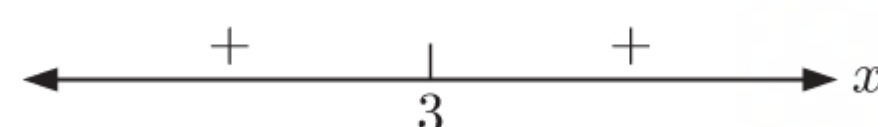


b $(x - 3)^2$ has zero 3.

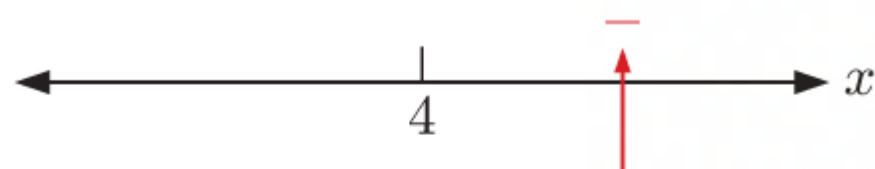


When $x = 4$ we have $(1)^2 > 0$,
so we put a + sign here.

As the factor is squared, the signs do not change.

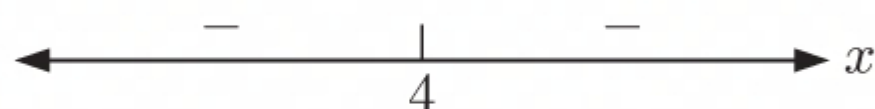


- c** $-(x-4)^2$ has zero 4.

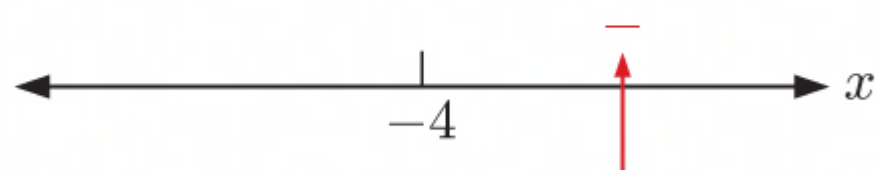


When $x = 5$ we have $-(1)^2 < 0$,
so we put a $-$ sign here.

As the factor is squared, the signs do not change.

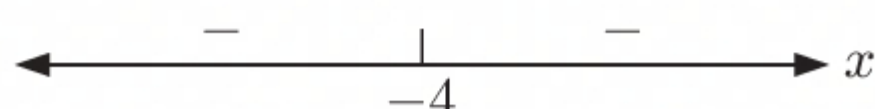


- e** $-3(x+4)^2$ has zero -4 .

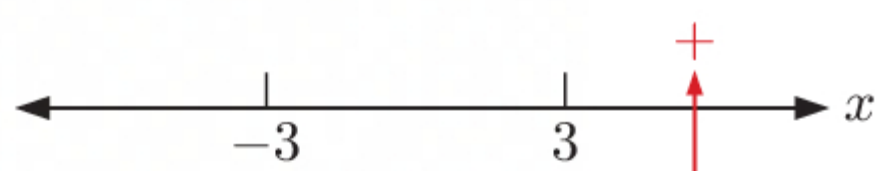


When $x = 0$ we have $-3(4)^2 < 0$,
so we put a $-$ sign here.

As the factor is squared, the signs do not change.



- 4 a** $x^2 - 9 = (x+3)(x-3)$
has zeros -3 and 3 .



When $x = 4$ we have $(7)(1) > 0$,
so we put a $+$ sign here.

As the factors are single, the signs alternate.

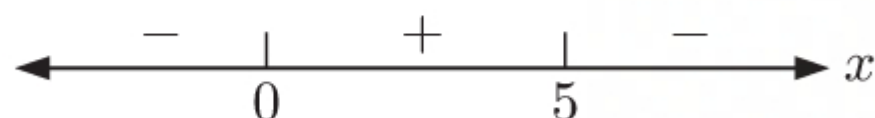


- c** $5x - x^2 = x(5-x)$ has zeros 0 and 5.

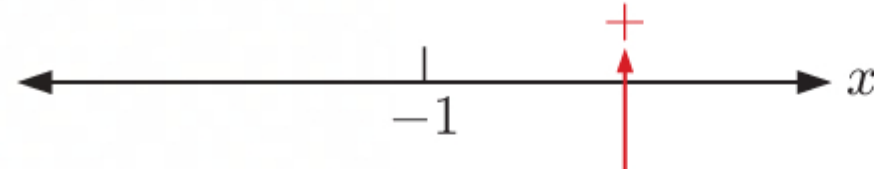


When $x = 6$ we have $(6)(-1) < 0$,
so we put a $-$ sign here.

As the factors are single, the signs alternate.

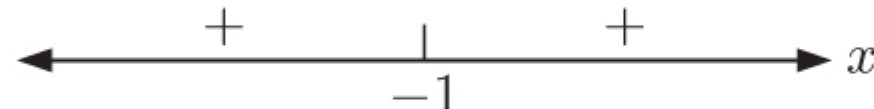


- d** $2(x+1)^2$ has zero -1 .

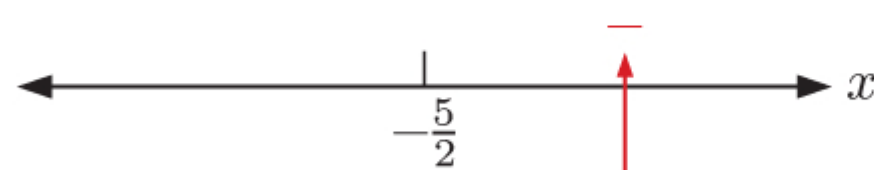


When $x = 0$ we have $2(1)^2 > 0$,
so we put a $+$ sign here.

As the factor is squared, the signs do not change.

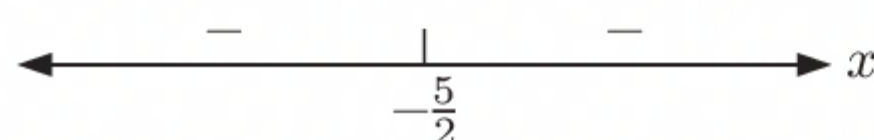


- f** $-\frac{1}{2}(2x+5)^2$ has zero $-\frac{5}{2}$.

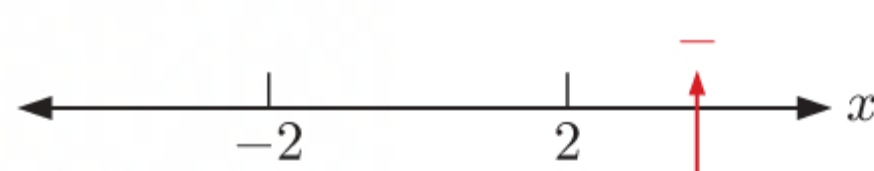


When $x = 0$ we have $-\frac{1}{2}(5)^2 < 0$,
so we put a $-$ sign here.

As the factor is squared, the signs do not change.

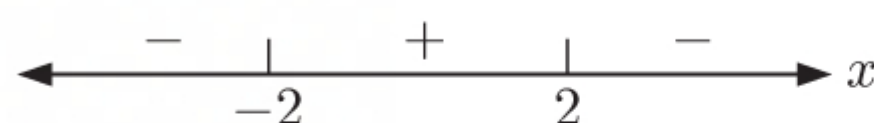


- b** $4 - x^2 = (2+x)(2-x)$
has zeros -2 and 2 .

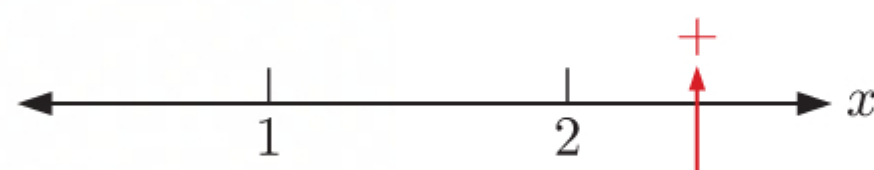


When $x = 3$ we have $(5)(-1) < 0$,
so we put a $-$ sign here.

As the factors are single, the signs alternate.

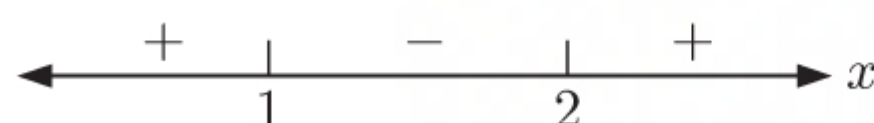


- d** $x^2 - 3x + 2 = (x-1)(x-2)$
has zeros 1 and 2.

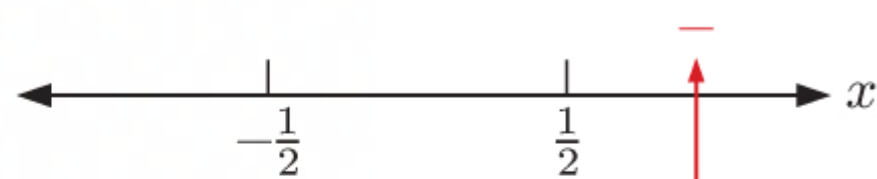


When $x = 3$ we have $(2)(1) > 0$,
so we put a $+$ sign here.

As the factors are single, the signs alternate.

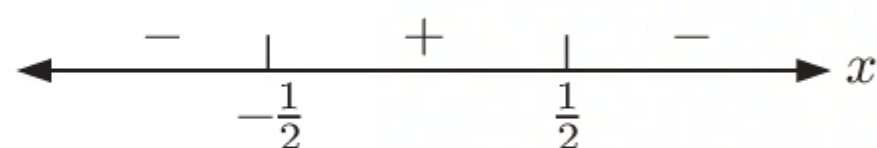


- e** $2 - 8x^2 = 2(1 + 2x)(1 - 2x)$
has zeros $-\frac{1}{2}$ and $\frac{1}{2}$.

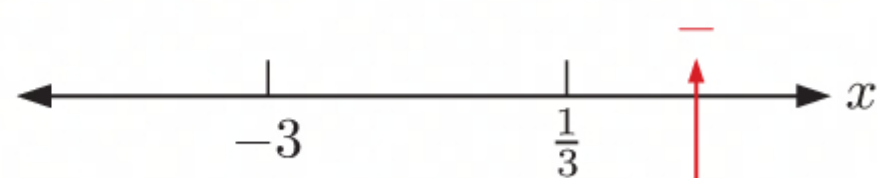


When $x = 1$ we have $2(3)(-1) < 0$,
so we put a $-$ sign here.

As the factors are single, the signs alternate.

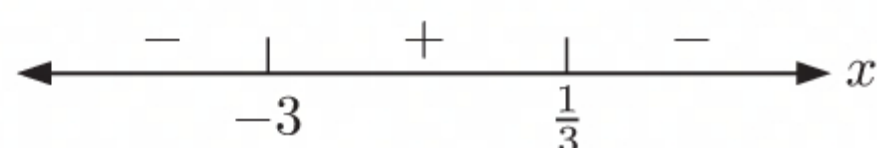


- g** $6 - 16x - 6x^2 = (6 + 2x)(1 - 3x)$
has zeros -3 and $\frac{1}{3}$.

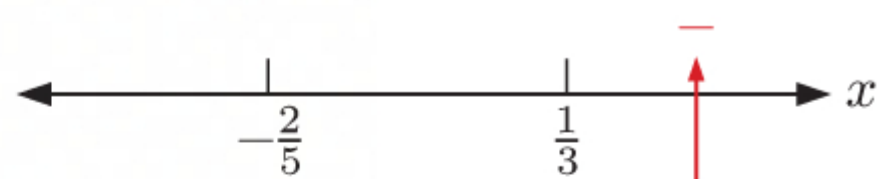


When $x = 1$ we have $(8)(-2) < 0$,
so we put a $-$ sign here.

As the factors are single, the signs alternate.

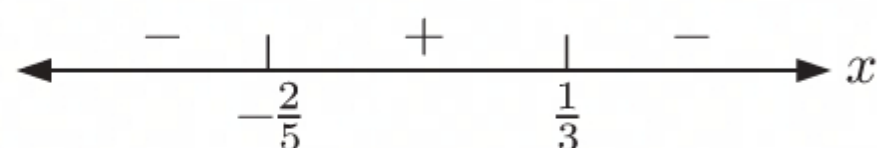


- i** $-15x^2 - x + 2 = (5x + 2)(-3x + 1)$
has zeros $-\frac{2}{5}$ and $\frac{1}{3}$.

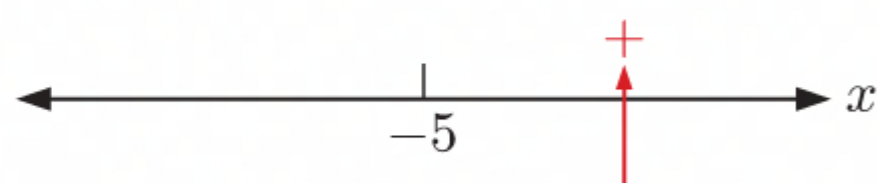


When $x = 1$ we have $(7)(-2) < 0$,
so we put a $-$ sign here.

As the factors are single, the signs alternate.

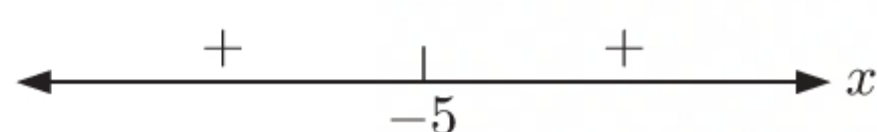


- 5 a** $x^2 + 10x + 25 = (x + 5)^2$ has zero -5 .

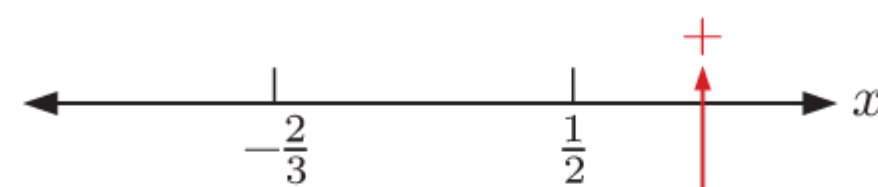


When $x = 0$ we have $(5)^2 > 0$,
so we put a $+$ sign here.

As the factor is squared, the signs do not change.

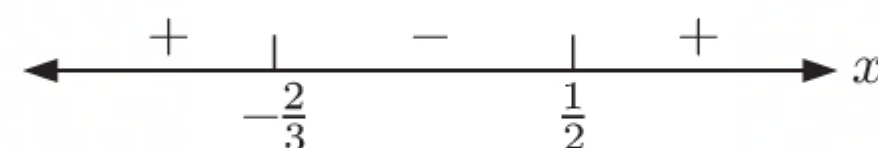


- f** $6x^2 + x - 2 = (3x + 2)(2x - 1)$
has zeros $-\frac{2}{3}$ and $\frac{1}{2}$.



When $x = 1$ we have $(5)(1) > 0$,
so we put a $+$ sign here.

As the factors are single, the signs alternate.

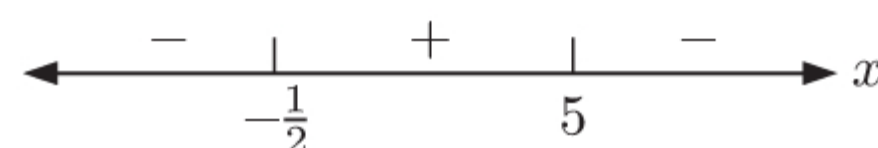


- h** $-2x^2 + 9x + 5 = (2x + 1)(5 - x)$
has zeros $-\frac{1}{2}$ and 5 .

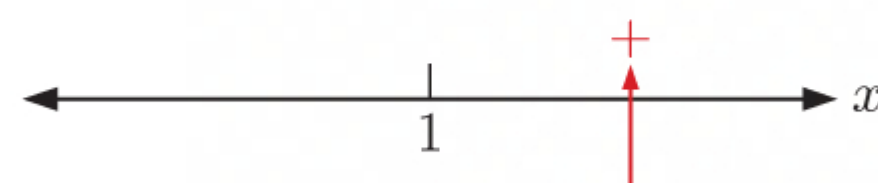


When $x = 6$ we have $(13)(-1) < 0$,
so we put a $-$ sign here.

As the factors are single, the signs alternate.

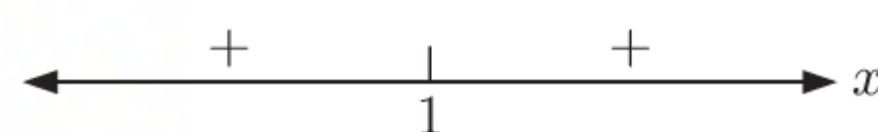


- b** $x^2 - 2x + 1 = (x - 1)^2$ has zero 1 .

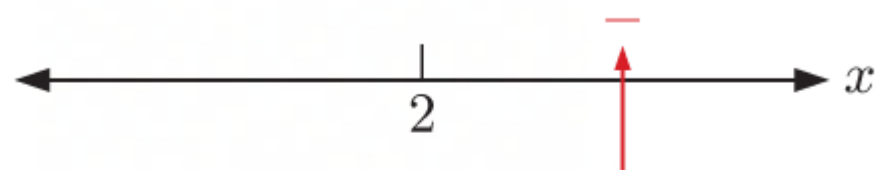


When $x = 2$ we have $(1)^2 > 0$,
so we put a $+$ sign here.

As the factor is squared, the signs do not change.

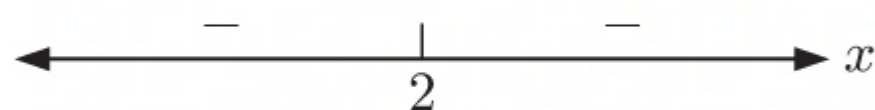


c $-x^2 + 4x - 4 = -(x - 2)^2$ has zero 2.

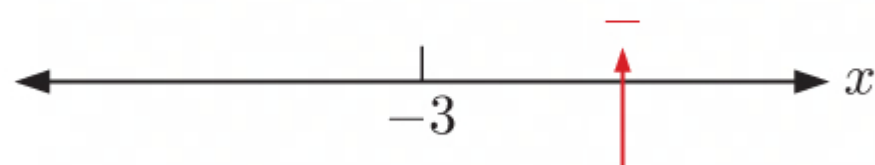


When $x = 3$ we have $-(1)^2 < 0$,
so we put a $-$ sign here.

As the factor is squared, the signs do not change.

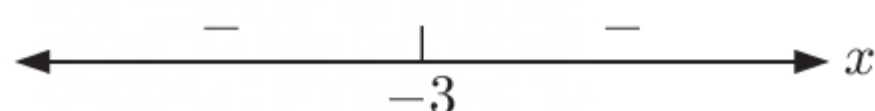


e $-x^2 - 6x - 9 = -(x + 3)^2$ has zero -3 .

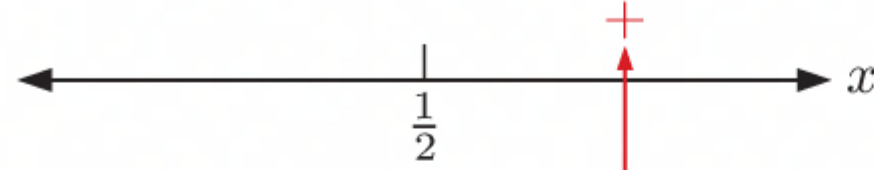


When $x = 0$ we have $-(3)^2 < 0$,
so we put a $-$ sign here.

As the factor is squared, the signs do not change.

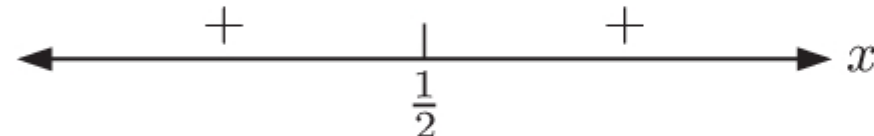


d $4x^2 - 4x + 1 = (2x - 1)^2$ has zero $\frac{1}{2}$.

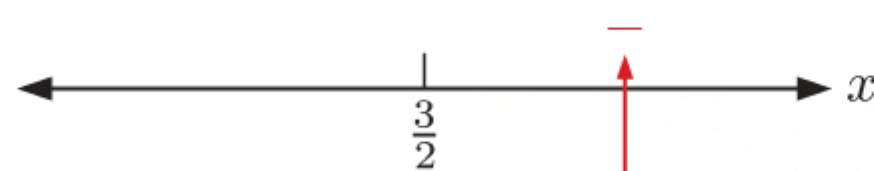


When $x = 1$ we have $(1)^2 > 0$,
so we put a $+$ sign here.

As the factor is squared, the signs do not change.

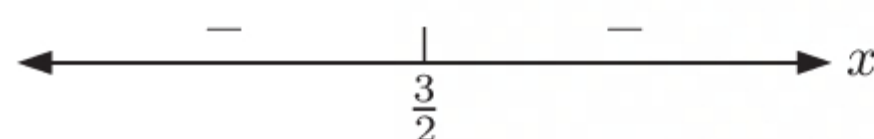


f $-4x^2 + 12x - 9 = -(2x - 3)^2$
has zero $\frac{3}{2}$.



When $x = 2$ we have $-(1)^2 < 0$,
so we put a $-$ sign here.

As the factor is squared, the signs do not change.



EXERCISE 2H.2

1 a $(x - 2)(x + 5) \leq 0$

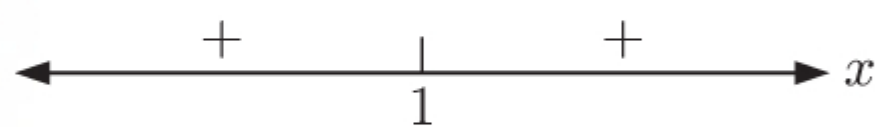
Sign diagram of LHS is



$\therefore -5 \leq x \leq 2$

c $(x - 1)^2 < 0$

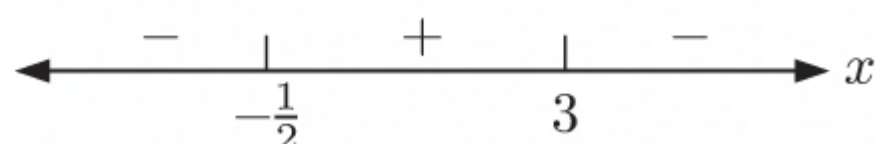
Sign diagram of LHS is



\therefore the inequality is not true for any real x .

e $(2x + 1)(3 - x) > 0$

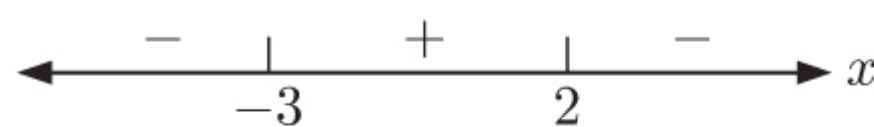
Sign diagram of LHS is



$\therefore -\frac{1}{2} < x < 3$

b $(2 - x)(x + 3) \geq 0$

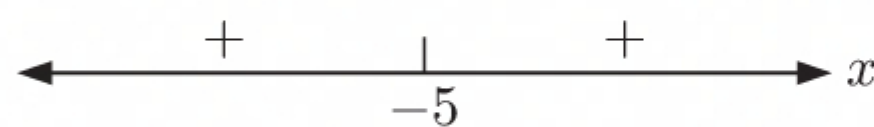
Sign diagram of LHS is



$\therefore -3 \leq x \leq 2$

d $(x + 5)^2 \geq 0$

Sign diagram of LHS is



\therefore the inequality is true for all real x .

f $(x - 4)(2x + 3) < 0$

Sign diagram of LHS is



$\therefore -\frac{3}{2} < x < 4$

2 a $x^2 - x \geq 0$

$\therefore x(x-1) \geq 0$

Sign diagram of LHS is

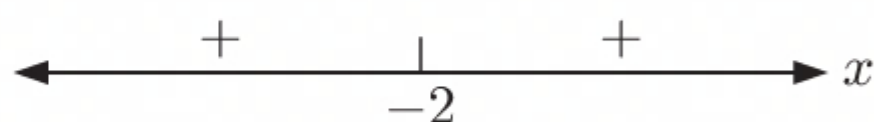


$\therefore x \leq 0 \text{ or } x \geq 1$

c $x^2 + 4x + 4 > 0$

$\therefore (x+2)^2 > 0$

Sign diagram of LHS is

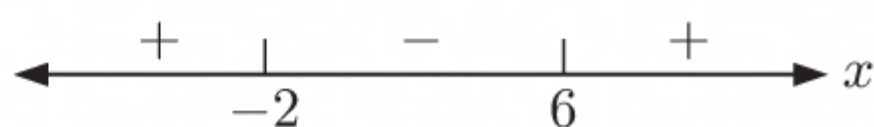


\therefore the inequality is true for all $x \neq -2$.

e $x^2 - 4x - 12 > 0$

$\therefore (x+2)(x-6) > 0$

Sign diagram of LHS is

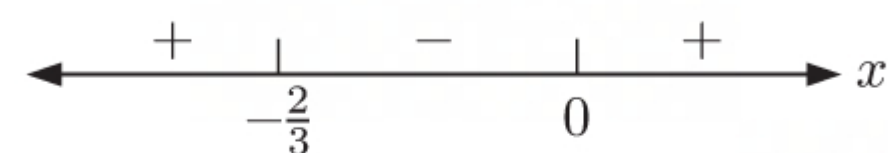


$\therefore x < -2 \text{ or } x > 6$

b $3x^2 + 2x < 0$

$\therefore x(3x+2) < 0$

Sign diagram of LHS is

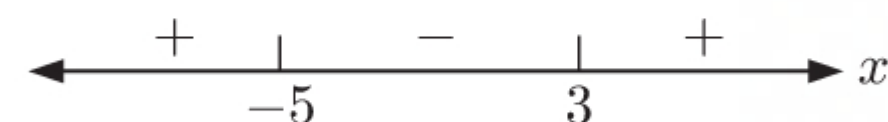


$\therefore -\frac{2}{3} < x < 0$

d $x^2 + 2x - 15 \leq 0$

$\therefore (x+5)(x-3) \leq 0$

Sign diagram of LHS is

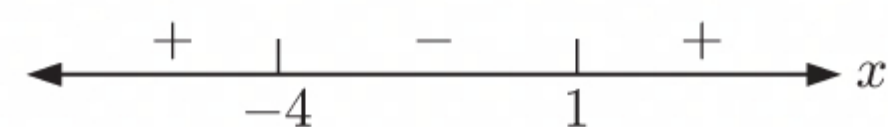


$\therefore -5 \leq x \leq 3$

f $3x^2 + 9x - 12 < 0$

$\therefore 3(x+4)(x-1) < 0$

Sign diagram of LHS is



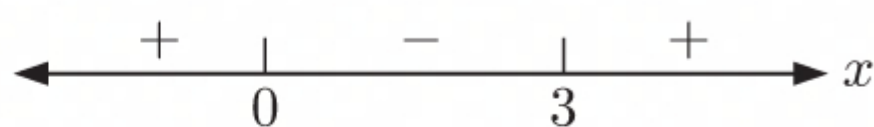
$\therefore -4 < x < 1$

3 a $x^2 \geq 3x$

$\therefore x^2 - 3x \geq 0$

$\therefore x(x-3) \geq 0$

Sign diagram of LHS is



$\therefore x \leq 0 \text{ or } x \geq 3$

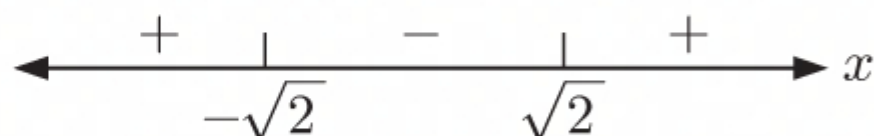
c $2x^2 \geq 4$

$\therefore 2x^2 - 4 \geq 0$

$\therefore 2(x^2 - 2) \geq 0$

$\therefore 2(x + \sqrt{2})(x - \sqrt{2}) \geq 0$

Sign diagram of LHS is



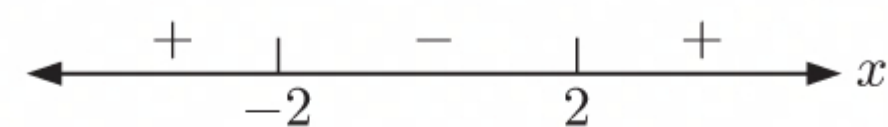
$\therefore x \leq -\sqrt{2} \text{ or } x \geq \sqrt{2}$

b $x^2 < 4$

$\therefore x^2 - 4 < 0$

$\therefore (x+2)(x-2) < 0$

Sign diagram of LHS is



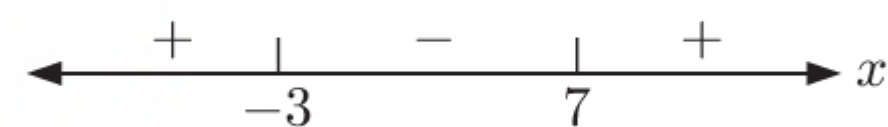
$\therefore -2 < x < 2$

d $x^2 - 21 \leq 4x$

$\therefore x^2 - 4x - 21 \leq 0$

$\therefore (x-7)(x+3) \leq 0$

Sign diagram of LHS is



$\therefore -3 \leq x \leq 7$

e $x^2 + 30 > 11x$

$$\therefore x^2 - 11x + 30 > 0$$

$$\therefore (x - 5)(x - 6) > 0$$

Sign diagram of LHS is



$$\therefore x < 5 \text{ or } x > 6$$

g $2x^2 \geq x + 3$

$$\therefore 2x^2 - x - 3 \geq 0$$

$$\therefore (2x - 3)(x + 1) \geq 0$$

Sign diagram of LHS is



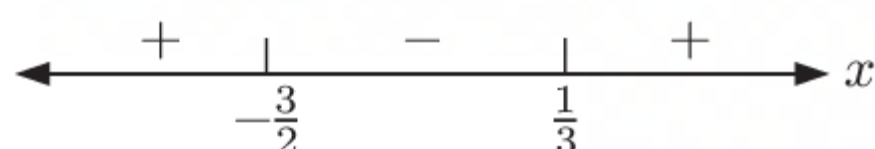
$$\therefore x \leq -1 \text{ or } x \geq \frac{3}{2}$$

i $6x^2 + 7x < 3$

$$\therefore 6x^2 + 7x - 3 < 0$$

$$\therefore (3x - 1)(2x + 3) < 0$$

Sign diagram of LHS is



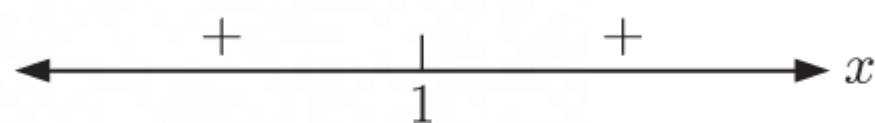
$$\therefore -\frac{3}{2} < x < \frac{1}{3}$$

k $2x^2 - 4x + 2 > 0$

$$\therefore 2(x^2 - 2x + 1) > 0$$

$$\therefore 2(x - 1)^2 > 0$$

Sign diagram of LHS is



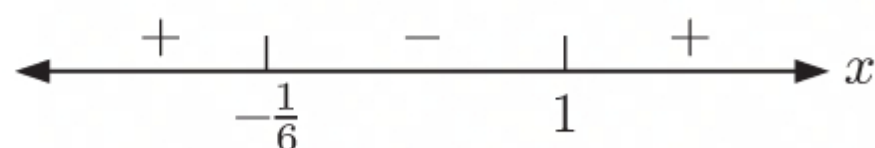
\therefore the inequality is true for all $x \neq 1$.

m $1 + 5x < 6x^2$

$$\therefore 6x^2 - 5x - 1 > 0$$

$$\therefore (6x + 1)(x - 1) > 0$$

Sign diagram of LHS is



$$\therefore x < -\frac{1}{6} \text{ or } x > 1$$

f $x + 42 < x^2$

$$\therefore x^2 - x - 42 > 0$$

$$\therefore (x + 6)(x - 7) > 0$$

Sign diagram of LHS is

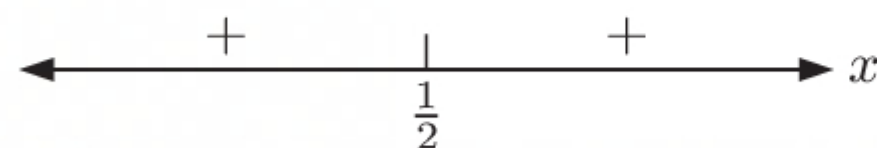


$$\therefore x < -6 \text{ or } x > 7$$

h $4x^2 - 4x + 1 < 0$

$$\therefore (2x - 1)^2 < 0$$

Sign diagram of LHS is



\therefore the inequality is not true for any real x .

j $3x^2 > 8(x + 2)$

$$\therefore 3x^2 > 8x + 16$$

$$\therefore 3x^2 - 8x - 16 > 0$$

$$\therefore (3x + 4)(x - 4) > 0$$

Sign diagram of LHS is



$$\therefore x < -\frac{4}{3} \text{ or } x > 4$$

l $6x^2 + 1 \leq 5x$

$$\therefore 6x^2 - 5x + 1 \leq 0$$

$$\therefore (3x - 1)(2x - 1) \leq 0$$

Sign diagram of LHS is



$$\therefore \frac{1}{3} \leq x \leq \frac{1}{2}$$

n $12x^2 \geq 5x + 2$

$$\therefore 12x^2 - 5x - 2 \geq 0$$

$$\therefore (4x + 1)(3x - 2) \geq 0$$

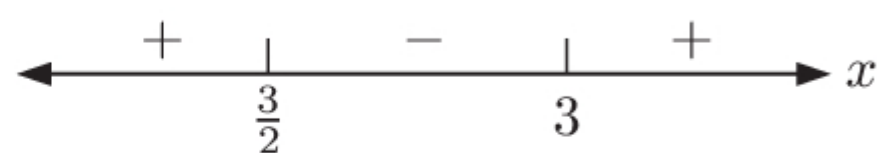
Sign diagram of LHS is



$$\therefore x \leq -\frac{1}{4} \text{ or } x \geq \frac{2}{3}$$

$$\begin{aligned} & 2x^2 + 9 > 9x \\ \therefore 2x^2 - 9x + 9 & > 0 \\ \therefore (2x - 3)(x - 3) & > 0 \end{aligned}$$

Sign diagram of LHS is



$$\therefore x < \frac{3}{2} \text{ or } x > 3$$

4 a $y = 2x^2 + kx - k$ has $a = 2$, $b = k$, $c = -k$

$$\begin{aligned} \therefore \Delta &= b^2 - 4ac \\ &= (k)^2 - 4(2)(-k) \\ &= k^2 + 8k \\ &= k(k + 8) \end{aligned}$$

So, Δ has sign diagram:



i The graph cuts the x -axis twice if $\Delta > 0$

$$\therefore k < -8 \text{ or } k > 0.$$

ii The graph touches the x -axis if $\Delta = 0$

$$\therefore k = -8 \text{ or } k = 0.$$

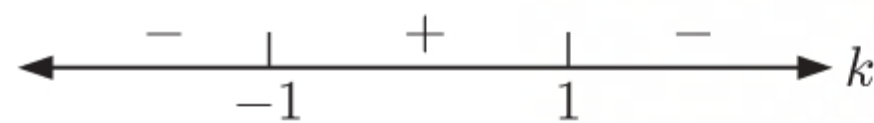
iii The graph misses the x -axis if $\Delta < 0$

$$\therefore -8 < k < 0.$$

b $y = kx^2 - 2x + k$ has $a = k$, $b = -2$, $c = k$

$$\begin{aligned} \therefore \Delta &= b^2 - 4ac \\ &= (-2)^2 - 4(k)(k) \\ &= 4 - 4k^2 \\ &= 4(1 - k^2) \\ &= 4(1 + k)(1 - k) \end{aligned}$$

So, Δ has sign diagram:



i The graph cuts the x -axis twice if $\Delta > 0$

$$\therefore -1 < k < 1, k \neq 0.$$

ii The graph touches the x -axis if $\Delta = 0$

$$\therefore k = -1 \text{ or } k = 1.$$

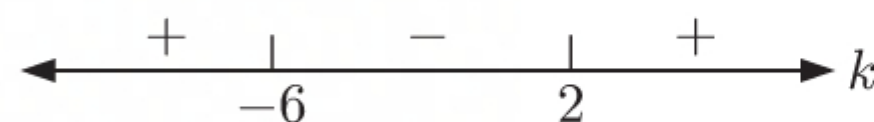
iii The graph misses the x -axis if $\Delta < 0$

$$\therefore k < -1 \text{ or } k > 1.$$

c $y = x^2 + (k + 2)x + 4$ has $a = 1$, $b = k + 2$, $c = 4$

$$\begin{aligned} \therefore \Delta &= b^2 - 4ac \\ &= (k + 2)^2 - 4(1)(4) \\ &= k^2 + 4k + 4 - 16 \\ &= k^2 + 4k - 12 \\ &= (k + 6)(k - 2) \end{aligned}$$

So, Δ has sign diagram:



i The graph cuts the x -axis twice if $\Delta > 0$

$$\therefore k < -6 \text{ or } k > 2.$$

ii The graph touches the x -axis if $\Delta = 0$

$$\therefore k = -6 \text{ or } k = 2.$$

iii The graph misses the x -axis if $\Delta < 0$

$$\therefore -6 < k < 2.$$

5 a $2x^2 + (k-2)x + 2$ has $a = 2$, $b = k-2$, $c = 2$

$$\begin{aligned}\therefore \Delta &= b^2 - 4ac \\ &= (k-2)^2 - 4(2)(2) \\ &= k^2 - 4k + 4 - 16 \\ &= k^2 - 4k - 12 \\ &= (k+2)(k-6)\end{aligned}$$

So, Δ has sign diagram:



$2x^2 + (k-2)x + 2 = 0$ has:

i two real roots if $\Delta > 0$

$$\therefore k < -2 \text{ or } k > 6$$

iii no real roots if $\Delta < 0$

$$\therefore -2 < k < 6.$$

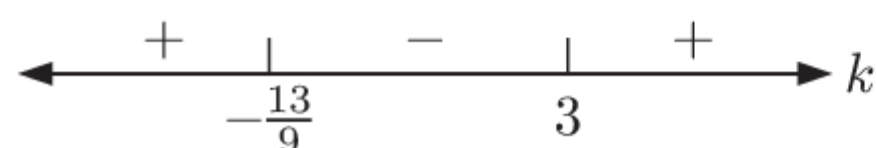
ii a repeated real root if $\Delta = 0$

$$\therefore k = -2 \text{ or } k = 6$$

b $x^2 + (3k-1)x + (2k+10)$ has $a = 1$, $b = 3k-1$, $c = 2k+10$

$$\begin{aligned}\therefore \Delta &= b^2 - 4ac \\ &= (3k-1)^2 - 4(1)(2k+10) \\ &= 9k^2 - 6k + 1 - 8k - 40 \\ &= 9k^2 - 14k - 39 \\ &= (9k+13)(k-3)\end{aligned}$$

So, Δ has sign diagram:



$x^2 + (3k-1)x + (2k+10) = 0$ has:

i two real roots if $\Delta > 0$

$$\therefore k < -\frac{13}{9} \text{ or } k > 3$$

iii no real roots if $\Delta < 0$

$$\therefore -\frac{13}{9} < k < 3.$$

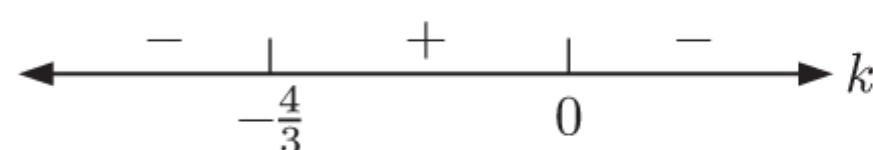
ii a repeated real root if $\Delta = 0$

$$\therefore k = -\frac{13}{9} \text{ or } k = 3$$

c $(k+1)x^2 + kx + k$ has $a = k+1$, $b = k$, $c = k$

$$\begin{aligned}\therefore \Delta &= b^2 - 4ac \\ &= (k)^2 - 4(k+1)(k) \\ &= k^2 - 4(k^2 + k) \\ &= k^2 - 4k^2 - 4k \\ &= -3k^2 - 4k \\ &= -k(3k+4)\end{aligned}$$

So, Δ has sign diagram:



$(k+1)x^2 + kx + k = 0$ has:

i two real roots if $\Delta > 0$

$$\therefore -\frac{4}{3} < k < 0, \quad k \neq -1$$

iii no real roots if $\Delta < 0$

$$\therefore k < -\frac{4}{3} \text{ or } k > 0.$$

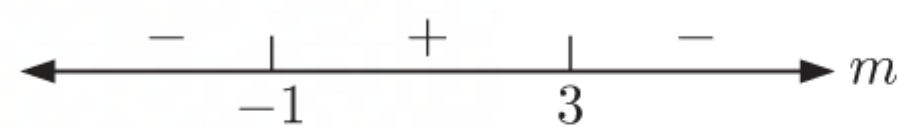
ii a repeated real root if $\Delta = 0$

$$\therefore k = -\frac{4}{3} \text{ or } k = 0$$

- 6 $(m-2)x^2 + 6x + 3m$ has $a = m-2$, $b = 6$, $c = 3m$

$$\begin{aligned}\therefore \Delta &= b^2 - 4ac \\ &= (6)^2 - 4(m-2)(3m) \\ &= 36 - 4(3m^2 - 6m) \\ &= -12m^2 + 24m + 36 \\ &= -12(m^2 - 2m - 3) \\ &= -12(m+1)(m-3)\end{aligned}$$

So, Δ has sign diagram:



- a $y = (m-2)x^2 + 6x + 3m$ is positive definite if $a > 0$ and $\Delta < 0$.

$$\begin{aligned}\text{Now, } a &> 0 & \text{and } \Delta &< 0 \\ \text{if } m-2 &> 0 & \text{if } m < -1 \text{ or } m > 3 \\ \therefore m &> 2\end{aligned}$$

\therefore the graph is positive definite if $m > 3$.

- b $y = (m-2)x^2 + 6x + 3m$ is negative definite if $a < 0$ and $\Delta < 0$.

$$\begin{aligned}\text{Now, } a &< 0 & \text{and } \Delta &< 0 \\ \text{if } m-2 &< 0 & \text{if } m < -1 \text{ or } m > 3 \\ \therefore m &< 2\end{aligned}$$

\therefore the graph is negative definite if $m < -1$.

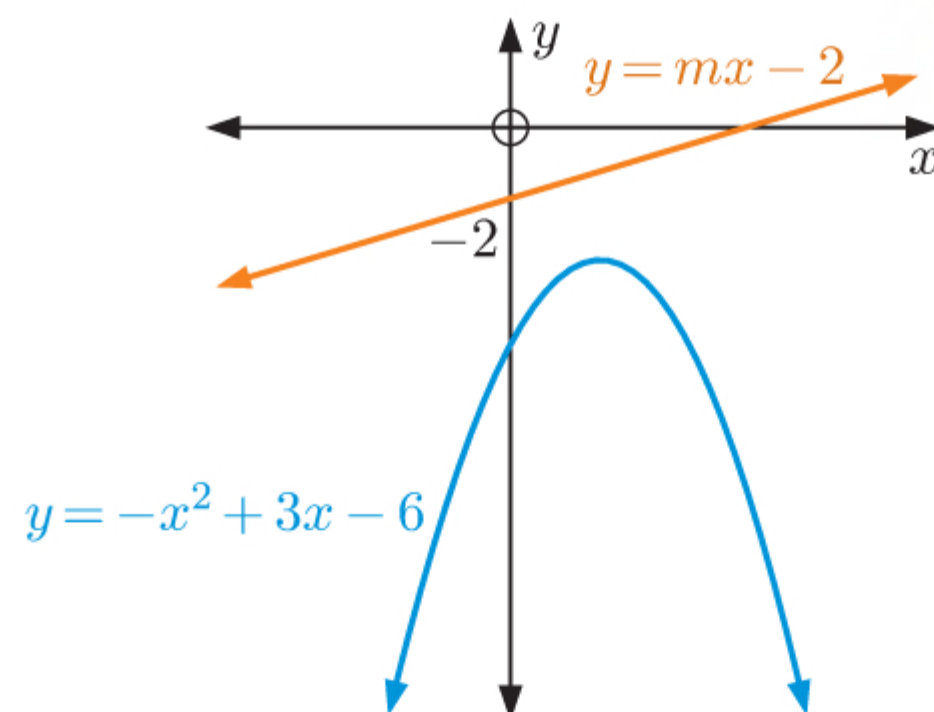
- 7 $y = -x^2 + 3x - 6$ meets $y = mx - 2$ where

$$-x^2 + 3x - 6 = mx - 2$$

$$\therefore x^2 + (m-3)x + 4 = 0$$

$$\begin{aligned}\text{Now } \Delta &= (m-3)^2 - 4(1)(4) \\ &= m^2 - 6m + 9 - 16 \\ &= m^2 - 6m - 7 \\ &= (m+1)(m-7)\end{aligned}$$

So, Δ has sign diagram:



- a The line meets the curve twice if $\Delta > 0$.

$$\therefore m < -1 \text{ or } m > 7$$

- b The line is a tangent to the curve if it *touches* the curve, $\Delta = 0$.

$$\therefore m = -1 \text{ or } m = 7$$

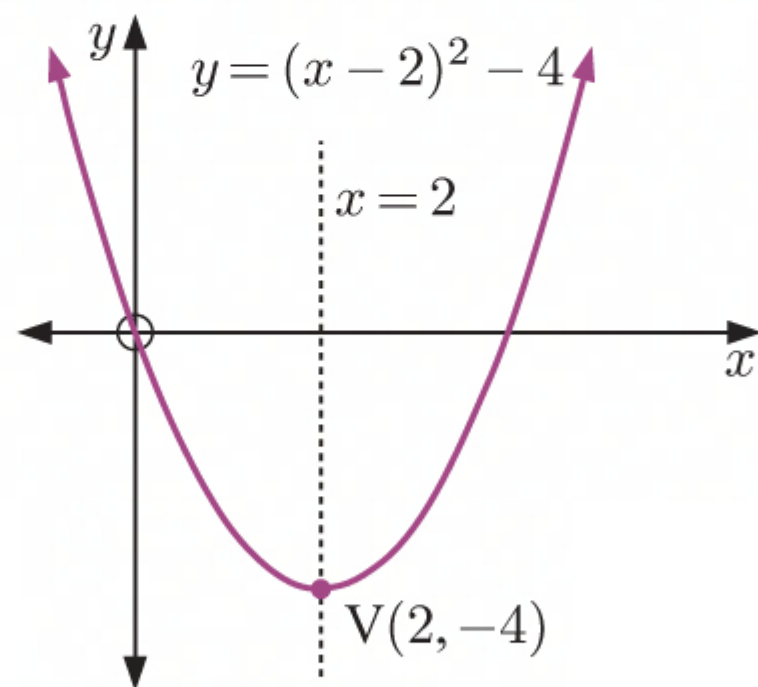
- c The line does not meet the curve if $\Delta < 0$.

$$\therefore -1 < m < 7$$

REVIEW SET 2A

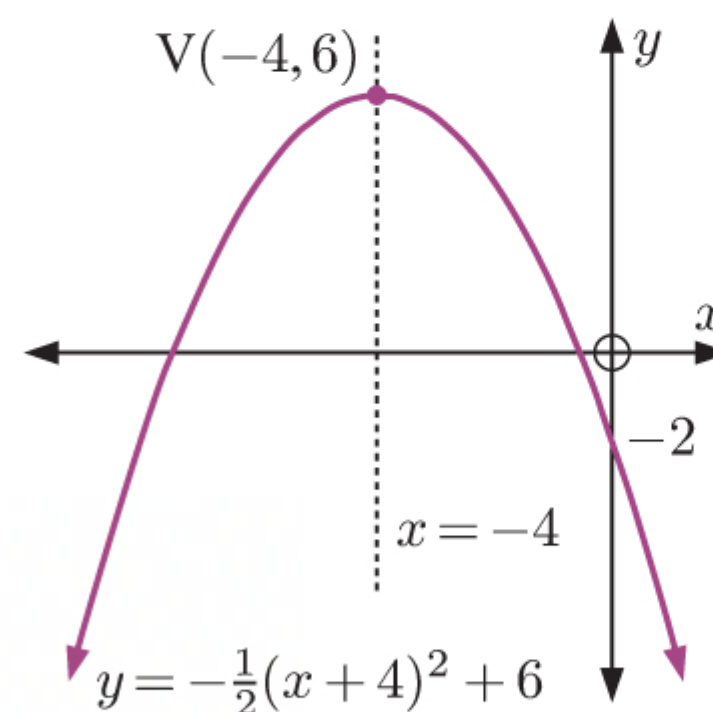
- 1 a** $y = (x - 2)^2 - 4$ has vertex $(2, -4)$ and axis of symmetry $x = 2$.

When $x = 0$, $y = (-2)^2 - 4 = 0$
so the y -intercept is 0.



- b** $y = -\frac{1}{2}(x + 4)^2 + 6$ has vertex $(-4, 6)$ and axis of symmetry $x = -4$.

When $x = 0$, $y = -\frac{1}{2}(4)^2 + 6 = -2$
so the y -intercept is -2 .



- 2 a** The graph touches the x -axis at 4, so the quadratic has the form $y = a(x - 4)^2$, $a \neq 0$.

When $x = 2$, $y = 12$
 $\therefore 12 = a(2 - 4)^2$
 $\therefore 12 = a(-2)^2$
 $\therefore a = 3$


The quadratic is $y = 3(x - 4)^2$
 $= 3(x^2 - 8x + 16)$
 $\therefore y = 3x^2 - 24x + 48$

- b** The vertex is $(-4, 1)$, so the quadratic has the form $y = a(x + 4)^2 + 1$, $a \neq 0$.

When $x = 1$, $y = 11$
 $\therefore 11 = a(1 + 4)^2 + 1$
 $= a(5)^2 + 1$
 $\therefore 25a = 10$
 $\therefore a = \frac{10}{25} = \frac{2}{5}$

The quadratic is $y = \frac{2}{5}(x + 4)^2 + 1$
 $= \frac{2}{5}(x^2 + 8x + 16) + 1$
 $\therefore y = \frac{2}{5}x^2 + \frac{16}{5}x + \frac{37}{5}$

- 3** $y = -2x^2 + 4x + 3$ has $a = -2$, $b = 4$, $c = 3$.

Since $a < 0$, the shape is 

The maximum value occurs when $x = \frac{-b}{2a} = \frac{-4}{2(-2)} = 1$

and $y = -2(1)^2 + 4(1) + 3$
 $= 5$

So, the maximum value of y is 5, occurring when $x = 1$.

$$\begin{aligned}
 \text{4 } y = x^2 - 3x \text{ meets } y = 3x^2 - 5x - 24 \\
 \text{where } x^2 - 3x = 3x^2 - 5x - 24 \\
 \therefore 2x^2 - 2x - 24 = 0 \\
 \therefore x^2 - x - 12 = 0 \\
 \therefore (x - 4)(x + 3) = 0 \\
 \therefore x = 4 \text{ or } -3
 \end{aligned}$$

$$\begin{aligned}
 \text{Substituting into } y = x^2 - 3x, \\
 \text{when } x = 4, \quad y = 4^2 - 3(4) \\
 = 4
 \end{aligned}$$

$$\begin{aligned}
 \text{and when } x = -3, \quad y = (-3)^2 - 3(-3) \\
 = 9 + 9 \\
 = 18
 \end{aligned}$$

\therefore the graphs meet at $(4, 4)$ and $(-3, 18)$.

$$\begin{aligned}
 \text{5 } y = -2x^2 + 5x + k \text{ has } a = -2, \quad b = 5, \quad c = k. \\
 \Delta = b^2 - 4ac \\
 = 5^2 - 4(-2)(k) \\
 = 25 + 8k
 \end{aligned}$$

The graph does not cut the x -axis if $\Delta < 0$

$$\begin{aligned}
 \therefore 25 + 8k < 0 \\
 \therefore 8k < -25 \\
 \therefore k < -\frac{25}{8} \\
 \therefore k < -3\frac{1}{8}
 \end{aligned}$$

$$\text{6 } 2x^2 - 3x + m = 0 \text{ has } a = 2, \quad b = -3, \quad c = m.$$

$$\begin{aligned}
 \therefore \Delta = b^2 - 4ac \\
 = (-3)^2 - 4(2)(m) \\
 = 9 - 8m
 \end{aligned}$$

$$\begin{aligned}
 \text{a There is a repeated root if } \Delta = 0 \\
 \therefore 9 - 8m = 0 \\
 \therefore m = \frac{9}{8}
 \end{aligned}$$

$$\begin{aligned}
 \text{b There are two distinct real roots if } \Delta > 0 \\
 \therefore 9 - 8m > 0 \\
 \therefore 8m < 9 \\
 \therefore m < \frac{9}{8}
 \end{aligned}$$

$$\begin{aligned}
 \text{c There are no real roots if } \Delta < 0 \\
 \therefore 9 - 8m < 0 \\
 \therefore 8m > 9 \\
 \therefore m > \frac{9}{8}
 \end{aligned}$$

- 7** Let the number be x , so its reciprocal is $\frac{1}{x}$.

$$\therefore x + \frac{1}{x} = 2\frac{1}{30} = \frac{61}{30}$$

$$\therefore x^2 + 1 = \frac{61}{30}x$$

$$\therefore 30x^2 + 30 = 61x$$

$$\therefore 30x^2 - 61x + 30 = 0$$

$$\therefore (5x - 6)(6x - 5) = 0$$

$$\therefore x = \frac{6}{5} \text{ or } \frac{5}{6}$$

So, the number is $\frac{6}{5}$ or $\frac{5}{6}$.

- 8** A line with y -intercept 10 will have an equation of the form $y = mx + 10$.

$y = 3x^2 + 7x - 2$ meets this line where $3x^2 + 7x - 2 = mx + 10$

$$\therefore 3x^2 + (7 - m)x - 12 = 0$$

For $y = mx + 10$ to be tangential to $y = 3x^2 + 7x - 2$, this equation must have exactly one solution, so there is a repeated root.

$$\therefore \Delta = 0$$

$$\therefore (7 - m)^2 - 4(3)(-12) = 0$$

$$\therefore 49 - 14m + m^2 + 144 = 0$$

$$\therefore m^2 - 14m + 193 = 0 \text{ which has discriminant } (-14)^2 - 4(1)(193) < 0$$

So, there are no real solutions for m .

\therefore no line with y -intercept 10 can be tangential to $y = 3x^2 + 7x - 2$.

9 a $y = 2x^2 + 6x - 3$

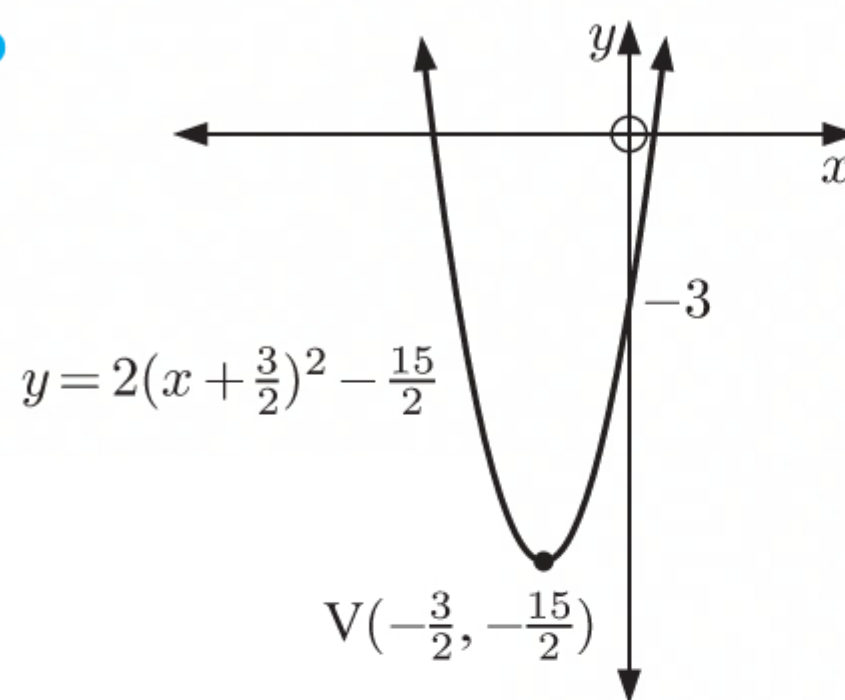
$$= 2\left[x^2 + 3x - \frac{3}{2}\right]$$

$$= 2\left[x^2 + 3x + \left(\frac{3}{2}\right)^2 - \frac{3}{2} - \left(\frac{3}{2}\right)^2\right]$$

$$= 2\left[\left(x + \frac{3}{2}\right)^2 - \frac{15}{4}\right]$$

$$\therefore y = 2\left(x + \frac{3}{2}\right)^2 - \frac{15}{2}$$

b



- 10 a** Since the vertex is $(2, -20)$, the quadratic has the form

$$y = a(x - 2)^2 - 20 \text{ where } a > 0$$

When $x = 5$, $y = 0$

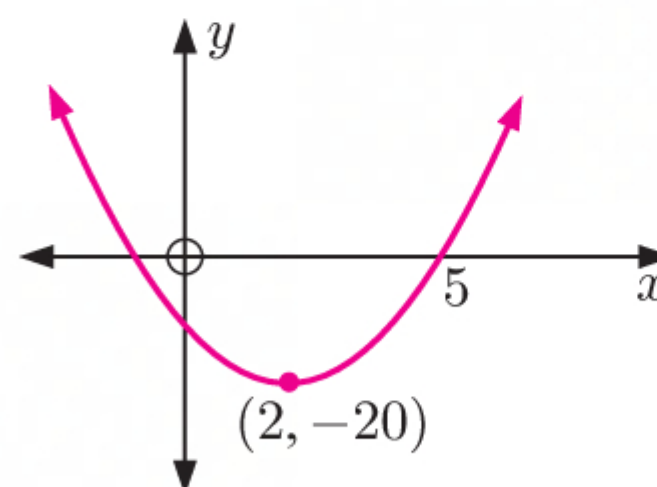
$$\therefore 0 = a(5 - 2)^2 - 20$$

$$= a(3)^2 - 20$$

$$\therefore 9a = 20$$

$$\therefore a = \frac{20}{9}$$

The quadratic is $y = \frac{20}{9}(x - 2)^2 - 20$.



- b** The axis of symmetry $x = 4$ lies midway between the x -intercepts.

\therefore the other x -intercept is 1.

\therefore the quadratic has the form

$$y = a(x - 1)(x - 7) \quad \text{where } a < 0$$

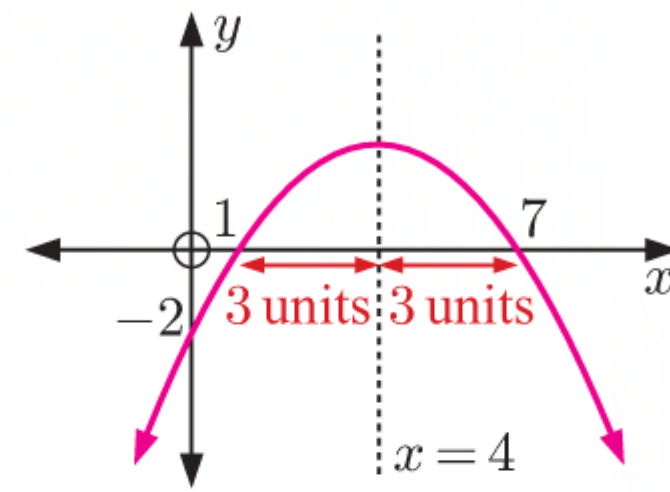
When $x = 0$, $y = -2$

$$\therefore -2 = a(-1)(-7)$$

$$\therefore -2 = 7a$$

$$\therefore a = -\frac{2}{7}$$

The quadratic is $y = -\frac{2}{7}(x - 1)(x - 7)$.



- c** The graph touches the x -axis at $x = -3$,
so $y = a(x + 3)^2$.

The graph is concave up, so $a > 0$.

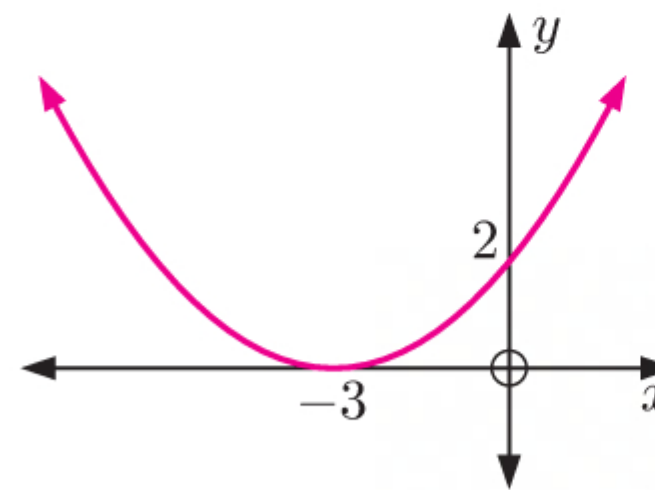
When $x = 0$, $y = 2$

$$\therefore 2 = a(3)^2$$

$$\therefore 2 = 9a$$

$$\therefore a = \frac{2}{9}$$

The quadratic is $y = \frac{2}{9}(x + 3)^2$.



11 $y = -x^2 + 2x$

$\therefore y = x(2 - x)$ has x -intercepts 0 and 2.

When $x = 0$, $y = 0(2 - 0)$
 $= 0$

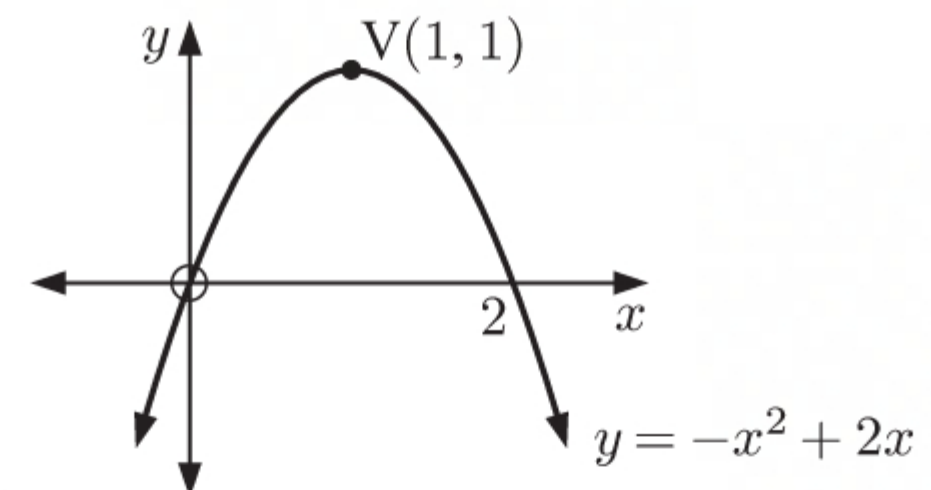
\therefore y -intercept is 0.

The axis of symmetry lies midway between the x -intercepts.

\therefore the axis of symmetry is $x = 1$.

When $x = 1$, $y = 1(2 - 1)$
 $= 1$

\therefore the vertex is $V(1, 1)$.



- 12** A line with gradient -3 will have an equation of the form $y = -3x + c$.

$y = 2x^2 - 5x + 1$ meets this line where $2x^2 - 5x + 1 = -3x + c$

$$\therefore 2x^2 - 2x + 1 - c = 0$$

If the graphs touch, this quadratic has $\Delta = 0$

$$\therefore (-2)^2 - 4(2)(1 - c) = 0$$

$$\therefore 4 - 8 + 8c = 0$$

$$\therefore 8c - 4 = 0$$

$$\therefore 8c = 4$$

$$\therefore c = \frac{1}{2}$$

$\therefore y = -3x + \frac{1}{2}$ is the line which is a tangent to $y = 2x^2 - 5x + 1$.

The y -intercept of the line is $\frac{1}{2}$.

13 $y = x^2 - 2x + k$ has $a = 1$, $b = -2$, $c = k$
 $\Delta = b^2 - 4ac$
 $= (-2)^2 - 4(1)(k)$
 $= 4 - 4k$

The graph cuts the x -axis twice if $\Delta > 0$

$$\therefore 4 - 4k > 0$$

$$\therefore 4k < 4$$

$$\therefore k < 1$$

14 a i The graph cuts the x -axis twice.
 $\therefore \Delta > 0$

ii The graph is concave down.
 $\therefore a < 0$

b i $y = a(x + m)(x + n)$ has x -intercepts
at $x = -m$ and $x = -n$.

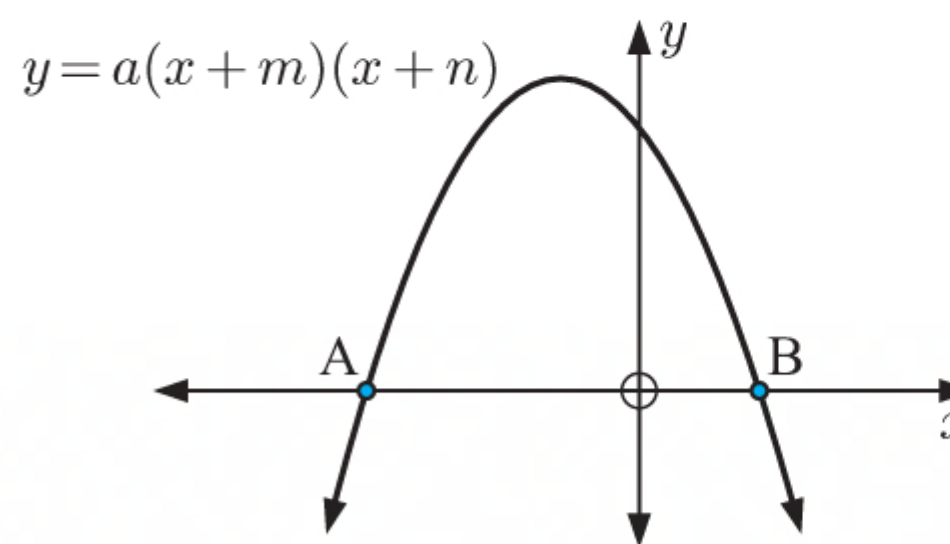
Now $m > n$

$$\therefore -m < -n$$

So, A is $(-m, 0)$ and B is $(-n, 0)$.

ii The axis of symmetry lies midway between the x -intercepts.

$$\therefore \text{the equation of the axis of symmetry is } x = \frac{-m + (-n)}{2} = \frac{-m - n}{2}.$$



15 Since the x -intercepts are 3 and -2 , $y = a(x - 3)(x + 2)$.

When $x = 0$, $y = 24$

$$\therefore 24 = a(-3)(2)$$

$$\therefore a = -4$$

The quadratic is $y = -4(x - 3)(x + 2)$

$$\therefore y = -4(x^2 - x - 6)$$

$$\therefore y = -4x^2 + 4x + 24$$

16 $y = mx - 10$ and $y = 3x^2 + 7x + 2$ meet where $mx - 10 = 3x^2 + 7x + 2$
 $\therefore 3x^2 + (7 - m)x + 12 = 0$

If the graphs touch, this quadratic has $\Delta = 0$

$$\therefore (7 - m)^2 - 4(3)(12) = 0$$


$$\therefore 49 - 14m + m^2 - 144 = 0$$

$$\therefore m^2 - 14m - 95 = 0$$

$$\therefore (m + 5)(m - 19) = 0$$

$$\therefore m = -5 \text{ or } 19$$

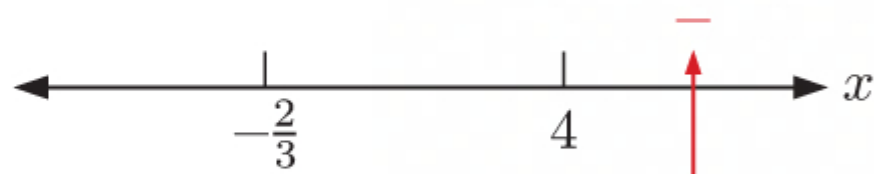
$\therefore y = mx - 10$ is a tangent to $y = 3x^2 + 7x + 2$ if $m = -5$ or $m = 19$.

- 17** $h = -4.9t^2 + 19.6t + 1.4$ is a quadratic with $a < 0$, so its shape is 
- So, at $t = \frac{-b}{2a}$ we have a maximum.

$$\therefore t = \frac{-19.6}{2(-4.9)} = 2, \text{ and when } t = 2, \quad h = -4.9(2)^2 + 19.6(2) + 1.4 = 21$$

The maximum height reached by the ball is 21 m, at $t = 2$ seconds.

- 18 a** $(3x + 2)(4 - x)$ has zeros $-\frac{2}{3}$ and 4.

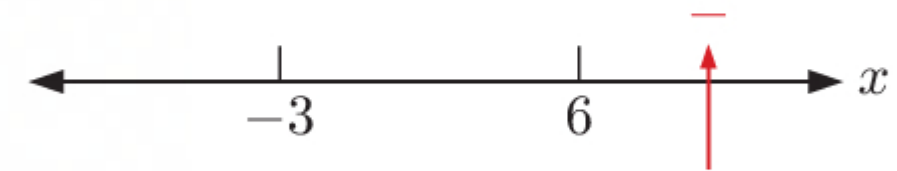


When $x = 5$ we have $(17)(-1) < 0$,
so we put a $-$ sign here.

As the factors are single, the signs alternate.

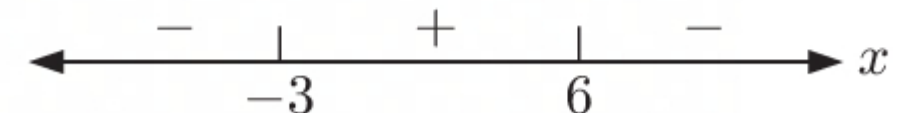


- b** $-x^2 + 3x + 18 = -(x + 3)(x - 6)$ has zeros -3 and 6.



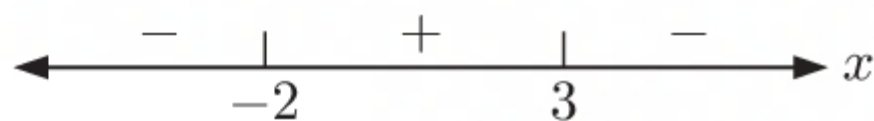
When $x = 7$ we have $-(10)(1) < 0$,
so we put a $-$ sign here.

As the factors are single, the signs alternate.



- 19 a** $(3 - x)(x + 2) < 0$

Sign diagram of LHS is

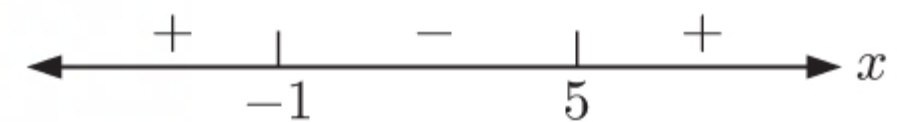


$$\therefore x < -2 \text{ or } x > 3$$

- b** $x^2 - 4x - 5 \leq 0$

$$\therefore (x + 1)(x - 5) \leq 0$$

Sign diagram of LHS is



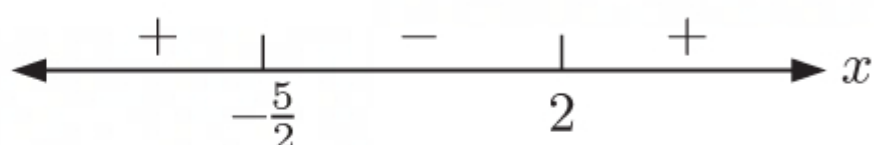
$$\therefore -1 \leq x \leq 5$$

- c** $2x^2 + x > 10$

$$\therefore 2x^2 + x - 10 > 0$$

$$\therefore (2x + 5)(x - 2) > 0$$

Sign diagram of LHS is



$$\therefore x < -\frac{5}{2} \text{ or } x > 2$$

- 20** $f(x) = x^2 + kx + (3k - 4)$ has $a = 1$, $b = k$, and $c = 3k - 4$.

$$\therefore \Delta = b^2 - 4ac$$

$$= k^2 - 4(1)(3k - 4)$$

$$= k^2 - 12k + 16$$

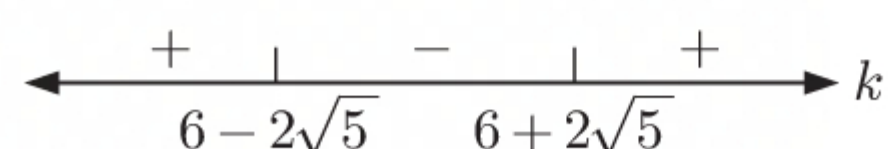
$$\Delta = 0 \text{ when } k = \frac{-(-12) \pm \sqrt{(-12)^2 - 4(1)(16)}}{2(1)}$$

$$= \frac{12 \pm \sqrt{80}}{2}$$

$$= \frac{12 \pm 4\sqrt{5}}{2}$$

$$= 6 \pm 2\sqrt{5}$$

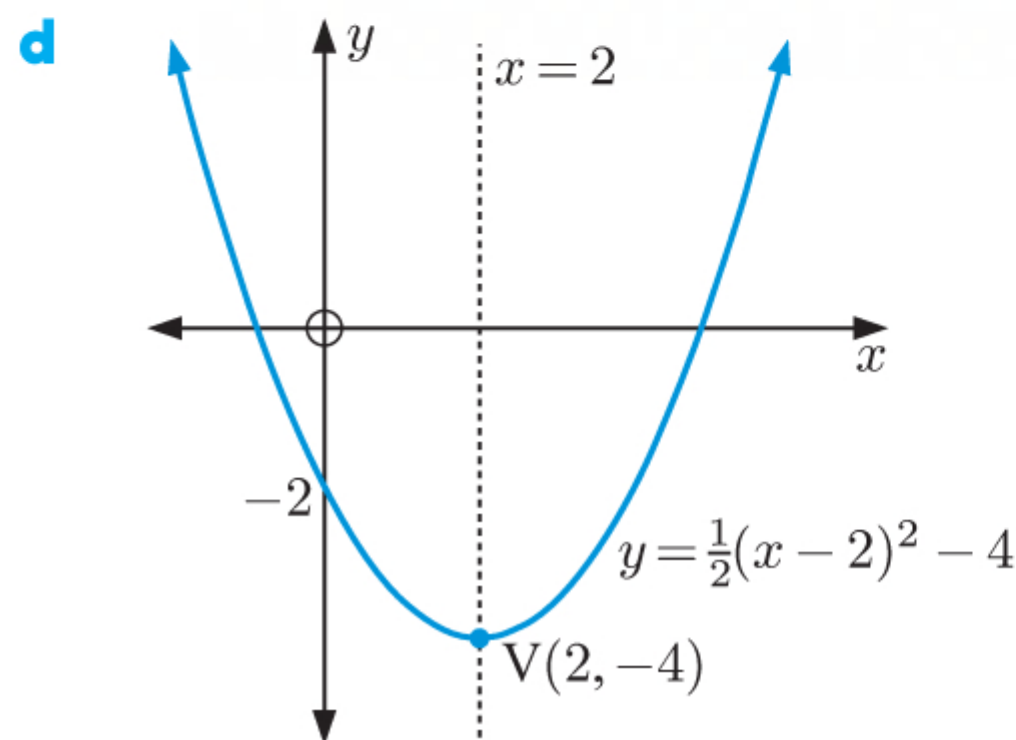
So, Δ has sign diagram:



- a** The function cuts the x -axis twice if $\Delta > 0$.
 $\therefore k < 6 - 2\sqrt{5}$ or $k > 6 + 2\sqrt{5}$.
- b** The function touches the x -axis if $\Delta = 0$.
 $\therefore k = 6 \pm 2\sqrt{5}$.
- c** The function misses the x -axis if $\Delta < 0$.
 $\therefore 6 - 2\sqrt{5} < k < 6 + 2\sqrt{5}$.

REVIEW SET 2B

- 1 a** The axis of symmetry is $x = 2$.
b The vertex is $(2, -4)$.
c When $x = 0$, $y = \frac{1}{2}(-2)^2 - 4$
 $= -2$
 \therefore the y -intercept is -2 .



- 2** $y = -3x^2 + 8x + 7$ has $a = -3$, $b = 8$, $c = 7$

$$\frac{-b}{2a} = \frac{-8}{2(-3)} = \frac{4}{3}$$

$$\begin{aligned}\text{When } x = \frac{4}{3}, \quad y &= -3\left(\frac{4}{3}\right)^2 + 8\left(\frac{4}{3}\right) + 7 \\ &= -\frac{16}{3} + \frac{32}{3} + 7 \\ &= \frac{37}{3}\end{aligned}$$

The axis of symmetry is $x = \frac{4}{3}$, and the vertex is $V\left(\frac{4}{3}, \frac{37}{3}\right)$ or $V\left(1\frac{1}{3}, 12\frac{1}{3}\right)$.

- 3 a** $y = 2x^2 + 3x - 7$
has $a = 2$, $b = 3$, $c = -7$
 $\Delta = b^2 - 4ac$
 $= (3)^2 - 4(2)(-7)$
 $= 65$

Since $\Delta > 0$, the graph cuts the x -axis twice.

Since $a > 0$, the graph is concave up.



- b** $y = -3x^2 - 7x + 4$
has $a = -3$, $b = -7$, $c = 4$
 $\Delta = b^2 - 4ac$
 $= (-7)^2 - 4(-3)(4)$
 $= 97$

Since $\Delta > 0$, the graph cuts the x -axis twice.

Since $a < 0$, the graph is concave down.



4 a $y = -2x^2 + 3x + 2$
 has $a = -2$, $b = 3$, $c = 2$
 $\Delta = b^2 - 4ac$
 $= 3^2 - 4(-2)(2)$
 $= 25$

Since $\Delta > 0$, the function is neither positive definite nor negative definite.

b $y = 3x^2 + x + 11$
 has $a = 3$, $b = 1$, $c = 11$
 $\Delta = b^2 - 4ac$
 $= 1^2 - 4(3)(11)$
 $= -131$

Since $\Delta < 0$, and since $a > 0$, the function is positive definite.

5 Since the vertex is $(2, 25)$, the quadratic has the form $y = a(x - 2)^2 + 25$, where $a \neq 0$.

When $x = 0$, $y = 1$

$$\therefore 1 = a(-2)^2 + 25$$

$$\therefore 1 = 4a + 25$$

$$\therefore 4a = -24$$

$$\therefore a = -6$$

The quadratic is $y = -6(x - 2)^2 + 25$.

6 a $y = 2x^2 + 4x - 1$ has $a = 2$, $b = 4$, $c = -1$

$$\frac{-b}{2a} = \frac{-4}{2(2)} = -1$$

The axis of symmetry is $x = -1$.

b When $x = -1$, $y = 2(-1)^2 + 4(-1) - 1$
 $= 2 - 4 - 1$
 $= -3$

\therefore the vertex is $(-1, -3)$.

c The y -intercept is -1 .

When $y = 0$, $2x^2 + 4x - 1 = 0$

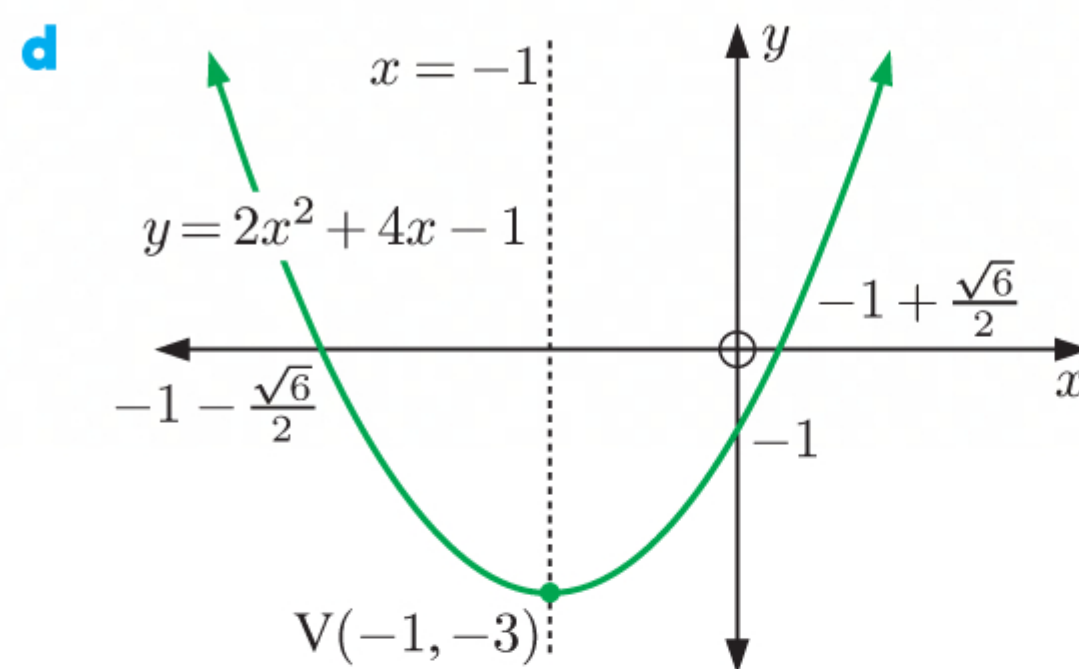
$$\therefore x = \frac{-4 \pm \sqrt{4^2 - 4(2)(-1)}}{2(2)}$$

$$= \frac{-4 \pm \sqrt{24}}{4}$$

$$= \frac{-4 \pm 2\sqrt{6}}{4}$$

$$= -1 \pm \frac{\sqrt{6}}{2}$$

\therefore the x -intercepts are $-1 \pm \frac{\sqrt{6}}{2}$.



7 a Since the x -intercepts are -5 and 1 , $y = a(x + 5)(x - 1)$.

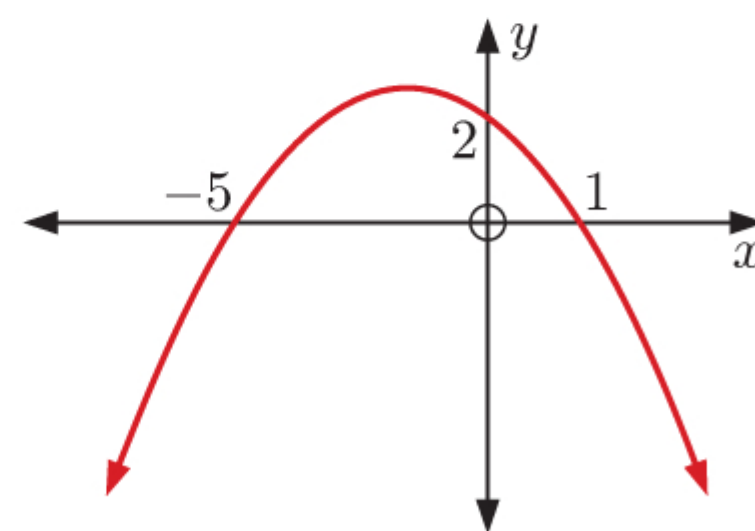
The graph is concave down, so $a < 0$.

When $x = 0$, $y = 2$

$$\therefore 2 = a(5)(-1)$$

$$\therefore a = -\frac{2}{5}$$

The quadratic is $y = -\frac{2}{5}(x + 5)(x - 1)$.



b The axis of symmetry is $x = \frac{-5+1}{2}$
 $\therefore x = -2$

When $x = -2$, $y = -\frac{2}{5}(-2+5)(-2-1)$
 $= -\frac{2}{5}(3)(-3)$
 $= \frac{18}{5} = 3\frac{3}{5}$

The vertex is $(-2, 3\frac{3}{5})$, and the axis of symmetry is $x = -2$.

8 a $y = 3x + c$ intersects the parabola $y = x^2 + x - 5$ where $x^2 + x - 5 = 3x + c$
 $\therefore x^2 - 2x - 5 - c = 0$

The graphs meet in two distinct points when this equation has two distinct real roots.

$\therefore \Delta > 0$
 $\therefore (-2)^2 - 4(1)(-5-c) > 0$
 $\therefore 4 + 20 + 4c > 0$
 $\therefore 4c > -24$
 $\therefore c > -6$

b Choose c such that $c > -6$, for example, $c = -2$.

The graphs meet where $x^2 + x - 5 = 3x - 2$
 $\therefore x^2 - 2x - 3 = 0$
 $\therefore (x+1)(x-3) = 0$
 $\therefore x = -1 \text{ or } 3$

Using the line $y = 3x - 2$, when $x = -1$, $y = 3(-1) - 2 = -5$
 and when $x = 3$, $y = 3(3) - 2 = 7$

\therefore the points of intersection are $(-1, -5)$ and $(3, 7)$.

9 a $y = 3x^2 + 4x + 7$
 has $a = 3$, $b = 4$, $c = 7$

Since $a > 0$, the shape is 


The minimum value occurs when

$$x = \frac{-b}{2a} = \frac{-4}{2(3)} = -\frac{2}{3}$$

and $y = 3(-\frac{2}{3})^2 + 4(-\frac{2}{3}) + 7$
 $= \frac{4}{3} - \frac{8}{3} + 7$
 $= \frac{17}{3} = 5\frac{2}{3}$

So, the minimum value of y is $5\frac{2}{3}$,
 occurring when $x = -\frac{2}{3}$.

b $y = -2x^2 - 5x + 2$
 has $a = -2$, $b = -5$, $c = 2$

Since $a < 0$, the shape is 

The maximum value occurs when

$$x = \frac{-b}{2a} = \frac{-(-5)}{2(-2)} = -\frac{5}{4}$$

and $y = -2(-\frac{5}{4})^2 - 5(-\frac{5}{4}) + 2$
 $= -\frac{25}{8} + \frac{25}{4} + 2$
 $= \frac{41}{8} = 5\frac{1}{8}$

So, the maximum value of y is $5\frac{1}{8}$,
 occurring when $x = -\frac{5}{4}$.

- 10 a** Since the graph cuts the x -axis at -2 and 3 , the quadratic has the form $y = a(x + 2)(x - 3)$.

When $x = -3$, $y = 18$

$$\therefore 18 = a(-3 + 2)(-3 - 3)$$

$$\therefore 18 = a(-1)(-6)$$

$$\therefore a = 3$$

The quadratic is $y = 3(x + 2)(x - 3)$

$$\therefore y = 3(x^2 - x - 6)$$

$$\therefore y = 3x^2 - 3x - 18$$

- b** When $x = 0$,

$$y = 3(0)^2 - 3(0) - 18$$

$$= -18$$

\therefore the y -intercept is -18 .

- c** $y = 3x^2 - 3x - 18$ has $a = 3$, $b = -3$, $c = -18$

$$\begin{aligned} \text{The axis of symmetry is } x &= \frac{-b}{2a} \\ &= \frac{-(-3)}{2(3)} \\ &= \frac{1}{2} \end{aligned}$$

$$\begin{aligned} \text{When } x = \frac{1}{2}, \quad y &= 3\left(\frac{1}{2}\right)^2 - 3\left(\frac{1}{2}\right) - 18 \\ &= \frac{3}{4} - \frac{3}{2} - 18 \\ &= -\frac{75}{4} = -18\frac{3}{4} \end{aligned}$$

The vertex is $\left(\frac{1}{2}, -18\frac{3}{4}\right)$.

- 11 a** The axis of symmetry is $x = 1$.

$$\therefore x = \frac{-b}{2a} = \frac{-m}{2(1)} = 1 \quad \text{and so } m = -2.$$

Now when $x = 1$, $y = 3$

$$\therefore 1^2 - 2(1) + n = 3$$

$$\therefore n = 4$$

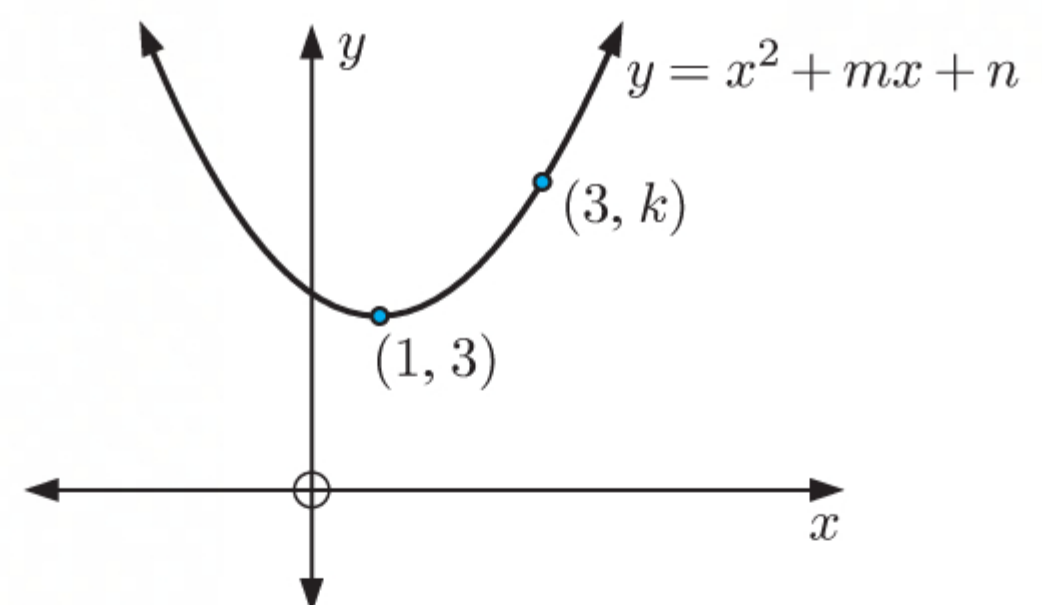
So, $m = -2$ and $n = 4$.

- b** $y = x^2 - 2x + 4$

When $x = 3$, $y = 3^2 - 2(3) + 4$

$$= 7$$

$$\therefore k = 7$$



- 12** Let the original piece of tinplate be x cm by x cm.

The volume of the folded box is 120 mL.

The volume $V = \text{length} \times \text{width} \times \text{height}$

$$\therefore 120 = (x - 8) \times (x - 8) \times 4$$

$$\therefore 120 = 4(x^2 - 16x + 64)$$

$$\therefore 120 = 4x^2 - 64x + 256$$

$$\therefore 4x^2 - 64x + 136 = 0$$

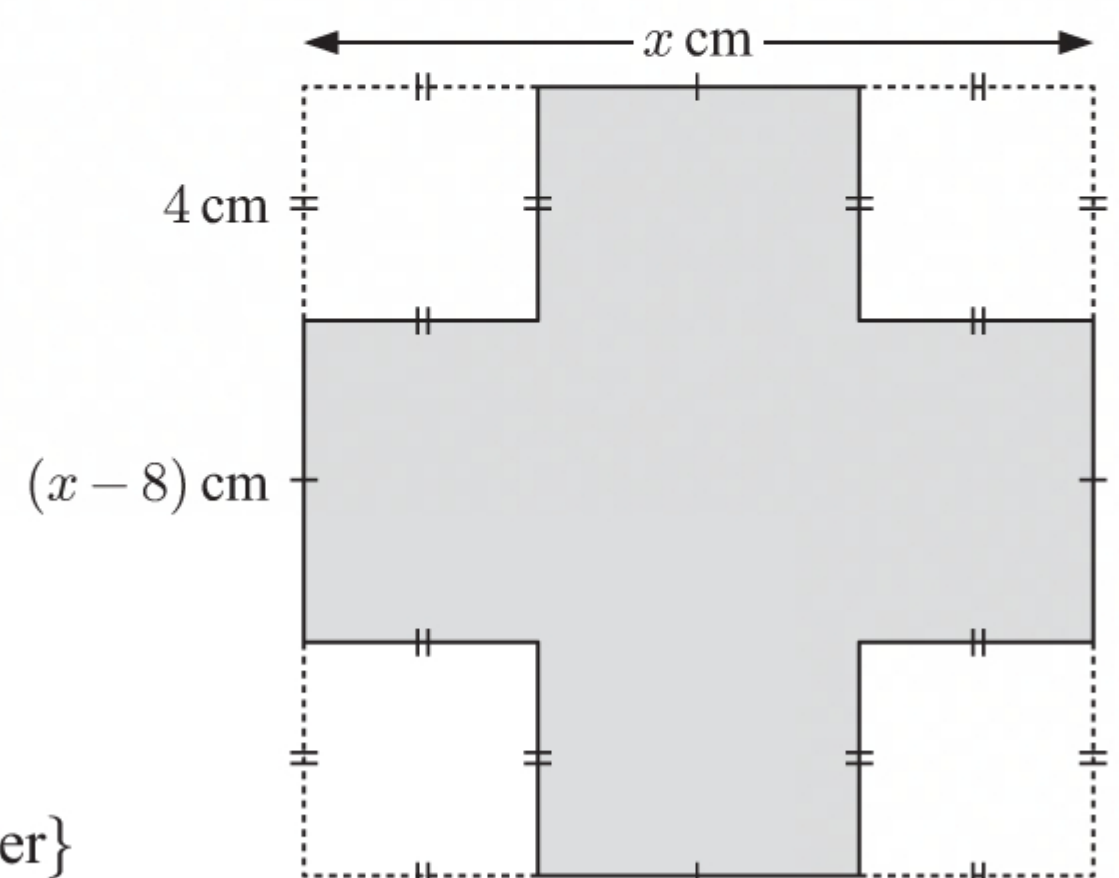
$$\therefore x \approx 13.5 \text{ or } 2.52$$

{using technology}

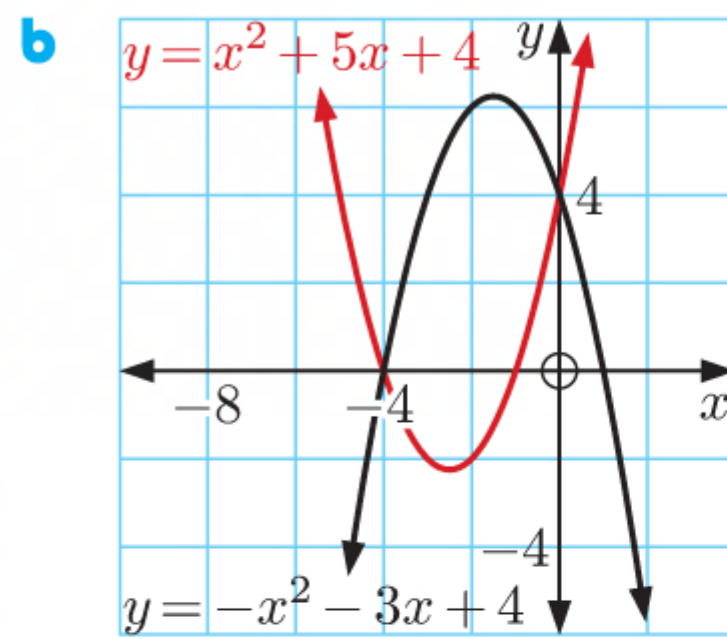
But $x > 8$ {4 cm squares are cut from each corner}

$$\therefore x \approx 13.5$$

\therefore the original piece of tinplate was about 13.5 cm square.



13 a $-x^2 - 3x + 4 = x^2 + 5x + 4$
 $\therefore 2x^2 + 8x = 0$
 $\therefore 2x(x + 4) = 0$
 $\therefore x = 0 \text{ or } -4$



c $x^2 + 5x + 4 > -x^2 - 3x + 4$ where the graph of $y = x^2 + 5x + 4$ is *above* the graph of $y = -x^2 - 3x + 4$.
 $\therefore x < -4 \text{ or } x > 0$

14 a The graph is concave up, so $a > 0$.

b The vertex has a positive x -coordinate.

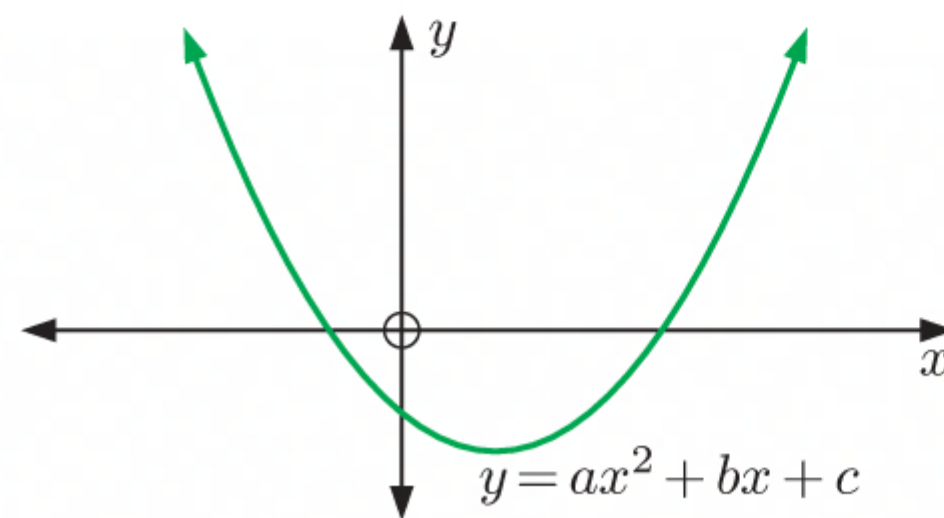
The x -coordinate of the vertex is $x = \frac{-b}{2a}$

so $\frac{-b}{2a} > 0$

$\therefore b < 0$ {since $a > 0$ }

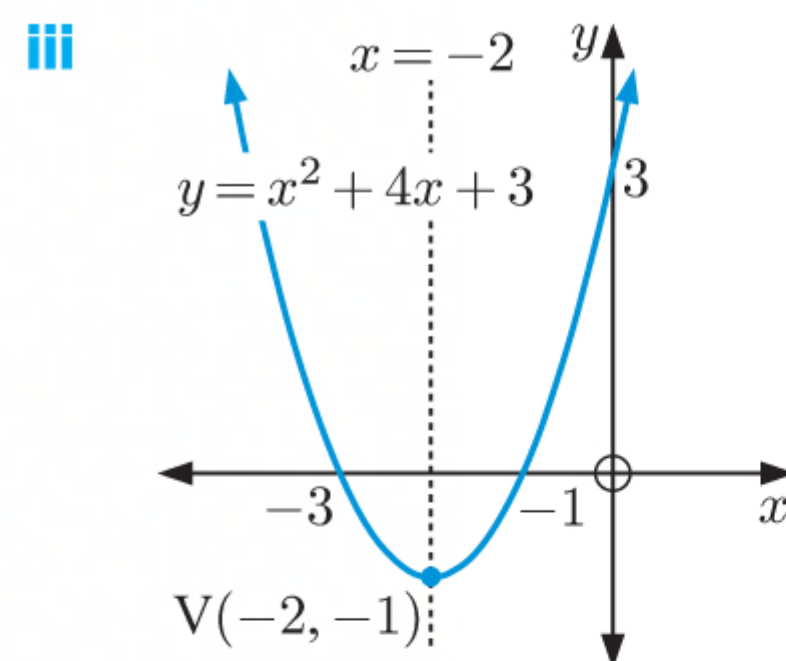
c The y -intercept is negative $\therefore c < 0$.

d There are two x -intercepts \therefore two real roots, so $\Delta > 0$.



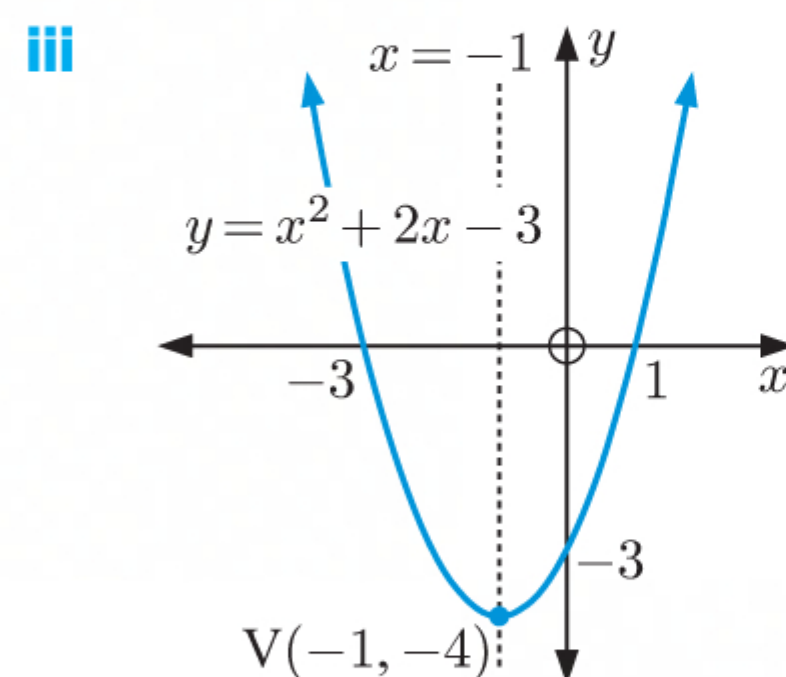
15 a i $y = x^2 + 4x + 3$
 $\therefore y = x^2 + 4x + 2^2 + 3 - 2^2$
 $\therefore y = (x + 2)^2 - 1$

ii $y = x^2 + 4x + 3$
 $\therefore y = (x + 3)(x + 1)$



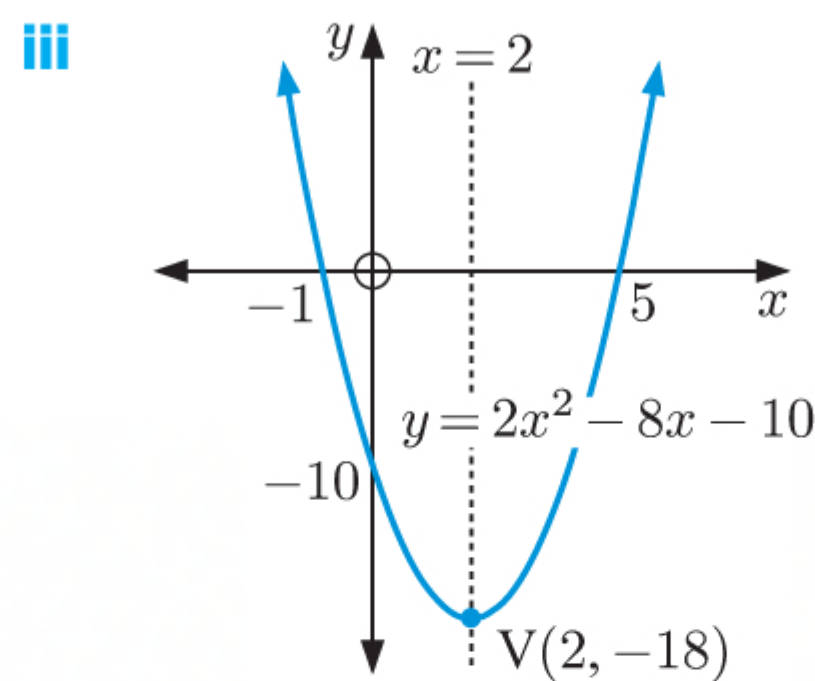
b i $y = x^2 + 2x - 3$
 $\therefore y = x^2 + 2x + 1^2 - 3 - 1^2$
 $\therefore y = (x + 1)^2 - 4$

ii $y = x^2 + 2x - 3$
 $\therefore y = (x + 3)(x - 1)$



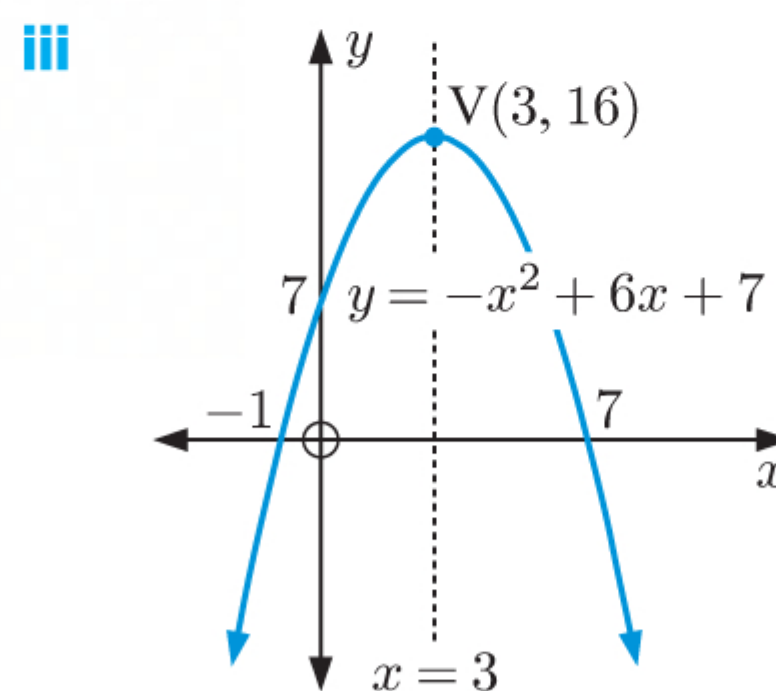
c i $y = 2x^2 - 8x - 10$
 $\therefore y = 2(x^2 - 4x - 5)$
 $\therefore y = 2[x^2 - 4x + (-2)^2 - 5 - (-2)^2]$
 $\therefore y = 2[(x - 2)^2 - 9]$
 $\therefore y = 2(x - 2)^2 - 18$

ii $y = 2x^2 - 8x - 10$
 $\therefore y = 2(x^2 - 4x - 5)$
 $\therefore y = 2(x - 5)(x + 1)$



d i $y = -x^2 + 6x + 7$
 $\therefore y = -(x^2 - 6x - 7)$
 $\therefore y = -[x^2 - 6x + (-3)^2 - 7 - (-3)^2]$
 $\therefore y = -[(x - 3)^2 - 16]$
 $\therefore y = -(x - 3)^2 + 16$

ii $y = -x^2 + 6x + 7$
 $\therefore y = -(x^2 - 6x - 7)$
 $\therefore y = -(x - 7)(x + 1)$



16 a $y = 9x^2 - kx + 4$ touches the x -axis.

\therefore there is a repeated root, $\Delta = 0$
 $\therefore (-k)^2 - 4(9)(4) = 0$
 $\therefore k^2 - 144 = 0$
 $\therefore k^2 = 144$
 $\therefore k = \pm 12$

b $y = 9x^2 - 12x + 4$ and $y = 9x^2 + 12x + 4$ meet where

$9x^2 - 12x + 4 = 9x^2 + 12x + 4$
 $\therefore 24x = 0$
 $\therefore x = 0$

Substituting $x = 0$ into either equation, we get $y = 4$.

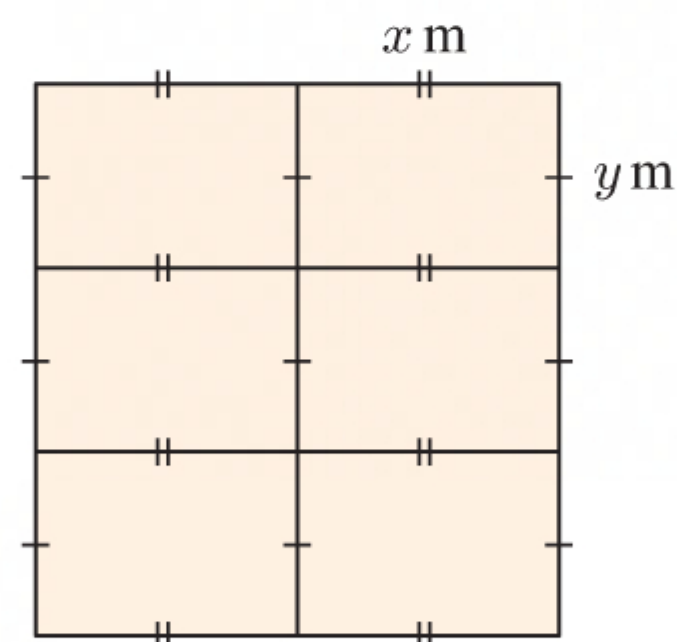
\therefore the two quadratic functions intersect at $(0, 4)$.

17 a The total length of fencing is $(8x + 9y)$ m


$\therefore 8x + 9y = 600$
 $\therefore 9y = 600 - 8x$
 $\therefore y = \frac{600 - 8x}{9}$

The area of each pen is $A = xy$

$= x \left(\frac{600 - 8x}{9} \right) \text{ m}^2$



$$\begin{aligned} \text{b } A &= x \left(\frac{600 - 8x}{9} \right) \\ &= \frac{600}{9}x - \frac{8}{9}x^2 \quad \text{which has } a = -\frac{8}{9}, \quad b = \frac{600}{9} \end{aligned}$$

Since $a < 0$, the shape is 

$\therefore A$ is maximised at the axis of symmetry, which is $x = \frac{-b}{2a}$

$$\therefore x = \frac{-\frac{600}{9}}{2(-\frac{8}{9})}$$

$$= \frac{600}{16}$$

$$\therefore x = \frac{75}{2}$$

$$\text{When } x = \frac{75}{2}, \quad y = \frac{600 - 8(\frac{75}{2})}{9} = 33\frac{1}{3}$$

\therefore for maximum area, each pen should be $37\frac{1}{2} \text{ m} \times 33\frac{1}{3} \text{ m}$.

$$\begin{aligned} \text{c } \text{The area of each pen in this case is } & 37\frac{1}{2} \times 33\frac{1}{3} \\ &= \frac{75}{2} \times \frac{100}{3} \\ &= 1250 \text{ m}^2 \end{aligned}$$

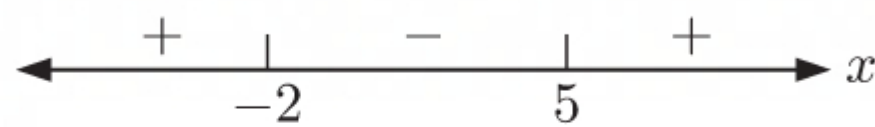
$$\text{18 a } x^2 - 3x - 10 = (x + 2)(x - 5)$$

has zeros -2 and 5 .

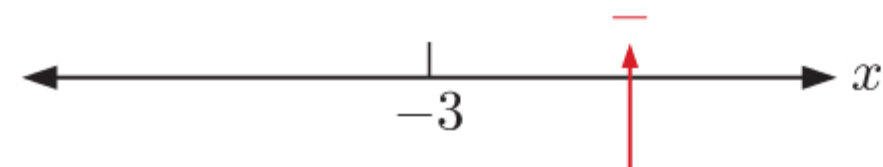


When $x = 6$ we have $(8)(1) > 0$,
so we put a $+$ sign here.

As the factors are single, the signs alternate.

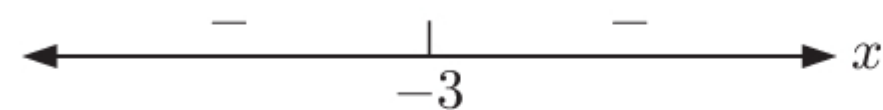


$$\text{b } -(x + 3)^2 \text{ has zero } -3.$$



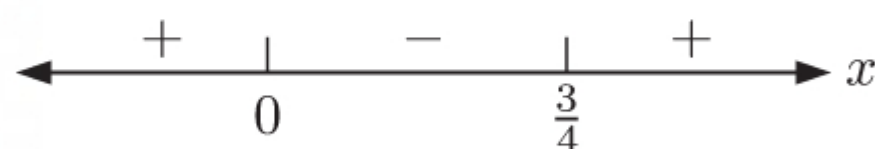
When $x = 0$ we have $-(3)^2 < 0$,
so we put a $-$ sign here.

As the factor is squared, the signs do not change.



$$\begin{aligned} \text{19 a } 4x^2 - 3x &< 0 \\ \therefore x(4x - 3) &< 0 \end{aligned}$$

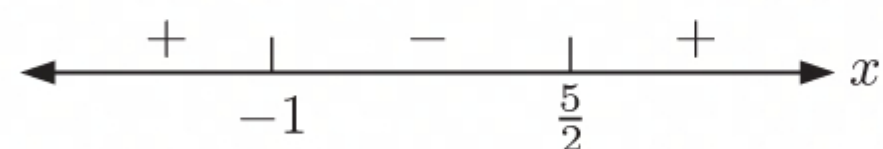
LHS has sign diagram



$$\therefore 0 < x < \frac{3}{4}$$

$$\begin{aligned} \text{b } 2x^2 - 3x - 5 &\geq 0 \\ \therefore (2x - 5)(x + 1) &\geq 0 \end{aligned}$$

LHS has sign diagram



$$\therefore x \leq -1 \text{ or } x \geq \frac{5}{2}$$

$$\text{c} \quad \frac{11}{3}x \leq 2x^2 + 1$$

$$\therefore 2x^2 - \frac{11}{3}x + 1 \geq 0$$

$$\therefore 6x^2 - 11x + 3 \geq 0$$

$$\therefore (3x - 1)(2x - 3) \geq 0$$

LHS has sign diagram



$$\therefore x \leq \frac{1}{3} \text{ or } x \geq \frac{3}{2}$$

20 $y = mx^2 + 5x + (m + 12)$ has $a = m$, $b = 5$, $c = m + 12$

$$\therefore \Delta = b^2 - 4ac$$

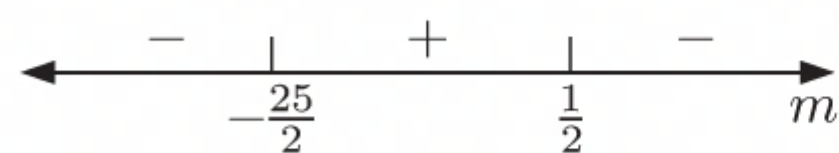
$$= (5)^2 - 4(m)(m + 12)$$

$$= 25 - 4m^2 - 48m$$

$$= -(4m^2 + 48m - 25)$$

$$= -(2m - 1)(2m + 25)$$

So, Δ has sign diagram:



a The function cuts the x -axis twice if $\Delta > 0$.

$$\therefore -\frac{25}{2} < m < \frac{1}{2}, \quad m \neq 0$$

b The function touches the x -axis if $\Delta = 0$.

$$\therefore m = -\frac{25}{2} \text{ or } m = \frac{1}{2}$$

c The function misses the x -axis if $\Delta < 0$.

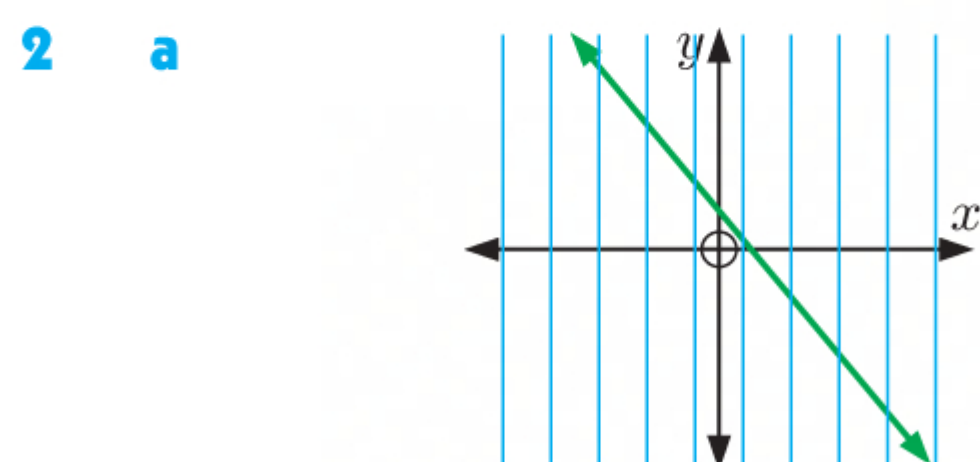
$$\therefore m < -\frac{25}{2} \text{ or } m > \frac{1}{2}$$

Chapter 3

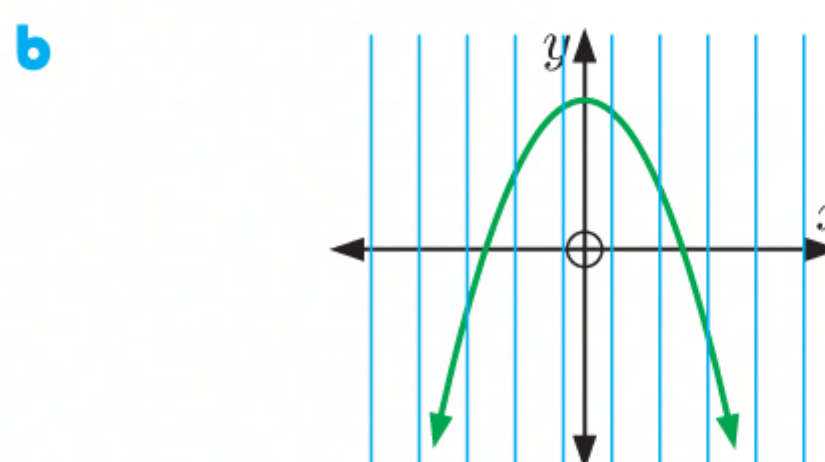
FUNCTIONS

EXERCISE 3A

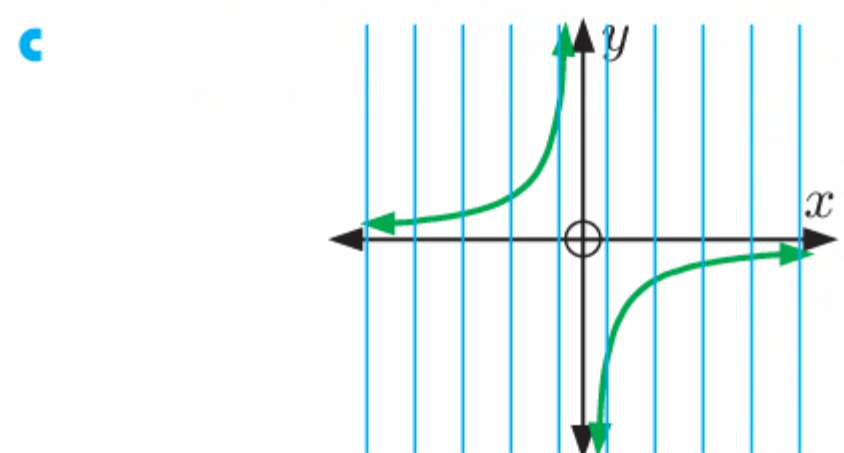
- 1 a $y = x^2 - 9$ is a function, since for any value of x there is at most one value of y .
 b $x + y = 9$ is a function, since for any value of x there is at most one value of y .
 c $x^2 + y^2 = 9$ is not a function. If $x^2 + y^2 = 9$, then $y = \pm\sqrt{9 - x^2}$. So, for example, for $x = 2$, $y = \pm\sqrt{5}$.



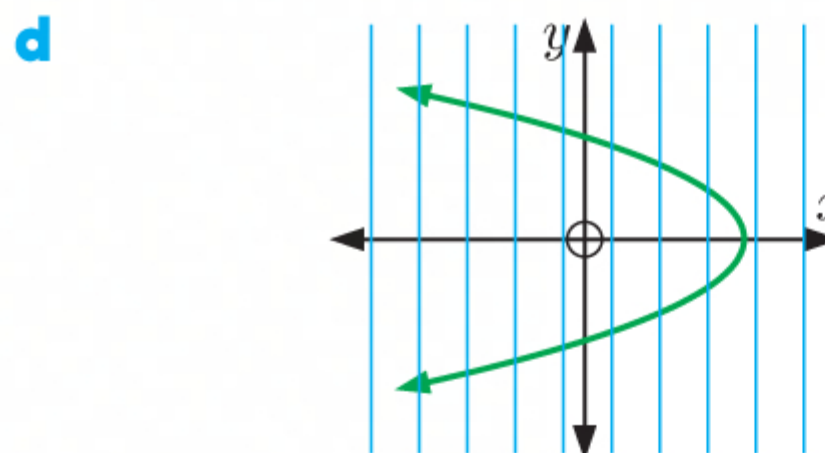
Each vertical line cuts the graph no more than once, so this relation is a function.



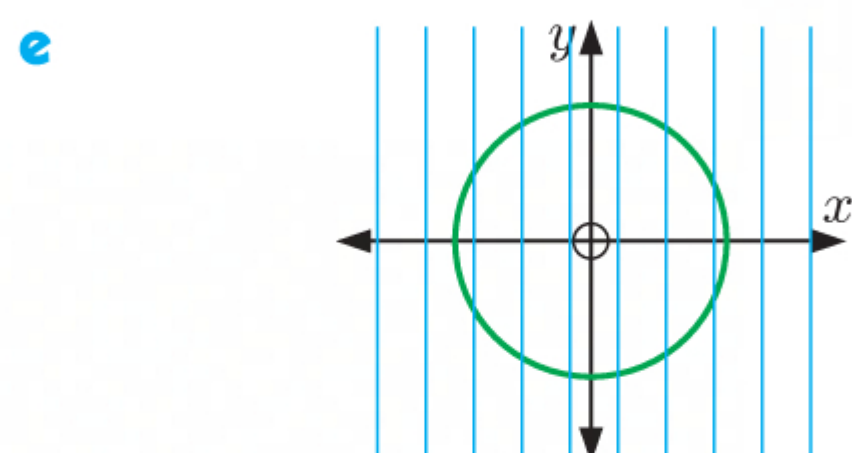
Each vertical line cuts the graph no more than once, so this relation is a function.



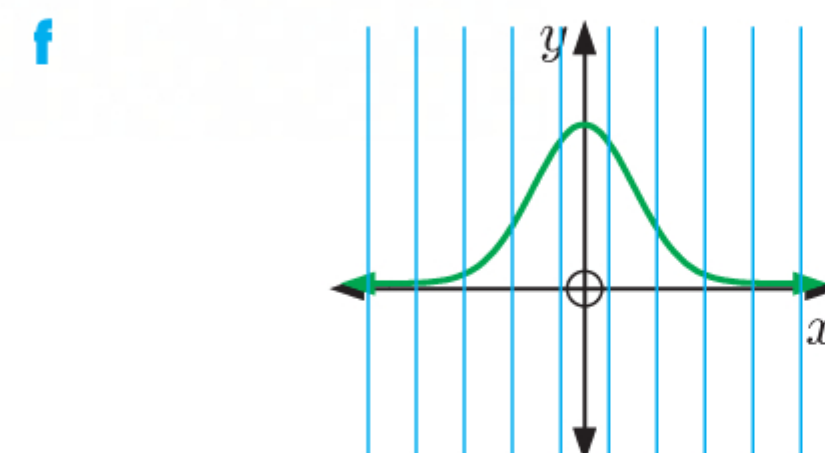
Each vertical line cuts the graph no more than once, so this relation is a function.



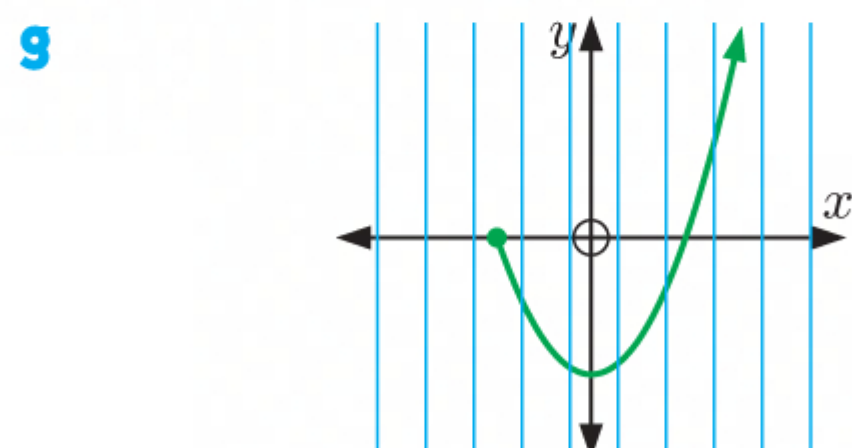
Some vertical lines cut the graph more than once, so this relation is not a function.



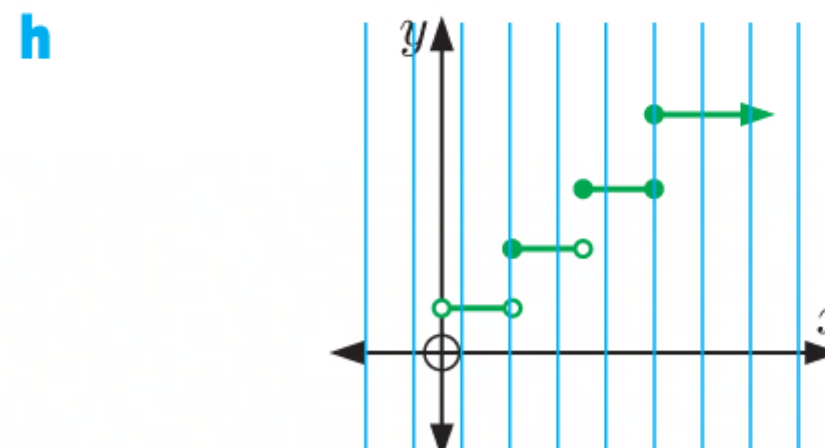
Some vertical lines cut the graph more than once, so this relation is not a function.



Each vertical line cuts the graph no more than once, so this relation is a function.



Each vertical line cuts the graph no more than once, so this relation is a function.

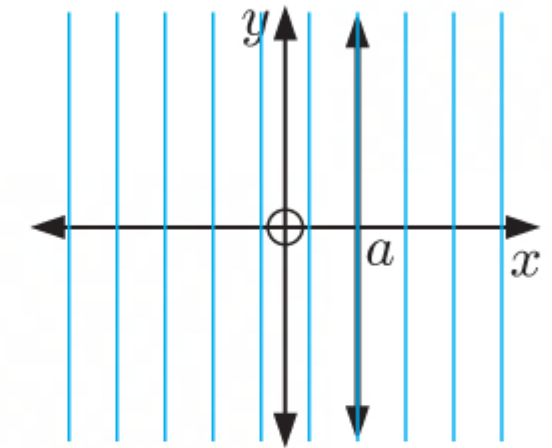


One vertical line cuts the graph more than once, so this relation is not a function.

- 3** It is not possible for a function to have more than one y -intercept. The y -axis is a vertical line, so if a relation has more than one y -intercept, then a vertical line cuts the relation more than once. So, such a relation cannot be a function.

- 4** No, the graph of a straight line will not be a function if it is a vertical line which has the form $x = a$ for some constant a .

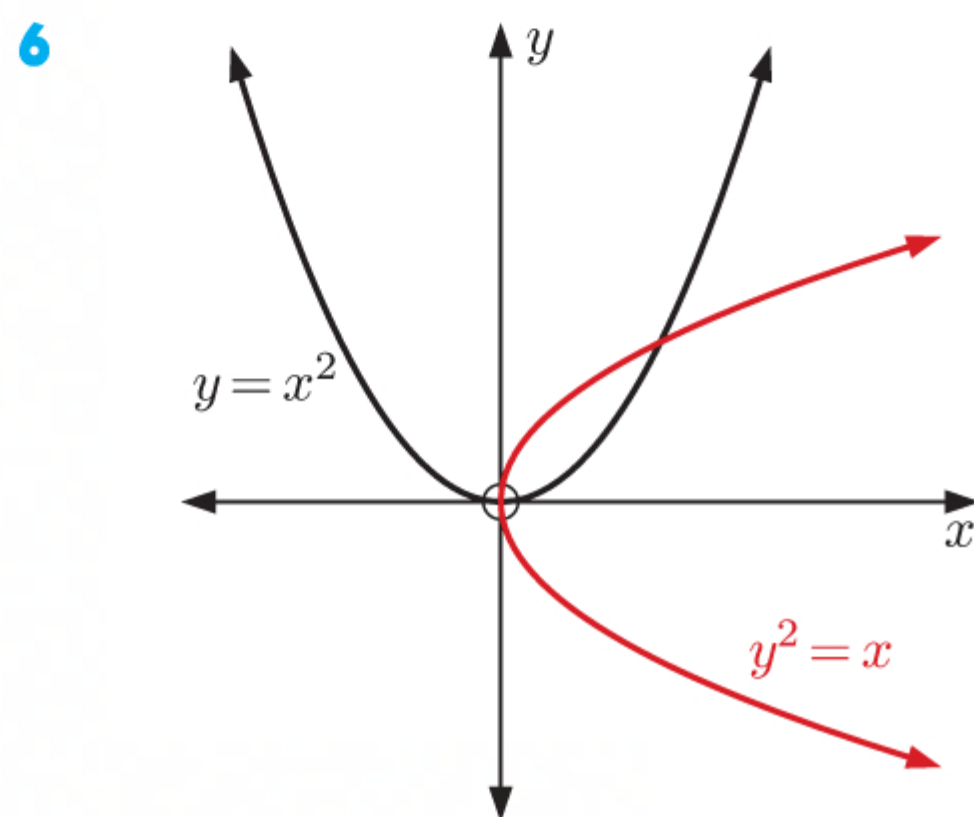
The vertical line through $x = a$ cuts the graph at every point, so the straight line $x = a$ does not pass the vertical line test and hence is not a function.



- 5** The relation between *age* and *cost* is not a function as there are two corresponding values of *cost* for a 2 year old child.

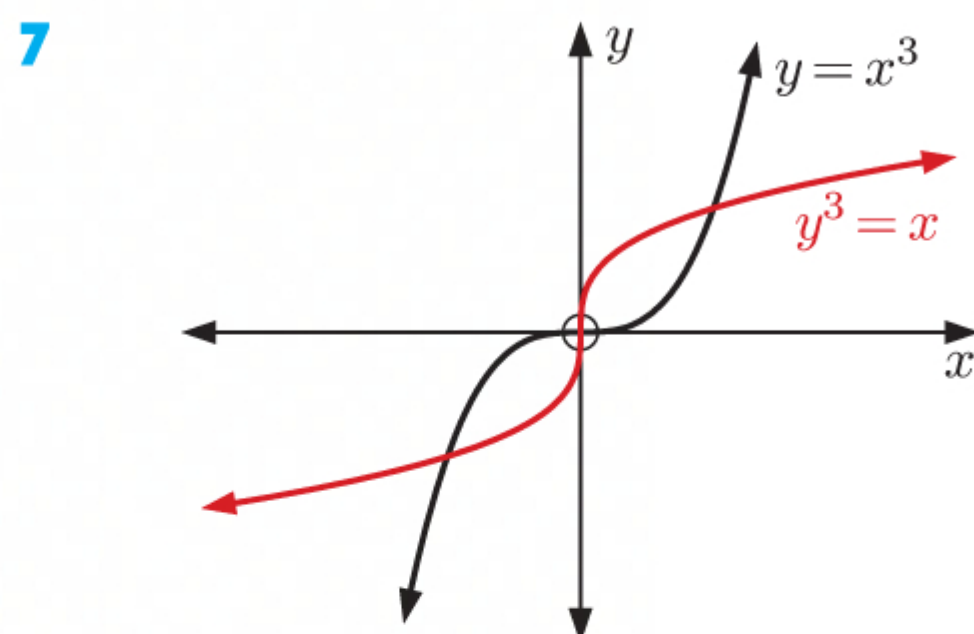
The categories 0 - 2 years and 2 - 16 years both include the age 2 years.

So, a ticket for a 2 year old child could cost either \$0 or \$20 according to the schedule.



- a** $y^2 = x$ is a relation but not a function.
 $y = x^2$ is a function (and a relation).
 $y^2 = x$ has a horizontal axis of symmetry (the x -axis).
 $y = x^2$ has a vertical axis of symmetry (the y -axis).
Both $y^2 = x$ and $y = x^2$ pass through $(0, 0)$ and $(1, 1)$.
 $y^2 = x$ is a rotation of $y = x^2$ clockwise through 90° about the origin or $y^2 = x$ is a reflection of $y = x^2$ in the line $y = x$.

- b** **i** The part of the graph of $y^2 = x$ in the first quadrant corresponds to $y = \sqrt{x}$.
ii $y = \sqrt{x}$ is a function as any vertical line cuts the graph at most once.



- a** Both curves are functions since any vertical line will cut each curve at most once.

b $y^3 = x$
 $\therefore y = x^{\frac{1}{3}}$
 $\therefore y = \sqrt[3]{x}$

EXERCISE 3B

1 $f(x) = 3x + 2$

a $f(0) = 3(0) + 2$ {replacing x with (0) }
 $= 0 + 2$
 $= 2$

b $f(2) = 3(2) + 2$ {replacing x with (2) }
 $= 6 + 2$
 $= 8$

c $f(-1) = 3(-1) + 2$ {replacing x with (-1) }
 $= -3 + 2$
 $= -1$

$$\begin{aligned} \text{d } f(-5) &= 3(-5) + 2 \quad \{\text{replacing } x \text{ with } (-5)\} \\ &= -15 + 2 \\ &= -13 \end{aligned}$$

$$\begin{aligned} \text{e } f(-\tfrac{1}{3}) &= 3(-\tfrac{1}{3}) + 2 \quad \{\text{replacing } x \text{ with } (-\tfrac{1}{3})\} \\ &= -1 + 2 \\ &= 1 \end{aligned}$$

$$2 \quad f(x) = 3x - x^2 + 2$$

$$\begin{aligned} \text{a } f(0) &= 3(0) - 0^2 + 2 \\ &= 0 - 0 + 2 \\ &= 2 \end{aligned}$$

$$\begin{aligned} \text{b } f(3) &= 3(3) - 3^2 + 2 \\ &= 9 - 9 + 2 \\ &= 2 \end{aligned}$$

$$\begin{aligned} \text{c } f(-3) &= 3(-3) - (-3)^2 + 2 \\ &= -9 - 9 + 2 \\ &= -16 \end{aligned}$$

$$\begin{aligned} \text{d } f(-7) &= 3(-7) - (-7)^2 + 2 \\ &= -21 - 49 + 2 \\ &= -68 \end{aligned}$$

$$\begin{aligned} \text{e } f(\tfrac{3}{2}) &= 3(\tfrac{3}{2}) - (\tfrac{3}{2})^2 + 2 \\ &= \tfrac{9}{2} - \tfrac{9}{4} + 2 \\ &= \tfrac{17}{4} \end{aligned}$$

$$3 \quad g(x) = x - \frac{4}{x}$$

$$\text{a } g(1) = 1 - \frac{4}{1} = -3$$

$$\text{b } g(4) = 4 - \frac{4}{4} = 3$$

$$\text{c } g(-1) = -1 - \frac{4}{(-1)} = 3$$

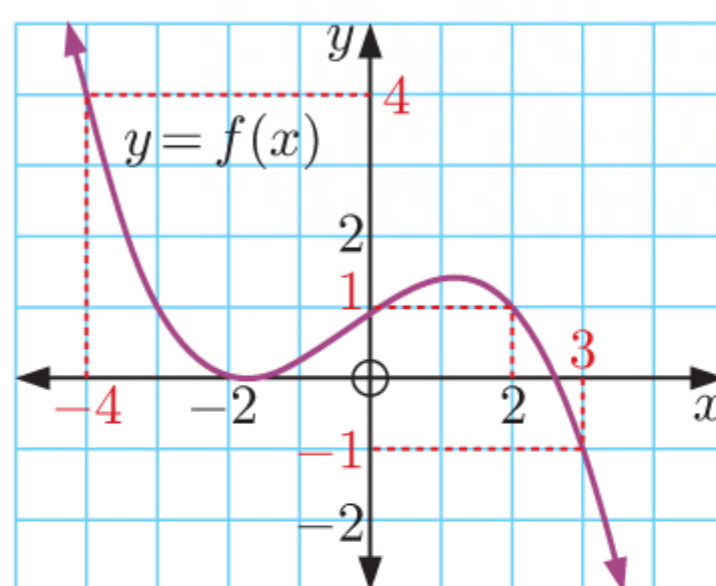
$$\text{d } g(-4) = -4 - \frac{4}{(-4)} = -3$$

$$\text{e } g(-\tfrac{1}{2}) = -\tfrac{1}{2} - \frac{4}{(-\frac{1}{2})} = -\tfrac{1}{2} + 8 = \tfrac{15}{2}$$

$$4 \quad \text{a } \text{i } f(2) = 1$$

$$\text{ii } f(3) = -1$$

$$\text{b } \text{When } y = f(x) = 4, \quad x = -4.$$



$$5 \quad G(x) = \frac{2x+3}{x-4}$$

$$\begin{aligned} \text{a } \text{i } G(2) &= \frac{2(2)+3}{2-4} \\ &= \frac{7}{-2} \\ &= -\frac{7}{2} \end{aligned}$$

$$\begin{aligned} \text{ii } G(0) &= \frac{2(0)+3}{0-4} \\ &= \frac{3}{-4} \\ &= -\frac{3}{4} \end{aligned}$$

$$\begin{aligned} \text{iii } G(-\tfrac{1}{2}) &= \frac{2(-\frac{1}{2})+3}{-\frac{1}{2}-4} \\ &= \frac{-1+3}{(-\frac{9}{2})} \\ &= \frac{2}{(-\frac{9}{2})} \\ &= -\frac{4}{9} \end{aligned}$$

b $G(x) = \frac{2x+3}{x-4}$ is undefined when $x-4=0$
 $\therefore x=4$

So, when $x=4$, $G(x)$ does not exist.

c $G(x) = -3$, so $\frac{2x+3}{x-4} = -3$
 $\therefore 2x+3 = -3(x-4)$
 $\therefore 2x+3 = -3x+12$
 $\therefore 5x = 9$
 $\therefore x = \frac{9}{5}$

6 $f(x) = 7 - 3x$

a $f(a) = 7 - 3a$

b $f(-a) = 7 - 3(-a)$
 $= 7 + 3a$

c $f(a+3) = 7 - 3(a+3)$
 $= 7 - 3a - 9$
 $= -3a - 2$

d $f(2a) = 7 - 3(2a)$
 $= 7 - 6a$

e $f(x+2) = 7 - 3(x+2)$
 $= 7 - 3x - 6$
 $= 1 - 3x$

f $f(x+h) = 7 - 3(x+h)$
 $= 7 - 3x - 3h$

7 $F(x) = 2x^2 + 3x - 1$

a $F(x+4)$
 $= 2(x+4)^2 + 3(x+4) - 1$
 $= 2(x^2 + 8x + 16) + 3x + 12 - 1$
 $= 2x^2 + 16x + 32 + 3x + 11$
 $= 2x^2 + 19x + 43$

b $F(2-x)$
 $= 2(2-x)^2 + 3(2-x) - 1$
 $= 2(4 - 4x + x^2) + 6 - 3x - 1$
 $= 8 - 8x + 2x^2 + 5 - 3x$
 $= 2x^2 - 11x + 13$

c $F(-x)$
 $= 2(-x)^2 + 3(-x) - 1$
 $= 2x^2 - 3x - 1$

d $F(x^2)$
 $= 2(x^2)^2 + 3(x^2) - 1$
 $= 2x^4 + 3x^2 - 1$

e $F(3x)$
 $= 2(3x)^2 + 3(3x) - 1$
 $= 2(9x^2) + 9x - 1$
 $= 18x^2 + 9x - 1$

f $F(x+h)$
 $= 2(x+h)^2 + 3(x+h) - 1$
 $= 2(x^2 + 2xh + h^2) + 3x + 3h - 1$
 $= 2x^2 + 4xh + 2h^2 + 3x + 3h - 1$
 $= 2x^2 + (4h+3)x + 2h^2 + 3h - 1$

8 $f(x) = x^2$

a $f(3x) = (3x)^2$
 $= 9x^2$

b $f\left(\frac{x}{2}\right) = \left(\frac{x}{2}\right)^2$
 $= \frac{x^2}{4}$

c $3f(x) = 3x^2$

d $2f(x-1) + 5 = 2(x-1)^2 + 5$
 $= 2(x^2 - 2x + 1) + 5$
 $= 2x^2 - 4x + 2 + 5$
 $= 2x^2 - 4x + 7$

9 $f(x) = \frac{1}{x}$

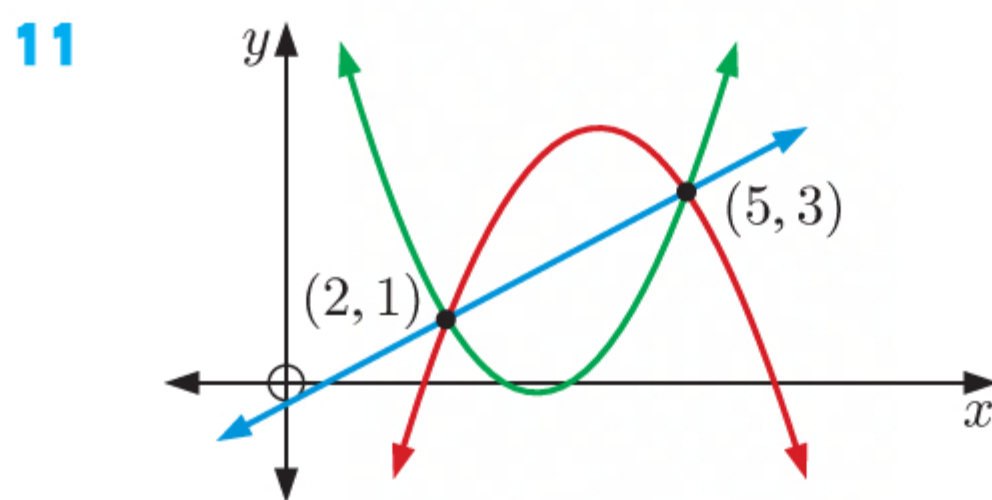
a $f(-x) = \frac{1}{(-x)}$
 $= -\frac{1}{x}$

b $f(\frac{1}{2}x) = \frac{1}{\frac{1}{2}x}$
 $= \frac{1}{(\frac{x}{2})}$
 $= \frac{2}{x}$

c $2f(x) + 3 = 2 \times \frac{1}{x} + 3$
 $= \frac{2}{x} + \frac{3x}{x}$
 $= \frac{2+3x}{x}$

d $3f(x-1) + 2 = 3 \times \frac{1}{x-1} + 2$
 $= \frac{3}{x-1} + \frac{2(x-1)}{x-1}$
 $= \frac{3+2x-2}{x-1}$
 $= \frac{2x+1}{x-1}$

- 10 f is the function which converts x into $f(x)$ whereas $f(x)$ is the value of the function at any value of x .



Note: Other answers are possible.

First sketch the straight line which passes through the points $(2, 1)$ and $(5, 3)$.

Then sketch two quadratic functions which also pass through the two points.

- 12 $f(x) = ax + b$ where $f(2) = 1$ and $f(-3) = 11$

So, $a(2) + b = 1$

and $a(-3) + b = 11$

$\therefore 2a + b = 1$

$\therefore -3a + b = 11$

$\therefore b = 1 - 2a \quad \dots (*)$

$\therefore -3a + (1 - 2a) = 11 \quad \{\text{using } (*)\}$

$\therefore -5a = 10$

$\therefore a = -2$

Substituting $a = -2$ into $(*)$ gives $b = 1 - 2(-2) = 5$.

So, $a = -2$, $b = 5$, and hence $f(x) = -2x + 5$.

13 a $P(t) = 5 + 10t$
 $\therefore P(3) = 5 + 10(3)$
 $= 35$

There are 35 L of petrol in the tank after 3 minutes.

b When $P(t) = 50$, then $5 + 10t = 50$
 $\therefore 10t = 45$
 $\therefore t = 4.5$

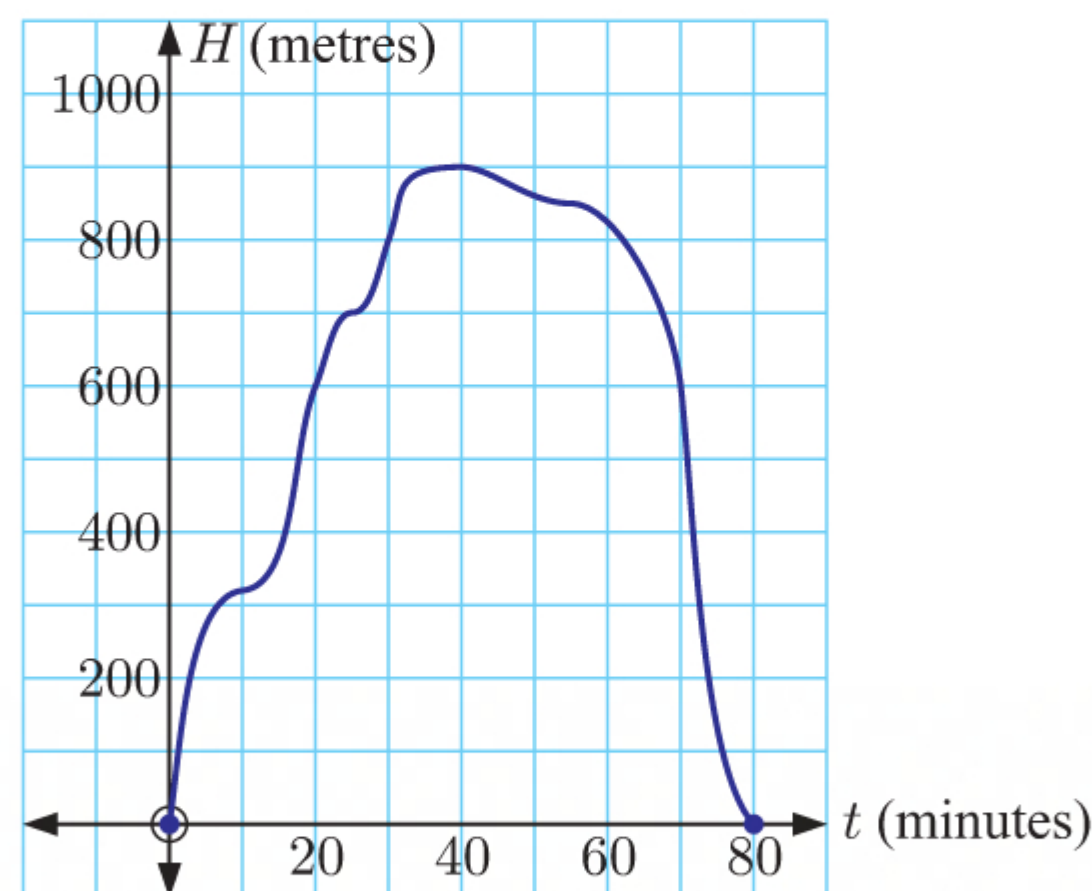
After $4\frac{1}{2}$ minutes, there are 50 L of petrol in the tank.

- c** When Samantha started to fill the tank, time $t = 0$.

$$\begin{aligned}\text{Now, when } t = 0, \quad P(0) &= 5 + 10(0) \\ &= 5\end{aligned}$$

There were 5 L of petrol in the tank when Samantha started to fill it.

- 14 a** $H(30) = 800$
After 30 minutes, the balloon is 800 m high.
- b** $H(t) = 600$ when $t = 20$ or 70 .
After 20 minutes and after 70 minutes the balloon is 600 m high.
- c** The height of the balloon was recorded for $0 \leq t \leq 80$ minutes.
- d** The range of heights recorded was 0 m to 900 m.



- 15** $f(x) = ax + \frac{b}{x}$ where $f(1) = 1$ and $f(2) = 5$

$$\text{So, } a(1) + \frac{b}{(1)} = 1$$

$$\therefore a + b = 1$$

$$\therefore b = 1 - a \quad \dots (*)$$

$$\text{and } a(2) + \frac{b}{(2)} = 5$$

$$\therefore 2a + \frac{b}{2} = 5$$

$$\therefore 4a + b = 10$$

$$\therefore 4a + (1 - a) = 10 \quad \{\text{using } (*)\}$$

$$\therefore 3a = 9$$

$$\therefore a = 3$$

Substituting $a = 3$ into $(*)$ gives $b = 1 - 3 = -2$.

So, $a = 3$, $b = -2$.

- 16** $T(x) = ax^2 + bx + c$ where $T(0) = -4$, $T(1) = -2$, and $T(2) = 6$

$$\text{So, } a(0)^2 + b(0) + c = -4$$

$$\therefore c = -4$$

$$\text{Now } a(1)^2 + b(1) - 4 = -2$$

$$\therefore a + b = 2$$

$$\therefore b = 2 - a \quad \dots (*)$$

$$\text{and } a(2)^2 + b(2) - 4 = 6$$

$$\therefore 4a + 2b = 10$$

$$\therefore 2a + b = 5$$

$$\therefore 2a + (2 - a) = 5 \quad \{\text{using } (*)\}$$

$$\therefore a = 3$$

Substituting $a = 3$ into $(*)$ gives $b = 2 - 3 = -1$.

So, $a = 3$, $b = -1$, and $c = -4$.

17 $V(t) = 9000 - 900t$

a $V(4) = 9000 - 900(4)$
 $= 9000 - 3600$
 $= 5400$

$V(4)$ is the value of the photocopier in pounds after 4 years.
 \therefore the value of the photocopier 4 years after purchase is £5400.

b $V(t) = 3600$, so $9000 - 900t = 3600$
 $\therefore 900t = 5400$
 $\therefore t = 6$

After 6 years, the value of the photocopier is £3600.

c The original purchase price is when $t = 0$.
Now, $V(0) = 9000 - 900(0)$
 $= 9000$

The original purchase price was £9000.

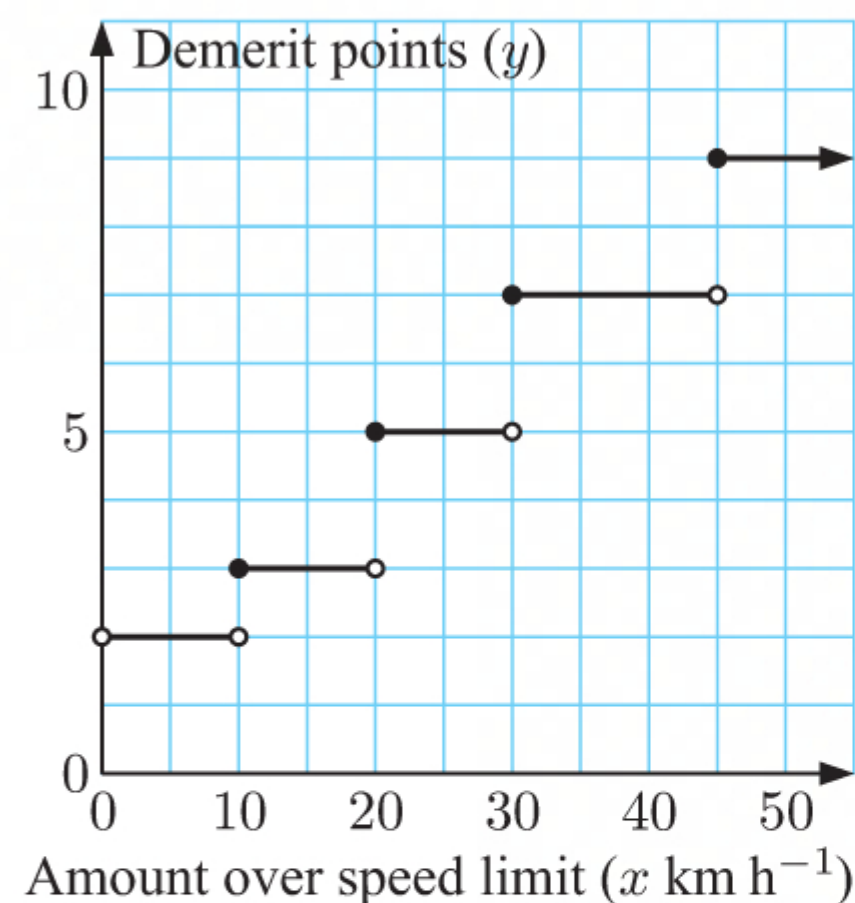
d We have $t \geq 0$ and $V \geq 0$ since time and value are always positive.
So, $9000 - 900t \geq 0$
 $\therefore 9000 \geq 900t$
 $\therefore t \leq 10$
So, $0 \leq t \leq 10$ years.

EXERCISE 3C

1

Amount over speed limit ($x \text{ km h}^{-1}$)	Demerit points (y)
$0 < x < 10$	2
$10 \leq x < 20$	3
$20 \leq x < 30$	5
$30 \leq x < 45$	7
$x \geq 45$	9

a



- b** The function is defined for x such that $x > 0$.
 \therefore the domain is $\{x \mid x > 0\}$.
The possible demerit points are 2, 3, 5, 7, and 9.
 \therefore the range is $\{y \mid y = 2, 3, 5, 7, \text{ or } 9\}$.

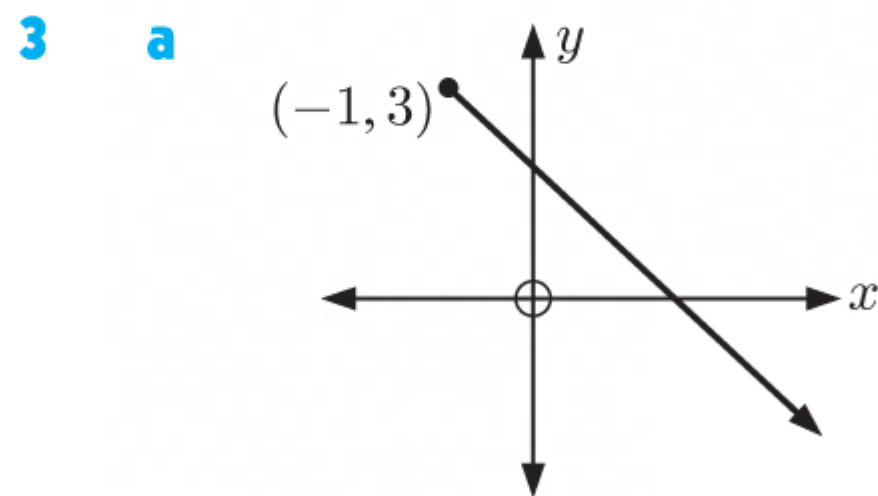
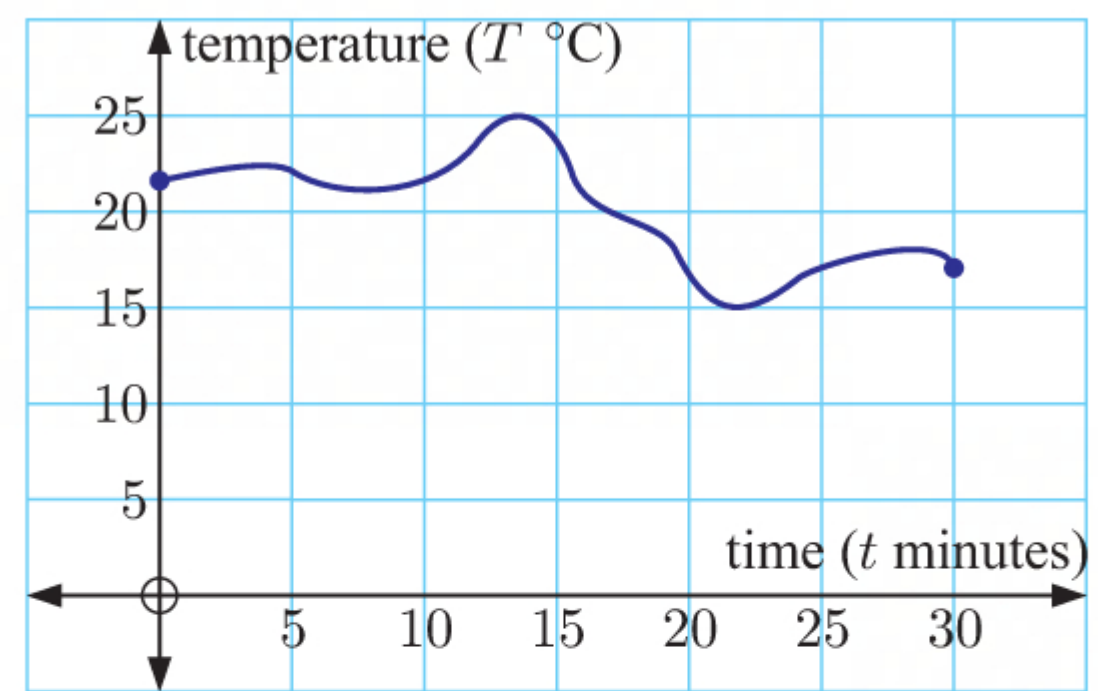
2 a At any moment in time there can be only one temperature, so the graph is a function.

b The temperature function is defined for all time t such that $0 \leq t \leq 30$ minutes.

\therefore the domain is $\{t \mid 0 \leq t \leq 30\}$.

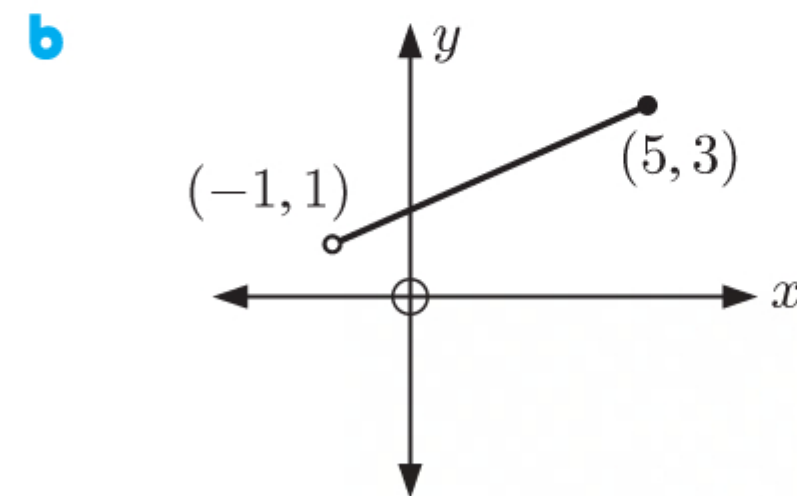
The recorded temperatures lie between $T = 15^\circ\text{C}$ and $T = 25^\circ\text{C}$.

\therefore the range is $\{T \mid 15 \leq T \leq 25\}$.



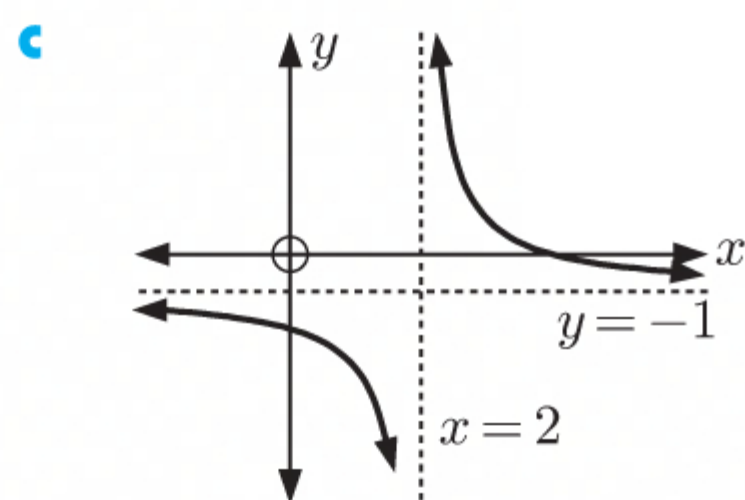
Domain is $\{x \mid x \geq -1\}$

Range is $\{y \mid y \leq 3\}$



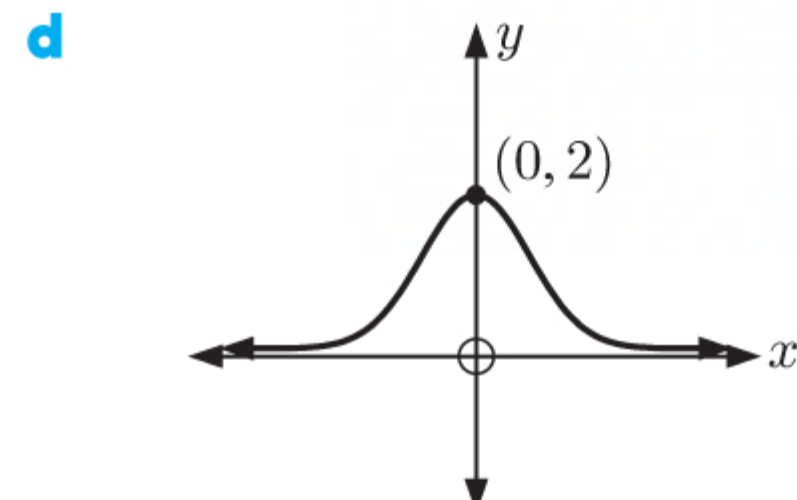
Domain is $\{x \mid -1 < x \leq 5\}$

Range is $\{y \mid 1 < y \leq 3\}$



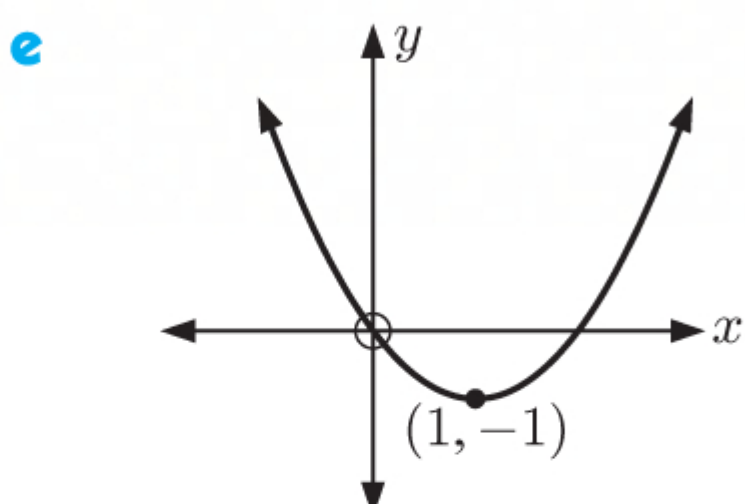
Domain is $\{x \mid x \neq 2\}$

Range is $\{y \mid y \neq -1\}$



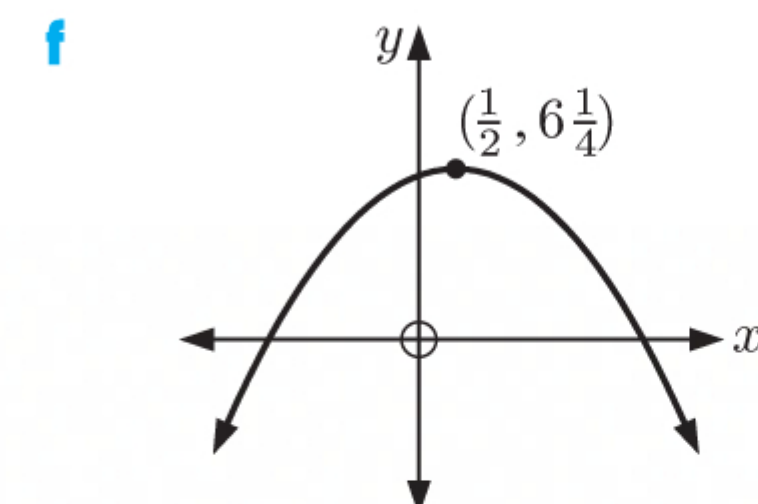
Domain is $\{x \mid x \in \mathbb{R}\}$

Range is $\{y \mid 0 < y \leq 2\}$



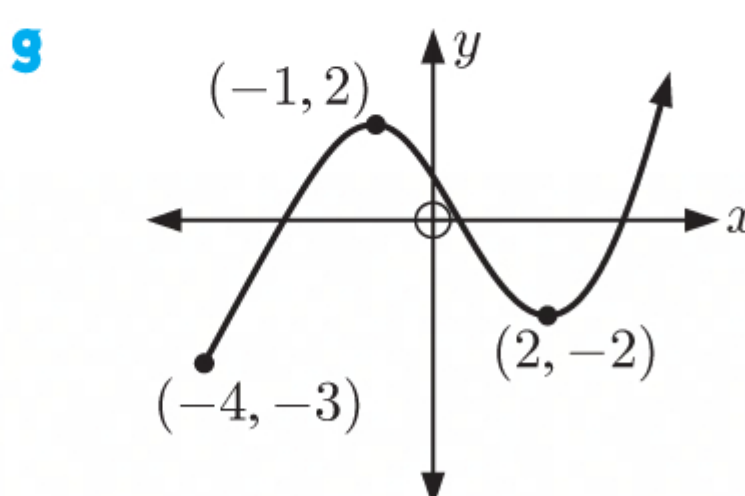
Domain is $\{x \mid x \in \mathbb{R}\}$

Range is $\{y \mid y \geq -1\}$



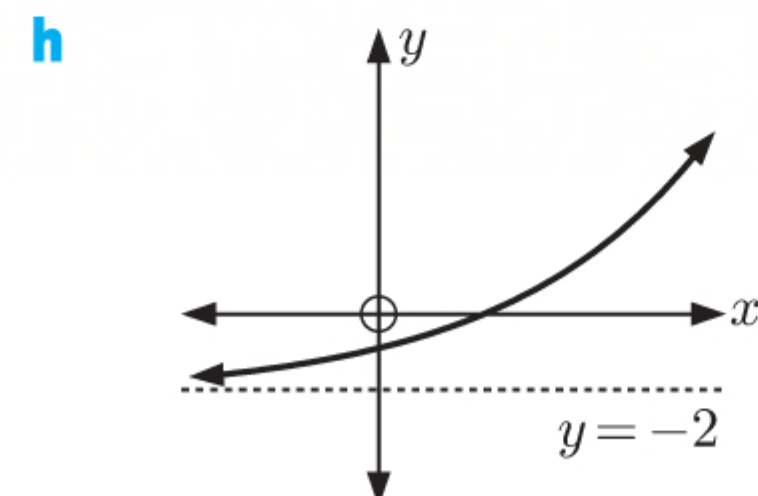
Domain is $\{x \mid x \in \mathbb{R}\}$

Range is $\{y \mid y \leq \frac{25}{4}\}$



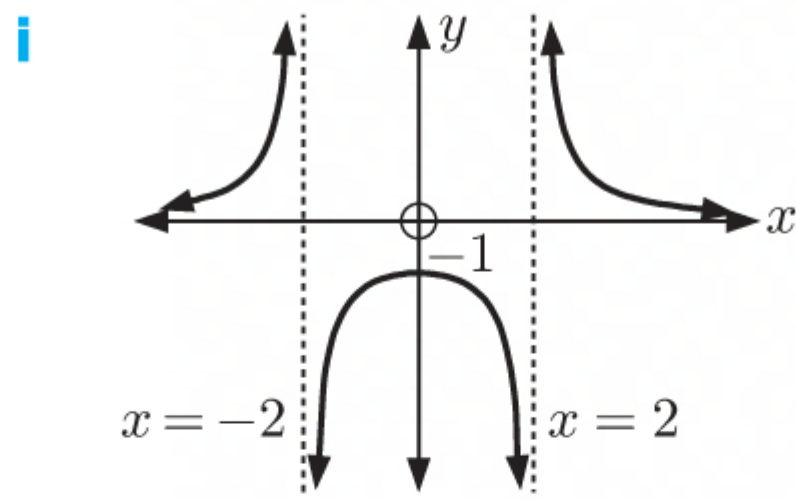
Domain is $\{x \mid x \geq -4\}$

Range is $\{y \mid y \geq -3\}$



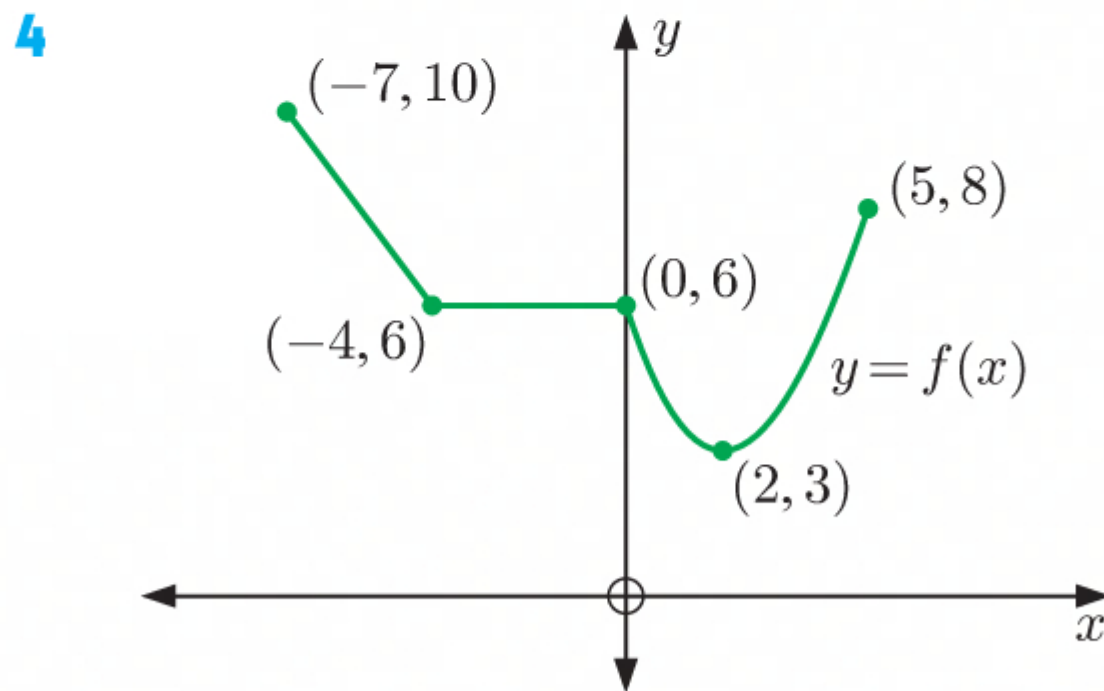
Domain is $\{x \mid x \in \mathbb{R}\}$

Range is $\{y \mid y > -2\}$



Domain is $\{x \mid x \neq \pm 2\}$

Range is $\{y \mid y \leq -1 \text{ or } y > 0\}$






From the graph:

Domain is $\{x \mid -7 \leq x \leq 5\}$

Range is $\{y \mid 3 \leq y \leq 10\}$

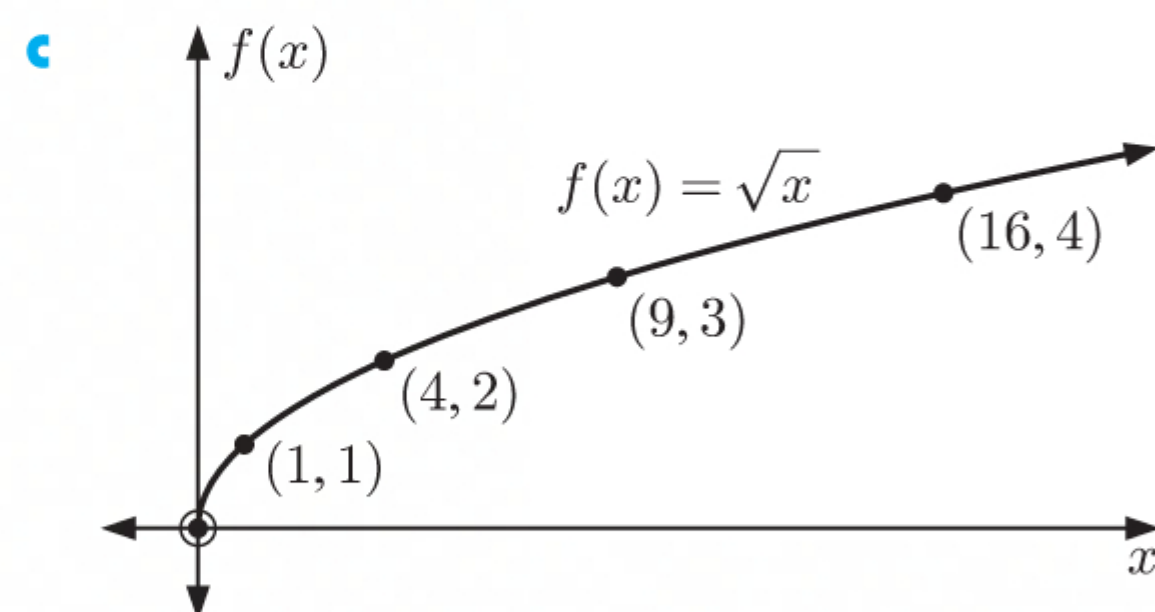
- a** $x = -5$ satisfies $-7 \leq x \leq 5$.
 \therefore “ -5 is in the domain of f ” is true.
- b** $y = 2$ does not satisfy $3 \leq y \leq 10$.
 \therefore “ 2 is in the range of f ” is false.
- c** $y = 9$ satisfies $3 \leq y \leq 10$.
 \therefore “ 9 is in the range of f ” is true.
- d** $x = \sqrt{2} \approx 1.41$ satisfies $-7 \leq x \leq 5$.
 \therefore “ $\sqrt{2}$ is in the domain of f ” is true.

- 5**
- a** $y = x^2$ has vertex $(0, 0)$ and shape ($a > 0$).
 \therefore the minimum y -value is 0 and there is no maximum y -value.
 \therefore the range is $\{y \mid y \geq 0\}$.
 - b** $y = -x^2$ has vertex $(0, 0)$ and shape ($a < 0$).
 \therefore the maximum y -value is 0 and there is no minimum y -value.
 \therefore the range is $\{y \mid y \leq 0\}$.
 - c** $y = x^2 + 2$ has vertex $(0, 2)$ and shape ($a > 0$).
 \therefore the minimum y -value is 2 and there is no maximum y -value.
 \therefore the range is $\{y \mid y \geq 2\}$.
 - d** $y = -2(x + 3)^2$ has vertex $(-3, 0)$ and shape ($a < 0$).
 \therefore the maximum y -value is 0 and there is no minimum y -value.
 \therefore the range is $\{y \mid y \leq 0\}$.
 - e** $y = 1 - (x - 2)^2 = -(x - 2)^2 + 1$ has vertex $(2, 1)$ and shape ($a < 0$).
 \therefore the maximum y -value is 1 and there is no minimum y -value.
 \therefore the range is $\{y \mid y \leq 1\}$.
 - f** $y = (2x + 1)^2 + 3$ has vertex $(-\frac{1}{2}, 3)$ and shape ($a > 0$).
 \therefore the minimum y -value is 3 and there is no maximum y -value.
 \therefore the range is $\{y \mid y \geq 3\}$.

- g** $y = x^2 - 7x + 10$
 $\therefore y = x^2 - 7x + \left(\frac{7}{2}\right)^2 + 10 - \left(\frac{7}{2}\right)^2$ {completing the square}
 $\therefore y = \left(x - \frac{7}{2}\right)^2 - \frac{9}{4}$ which has vertex $\left(\frac{7}{2}, -\frac{9}{4}\right)$ and shape  ($a > 0$).
 \therefore the minimum y -value is $-\frac{9}{4}$ and there is no maximum y -value.
 \therefore the range is $\{y \mid y \geq -\frac{9}{4}\}$.
- h** $y = -x^2 + 2x + 8$
 $\therefore y = -(x^2 - 2x - 8)$
 $\therefore y = -(x^2 - 2x + (-1)^2 - 8 - (-1)^2)$ {completing the square}
 $\therefore y = -[(x - 1)^2 - 9]$
 $\therefore y = -(x - 1)^2 + 9$ which has vertex $(1, 9)$ and shape  ($a < 0$).
 \therefore the maximum y -value is 9 and there is no minimum y -value.
 \therefore the range is $\{y \mid y \leq 9\}$.
- i** $f : x \mapsto 5x - 3x^2$
 $\therefore y = 5x - 3x^2$
 $\therefore y = -3\left(x^2 - \frac{5}{3}x\right)$
 $\therefore y = -3\left(x^2 - \frac{5}{3}x + \left(-\frac{5}{6}\right)^2 - \left(-\frac{5}{6}\right)^2\right)$ {completing the square}
 $\therefore y = -3\left[\left(x - \frac{5}{6}\right)^2 - \frac{25}{36}\right]$
 $\therefore y = -3\left(x - \frac{5}{6}\right)^2 + \frac{25}{12}$ which has vertex $\left(\frac{5}{6}, \frac{25}{12}\right)$ and shape  ($a < 0$).
 \therefore the maximum y -value is $\frac{25}{12}$ and there is no minimum y -value.
 \therefore the range is $\{y \mid y \leq \frac{25}{12}\}$.

6 $f(x) = \sqrt{x}$

- a** \sqrt{x} is defined when $x \geq 0$.
 \therefore the domain is $\{x \mid x \geq 0\}$.



b

x	0	1	4	9	16
$f(x)$	0	1	2	3	4

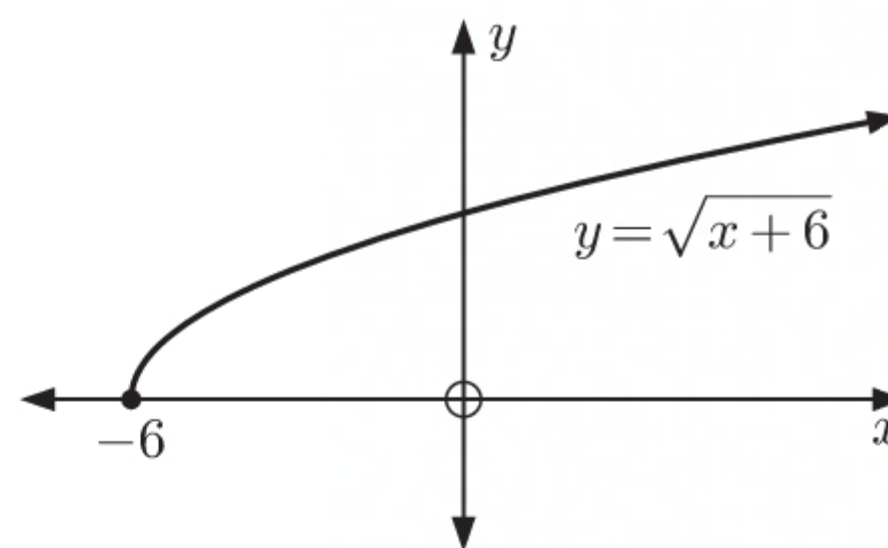
- d** A square root cannot be negative.
 \therefore the range is $\{y \mid y \geq 0\}$.

7 a $\sqrt{x+6}$ is defined when $x+6 \geq 0$
 $\therefore x \geq -6$

\therefore the domain is $\{x \mid x \geq -6\}$.

A square root cannot be negative.

\therefore the range is $\{y \mid y \geq 0\}$.

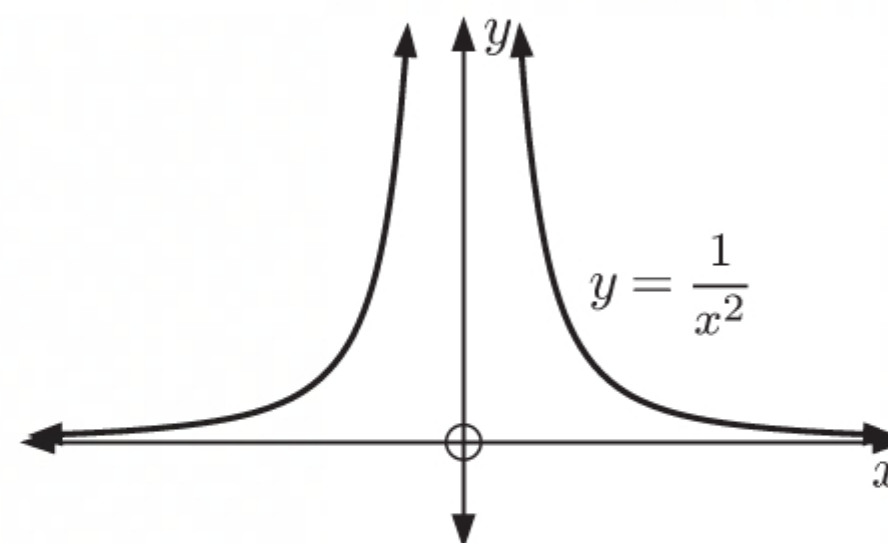


b $\frac{1}{x^2}$ is defined when $x^2 \neq 0$
 $\therefore x \neq 0$

\therefore the domain is $\{x \mid x \neq 0\}$.

$y = f(x)$ is always positive and never zero.

\therefore the range is $\{y \mid y > 0\}$.



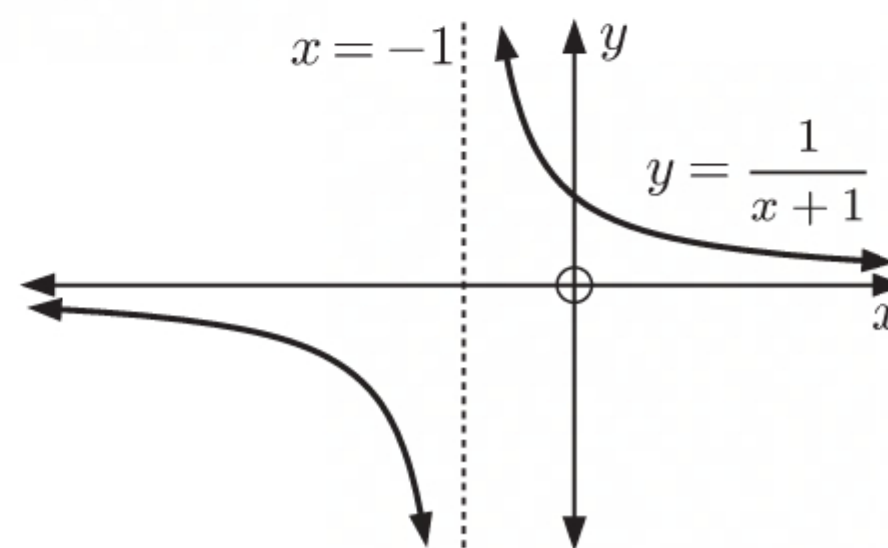
c $\frac{1}{x+1}$ is defined when $x+1 \neq 0$
 $\therefore x \neq -1$

\therefore the domain is $\{x \mid x \neq -1\}$.

No matter how large or small x is,

$y = f(x)$ is never zero.

\therefore the range is $\{y \mid y \neq 0\}$.

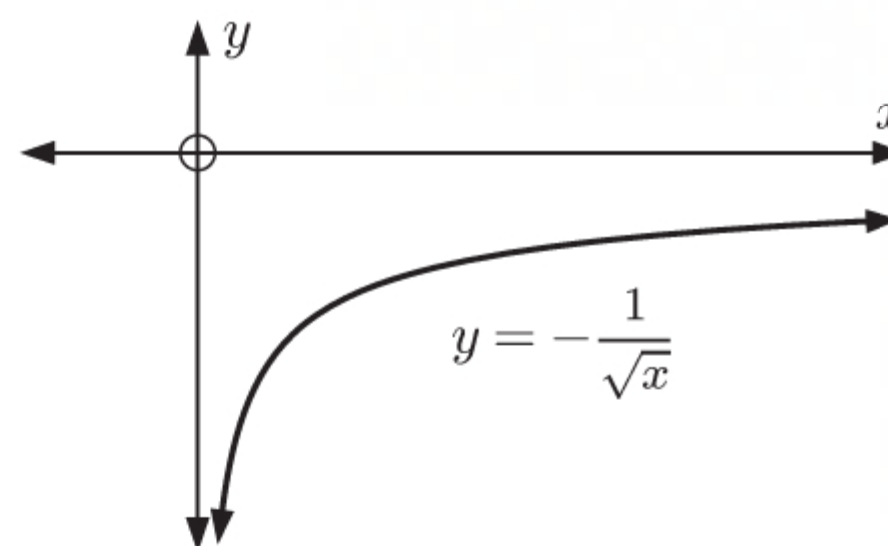


d $-\frac{1}{\sqrt{x}}$ is defined when $x > 0$

\therefore the domain is $\{x \mid x > 0\}$.

$y = f(x)$ is always negative and never zero.

\therefore the range is $\{y \mid y < 0\}$.



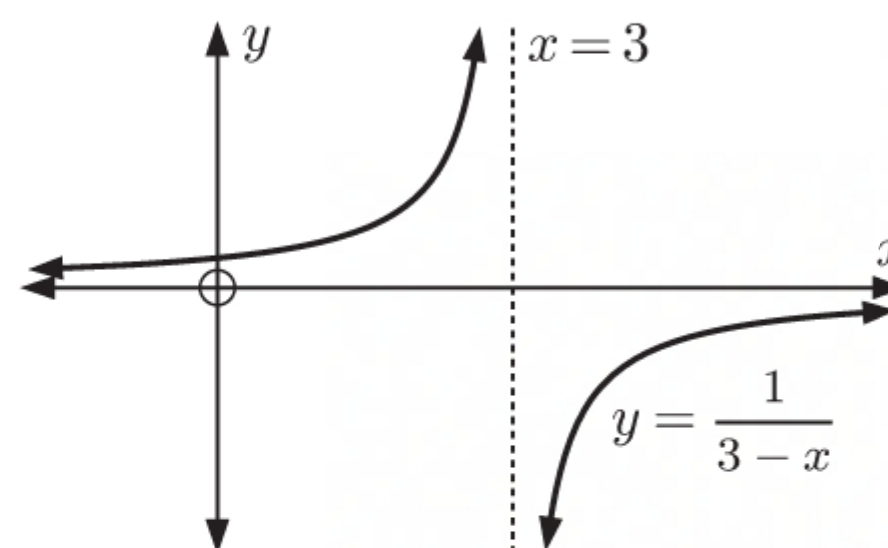
e $\frac{1}{3-x}$ is defined when $3-x \neq 0$
 $\therefore x \neq 3$

\therefore the domain is $\{x \mid x \neq 3\}$.

No matter how large or small x is,

$y = f(x)$ is never zero.

\therefore the range is $\{y \mid y \neq 0\}$.

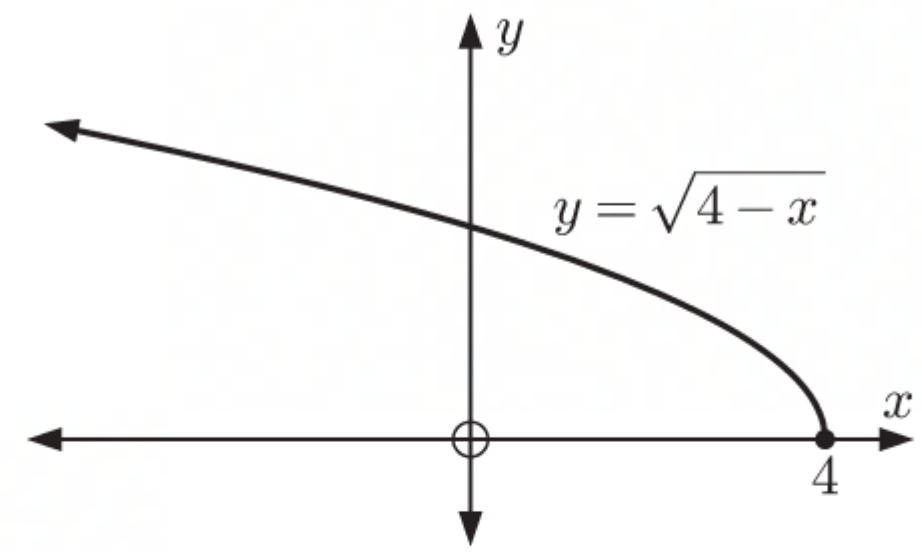


f $\sqrt{4-x}$ is defined when $4-x \geq 0$
 $\therefore x \leq 4$

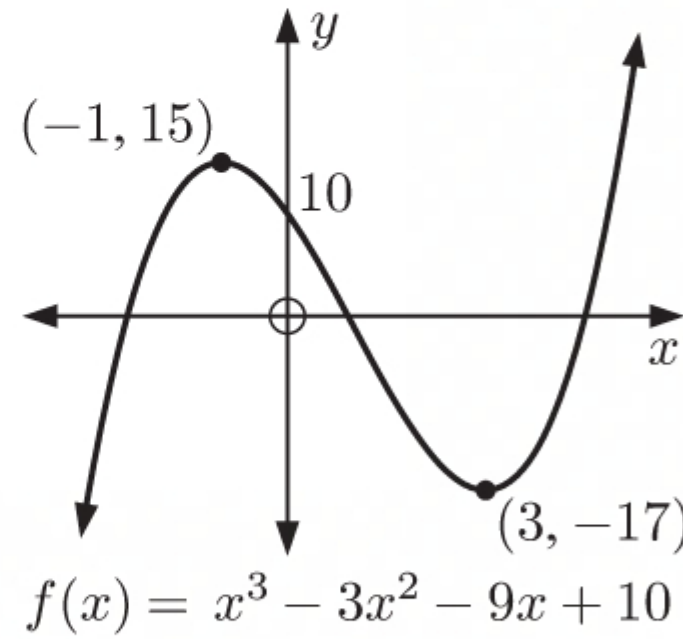
\therefore the domain is $\{x \mid x \leq 4\}$.

A square root cannot be negative.

\therefore the range is $\{y \mid y \geq 0\}$.



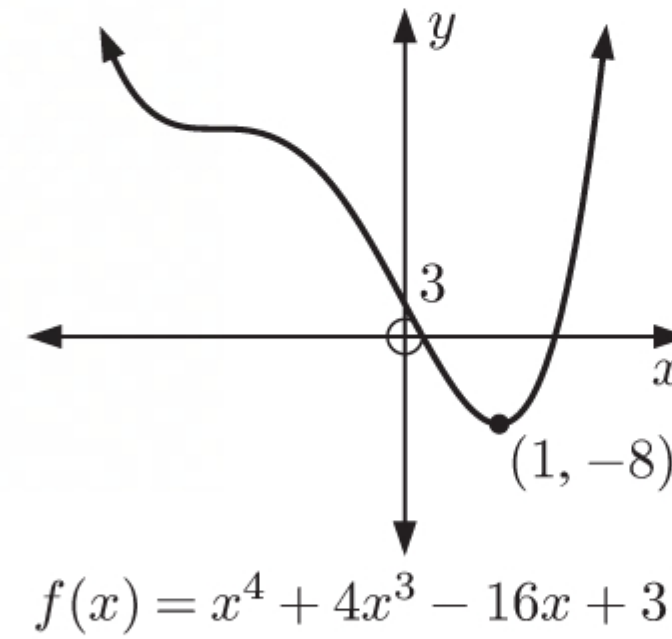
8 a



The domain is $\{x \mid x \in \mathbb{R}\}$.

The range is $\{y \mid y \in \mathbb{R}\}$.

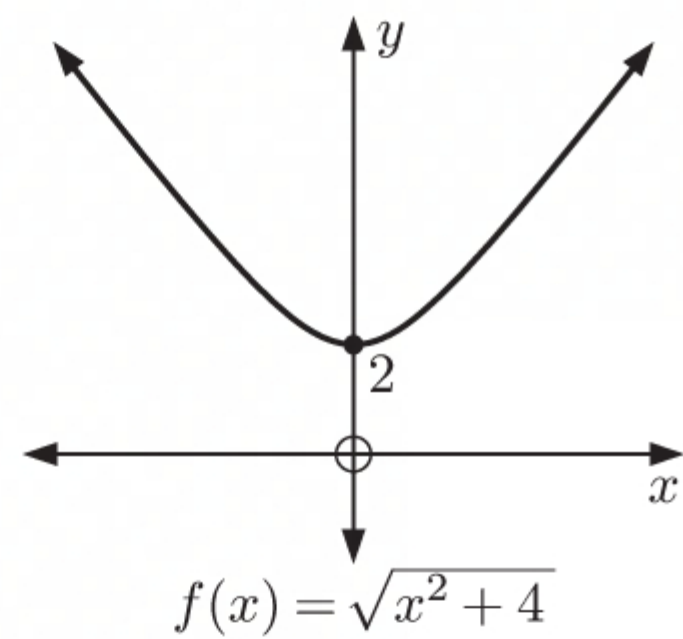
b



The domain is $\{x \mid x \in \mathbb{R}\}$.

The range is $\{y \mid y \geq -8\}$.

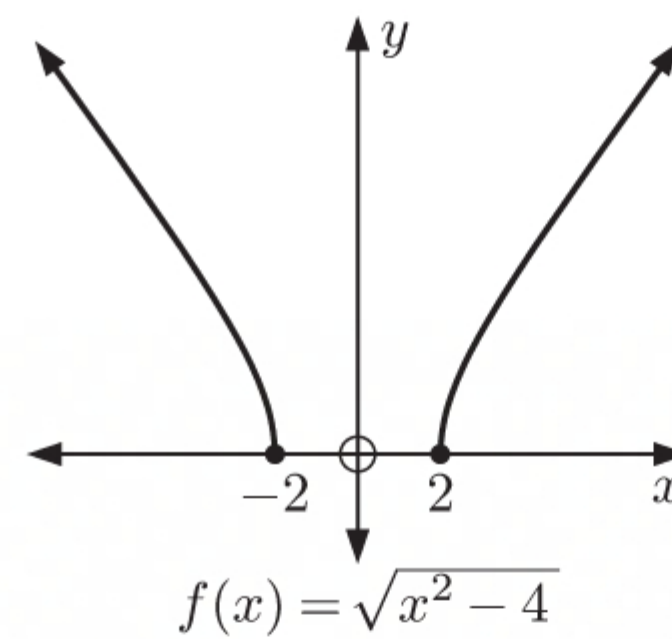
c



The domain is $\{x \mid x \in \mathbb{R}\}$.

The range is $\{y \mid y \geq 2\}$.

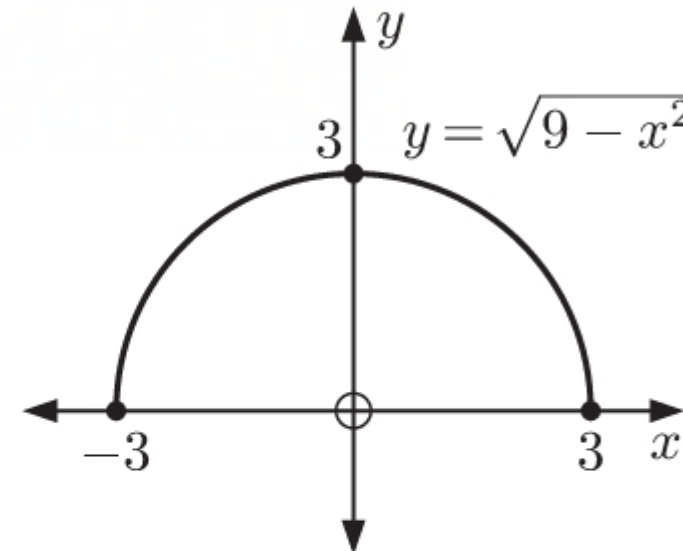
d



The domain is $\{x \mid x \leq -2 \text{ or } x \geq 2\}$.

The range is $\{y \mid y \geq 0\}$.

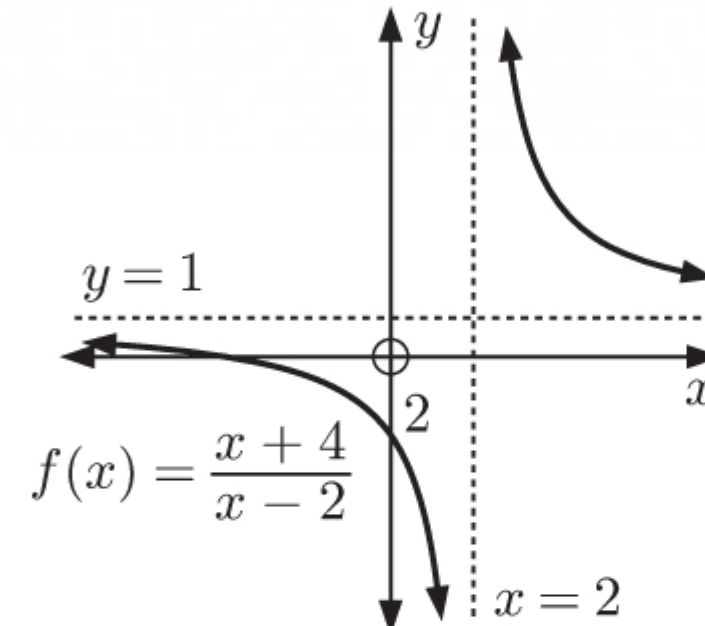
e



The domain is $\{x \mid -3 \leq x \leq 3\}$.

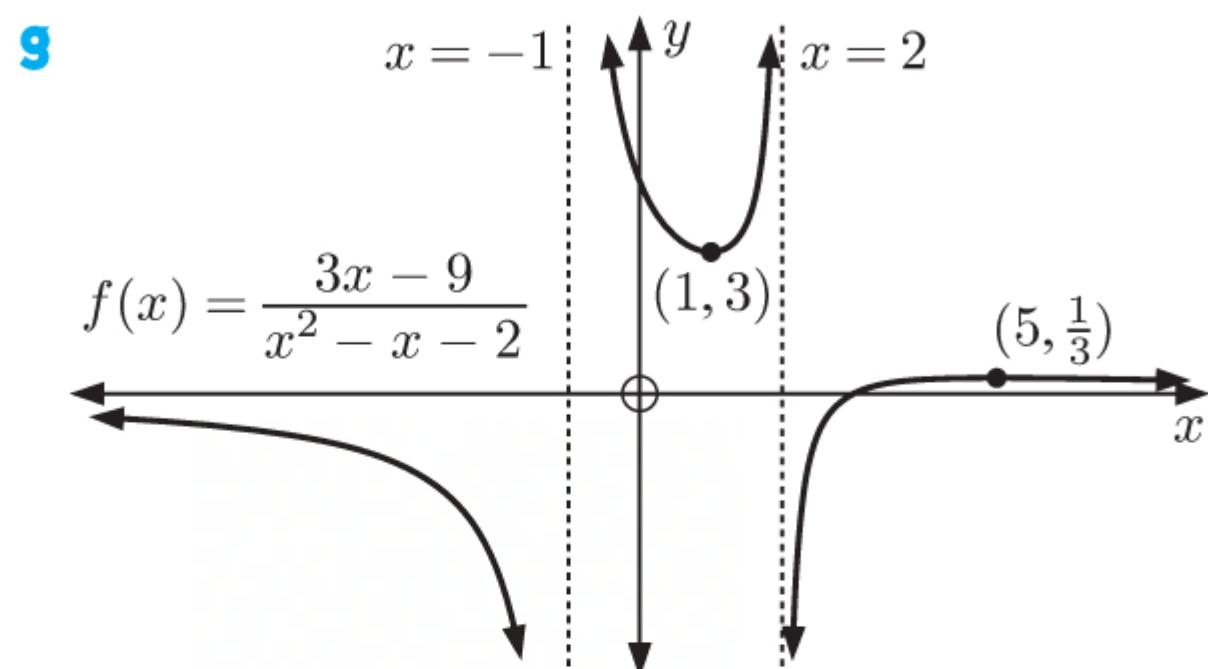
The range is $\{y \mid 0 \leq y \leq 3\}$.

f



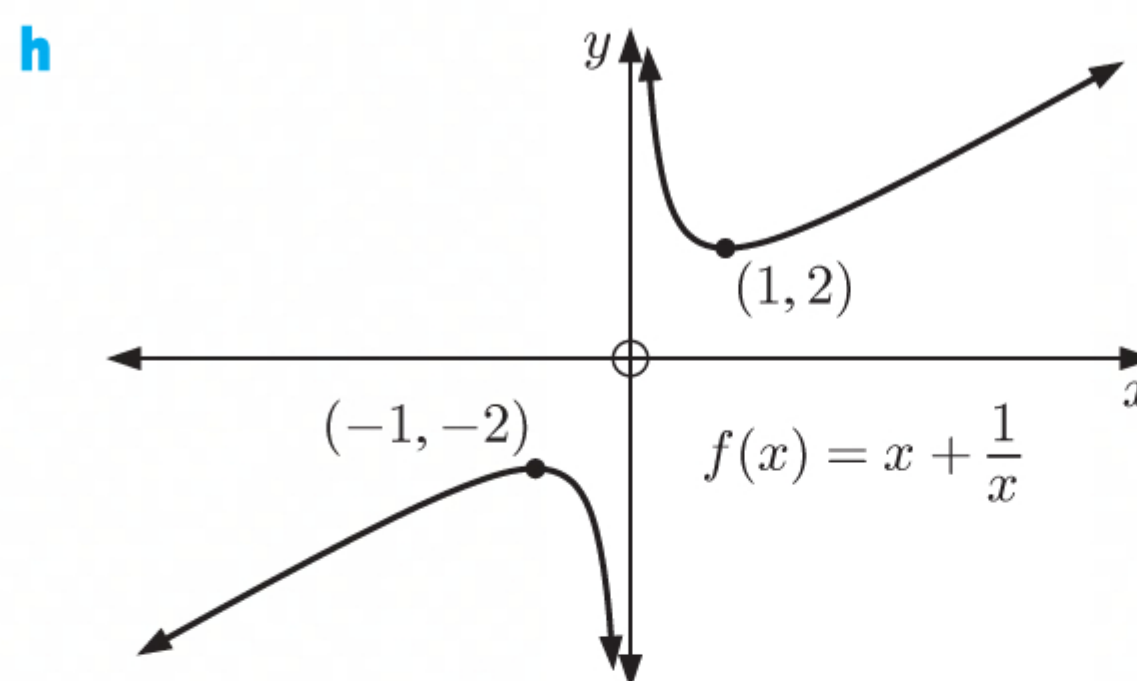
The domain is $\{x \mid x \neq 2\}$.

The range is $\{y \mid y \neq 1\}$.



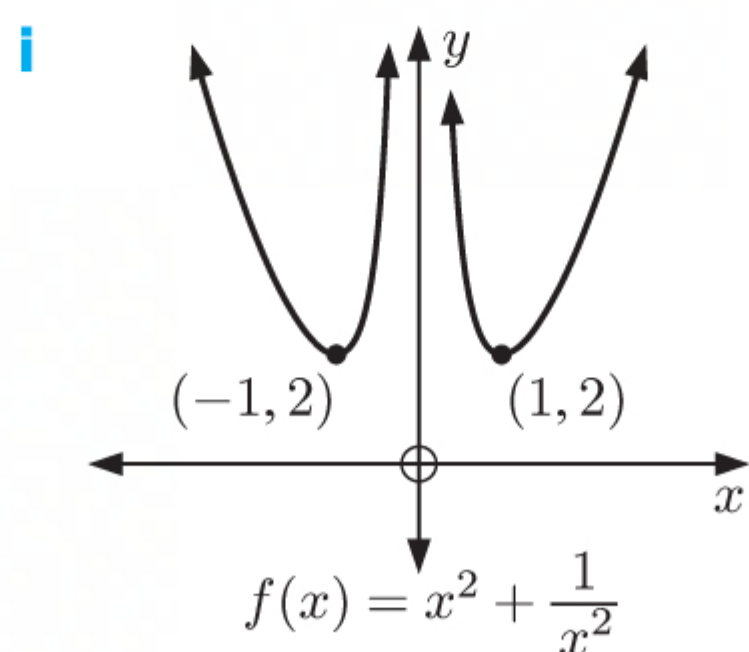
The domain is $\{x \mid x \neq -1 \text{ or } 2\}$.

The range is $\{y \mid y \leq \frac{1}{3} \text{ or } y \geq 3\}$.



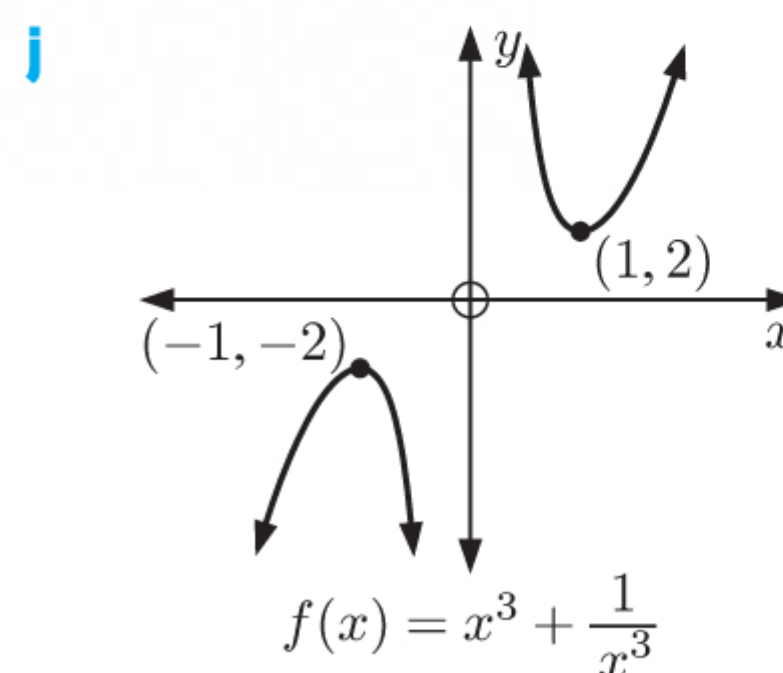
The domain is $\{x \mid x \neq 0\}$.

The range is $\{y \mid y \leq -2 \text{ or } y \geq 2\}$.



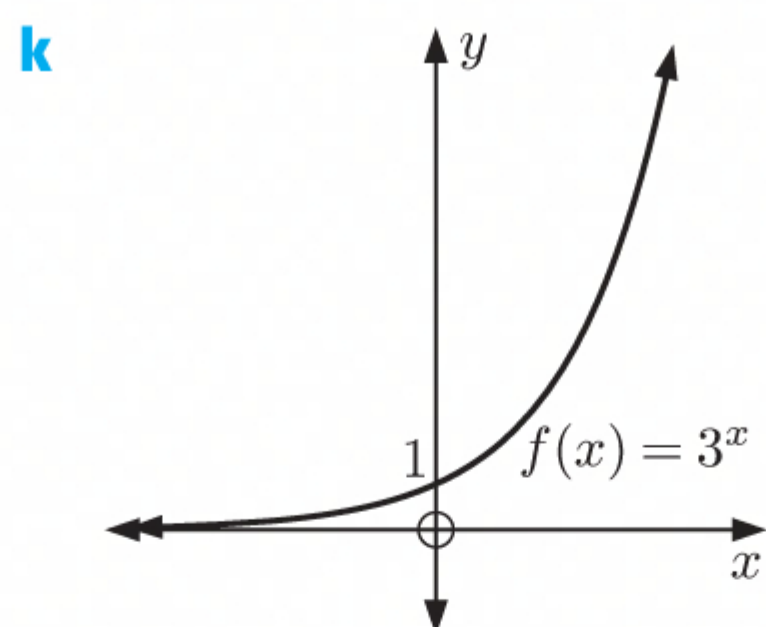
The domain is $\{x \mid x \neq 0\}$.

The range is $\{y \mid y \geq 2\}$.



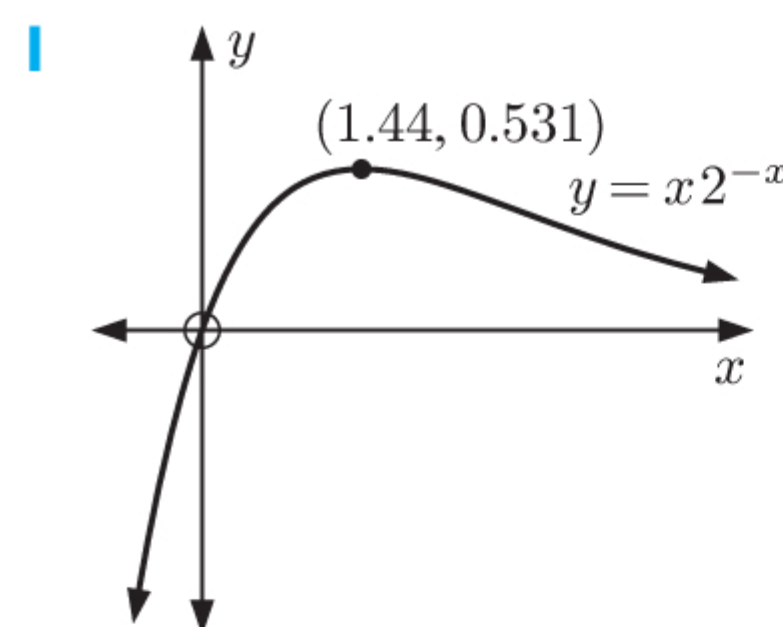
The domain is $\{x \mid x \neq 0\}$.

The range is $\{y \mid y \leq -2 \text{ or } y \geq 2\}$.



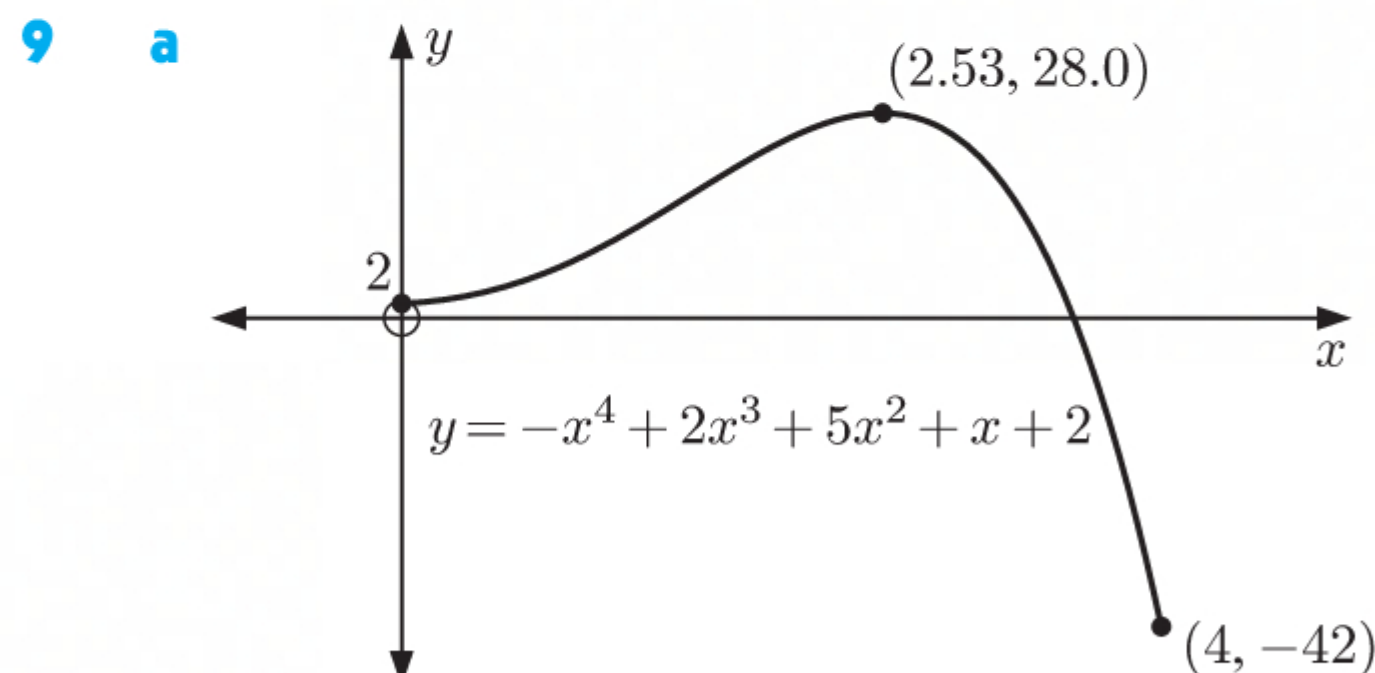
The domain is $\{x \mid x \in \mathbb{R}\}$.

The range is $\{y \mid y > 0\}$.

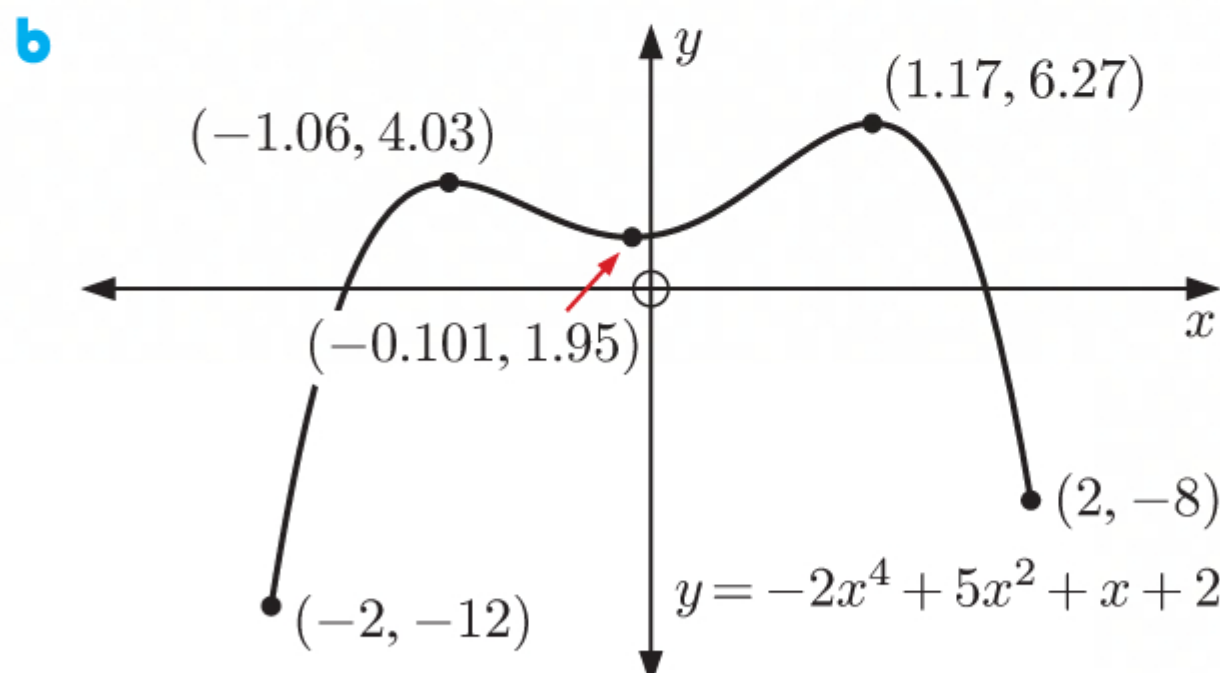


The domain is $\{x \mid x \in \mathbb{R}\}$.

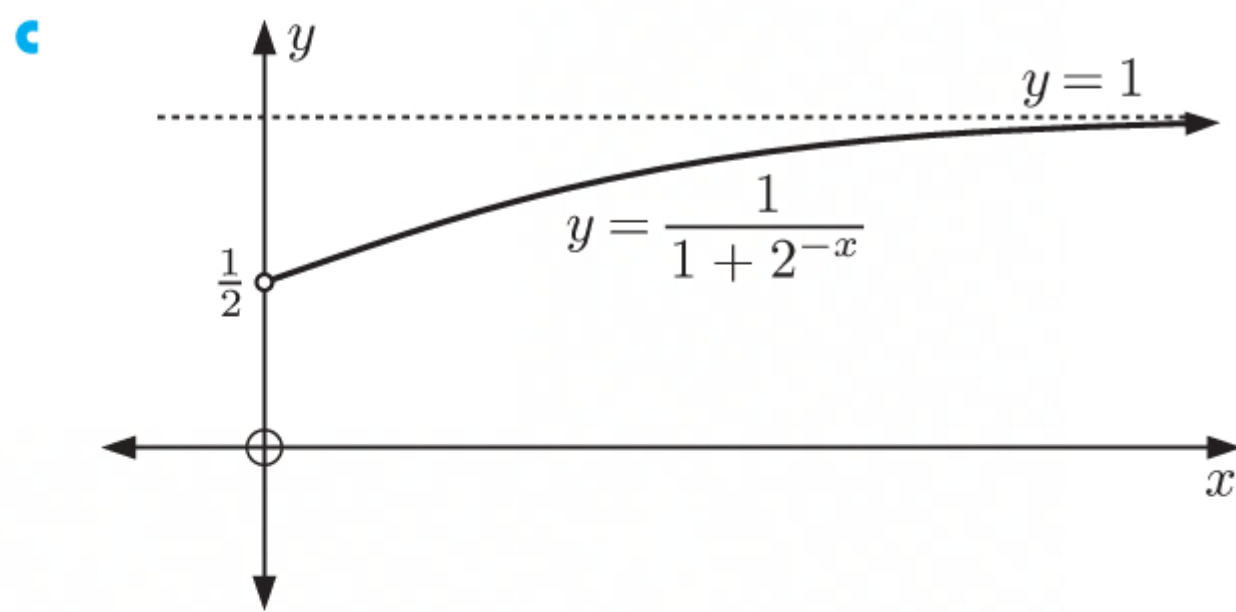
The range is $\{y \mid y \leq 0.531\}$.



The range is $\{y \mid -42 \leq y \leq 28.0\}$.

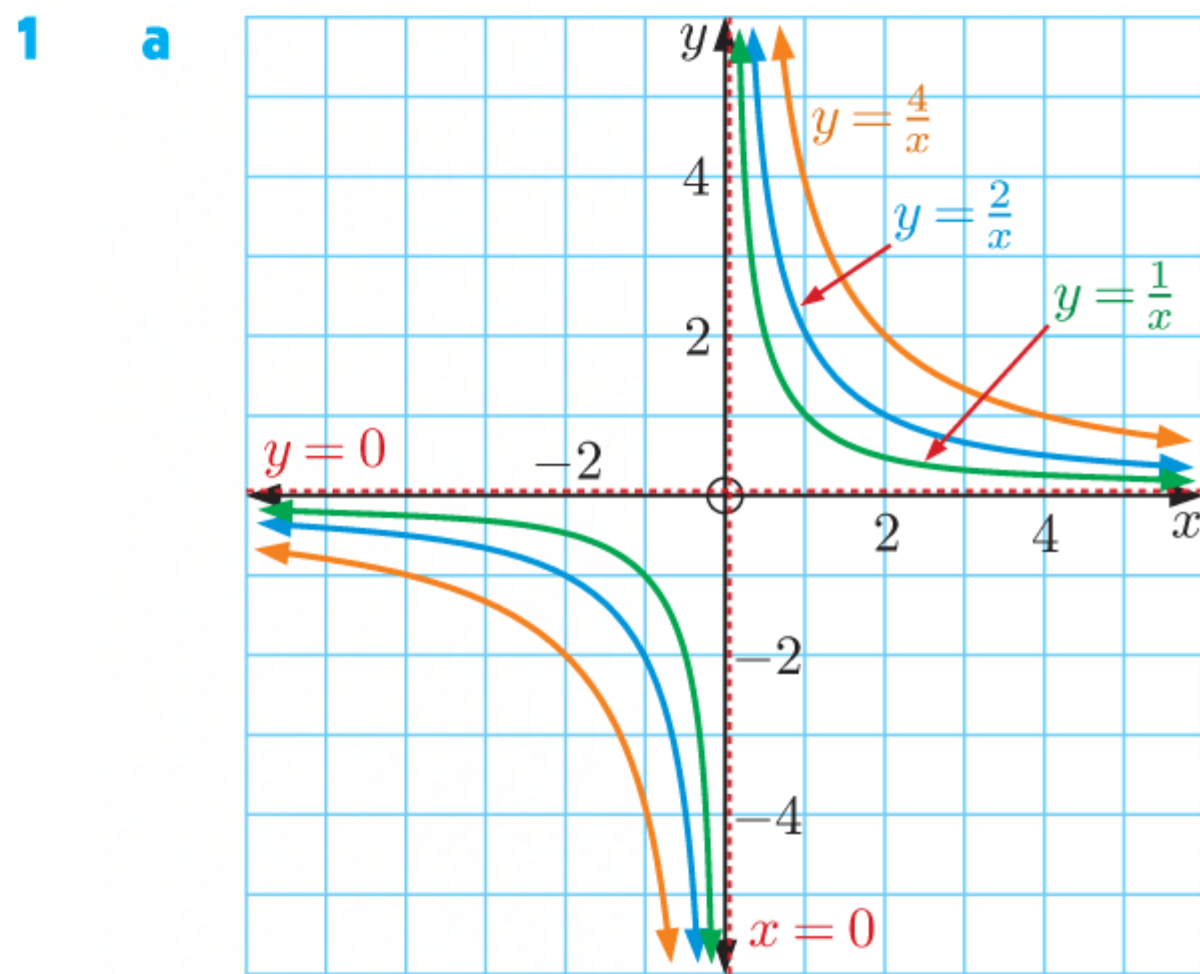


The range is $\{y \mid -12 \leq y \leq 6.27\}$.



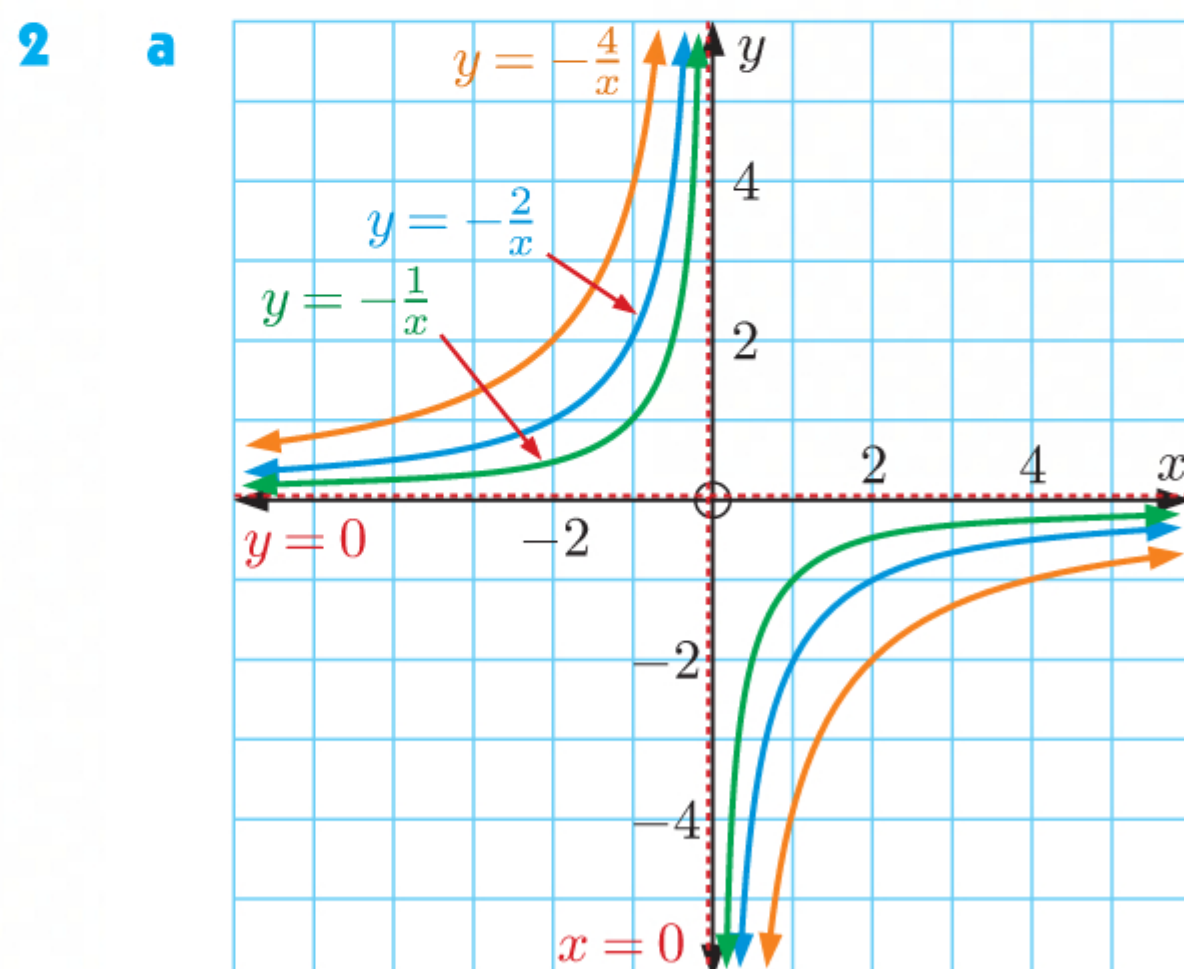
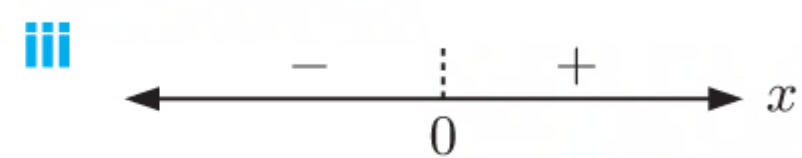
The range is $\{y \mid \frac{1}{2} < y < 1\}$.

EXERCISE 3D.1



b $y = \frac{k}{x}, k > 0$

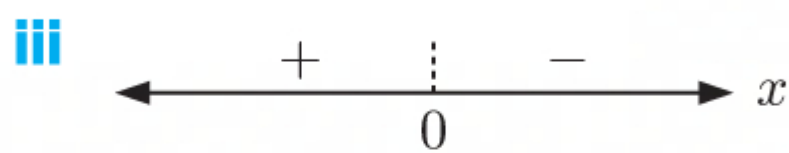
- i** As k becomes larger the graphs move further from the origin.
- ii** The graph lies in quadrants 1 and 3.



b $y = \frac{k}{x}, k < 0$

i As $|k|$ becomes larger, the graphs move further from the origin.

ii The graph lies in quadrants 2 and 4.



3 $y = \frac{k}{x}, k \neq 0$

a $\frac{k}{x}$ is defined when $x \neq 0$.
 \therefore the domain is $\{x \mid x \neq 0\}$.

c For $k > 0$, as $x \rightarrow 0^-$, $\frac{k}{x} \rightarrow -\infty$
 and as $x \rightarrow 0^+$, $\frac{k}{x} \rightarrow \infty$.

For $k < 0$, as $x \rightarrow 0^-$, $\frac{k}{x} \rightarrow \infty$
 and as $x \rightarrow 0^+$, $\frac{k}{x} \rightarrow -\infty$.

\therefore the vertical asymptote is $x = 0$.

b No matter how large or small x is, $y = \frac{k}{x}$ is never zero.
 \therefore the range is $\{y \mid y \neq 0\}$.

d For $k > 0$, as $x \rightarrow \infty$, $\frac{k}{x} \rightarrow 0^+$
 and as $x \rightarrow -\infty$, $\frac{k}{x} \rightarrow 0^-$.

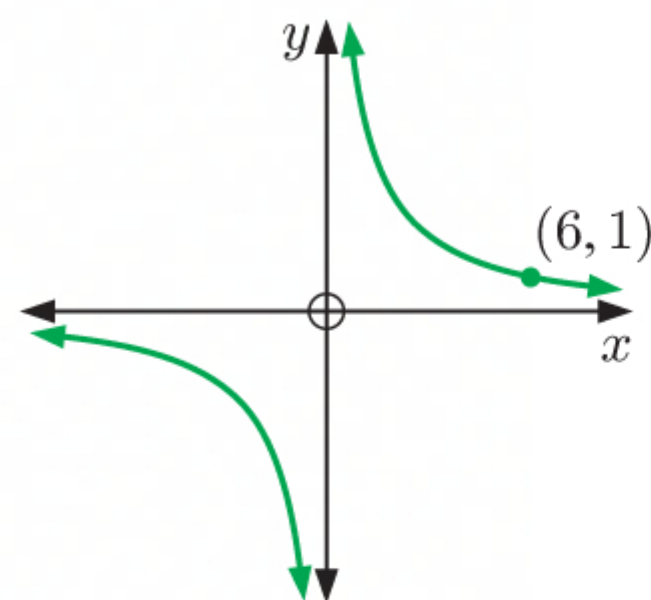
For $k < 0$, as $x \rightarrow \infty$, $\frac{k}{x} \rightarrow 0^-$
 and as $x \rightarrow -\infty$, $\frac{k}{x} \rightarrow 0^+$.

\therefore the horizontal asymptote is $y = 0$.

4 a Let the function be $y = \frac{k}{x}$.

When $x = 6$, $y = 1$, so $1 = \frac{k}{6}$
 $\therefore k = 6$

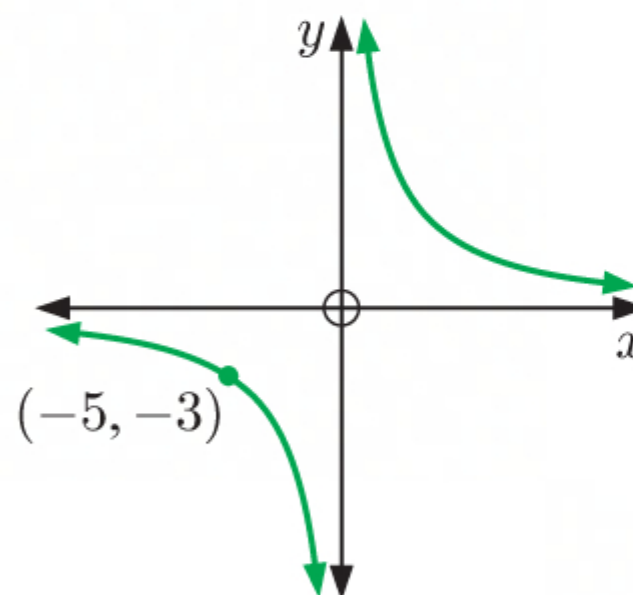
So, the function is $y = \frac{6}{x}$.



b Let the function be $y = \frac{k}{x}$.

When $x = -5$, $y = -3$, so $-3 = \frac{k}{-5}$
 $\therefore k = 15$

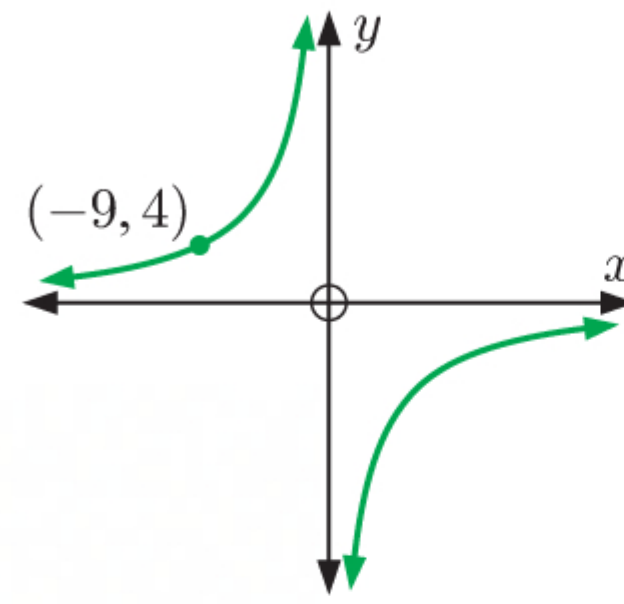
So, the function is $y = \frac{15}{x}$.



- c** Let the function be $y = \frac{k}{x}$.

When $x = -9$, $y = 4$, so $4 = \frac{k}{-9}$
 $\therefore k = -36$

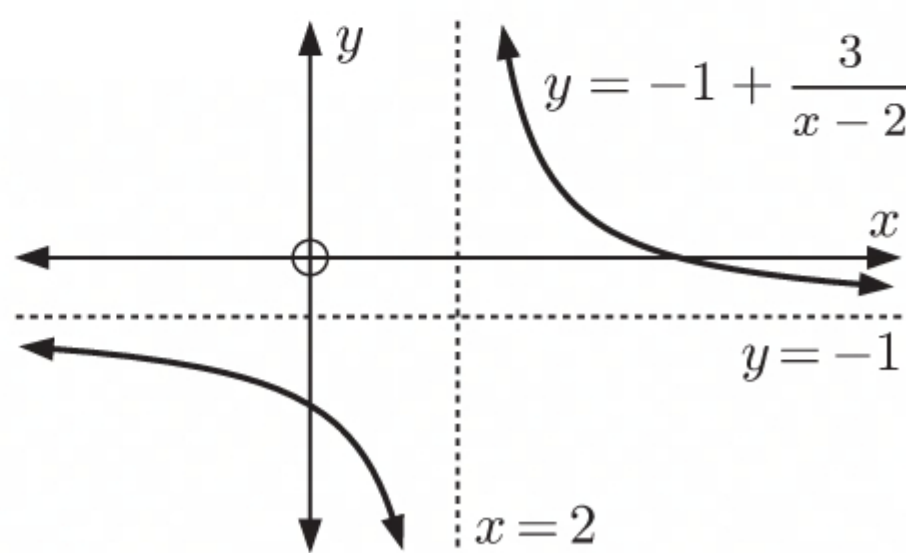
So, the function is $y = -\frac{36}{x}$.



INVESTIGATION

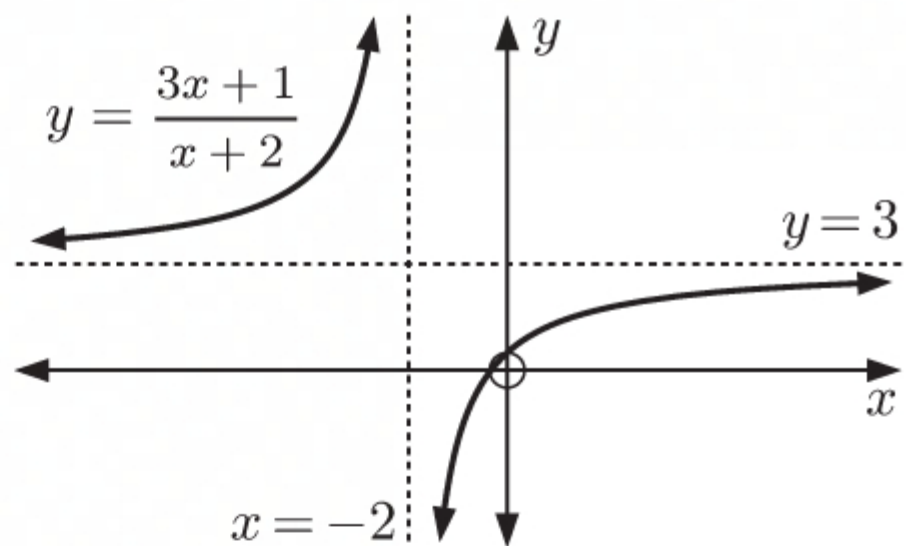
RATIONAL FUNCTIONS

1 a



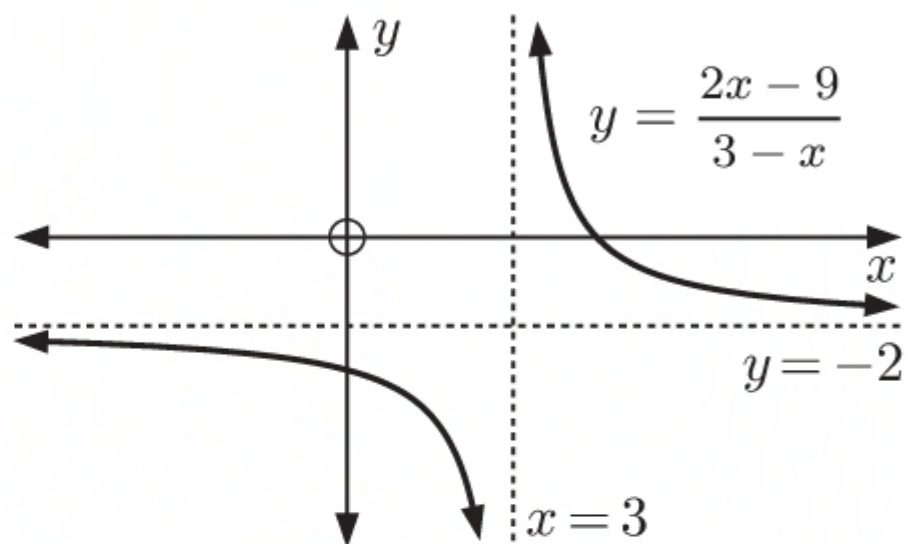
- i** The domain is $\{x \mid x \neq 2\}$.
- ii** The vertical asymptote is $x = 2$.
The horizontal asymptote is $y = -1$.

b



- i** The domain is $\{x \mid x \neq -2\}$.
- ii** The vertical asymptote is $x = -2$.
The horizontal asymptote is $y = 3$.

c



- i** The domain is $\{x \mid x \neq 3\}$.
- ii** The vertical asymptote is $x = 3$.
The horizontal asymptote is $y = -2$.

2 $y = \frac{b}{cx+d} + a$ where $b, c \neq 0$

- a** The horizontal asymptote is $y = a$.
- b** The vertical asymptote is $cx + d = 0$

$$\therefore x = -\frac{d}{c}$$

3 $y = \frac{ax+b}{cx+d}$ where $c \neq 0$

a The vertical asymptote is $cx + d = 0$

$$\therefore x = -\frac{d}{c}$$

b If $|x|$ is very large, then $\frac{ax+b}{cx+d} \approx \frac{ax}{cx} \approx \frac{a}{c}$

So, the graph approaches the line $y = \frac{a}{c}$, but never reaches it.

This tells us that the horizontal asymptote is $y = \frac{a}{c}$.

EXERCISE 3D.2

1 a $f(x) = \frac{3}{x-2}$

i The vertical asymptote is $x = 2$.
The horizontal asymptote is $y = 0$.

ii The domain is $\{x \mid x \neq 2\}$.
The range is $\{y \mid y \neq 0\}$.

iii When $x = 0$, $y = \frac{3}{-2} = -\frac{3}{2}$

\therefore the y -intercept is $-\frac{3}{2}$.

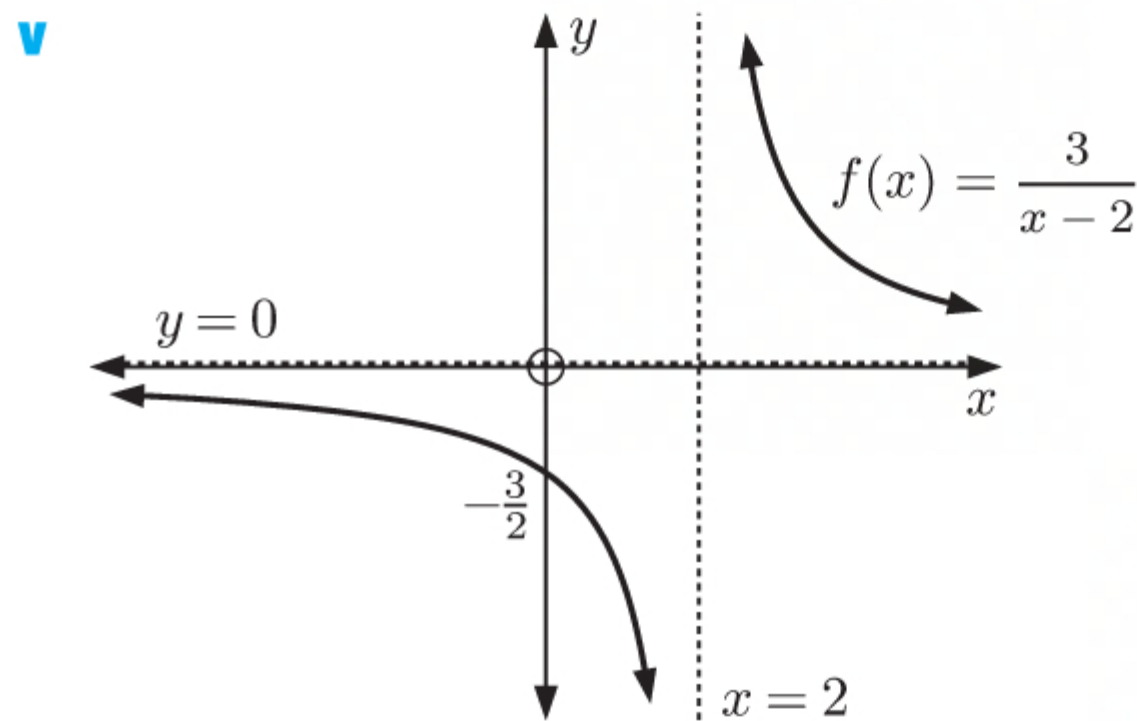
$y \neq 0$, so there is no x -intercept.

iv As $x \rightarrow 2^-$, $f(x) \rightarrow -\infty$

As $x \rightarrow 2^+$, $f(x) \rightarrow \infty$

As $x \rightarrow -\infty$, $f(x) \rightarrow 0^-$

As $x \rightarrow \infty$, $f(x) \rightarrow 0^+$



b $f : x \mapsto 2 + \frac{1}{x-3}$

i The vertical asymptote is $x = 3$.
The horizontal asymptote is $y = 2$.

ii The domain is $\{x \mid x \neq 3\}$.
The range is $\{y \mid y \neq 2\}$.

iii When $y = 0$, $2 + \frac{1}{x-3} = 0$
 $\therefore 2(x-3) + 1 = 0$
 $\therefore 2x - 6 + 1 = 0$
 $\therefore 2x = 5$
 $\therefore x = \frac{5}{2}$

When $x = 0$, $y = 2 + \frac{1}{-3} = \frac{5}{3}$

So, the x -intercept is $\frac{5}{2}$, and the y -intercept is $\frac{5}{3}$.

iv As $x \rightarrow 3^-$, $f(x) \rightarrow -\infty$
 As $x \rightarrow 3^+$, $f(x) \rightarrow \infty$
 As $x \rightarrow -\infty$, $f(x) \rightarrow 2^-$
 As $x \rightarrow \infty$, $f(x) \rightarrow 2^+$

c $f(x) = 2 - \frac{3}{x+1}$

i The vertical asymptote is $x = -1$.
The horizontal asymptote is $y = 2$.

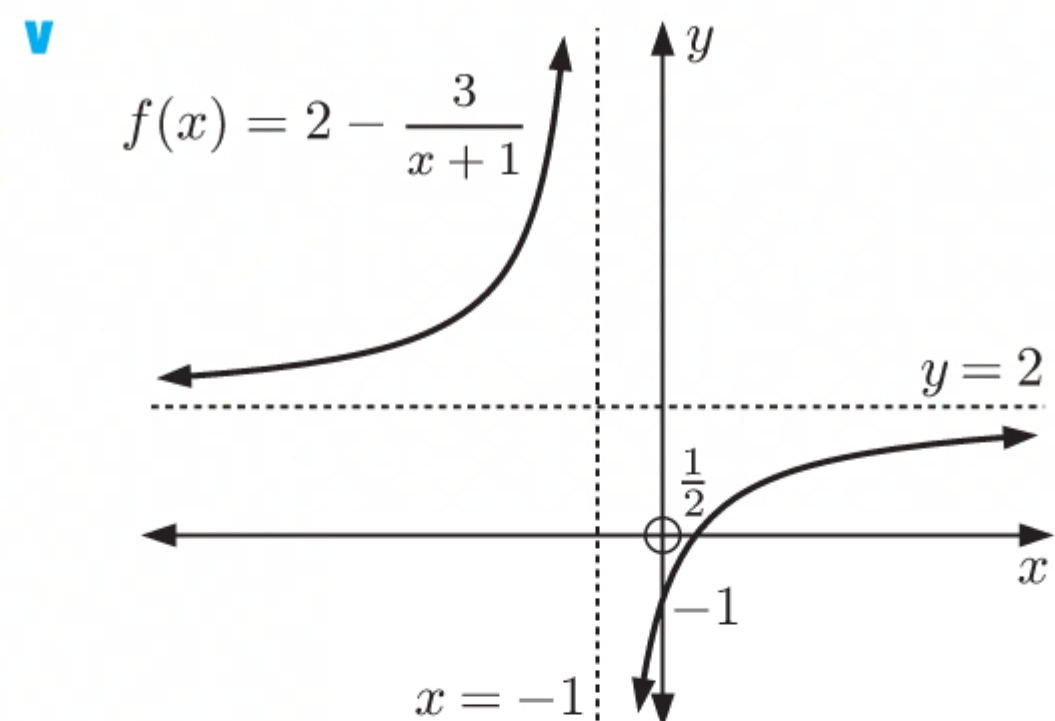
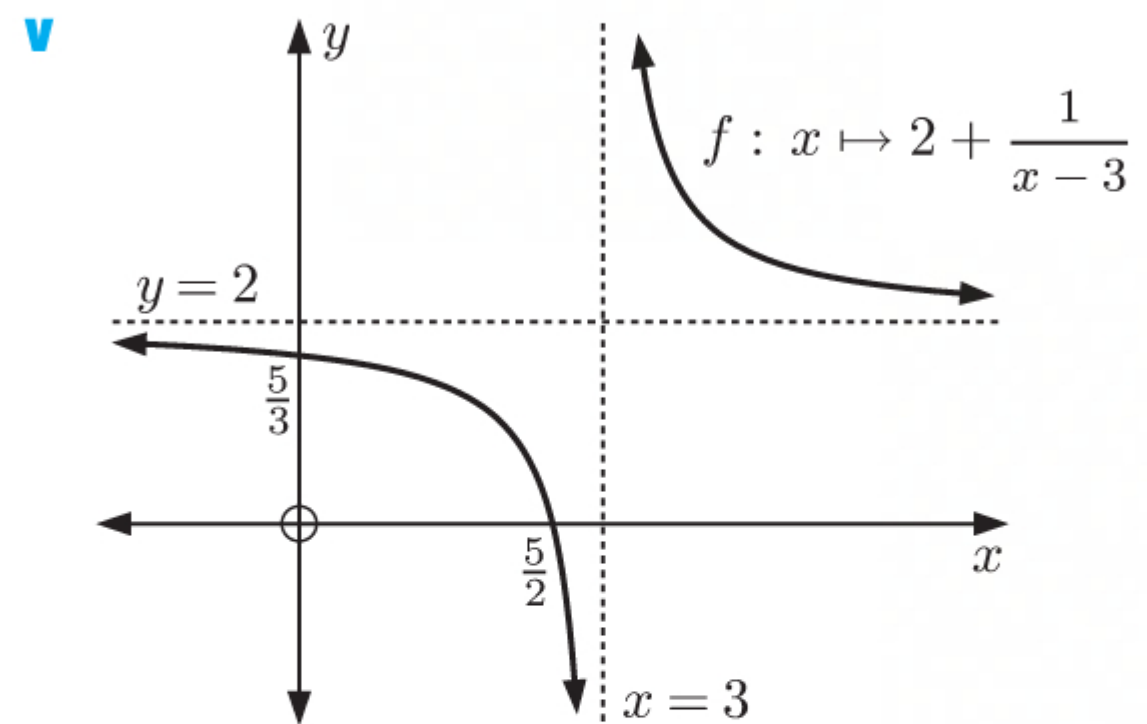
ii The domain is $\{x \mid x \neq -1\}$.
The range is $\{y \mid y \neq 2\}$.

iii When $y = 0$, $2 - \frac{3}{x+1} = 0$
 $\therefore 2(x+1) - 3 = 0$
 $\therefore 2x + 2 - 3 = 0$
 $\therefore 2x = 1$
 $\therefore x = \frac{1}{2}$

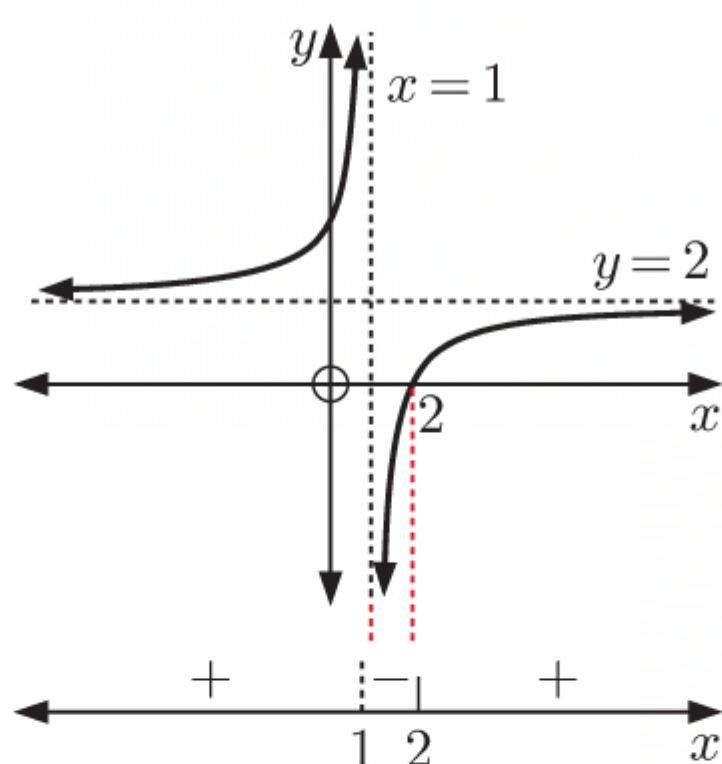
When $x = 0$, $y = 2 - \frac{3}{1} = -1$

So, the x -intercept is $\frac{1}{2}$, and the y -intercept is -1 .

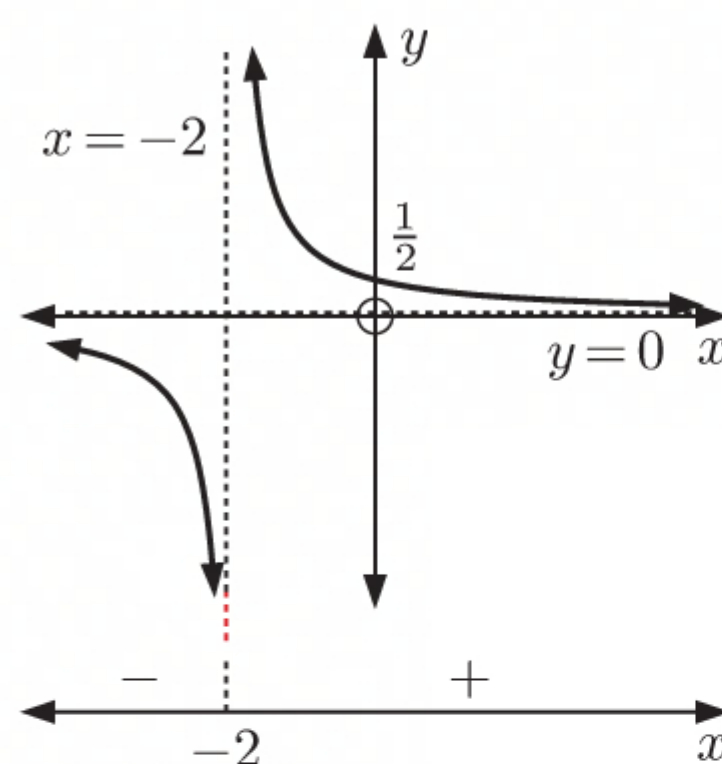
iv As $x \rightarrow -1^-$, $f(x) \rightarrow \infty$
 As $x \rightarrow -1^+$, $f(x) \rightarrow -\infty$
 As $x \rightarrow -\infty$, $f(x) \rightarrow 2^+$
 As $x \rightarrow \infty$, $f(x) \rightarrow 2^-$



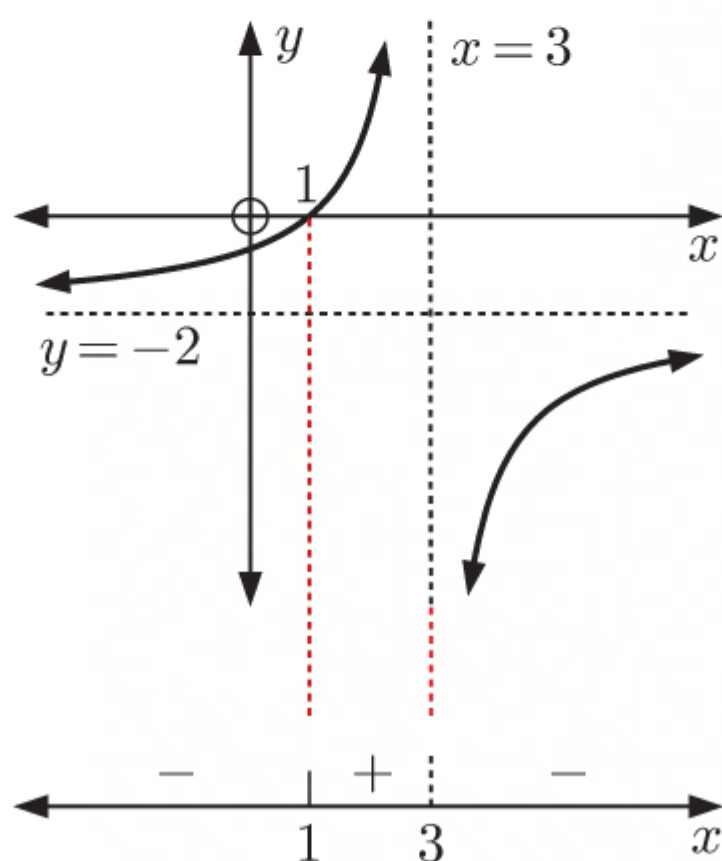
2 a



b



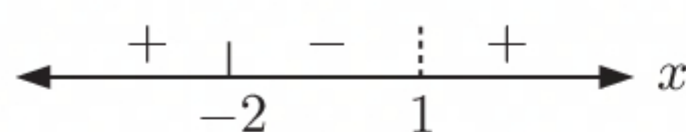
c



3 a $\frac{x+2}{x-1}$ is zero when $x = -2$ and undefined when $x = 1$.



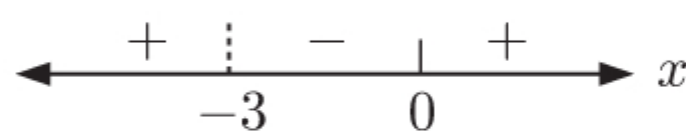
Since $(x+2)$ and $(x-1)$ are single factors, the signs alternate.



b $\frac{x}{x+3}$ is zero when $x = 0$ and undefined when $x = -3$.



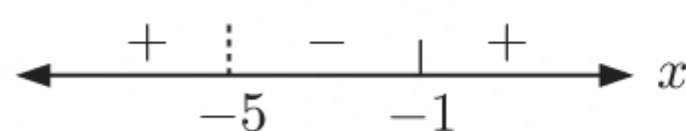
Since x and $(x+3)$ are single factors, the signs alternate.



c $\frac{x+1}{x+5}$ is zero when $x = -1$ and undefined when $x = -5$.



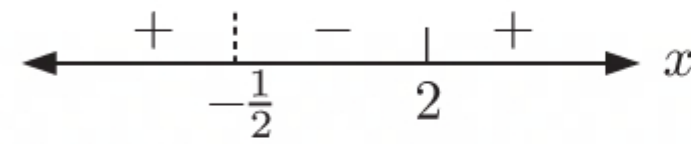
Since $(x+1)$ and $(x+5)$ are single factors, the signs alternate.



- d** $\frac{x-2}{2x+1}$ is zero when $x = 2$ and undefined when $x = -\frac{1}{2}$.



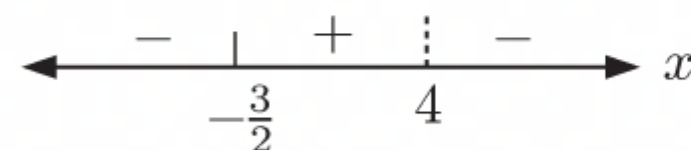
Since $(x-2)$ and $(2x+1)$ are single factors, the signs alternate.



- e** $\frac{2x+3}{4-x}$ is zero when $x = -\frac{3}{2}$ and undefined when $x = 4$.



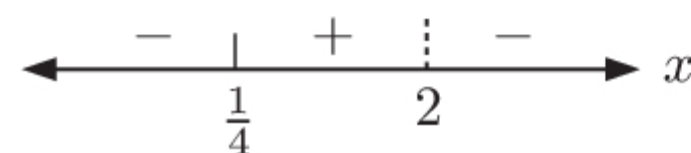
Since $(2x+3)$ and $(4-x)$ are single factors, the signs alternate.



- f** $\frac{4x-1}{2-x}$ is zero when $x = \frac{1}{4}$ and undefined when $x = 2$.



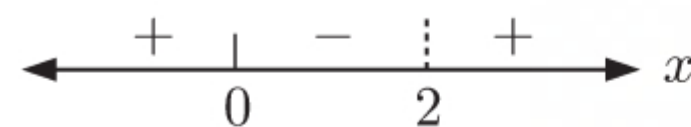
Since $(4x-1)$ and $(2-x)$ are single factors, the signs alternate.



- g** $\frac{3x}{x-2}$ is zero when $x = 0$ and undefined when $x = 2$.



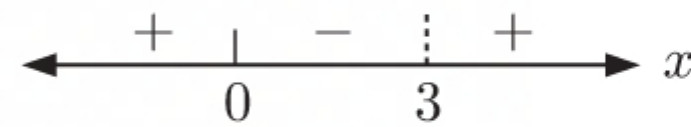
Since $3x$ and $(x-2)$ are single factors, the signs alternate.



- h** $\frac{-8x}{3-x}$ is zero when $x = 0$ and undefined when $x = 3$.



Since $-8x$ and $(3-x)$ are single factors, the signs alternate.



- 4 a** $f(x) = \frac{x}{x-1}$

i The vertical asymptote is $x = 1$.

ii $f(0) = \frac{0}{-1} = 0$, so the y -intercept is 0.

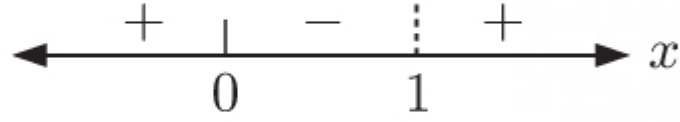
$f(x) = 0$ when $x = 0$

\therefore the x -intercept is 0.

$$\begin{aligned} \text{iii } f(x) &= \frac{x}{x-1} \\ &= \frac{(x-1)+1}{x-1} \\ &= 1 + \frac{1}{x-1} \end{aligned}$$

\therefore the horizontal asymptote is $y = 1$.

iv



v As $x \rightarrow 1^-$, $f(x) \rightarrow -\infty$

As $x \rightarrow 1^+$, $f(x) \rightarrow \infty$

As $x \rightarrow -\infty$, $f(x) \rightarrow 1^-$

As $x \rightarrow \infty$, $f(x) \rightarrow 1^+$

b $f : x \mapsto \frac{x+3}{x-2}$

i The vertical asymptote is $x = 2$.

ii $f(0) = \frac{3}{-2} = -\frac{3}{2}$, so the y -intercept is $-\frac{3}{2}$.

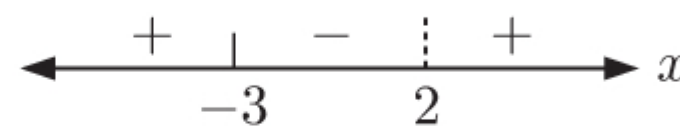
$$\begin{aligned} f(x) = 0 \text{ when } x+3 &= 0 \\ \therefore x &= -3 \end{aligned}$$

\therefore the x -intercept is -3 .

$$\begin{aligned} \text{iii } f(x) &= \frac{x+3}{x-2} \\ &= \frac{(x-2)+5}{x-2} \\ &= 1 + \frac{5}{x-2} \end{aligned}$$

\therefore the horizontal asymptote is $y = 1$.

iv



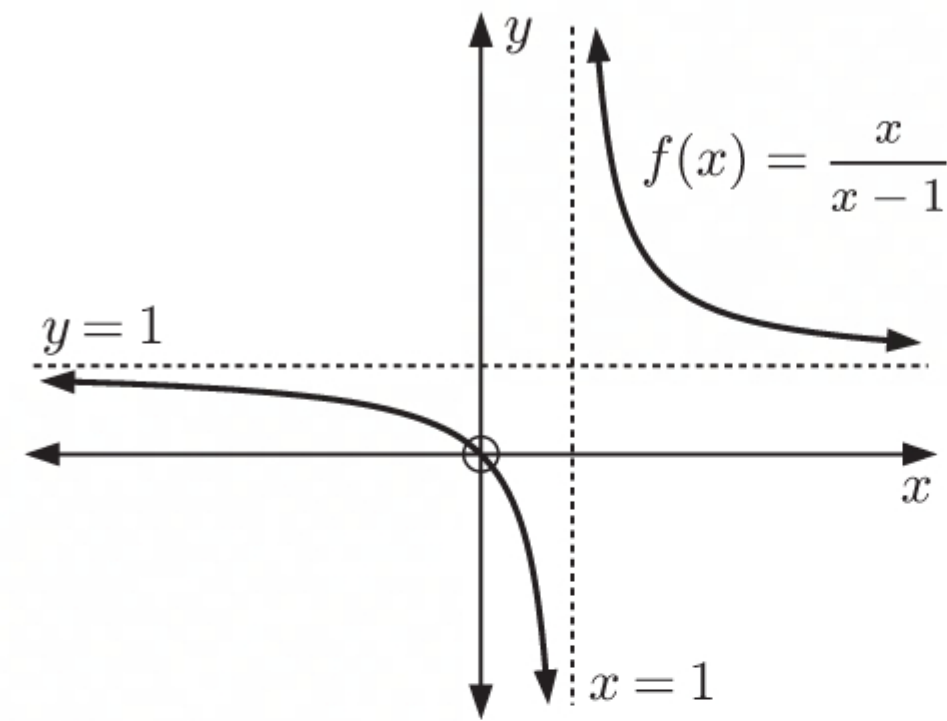
v As $x \rightarrow 2^-$, $f(x) \rightarrow -\infty$

As $x \rightarrow 2^+$, $f(x) \rightarrow \infty$

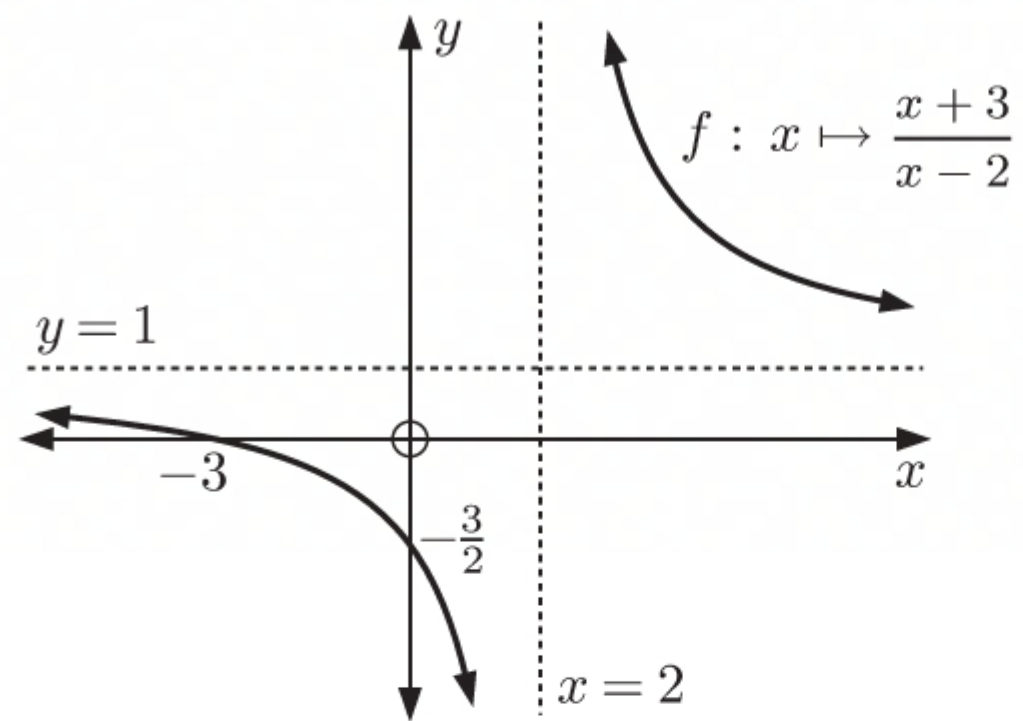
As $x \rightarrow -\infty$, $f(x) \rightarrow 1^-$

As $x \rightarrow \infty$, $f(x) \rightarrow 1^+$

vi



vi



c $f(x) = \frac{3x-1}{x+2}$

i The vertical asymptote is $x = -2$.

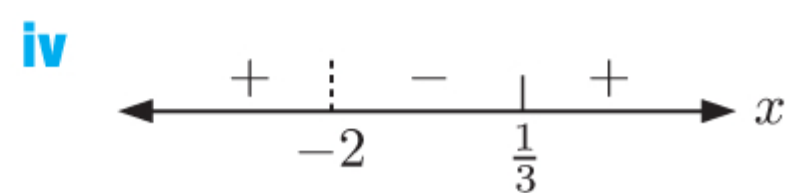
ii $f(0) = \frac{-1}{2} = -\frac{1}{2}$, so the y -intercept is $-\frac{1}{2}$.

$$\begin{aligned} f(x) = 0 \quad \text{when} \quad 3x - 1 &= 0 \\ \therefore 3x &= 1 \\ \therefore x &= \frac{1}{3} \end{aligned}$$

\therefore the x -intercept is $\frac{1}{3}$.

iii
$$\begin{aligned} f(x) &= \frac{3x-1}{x+2} \\ &= \frac{3(x+2)-7}{x+2} \\ &= 3 - \frac{7}{x+2} \end{aligned}$$

\therefore the horizontal asymptote is $y = 3$.



v As $x \rightarrow -2^-$, $f(x) \rightarrow \infty$

As $x \rightarrow -2^+$, $f(x) \rightarrow -\infty$

As $x \rightarrow -\infty$, $f(x) \rightarrow 3^+$

As $x \rightarrow \infty$, $f(x) \rightarrow 3^-$

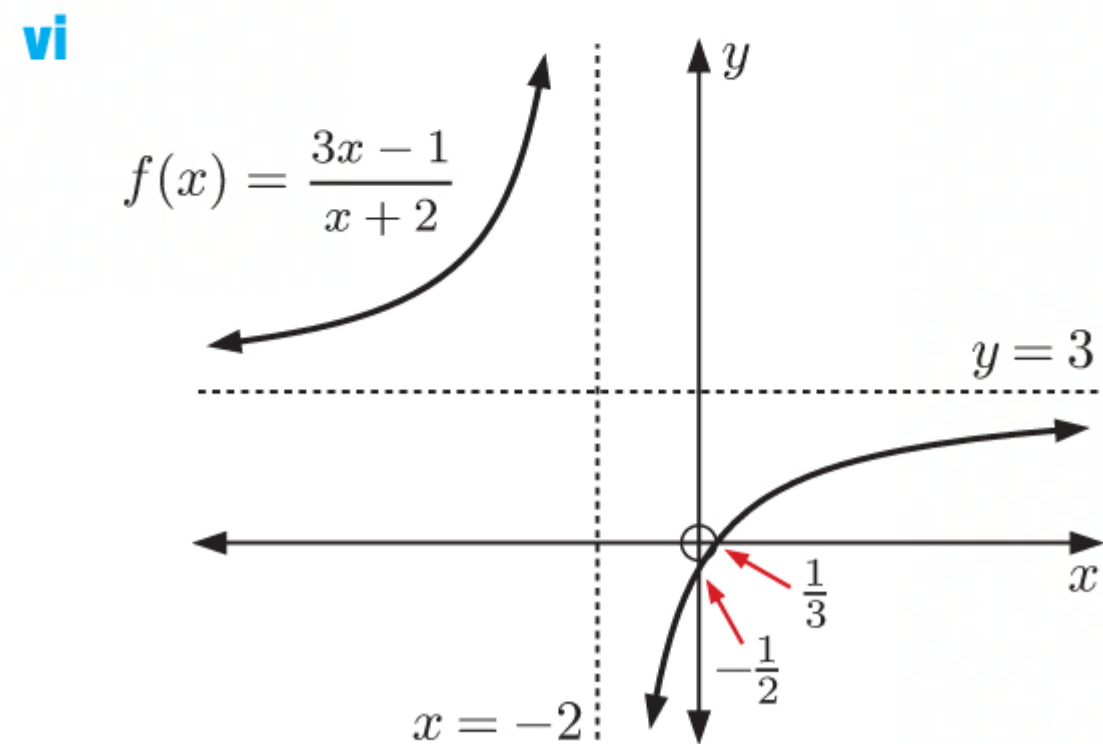
d $f(x) = -\frac{2x+1}{x-3}$

i The vertical asymptote is $x = 3$.

ii $f(0) = -\frac{1}{(-3)} = \frac{1}{3}$, so the y -intercept is $\frac{1}{3}$.

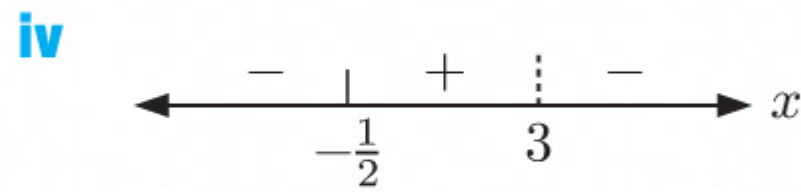
$$\begin{aligned} f(x) = 0 \quad \text{when} \quad 2x + 1 &= 0 \\ \therefore 2x &= -1 \\ \therefore x &= -\frac{1}{2} \end{aligned}$$

\therefore the x -intercept is $-\frac{1}{2}$.



$$\begin{aligned}
 \text{iii } f(x) &= -\frac{2x+1}{x-3} \\
 &= -\frac{2(x-3)+7}{x-3} \\
 &= -2 - \frac{7}{x-3}
 \end{aligned}$$

\therefore the horizontal asymptote is $y = -2$.



- v As $x \rightarrow 3^-$, $f(x) \rightarrow \infty$
 As $x \rightarrow 3^+$, $f(x) \rightarrow -\infty$
 As $x \rightarrow -\infty$, $f(x) \rightarrow -2^+$
 As $x \rightarrow \infty$, $f(x) \rightarrow -2^-$

e $f: x \mapsto \frac{2x+4}{3-x}$

- i The vertical asymptote has equation $x = 3$.

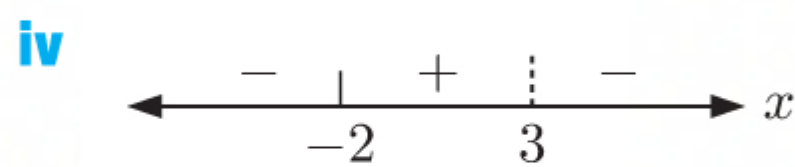
ii $f(0) = \frac{4}{3}$, so the y -intercept is $\frac{4}{3}$.

$$\begin{aligned}
 f(x) = 0 \quad \text{when} \quad 2x+4 &= 0 \\
 \therefore 2x &= -4 \\
 \therefore x &= -2
 \end{aligned}$$

\therefore the x -intercept is -2 .

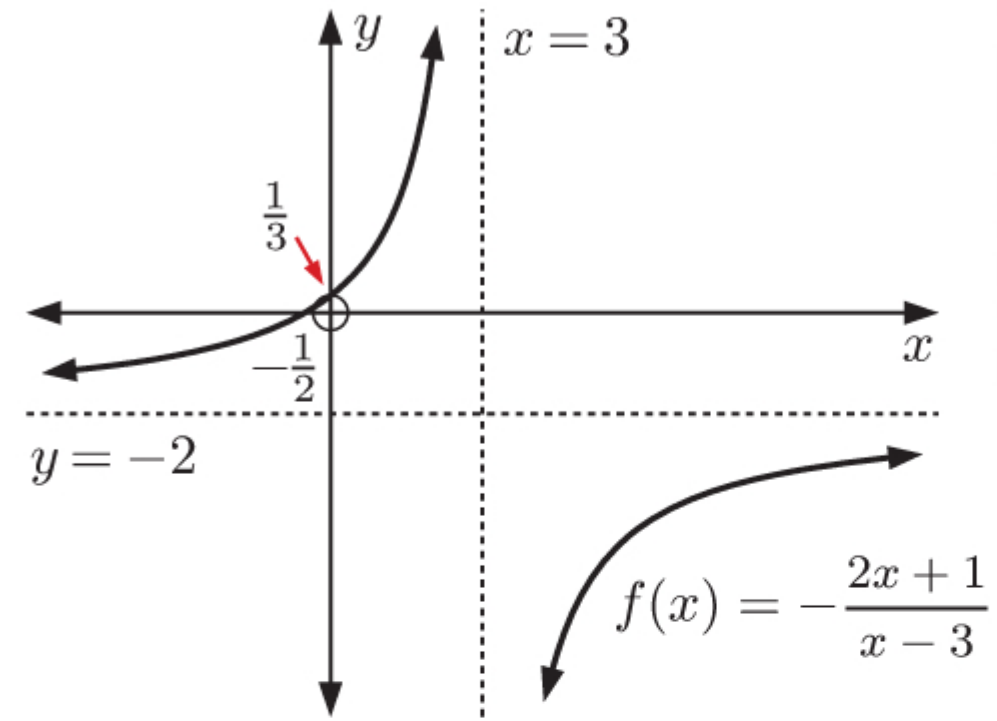
$$\begin{aligned}
 \text{iii } f(x) &= \frac{2x+4}{3-x} \\
 &= \frac{-2(3-x)+10}{3-x} \\
 &= -2 + \frac{10}{3-x}
 \end{aligned}$$

\therefore the horizontal asymptote is $y = -2$.

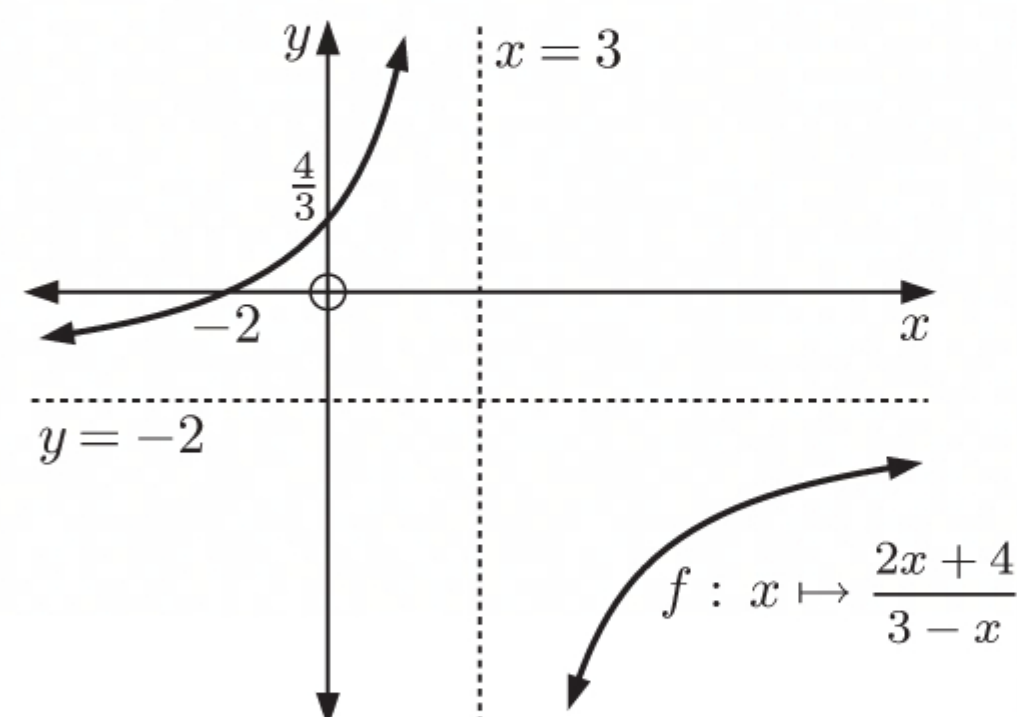


- v As $x \rightarrow 3^-$, $f(x) \rightarrow \infty$
 As $x \rightarrow 3^+$, $f(x) \rightarrow -\infty$
 As $x \rightarrow -\infty$, $f(x) \rightarrow -2^+$
 As $x \rightarrow \infty$, $f(x) \rightarrow -2^-$

vi



vi



f $f(x) = \frac{x+3}{2x-1}$

i The vertical asymptote has equation $x = \frac{1}{2}$.

ii $f(0) = \frac{3}{-1} = -3$, so the y -intercept is -3 .

$$f(x) = 0 \text{ when } x+3 = 0$$

$$\therefore x = -3$$

\therefore the x -intercept is -3 .

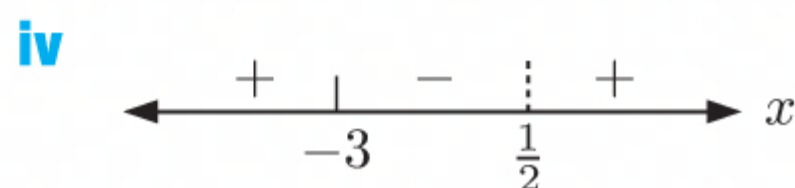
iii $f(x) = \frac{x+3}{2x-1}$

$$= \frac{\frac{1}{2}(2x-1) + \frac{7}{2}}{2x-1}$$

$$= \frac{1}{2} + \frac{7}{2(2x-1)}$$

$$= \frac{1}{2} + \frac{7}{4x-2}$$

\therefore the horizontal asymptote is $y = \frac{1}{2}$.

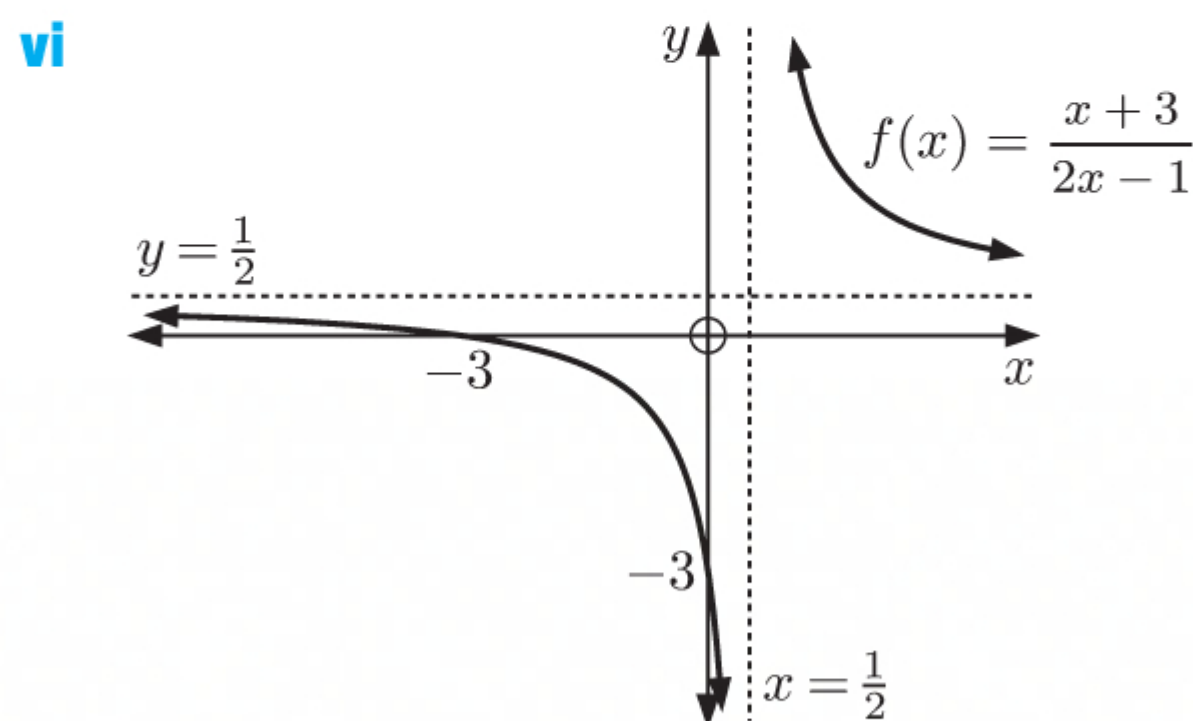


v As $x \rightarrow \frac{1}{2}^-$, $f(x) \rightarrow -\infty$

As $x \rightarrow \frac{1}{2}^+$, $f(x) \rightarrow \infty$

As $x \rightarrow -\infty$, $f(x) \rightarrow \frac{1}{2}^-$

As $x \rightarrow \infty$, $f(x) \rightarrow \frac{1}{2}^+$



5 $y = \frac{ax+b}{cx+d}$ where a , b , c , and d are constants and $c \neq 0$.

a $\frac{ax+b}{cx+d}$ is undefined when $cx+d = 0$

$$\therefore x = -\frac{d}{c}$$

\therefore the domain is $\{x \mid x \neq -\frac{d}{c}\}$.

b The vertical asymptote is $x = -\frac{d}{c}$.

c When $x = 0$, $y = \frac{b}{d}$, $d \neq 0$

\therefore the y -intercept is $\frac{b}{d}$, $d \neq 0$.

When $y = 0$, $ax+b = 0$

$$\therefore x = -\frac{b}{a}, a \neq 0$$

\therefore the x -intercept is $-\frac{b}{a}$, $a \neq 0$.

$$\begin{aligned} \text{d } \frac{ax+b}{cx+d} &= \frac{\frac{a}{c}(cx+d) + b - \frac{ad}{c}}{cx+d} \\ &= \frac{a}{c} + \frac{b - \frac{ad}{c}}{cx+d} \end{aligned}$$

$$\text{As } |x| \rightarrow \infty, \quad \frac{a}{c} + \frac{b - \frac{ad}{c}}{cx+d} \rightarrow \frac{a}{c} + 0 = \frac{a}{c}$$

\therefore the horizontal asymptote is $y = \frac{a}{c}$.

EXERCISE 3E

1 $f(x) = 2x + 3$ and $g(x) = 1 - x$

$$\begin{aligned} \text{a } (f \circ g)(x) &= f(g(x)) \\ &= f(1 - x) \\ &= 2(1 - x) + 3 \\ &= 2 - 2x + 3 \\ &= 5 - 2x \end{aligned}$$

$$\begin{aligned} \text{c } (f \circ g)(-3) &= 5 - 2(-3) \quad \{\text{using a}\} \\ &= 11 \end{aligned}$$

$$\begin{aligned} \text{b } (g \circ f)(x) &= g(f(x)) \\ &= g(2x + 3) \\ &= 1 - (2x + 3) \\ &= 1 - 2x - 3 \\ &= -2x - 2 \end{aligned}$$

$$\begin{aligned} \text{d } (g \circ f)(0) &= -2(0) - 2 \quad \{\text{using b}\} \\ &= -2 \end{aligned}$$

2 $f(x) = -2x$ and $g(x) = 1 + x^2$

$$\begin{aligned} \text{a } (f \circ g)(x) &= f(g(x)) \\ &= f(1 + x^2) \\ &= -2(1 + x^2) \\ &= -2 - 2x^2 \end{aligned}$$

$$\begin{aligned} \text{c } (f \circ g)(2) &= -2 - 2(2)^2 \quad \{\text{using a}\} \\ &= -10 \end{aligned}$$

$$\begin{aligned} \text{b } (g \circ f)(x) &= g(f(x)) \\ &= g(-2x) \\ &= 1 + (-2x)^2 \\ &= 1 + 4x^2 \end{aligned}$$

$$\begin{aligned} \text{d } (f \circ f)(-1) &= f(f(-1)) \\ &= f(-2(-1)) \\ &= f(2) \\ &= -2(2) \\ &= -4 \end{aligned}$$

3 $f(x) = 3 - x^2$ and $g(x) = 2x + 4$

$$\begin{aligned} \text{a } (f \circ g)(x) &= f(g(x)) \\ &= f(2x + 4) \\ &= 3 - (2x + 4)^2 \\ &= 3 - 4x^2 - 16x - 16 \\ &= -4x^2 - 16x - 13 \end{aligned}$$

$$\begin{aligned} \text{b } (g \circ f)(x) &= g(f(x)) \\ &= g(3 - x^2) \\ &= 2(3 - x^2) + 4 \\ &= 6 - 2x^2 + 4 \\ &= 10 - 2x^2 \end{aligned}$$

$$\begin{aligned}
 \text{c } (g \circ g)\left(\frac{1}{2}\right) &= g\left(g\left(\frac{1}{2}\right)\right) \\
 &= g\left(2\left(\frac{1}{2}\right) + 4\right) \\
 &= g(5) \\
 &= 2(5) + 4 \\
 &= 14
 \end{aligned}$$

$$\begin{aligned}
 \text{d } (f \circ f)\left(-\frac{1}{2}\right) &= f\left(f\left(-\frac{1}{2}\right)\right) \\
 &= f\left(3 - \left(-\frac{1}{2}\right)^2\right) \\
 &= f\left(\frac{11}{4}\right) \\
 &= 3 - \left(\frac{11}{4}\right)^2 \\
 &= 3 - \frac{121}{16} \\
 &= -\frac{73}{16}
 \end{aligned}$$

$$4 \quad f(x) = \sqrt{6-x} \quad \text{and} \quad g(x) = 5x - 7$$

$$\begin{aligned}
 \text{a } (g \circ g)(x) &= g(g(x)) \\
 &= g(5x - 7) \\
 &= 5(5x - 7) - 7 \\
 &= 25x - 35 - 7 \\
 &= 25x - 42
 \end{aligned}$$

$$\begin{aligned}
 \text{b } (f \circ g)(1) &= f(g(1)) \\
 &= f(5(1) - 7) \\
 &= f(-2) \\
 &= \sqrt{6 - (-2)} \\
 &= \sqrt{8}
 \end{aligned}$$

$$\begin{aligned}
 \text{c } (g \circ f)(6) &= g(f(6)) \\
 &= g(\sqrt{6-6}) \\
 &= g(0) \\
 &= 5(0) - 7 \\
 &= -7
 \end{aligned}$$

$$\begin{aligned}
 \text{d } (f \circ f)(2) &= f(f(2)) \\
 &= f(\sqrt{6-2}) \\
 &= f(2) \\
 &= \sqrt{6-2} \\
 &= 2
 \end{aligned}$$

$$5 \quad f(x) = x^2 + 1 \quad \text{and} \quad g(x) = 3 - x$$

$$\begin{aligned}
 \text{a } \quad \text{i } (f \circ g)(x) &= f(g(x)) \\
 &= f(3 - x) \\
 &= (3 - x)^2 + 1 \\
 &= 9 - 6x + x^2 + 1 \\
 &= x^2 - 6x + 10
 \end{aligned}$$

$$\begin{aligned}
 \text{ii } (g \circ f)(x) &= g(f(x)) \\
 &= g(x^2 + 1) \\
 &= 3 - (x^2 + 1) \\
 &= 3 - x^2 - 1 \\
 &= 2 - x^2
 \end{aligned}$$

$$\begin{aligned}
 \text{b } (g \circ f)(x) &= f(x) \\
 \therefore 2 - x^2 &= x^2 + 1 \\
 \therefore 2x^2 &= 1 \\
 \therefore x^2 &= \frac{1}{2} \\
 \therefore x &= \pm \frac{1}{\sqrt{2}}
 \end{aligned}$$

6 $f(x) = 9 - \sqrt{x}$ and $g(x) = x^2 + 4$

a $(f \circ g)(x) = f(g(x))$
 $= f(x^2 + 4)$
 $= 9 - \sqrt{x^2 + 4}$

$x^2 + 4 \geq 0$, so $\sqrt{x^2 + 4}$ is defined for every value of x .

\therefore domain is $\{x \mid x \in \mathbb{R}\}$

$$-\sqrt{x^2 + 4} \leq -2$$

$$\therefore 9 - \sqrt{x^2 + 4} \leq 7$$

\therefore range is $\{y \mid y \leq 7\}$

c $(f \circ f)(x) = f(f(x))$
 $= f(9 - \sqrt{x})$
 $= 9 - \sqrt{9 - \sqrt{x}}$

\sqrt{x} is defined when $x \geq 0$,

and $\sqrt{9 - \sqrt{x}}$ is defined when $\sqrt{x} \leq 9$
 $\therefore x \leq 81$

\therefore domain is $\{x \mid 0 \leq x \leq 81\}$

$$-3 \leq -\sqrt{9 - \sqrt{x}} \leq 0$$

$$\therefore 6 \leq 9 - \sqrt{9 - \sqrt{x}} \leq 9$$

\therefore range is $\{y \mid 6 \leq y \leq 9\}$

7 $f(x) = 1 - 2x$ and $g(x) = 3x + 5$

a $f(g(x)) = f(3x + 5)$
 $= 1 - 2(3x + 5)$
 $= 1 - 6x - 10$
 $= -6x - 9$

b $(f \circ g)(x) = f(x + 3)$
 $\therefore f(g(x)) = 1 - 2(x + 3)$
 $\therefore -6x - 9 = 1 - 2x - 6$
 $\therefore -4x = 4$
 $\therefore x = -1$

8 $f : x \mapsto 2x - x^2$ and $g : x \mapsto 1 + 3x$

a i $(f \circ g)(x) = f(g(x))$
 $= f(1 + 3x)$
 $= 2(1 + 3x) - (1 + 3x)^2$
 $= 2 + 6x - 1 - 6x - 9x^2$
 $= 1 - 9x^2$

ii $(g \circ f)(x) = g(f(x))$
 $= g(2x - x^2)$
 $= 1 + 3(2x - x^2)$
 $= 1 + 6x - 3x^2$

b $(f \circ g)(x) = 3(g \circ f)(x)$
 $\therefore 1 - 9x^2 = 3(1 + 6x - 3x^2)$
 $\therefore 1 - 9x^2 = 3 + 18x - 9x^2$
 $\therefore -2 = 18x$
 $\therefore x = -\frac{1}{9}$

9 a $f(x) = \frac{1}{x}$ and $g(x) = x - 3$

$$\begin{aligned}\therefore (f \circ g)(x) &= f(g(x)) \\ &= f(x - 3) \\ &= \frac{1}{x - 3}\end{aligned}$$

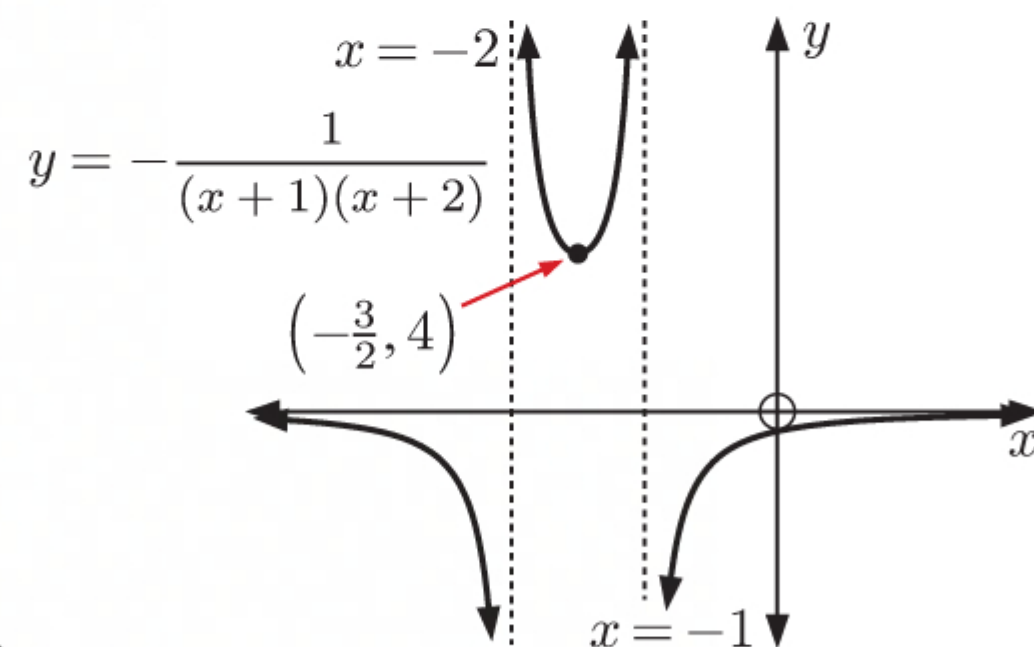
\therefore domain is $\{x \mid x \neq 3\}$ and range is $\{y \mid y \neq 0\}$.

b $f(x) = -\frac{1}{x}$ and $g(x) = x^2 + 3x + 2$

$$\begin{aligned}\therefore (f \circ g)(x) &= f(g(x)) \\ &= f(x^2 + 3x + 2) \\ &= -\frac{1}{x^2 + 3x + 2} \\ &= -\frac{1}{(x + 1)(x + 2)}\end{aligned}$$

\therefore domain is $\{x \mid x \neq -1 \text{ or } -2\}$

From the graph, the range is $\{y \mid y \geq 4 \text{ or } y < 0\}$.



10 a $ax + b = cx + d$ is true for all x

When $x = 0$, $a(0) + b = c(0) + d$
 $\therefore b = d \dots (*)$

When $x = 1$, $a(1) + b = c(1) + d$

$$\begin{aligned}\therefore a + b &= c + d \\ \therefore a + d &= c + d \quad \{\text{using } (*)\} \\ \therefore a &= c\end{aligned}$$

b $f(x) = 2x + 3$ and $g(x) = ax + b$

$$\begin{aligned}(f \circ g)(x) &= x \text{ for all } x \\ \therefore f(g(x)) &= x \\ \therefore f(ax + b) &= x \\ \therefore 2(ax + b) + 3 &= x \\ \therefore 2ax + (2b + 3) &= x \\ \therefore 2a &= 1 \text{ and } 2b + 3 = 0 \quad \{\text{using a}\} \\ \therefore a &= \frac{1}{2} \text{ and } 2b = -3 \\ \text{So, } a &= \frac{1}{2} \text{ and } b = -\frac{3}{2} \text{ as required.}\end{aligned}$$

c If $(g \circ f)(x) = x$ for all x
 then $g(f(x)) = x$
 $\therefore g(2x + 3) = x$
 $\therefore a(2x + 3) + b = x$
 $\therefore 2ax + (3a + b) = x$
 $\therefore 2a = 1 \text{ and } 3a + b = 0 \quad \{\text{using a}\}$
 $\therefore a = \frac{1}{2} \text{ and } b = -3a$
 So, $a = \frac{1}{2} \text{ and } b = -\frac{3}{2}$
 \therefore the result in **b** is also true if
 $(g \circ f)(x) = x$ for all x .

11 $f(x) = \sqrt{1 - x}$ and $g(x) = x^2$

a $(f \circ g)(x) = f(g(x))$ **b** $(f \circ g)(x) = \sqrt{1 - x^2}$ is defined when $1 - x^2 \geq 0$

$$\begin{aligned}&= f(x^2) \\ &= \sqrt{1 - x^2}\end{aligned}$$

$$\begin{aligned}\therefore x^2 &\leq 1 \\ \therefore -1 &\leq x \leq 1\end{aligned}$$

\therefore the domain is $\{x \mid -1 \leq x \leq 1\}$

$y = (f \circ g)(x)$ is always positive and ≤ 1 as $-1 \leq x \leq 1$.

\therefore the range is $\{y \mid 0 \leq y \leq 1\}$.

12 a Since $(f \circ g)(x) = f(g(x))$, x will only be in the domain of $(f \circ g)$ if:

- it is in the domain of g , so $g(x)$ exists, and
- the value $g(x)$ is in the domain of f , so $f(g(x))$ exists.

Now $g(x)$ is in the range of g , so $(f \circ g)(x)$ will only be defined when $R_g \cap D_f \neq \emptyset$.

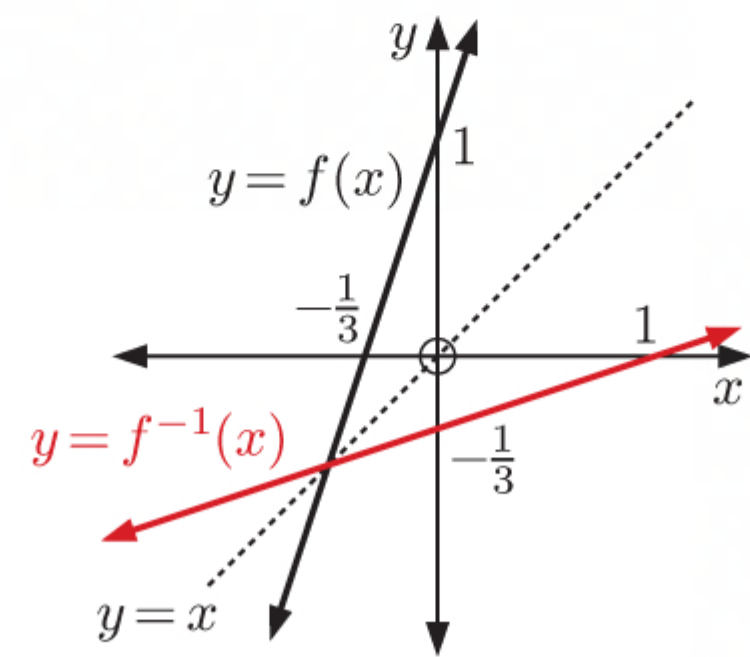
b The domain of $(f \circ g)(x)$ is $\{x \mid x \in D_g \text{ and } g(x) \in D_f\}$.

EXERCISE 3F

1 a $f : x \mapsto 3x + 1$

i $f(x) = 3x + 1$ passes through $(0, 1)$ and $(-\frac{1}{3}, 0)$.

$\therefore f^{-1}(x)$ passes through $(1, 0)$ and $(0, -\frac{1}{3})$.



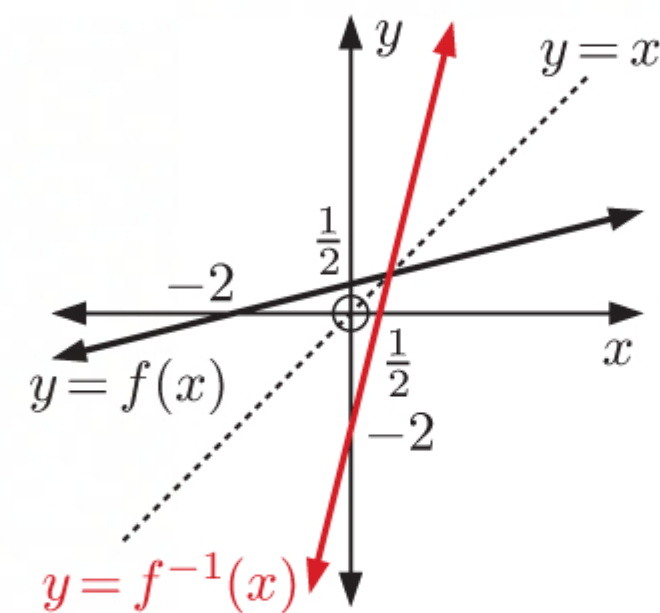
ii $y = f^{-1}(x)$ has gradient $\frac{-\frac{1}{3} - 0}{0 - 1} = \frac{1}{3}$
 Its equation is $\frac{y - 0}{x - 1} = \frac{1}{3}$
 $\therefore y = \frac{x - 1}{3}$
 $\therefore f^{-1}(x) = \frac{x - 1}{3}$

iii f is $y = 3x + 1$,
 $\therefore f^{-1}$ is $x = 3y + 1$
 $\therefore x - 1 = 3y$
 $\therefore \frac{x - 1}{3} = y$
 $\therefore f^{-1}(x) = \frac{x - 1}{3}$

b $f : x \mapsto \frac{x + 2}{4}$

i $f(x) = \frac{x + 2}{4}$ passes through $(0, \frac{1}{2})$ and $(-2, 0)$.

$\therefore f^{-1}(x)$ passes through $(\frac{1}{2}, 0)$ and $(0, -2)$.



ii $y = f^{-1}(x)$ has gradient $\frac{-2 - 0}{0 - \frac{1}{2}} = 4$
 Its equation is $\frac{y - 0}{x - \frac{1}{2}} = 4$
 $\therefore y = 4x - 2$
 $\therefore f^{-1}(x) = 4x - 2$

iii f is $y = \frac{x + 2}{4}$,
 $\therefore f^{-1}$ is $x = \frac{y + 2}{4}$
 $\therefore 4x = y + 2$
 $\therefore 4x - 2 = y$
 $\therefore f^{-1}(x) = 4x - 2$

2 a $f : x \mapsto 2x + 5$

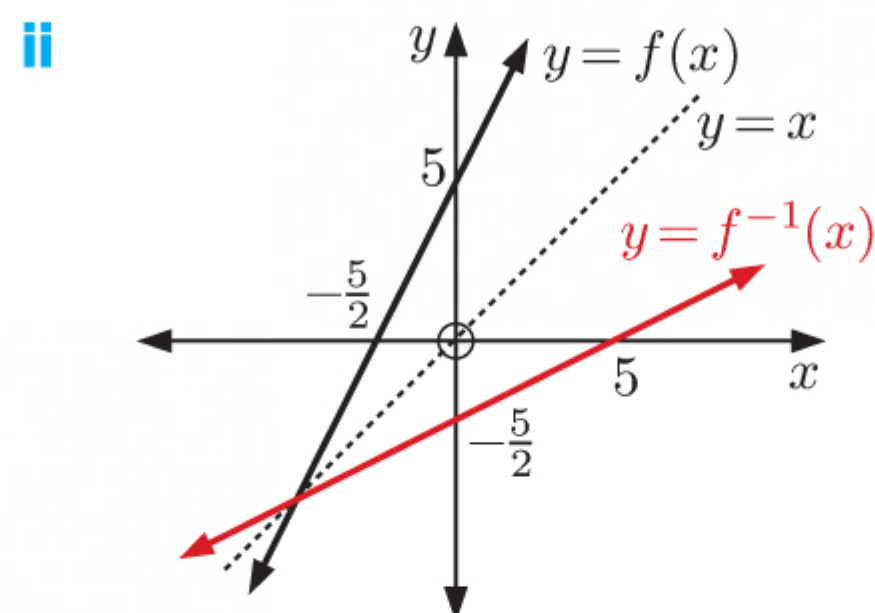
i f is $y = 2x + 5$

$\therefore f^{-1}$ is $x = 2y + 5$

$\therefore x - 5 = 2y$

$\therefore \frac{x - 5}{2} = y$

$\therefore f^{-1}(x) = \frac{x - 5}{2}$



iii $(f^{-1} \circ f)(x) = f^{-1}(f(x))$
 $= f^{-1}(2x + 5)$
 $= \frac{2x + 5 - 5}{2}$
 $= \frac{2x}{2}$
 $= x$

$(f \circ f^{-1})(x) = f(f^{-1}(x))$
 $= f\left(\frac{x - 5}{2}\right)$
 $= 2\left(\frac{x - 5}{2}\right) + 5$
 $= x - 5 + 5$
 $= x$

$\therefore (f^{-1} \circ f)(x) = (f \circ f^{-1})(x) = x$ as required

b $f : x \mapsto \frac{3 - 2x}{4}$

i f is $y = \frac{3 - 2x}{4}$

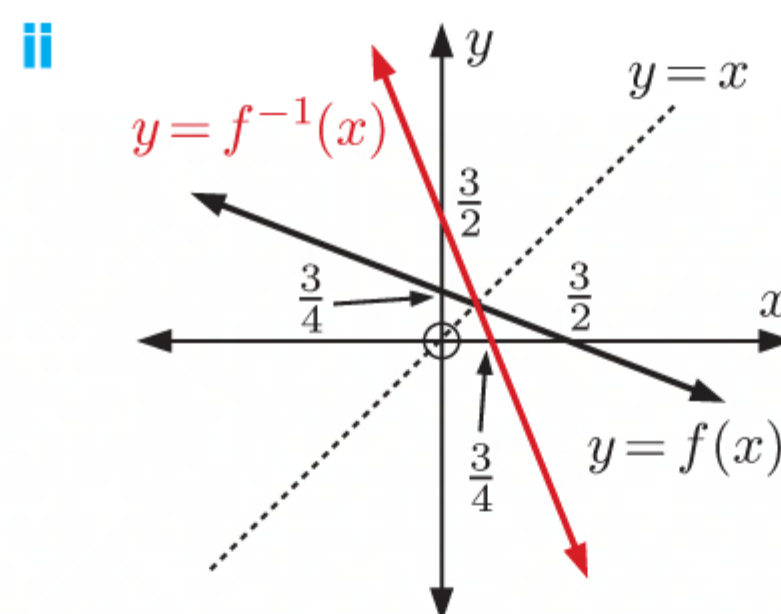
$\therefore f^{-1}$ is $x = \frac{3 - 2y}{4}$

$\therefore 4x = 3 - 2y$

$\therefore 4x - 3 = -2y$

$\therefore -2x + \frac{3}{2} = y$

$\therefore f^{-1}(x) = -2x + \frac{3}{2}$



iii $(f^{-1} \circ f)(x) = f^{-1}(f(x))$
 $= f^{-1}\left(\frac{3 - 2x}{4}\right)$
 $= -2\left(\frac{3 - 2x}{4}\right) + \frac{3}{2}$
 $= -\frac{3}{2} + x + \frac{3}{2}$
 $= x$

$(f \circ f^{-1})(x) = f(f^{-1}(x))$
 $= f\left(-2x + \frac{3}{2}\right)$
 $= \frac{3 - 2(-2x + \frac{3}{2})}{4}$
 $= \frac{3 + 4x - 3}{4}$
 $= \frac{4x}{4}$
 $= x$

$\therefore (f^{-1} \circ f)(x) = (f \circ f^{-1})(x) = x$ as required

c $f : x \mapsto x + 3$

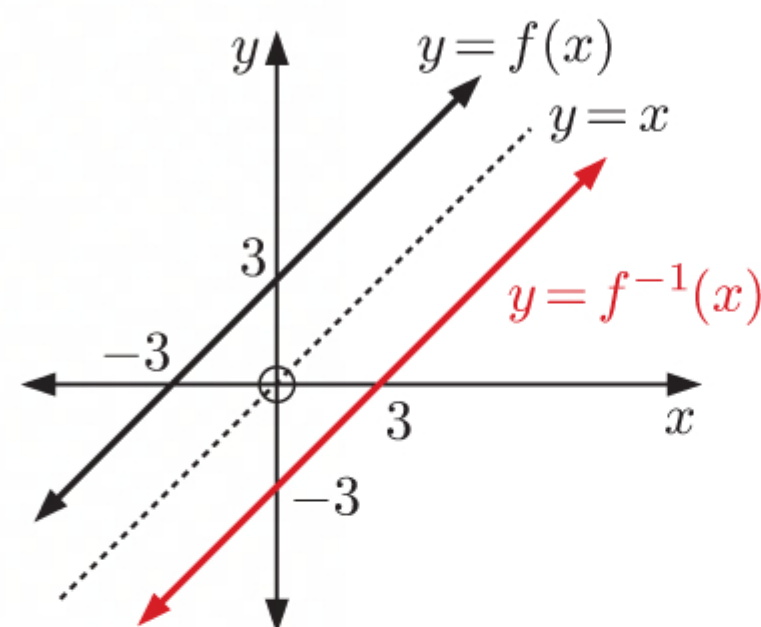
i f is $y = x + 3$

$\therefore f^{-1}$ is $x = y + 3$

$\therefore x - 3 = y$

$\therefore f^{-1}(x) = x - 3$

ii



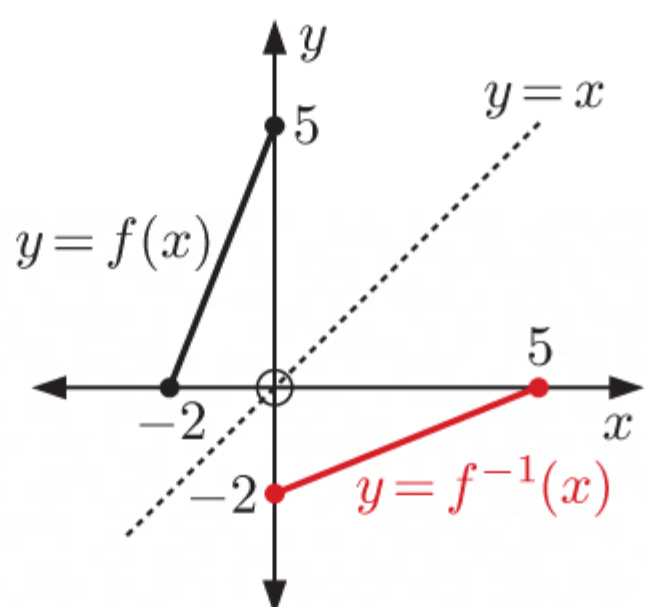
iii $(f^{-1} \circ f)(x) = f^{-1}(f(x))$
 $= f^{-1}(x + 3)$
 $= x + 3 - 3$
 $= x$

$(f \circ f^{-1})(x) = f(f^{-1}(x))$
 $= f(x - 3)$
 $= x - 3 + 3$
 $= x$

$\therefore (f^{-1} \circ f)(x) = (f \circ f^{-1})(x) = x$ as required

3 a $f(x)$ passes through $(0, 5)$ and $(-2, 0)$.

$\therefore f^{-1}(x)$ passes through $(5, 0)$ and $(0, -2)$.



f : Domain is $\{x \mid -2 \leq x \leq 0\}$

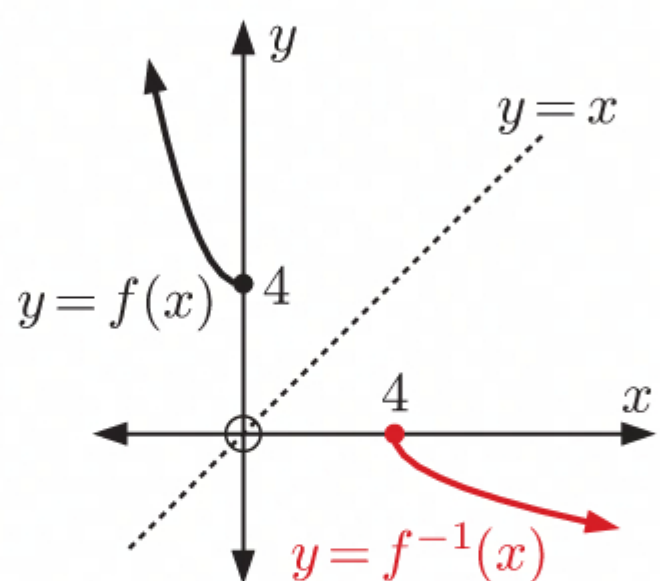
Range is $\{y \mid 0 \leq y \leq 5\}$

f^{-1} : Domain is $\{x \mid 0 \leq x \leq 5\}$

Range is $\{y \mid -2 \leq y \leq 0\}$

b $f(x)$ passes through $(0, 4)$.

$\therefore f^{-1}(x)$ passes through $(4, 0)$.



f : Domain is $\{x \mid x \leq 0\}$

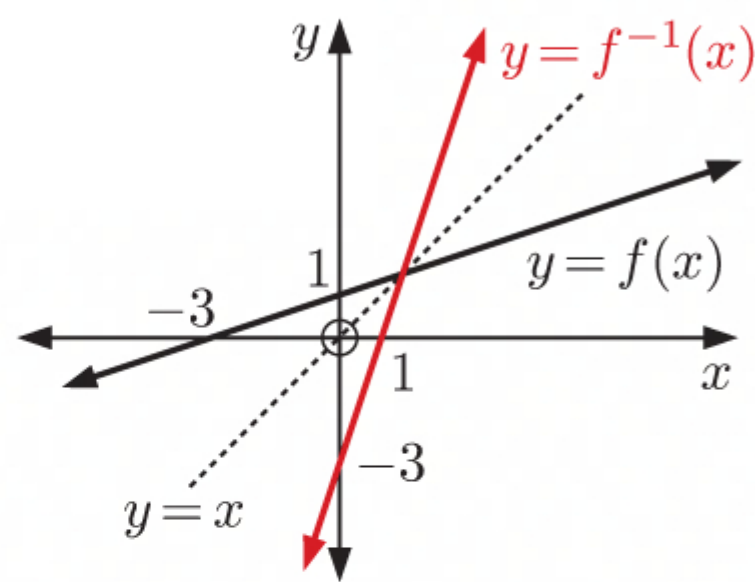
Range is $\{y \mid y \geq 4\}$

f^{-1} : Domain is $\{x \mid x \geq 4\}$

Range is $\{y \mid y \leq 0\}$

- c $f(x)$ passes through $(0, 1)$ and $(-3, 0)$.

$\therefore f^{-1}(x)$ passes through $(1, 0)$ and $(0, -3)$.



f : Domain is $\{x \mid x \in \mathbb{R}\}$

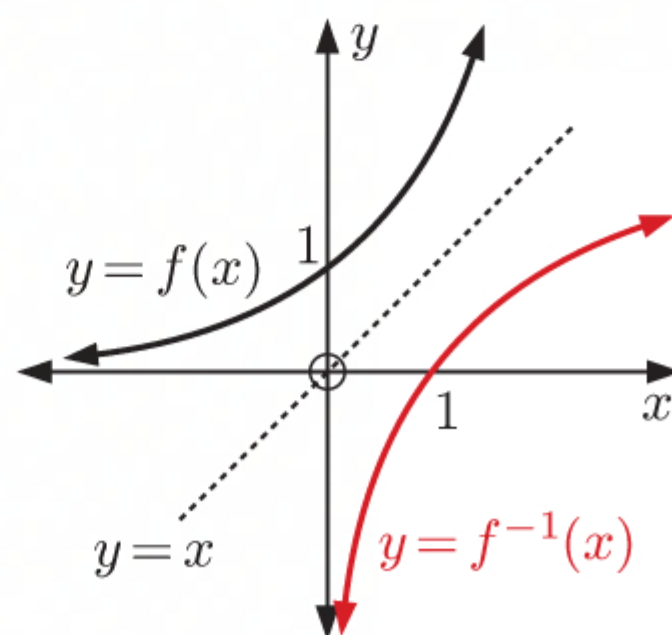
Range is $\{y \mid y \in \mathbb{R}\}$

f^{-1} : Domain is $\{x \mid x \in \mathbb{R}\}$

Range is $\{y \mid y \in \mathbb{R}\}$

- d $f(x)$ passes through $(0, 1)$.

$\therefore f^{-1}(x)$ passes through $(1, 0)$.



f : Domain is $\{x \mid x \in \mathbb{R}\}$

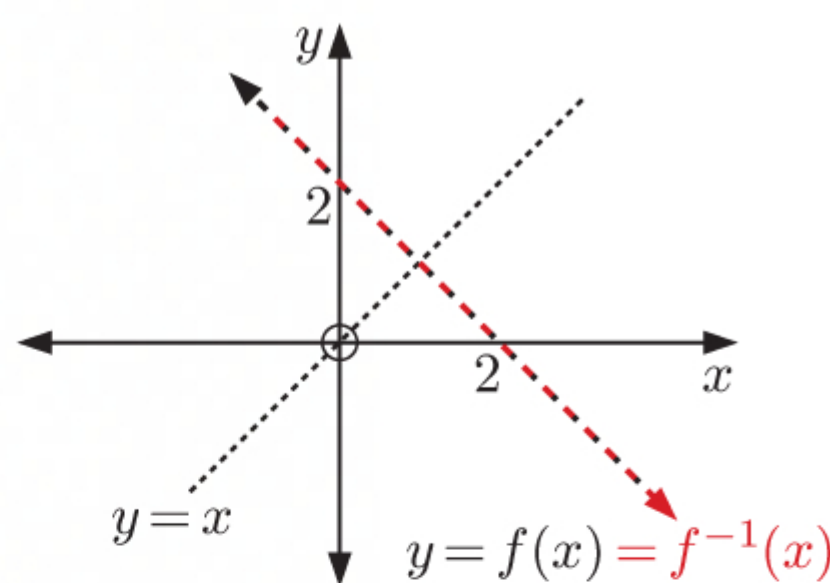
Range is $\{y \mid y > 0\}$

f^{-1} : Domain is $\{x \mid x > 0\}$

Range is $\{y \mid y \in \mathbb{R}\}$

- e $f(x)$ passes through $(0, 2)$ and $(2, 0)$.

$\therefore f^{-1}(x)$ passes through $(2, 0)$ and $(0, 2)$.



f : Domain is $\{x \mid x \in \mathbb{R}\}$

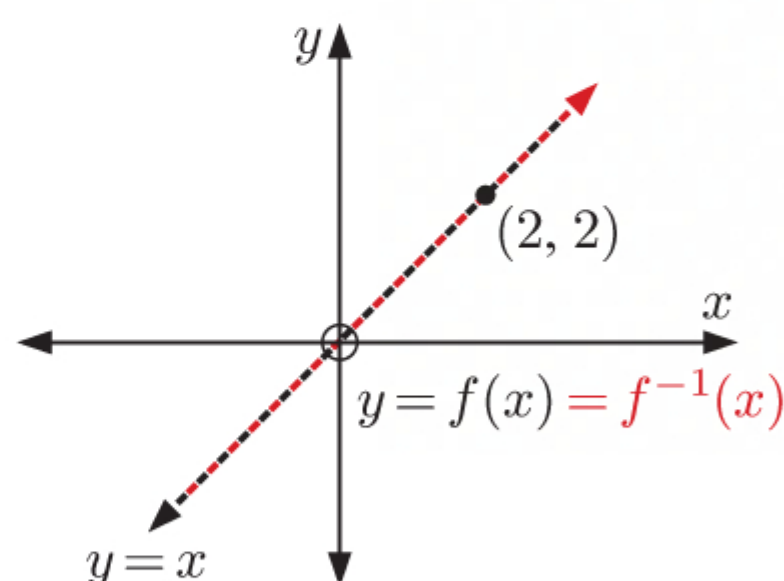
Range is $\{y \mid y \in \mathbb{R}\}$

f^{-1} : Domain is $\{x \mid x \in \mathbb{R}\}$

Range is $\{y \mid y \in \mathbb{R}\}$

- f $f(x)$ passes through $(0, 0)$ and $(2, 2)$.

$\therefore f^{-1}(x)$ passes through $(0, 0)$ and $(2, 2)$.



f : Domain is $\{x \mid x \in \mathbb{R}\}$

Range is $\{y \mid y \in \mathbb{R}\}$

f^{-1} : Domain is $\{x \mid x \in \mathbb{R}\}$

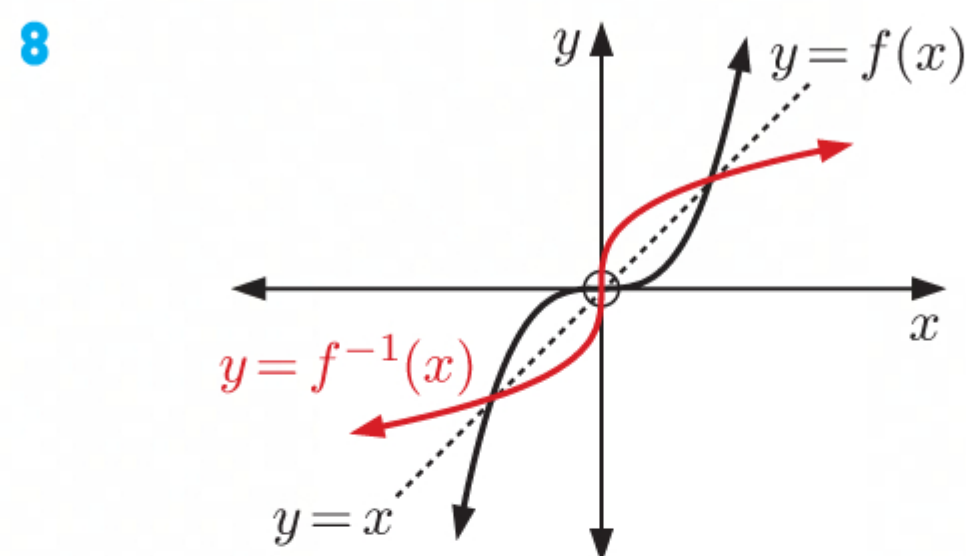
Range is $\{y \mid y \in \mathbb{R}\}$

$$\begin{array}{ll}
 \text{4} & f \text{ is } y = 2x - 5 \\
 & \therefore f^{-1} \text{ is } x = 2y - 5 \\
 & \therefore x + 5 = 2y \\
 & \therefore \frac{x+5}{2} = y \\
 & \therefore f^{-1}(x) = \frac{x+5}{2} \\
 & f^{-1} \text{ is } y = \frac{x+5}{2} \\
 & \therefore (f^{-1})^{-1} \text{ is } x = \frac{y+5}{2} \\
 & \therefore 2x = y + 5 \\
 & \therefore 2x - 5 = y \\
 & \therefore (f^{-1})^{-1}(x) = 2x - 5 = f(x)
 \end{array}$$

$$\begin{array}{l}
 \text{5} \quad f \text{ is } y = 3 - x \\
 \therefore f^{-1} \text{ is } x = 3 - y \\
 \therefore y = 3 - x \\
 \therefore f^{-1}(x) = 3 - x = f(x) \text{ and the function is its own inverse.}
 \end{array}$$

- 6 a** For $\{(1, 2), (2, 4), (3, 5)\}$, there is at most one x -value corresponding to each y -value. So, the function is one-to-one and hence has an inverse. The inverse function is $\{(2, 1), (4, 2), (5, 3)\}$.
- b** For $\{(-1, 3), (0, 2), (1, 3)\}$, there are two x -values corresponding to the y -value of 3. So, the function is many-to-one and hence does not have an inverse.
- c** For $\{(2, 1), (-1, 0), (0, 2), (1, 3)\}$, there is at most one x -value corresponding to each y -value. So, the function is one-to-one and hence has an inverse. The inverse function is $\{(1, 2), (0, -1), (2, 0), (3, 1)\}$.
- d** For $\{(-1, -1), (0, 0), (1, 1)\}$, there is at most one x -value corresponding to each y -value. So, the function is one-to-one and hence has an inverse. The inverse function is $\{(-1, -1), (0, 0), (1, 1)\}$.

- 7** The range of $H^{-1}(x)$ is the domain of $H(x)$.
 \therefore the range of $H^{-1}(x)$ is $\{y \mid -2 \leq y < 3\}$.



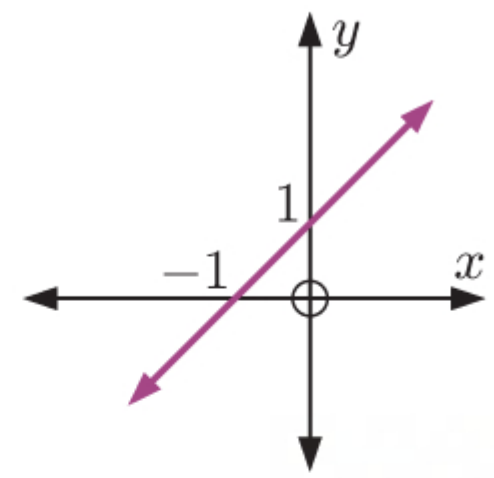
$$\begin{array}{l}
 \text{9} \quad f \text{ is } y = \frac{1}{x}, \quad x \neq 0 \\
 \therefore f^{-1} \text{ is } x = \frac{1}{y}, \quad y \neq 0 \\
 \therefore y = \frac{1}{x}
 \end{array}$$

$$\text{So, } f^{-1}(x) = \frac{1}{x} = f(x)$$

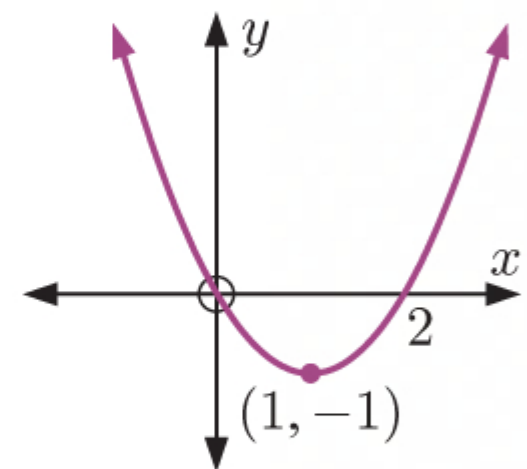
$\therefore f$ is self-inverse, as required.

- 10 a** The inverse function must also be a function and must therefore satisfy the vertical line test, which it can only do if the original function satisfies the horizontal line test.

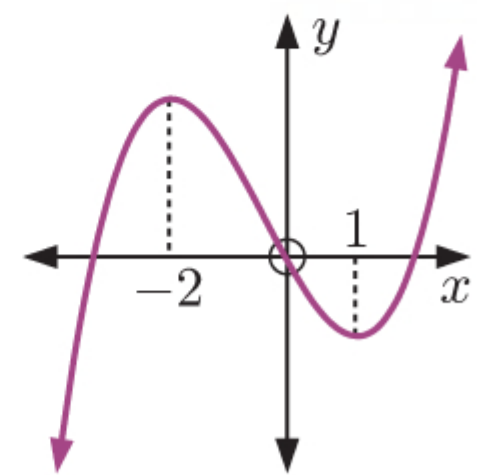
- b** **i** This graph satisfies the horizontal line test and therefore has an inverse function.



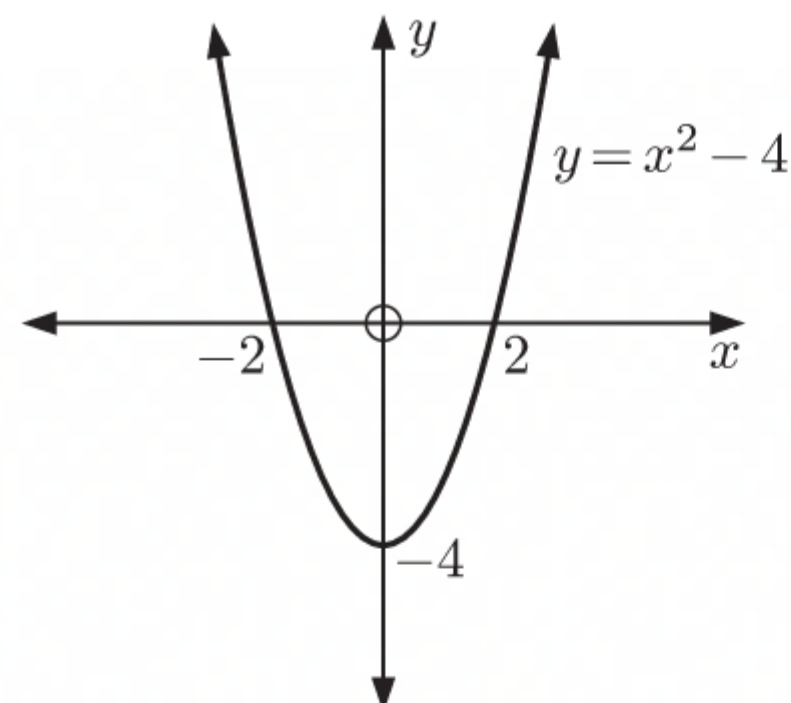
- ii** This graph fails the horizontal line test and therefore does not have an inverse function.



- iii** This graph fails the horizontal line test and therefore does not have an inverse function.

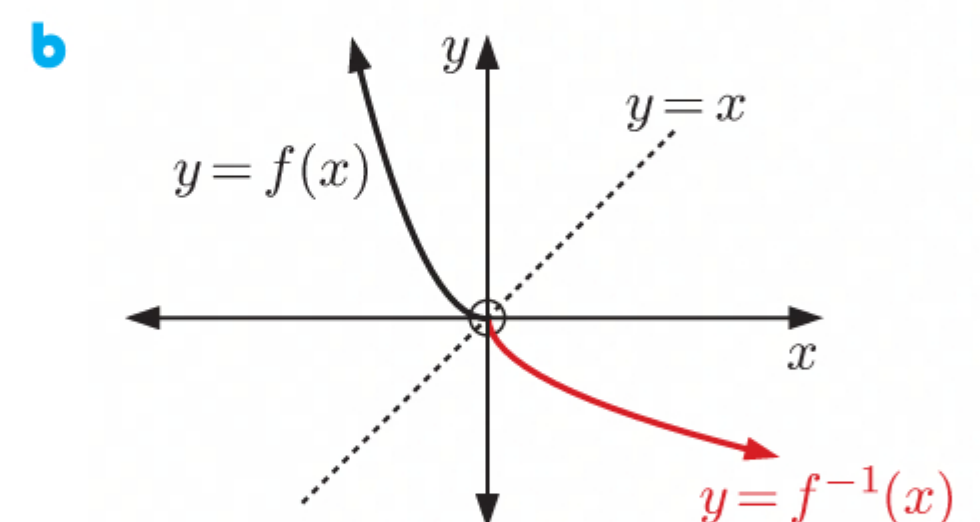


- 11** $f : x \mapsto x^2 - 4$ is many-to-one so it does not satisfy the horizontal line test.
 \therefore it does not have an inverse.



- 12** $f : x \mapsto x^2, x \leq 0$

- a** f is $y = x^2, x \leq 0$
 $\therefore f^{-1}$ is $x = y^2, y \leq 0$
 $\therefore y = \pm\sqrt{x}, y \leq 0$
 $\therefore y = -\sqrt{x}$ {as $y \leq 0$ and $-\sqrt{x} \leq 0$ }
 So, $f^{-1}(x) = -\sqrt{x}$



13 $f : x \mapsto 2x + 5$ and $g : x \mapsto \frac{8-x}{2}$

a g is $y = \frac{8-x}{2}$
 $\therefore g^{-1}$ is $x = \frac{8-y}{2}$
 $\therefore 2x = 8 - y$
 $\therefore y = 8 - 2x$

So, $g^{-1}(x) = 8 - 2x$

c f is $y = 2x + 5$
 $\therefore f^{-1}$ is $x = 2y + 5$
 $\therefore 2y = x - 5$
 $\therefore y = \frac{x-5}{2}$
 So, $f^{-1}(x) = \frac{x-5}{2}$

$$\begin{aligned} \therefore f^{-1}(-3) &= \frac{(-3)-5}{2} & \text{and} & \quad g^{-1}(6) = 8 - 2(6) \\ &= \frac{-8}{2} & & \quad = 8 - 12 \\ &= -4 & & \quad = -4 \end{aligned}$$

$$\begin{aligned} \therefore f^{-1}(-3) - g^{-1}(6) &= -4 - (-4) \\ &= 0 \quad \text{as required} \end{aligned}$$

b $g(x) = -1$
 $\therefore g^{-1}(g(x)) = g^{-1}(-1)$
 $\therefore x = 8 - 2(-1)$
 $\therefore x = 10$

d $(f \circ g^{-1})(x) = 9$
 $\therefore f(g^{-1}(x)) = 9$
 $\therefore f(8 - 2x) = 9$
 $\therefore 2(8 - 2x) + 5 = 9$
 $\therefore 16 - 4x + 5 = 9$
 $\therefore -4x = -12$
 $\therefore x = 3$

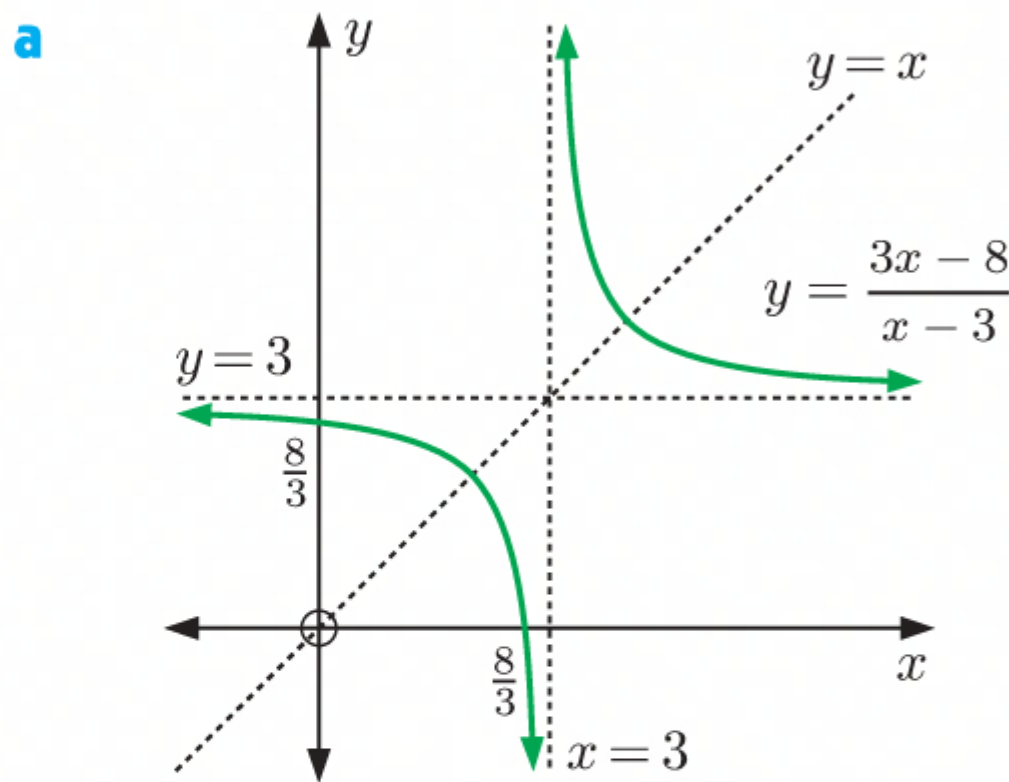
14 $f : x \mapsto 2x$ and $g : x \mapsto 4x - 3$

$$\begin{aligned} f \text{ is } y &= 2x & g \text{ is } y &= 4x - 3 \\ \therefore f^{-1} \text{ is } x &= 2y & \therefore g^{-1} \text{ is } x &= 4y - 3 \\ &\therefore y = \frac{x}{2} & &\therefore 4y = x + 3 \\ \therefore f^{-1}(x) &= \frac{x}{2} & &\therefore y = \frac{x+3}{4} \\ & & &\therefore g^{-1}(x) = \frac{x+3}{4} \end{aligned}$$

$$\begin{aligned} (g \circ f)(x) &= g(f(x)) & g \circ f \text{ is } y &= 8x - 3 \quad \{\text{using } (*)\} \\ &= g(2x) & \therefore (g \circ f)^{-1} \text{ is } x &= 8y - 3 \\ &= 4(2x) - 3 & &\therefore y = \frac{x+3}{8} \\ \therefore (g \circ f)(x) &= 8x - 3 \quad \dots (*) & \therefore (g \circ f)^{-1}(x) &= \frac{x+3}{8} \end{aligned}$$

Now $(f^{-1} \circ g^{-1})(x) = f^{-1}(g^{-1}(x))$
 $= f^{-1}\left(\frac{x+3}{4}\right)$
 $= \frac{\left(\frac{x+3}{4}\right)}{2}$
 $= \frac{x+3}{8}$
 $= (g \circ f)^{-1}(x) \quad \text{as required}$

15 $f : x \mapsto \frac{3x-8}{x-3}, \quad x \neq 3$



$y = \frac{3x-8}{x-3}$ is symmetrical about $y = x$

$\therefore f$ is a self-inverse function.

$$\therefore f^{-1}(x) = \frac{3x-8}{x-3} = f(x)$$

b f is $y = \frac{3x-8}{x-3}$
 $\therefore f^{-1}$ is $x = \frac{3y-8}{y-3}$
 $\therefore x(y-3) = 3y-8$
 $\therefore xy-3x = 3y-8$
 $\therefore y(x-3) = 3x-8$
 $\therefore y = \frac{3x-8}{x-3}$
 So, $f^{-1}(x) = \frac{3x-8}{x-3} = f(x)$

16 a $f(x)$ passes through $A(x, f(x))$, so $f^{-1}(x)$ passes through $B(f(x), x)$.

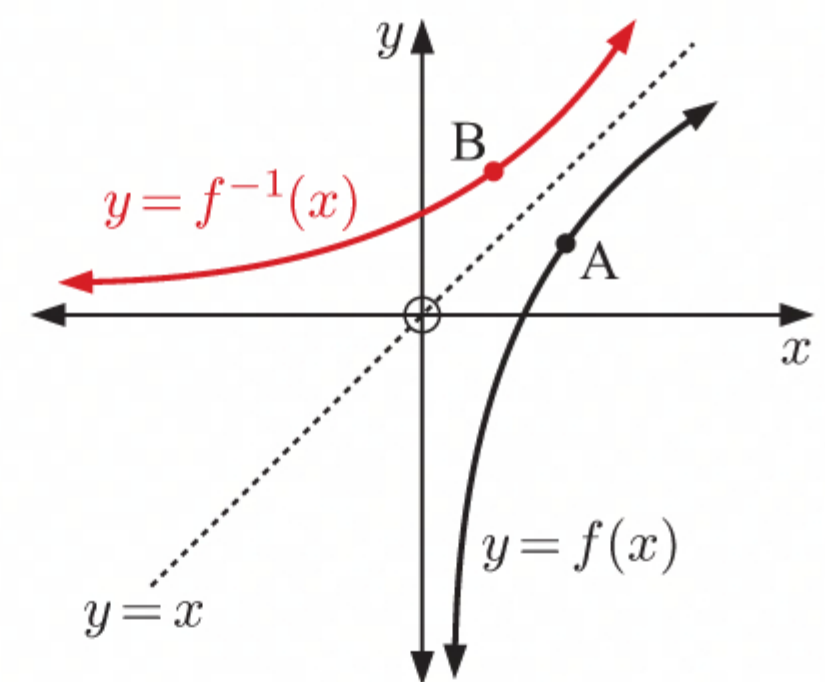
b Substituting the coordinates of $B(f(x), x)$ into $y = f^{-1}(x)$ gives $x = f^{-1}(f(x))$.

$$\therefore f^{-1}(f(x)) = x \text{ as required}$$

c B has coordinates $(x, f^{-1}(x))$ since it lies on $y = f^{-1}(x)$, so A has coordinates $(f^{-1}(x), x)$ as $f(x)$ is the inverse of $f^{-1}(x)$.

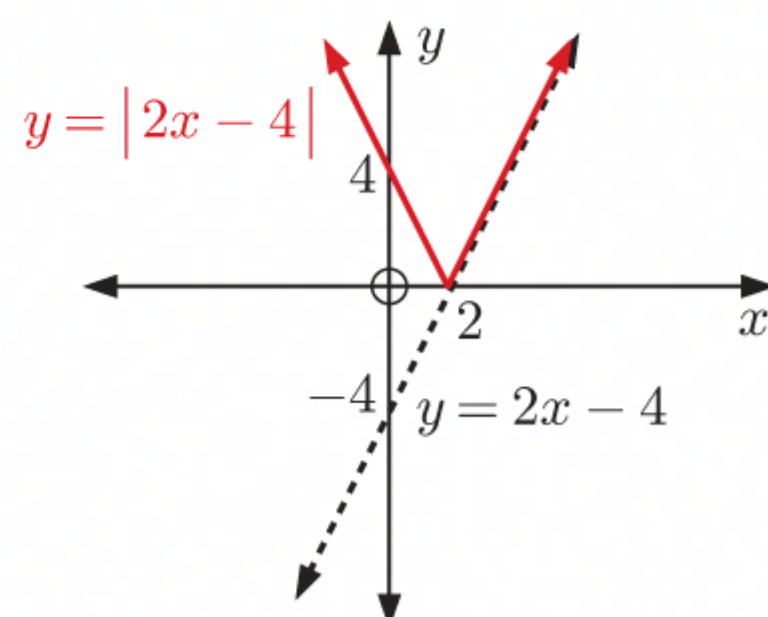
Substituting the coordinates of $A(f^{-1}(x), x)$ into $y = f(x)$ gives $x = f(f^{-1}(x))$.

$$\therefore f(f^{-1}(x)) = x \text{ as required}$$

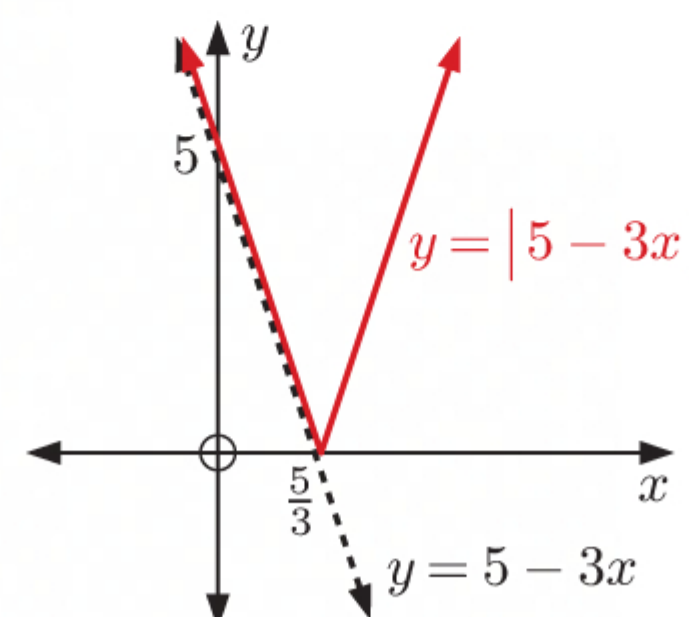


EXERCISE 3G

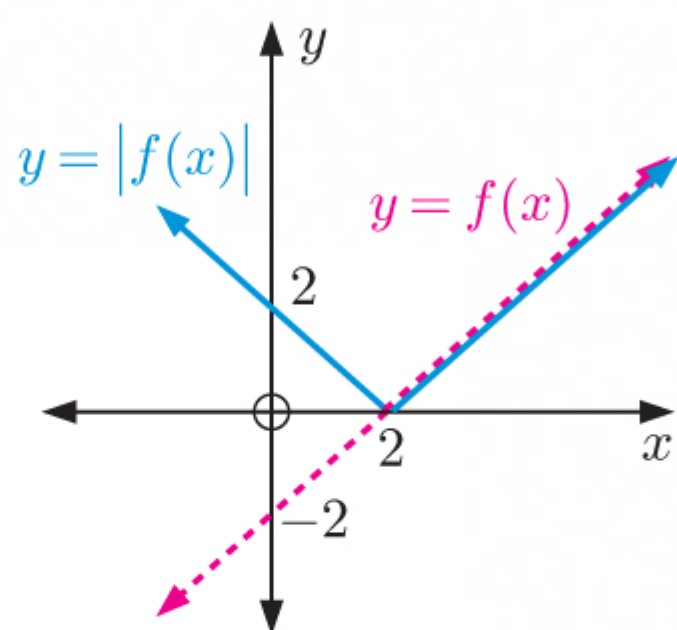
1 a The graph is unchanged for $f(x) \geq 0$ and reflected in the x -axis for $f(x) < 0$.



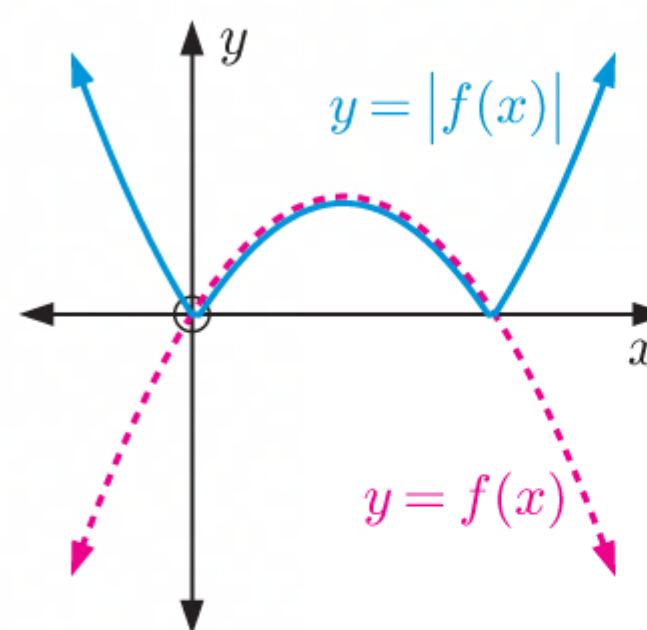
b The graph is unchanged for $f(x) \geq 0$ and reflected in the x -axis for $f(x) < 0$.



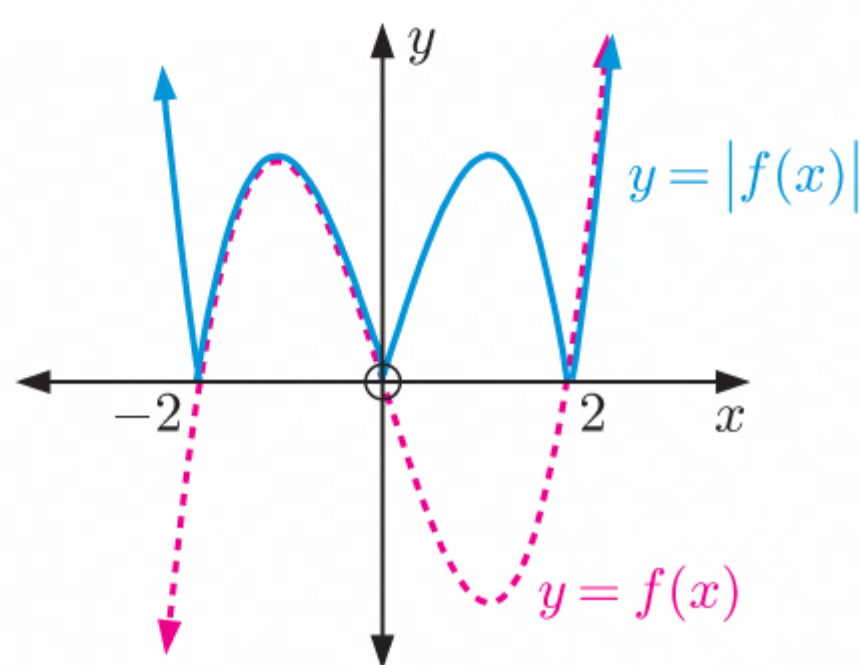
- 2 a** The graph is unchanged for $f(x) \geq 0$ and reflected in the x -axis for $f(x) < 0$.



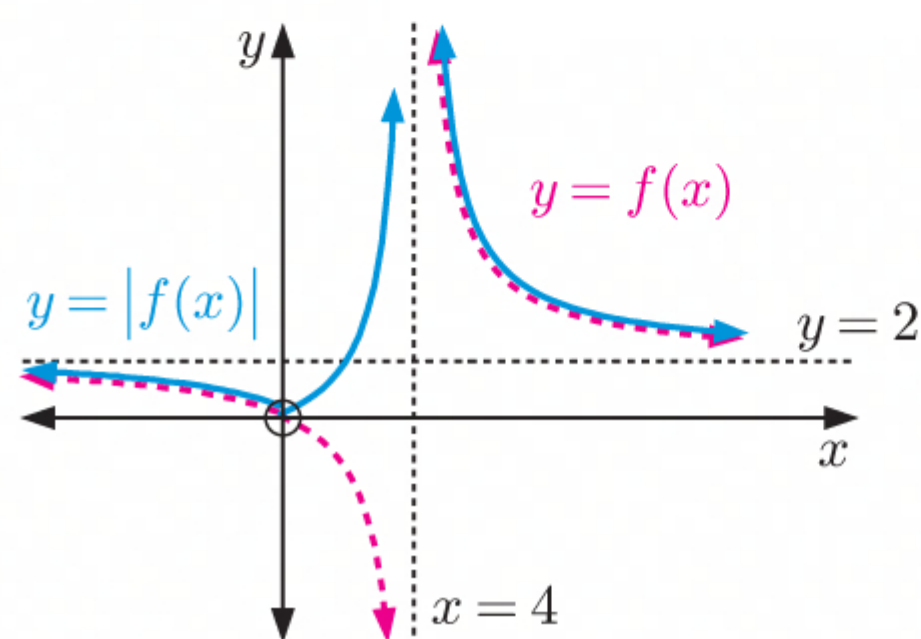
- b** The graph is unchanged for $f(x) \geq 0$ and reflected in the x -axis for $f(x) < 0$.



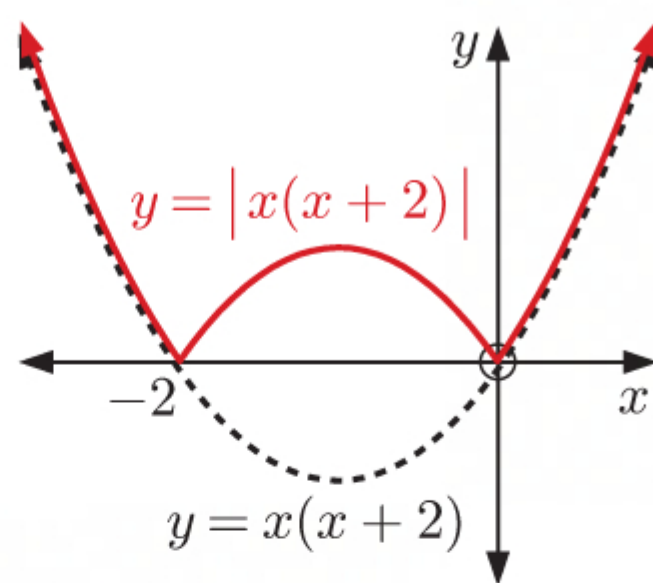
- c** The graph is unchanged for $f(x) \geq 0$ and reflected in the x -axis for $f(x) < 0$.



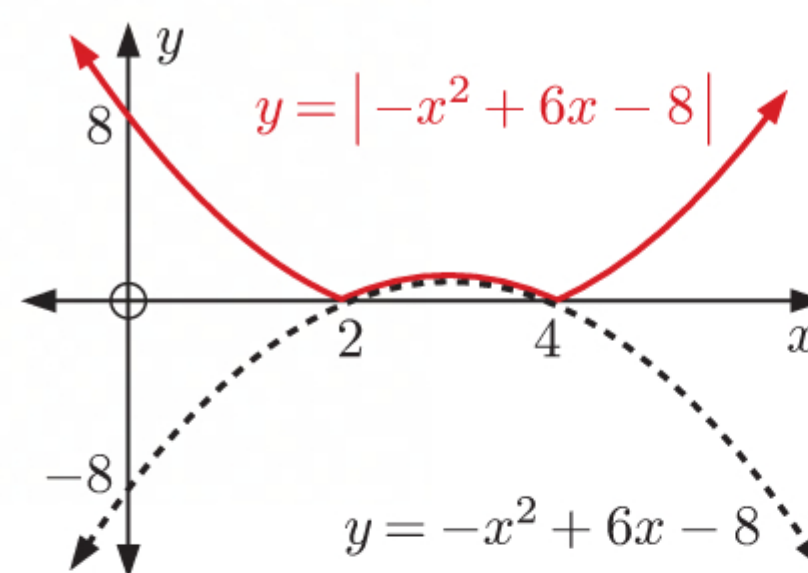
- d** The graph is unchanged for $f(x) \geq 0$ and reflected in the x -axis for $f(x) < 0$.



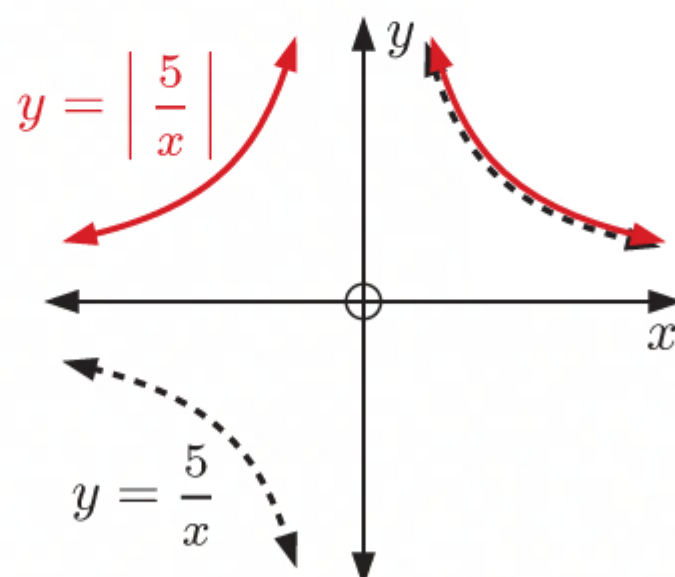
- 3 a** The graph is unchanged for $f(x) \geq 0$ and reflected in the x -axis for $f(x) < 0$.



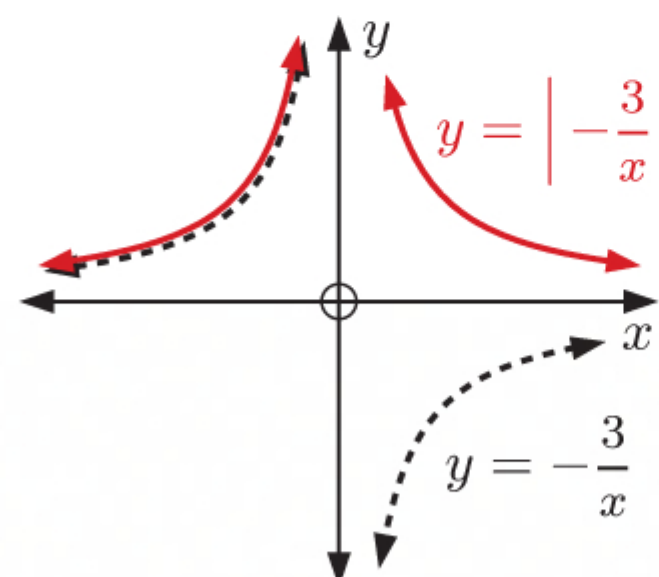
- b** The graph is unchanged for $f(x) \geq 0$ and reflected in the x -axis for $f(x) < 0$.



- 4 a** The graph is unchanged for $f(x) \geq 0$ and reflected in the x -axis for $f(x) < 0$.

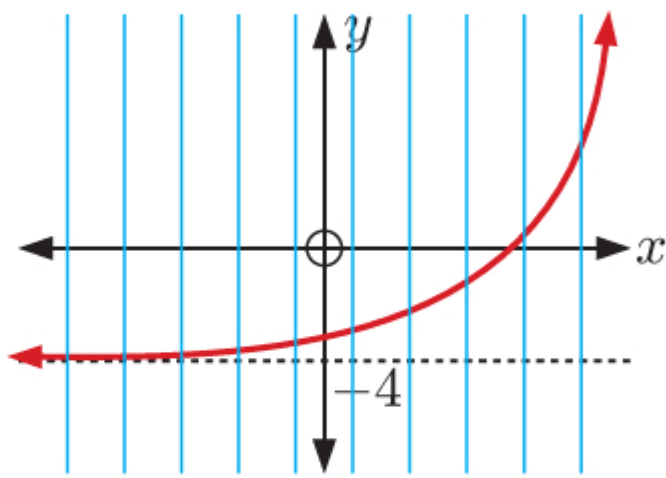


- b** The graph is unchanged for $f(x) \geq 0$ and reflected in the x -axis for $f(x) < 0$.



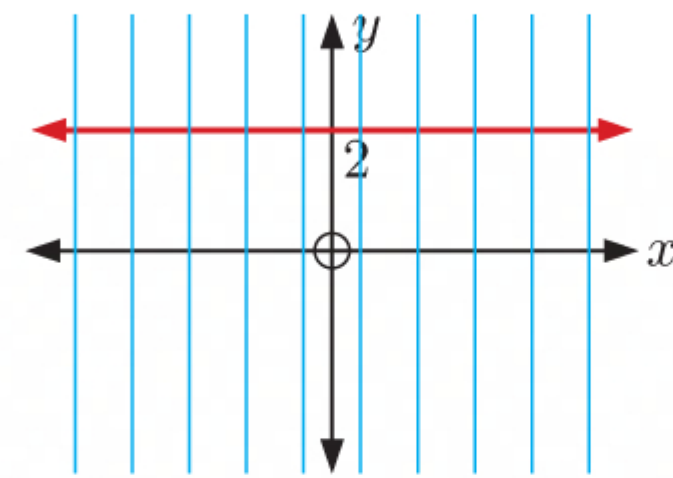
REVIEW SET 3A

1 a



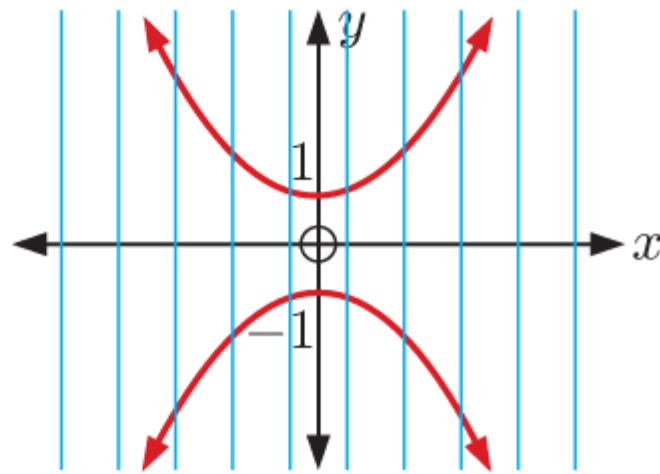
- i Domain is $\{x \mid x \in \mathbb{R}\}$.
- ii Range is $\{y \mid y > -4\}$.
- iii Each vertical line cuts the graph at most once, so the graph shows a function.

b



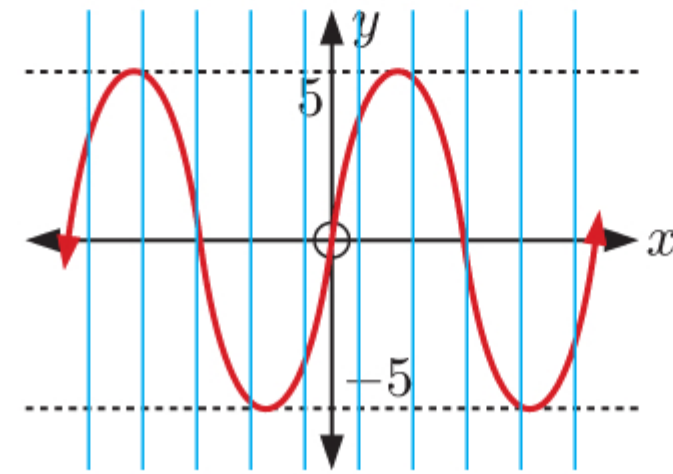
- i Domain is $\{x \mid x \in \mathbb{R}\}$.
- ii Range is $\{y \mid y = 2\}$.
- iii Each vertical line cuts the graph at most once, so the graph shows a function.

c



- i Domain is $\{x \mid x \in \mathbb{R}\}$.
- ii Range is $\{y \mid y \leq -1 \text{ or } y \geq 1\}$.
- iii All vertical lines cut the graph more than once, so the graph does not show a function.

d



- i Domain is $\{x \mid x \in \mathbb{R}\}$.
- ii Range is $\{y \mid -5 \leq y \leq 5\}$.
- iii Each vertical line cuts the graph at most once, so the graph shows a function.

2 $f(x) = 2x - x^2$

$$\begin{aligned} \text{a } f(2) &= 2(2) - 2^2 \\ &= 4 - 4 \\ &= 0 \end{aligned}$$

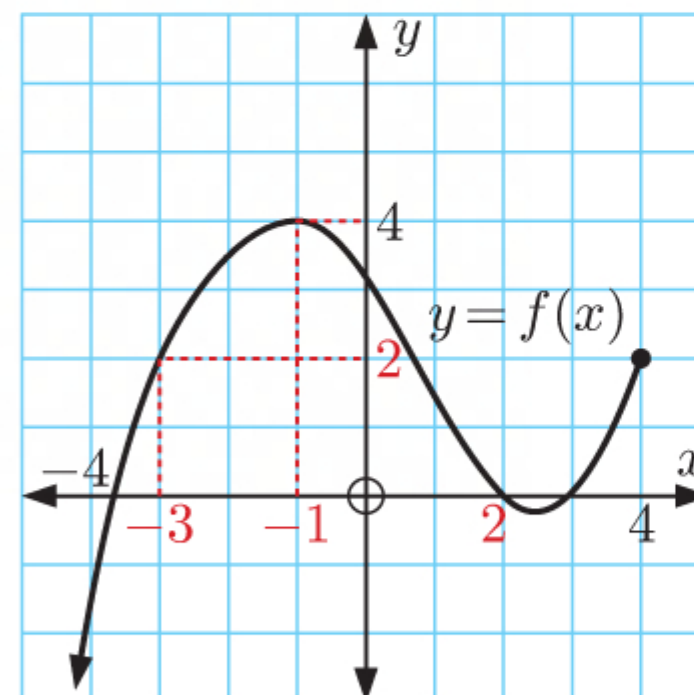
$$\begin{aligned} \text{b } f(-3) &= 2(-3) - (-3)^2 \\ &= -6 - 9 \\ &= -15 \end{aligned}$$

$$\begin{aligned} \text{c } f(-\tfrac{1}{2}) &= 2(-\tfrac{1}{2}) - (-\tfrac{1}{2})^2 \\ &= -1 - \tfrac{1}{4} \\ &= -\tfrac{5}{4} \end{aligned}$$

3 a i $f(-3) = 2$

ii $f(2) = 0$

b When $y = f(x) = 4$, $x = -1$.



4 $f(x) = ax + b$ where $f(1) = 7$ and $f(3) = -5$

So, $a(1) + b = 7$ and $a(3) + b = -5$

$\therefore a + b = 7$ $\therefore 3a + b = -5$

$\therefore b = 7 - a$ (*) $\therefore 3a + (7 - a) = -5$ {using (*)}

$\therefore 2a = -12$

$\therefore a = -6$

Substituting $a = -6$ into (*) gives $b = 7 - (-6) = 13$

So, $a = -6$, $b = 13$.

5 a $h(x) = 7 - 3x$

$\therefore h(2x - 1) = 7 - 3(2x - 1)$

$= 7 - 6x + 3$

$= 10 - 6x$

b $h(2x - 1) = -2$

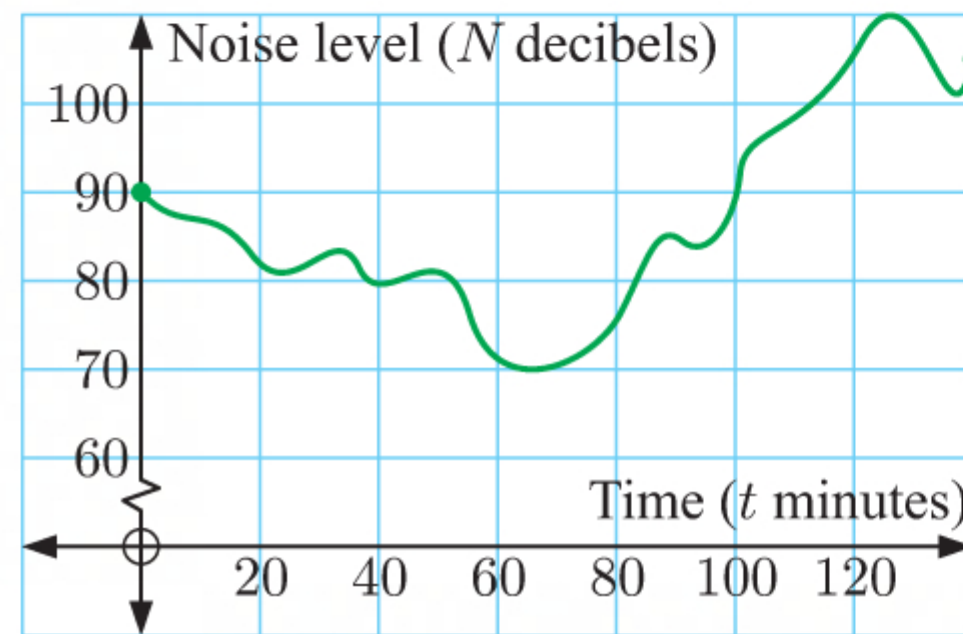
$\therefore 10 - 6x = -2$ {using **a**}

$\therefore -6x = -12$

$\therefore x = 2$

6 The domain is $\{t \mid 0 \leq t \leq 140\}$.

The range is $\{N \mid 70 \leq N \leq 110\}$.

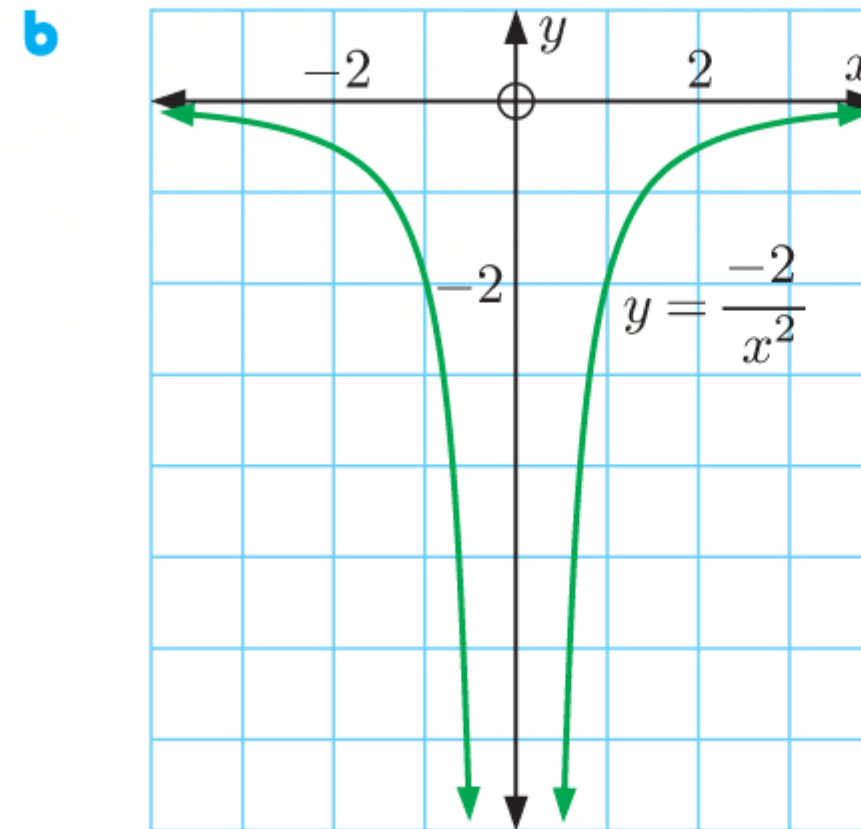


7 $f(x) = \frac{-2}{x^2}$

a $f(x) = \frac{-2}{x^2}$ is undefined when $x^2 = 0$
 $\therefore x = 0$

c The domain is $\{x \mid x \neq 0\}$.

The range is $\{y \mid y < 0\}$.



8 $f(x) = x^2$ and $g(x) = 1 - 6x$

a $f(-3) = (-3)^2$ and $g(-\frac{4}{3}) = 1 - 6(-\frac{4}{3})$
 $= 9$ $= 1 + 8$
 $= 9$

$\therefore f(-3) = g(-\frac{4}{3})$ as required

b $g(x) = f(5)$

$\therefore 1 - 6x = 5^2$

$\therefore 1 - 6x = 25$

$\therefore -6x = 24$

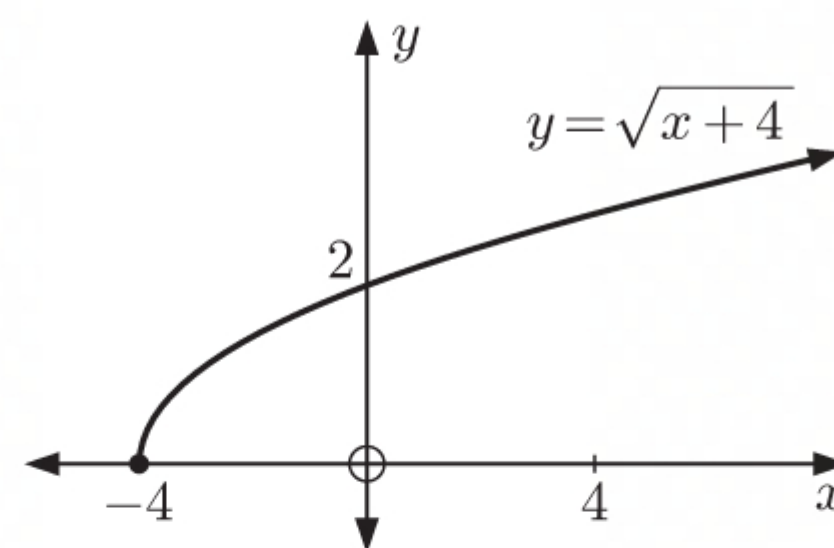
$\therefore x = -4$


9 a $y = \sqrt{x+4}$ is defined when $x+4 \geq 0$
 $\therefore x \geq -4$

\therefore the domain is $\{x \mid x \geq -4\}$.

A square root cannot be negative.

\therefore the range is $\{y \mid y \geq 0\}$.

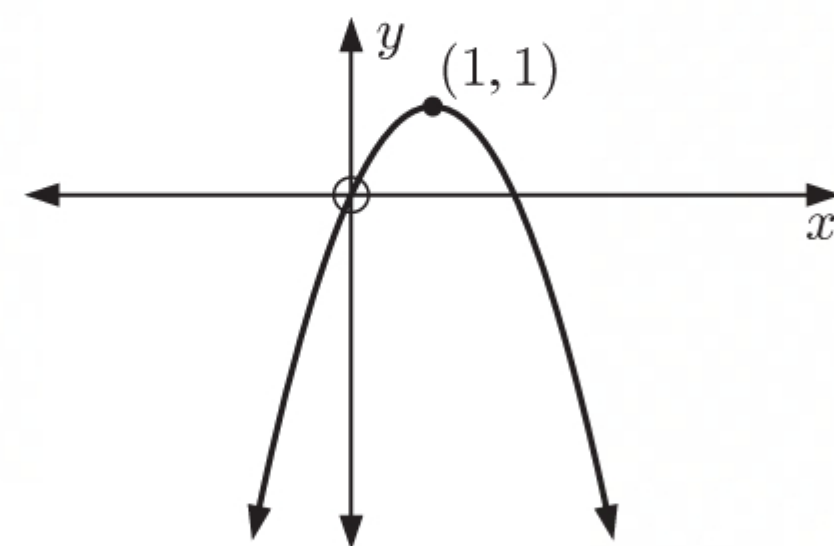


b $y = -(1-x)^2 + 1$ is a quadratic with vertex $(1, 1)$
 and shape  ($a < 0$).

\therefore the domain is $\{x \mid x \in \mathbb{R}\}$.

\therefore the maximum y -value is 1 and there is no minimum y -value.

\therefore the range is $\{y \mid y \leq 1\}$.



c $y = 2x^2 - 3x + 1$
 $\therefore y = 2(x^2 - \frac{3}{2}x + \frac{1}{2})$
 $\therefore y = 2(x^2 - \frac{3}{2}x + (-\frac{3}{4})^2 + \frac{1}{2} - (-\frac{3}{4})^2)$
 $\therefore y = 2[(x - \frac{3}{2})^2 - \frac{1}{16}]$
 $\therefore y = 2(x - \frac{3}{2})^2 - \frac{1}{8}$

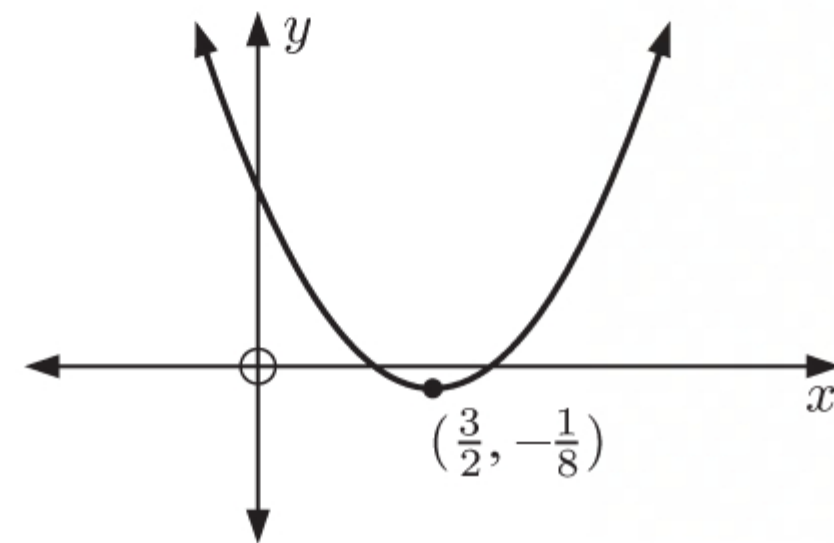
which is a quadratic with vertex $(\frac{3}{2}, -\frac{1}{8})$

and shape  ($a > 0$).

\therefore the domain is $\{x \mid x \in \mathbb{R}\}$

\therefore the minimum y -value is $-\frac{1}{8}$ and there is no maximum y -value.

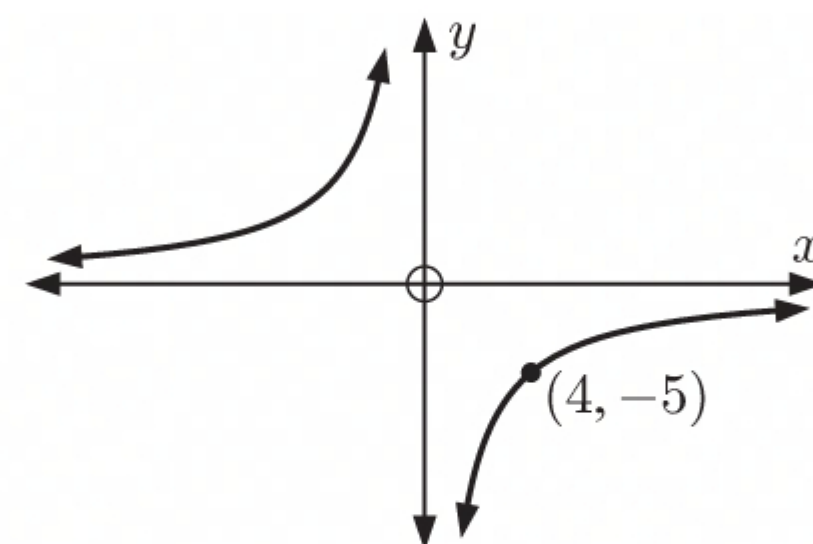
\therefore the range is $\{y \mid y \geq -\frac{1}{8}\}$.



10 a Let the function be $y = \frac{k}{x}$.

When $x = 4$, $y = -5$, so $-5 = \frac{k}{4}$
 $\therefore k = -20$

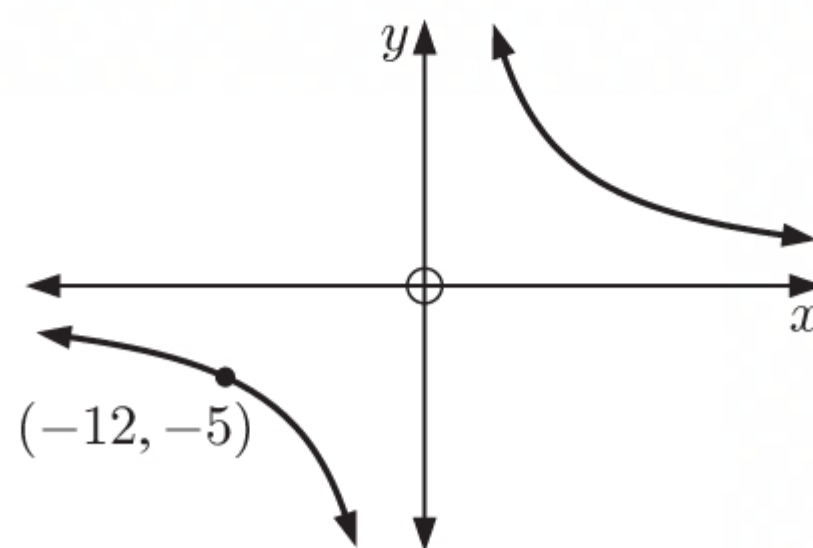
\therefore the function is $y = -\frac{20}{x}$.



b Let the function be $y = \frac{k}{x}$.

When $x = -12$, $y = -5$, so $-5 = \frac{k}{-12}$
 $\therefore k = 60$

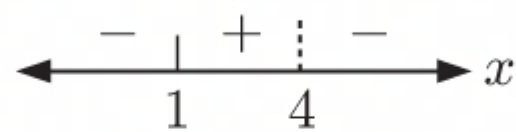
\therefore the function is $y = \frac{60}{x}$.



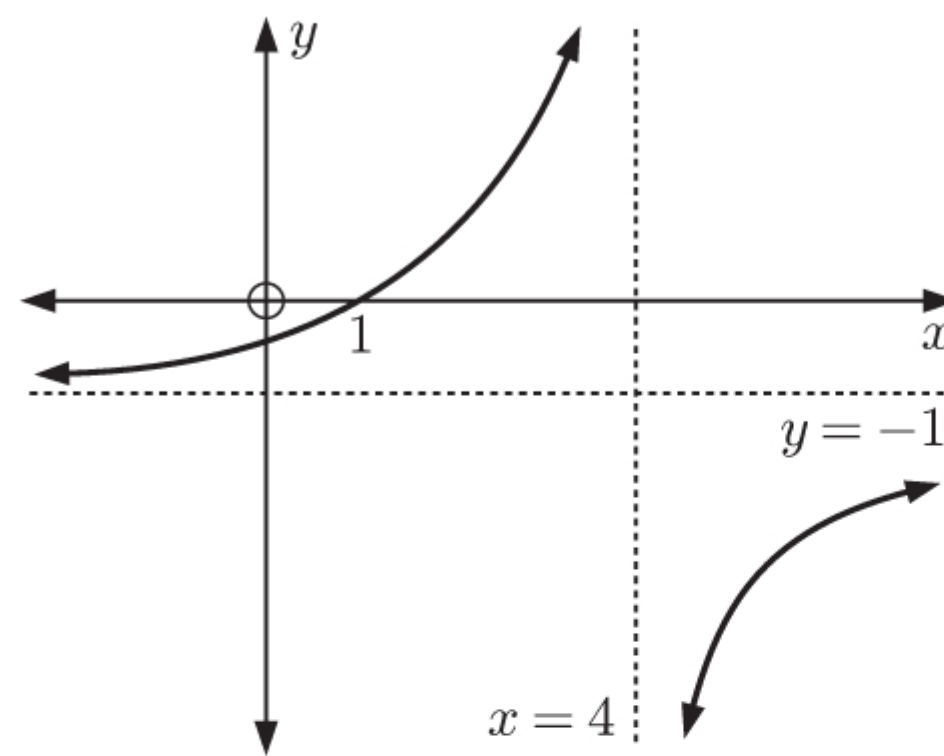
- 11** The domain is $\{x \mid x \neq 4\}$ so there must be a vertical asymptote at $x = 4$.

The range is $\{y \mid y \neq -1\}$ so there must be a horizontal asymptote at $y = -1$.

The sign diagram is



A graph with all of these properties is shown alongside.



12 $f : x \mapsto \frac{4x+1}{2-x}$

$$\begin{aligned} \text{a } f(x) &= \frac{4x+1}{2-x} \\ &= \frac{-4(2-x)+9}{2-x} \\ &= -4 + \frac{9}{2-x} \end{aligned}$$

\therefore the vertical asymptote is $x = 2$ and the horizontal asymptote is $y = -4$.

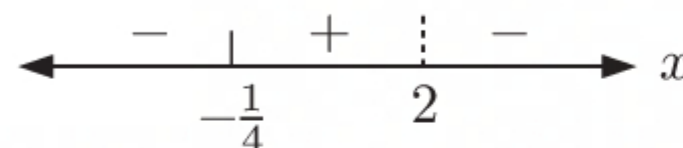
- b** The domain is $\{x \mid x \neq 2\}$. The range is $\{y \mid y \neq -4\}$.

$$\begin{aligned} \text{c } f(x) = 0 \text{ when } 4x+1 &= 0 \\ \therefore 4x &= -1 \\ \therefore x &= -\frac{1}{4} \end{aligned}$$

\therefore the x -intercept is $-\frac{1}{4}$.



Since $(4x+1)$ and $(2-x)$ are single factors, the signs alternate.



As $x \rightarrow 2^-$, $y \rightarrow \infty$

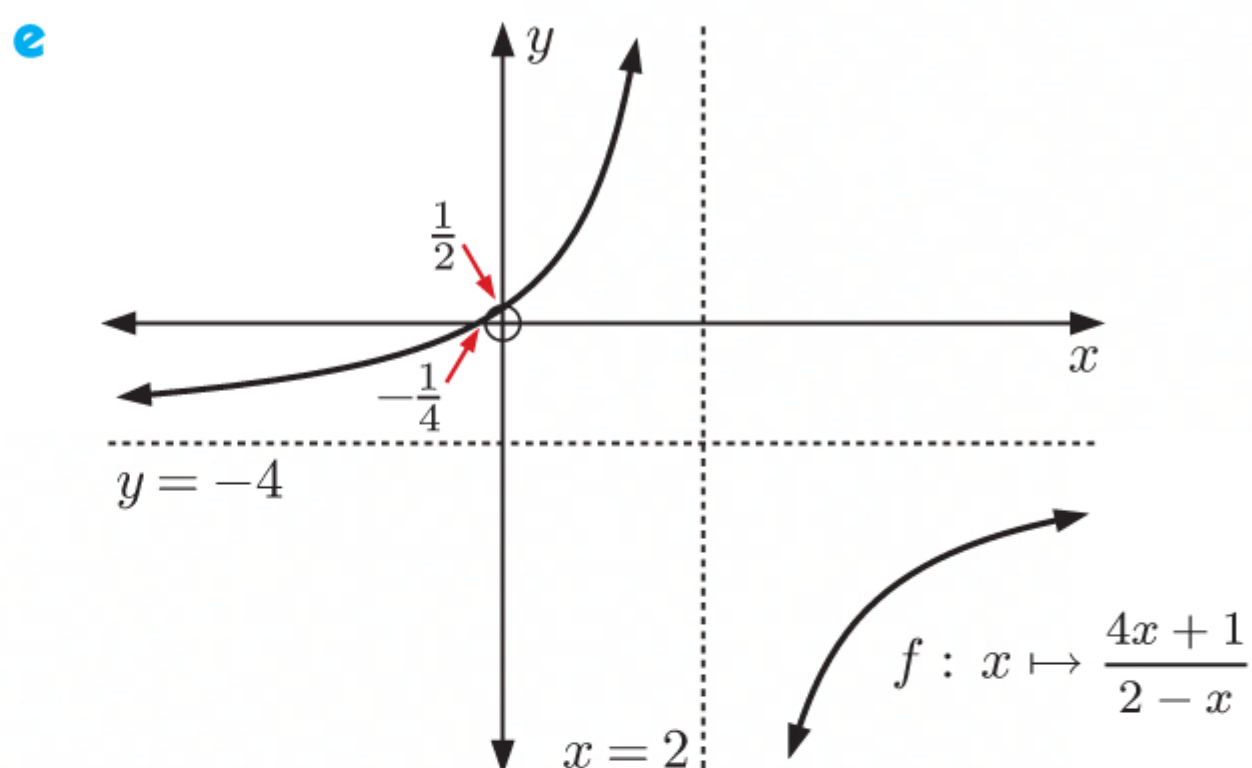
As $x \rightarrow 2^+$, $y \rightarrow -\infty$

As $x \rightarrow -\infty$, $y \rightarrow -4^+$

As $x \rightarrow \infty$, $y \rightarrow -4^-$

- d** $f(0) = \frac{1}{2} \therefore$ the y -intercept is $\frac{1}{2}$.

\therefore the x -intercept is $-\frac{1}{4}$ {from **b**} and the y -intercept is $\frac{1}{2}$.



13 $f(x) = 2x - 3$ and $g(x) = x^2 + 2$

a $(f \circ g)(x) = f(g(x))$
 $= f(x^2 + 2)$
 $= 2(x^2 + 2) - 3$
 $= 2x^2 + 4 - 3$
 $= 2x^2 + 1$

b $(g \circ f)(x) = g(f(x))$
 $= g(2x - 3)$
 $= (2x - 3)^2 + 2$
 $= 4x^2 - 12x + 9 + 2$
 $= 4x^2 - 12x + 11$

c $(f \circ f)(2) = f(f(2))$
 $= f(2(2) - 3)$
 $= f(1)$
 $= 2(1) - 3$
 $= -1$

14 $f(x) = 2x - 5$ and $g(x) = 3x + 1$

a $(f \circ g)(x) = f(g(x))$
 $= f(3x + 1)$
 $= 2(3x + 1) - 5$
 $= 6x + 2 - 5$
 $= 6x - 3$

b $(f \circ g)(x) = f(x + 3)$
 $\therefore 6x - 3 = 2(x + 3) - 5$
 $\therefore 6x - 3 = 2x + 6 - 5$
 $\therefore 4x = 4$
 $\therefore x = 1$

15 $f(x) = 1 - 2x$ and $g(x) = \sqrt{x}$

a $(f \circ g)(x) = f(g(x))$
 $= f(\sqrt{x})$
 $= 1 - 2\sqrt{x}$

b $(g \circ f)(x) = g(f(x))$
 $= g(1 - 2x)$
 $= \sqrt{1 - 2x}$

c $(g \circ g)(81) = g(g(81))$
 $= g(\sqrt{81})$
 $= g(9)$
 $= \sqrt{9}$
 $= 3$

16 $f(x) = ax + b$ where $f(2) = 1$ and $f^{-1}(3) = 4$

f is $y = ax + b$

$\therefore f^{-1}$ is $x = ay + b$

$\therefore ay = x - b$

$\therefore y = \frac{x - b}{a}$

So, $f^{-1}(x) = \frac{x - b}{a}$

$\therefore a(2) + b = 1$

$\therefore 2a + b = 1$

$\therefore b = 1 - 2a \dots (*)$

and

$\frac{3 - b}{a} = 4$

$\therefore 3 - b = 4a$

$\therefore 3 - (1 - 2a) = 4a$ {using (*)}

$\therefore 3 - 1 + 2a = 4a$

$\therefore 2a = 2$

$\therefore a = 1$

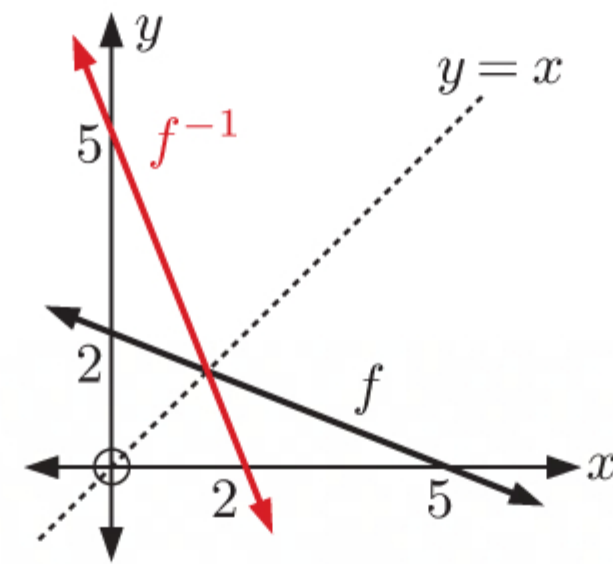
Substituting $a = 1$ into (*) gives $b = 1 - 2(1) = -1$.

So, $a = 1$, $b = -1$.

17 a f passes through $(0, 2)$ and $(5, 0)$.

$\therefore f^{-1}$ passes through $(2, 0)$ and $(0, 5)$.

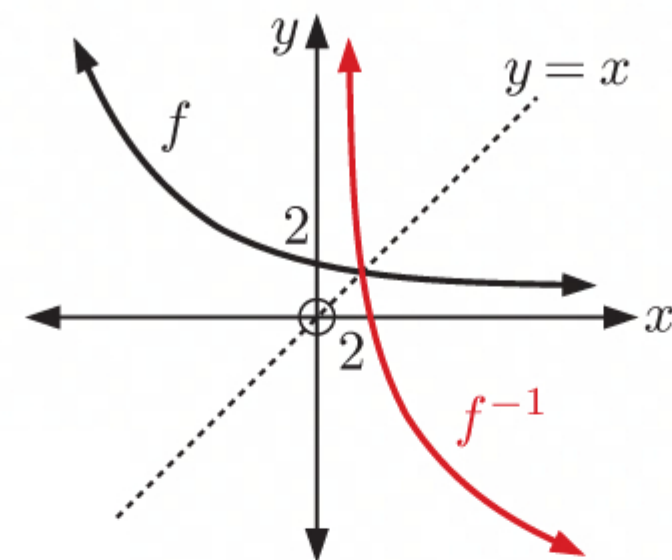
The graphs are reflections of each other in the line $y = x$.



b f passes through $(0, 2)$.

$\therefore f^{-1}$ passes through $(2, 0)$.

The graphs are reflections of each other in the line $y = x$.



18 a f is $y = 4x + 2$

$\therefore f^{-1}$ is $x = 4y + 2$

$\therefore y = \frac{x - 2}{4}$

$\therefore f^{-1}(x) = \frac{x - 2}{4}$

b f is $y = \frac{3 - 5x}{4}$

$\therefore f^{-1}$ is $x = \frac{3 - 5y}{4}$

$\therefore 4x = 3 - 5y$

$\therefore y = \frac{3 - 4x}{5}$

$\therefore f^{-1}(x) = \frac{3 - 4x}{5}$

19 f is $y = 3x + 6$

$\therefore f^{-1}$ is $x = 3y + 6$

$\therefore y = \frac{x-6}{3}$

$\therefore f^{-1}(x) = \frac{x-6}{3}$

h is $y = \frac{x}{3}$

$\therefore h^{-1}$ is $x = \frac{y}{3}$

$\therefore y = 3x$

$\therefore h^{-1}(x) = 3x$

$(h \circ f)(x) = h(f(x))$

$= h(3x + 6)$

$= \frac{3x+6}{3}$

$= x + 2$

$\therefore (h \circ f)(x) = x + 2$

$h \circ f$ is $y = x + 2$

$\therefore (h \circ f)^{-1}$ is $x = y + 2$

$\therefore y = x - 2$

$\therefore (h \circ f)^{-1}(x) = x - 2$

Now $(f^{-1} \circ h^{-1})(x) = f^{-1}(h^{-1}(x))$

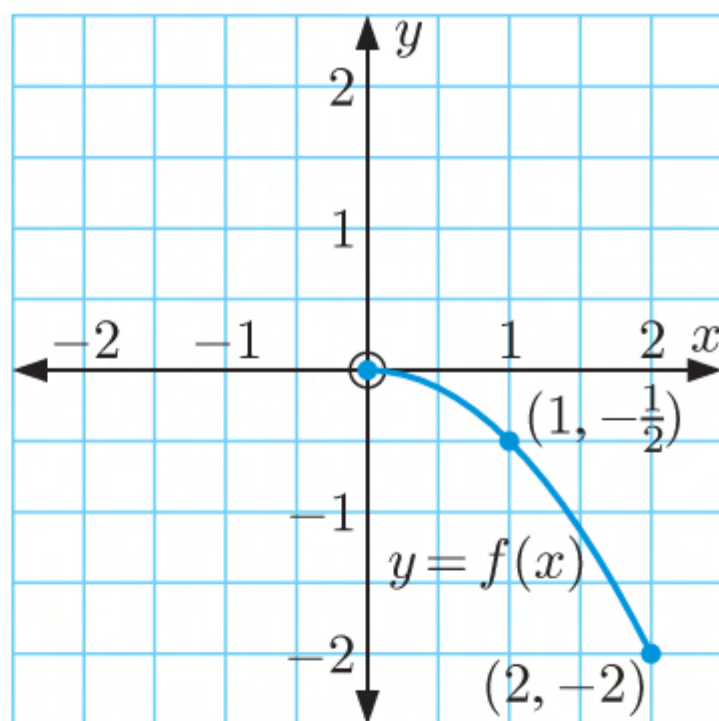
$= f^{-1}(3x)$

$= \frac{3x-6}{3}$

$= x - 2$

$\therefore (f^{-1} \circ h^{-1})(x) = (h \circ f)^{-1}(x)$ as required

20 a



b The domain of f is $\{x \mid 0 \leq x \leq 2\}$.

\therefore the range of f^{-1} is $\{y \mid 0 \leq y \leq 2\}$.

c i $f(x) = -\frac{3}{2}$

$\therefore -\frac{1}{2}x^2 = -\frac{3}{2}$

$\therefore x^2 = 3$

$\therefore x = \sqrt{3} \quad \{0 \leq x \leq 2\}$

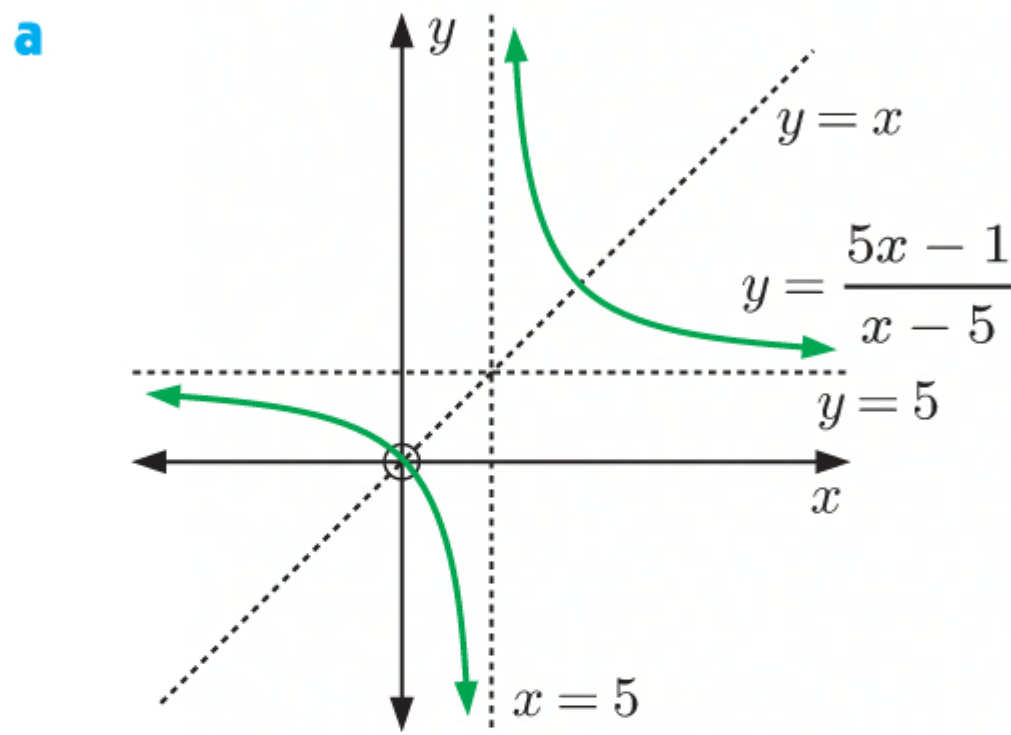
ii $f^{-1}(x) = 1$

$\therefore f(f^{-1}(x)) = f(1)$

$\therefore x = -\frac{1}{2}(1)^2$

$\therefore x = -\frac{1}{2}$

21 $f : x \mapsto \frac{5x-1}{x-5}, \quad x \neq 5$



$y = \frac{5x-1}{x-5}$ is symmetrical about $y = x$
 $\therefore f$ is a self-inverse function.

b f is $y = \frac{5x-1}{x-5}$
 $\therefore f^{-1}$ is $x = \frac{5y-1}{y-5}$
 $\therefore xy - 5x = 5y - 1$
 $\therefore xy - 5y = 5x - 1$
 $\therefore y(x-5) = 5x - 1$
 $\therefore y = \frac{5x-1}{x-5}$
 $\therefore f^{-1}(x) = \frac{5x-1}{x-5} = f(x)$
 $\therefore f$ is a self-inverse function.

22 $f : x \mapsto \sqrt{x}$ and $g : x \mapsto 3+x$

a f is $y = \sqrt{x}, \quad x \geq 0$
 $\therefore f^{-1}$ is $x = \sqrt{y}, \quad y \geq 0$
 $\therefore y = x^2$
 So, $f^{-1}(x) = x^2, \quad x \geq 0$

$$\begin{aligned} f^{-1}(2) \times g^{-1}(2) &= (2)^2 \times (2-3) \\ &= 4 \times -1 \\ &= -4 \end{aligned}$$

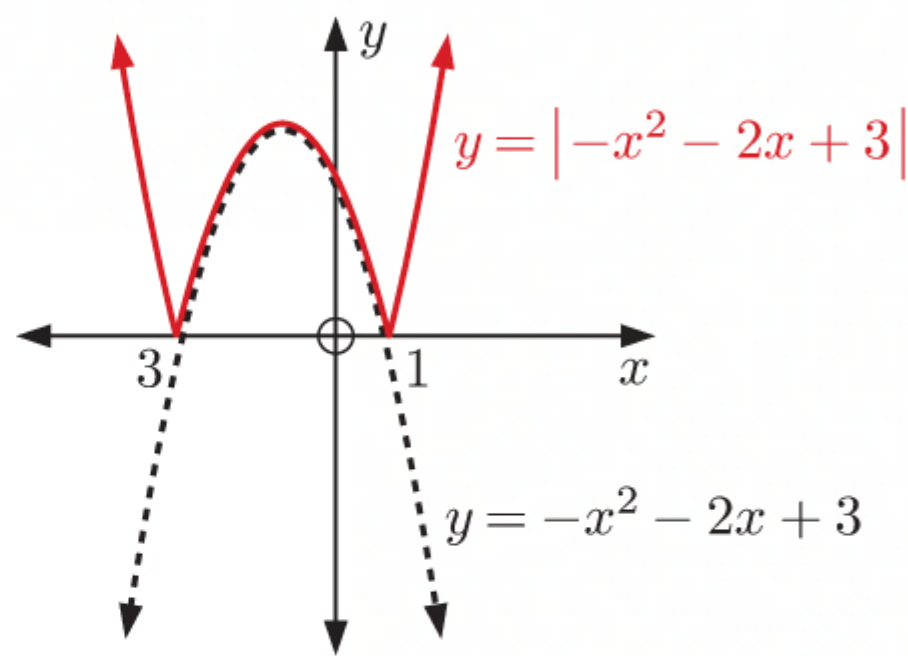
b $(f \circ g)(x) = f(g(x))$
 $= f(3+x)$
 $= \sqrt{3+x}$
 $\therefore (f \circ g)(x) = \sqrt{3+x} \quad \dots (*)$

$$\begin{aligned} \therefore (f \circ g)^{-1}(2) &= (2)^2 - 3 \\ &= 4 - 3 \\ &= 1 \end{aligned}$$

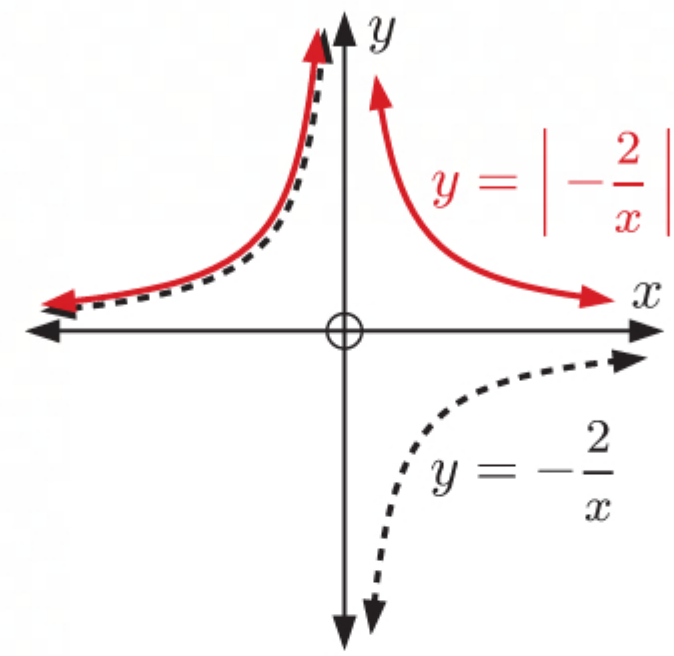
g is $y = 3+x$
 $\therefore g^{-1}$ is $x = 3+y$
 $\therefore y = x-3$
 So, $g^{-1}(x) = x-3$

$f \circ g$ is $y = \sqrt{3+x}, \quad x \geq -3$
 {using (*)}
 $\therefore (f \circ g)^{-1}$ is $x = \sqrt{3+y}, \quad y \geq -3$
 $\therefore x^2 = 3+y$
 $\therefore y = x^2 - 3, \quad x \geq 0$
 So, $(f \circ g)^{-1}(x) = x^2 - 3$

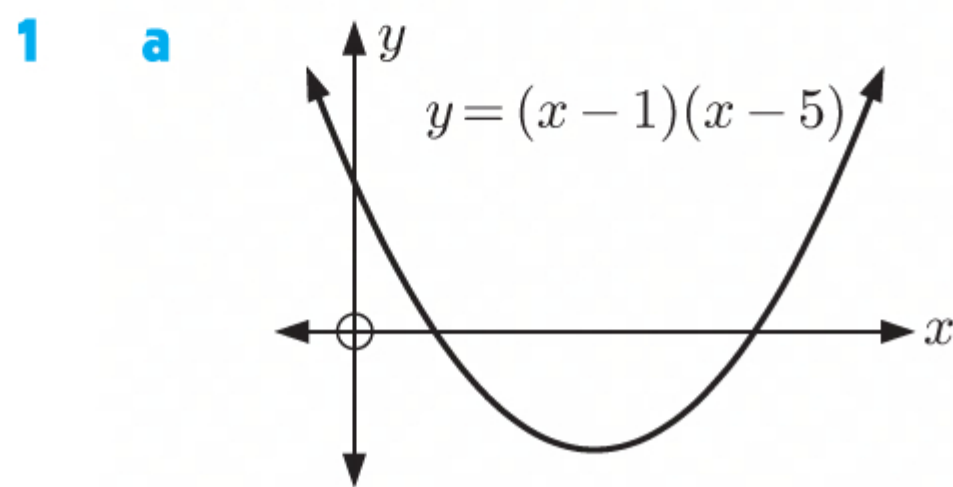
- 23 a** The graph is unchanged for $f(x) \geq 0$ and reflected in the x -axis for $f(x) < 0$.



- b** The graph is unchanged for $f(x) \geq 0$ and reflected in the x -axis for $f(x) < 0$.



REVIEW SET 3B

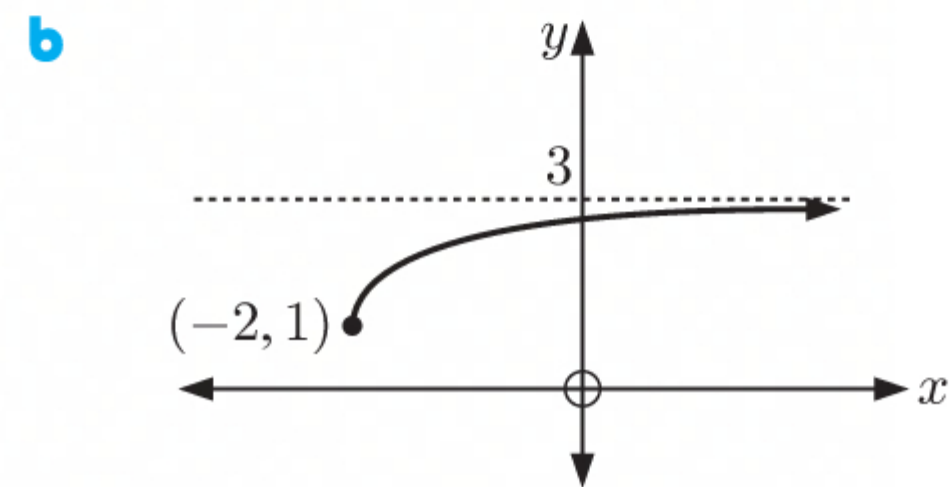


The minimum y -value occurs at $x = 3$.

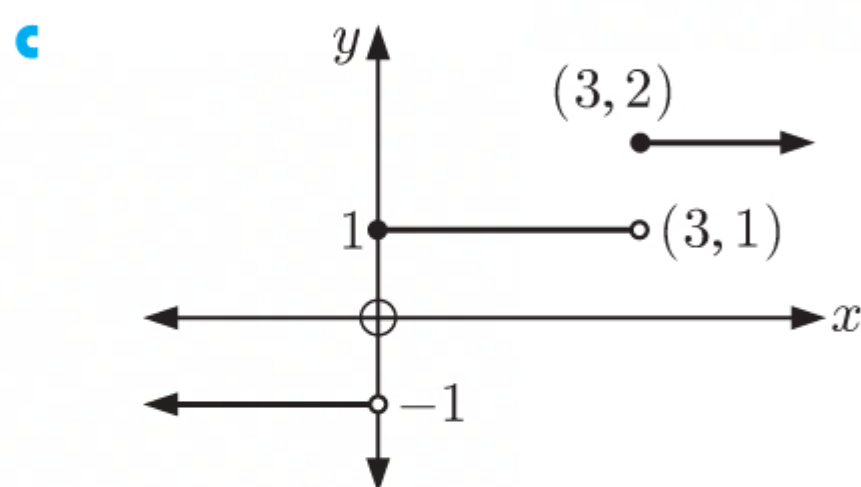
$$\begin{aligned}\text{When } x = 3, \quad y &= (3-1)(3-5) \\ &= (2)(-2) \\ &= -4\end{aligned}$$

So, the minimum y -value is -4 and there is no maximum y -value.

\therefore the domain is $\{x \mid x \in \mathbb{R}\}$,
and the range is $\{y \mid y \geq -4\}$.



The domain is $\{x \mid x \geq -2\}$,
and the range is $\{y \mid 1 \leq y < 3\}$.



x can take any value.

\therefore the domain is $\{x \mid x \in \mathbb{R}\}$.

The possible values of y are -1 , 1 , or 2 .

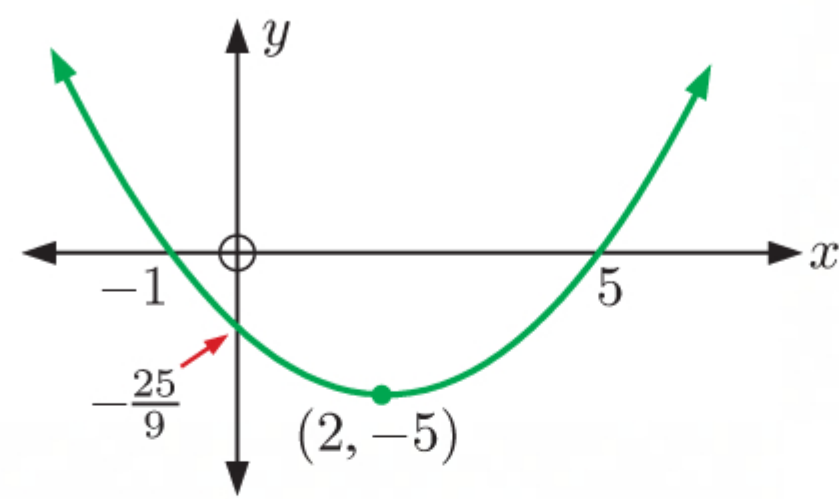
\therefore the range is $\{y \mid y = -1, 1, \text{ or } 2\}$.

2 $g(x) = x^2 - 3x$

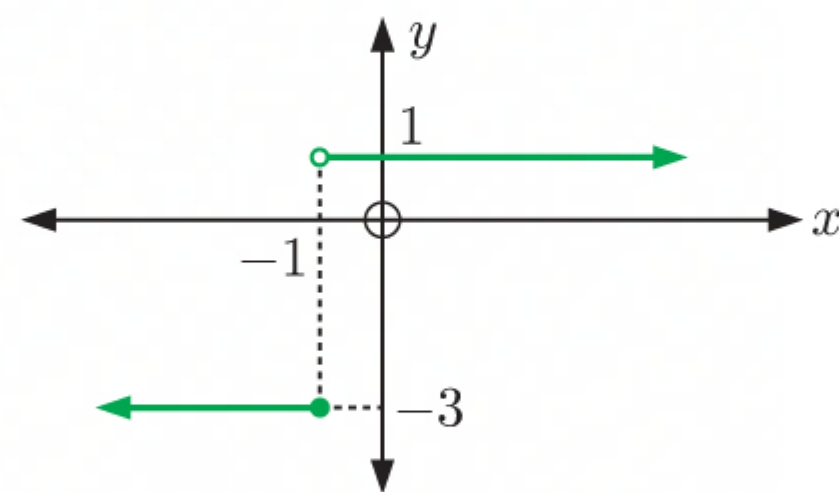
a
$$\begin{aligned}g(x+1) &= (x+1)^2 - 3(x+1) \\ &= x^2 + 2x + 1 - 3x - 3 \\ &= x^2 - x - 2\end{aligned}$$

b
$$\begin{aligned}g(4x) &= (4x)^2 - 3(4x) \\ &= 16x^2 - 12x\end{aligned}$$

- 3 a i** Domain is $\{x \mid x \in \mathbb{R}\}$.
Range is $\{y \mid y \geq -5\}$.
- ii** x -intercepts are -1 and 5 , y -intercept is $-\frac{25}{9}$
- iii** The graph passes the vertical line test, so it is therefore a function.



- b i** Domain is $\{x \mid x \in \mathbb{R}\}$.
Range is $\{y \mid y = 1 \text{ or } -3\}$.
- ii** There are no x -intercepts, y -intercept is 1 .
- iii** The graph passes the vertical line test, so it is therefore a function.



- 4 a** $x + 2y = 10$ is a function, since for any value of x there is at most one value of y .

- b** $x + y^2 = 10$ is not a function.

If $x + y^2 = 10$, then $y = \pm\sqrt{10 - x}$. So, for example, for $x = 1$, $y = \pm 3$.

5 $f(x) = \frac{3x - 1}{x + 2}$

a i $f(-1) = \frac{3(-1) - 1}{-1 + 2}$
 $= -4$

ii $f(0) = \frac{3(0) - 1}{0 + 2}$
 $= -\frac{1}{2}$

iii $f(5) = \frac{3(5) - 1}{5 + 2}$
 $= \frac{14}{7}$
 $= 2$

b $f(x) = \frac{3x - 1}{x + 2}$ is undefined when $x = -2$.

c $f(x - 1) = \frac{3(x - 1) - 1}{(x - 1) + 2}$
 $= \frac{3x - 3 - 1}{x + 1}$
 $= \frac{3x - 4}{x + 1}$

d If $f(x) = 4$, then $\frac{3x - 1}{x + 2} = 4$
 $\therefore 3x - 1 = 4(x + 2)$
 $\therefore 3x - 1 = 4x + 8$
 $\therefore x = -9$

6 a $f(x) = x^2 + 3$
 $\therefore f(-3) = (-3)^2 + 3$
 $= 9 + 3$
 $= 12$

b $f(x) = 4$
 $\therefore x^2 + 3 = 4$
 $\therefore x^2 = 1$
 $\therefore x = \pm 1$

7 a $f(x) = 10 + \frac{3}{2x-1}$ is undefined when $2x - 1 = 0$
 $\therefore 2x = 1$
 $\therefore x = \frac{1}{2}$

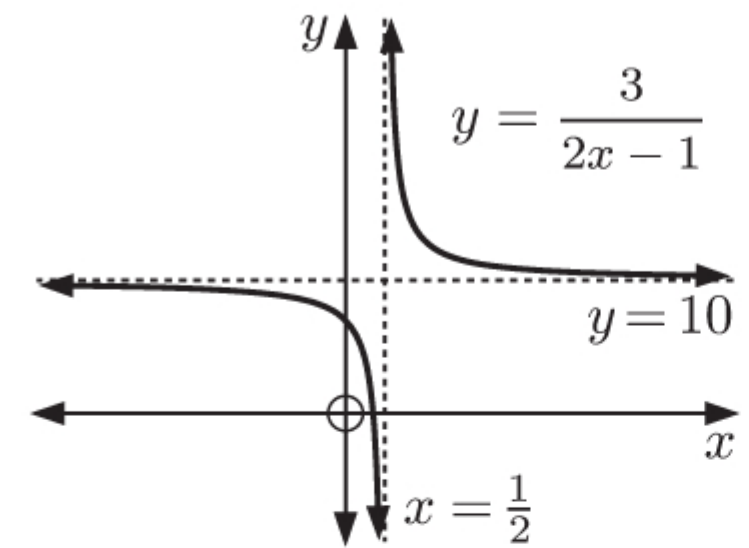
\therefore the domain is $\{x \mid x \neq \frac{1}{2}\}$.

No matter how large or small x is,

$y = \frac{3}{2x-1}$ is never zero.

$\therefore y = f(x)$ is never 10.

\therefore the range is $\{y \mid y \neq 10\}$.

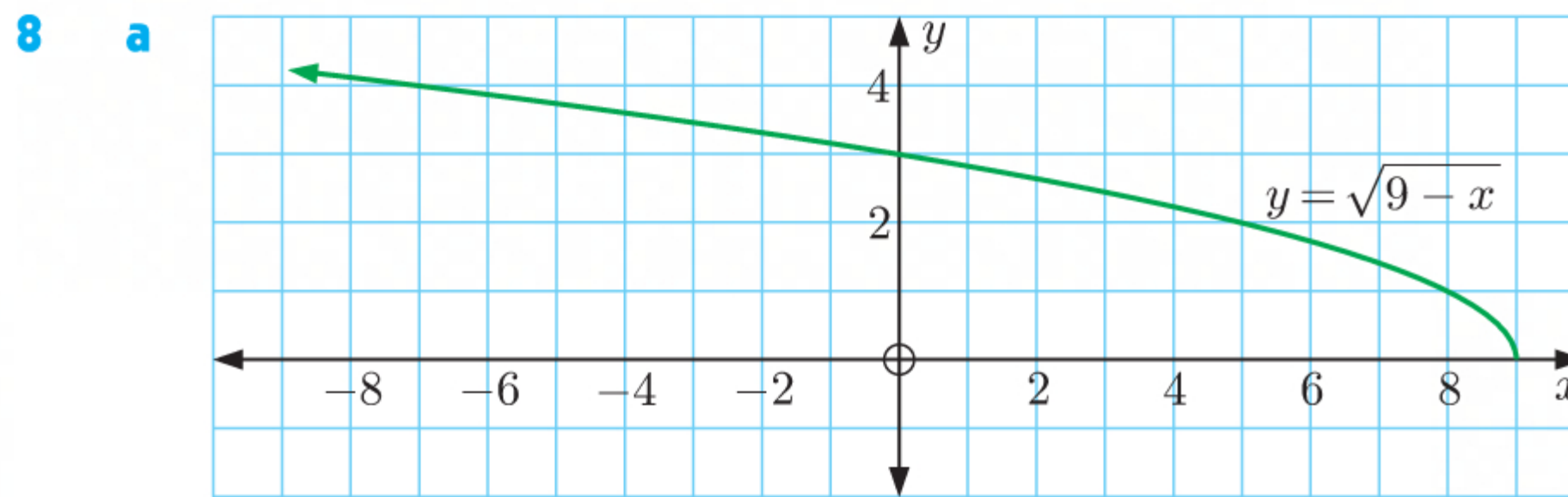
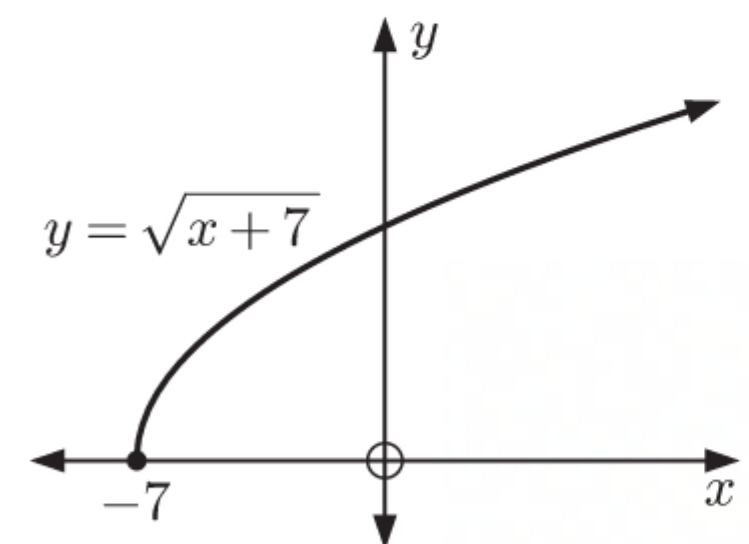


b $f(x) = \sqrt{x+7}$ is defined when $x + 7 \geq 0$
 $\therefore x \geq -7$

\therefore the domain is $\{x \mid x \geq -7\}$.

A square root cannot be negative.

\therefore the range is $\{y \mid y \geq 0\}$.



b The graph passes the vertical line test. There is at most one value of y for any one value of x .
 \therefore the relation is a function.

c $\sqrt{9-x}$ is defined when $9 - x \geq 0$
 $\therefore x \leq 9$

\therefore the domain is $\{x \mid x \leq 9\}$.

A square root cannot be negative.

\therefore the range is $\{y \mid y \geq 0\}$.

9 $f(x) = ax^2 + bx + c$ where $f(0) = 5$, $f(-2) = 21$, and $f(3) = -4$

So, $a(0)^2 + b(0) + c = 5$

$\therefore c = 5$

Now $a(-2)^2 + b(-2) + 5 = 21$

$\therefore 4a - 2b + 5 = 21$

$\therefore 4a - 2b = 16$

$\therefore 2a - b = 8$

$\therefore b = 2a - 8 \dots (*)$

and $a(3)^2 + b(3) + 5 = -4$

$\therefore 9a + 3b = -9$

$\therefore 3a + b = -3$

$\therefore 3a + (2a - 8) = -3$ {using (*)}

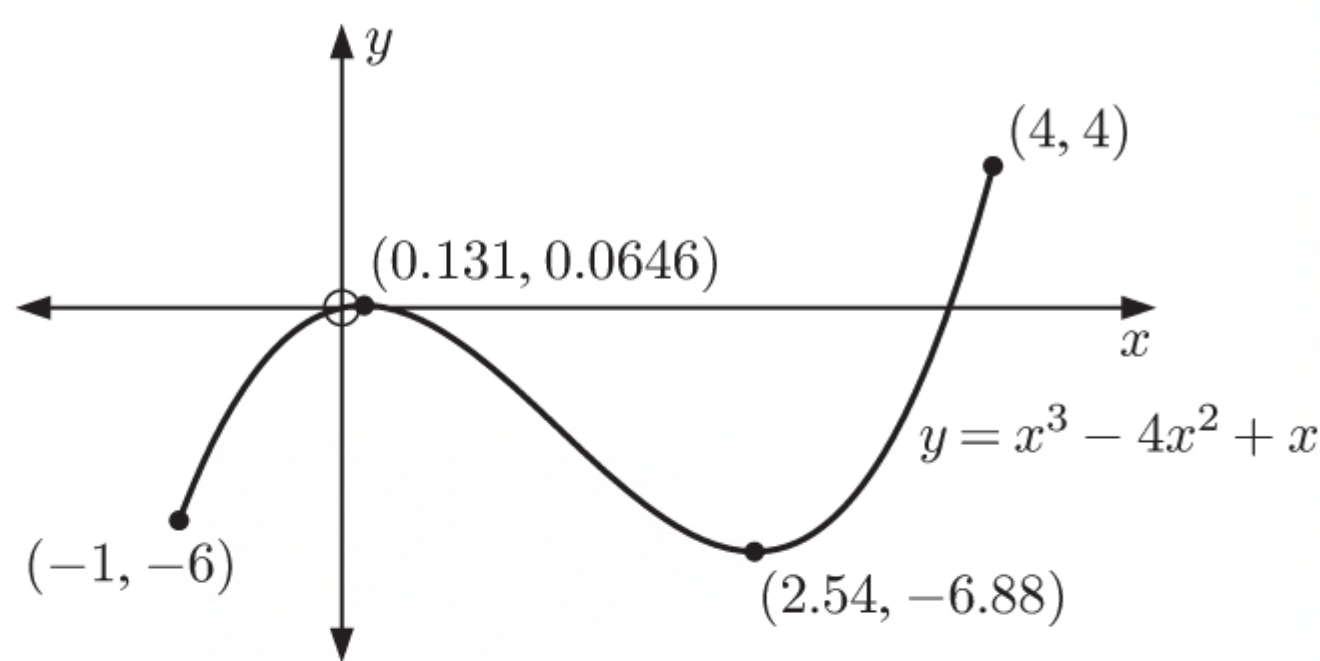
$\therefore 5a = 5$

$\therefore a = 1$

Substituting $a = 1$ into (*) gives $b = 2(1) - 8 = -6$.

So, $a = 1$, $b = -6$, $c = 5$.

10

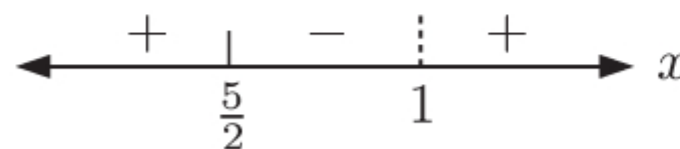


The range is $\{y \mid -6.88 \leq y \leq 4\}$.

- 11 a $\frac{2x-5}{x-4}$ is zero when $x = \frac{5}{2}$ and undefined when $x = 4$.



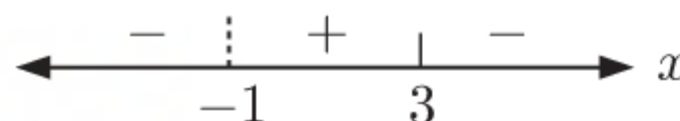
Since $(2x-5)$ and $(x-4)$ are single factors, the signs alternate.



- b $\frac{3-x}{x+1}$ is zero when $x = 3$ and undefined when $x = -1$.



Since $(3-x)$ and $(x+1)$ are single factors, the signs alternate.



- 12 $f(x) = -1 + \frac{3}{x+2}$

- a The vertical asymptote is $x = -2$.
The horizontal asymptote is $y = -1$.

- b The domain is $\{x \mid x \neq -2\}$.
The range is $\{y \mid y \neq -1\}$.

- c $f(0) = -1 + \frac{3}{2} = \frac{1}{2}$
 \therefore the y -intercept is $\frac{1}{2}$.

$$f(x) = 0 \text{ when } -1 + \frac{3}{x+2} = 0$$

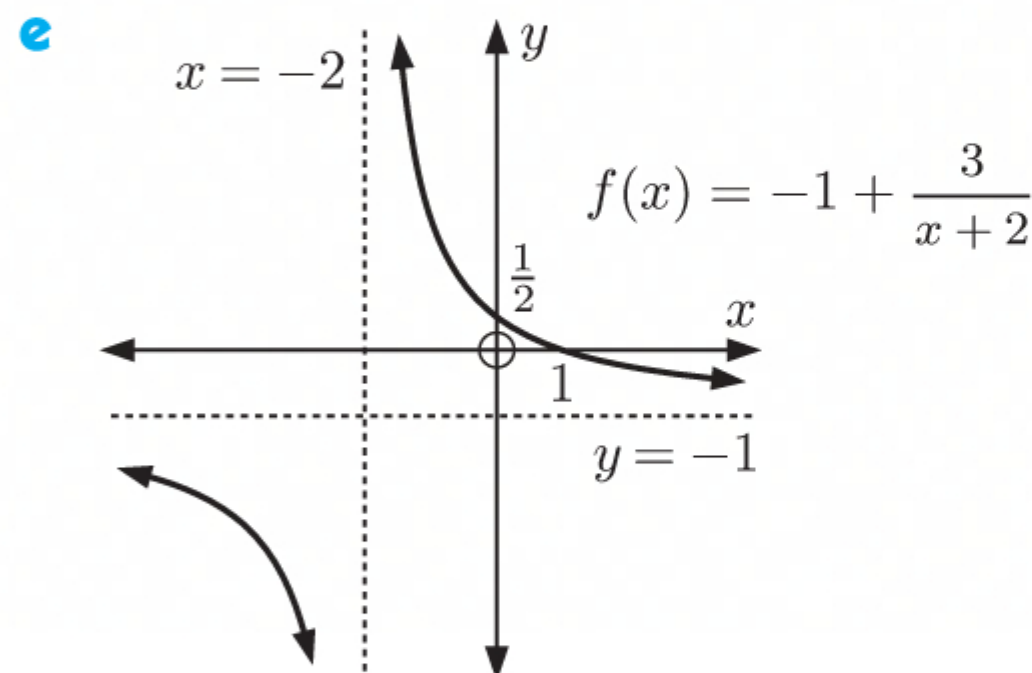
$$\therefore \frac{3}{x+2} = 1$$

$$\therefore 3 = x + 2$$

$$\therefore x = 1$$

\therefore the x -intercept is 1.

- d As $x \rightarrow -2^-$, $f(x) \rightarrow -\infty$
As $x \rightarrow -2^+$, $f(x) \rightarrow \infty$
As $x \rightarrow -\infty$, $f(x) \rightarrow -1^-$
As $x \rightarrow \infty$, $f(x) \rightarrow -1^+$



13 $f(x) = 3 - x^2$ and $g(x) = 2x - 1$

a $(f \circ g)(x) = f(g(x))$
 $= f(2x - 1)$
 $= 3 - (2x - 1)^2$
 $= 3 - 4x^2 + 4x - 1$
 $= -4x^2 + 4x + 2$

b $(g \circ f)(x) = g(f(x))$
 $= g(3 - x^2)$
 $= 2(3 - x^2) - 1$
 $= 6 - 2x^2 - 1$
 $= 5 - 2x^2$

c $(f \circ f)(-2) = f(f(-2))$
 $= f(3 - (-2)^2)$
 $= f(-1)$
 $= 3 - (-1)^2$
 $= 2$

14 $f(x) = \frac{1}{x^2}$ and $g(x) = x^2 - 4x + 3$

$$(f \circ g)(x) = f(g(x))$$

$$= f(x^2 - 4x + 3)$$

$$= \frac{1}{(x^2 - 4x + 3)^2}$$

$$= \frac{1}{[(x - 3)(x - 1)]^2}$$

\therefore the domain is $\{x \mid x \neq 3 \text{ or } 1\}$ and the range is $\{y \mid y > 0\}$.

15 $f(x) = 3x + 5$ and $g(x) = 2x^2 - x$

a i $(f \circ g)(x) = f(g(x))$
 $= f(2x^2 - x)$
 $= 3(2x^2 - x) + 5$
 $= 6x^2 - 3x + 5$

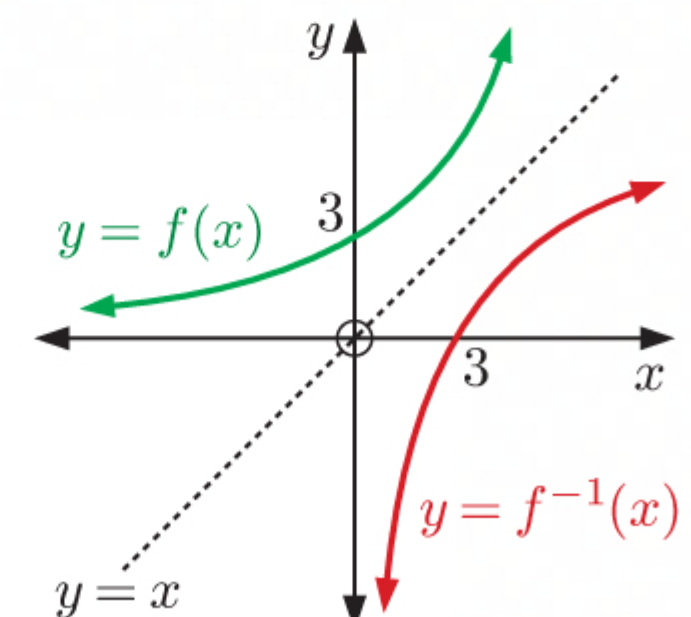
ii $(g \circ f)(x) = g(f(x))$
 $= g(3x + 5)$
 $= 2(3x + 5)^2 - (3x + 5)$
 $= 2(9x^2 + 30x + 25) - 3x - 5$
 $= 18x^2 + 60x + 50 - 3x - 5$
 $= 18x^2 + 57x + 45$

b $3(f \circ g)(x) = (g \circ f)(x)$
 $\therefore 3(6x^2 - 3x + 5) = 18x^2 + 57x + 45$
 $\therefore 18x^2 - 9x + 15 = 18x^2 + 57x + 45$
 $\therefore -30 = 66x$
 $\therefore x = -\frac{5}{11}$

16 a f passes through $(0, 3)$.

$\therefore f^{-1}$ passes through $(3, 0)$.

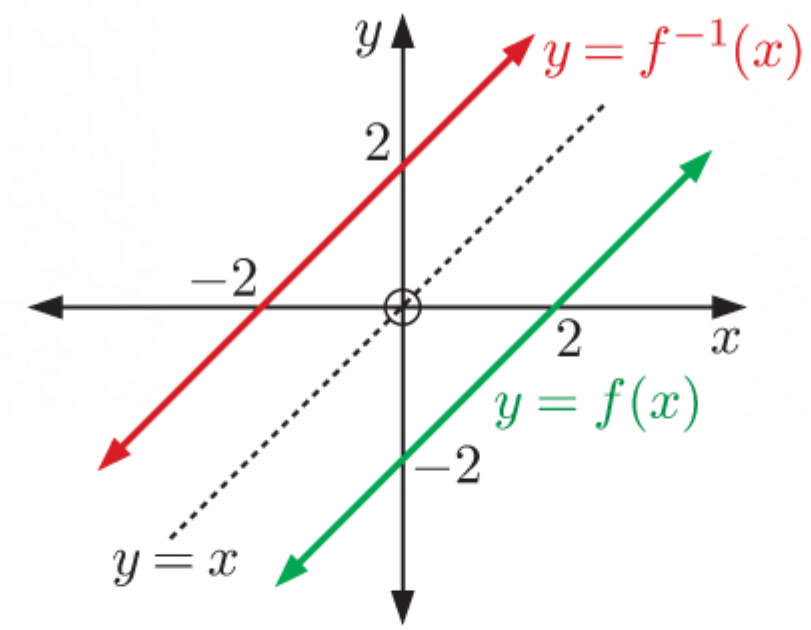
The graphs are reflections of each other in the line $y = x$.



b f passes through $(0, -2)$ and $(2, 0)$.

$\therefore f^{-1}$ passes through $(-2, 0)$ and $(0, 2)$.

The graphs are reflections of each other in the line $y = x$.



17 a f is $y = 7 - 4x$,

$\therefore f^{-1}$ is $x = 7 - 4y$

$\therefore x - 7 = -4y$

$\therefore \frac{x - 7}{-4} = y$

$\therefore \frac{7 - x}{4} = y$

$\therefore f^{-1}(x) = \frac{7 - x}{4}$

b f is $y = \frac{3 + 2x}{5}$,

$\therefore f^{-1}$ is $x = \frac{3 + 2y}{5}$

$\therefore 5x = 3 + 2y$

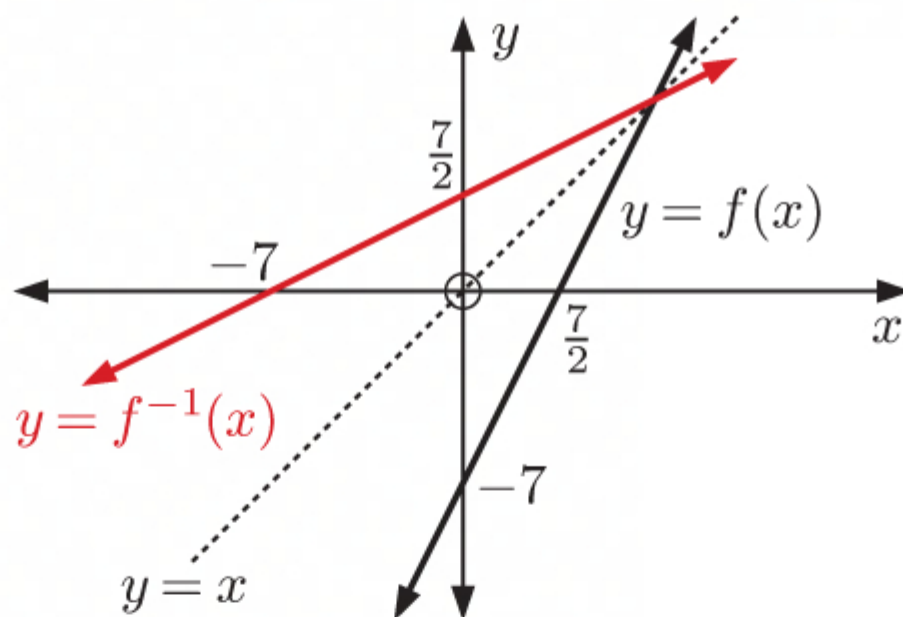
$\therefore 5x - 3 = 2y$

$\therefore \frac{5x - 3}{2} = y$

$\therefore f^{-1}(x) = \frac{5x - 3}{2}$

18 $f : x \mapsto 2x - 7$

a



b f is $y = 2x - 7$

$\therefore f^{-1}$ is $x = 2y - 7$

$\therefore x + 7 = 2y$

$\therefore \frac{x + 7}{2} = y$

$\therefore f^{-1}(x) = \frac{x + 7}{2}$

c $(f \circ f^{-1})(x) = f(f^{-1}(x))$

$$= f\left(\frac{x + 7}{2}\right)$$

$$= 2\left(\frac{x + 7}{2}\right) - 7$$

$$= x + 7 - 7$$

$$= x$$

$\therefore (f \circ f^{-1})(x) = (f^{-1} \circ f)(x) = x$ as required

$(f^{-1} \circ f)(x) = f^{-1}(f(x))$

$$= f^{-1}(2x - 7)$$

$$= \frac{2x - 7 + 7}{2}$$

$$= \frac{2x}{2}$$

$$= x$$

19 f is $y = 5x - 2$

$\therefore f^{-1}$ is $x = 5y - 2$

$$\therefore y = \frac{x+2}{5}$$

$$\therefore f^{-1}(x) = \frac{x+2}{5}$$

$$\begin{aligned}(h \circ f)(x) &= h(f(x)) \\ &= h(5x - 2) \\ &= \frac{3(5x - 2)}{4} \\ &= \frac{15x - 6}{4}\end{aligned}$$

$$\therefore (h \circ f)(x) = \frac{15x - 6}{4} \quad \dots (*)$$

$$\begin{aligned}\text{Now } (f^{-1} \circ h^{-1})(x) &= f^{-1}(h^{-1}(x)) \\ &= f^{-1}\left(\frac{4x}{3}\right) \\ &= \frac{\frac{4x}{3} + 2}{5} \\ &= \frac{4x + 6}{15}\end{aligned}$$

So, $(f^{-1} \circ h^{-1})(x) = (h \circ f)^{-1}(x)$ as required.

h is $y = \frac{3x}{4}$

$\therefore h^{-1}$ is $x = \frac{3y}{4}$

$$\therefore y = \frac{4x}{3}$$

$$\therefore h^{-1}(x) = \frac{4x}{3}$$

$h \circ f$ is $y = \frac{15x - 6}{4}$ {using (*)}

$\therefore (h \circ f)^{-1}$ is $x = \frac{15y - 6}{4}$

$$\therefore 4x = 15y - 6$$

$$\therefore y = \frac{4x + 6}{15}$$

$$\therefore (h \circ f)^{-1}(x) = \frac{4x + 6}{15}$$

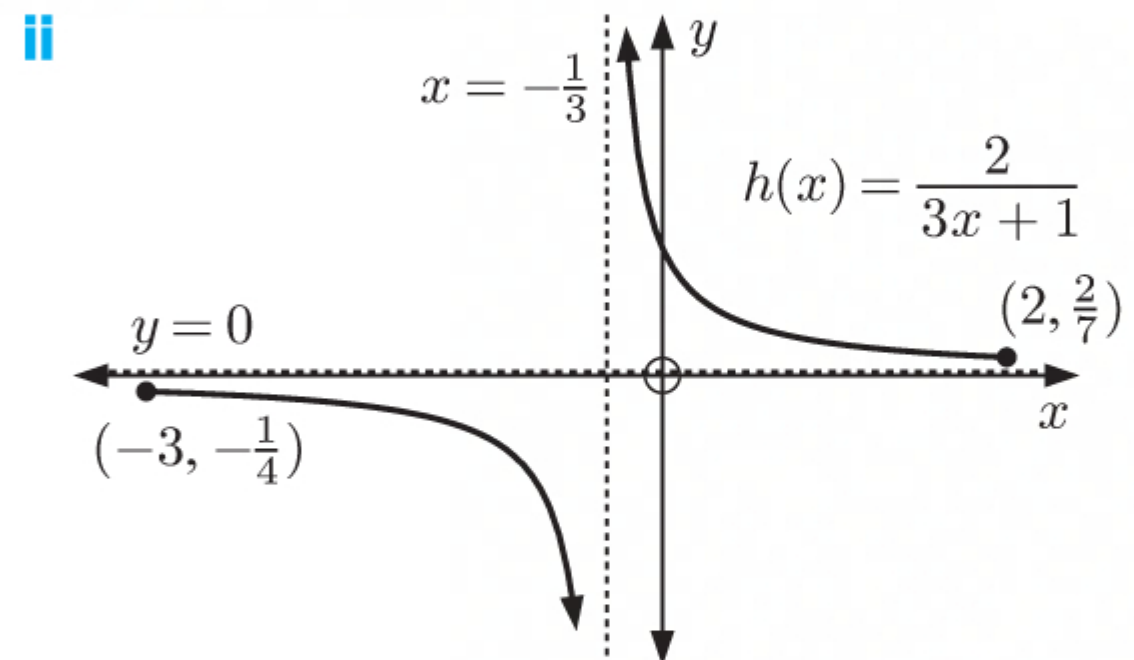
20 $f(x) = 3x + 1$ and $g(x) = \frac{2}{x}$

a $(g \circ f)(x) = g(f(x))$
 $= g(3x + 1)$
 $= \frac{2}{3x + 1}$

b $(g \circ f)(x) = -4$
 $\therefore \frac{2}{3x + 1} = -4$
 $\therefore -4(3x + 1) = 2$
 $\therefore -12x - 4 = 2$
 $\therefore -12x = 6$
 $\therefore x = -\frac{1}{2}$

c $h(x) = \frac{2}{3x + 1}, \quad x \neq -\frac{1}{3}$

- i** The vertical asymptote is $x = -\frac{1}{3}$.
 The horizontal asymptote is $y = 0$.



- iii** The range of h is $\{y \mid y \leq -\frac{1}{4} \text{ or } y \geq \frac{2}{7}\}$.

21 $f(x) = 2x + 11$ and $g(x) = x^2$

f is $y = 2x + 11$

$\therefore f^{-1}$ is $x = 2y + 11$

$\therefore 2y = x - 11$

$\therefore y = \frac{x - 11}{2}$

So, $f^{-1}(x) = \frac{x - 11}{2}$ (*)

$(g \circ f^{-1})(3) = g(f^{-1}(3))$

$= g\left(\frac{3 - 11}{2}\right)$ {using (*)}

$= g(-4)$

$= (-4)^2$

$= 16$

22 $f(x) = \frac{ax + 3}{x - b}$

a $f(x)$ has vertical asymptote $x = -1$, so the function is undefined when $x = -1$.

$\therefore -1 - b = 0$

$\therefore b = -1$

$f(x) = \frac{ax + 3}{x - 1}$

$= \frac{a(x - 1) + a + 3}{x - 1}$

$= a + \frac{a + 3}{x - 1}$ which has horizontal asymptote $y = 2$

$\therefore a = 2$

So, $a = 2$, $b = -1$.

b $f(x)$ has domain $\{x \mid x \neq -1\}$ and range $\{y \mid y \neq 2\}$.

$\therefore f^{-1}(x)$ has domain $\{x \mid x \neq 2\}$ and range $\{y \mid y \neq -1\}$.

23 $f : x \mapsto 2x + 1$, $g : x \mapsto \frac{x + 1}{x - 2}$

a $(f \circ g)(x) = f(g(x))$

$= f\left(\frac{x + 1}{x - 2}\right)$

$= 2\left(\frac{x + 1}{x - 2}\right) + 1$

$= \frac{2x + 2}{x - 2} + \frac{x - 2}{x - 2}$

$= \frac{3x}{x - 2}$

b g is $y = \frac{x + 1}{x - 2}$

$\therefore g^{-1}$ is $x = \frac{y + 1}{y - 2}$

$\therefore x(y - 2) = y + 1$

$\therefore xy - 2x = y + 1$

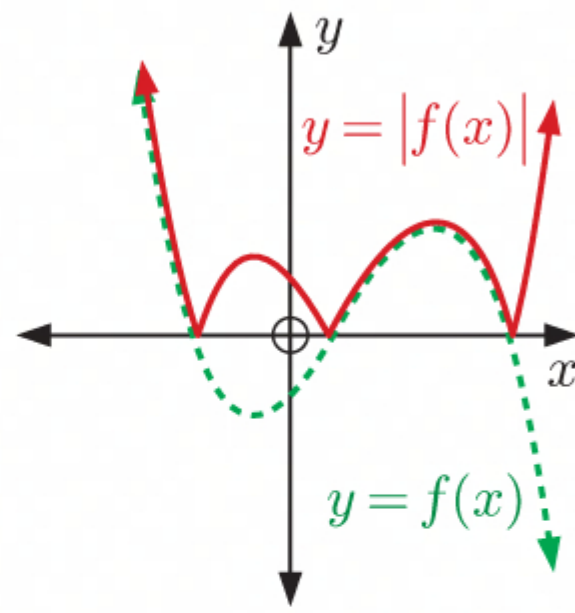
$\therefore xy - y = 2x + 1$

$\therefore y(x - 1) = 2x + 1$

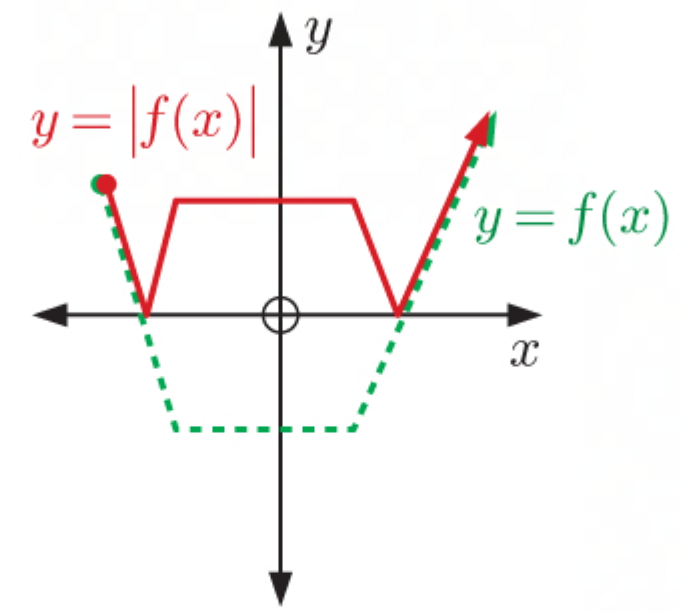
$\therefore y = \frac{2x + 1}{x - 1}$

So, $g^{-1}(x) = \frac{2x + 1}{x - 1}$

- 24 a** The graph is unchanged for $f(x) \geq 0$ and reflected in the x -axis for $f(x) < 0$.



- b** The graph is unchanged for $f(x) \geq 0$ and reflected in the x -axis for $f(x) < 0$.



Chapter 4

TRANSFORMATIONS OF FUNCTIONS

INVESTIGATION 1

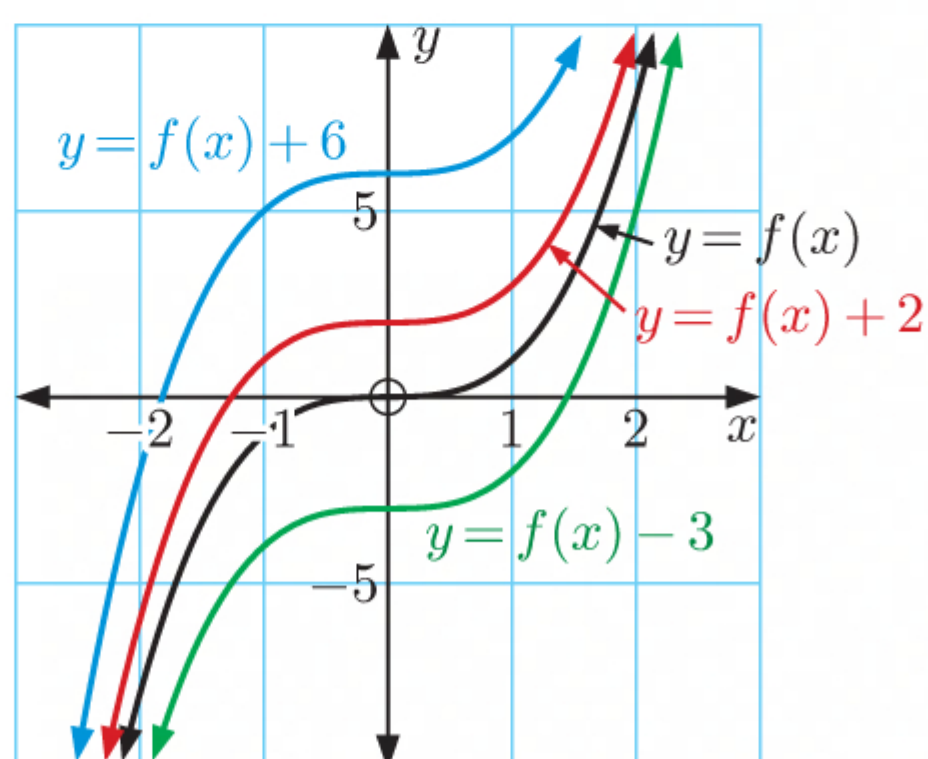
TRANSLATIONS

1 $f(x) = x^3$

a i $f(x) + 2 = x^3 + 2$

ii $f(x) - 3 = x^3 - 3$

iii $f(x) + 6 = x^3 + 6$



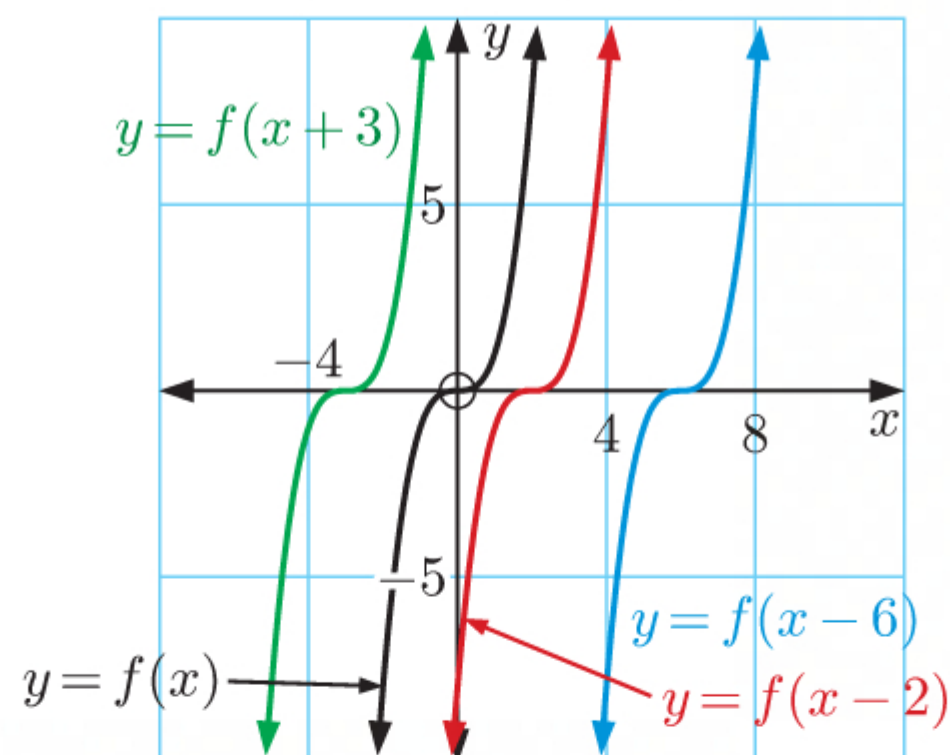
For $y = f(x) + b$, the effect of b is to **translate** the graph **vertically** through b units.

- If $b > 0$ it moves **upwards**.
- If $b < 0$ it moves **downwards**.

b i $f(x - 2) = (x - 2)^3$

ii $f(x + 3) = (x + 3)^3$

iii $f(x - 6) = (x - 6)^3$



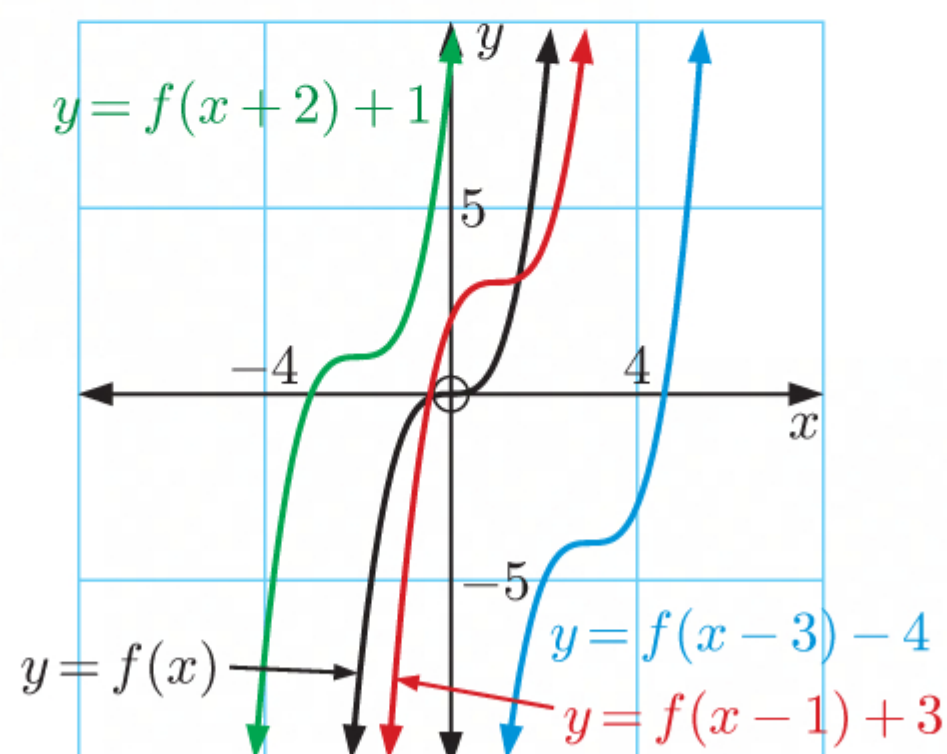
For $y = f(x - a)$, the effect of a is to **translate** the graph **horizontally** through a units.

- If $a > 0$ it moves to the **right**.
- If $a < 0$ it moves to the **left**.

c i $f(x - 1) + 3 = (x - 1)^3 + 3$

ii $f(x + 2) + 1 = (x + 2)^3 + 1$

iii $f(x - 3) - 4 = (x - 3)^3 - 4$

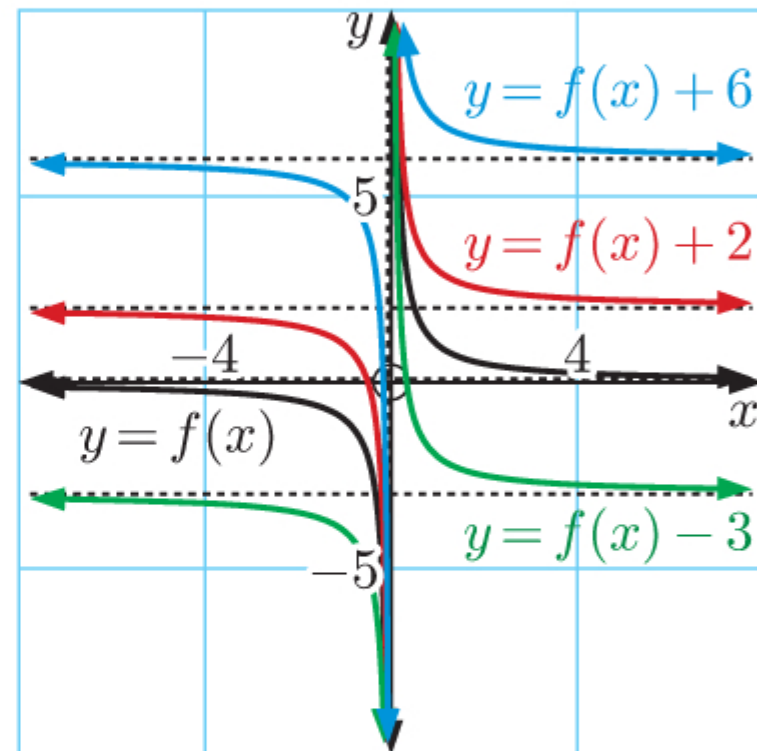


2 $f(x) = \frac{1}{x}$

a i $f(x) + 2 = \frac{1}{x} + 2$

ii $f(x) - 3 = \frac{1}{x} - 3$

iii $f(x) + 6 = \frac{1}{x} + 6$



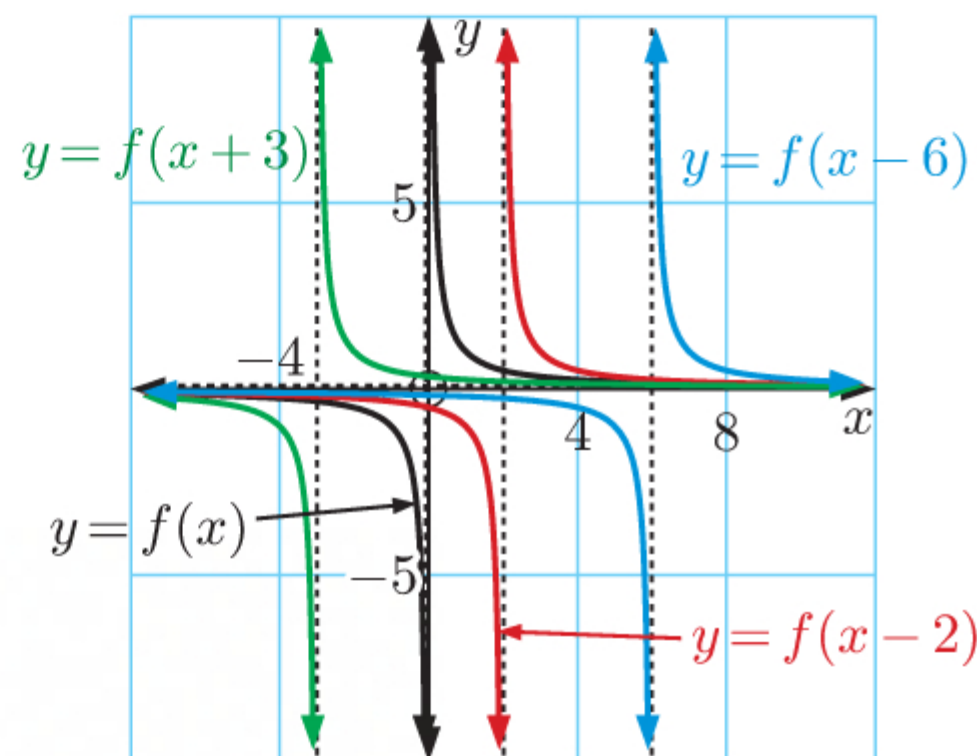
For $y = f(x) + b$, the effect of b is to **translate** the graph **vertically** through b units.

- If $b > 0$ it moves **upwards**.
- If $b < 0$ it moves **downwards**.

b i $f(x - 2) = \frac{1}{x - 2}$

ii $f(x + 3) = \frac{1}{x + 3}$

iii $f(x - 6) = \frac{1}{x - 6}$



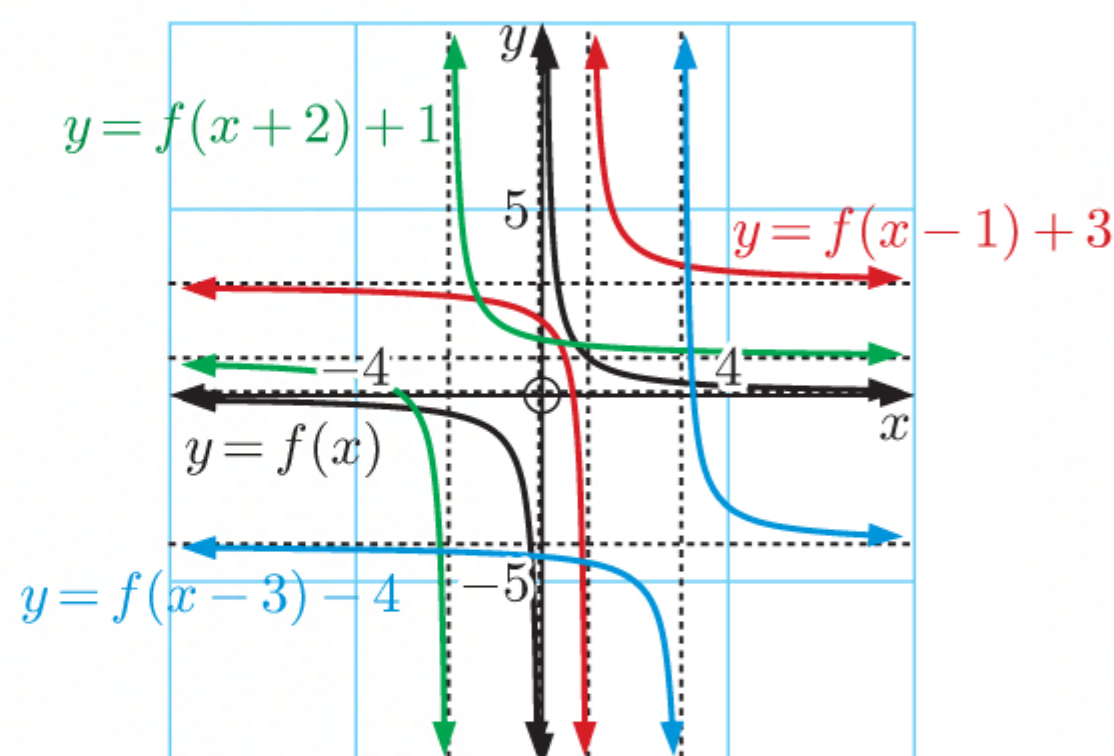
For $y = f(x - a)$, the effect of a is to **translate** the graph **horizontally** through a units.

- If $a > 0$ it moves to the **right**.
- If $a < 0$ it moves to the **left**.

c i $f(x - 1) + 3 = \frac{1}{x - 1} + 3$

ii $f(x + 2) + 1 = \frac{1}{x + 2} + 1$

iii $f(x - 3) - 4 = \frac{1}{x - 3} - 4$



3 a A translation b units vertically will map $y = f(x)$ onto $y = f(x) + b$.

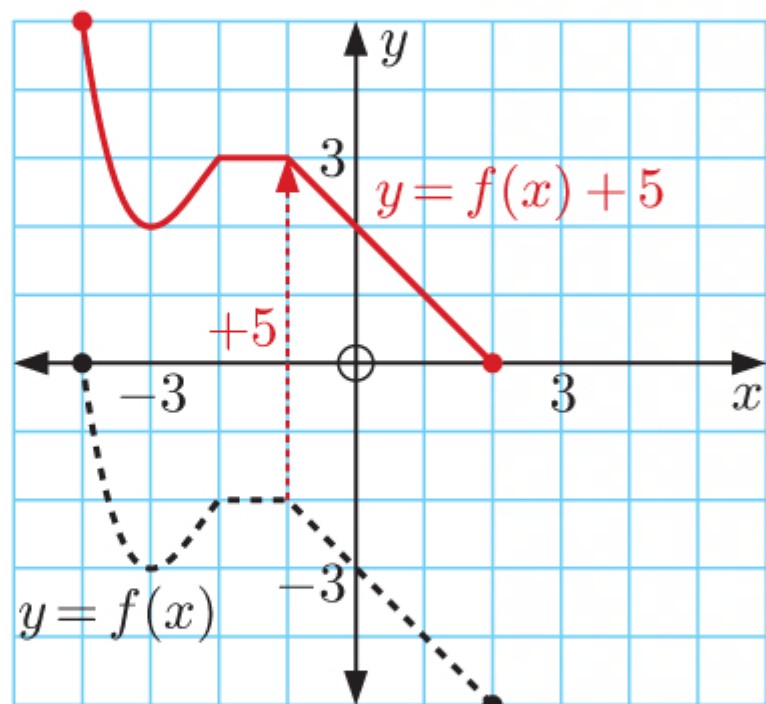
b A translation a units horizontally will map $y = f(x)$ onto $y = f(x - a)$.

c A translation a units horizontally and a translation b units vertically will map $y = f(x)$ onto $y = f(x - a) + b$.

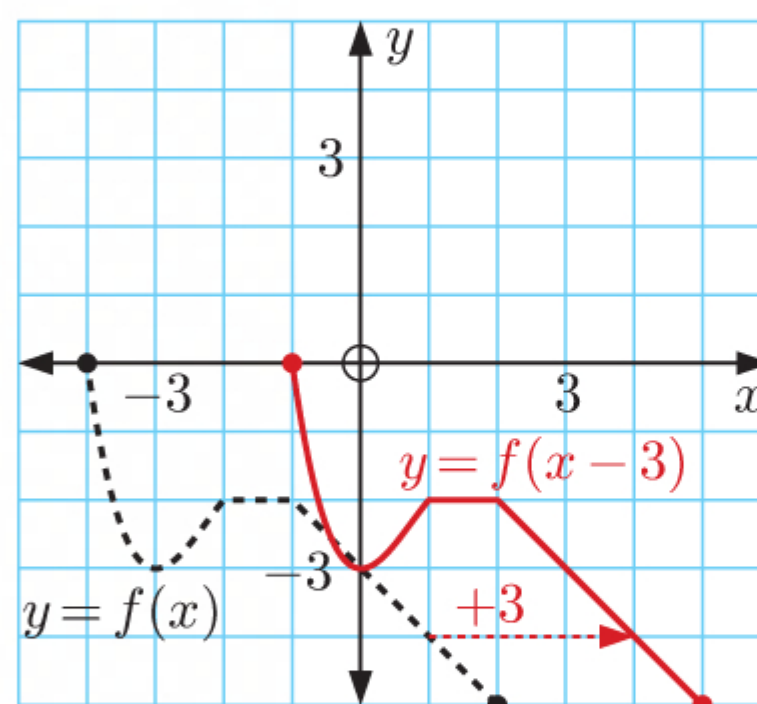
4 No, none of these transformations change the *shape* of the graph.

EXERCISE 4A

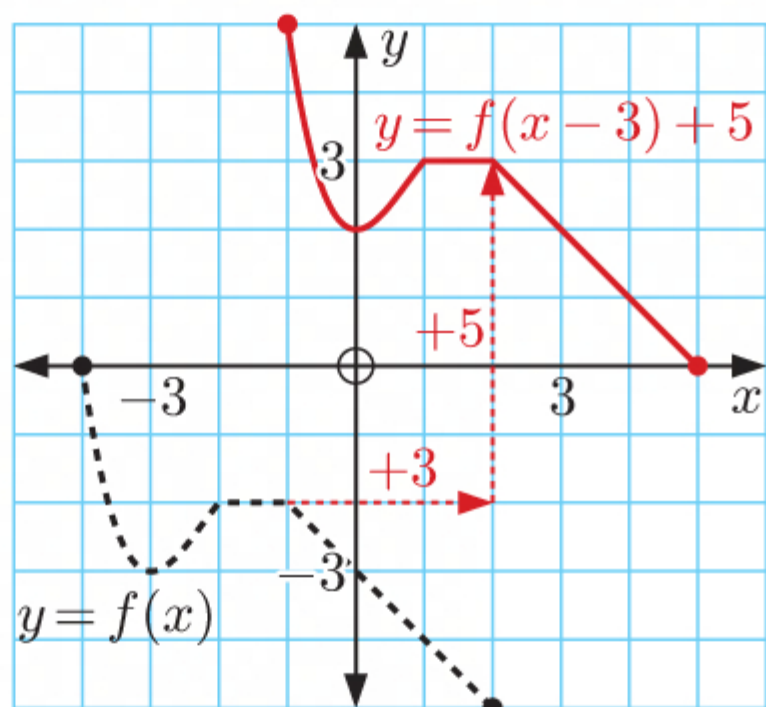
- 1 a The graph of $y = f(x) + 5$ is found by translating $y = f(x)$ 5 units upwards.



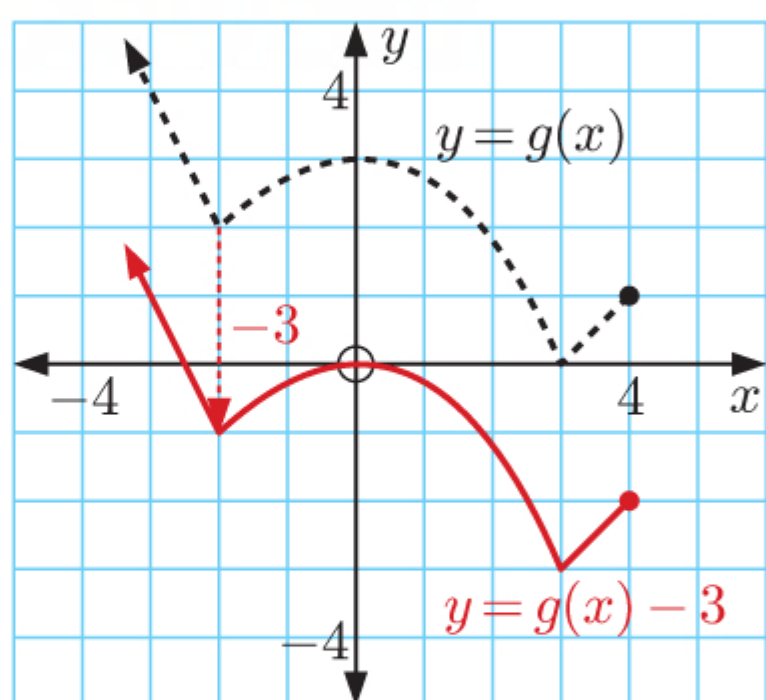
- b The graph of $y = f(x - 3)$ is found by translating $y = f(x)$ 3 units to the right.



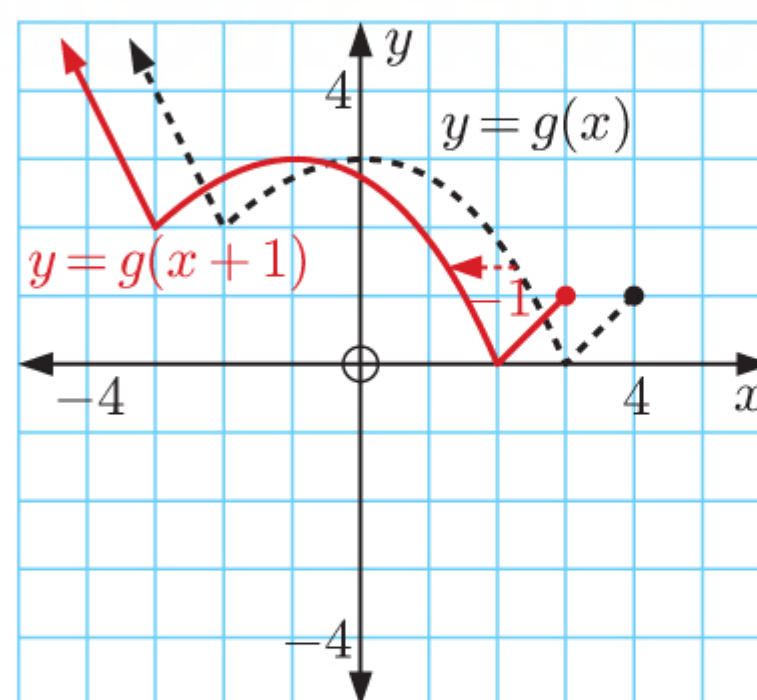
- c The graph of $y = f(x - 3) + 5$ is found by translating $y = f(x)$ 3 units to the right and 5 units upwards.



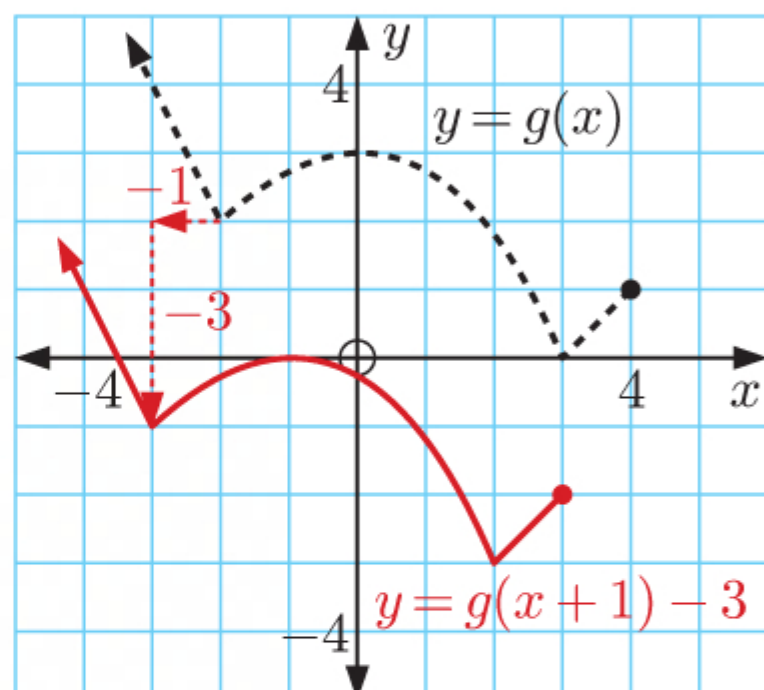
- 2 a The graph of $y = g(x) - 3$ is found by translating $y = g(x)$ 3 units downwards.



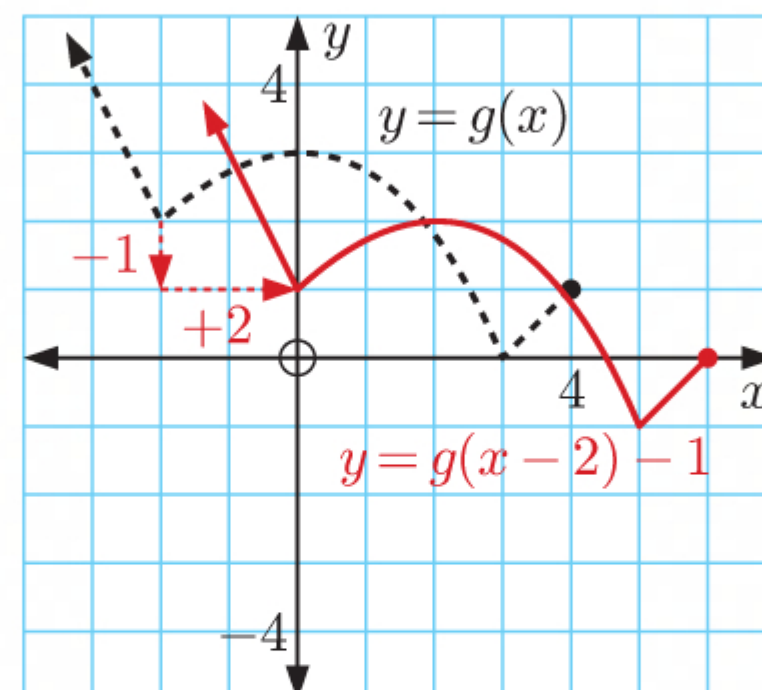
- b The graph of $y = g(x + 1)$ is found by translating $y = g(x)$ 1 unit to the left.



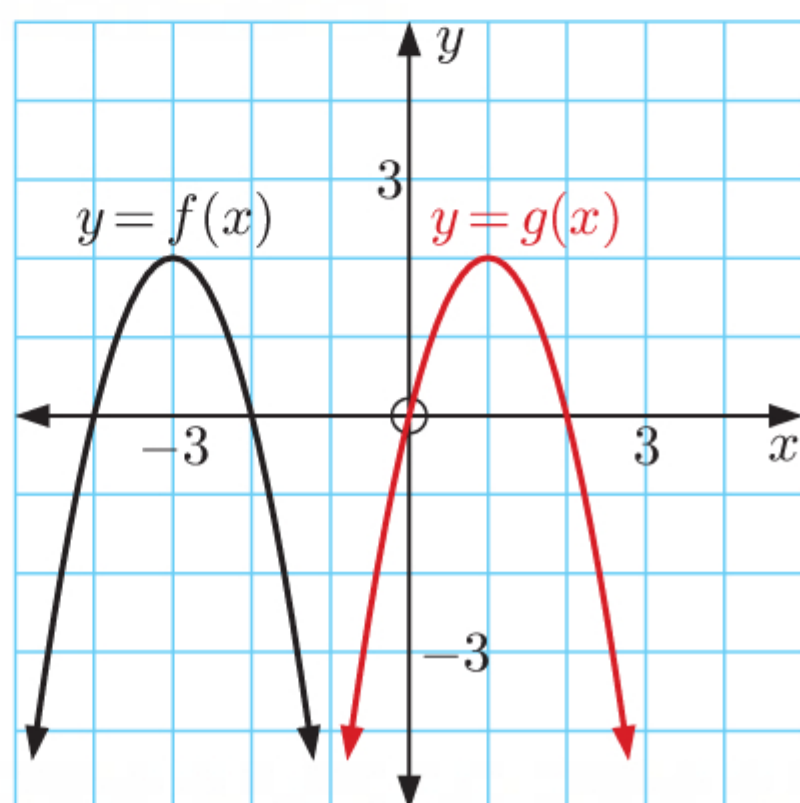
- c** The graph of $y = g(x + 1) - 3$ is found by translating $y = g(x)$ 1 unit to the left and 3 units downwards.



- d** The graph of $y = g(x - 2) - 1$ is found by translating $y = g(x)$ 2 units to the right and 1 unit downwards.



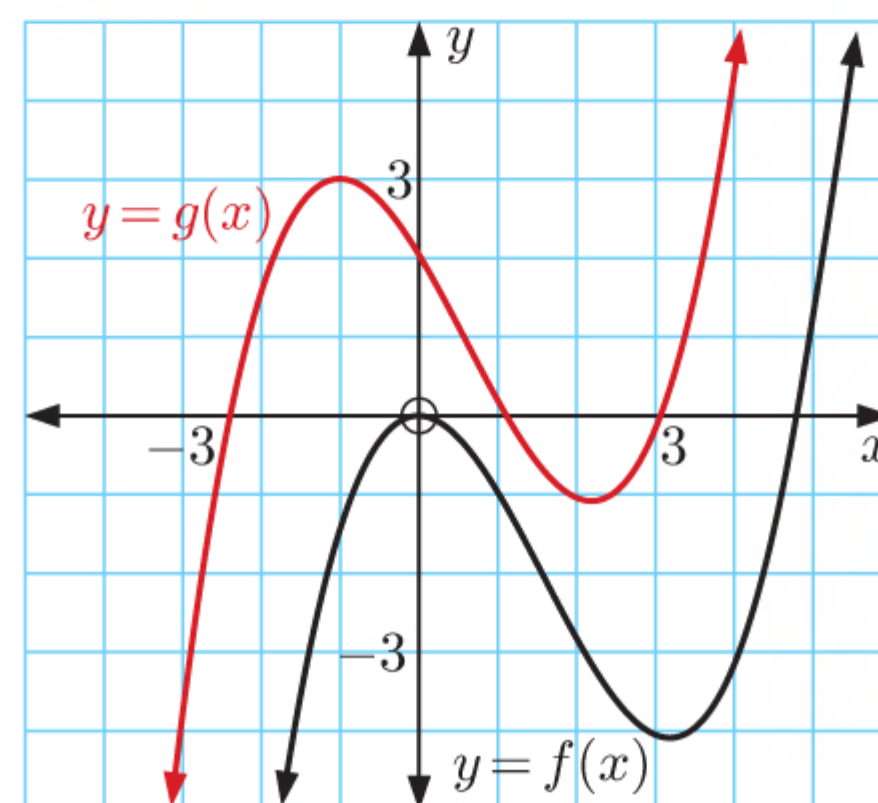
3 a



The graph of $y = f(x)$ has been translated 4 units to the right to result in $y = g(x)$.

So, $g(x) = f(x - 4)$.

b



The graph of $y = f(x)$ has been translated 1 unit to the left, and 3 units upwards, to result in $y = g(x)$.

So, $g(x) = f(x + 1) + 3$.

- 4 a** The graph of $y = g(x)$ is found by translating $y = f(x)$ 4 units downwards.

$$\therefore g(x) = f(x) - 4$$

$$\therefore g(x) = (2x + 3) - 4 \quad \{\text{since } f(x) = 2x + 3\}$$

$$\therefore g(x) = 2x - 1$$

- b** The graph of $y = g(x)$ is found by translating $y = f(x)$ 2 units to the left.

$$\therefore g(x) = f(x + 2)$$

$$\therefore g(x) = 3(x + 2) - 4 \quad \{\text{since } f(x) = 3x - 4\}$$

$$\therefore g(x) = 3x + 2$$

- c** The graph of $y = g(x)$ is found by translating $y = f(x)$ 3 units upwards.

$$\therefore g(x) = f(x) + 3$$

$$\therefore g(x) = (-x^2 + 5x - 7) + 3 \quad \{\text{since } f(x) = -x^2 + 5x - 7\}$$

$$\therefore g(x) = -x^2 + 5x - 4$$

- d** The graph of $y = g(x)$ is found by translating $y = f(x)$ 5 units to the right.

$$\therefore g(x) = f(x - 5)$$

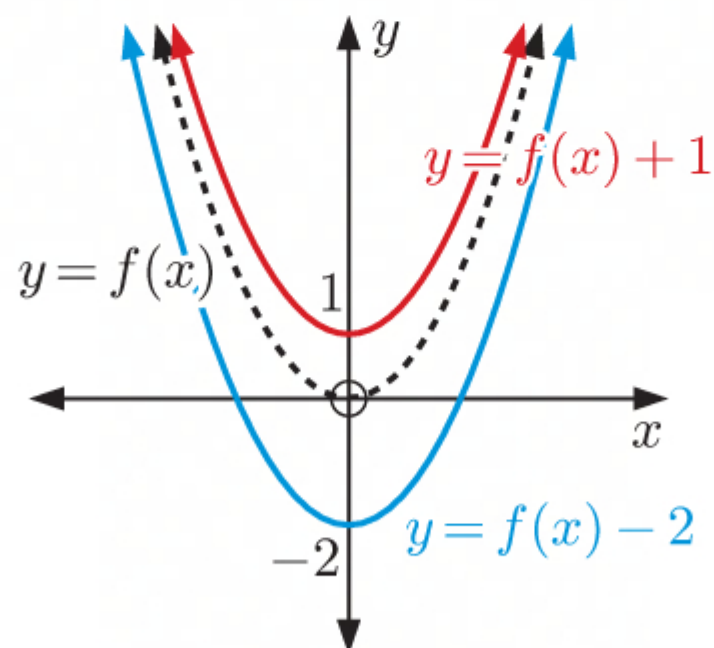
$$\therefore g(x) = (x - 5)^2 + 4(x - 5) - 1 \quad \{\text{since } f(x) = x^2 + 4x - 1\}$$

$$\therefore g(x) = x^2 - 10x + 25 + 4x - 20 - 1$$

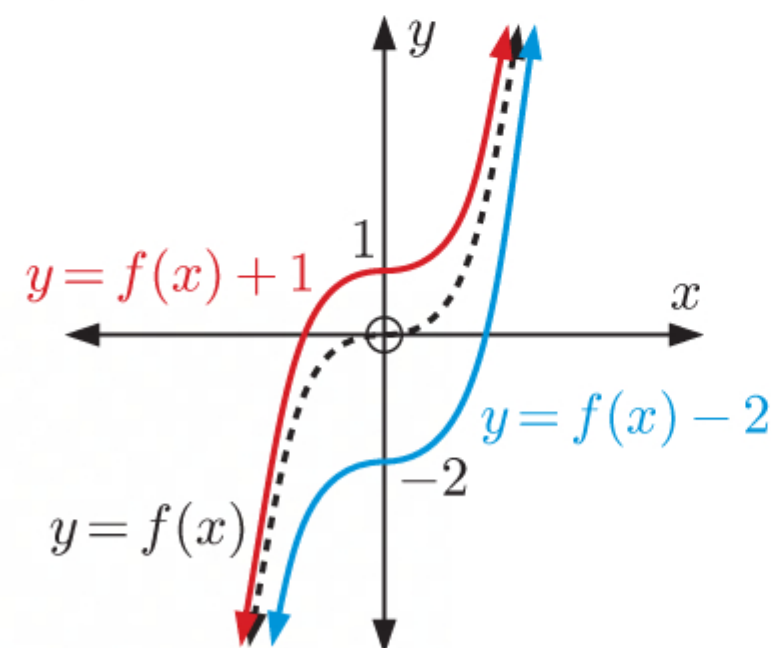
$$\therefore g(x) = x^2 - 6x + 4$$

- 5** $y = f(x) + 1$ is found by translating $y = f(x)$ 1 unit upwards, $y = f(x) - 2$ is found by translating $y = f(x)$ 2 units downwards.

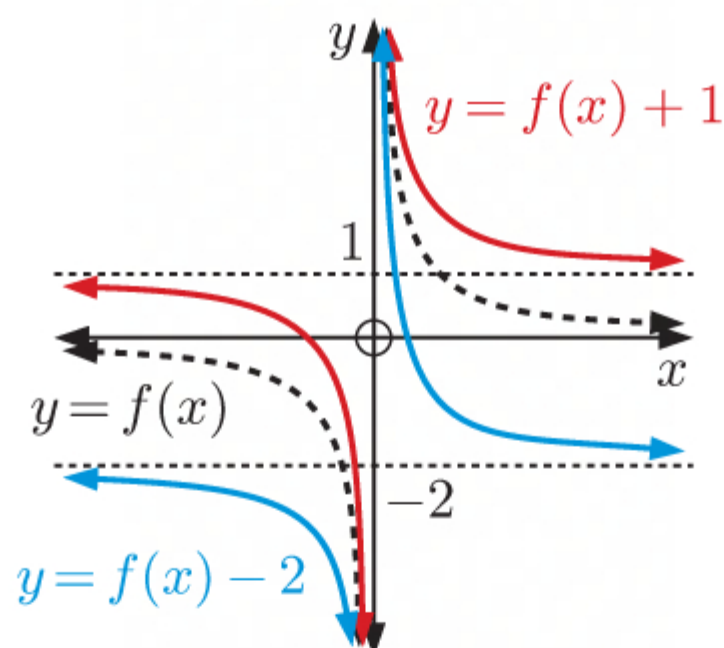
a $f(x) = x^2$



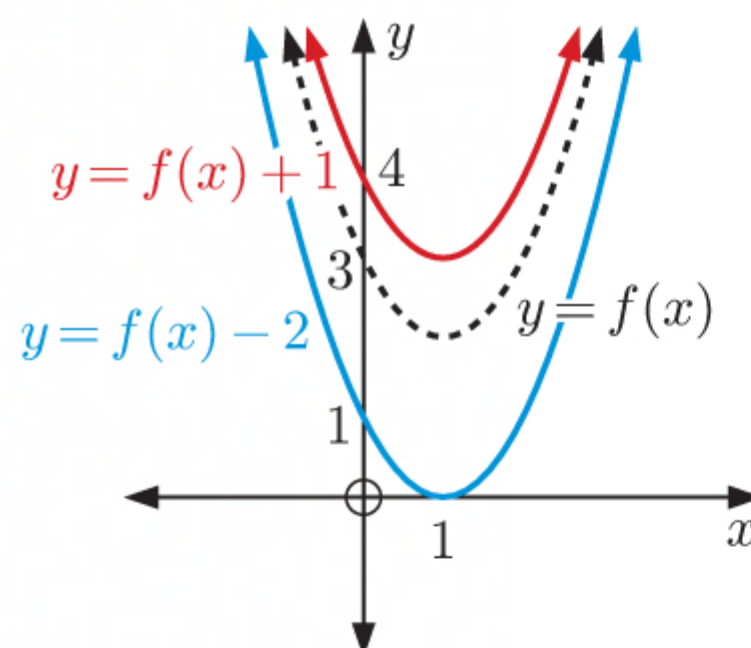
b $f(x) = x^3$



c $f(x) = \frac{1}{x}$

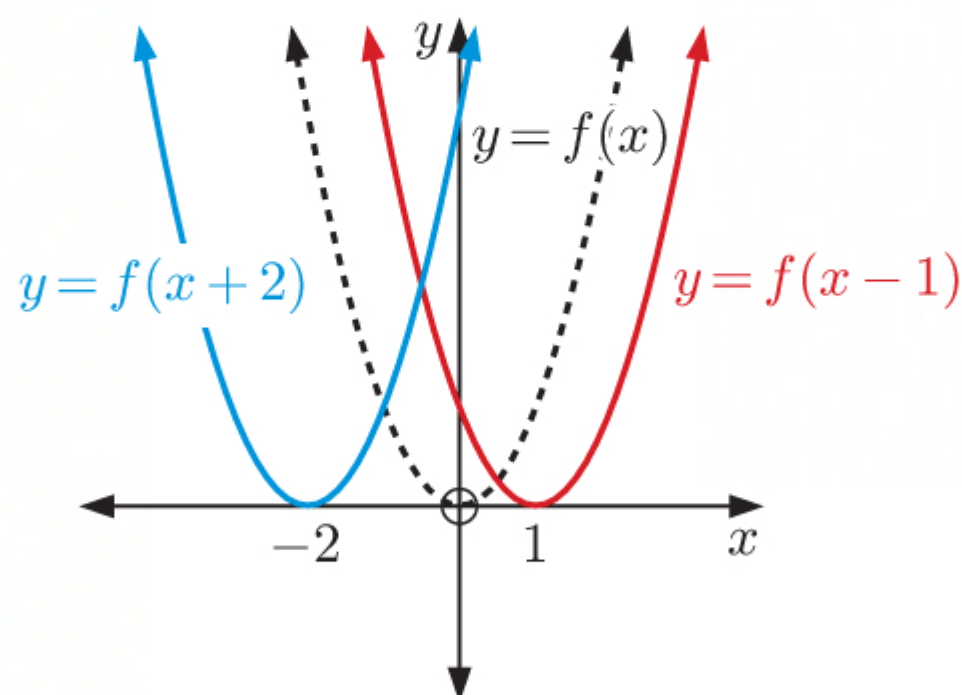


d $f(x) = (x - 1)^2 + 2$

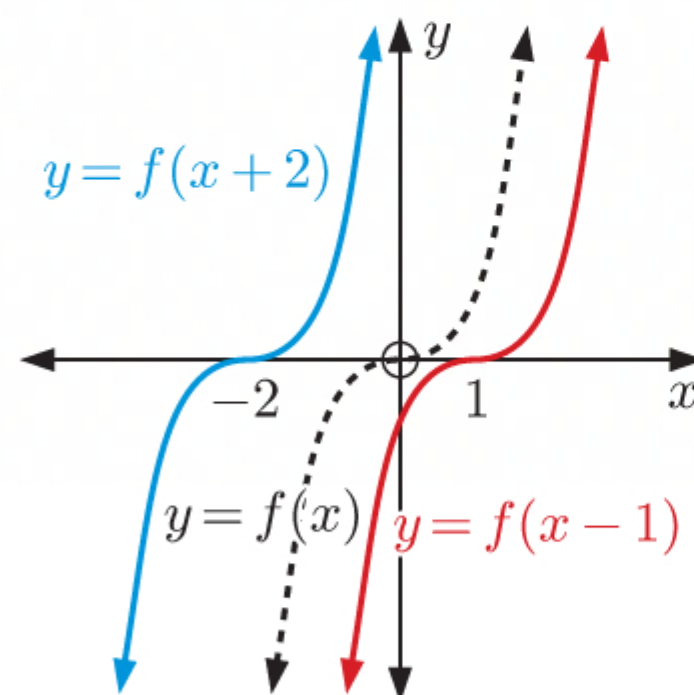


- 6** $y = f(x - 1)$ is found by translating $y = f(x)$ 1 unit to the right, $y = f(x + 2)$ is found by translating $y = f(x)$ 2 units to the left.

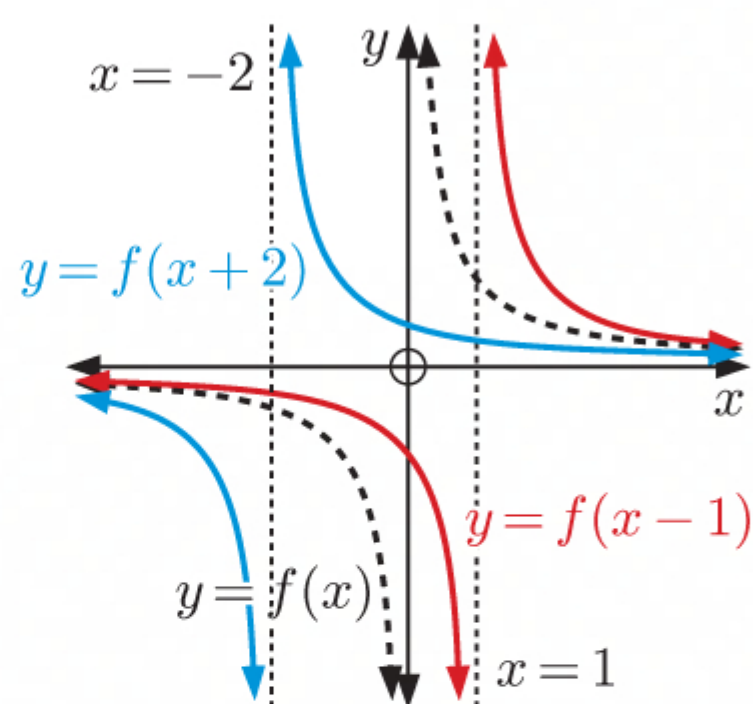
a $f(x) = x^2$



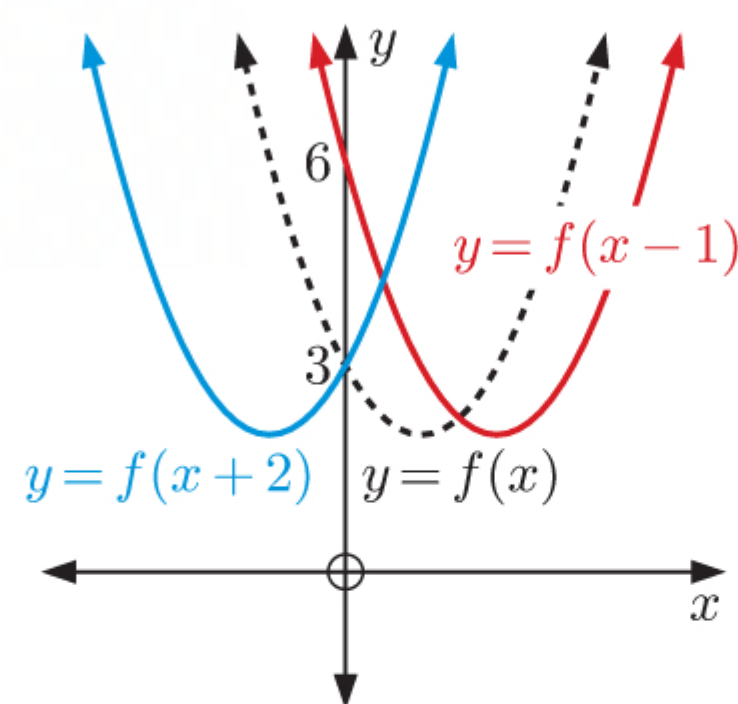
b $f(x) = x^3$



c $f(x) = \frac{1}{x}$

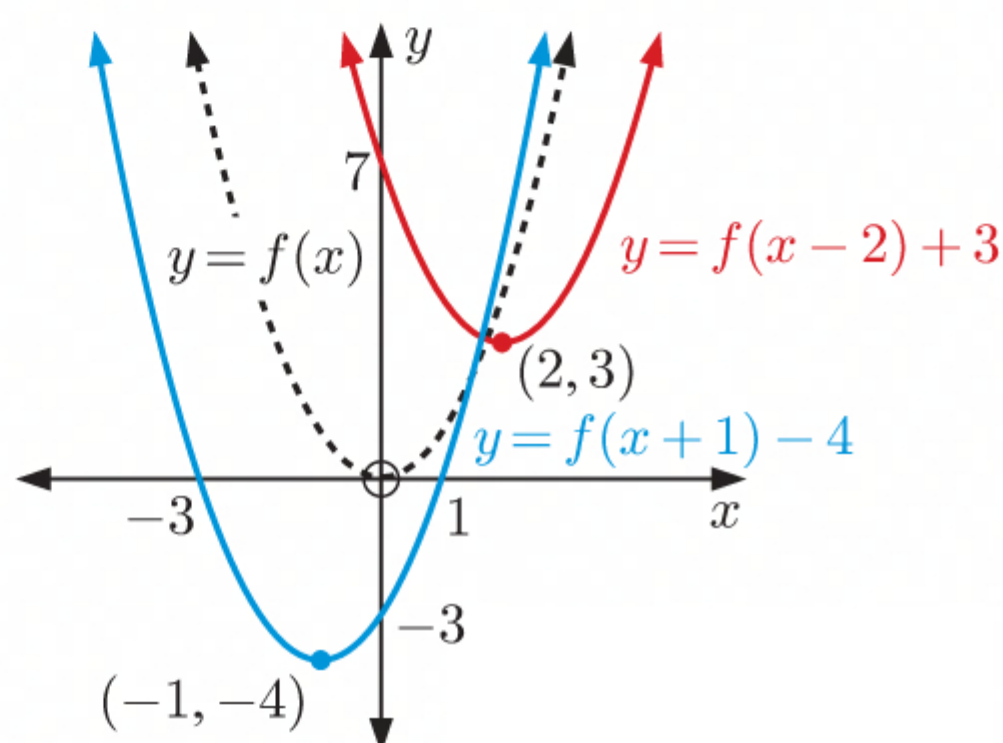


d $f(x) = (x-1)^2 + 2$

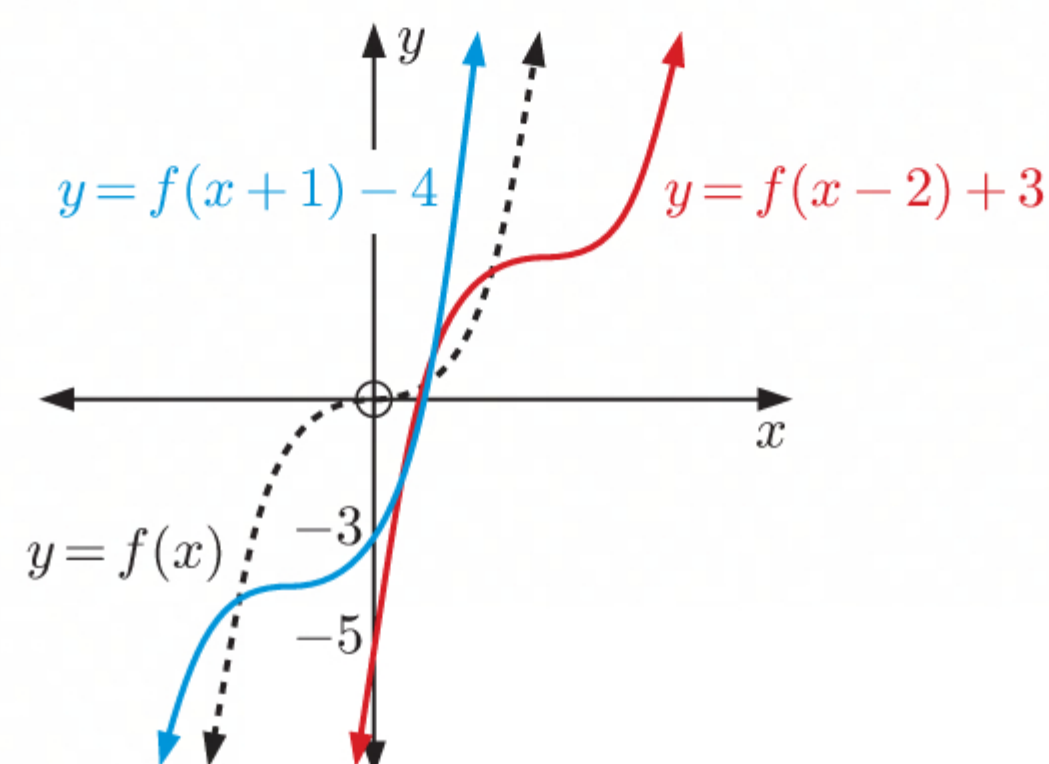


- 7 $y = f(x-2) + 3$ is found by translating $y = f(x)$ 2 units to the right and 3 units upwards,
 $y = f(x+1) - 4$ is found by translating $y = f(x)$ 1 unit to the left and 4 units downwards.

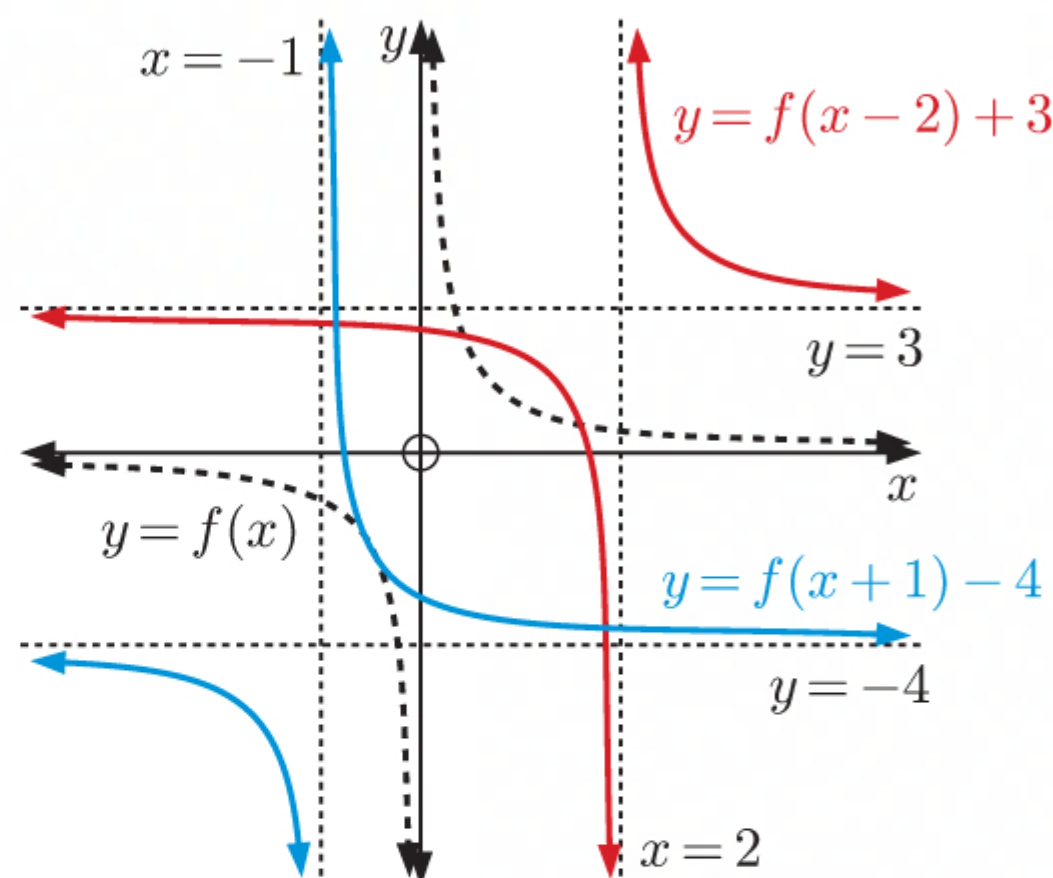
a $f(x) = x^2$



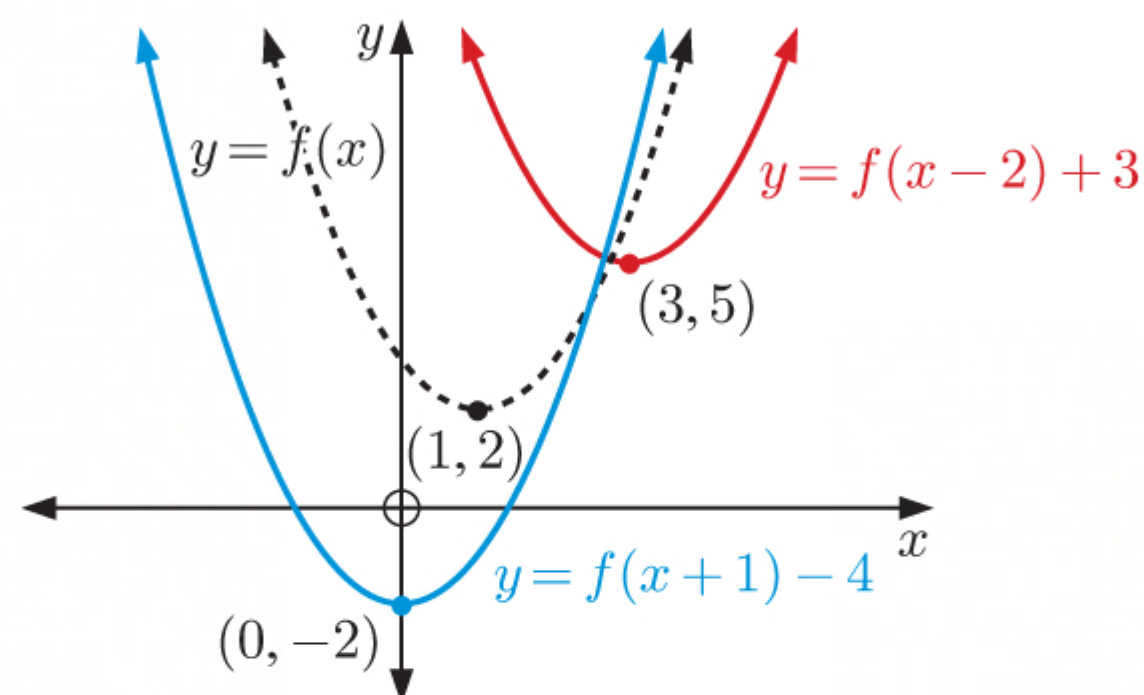
b $f(x) = x^3$



c $f(x) = \frac{1}{x}$



d $f(x) = (x-1)^2 + 2$



- 8 The graph of $y = g(x) = f(x-3) - 4$ is a translation of $y = f(x)$ 3 units to the right and 4 units downwards.

So, $y = f(x)$ has been translated by the vector $\begin{pmatrix} 3 \\ -4 \end{pmatrix}$.

The point $(-2, -5)$ on the graph of $y = f(x)$ will therefore be translated by the vector $\begin{pmatrix} 3 \\ -4 \end{pmatrix}$, to give the point $(-2+3, -5-4)$, or $(1, -9)$, on the graph of $y = g(x)$.

- 9 a** The graph of $y = g(x) = f(x) - 3$ is a translation of $y = f(x)$ 3 units downwards.
So, the graph of $y = g(x)$ has y -intercept $2 - 3 = -1$.
There is not enough information to determine the x -intercepts.
- b** The graph of $y = h(x) = f(x - 1)$ is a translation of $y = f(x)$ 1 unit to the right.
So, the graph of $y = h(x)$ has x -intercepts $-3 + 1 = -2$ and $4 + 1 = 5$.
There is not enough information to determine the y -intercept.
- c** The graph of $y = j(x) = f(x + 2) - 4$ is a translation of $y = f(x)$ 2 units to the left and 4 units downwards.
There is not enough information to determine the x or the y -intercepts.
- 10** $g(x) = f(x - 3) = (x - 3)^2 - 2(x - 3) + 2$ {since $f(x) = x^2 - 2x + 2$ }
 $= x^2 - 6x + 9 - 2x + 6 + 2$
 $= x^2 - 8x + 17$
- 11** $g(x) = (x - 3)^2 + 2$ is found by translating $f(x) = x^2$ 3 units to the right and 2 units upwards.
- a** The points on $y = f(x)$ are translated by $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$ to find the image points on $y = g(x)$.
- i** $(0, 0)$ is translated to $(3, 2)$
 - ii** $(-3, 9)$ is translated to $(-3 + 3, 9 + 2)$, or $(0, 11)$
 - iii** $(2, 4)$ is translated to $(2 + 3, 4 + 2)$, or $(5, 6)$
- b** The points on $y = g(x)$ are translated by $\begin{pmatrix} -3 \\ -2 \end{pmatrix}$ to find the corresponding points on $y = f(x)$.
- i** $(1, 6)$ corresponds to $(1 - 3, 6 - 2)$, or $(-2, 4)$
 - ii** $(-2, 27)$ corresponds to $(-2 - 3, 27 - 2)$, or $(-5, 25)$
 - iii** $(1\frac{1}{2}, 4\frac{1}{4})$ corresponds to $(1\frac{1}{2} - 3, 4\frac{1}{4} - 2)$, or $(-1\frac{1}{2}, 2\frac{1}{4})$

INVESTIGATION 2

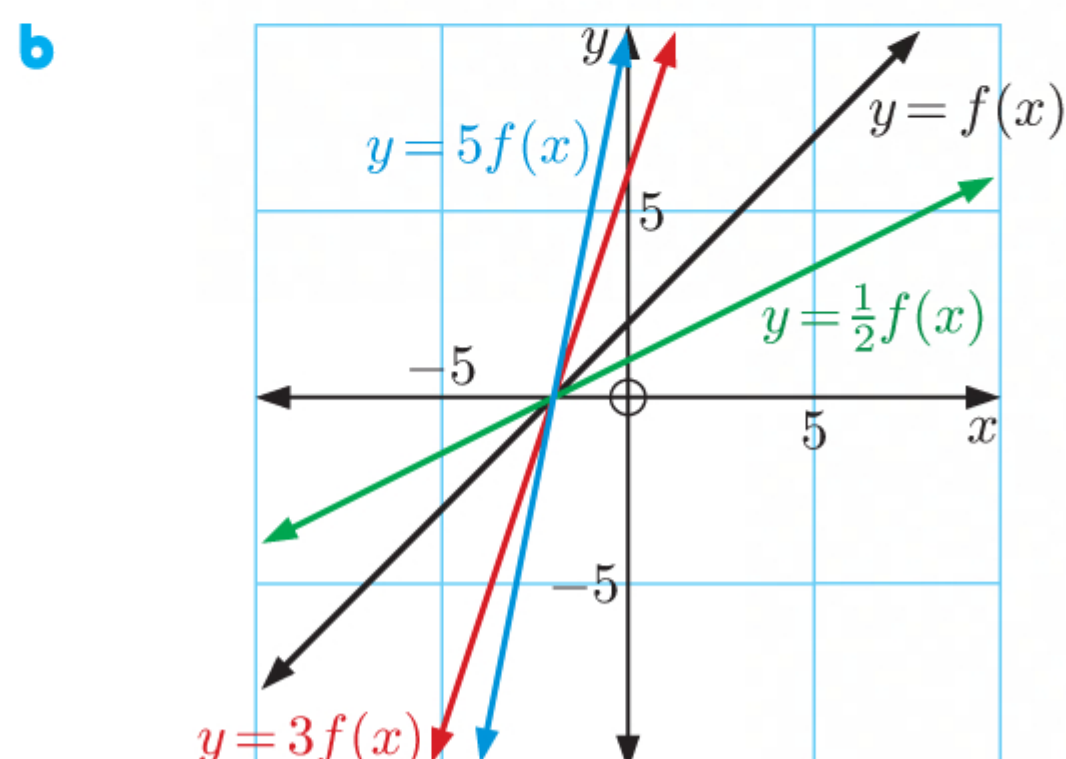
STRETCHES

1 $f(x) = x + 2$

a i $3f(x) = 3(x + 2)$
 $= 3x + 6$

ii $\frac{1}{2}f(x) = \frac{1}{2}(x + 2)$
 $= \frac{1}{2}x + 1$

iii $5f(x) = 5(x + 2)$
 $= 5x + 10$



- c** All transformations of the form $pf(x)$, $p > 0$ do not move the point $(-2, 0)$.
 $\therefore (-2, 0)$ is *invariant*.

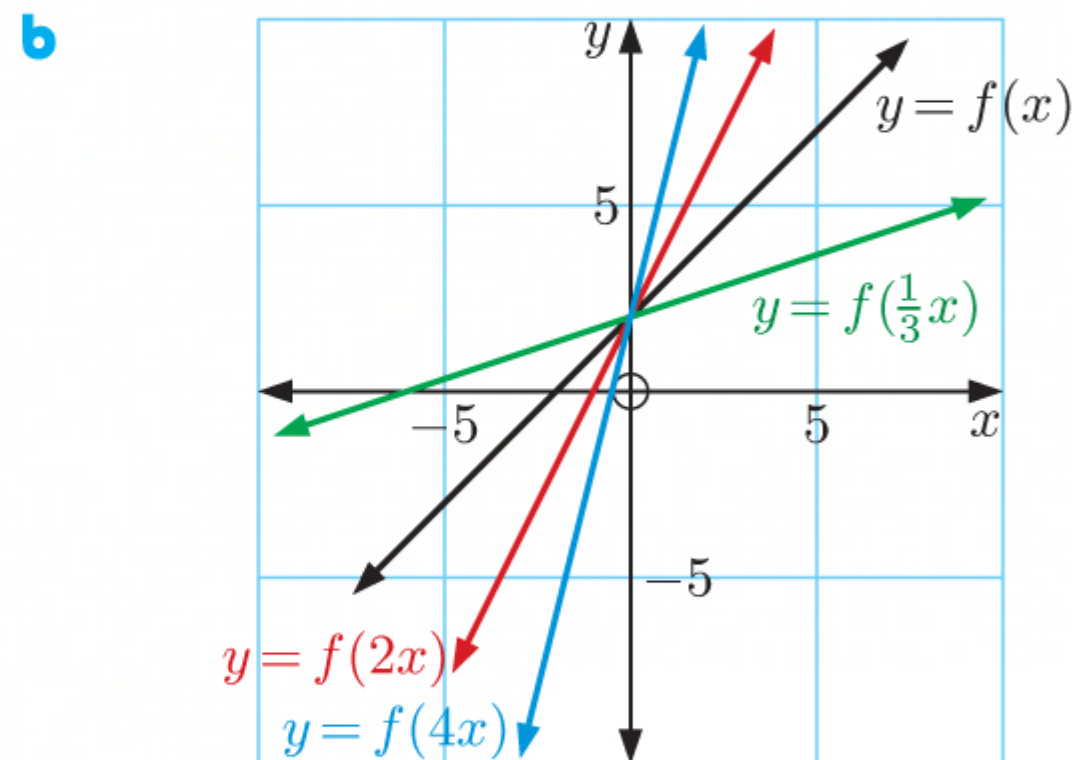
- d** For the transformation $y = pf(x)$, each point becomes p times its previous distance from the x -axis.

2 $f(x) = x + 2$

a **i** $f(2x) = 2x + 2$

ii $f(\frac{1}{3}x) = \frac{1}{3}x + 2$

iii $f(4x) = 4x + 2$

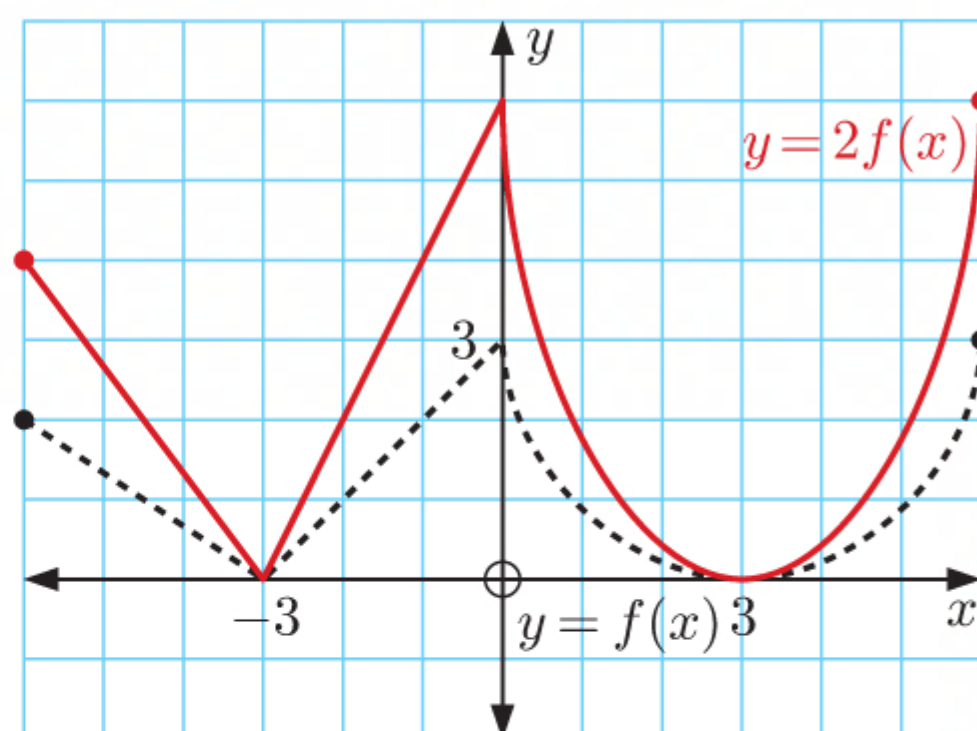


- c** All transformations of the form $f(qx)$, $q > 0$ do not move the point $(0, 2)$.
 $\therefore (0, 2)$ is *invariant*.

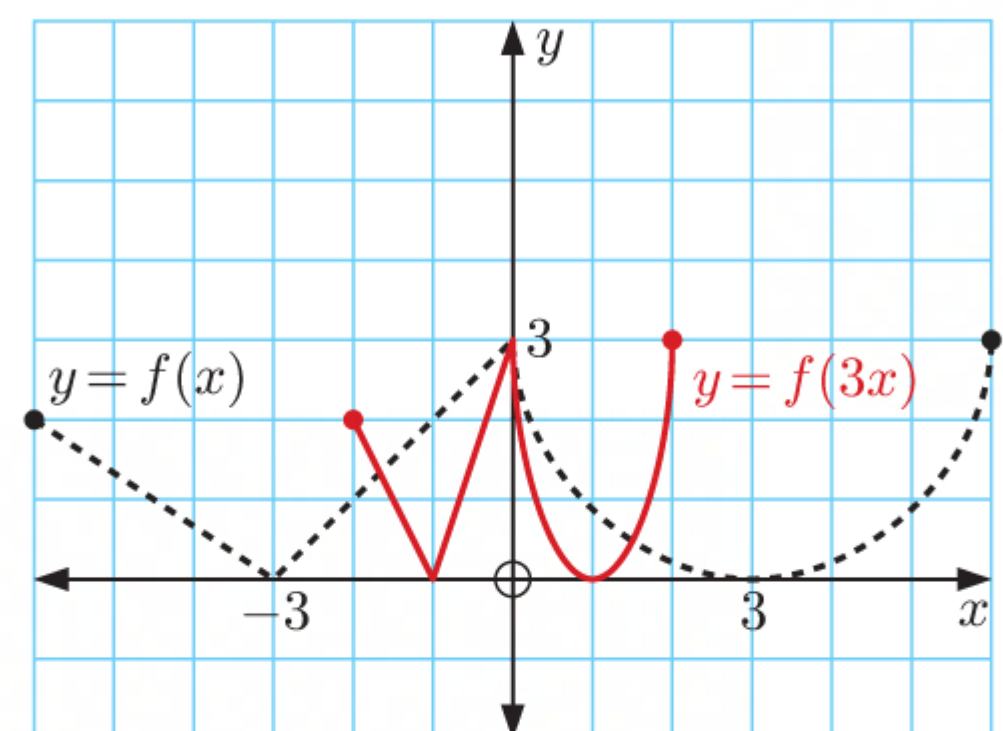
- d** For the transformation $y = f(qx)$, each point becomes $\frac{1}{q}$ times its previous distance from the y -axis.

EXERCISE 4B

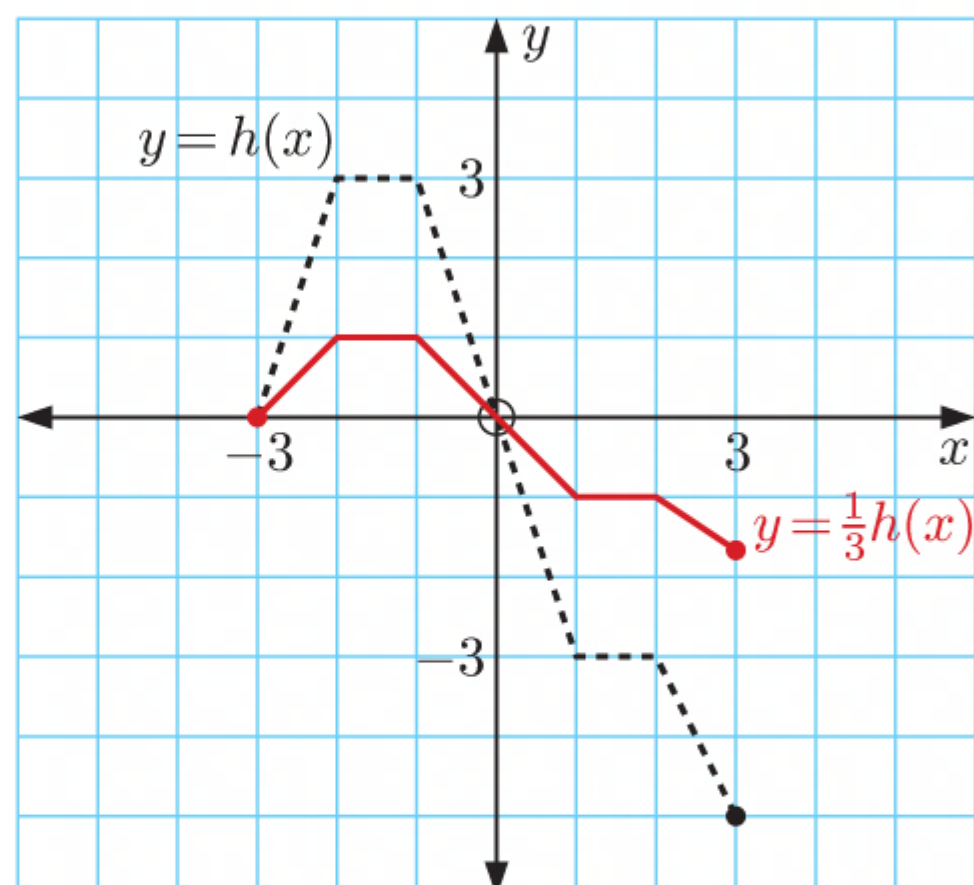
- 1 a** The graph of $y = 2f(x)$ is a vertical stretch of $y = f(x)$ with scale factor 2.



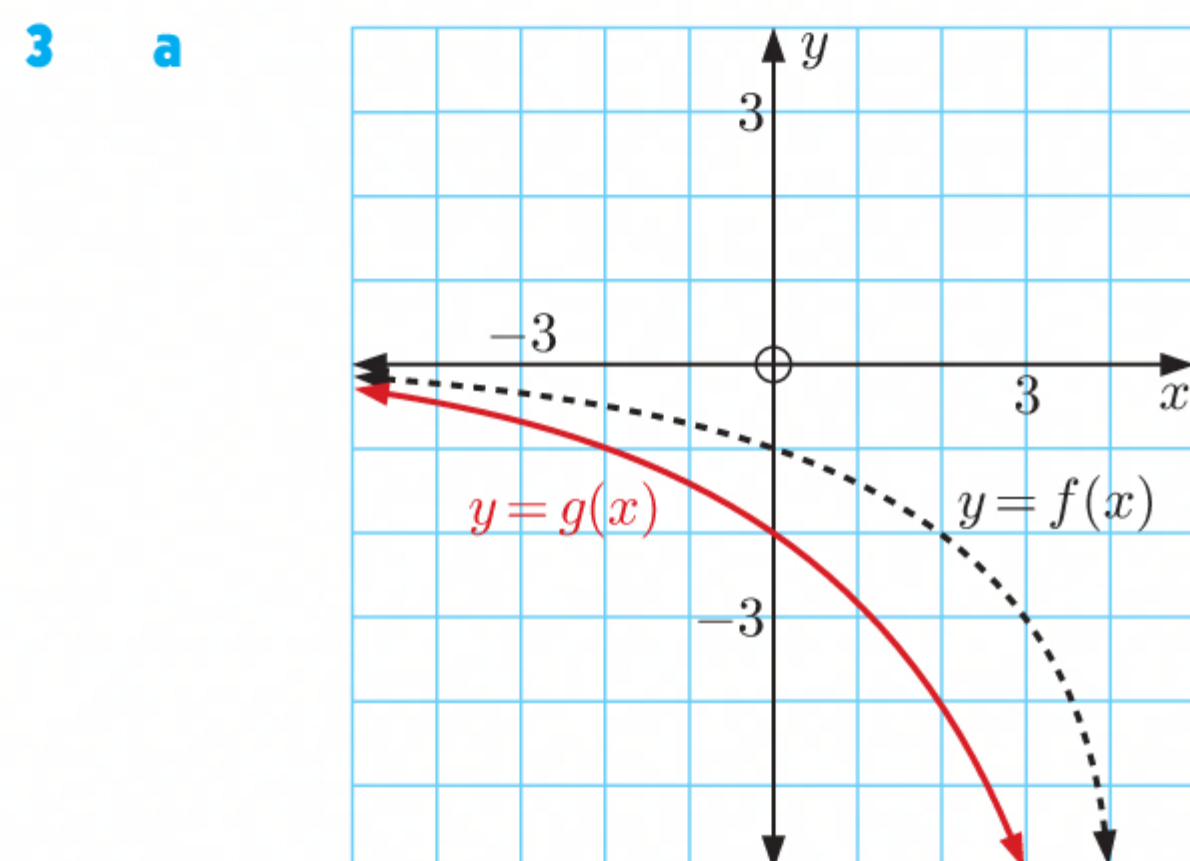
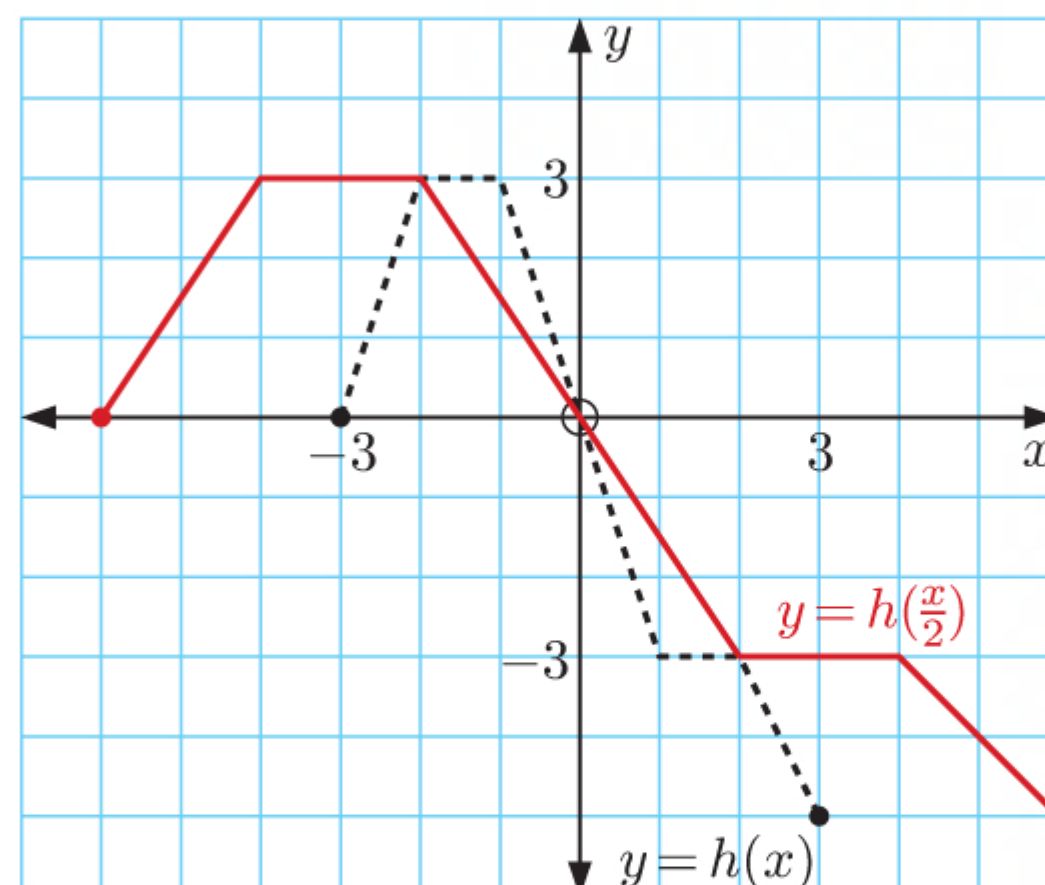
- b** The graph of $y = f(3x)$ is a horizontal stretch of $y = f(x)$ with scale factor $\frac{1}{3}$.



- 2 a** The graph of $y = \frac{1}{3}h(x)$ is a vertical stretch of $y = h(x)$ with scale factor $\frac{1}{3}$.

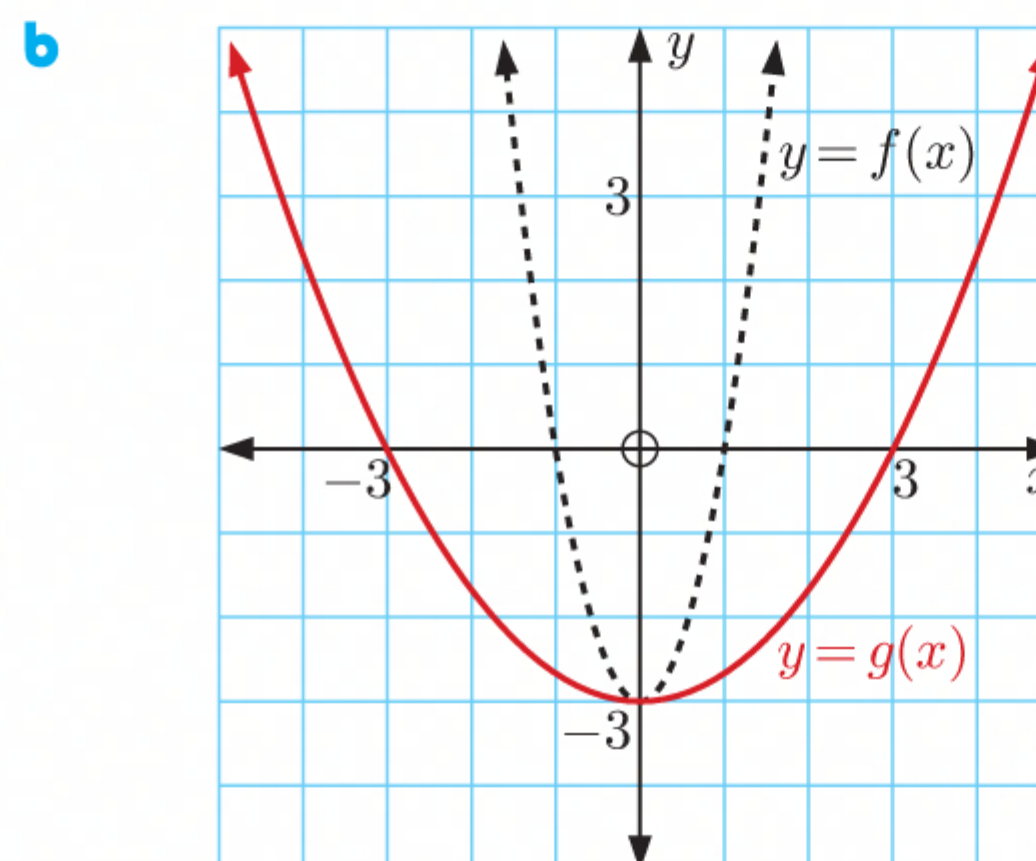


- b** The graph of $y = h\left(\frac{x}{2}\right)$ is a horizontal stretch of $y = h(x)$ with scale factor 2.



The graph of $y = f(x)$ has been vertically stretched with scale factor 2 to give $y = g(x)$.

So, $g(x) = 2f(x)$.



The graph of $y = f(x)$ has been horizontally stretched with scale factor 3 to give $y = g(x)$.

So, $g(x) = f\left(\frac{x}{3}\right)$.

- 4** Let the original linear function be $y = f(x) = mx + a$.

When $y = f(x)$ is vertically stretched with scale factor c , the resulting function is

$$y = c(f(x))$$

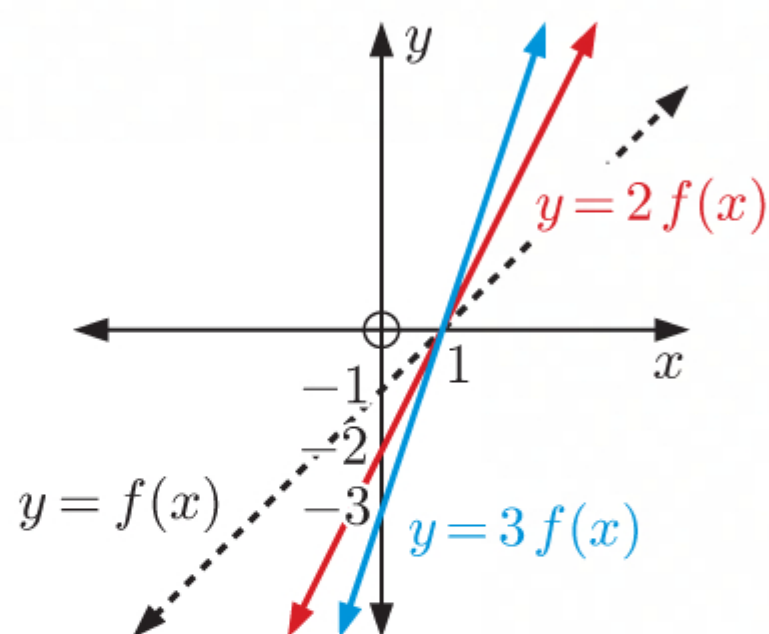
$$\therefore y = c(mx + a)$$

$$\therefore y = cmx + ac$$

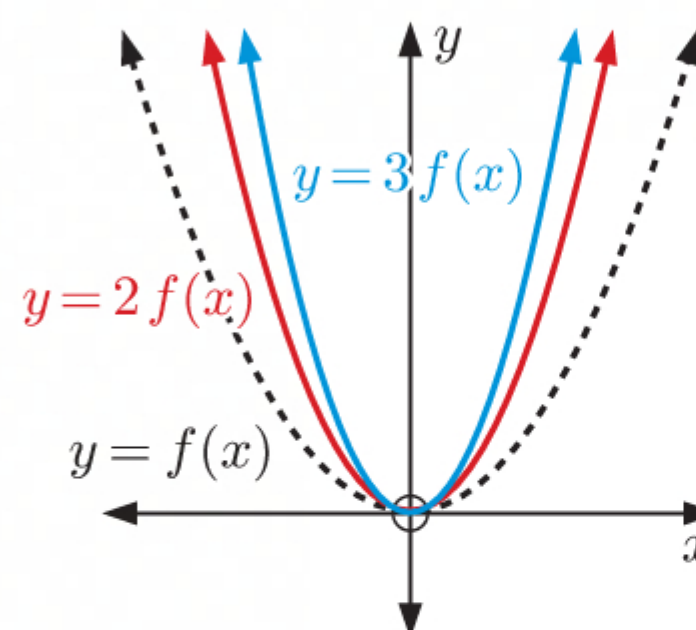
So, the resulting line has gradient cm .

- 5** The graphs of $y = 2f(x)$ and $y = 3f(x)$ are vertical stretches of $y = f(x)$ with scale factors 2 and 3, respectively.

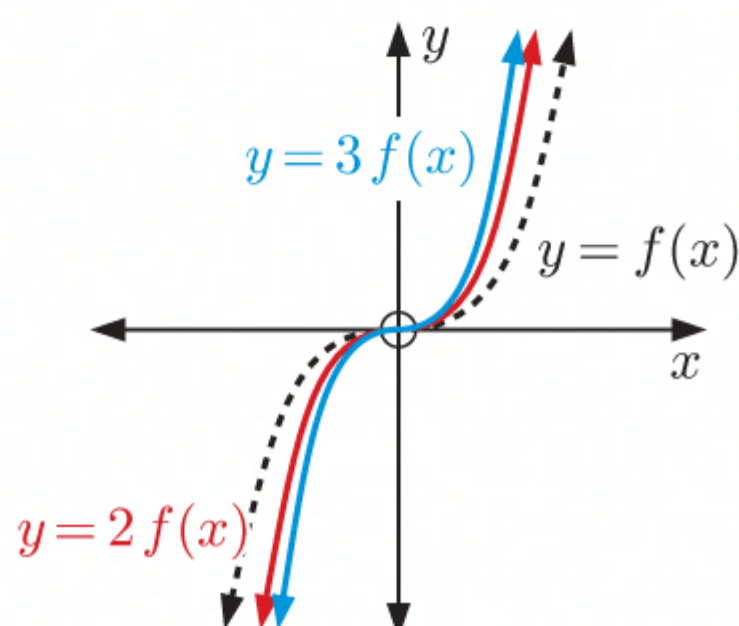
a $f(x) = x - 1$



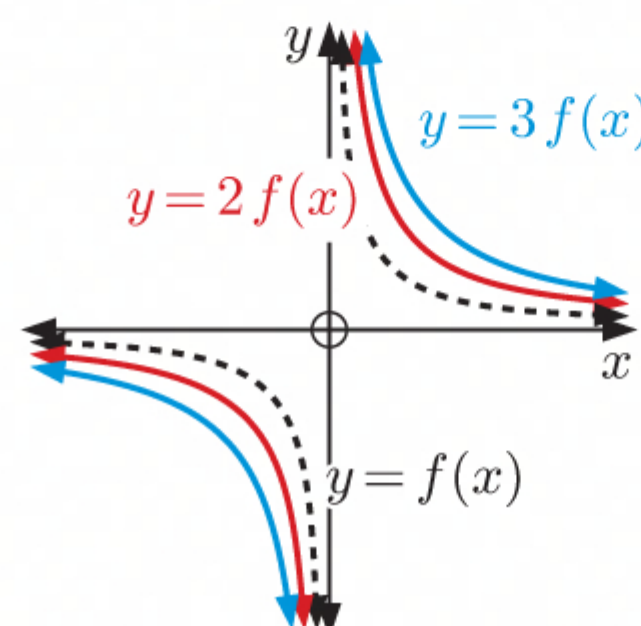
b $f(x) = x^2$



c $f(x) = x^3$

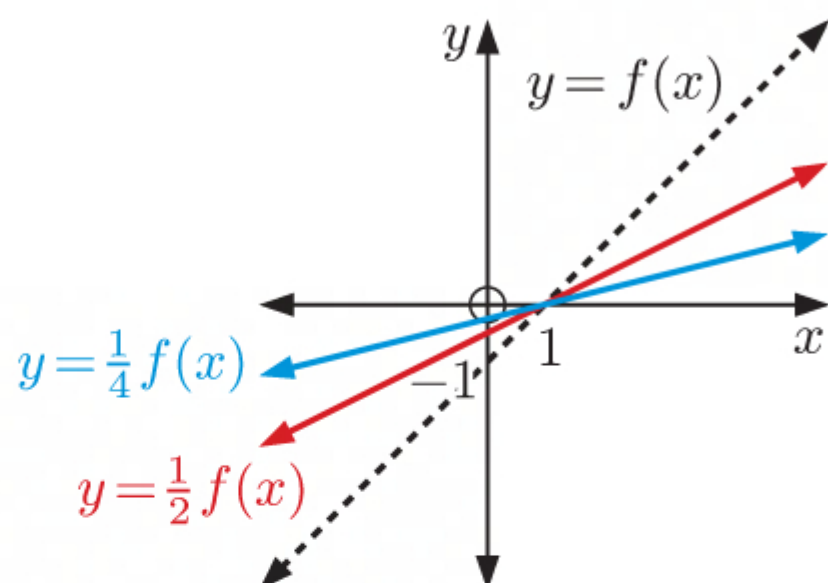


d $f(x) = \frac{1}{x}$

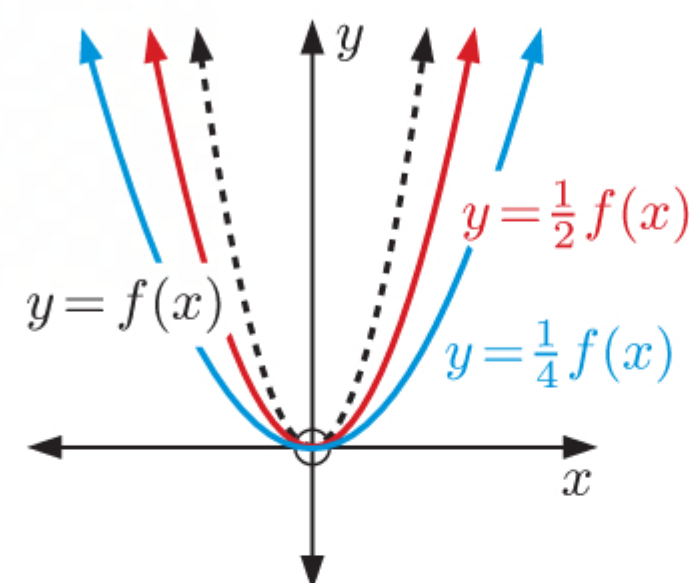


- 6** The graphs of $y = \frac{1}{2}f(x)$ and $y = \frac{1}{4}f(x)$ are vertical stretches of $y = f(x)$ with scale factors $\frac{1}{2}$ and $\frac{1}{4}$, respectively.

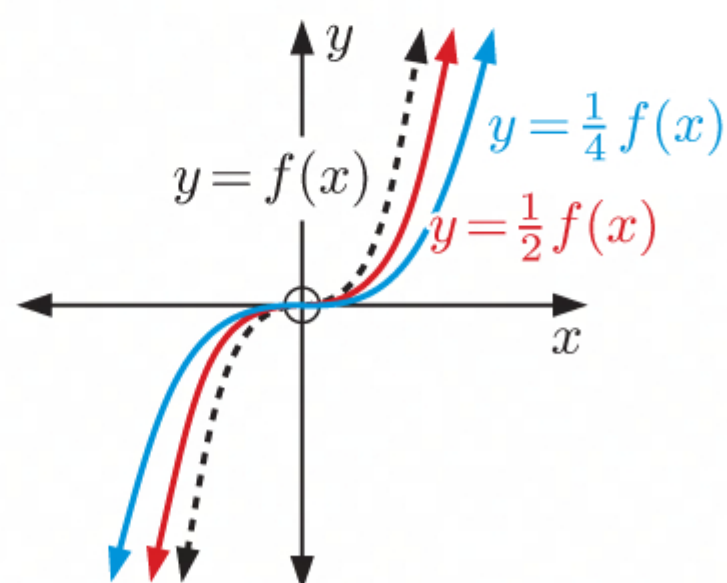
a $f(x) = x - 1$



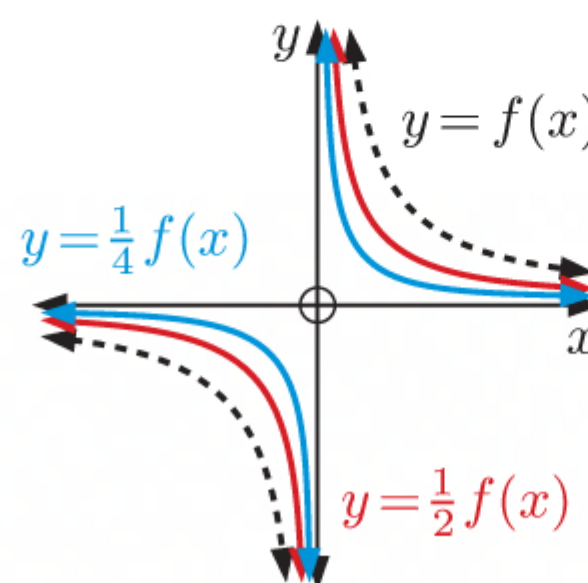
b $f(x) = x^2$



c $f(x) = x^3$

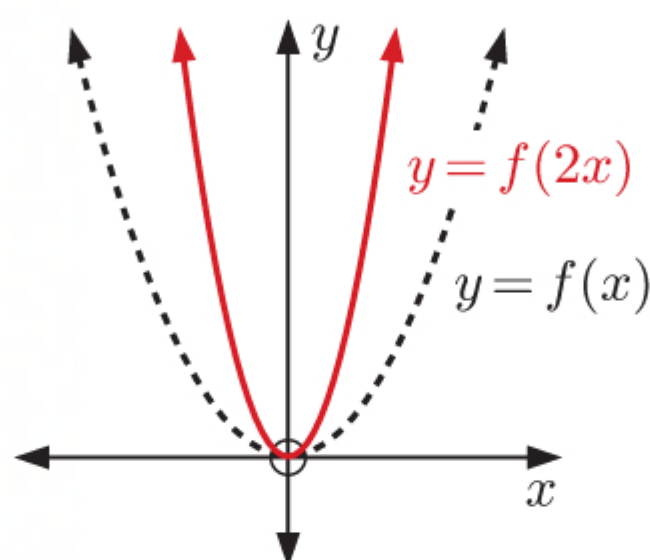


d $f(x) = \frac{1}{x}$

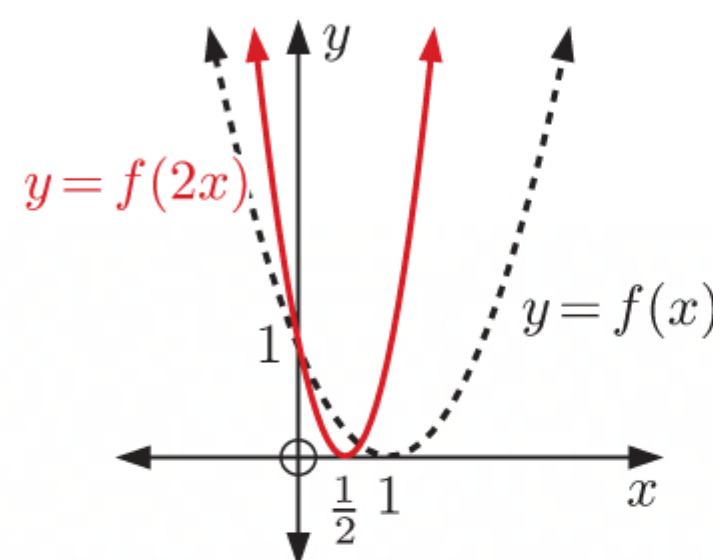


- 7** The graph of $y = f(2x)$ is a horizontal stretch of $y = f(x)$ with scale factor $\frac{1}{2}$.

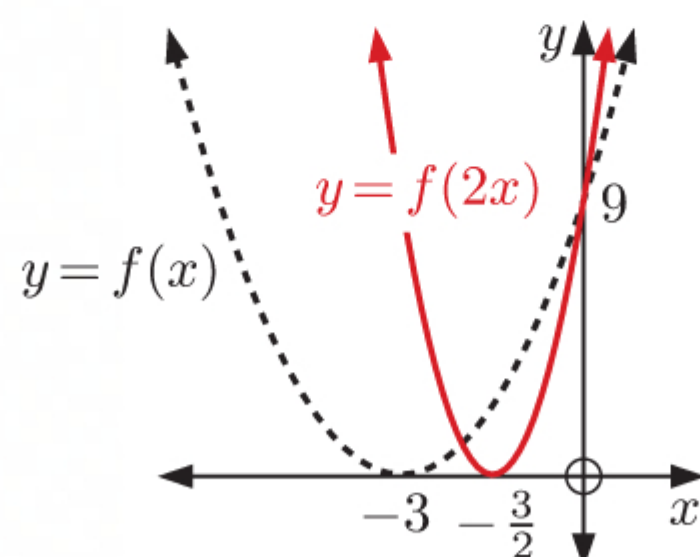
a $y = x^2$



b $y = (x - 1)^2$

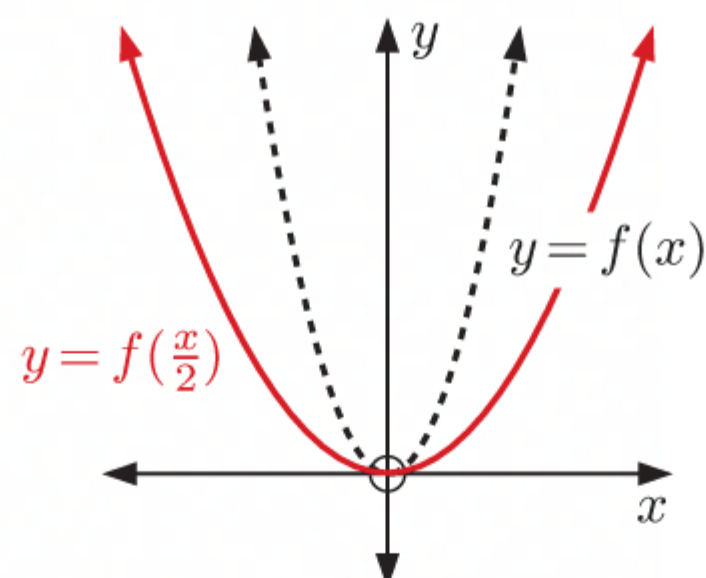


c $y = (x + 3)^2$

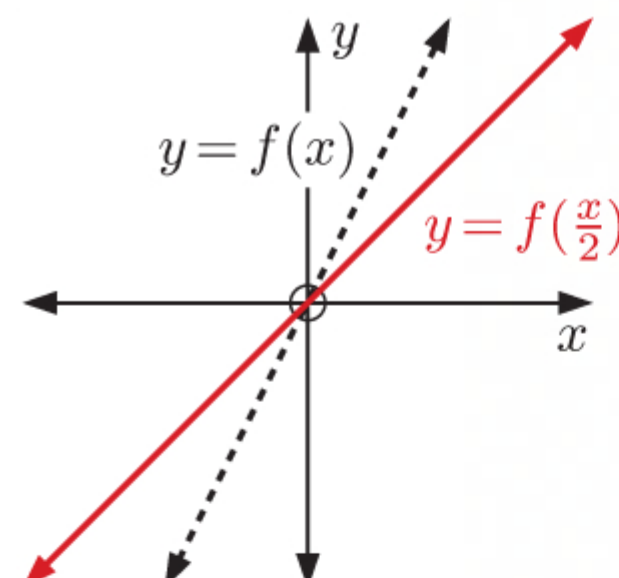


- 8** The graph of $y = f\left(\frac{x}{2}\right)$ is a horizontal stretch of $y = f(x)$ with scale factor 2.

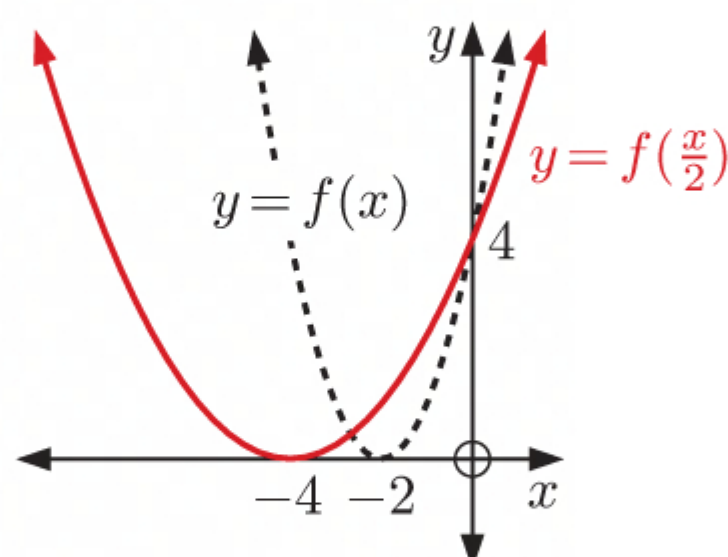
a $y = x^2$



b $y = 2x$



c $y = (x + 2)^2$



9 $g(x) = f(5x)$

The graph of $y = g(x)$ is a horizontal stretch of $y = f(x)$ with scale factor $\frac{1}{5}$.

- a** Each point on $y = g(x)$ is $\frac{1}{5}$ times the distance that $y = f(x)$ is from the y -axis.

The point (10, 25) on $y = f(x)$ is 10 units from the y -axis. The corresponding point on $y = g(x)$, which is $\frac{1}{5} \times 10 = 2$ units from the y -axis, is (2, 25).

- b** Each point on $y = f(x)$ is 5 times the distance that $y = g(x)$ is from the y -axis.
The point $(-5, -15)$ on $y = g(x)$ is 5 units from the y -axis. The corresponding point on $y = f(x)$, which is $5 \times 5 = 25$ units from the y -axis, is $(-25, -15)$.

- 10 a** The graph of $y = g(x)$ is a vertical stretch of $y = f(x)$ with scale factor 2.

$$\therefore g(x) = 2f(x)$$

$$\therefore g(x) = 2(x^2 + 2) \quad \{\text{since } f(x) = x^2 + 2\}$$

$$\therefore g(x) = 2x^2 + 4$$

- b** The graph of $y = g(x)$ is a horizontal stretch of $y = f(x)$ with scale factor 3.

$$\therefore g(x) = f\left(\frac{x}{3}\right)$$

$$\therefore g(x) = 5 - 3\left(\frac{x}{3}\right) \quad \{\text{since } f(x) = 5 - 3x\}$$

$$\therefore g(x) = 5 - x$$

- c** The graph of $y = g(x)$ is a vertical dilation of $y = f(x)$ with scale factor $\frac{1}{4}$.

$$\therefore g(x) = \frac{1}{4}f(x)$$

$$\therefore g(x) = \frac{1}{4}(x^3 + 8x^2 - 2) \quad \{\text{since } f(x) = x^3 + 8x^2 - 2\}$$

$$\therefore g(x) = \frac{1}{4}x^3 + 2x^2 - \frac{1}{2}$$

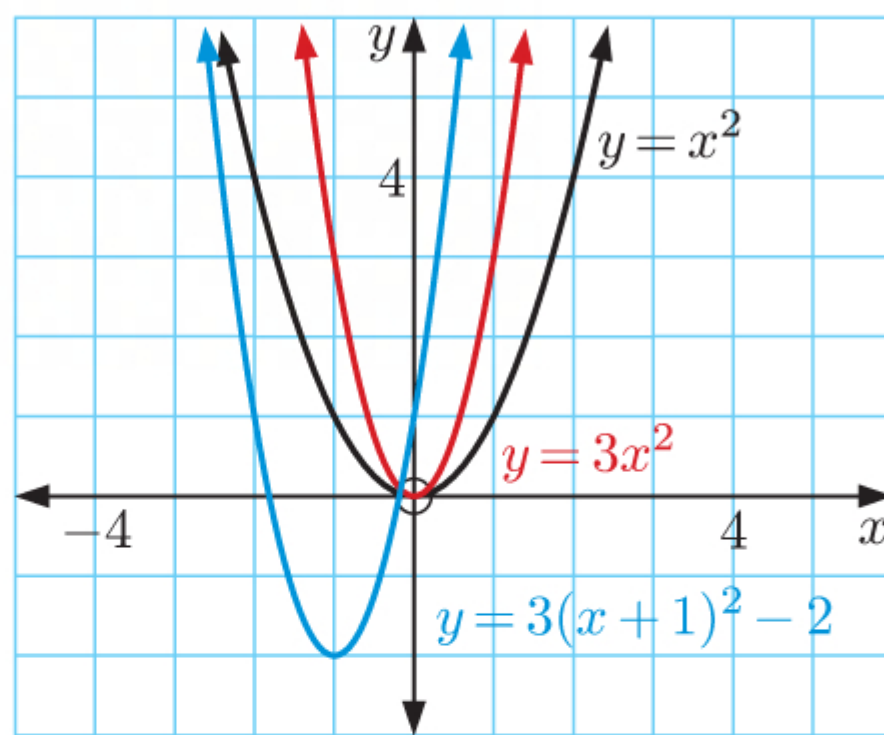
- d** The graph of $y = g(x)$ is a horizontal dilation of $y = f(x)$ with scale factor $\frac{1}{2}$.

$$\therefore g(x) = f(2x)$$

$$\therefore g(x) = 2(2x)^2 + (2x) - 3 \quad \{\text{since } f(x) = 2x^2 + x - 3\}$$

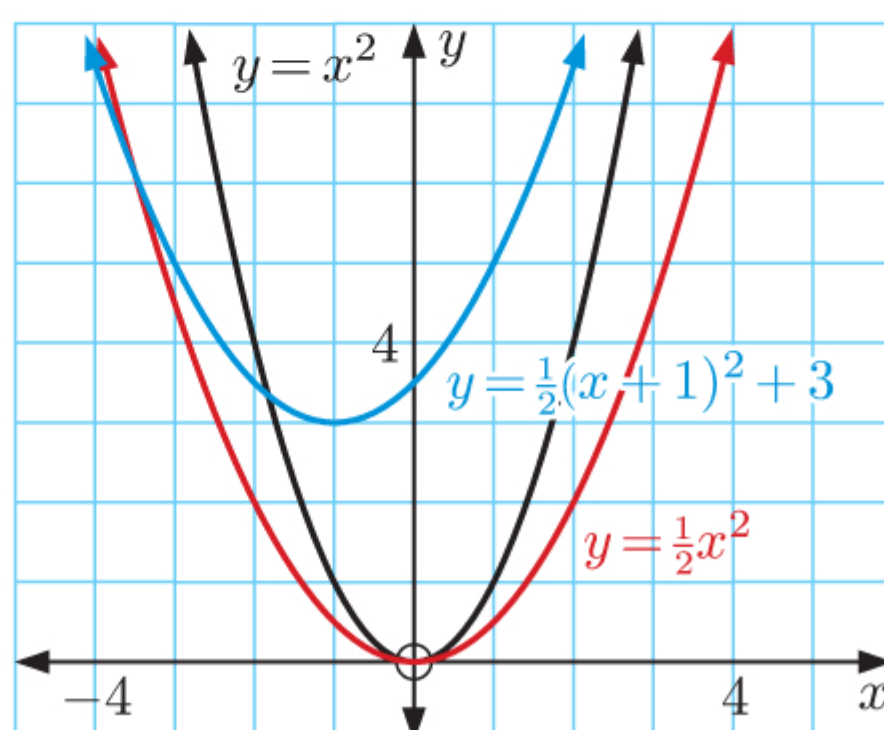
$$\therefore g(x) = 8x^2 + 2x - 3$$

11

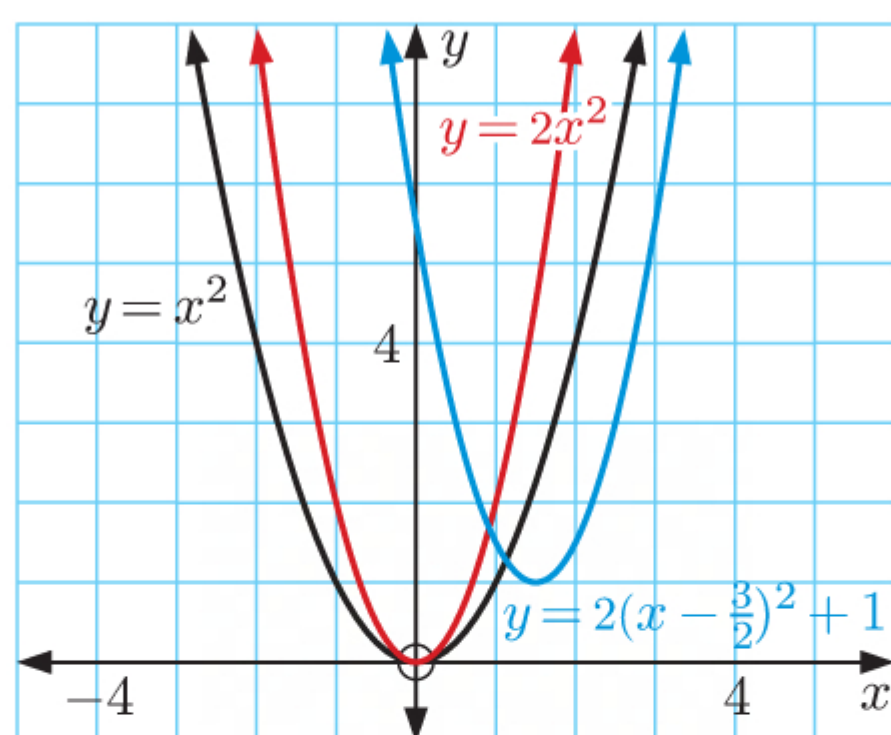


$y = x^2$ is transformed to $y = 3(x+1)^2 - 2$ by vertically stretching with scale factor 3 and then translating through $\begin{pmatrix} -1 \\ -2 \end{pmatrix}$.

12

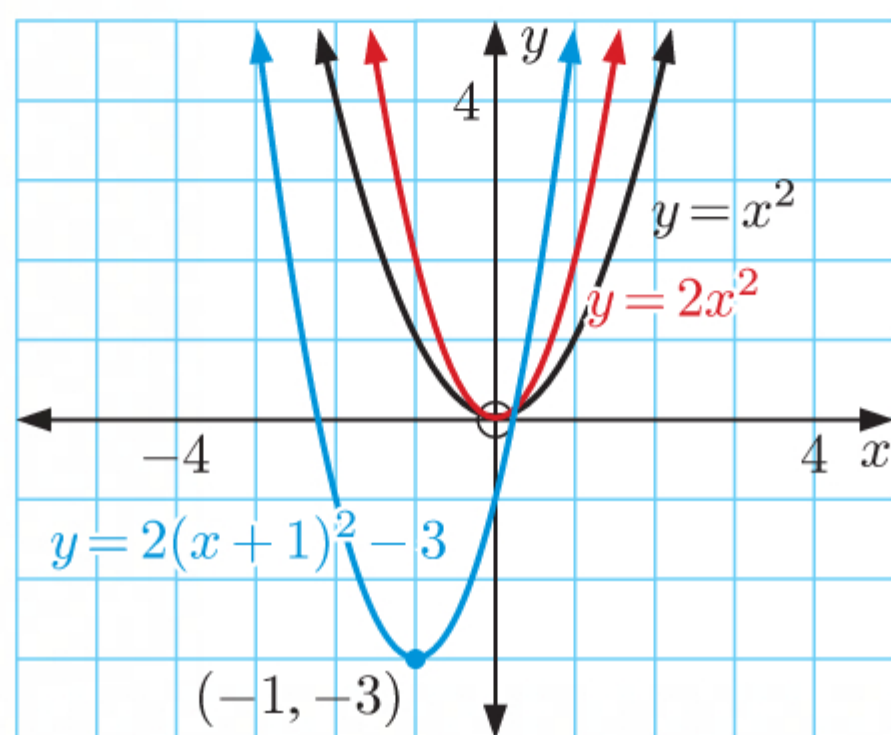


$y = x^2$ is transformed to $y = \frac{1}{2}(x+1)^2 + 3$ by vertically stretching with scale factor $\frac{1}{2}$ and then translating through $\begin{pmatrix} -1 \\ 3 \end{pmatrix}$.

13

$y = x^2$ is transformed to $y = 2(x - \frac{3}{2})^2 + 1$ by vertically stretching with scale factor 2 and then translating through $\begin{pmatrix} \frac{3}{2} \\ 1 \end{pmatrix}$.

- 14** We vertically stretch $y = x^2$ with scale factor 2 to give $y = 2x^2$. We then translate $y = 2x^2$ through $\begin{pmatrix} -1 \\ -3 \end{pmatrix}$ to give $y = 2(x + 1)^2 - 3$.



- 15** **a** $y = f(2x)$ is a horizontal stretch of $y = f(x)$ with scale factor $\frac{1}{2}$.
 $y = 3f(2x)$ is a vertical stretch of $y = f(2x)$ with scale factor 3.
 To map $y = f(x)$ onto $y = g(x)$ we apply a horizontal stretch with scale factor $\frac{1}{2}$, then a vertical stretch with scale factor 3.
- b** The points on $y = f(x)$ become $\frac{1}{2}$ times their distance from the y -axis, and 3 times their distance from the x -axis.
- i** The image of $(3, -5)$ on $f(x)$ is $(3 \times \frac{1}{2}, -5 \times 3)$, or $(\frac{3}{2}, -15)$.
 - ii** The image of $(1, 2)$ on $f(x)$ is $(1 \times \frac{1}{2}, 2 \times 3)$, or $(\frac{1}{2}, 6)$.
 - iii** The image of $(-2, 1)$ on $f(x)$ is $(-2 \times \frac{1}{2}, 1 \times 3)$, or $(-1, 3)$.
- c** We multiply the distance from the x -axis by $\frac{1}{3}$ and the distance from the y -axis by 2 to find the corresponding point on $y = f(x)$.
- i** $(2 \times 2, 1 \times \frac{1}{3})$ or $(4, \frac{1}{3})$ is the point on $y = f(x)$ which maps onto $(2, 1)$.
 - ii** $(-3 \times 2, 2 \times \frac{1}{3})$ or $(-6, \frac{2}{3})$ is the point on $y = f(x)$ which maps onto $(-3, 2)$.
 - iii** $(-7 \times 2, 3 \times \frac{1}{3})$ or $(-14, 1)$ is the point on $y = f(x)$ which maps onto $(-7, 3)$.

- 16 a** Under a vertical dilation with scale factor $\frac{1}{2}$, $f(x)$ becomes $\frac{1}{2}f(x)$.

$$\therefore y = \frac{1}{x} \text{ becomes } y = \frac{1}{2} \left(\frac{1}{x} \right) = \frac{1}{2x}.$$

- b** Under a horizontal dilation with scale factor 3, $f(x)$ becomes $f(\frac{1}{3}x)$.

$$\therefore y = \frac{1}{x} \text{ becomes } y = \frac{1}{\frac{1}{3}x} = \frac{3}{x}.$$

- c** Under a horizontal translation of -3 , $f(x)$ becomes $f(x+3)$.

$$\therefore y = \frac{1}{x} \text{ becomes } y = \frac{1}{x+3}.$$

- d** Under a vertical translation of 4, $f(x)$ becomes $f(x) + 4$.

$$\begin{aligned} \therefore y = \frac{1}{x} \text{ becomes } y &= \frac{1}{x} + 4 \\ \therefore y &= \frac{1}{x} + \frac{4x}{x} \\ \therefore y &= \frac{4x+1}{x} \end{aligned}$$

- 17 a** Under a vertical stretch with scale factor 3, $f(x)$ becomes $3f(x)$.

$$\therefore y = \frac{1}{x} \text{ becomes } y = 3 \left(\frac{1}{x} \right) = \frac{3}{x}.$$

Under a translation of $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$, $f(x)$ becomes $f(x-1) - 1$.

$$\therefore y = \frac{3}{x} \text{ becomes } y = \frac{3}{x-1} - 1.$$

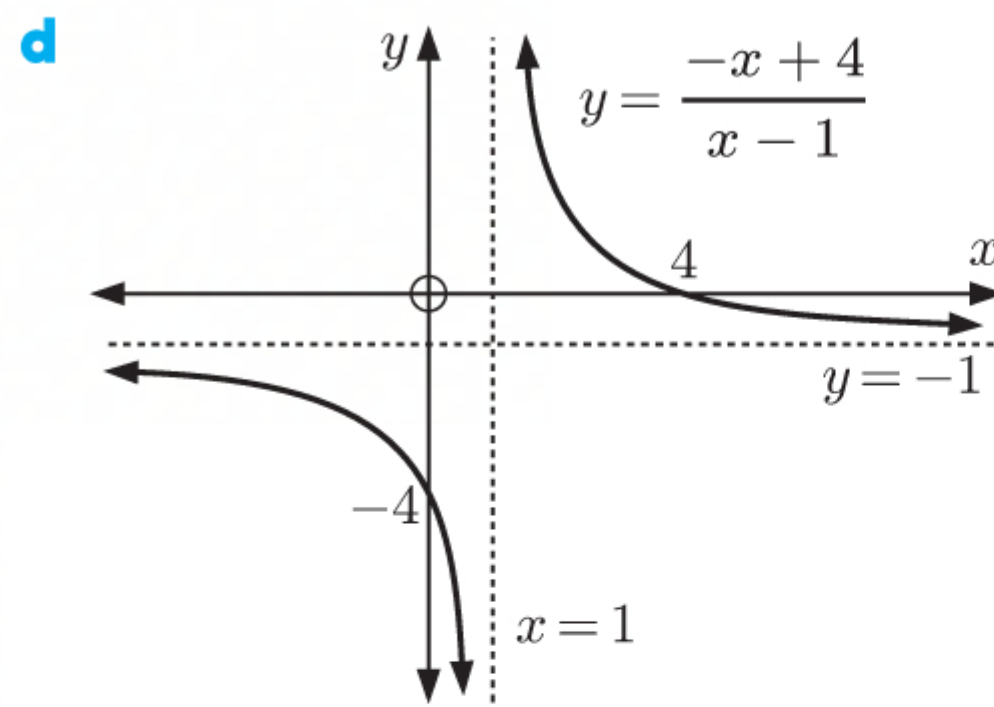
$$\begin{aligned} \text{So, } y = \frac{1}{x} \text{ becomes } g(x) &= \frac{3}{x-1} - 1 \\ &= \frac{3 - (x-1)}{x-1} \\ &= \frac{3 - x + 1}{x-1} \\ &= \frac{-x+4}{x-1} \end{aligned}$$

- b** The asymptotes of $y = \frac{1}{x}$ are $x = 0$ and $y = 0$.

These are unchanged by the vertical stretch, and shifted $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ by the translation.

\therefore the vertical asymptote is $x = 1$ and the horizontal asymptote is $y = -1$.

- c The domain is $\{x \mid x \neq 1\}$.
The range is $\{y \mid y \neq -1\}$.



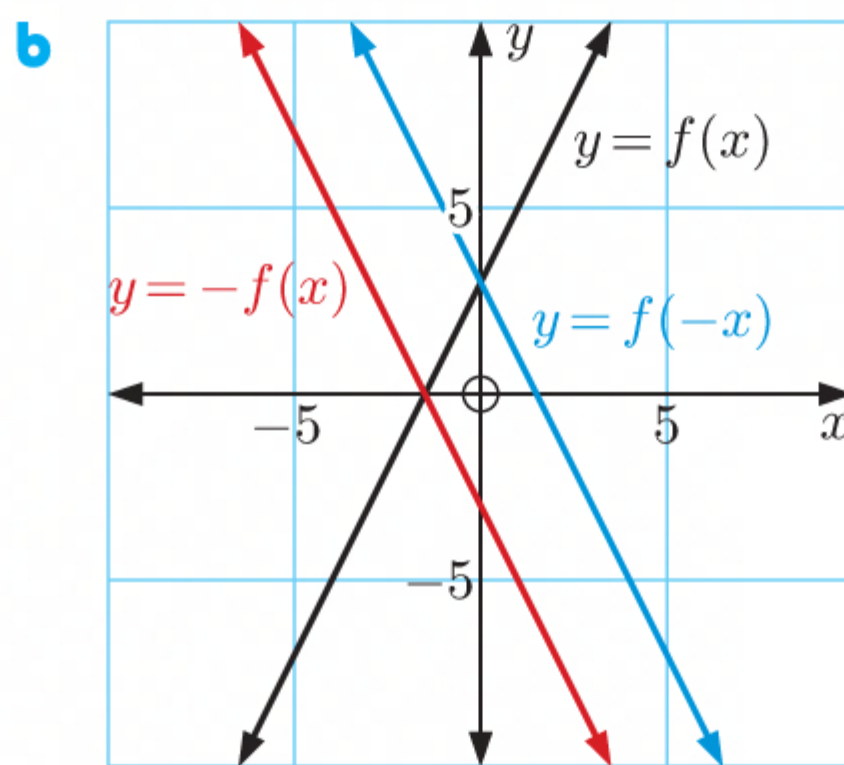
INVESTIGATION 3

REFLECTIONS

1 $f(x) = 2x + 3$

a i $-f(x) = -(2x + 3)$
 $= -2x - 3$

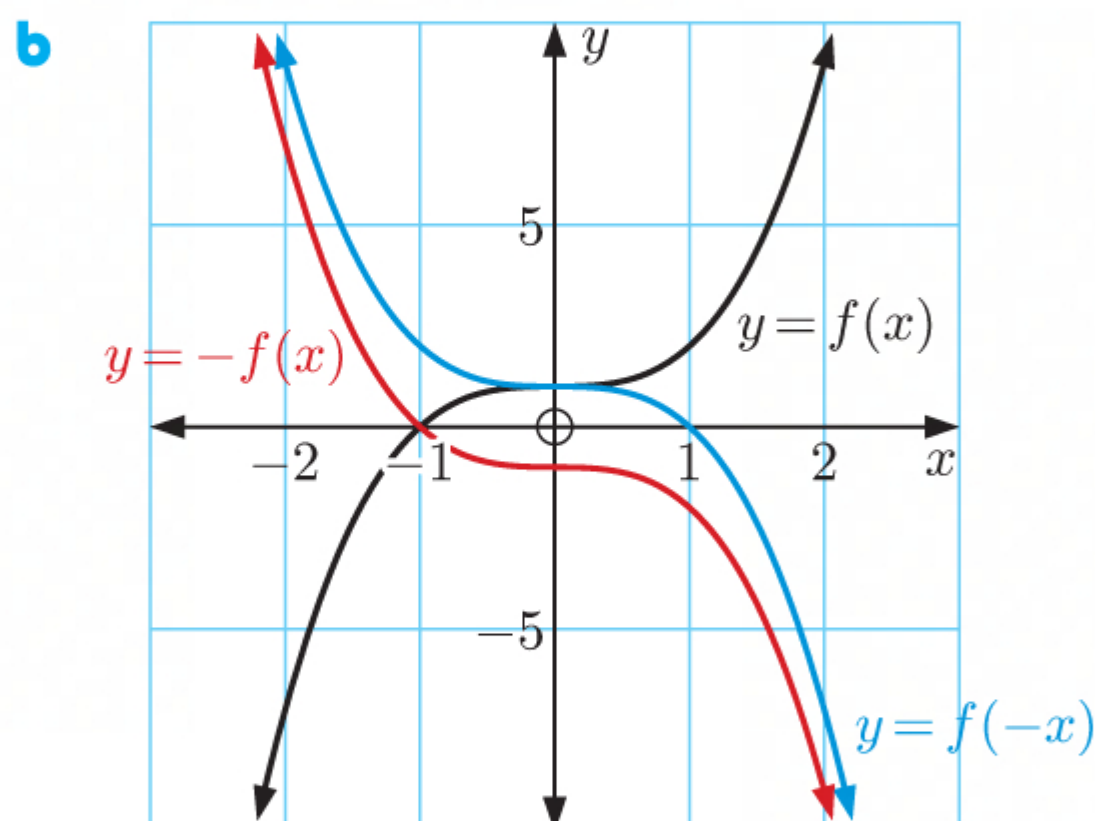
ii $f(-x) = 2(-x) + 3$
 $= -2x + 3$



2 $f(x) = x^3 + 1$

a i $-f(x) = -(x^3 + 1)$
 $= -x^3 - 1$

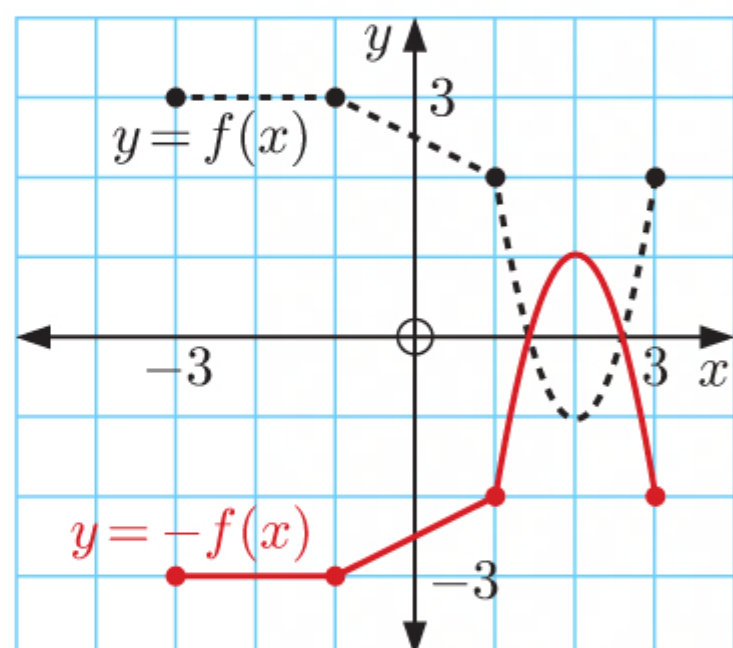
ii $f(-x) = (-x)^3 + 1$
 $= -x^3 + 1$



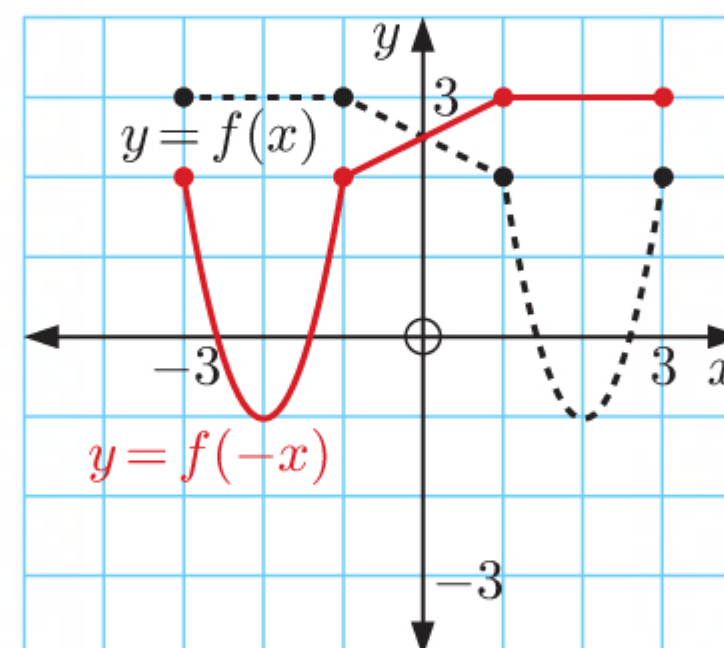
- 3 a A reflection of the graph in the **x-axis** moves $y = f(x)$ to $y = -f(x)$.
b A reflection of the graph in the **y-axis** moves $y = f(x)$ to $y = f(-x)$.

EXERCISE 4C

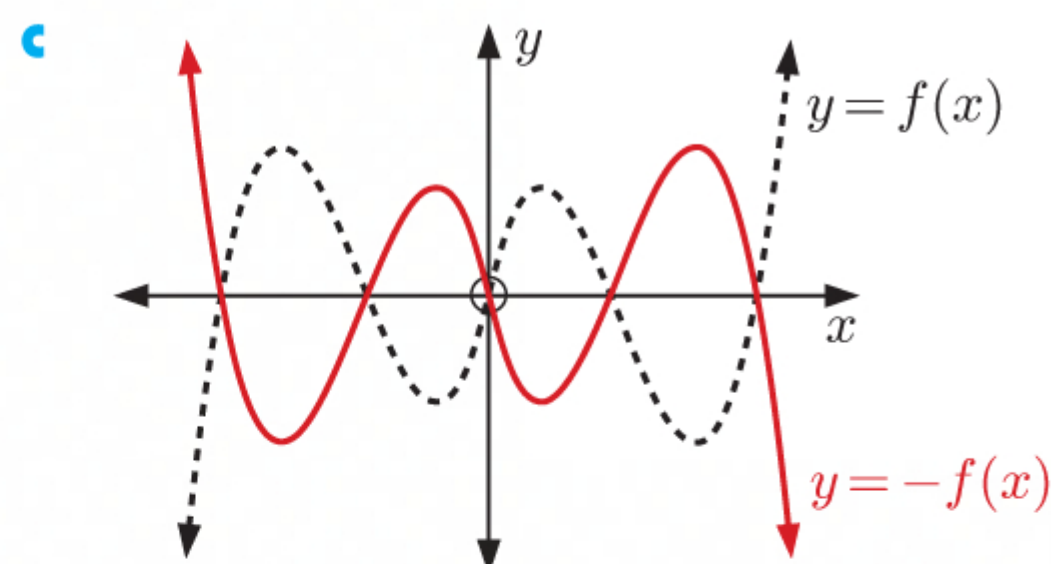
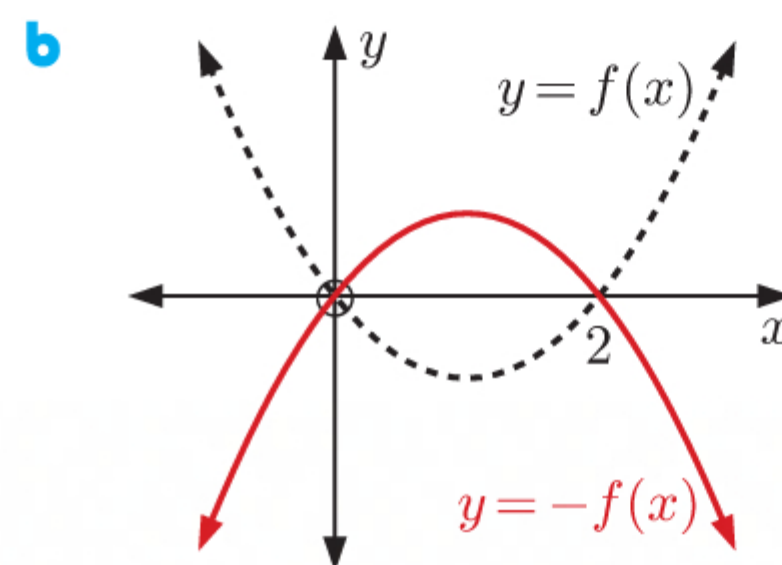
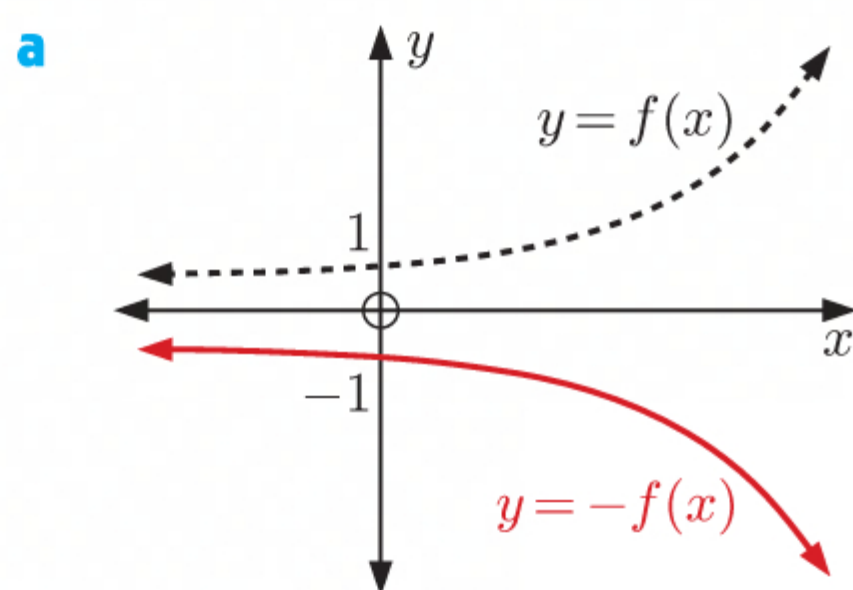
- 1 a The graph of $y = -f(x)$ is found by reflecting $y = f(x)$ in the x -axis.



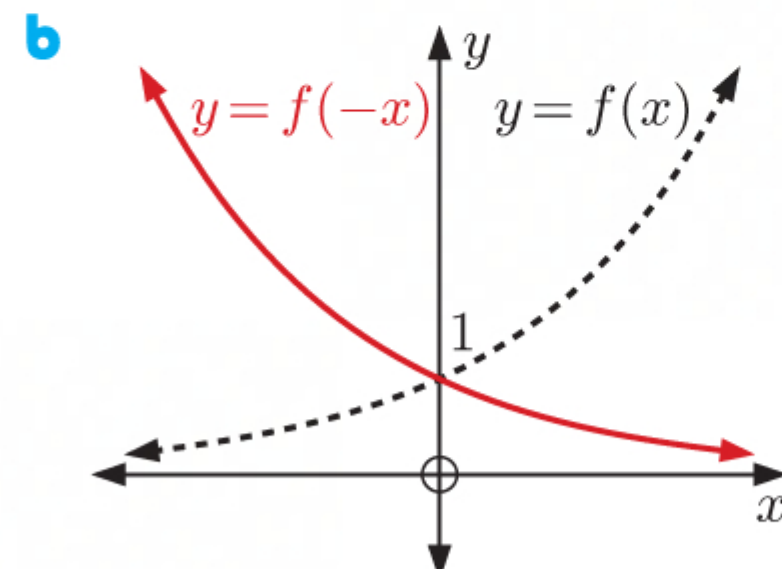
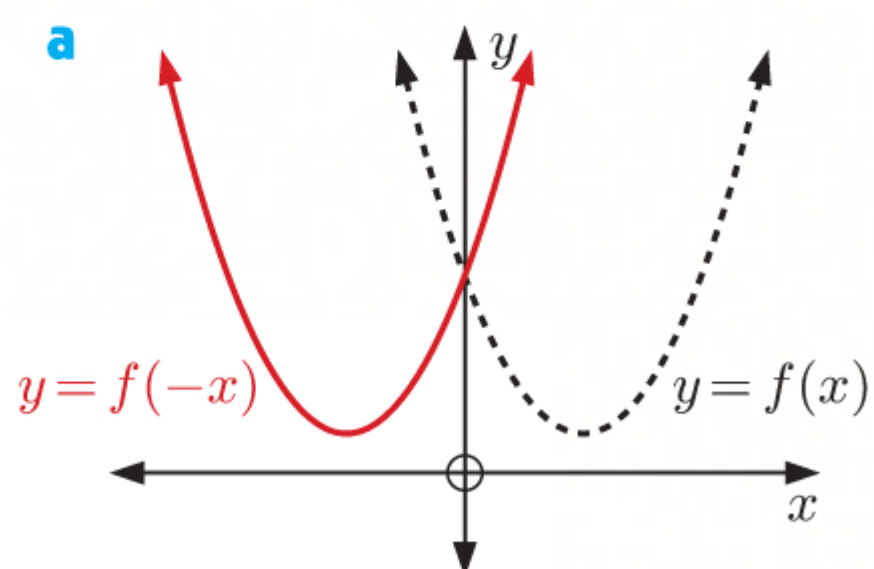
- b The graph of $y = f(-x)$ is found by reflecting $y = f(x)$ in the y -axis.

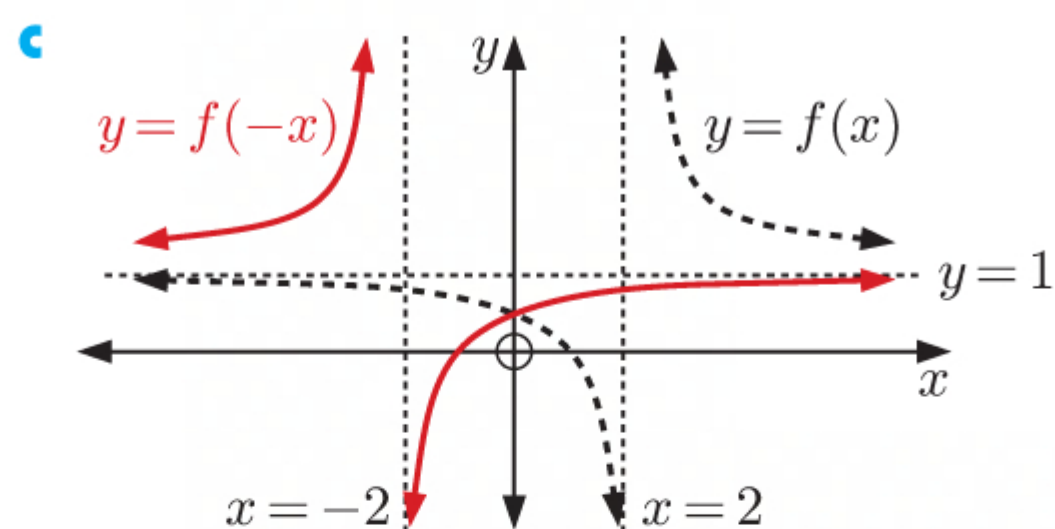


- 2 The graph of $y = -f(x)$ is found by reflecting $y = f(x)$ in the x -axis.



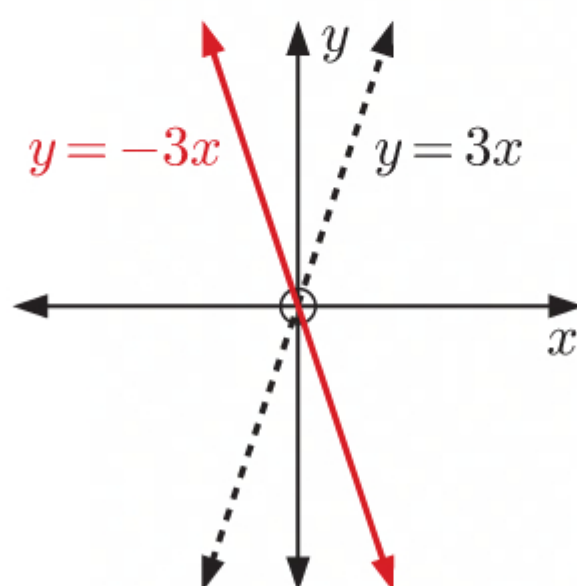
- 3 The graph of $y = f(-x)$ is found by reflecting $y = f(x)$ in the y -axis.



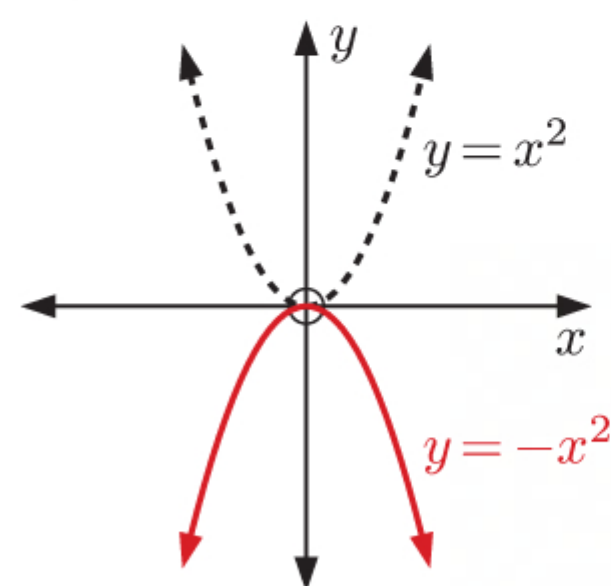


4 The graph of $y = -f(x)$ is found by reflecting $y = f(x)$ in the x -axis.

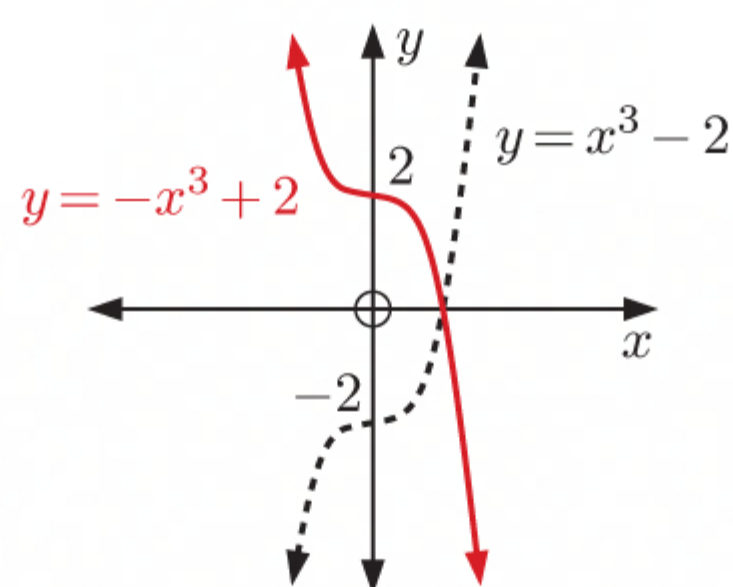
a $f(x) = 3x$



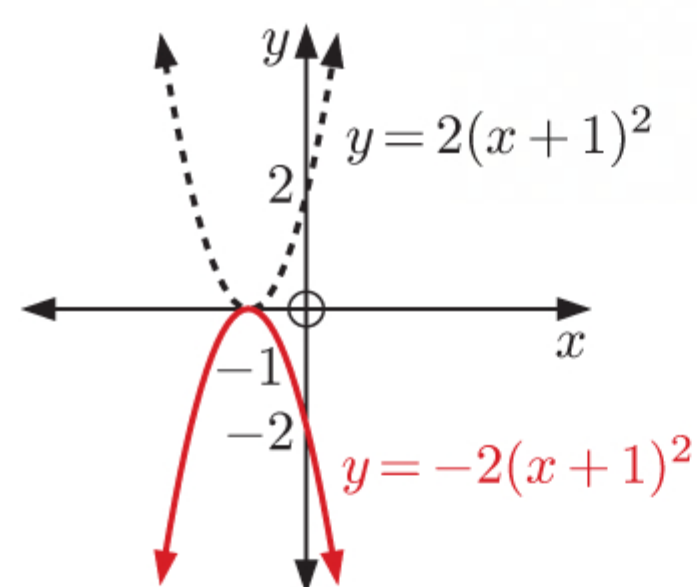
b $f(x) = x^2$



c $f(x) = x^3 - 2$



d $f(x) = 2(x + 1)^2$



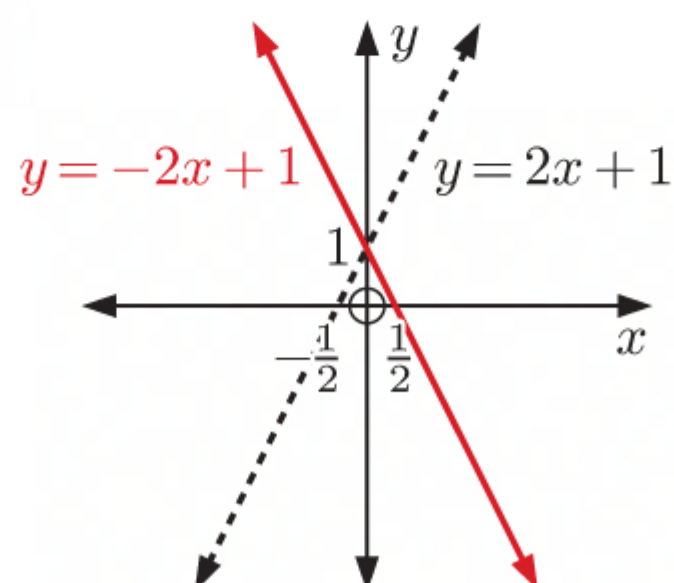
5 a i $f(x) = 2x + 1$
 $\therefore f(-x) = 2(-x) + 1$
 $= -2x + 1$

iii $f(x) = x^3$
 $\therefore f(-x) = (-x)^3$
 $= -x^3$

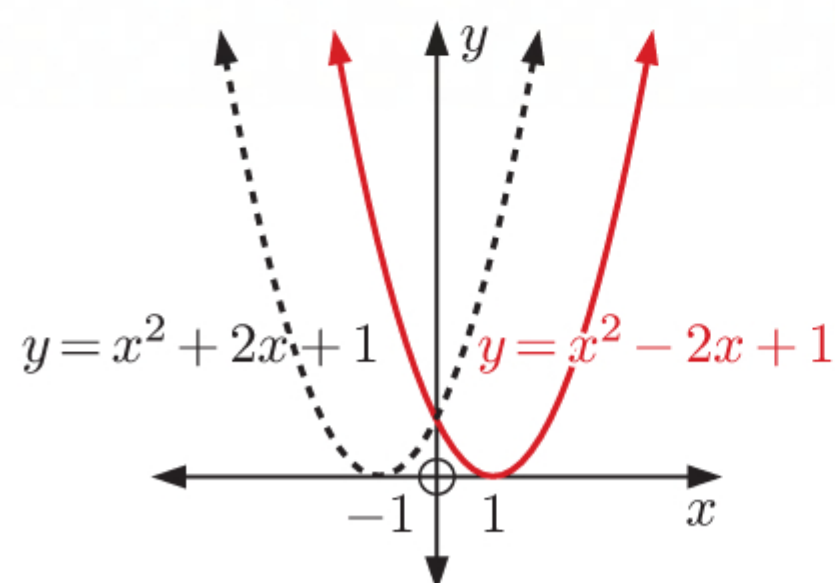
ii $f(x) = x^2 + 2x + 1$
 $\therefore f(-x) = (-x)^2 + 2(-x) + 1$
 $= x^2 - 2x + 1$

b The graph of $y = f(-x)$ is found by reflecting $y = f(x)$ in the y -axis.

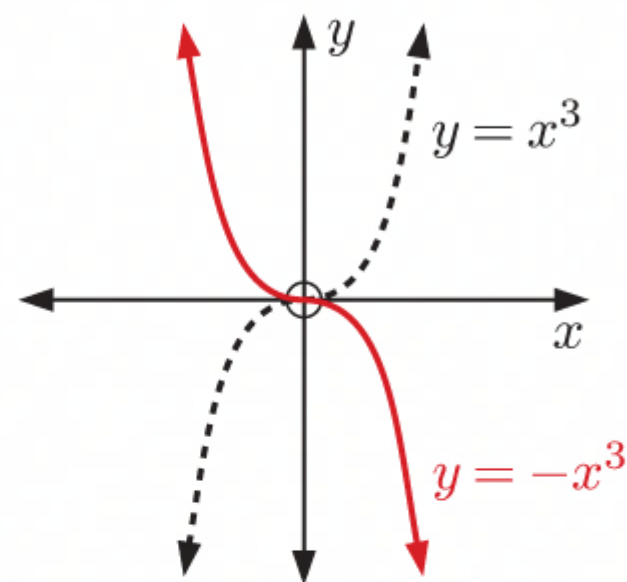
i $f(x) = 2x + 1$



ii $f(x) = x^2 + 2x + 1$



iii $f(x) = x^3$



6 $f(x) = x^4 - 2x^3 - 3x^2 + 5x - 7$

$$\begin{aligned}\therefore g(x) &= f(-x) \quad \{\text{reflected in the } y\text{-axis}\} \\ &= (-x)^4 - 2(-x)^3 - 3(-x)^2 + 5(-x) - 7 \\ &= x^4 + 2x^3 - 3x^2 - 5x - 7\end{aligned}$$

- 7 To transform $y = f(x)$ to $g(x) = -f(x)$, we reflect $y = f(x)$ in the x -axis.
By doing this, the x -coordinate stays the same, and we take the negative of the y -coordinate.

- a
 - i The point $(3, 0)$ on $y = f(x)$ will be transformed to $(3, 0)$ on $y = g(x)$.
 - ii The point $(2, -1)$ on $y = f(x)$ will be transformed to $(2, 1)$ on $y = g(x)$.
 - iii The point $(-3, 2)$ on $y = f(x)$ will be transformed to $(-3, -2)$ on $y = g(x)$.
- b
 - i The point on $y = f(x)$ which has been transformed to $(7, -1)$ on $y = g(x)$ is $(7, 1)$.
 - ii The point on $y = f(x)$ which has been transformed to $(-5, 0)$ on $y = g(x)$ is $(-5, 0)$.
 - iii The point on $y = f(x)$ which has been transformed to $(-3, -2)$ on $y = g(x)$ is $(-3, 2)$.

- 8 To transform $y = f(x)$ to $h(x) = f(-x)$, we reflect $y = f(x)$ in the y -axis.
By doing this, the y -coordinate stays the same, and we take the negative of the x -coordinate.

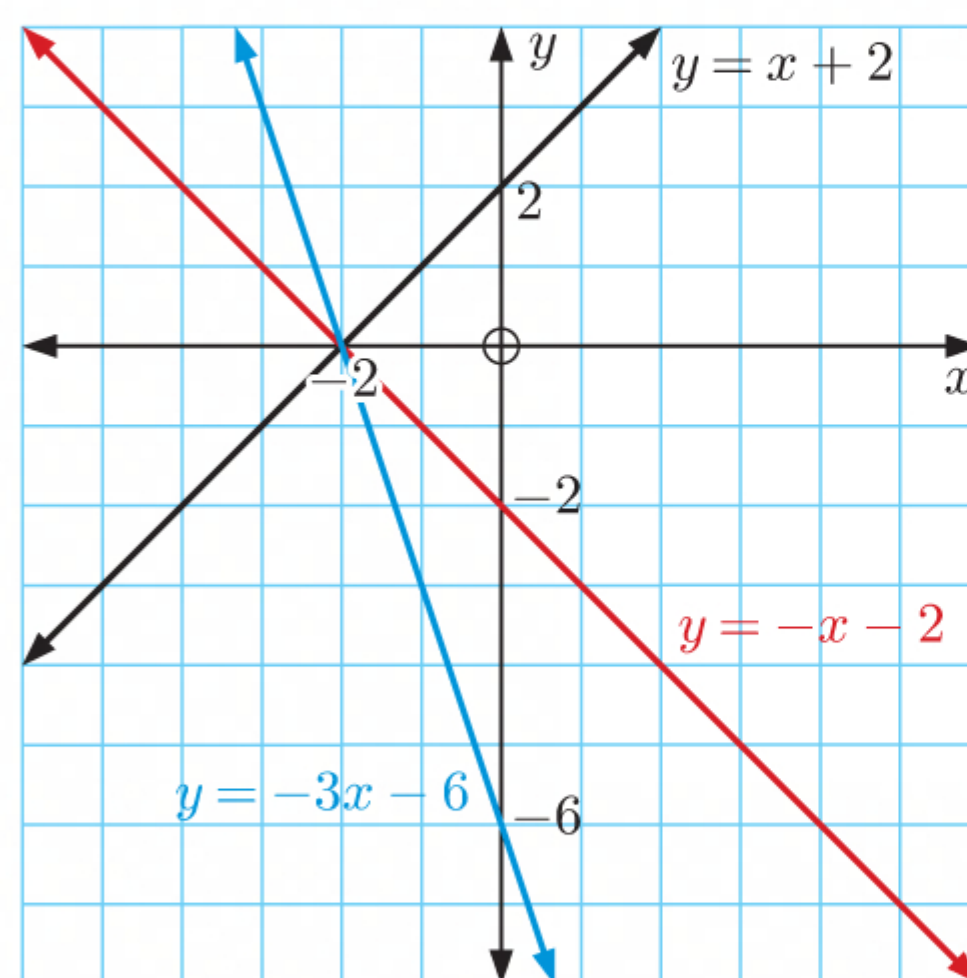
- a
 - i The point $(2, -1)$ on $y = f(x)$ will be transformed to $(-2, -1)$ on $y = h(x)$.
 - ii The point $(0, 3)$ on $y = f(x)$ will be transformed to $(0, 3)$ on $y = h(x)$.
 - iii The point $(-1, 2)$ on $y = f(x)$ will be transformed to $(1, 2)$ on $y = h(x)$.
- b
 - i The point on $y = f(x)$ which has been transformed to $(5, -4)$ on $y = h(x)$ is $(-5, -4)$.
 - ii The point on $y = f(x)$ which has been transformed to $(0, 3)$ on $y = h(x)$ is $(0, 3)$.
 - iii The point on $y = f(x)$ which has been transformed to $(2, 3)$ on $y = h(x)$ is $(-2, 3)$.

- 9 a To transform $y = f(x)$ to $g(x) = -f(-x)$, we first reflect $y = f(x)$ in the y -axis, and then reflect it in the x axis.
- b By reflecting in the x -axis and y -axis, we take the negative of the x -coordinate and the negative of the y -coordinate.
The point $(3, -7)$ on $y = f(x)$ will be transformed to $(-3, 7)$.
- c The point on $y = f(x)$ which has been transformed to $(-5, -1)$ on $y = g(x)$ is $(5, 1)$.

10 $f(x) = x + 2$

- a** To transform $y = f(x)$ to $y = -f(x)$, we reflect $y = f(x)$ in the x -axis.
b To transform $y = -f(x)$ to $y = -3f(x)$, we stretch $y = -f(x)$ vertically with scale factor 3.

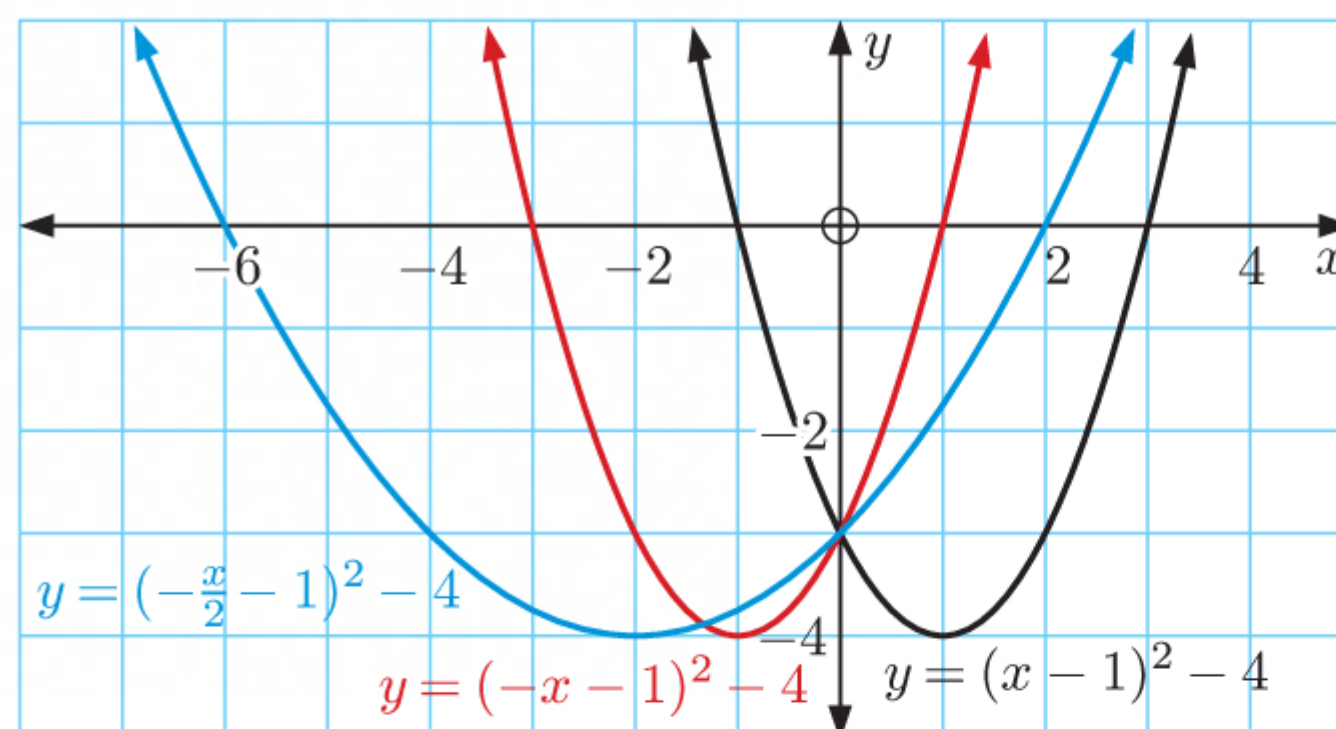
c $f(x) = x + 2$
 $\therefore -f(x) = -(x + 2) = -x - 2$
 and $-3f(x) = 3(-x - 2) = -3x - 6$



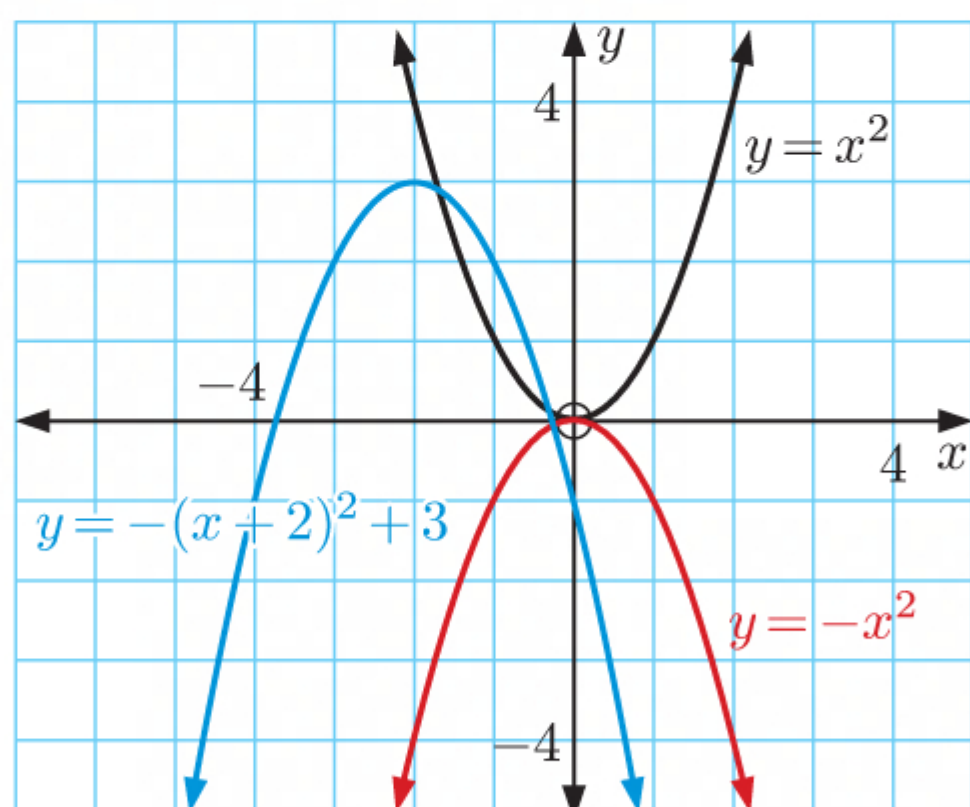
11 $f(x) = (x - 1)^2 - 4$

- a** To transform $y = f(x)$ to $y = f(-x)$, we reflect $y = f(x)$ in the y -axis.
b To transform $y = f(-x)$ to $y = f(-\frac{1}{2}x)$, we stretch $y = f(-x)$ horizontally with scale factor 2.

c $f(x) = (x - 1)^2 - 4$
 $\therefore f(-x) = (-x - 1)^2 - 4$
 and $f(-\frac{1}{2}x) = \left(-\frac{x}{2} - 1\right)^2 - 4$



12

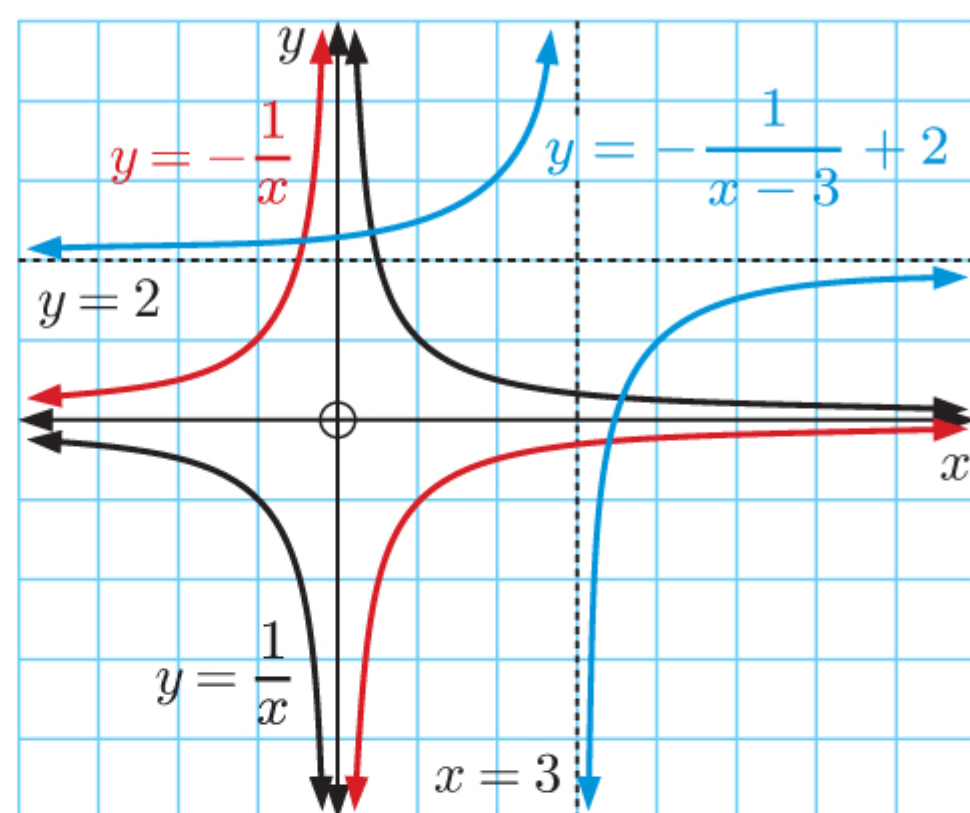


To transform $y = x^2$ to $y = -x^2$, we reflect $y = x^2$ in the x -axis.

To transform $y = -x^2$ to $y = -(x + 2)^2 + 3$,

we translate $y = -x^2$ through $\begin{pmatrix} -2 \\ 3 \end{pmatrix}$.

13



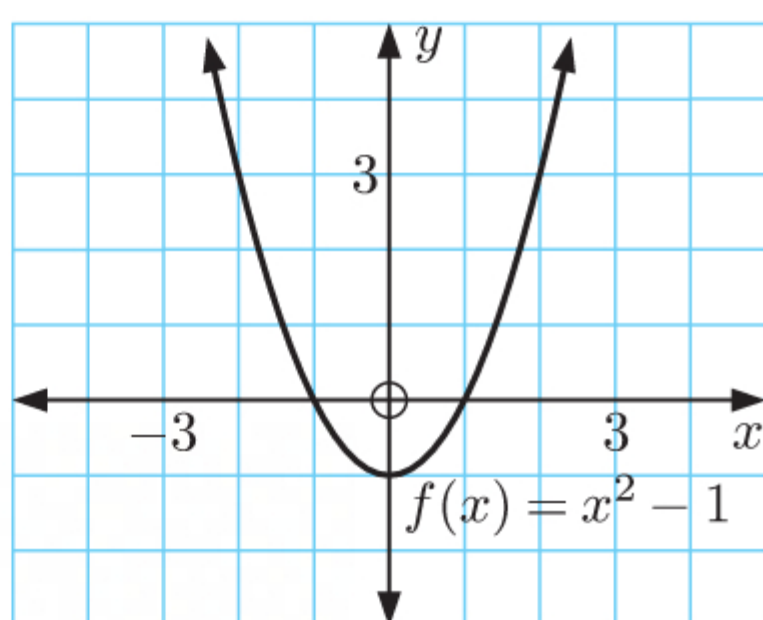
To transform $y = \frac{1}{x}$ to $y = -\frac{1}{x}$, we reflect $y = \frac{1}{x}$ in the x -axis.

To transform $y = -\frac{1}{x}$ to $y = -\frac{1}{x-3} + 2$, we translate $y = -\frac{1}{x}$ through $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$.

EXERCISE 4D

1 $f(x) = x^2 - 1$

a



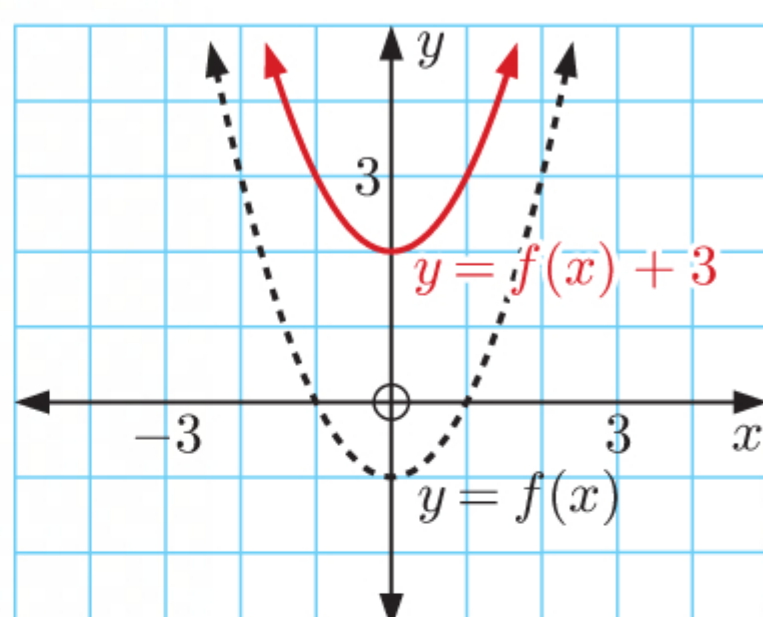
$$f(0) = (0)^2 - 1 = -1$$

\therefore the y -intercept is -1 .

$$\begin{aligned} \text{When } f(x) = 0, \quad x^2 - 1 &= 0 \\ x^2 &= 1 \\ x &= \pm 1 \end{aligned}$$

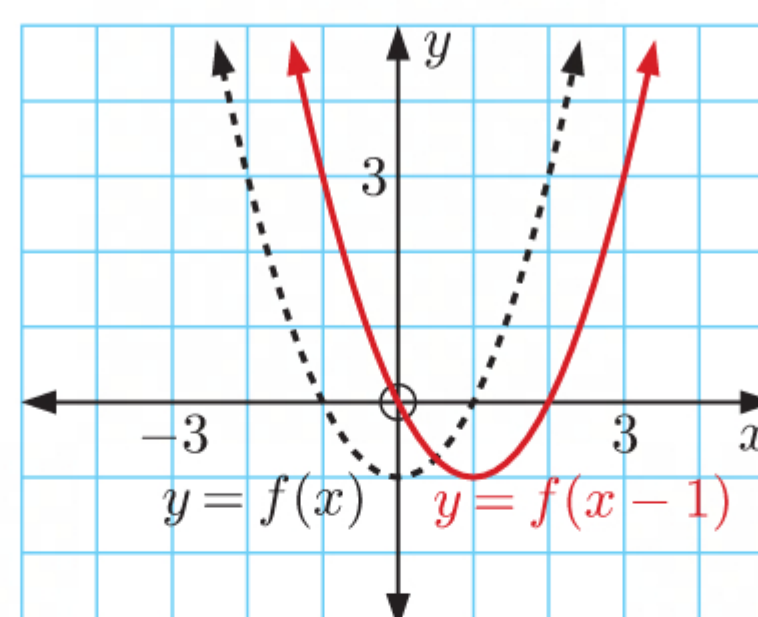
\therefore the x -intercepts are 1 and -1 .

b i



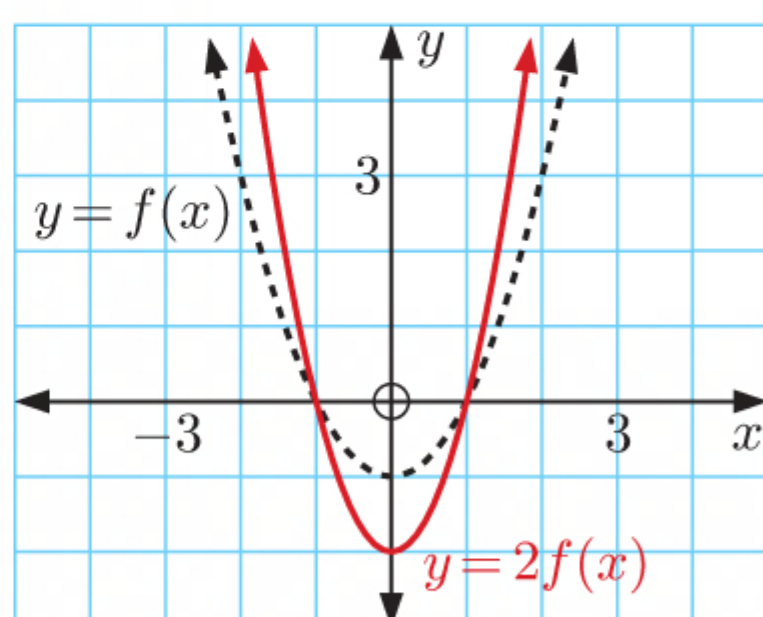
$y = f(x)$ has been translated 3 units upwards.

ii



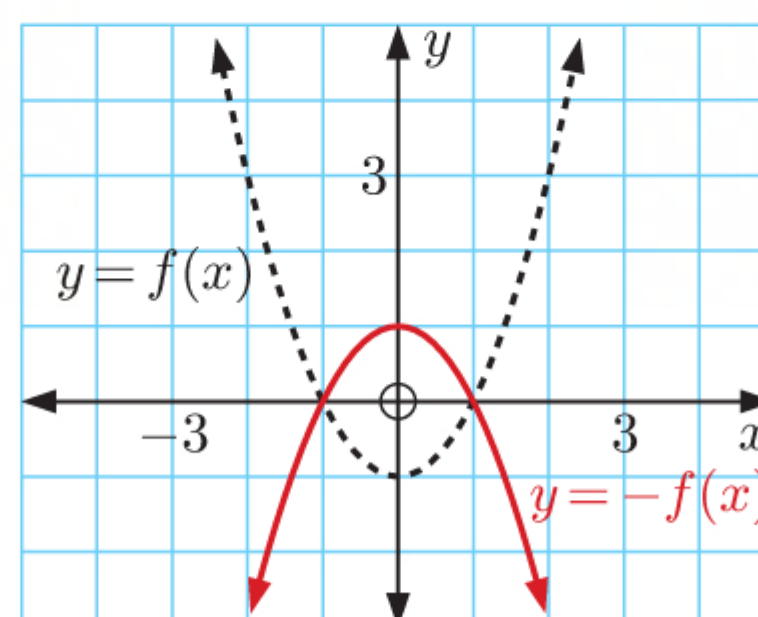
$y = f(x)$ has been translated 1 unit to the right.

iii



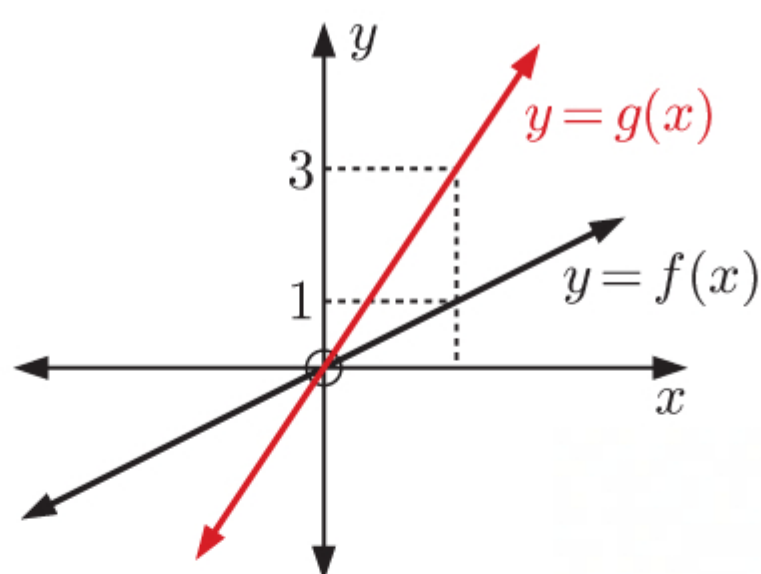
$y = f(x)$ has been vertically stretched with scale factor 2.

iv



$y = f(x)$ has been reflected in the x -axis.

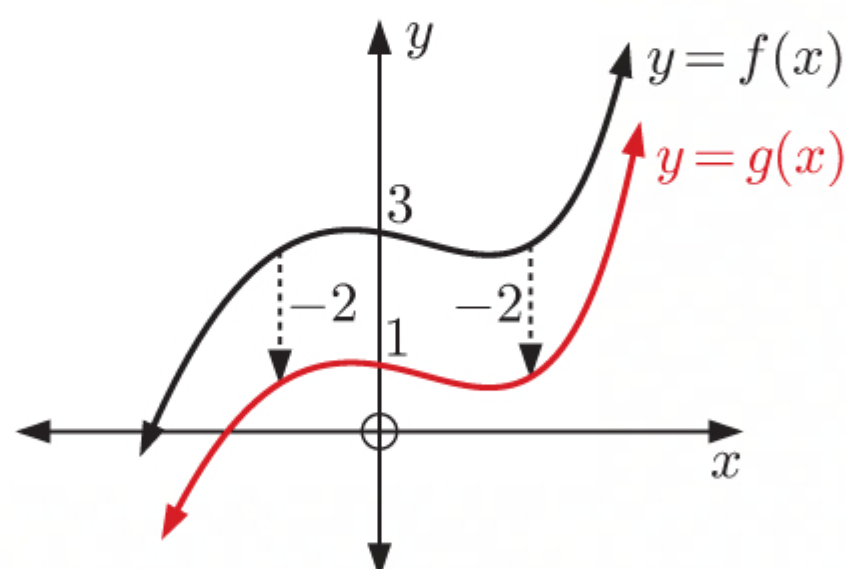
2 a



i $y = f(x)$ has been stretched vertically with scale factor 3.

ii $g(x) = 3f(x)$

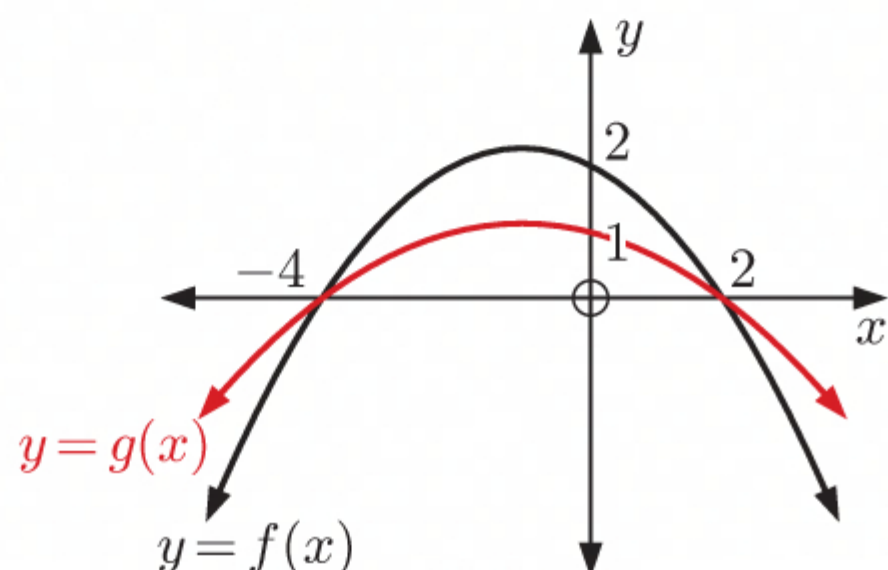
b



i $y = f(x)$ has been translated through $\begin{pmatrix} 0 \\ -2 \end{pmatrix}$.

ii $g(x) = f(x) - 2$

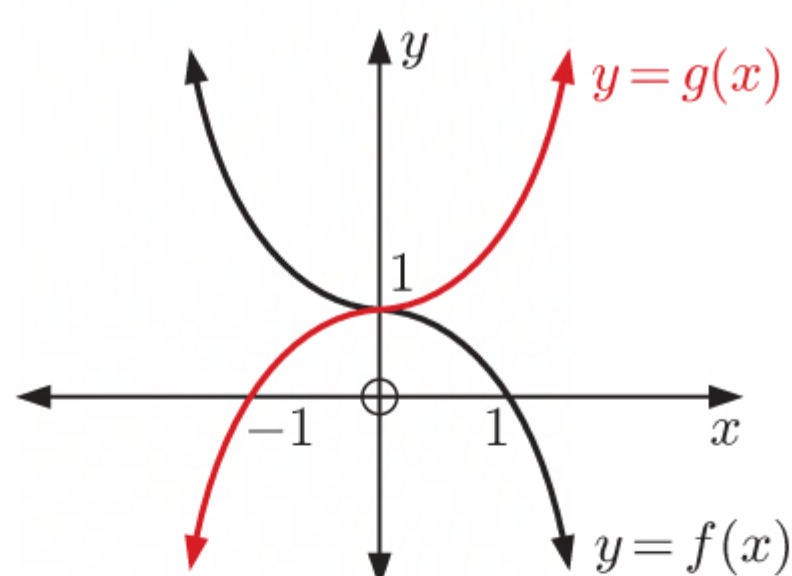
c



i $y = f(x)$ has been stretched vertically with scale factor $\frac{1}{2}$.

ii $g(x) = \frac{1}{2}f(x)$

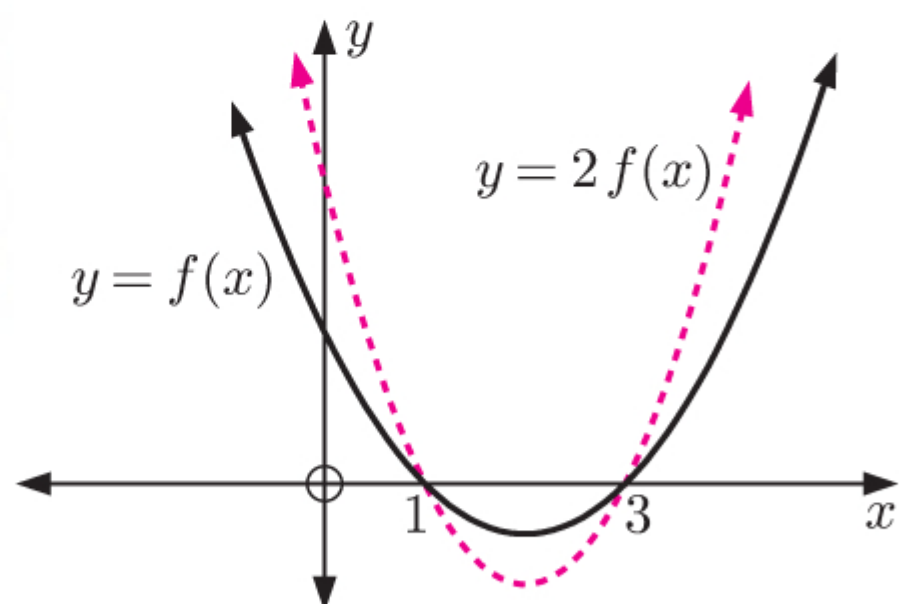
d



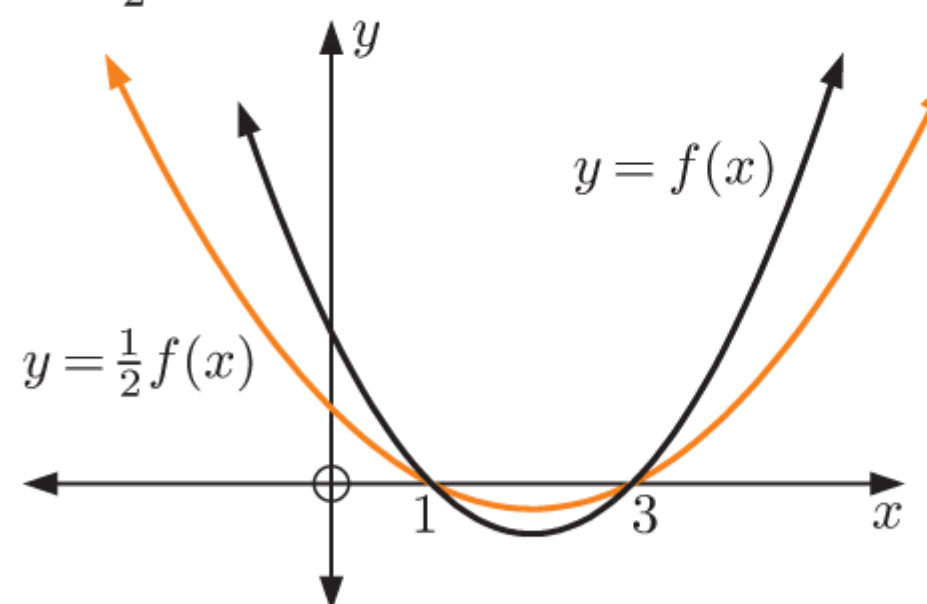
i $y = f(x)$ has been reflected in the y -axis.

ii $g(x) = f(-x)$

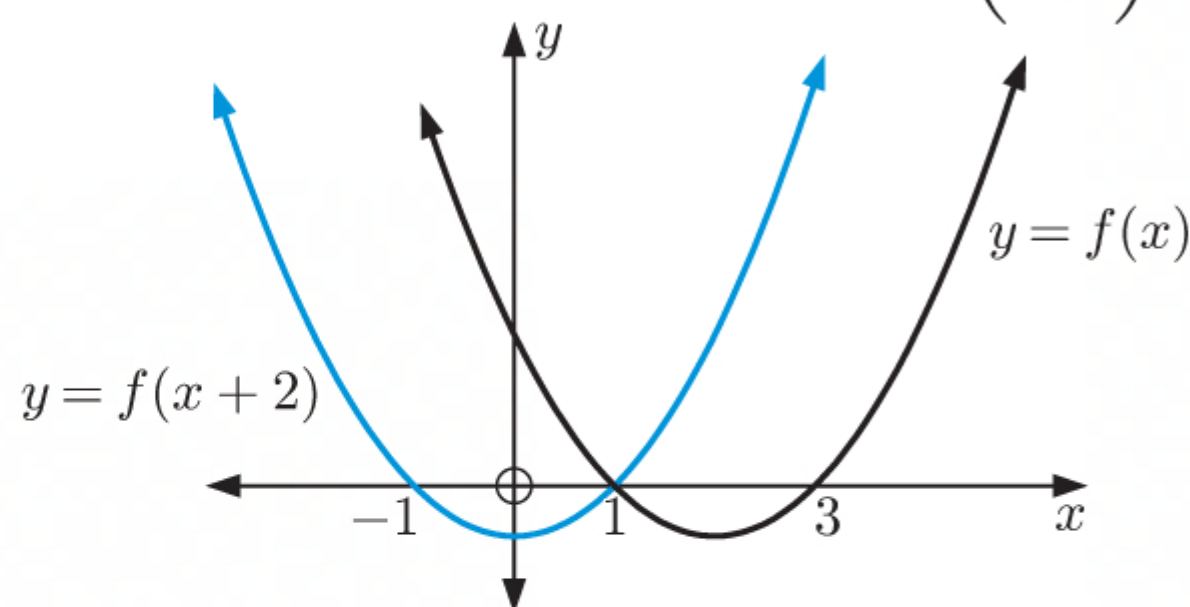
3 a To transform $y = f(x)$ to $y = 2f(x)$, we vertically stretch $y = f(x)$ with scale factor 2.



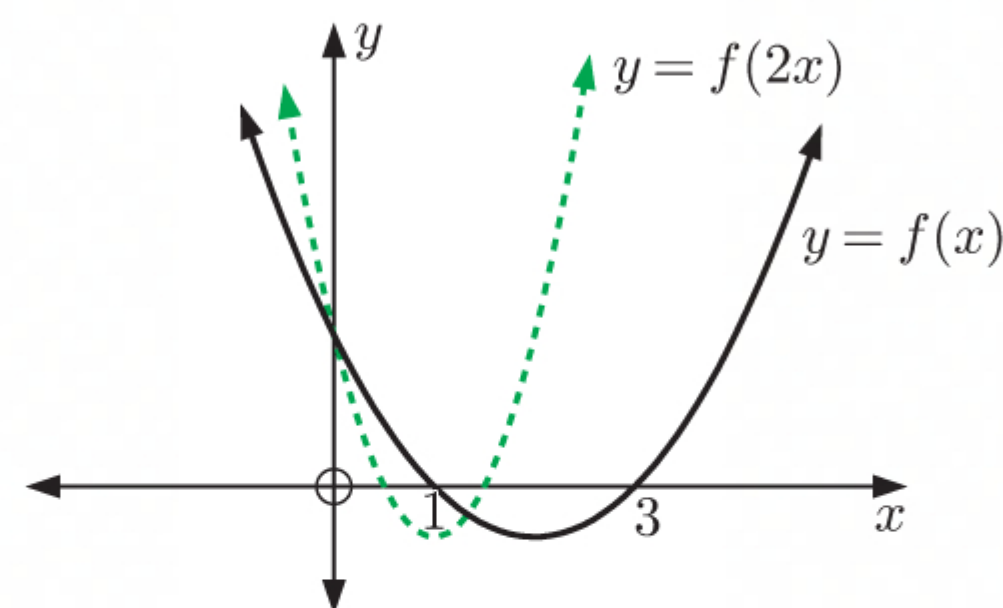
b To transform $y = f(x)$ to $y = \frac{1}{2}f(x)$, we vertically stretch $y = f(x)$ with scale factor $\frac{1}{2}$.



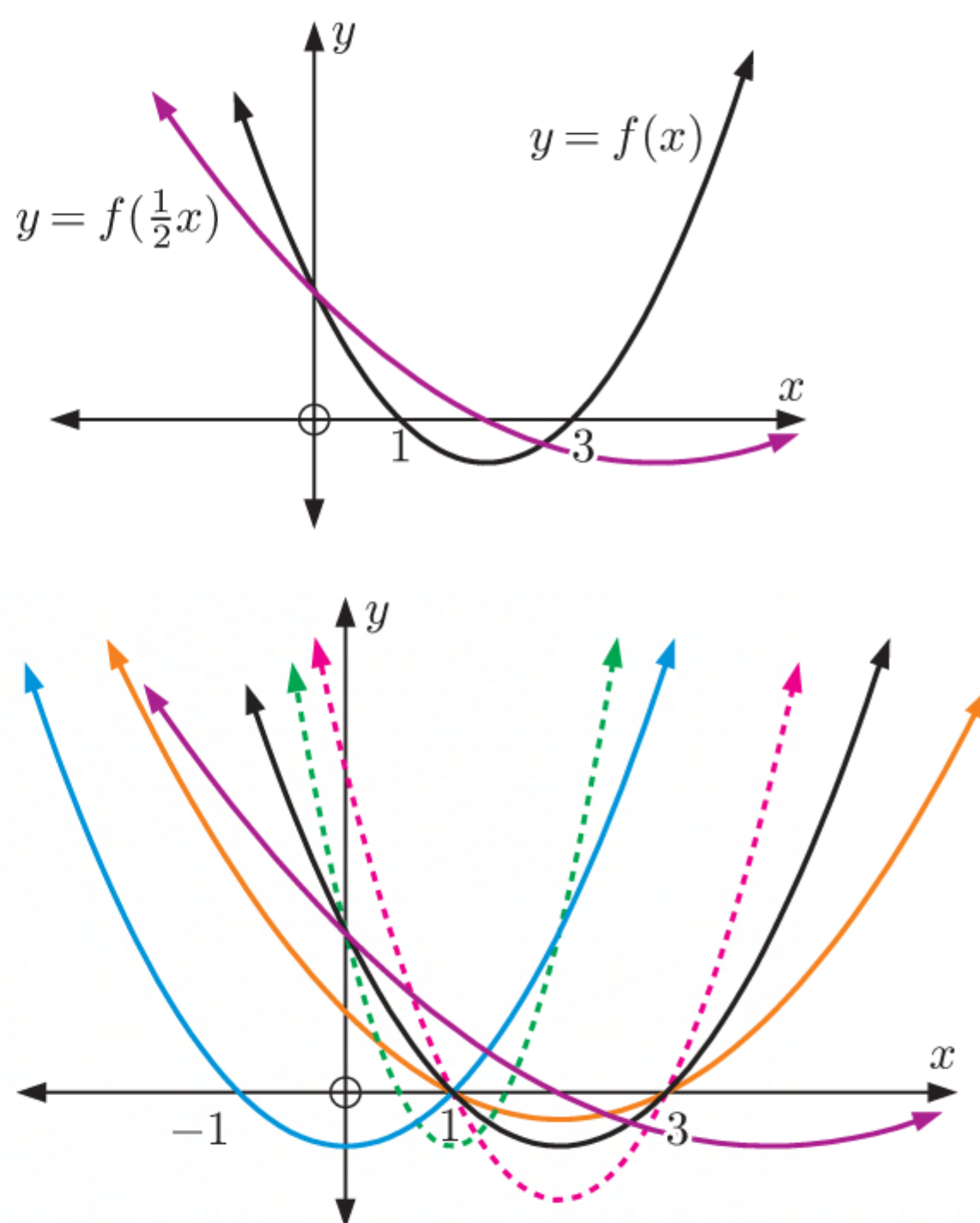
- c** To transform $y = f(x)$ to $y = f(x+2)$, we translate $y = f(x)$ through $\begin{pmatrix} -2 \\ 0 \end{pmatrix}$.



- d** To transform $y = f(x)$ to $y = f(2x)$, we horizontally stretch $y = f(x)$ with scale factor $\frac{1}{2}$.

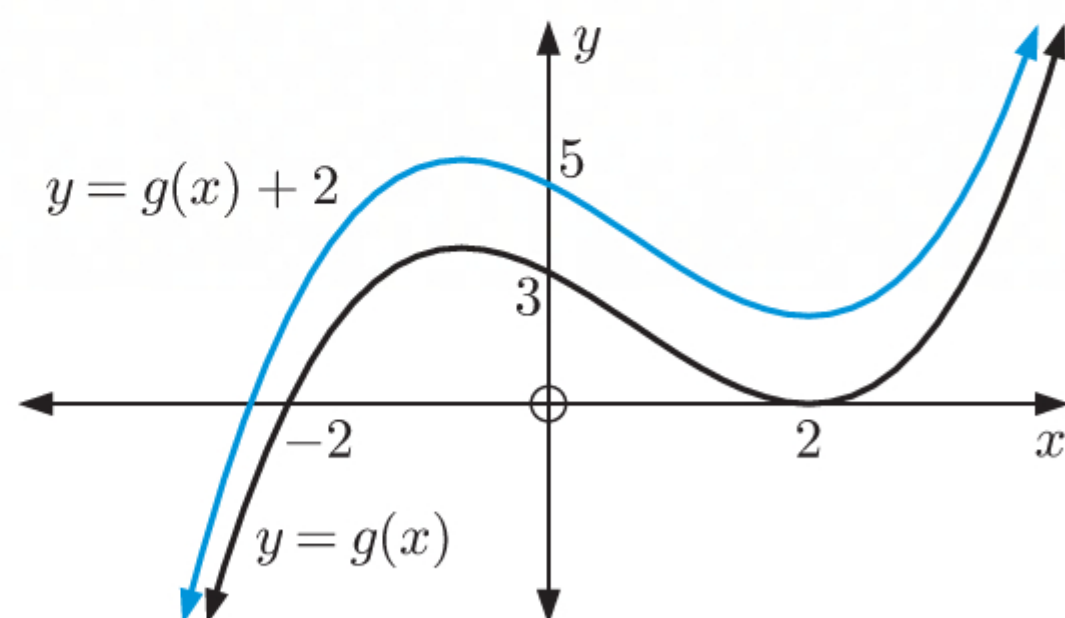


- e** To transform $y = f(x)$ to $y = f(\frac{1}{2}x)$, we horizontally stretch $y = f(x)$ with scale factor 2.

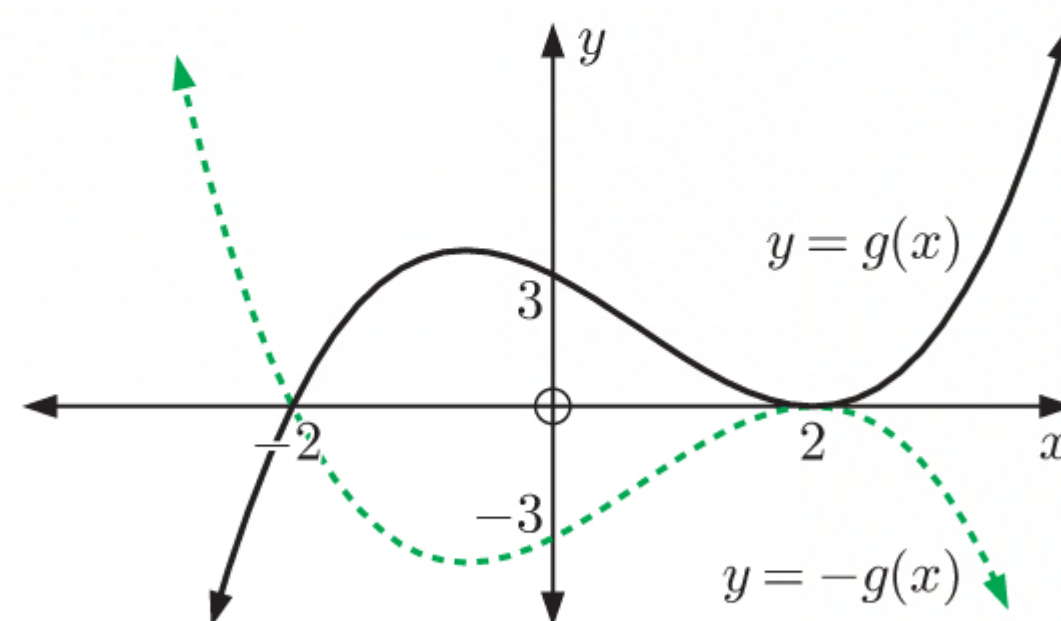


- $\longleftrightarrow y = f(x)$
 $\dashrightarrow y = 2f(x)$
 $\dashleftarrow y = \frac{1}{2}f(x)$
 $\longleftrightarrow y = f(x+2)$
 $\dashrightarrow y = f(2x)$
 $\dashleftarrow y = f(\frac{1}{2}x)$

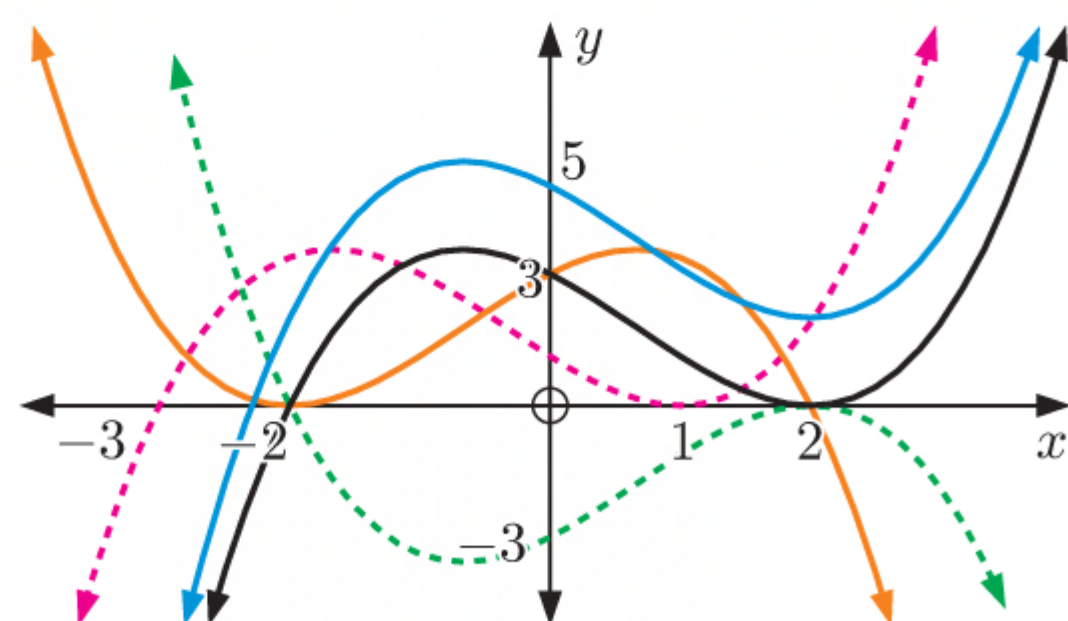
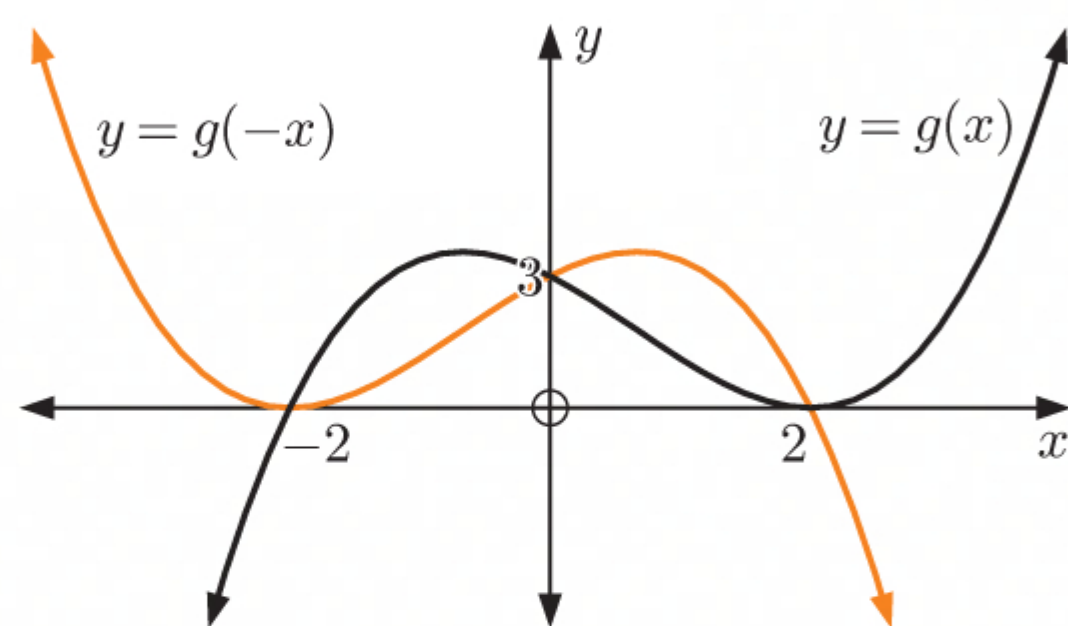
- 4 a** To transform $y = g(x)$ to $y = g(x) + 2$, we translate $y = g(x)$ through $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$.



- b** To transform $y = g(x)$ to $y = -g(x)$, we reflect $y = g(x)$ in the x -axis.

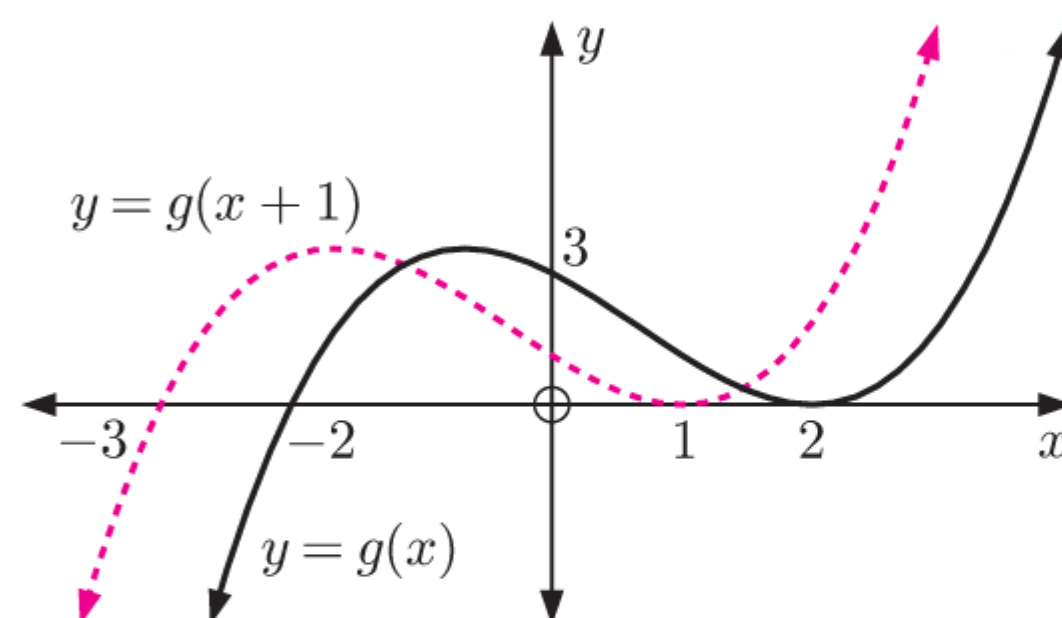


- c** To transform $y = g(x)$ to $y = g(-x)$, we reflect $y = g(x)$ in the y -axis.

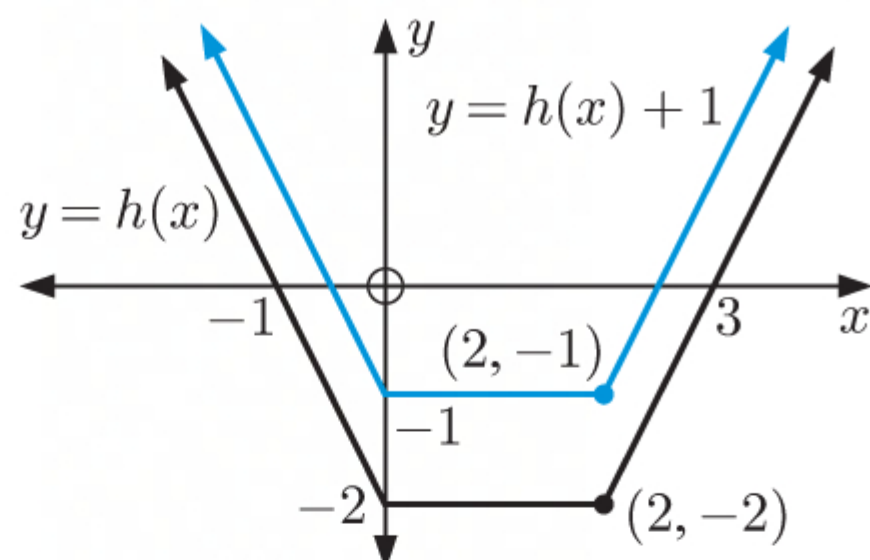


- \longleftrightarrow $y = g(x)$
 \longleftrightarrow $y = g(x) + 2$
 \longleftrightarrow $y = -g(x)$
 \longleftrightarrow $y = g(-x)$
 \longleftrightarrow $y = g(x + 1)$

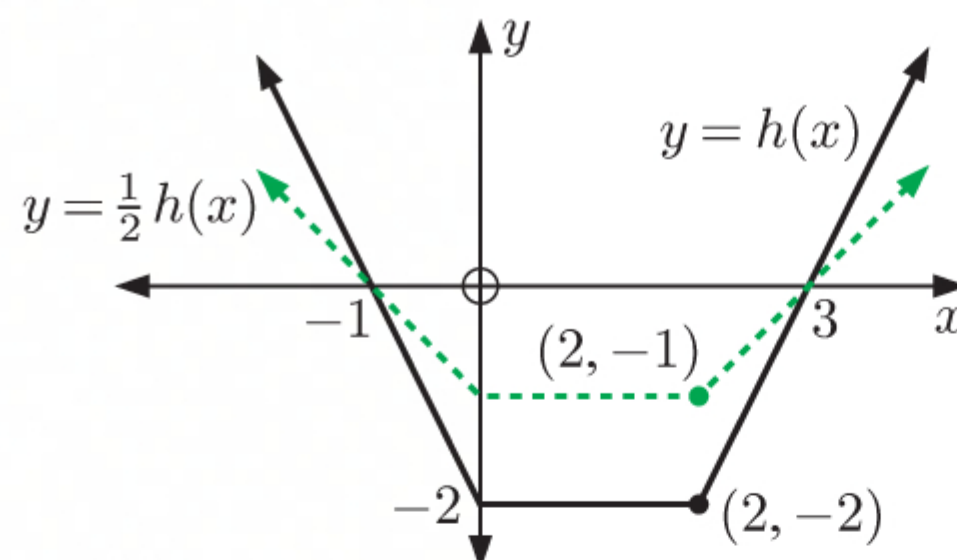
- d** To transform $y = g(x)$ to $y = g(x + 1)$, we translate $y = g(x)$ through $\begin{pmatrix} -1 \\ 0 \end{pmatrix}$.



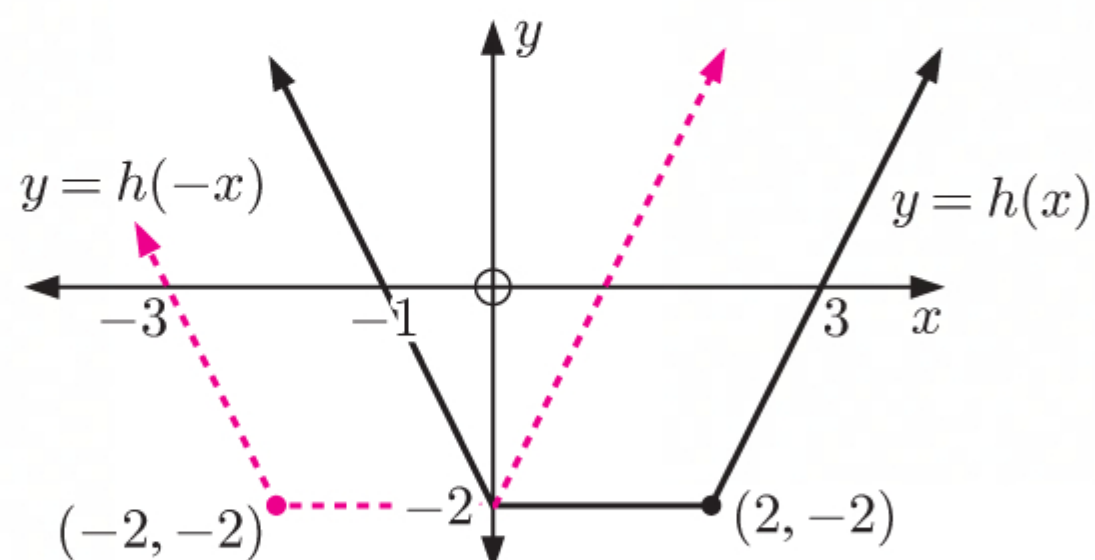
- 5 a** To transform $y = h(x)$ to $y = h(x) + 1$, we translate $y = h(x)$ through $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$.



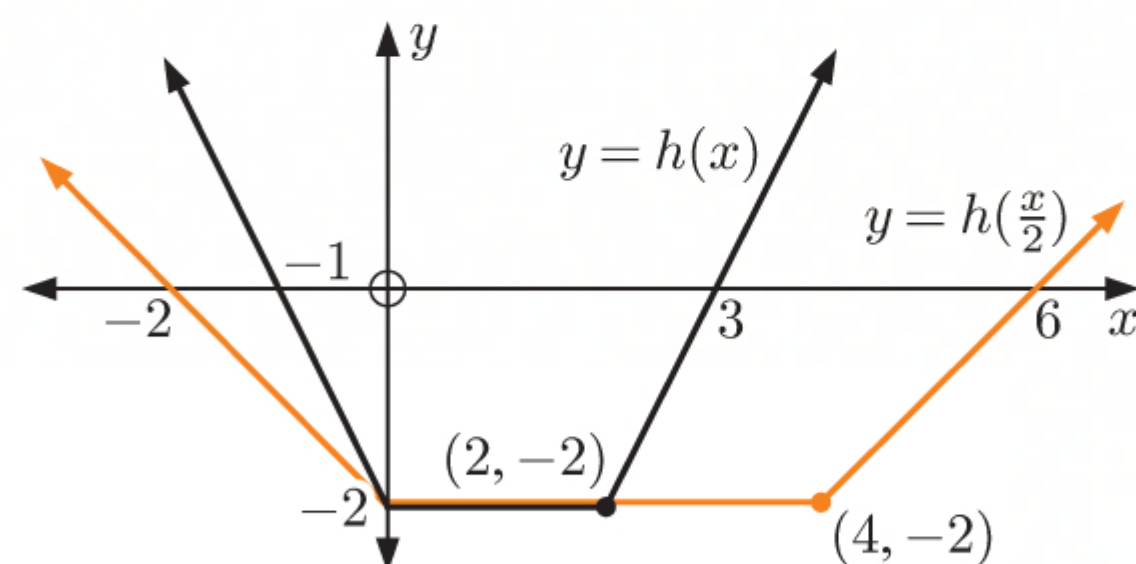
- b** To transform $y = h(x)$ to $y = \frac{1}{2}h(x)$, we vertically stretch $y = h(x)$ with scale factor $\frac{1}{2}$.

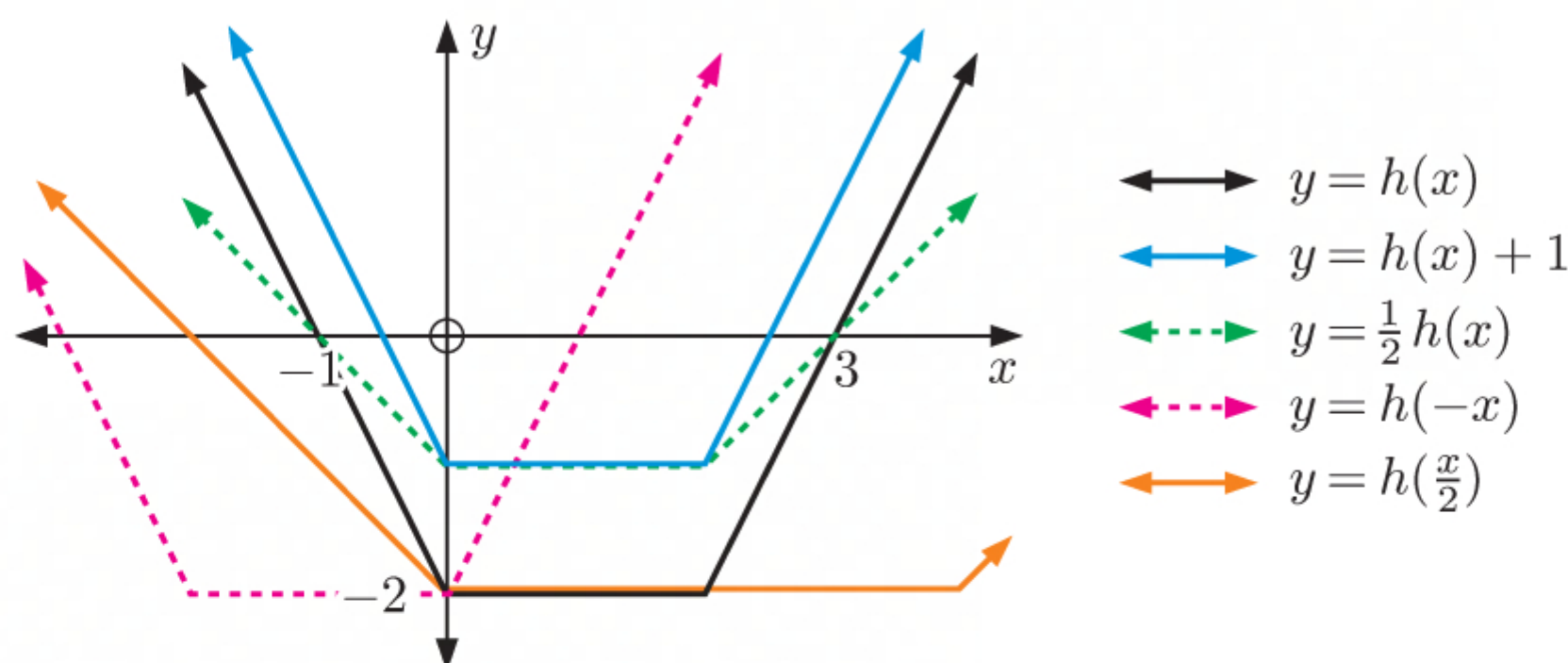


- c** To transform $y = h(x)$ to $y = h(-x)$, we reflect $y = h(x)$ in the y -axis.

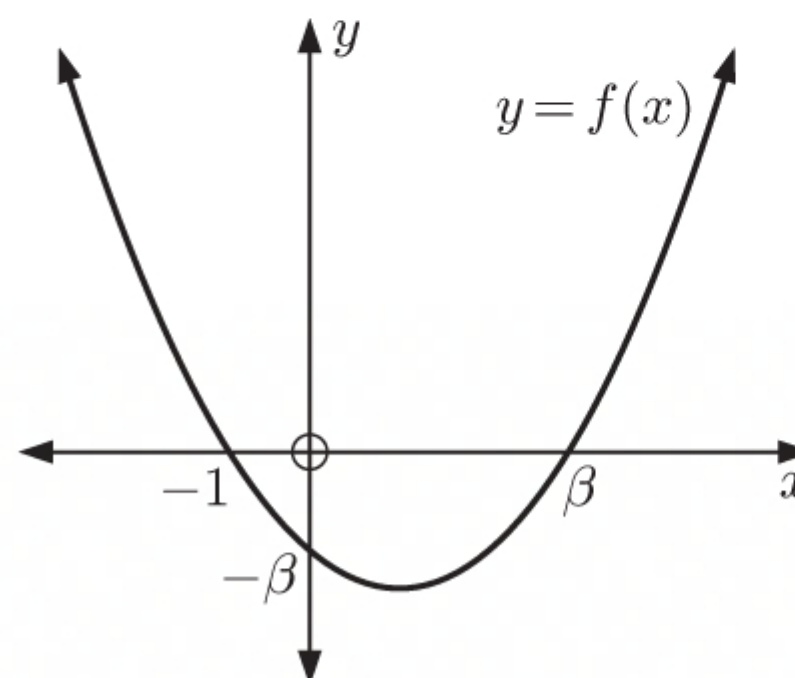


- d** To transform $y = h(x)$ to $y = h\left(\frac{x}{2}\right)$, we horizontally stretch $y = h(x)$ with scale factor 2.



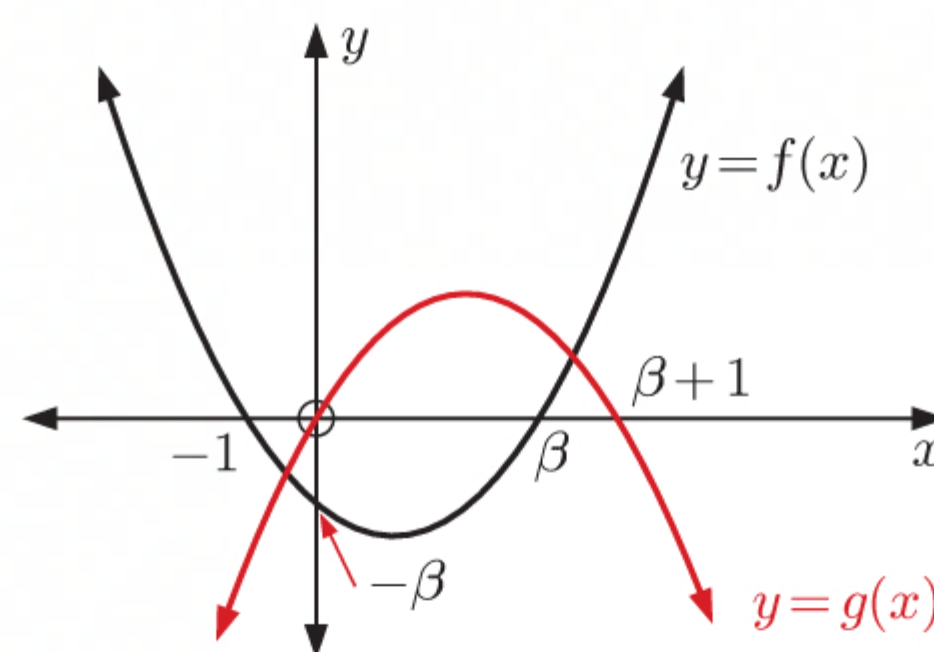


- 6 a $f(x) = (x+1)(x-\beta)$
 $= 0$ when $x = -1$ or $x = \beta$
 \therefore the x -intercepts are -1 and β .
 $f(0) = (0+1)(0-\beta)$
 $= (1)(-\beta)$
 $= -\beta$
 \therefore the y -intercept is $-\beta$.



- b To transform $f(x)$ to $g(x) = -f(x-1)$, we reflect $y = f(x)$ in the x -axis, then translate it through $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

- c The x -intercepts of $y = g(x)$ are the x -intercepts of $y = f(x)$ translated 1 unit to the right.
 \therefore the x -intercepts are 0 and $\beta + 1$.



Now $g(0) = -f(0-1)$
 $= -f(-1)$
 $= -(-1+1)(-1-\beta)$
 $= -(0)(-1-\beta)$
 $= 0$

\therefore the y -intercept is 0 .

7 a $f(x) \xrightarrow{\text{translation } \begin{pmatrix} 4 \\ -1 \end{pmatrix}} f(x-4) - 1 \xrightarrow{\text{reflection in } y\text{-axis}} f(-x-4) - 1$

The resulting function is $f(-x-4) - 1$.

b $f(x) \xrightarrow{\text{reflection in } y\text{-axis}} f(-x) \xrightarrow{\text{translation } \begin{pmatrix} 4 \\ -1 \end{pmatrix}} f(-(x-4)) - 1$

The resulting function is $f(-x+4) - 1$.

$$\text{c } f(x) \xrightarrow[\text{translation } \begin{pmatrix} -2 \\ 1 \end{pmatrix}]{\text{vertical stretch scale factor } \frac{1}{2}} f(x+2)+1 \xrightarrow{\text{vertical stretch scale factor } \frac{1}{2}} \frac{1}{2}(f(x+2)+1)$$

The resulting function is $\frac{1}{2}f(x+2)+\frac{1}{2}$.

$$\text{d } f(x) \xrightarrow[\text{translation } \begin{pmatrix} -2 \\ 1 \end{pmatrix}]{\text{vertical stretch scale factor } \frac{1}{2}} \frac{1}{2}f(x) \xrightarrow{\text{vertical stretch scale factor } \frac{1}{2}} \frac{1}{2}f(x+2)+1$$

The resulting function is $\frac{1}{2}f(x+2)+1$.

$$\text{8 a } f(x) \xrightarrow[\text{translation } \begin{pmatrix} -1 \\ 3 \end{pmatrix}]{\text{reflection in } x\text{-axis}} -f(x) \xrightarrow{\text{translation } \begin{pmatrix} -1 \\ 3 \end{pmatrix}} -f(x+1)+3$$

A reflection in the x -axis, then a translation through $\begin{pmatrix} -1 \\ 3 \end{pmatrix}$ maps $y = f(x)$ onto $y = -f(x+1)+3$.

$$\text{b } f(x) \xrightarrow[\text{translation } \begin{pmatrix} 0 \\ -7 \end{pmatrix}]{\text{horizontal stretch scale factor } 2} f\left(\frac{1}{2}x\right) \xrightarrow{\text{translation } \begin{pmatrix} 0 \\ -7 \end{pmatrix}} f\left(\frac{1}{2}x\right)-7$$

A horizontal stretch with scale factor 2, then a translation through $\begin{pmatrix} 0 \\ -7 \end{pmatrix}$ maps $y = f(x)$ onto $y = f\left(\frac{1}{2}x\right)-7$.

$$\text{c } f(x) \xrightarrow[\text{translation } \begin{pmatrix} 1 \\ 0 \end{pmatrix}]{\text{horizontal stretch scale factor } \frac{1}{3}} f(3x) \xrightarrow{\text{translation } \begin{pmatrix} 1 \\ 0 \end{pmatrix}} f(3(x-1))$$

A horizontal stretch with scale factor $\frac{1}{3}$, then a translation through $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, maps $y = f(x)$ onto $y = f(3(x-1))$.

$$\text{d } f(x) \xrightarrow[\text{translation } \begin{pmatrix} 4 \\ -1 \end{pmatrix}]{\text{vertical stretch scale factor } 2} 2f(x) \xrightarrow[\text{translation } \begin{pmatrix} 4 \\ -1 \end{pmatrix}]{\text{horizontal stretch scale factor } 4} 2f\left(\frac{1}{4}x\right) \xrightarrow{\text{translation } \begin{pmatrix} 4 \\ -1 \end{pmatrix}} -1+2f\left(\frac{1}{4}(x-4)\right)$$

A vertical stretch with scale factor 2, a horizontal stretch with scale factor 4, then a translation through $\begin{pmatrix} 4 \\ -1 \end{pmatrix}$, maps $y = f(x)$ onto $y = -1+2f\left(\frac{1}{4}(x-4)\right)$.

$$\text{e } f(x) \xrightarrow[\text{translation } \begin{pmatrix} 1 \\ 5 \end{pmatrix}]{\text{vertical stretch scale factor } 2} 2f(x) \xrightarrow[\text{translation } \begin{pmatrix} 1 \\ 5 \end{pmatrix}]{\text{horizontal stretch scale factor } \frac{1}{3}} 2f(3x) \xrightarrow{\text{translation } \begin{pmatrix} 1 \\ 5 \end{pmatrix}} 5+2f(3(x-1))$$

A vertical stretch with scale factor 2, a horizontal stretch with scale factor $\frac{1}{3}$, then a translation through $\begin{pmatrix} 1 \\ 5 \end{pmatrix}$ maps $y = f(x)$ onto $y = 5+2f(3(x-1))$.

f

$$f(x) \xrightarrow{\text{reflection in } x\text{-axis}} -f(x) \xrightarrow{\text{vertical stretch scale factor 4}} -4f(x) \xrightarrow{\text{horizontal stretch scale factor 2}} -4f\left(\frac{1}{2}x\right) \xrightarrow{\text{translation } \begin{pmatrix} -3 \\ -1 \end{pmatrix}} -4f\left(\frac{1}{2}(x+3)\right) - 1$$

A reflection in the x -axis, a vertical stretch with scale factor 4, a horizontal stretch with scale factor 2, then a translation through $\begin{pmatrix} -3 \\ -1 \end{pmatrix}$ maps $y = f(x)$ onto $y = -4f\left(\frac{1}{2}(x+3)\right) - 1$.

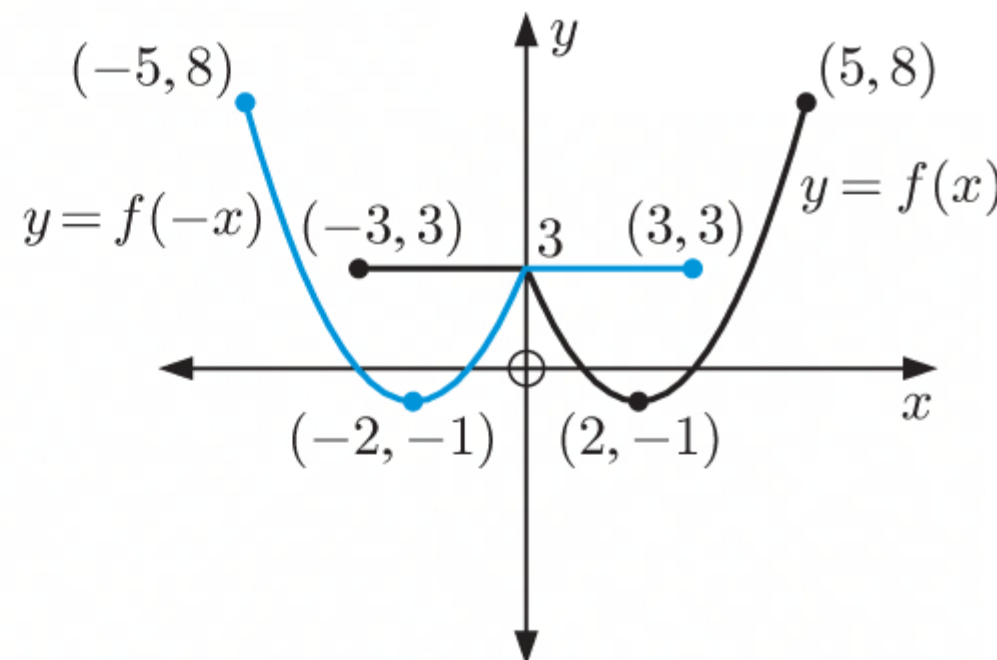
- 9 a** The graph is stretched vertically with scale factor $|a|$, and reflected in the x -axis. It is then translated h units horizontally and k units vertically.

- b** The function has shape  after it is reflected in the x -axis.

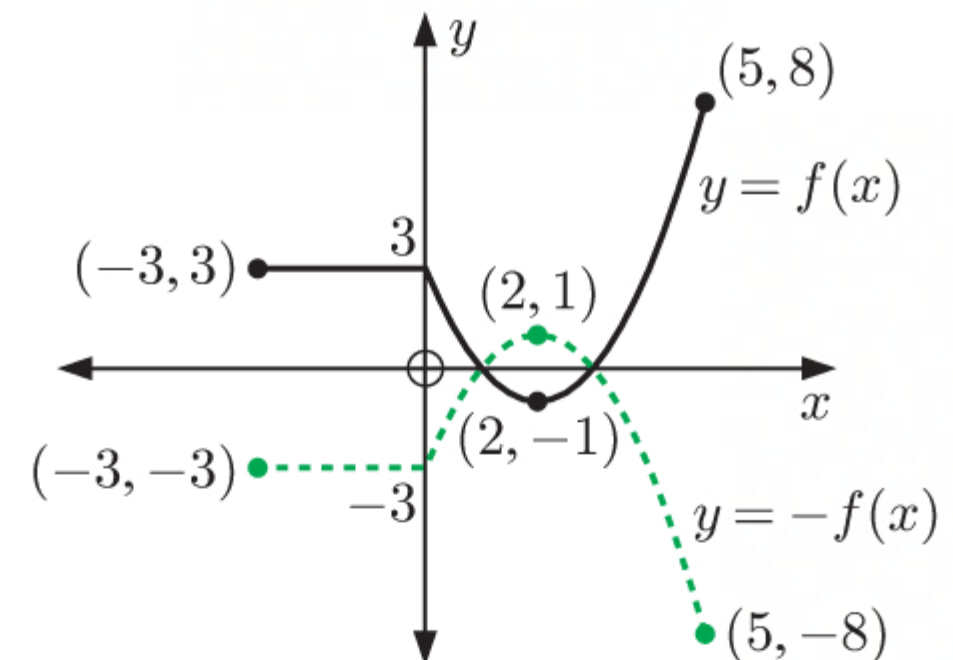
The function has vertex (h, k) , and y -intercept $ah^2 + k$.

REVIEW SET 4A

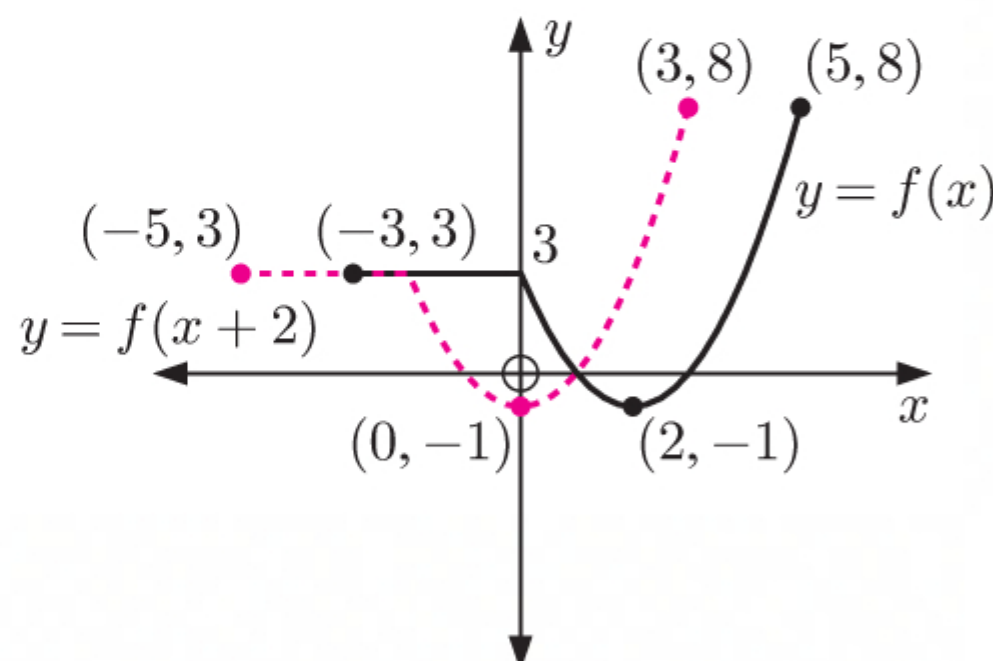
- 1 a** To transform $y = f(x)$ to $y = f(-x)$, we reflect $y = f(x)$ in the y -axis.



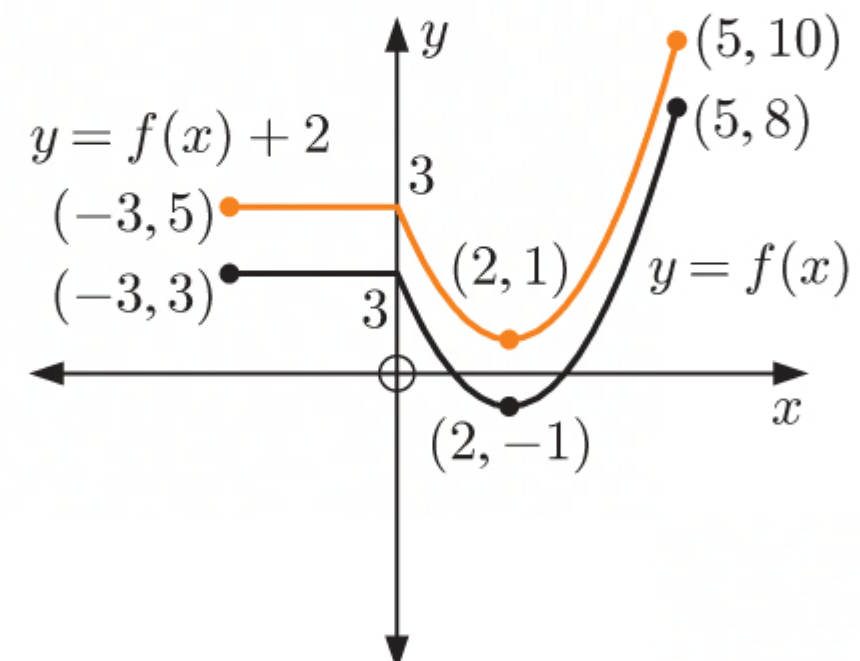
- b** To transform $y = f(x)$ to $y = -f(x)$, we reflect $y = f(x)$ in the x -axis.

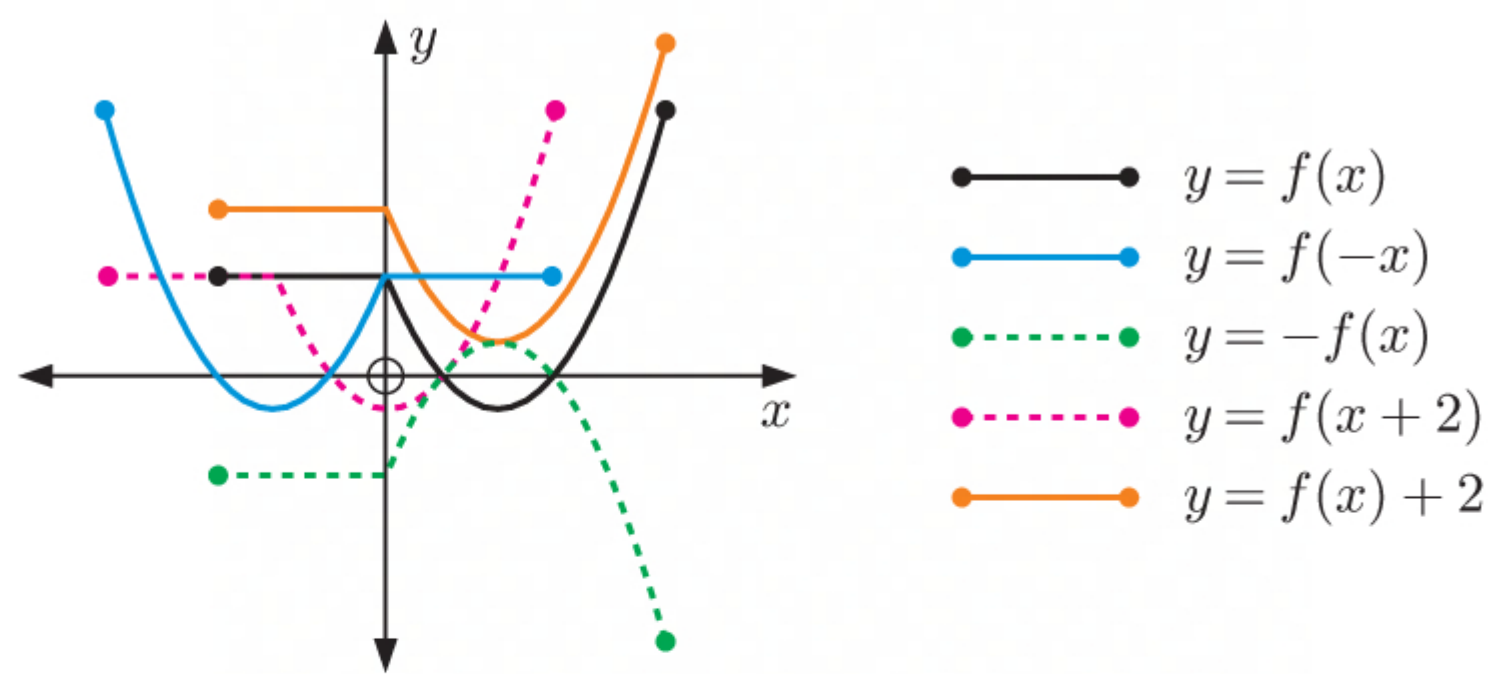


- c** To transform $y = f(x)$ to $y = f(x+2)$, we translate $y = f(x)$ through $\begin{pmatrix} -2 \\ 0 \end{pmatrix}$.



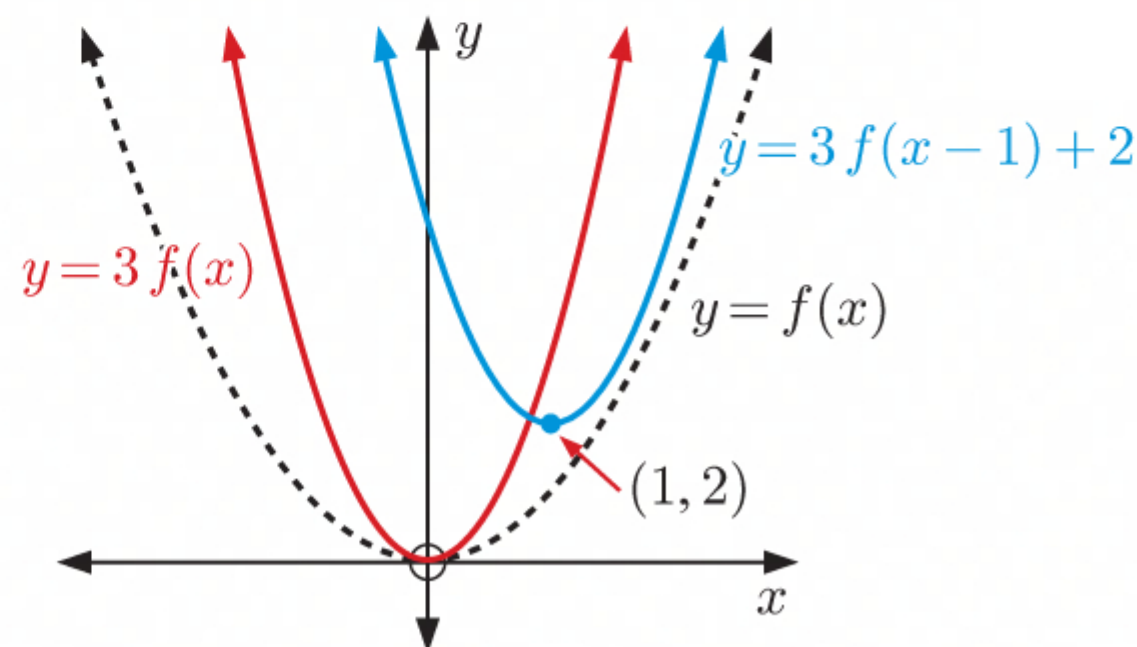
- d** To transform $y = f(x)$ to $y = f(x) + 2$, we translate $y = f(x)$ through $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$.





2 $y = 3f(x)$ is a vertical stretch of $y = f(x)$ with scale factor 3.

$y = 3f(x-1) + 2$ is a translation of $y = 3f(x)$ through $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$.



3 a $g(x) = f(x) - 3$
 $= 4x - 7 - 3$
 $= 4x - 10$

b $g(x) = 5f(x)$
 $= 5(x^2 + 6)$
 $= 5x^2 + 30$

c $g(x) = f(x+4)$
 $= 7 - 3(x+4)$
 $= 7 - 3x - 12$
 $= -3x - 5$

d $g(x) = f\left(\frac{1}{3}x\right)$
 $= 2\left(\frac{1}{3}x\right)^2 - \left(\frac{1}{3}x\right) + 4$
 $= \frac{2}{9}x^2 - \frac{1}{3}x + 4$

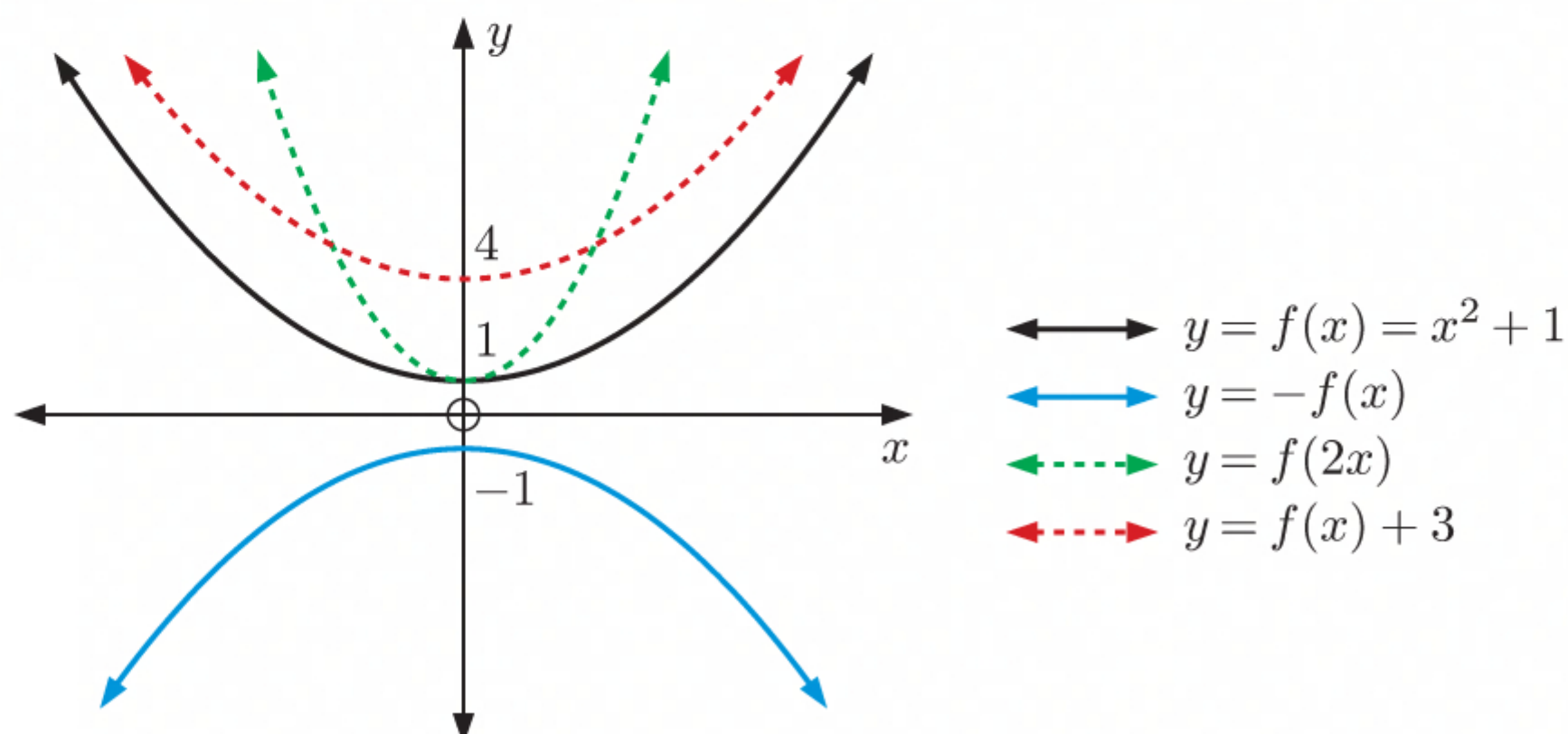
e $g(x) = f(-x)$
 $= (-x)^3$
 $= -x^3$

4 $f(x) = x^2 + 1$

a To transform $y = f(x)$ to $y = -f(x)$, we reflect $y = f(x)$ in the x -axis.


b To transform $y = f(x)$ to $y = f(2x)$, we horizontally stretch $y = f(x)$ with scale factor $\frac{1}{2}$.


c To transform $y = f(x)$ to $y = f(x) + 3$, we translate $y = f(x)$ through $\begin{pmatrix} 0 \\ 3 \end{pmatrix}$.



- 5** $f(x)$ has domain $\{x \mid -2 \leq x \leq 3\}$ and range $\{y \mid -1 \leq y \leq 7\}$.
 $g(x) = f(x+3) - 4$ translates every point on $y = f(x)$ 3 units to the left and 4 units downwards.
 $\therefore g(x)$ has domain $\{x \mid -5 \leq x \leq 0\}$ and range $\{y \mid -5 \leq y \leq 3\}$.

6 a $g(x) = f(x-2) + 4$
 $= [(x-2) + 1]^2 + 4 + 4$
 $= (x-1)^2 + 8$

- b i** $f(x) = (x+1)^2 + 4$ has vertex $(-1, 4)$ and shape  ($a > 0$).
 \therefore the minimum value is 4.
 \therefore the range of $f(x)$ is $\{y \mid y \geq 4\}$.

- ii** $g(x) = (x-1)^2 + 8$ has vertex $(1, 8)$ and shape  ($a > 0$).
 \therefore the minimum value is 8.
 \therefore the range is $\{y \mid y \geq 8\}$.

- 7** $f(x) = 3x^2 - x + 4$ is transformed to $g(x)$ by translating through $\begin{pmatrix} -1 \\ 3 \end{pmatrix}$.
 $\therefore g(x) = f(x+1) + 3$
 $= 3(x+1)^2 - (x+1) + 4 + 3$
 $= 3(x^2 + 2x + 1) - x - 1 + 4 + 3$
 $= 3x^2 + 6x + 3 - x + 6$
 $= 3x^2 + 5x + 9$

8 a $f(x) = 3x + 2$

- i** When the graph of $y = f(x)$ is translated 2 units to the left, the resulting function is

$$\begin{aligned} f(x+2) &= 3(x+2) + 2 \\ &= 3x + 6 + 2 \\ &= 3x + 8 \end{aligned}$$

- ii** When the graph of $y = f(x)$ is translated 6 units upwards, the resulting function is

$$\begin{aligned} f(x) + 6 &= 3x + 2 + 6 \\ &= 3x + 8 \end{aligned}$$

b $f(x) = ax + b, \quad a > 0$

When the graph of $y = f(x)$ is translated k units to the left, the resulting function is

$$\begin{aligned} f(x+k) &= a(x+k) + b \\ &= ax + ak + b \end{aligned}$$

When the graph of $y = f(x)$ is translated ak units upwards, the resulting function is

$$\begin{aligned} f(x) + ak &= ax + b + ak \\ &= f(x+k) \end{aligned}$$

So the resulting line is the same.

9 a $f(x) \xrightarrow[\text{reflection in } x\text{-axis}]{\text{translation } \begin{pmatrix} -2 \\ 3 \end{pmatrix}} -f(x) \xrightarrow{\text{translation } \begin{pmatrix} -2 \\ 3 \end{pmatrix}} -f(x+2) + 3$

The resulting function is $-f(x+2) + 3$.

$$\text{b } f(x) \xrightarrow{\text{translation } \begin{pmatrix} 4 \\ -1 \end{pmatrix}} f(x-4) - 1 \xrightarrow{\text{vertical stretch scale factor 2}} 2[f(x-4) - 1]$$

The resulting function is $2f(x-4) - 2$.

- 10 a** The graph of $y = f(x+4)$ is a translation of $y = f(x)$ 4 units to the left.
So, the graph of $y = f(x+4)$ will have x -intercepts $-5 - 4 = -9$ and $1 - 4 = -3$.
There is not enough information to determine the y -intercept.
- b** The graph of $y = 3f(x)$ is a vertical stretch of $y = f(x)$ with scale factor 3.
Each point on the graph of $y = f(x)$ becomes 3 times its previous distance from the x -axis.
So, the graph of $y = 3f(x)$ will have x -intercepts -5 and 1 (unchanged), and y -intercept $3 \times -3 = -9$.
- c** The graph of $y = f\left(\frac{x}{2}\right)$ is a horizontal stretch of $y = f(x)$ with scale factor 2.
Each point on the graph of $y = f(x)$ becomes 2 times its previous distance from the y -axis.
So, the graph of $y = f\left(\frac{x}{2}\right)$ will have x -intercepts $-5 \times 2 = -10$ and $1 \times 2 = 2$, and y -intercept -3 (unchanged).
- d** The graph of $y = -f(x)$ is a reflection of the graph of $y = f(x)$ in the x -axis.
The y -coordinate of each point on $y = f(x)$ becomes negative.
So, the graph of $y = -f(x)$ has x -intercepts -5 and 1 (unchanged), and y -intercept $-3 \times -1 = 3$.

$$\text{11 } \frac{1}{x} \xrightarrow{\text{translation } \begin{pmatrix} -1 \\ 2 \end{pmatrix}} \frac{1}{x+1} + 2 \xrightarrow{\text{reflection in } y\text{-axis}} \frac{1}{1-x} + 2$$

The resulting function is $g(x) = \frac{1}{1-x} + 2$.

$$\text{a } g(x) = \frac{1}{1-x} + 2$$

$$= \frac{1}{1-x} + \frac{2(1-x)}{(1-x)}$$

$$= \frac{1 + 2(1-x)}{1-x}$$

$$= \frac{1 + 2 - 2x}{1-x}$$

$$= \frac{3 - 2x}{1-x}$$

$$= \frac{2x - 3}{x - 1}$$

$$\text{b } \text{The asymptotes of } y = \frac{1}{x} \text{ are } x = 0 \text{ and } y = 0.$$

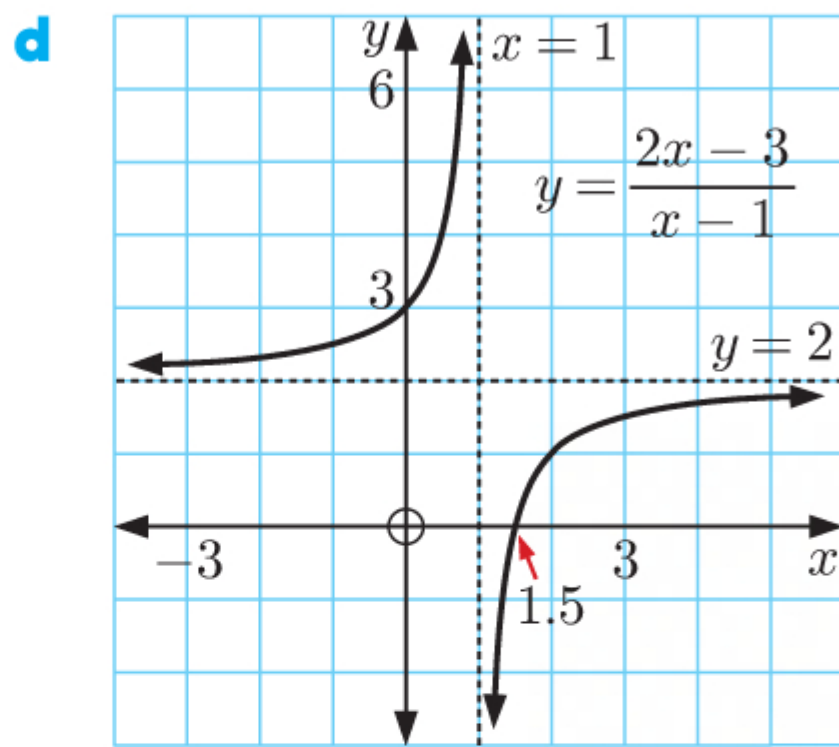
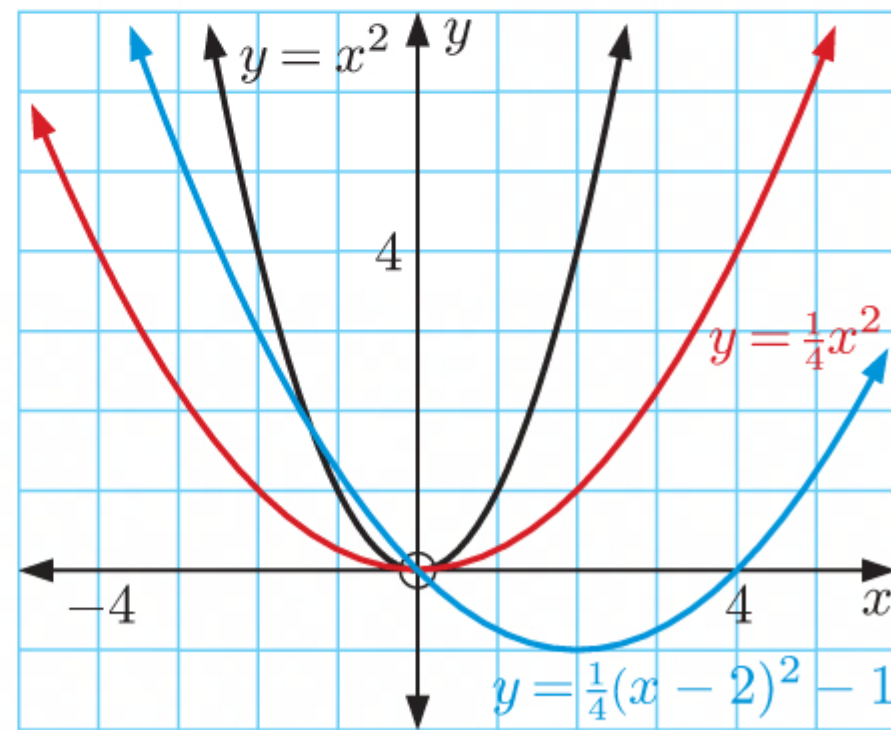
These are translated $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$ and reflected in the y -axis.

\therefore the vertical asymptote is $x = 1$ and the horizontal asymptote is $y = 2$.

$$\text{c } g(x) = \frac{1}{1-x} + 2$$

The domain of $g(x)$ is $\{x \mid x \neq 1\}$.

The range of $g(x)$ is $\{y \mid y \neq 2\}$.

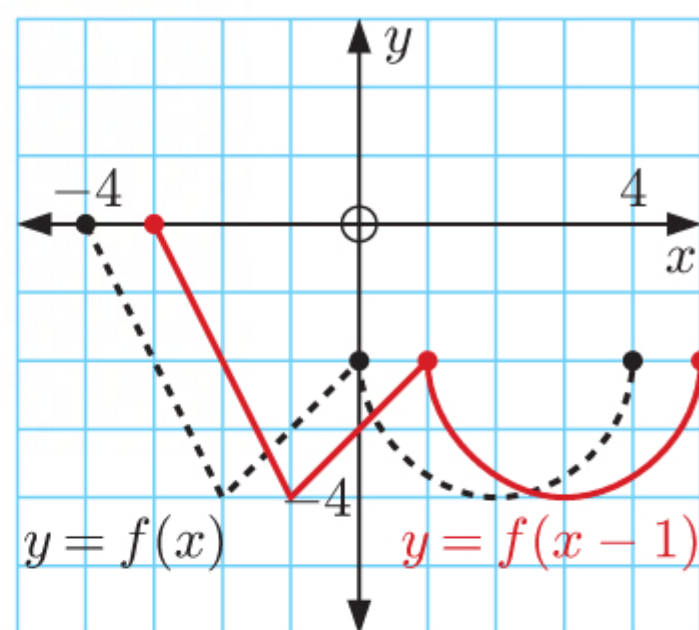
**12**

To transform $y = x^2$ to $y = \frac{1}{4}x^2$, we vertically stretch $y = x^2$ with scale factor $\frac{1}{4}$.

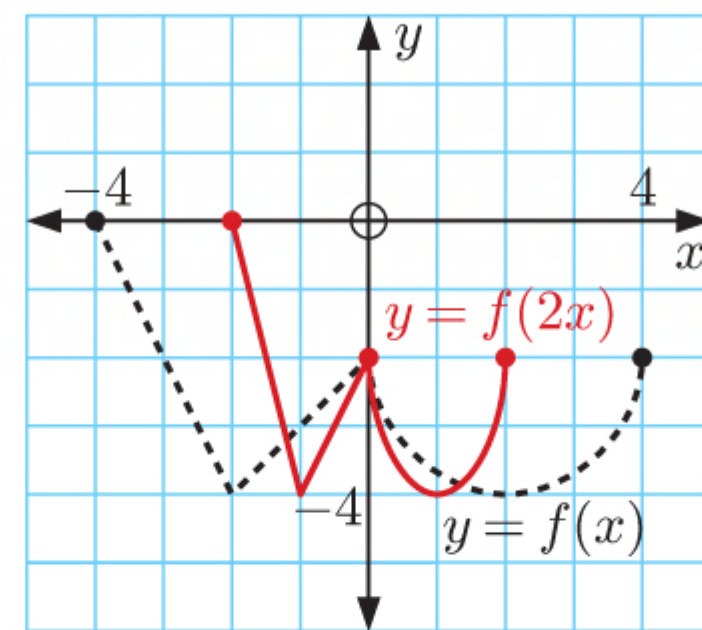
To transform $y = \frac{1}{4}x^2$ to $y = \frac{1}{4}(x-2)^2 - 1$, we translate $y = \frac{1}{4}x^2$ through $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$.

REVIEW SET 4B

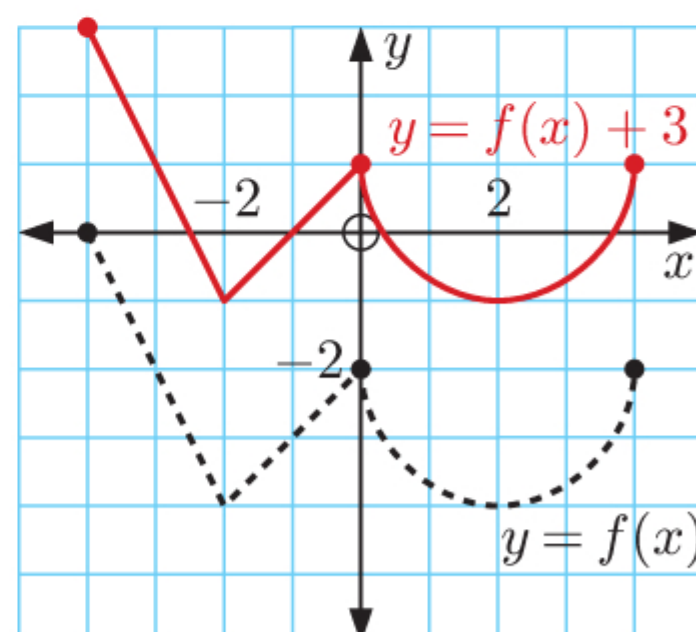
- 1 a** The graph of $y = f(x-1)$ is found by translating $y = f(x)$ 1 unit to the right.



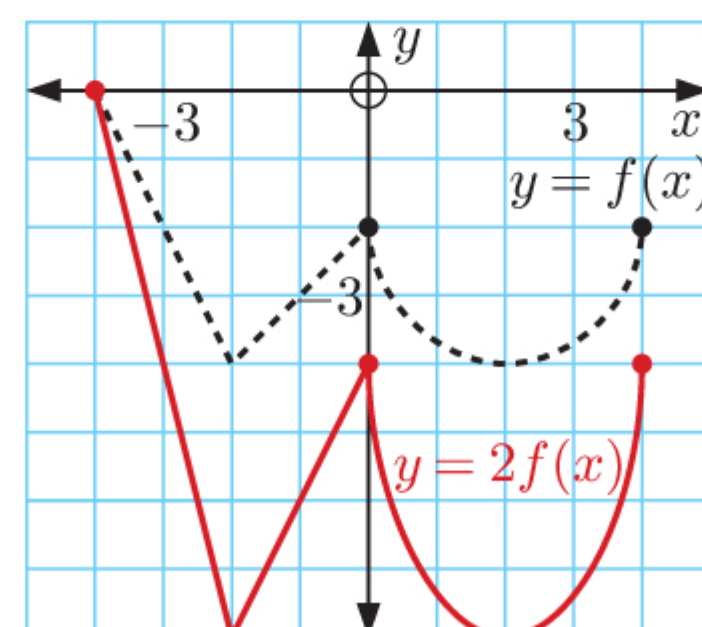
- b** The graph of $y = f(2x)$ is found by horizontally stretching $y = f(x)$ with scale factor $\frac{1}{2}$.



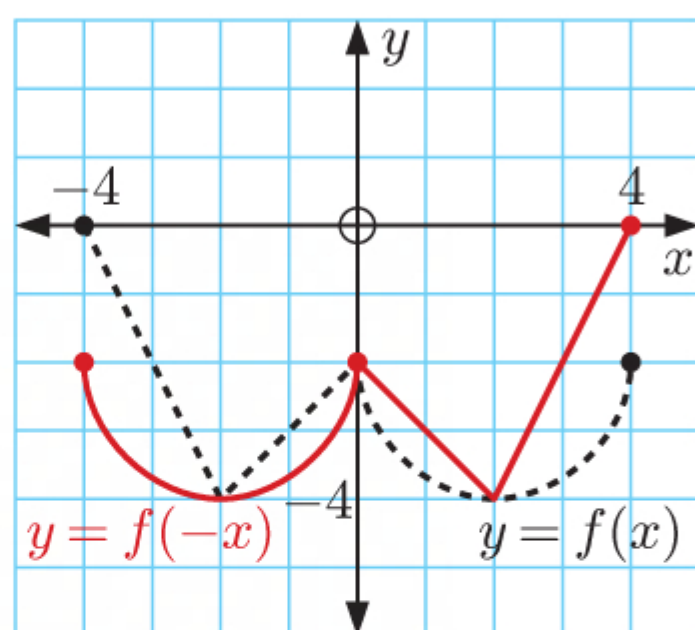
- c** The graph of $y = f(x) + 3$ is found by translating $y = f(x)$ 3 units upwards.



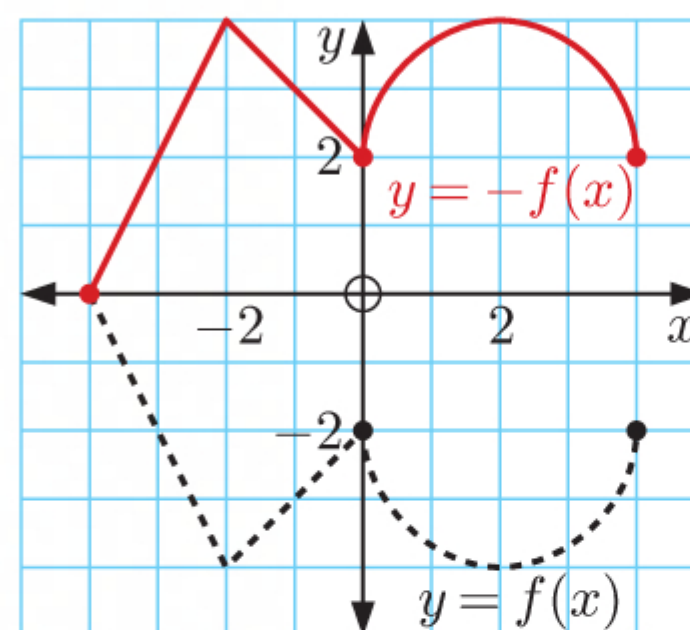
- d** The graph of $y = 2f(x)$ is found by vertically stretching $y = f(x)$ with scale factor 2.



- e** The graph of $y = f(-x)$ is found by reflecting $y = f(x)$ in the y -axis.



- f** The graph of $y = -f(x)$ is found by reflecting $y = f(x)$ in the x -axis.



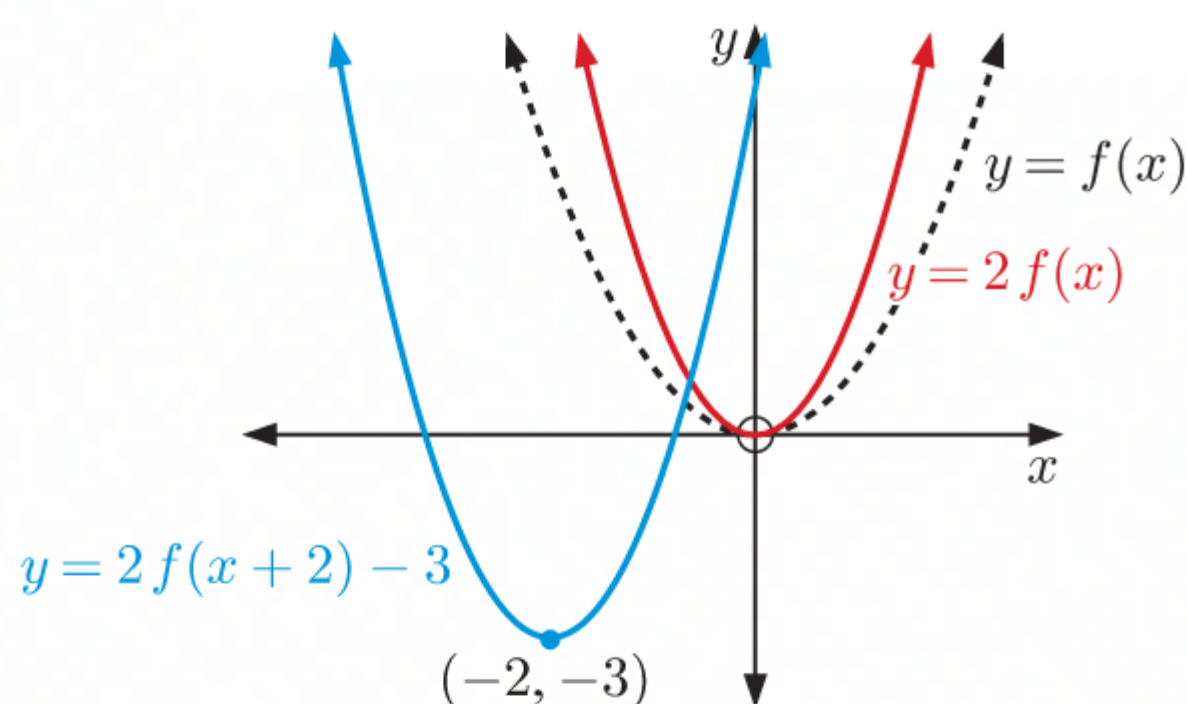
2 a $g(x) = -f(x)$
 $= -(x^2 - 3x)$
 $= 3x - x^2$

b $g(x) = f(x) + 2$
 $= 14 - x + 2$
 $= 16 - x$

c $g(x) = f\left(\frac{1}{4}x\right)$
 $= \frac{1}{3}\left(\frac{1}{4}x\right) + 2$
 $= \frac{1}{12}x + 2$

- 3** $y = 2f(x)$ is a vertical stretch of $y = f(x)$ with scale factor 2.

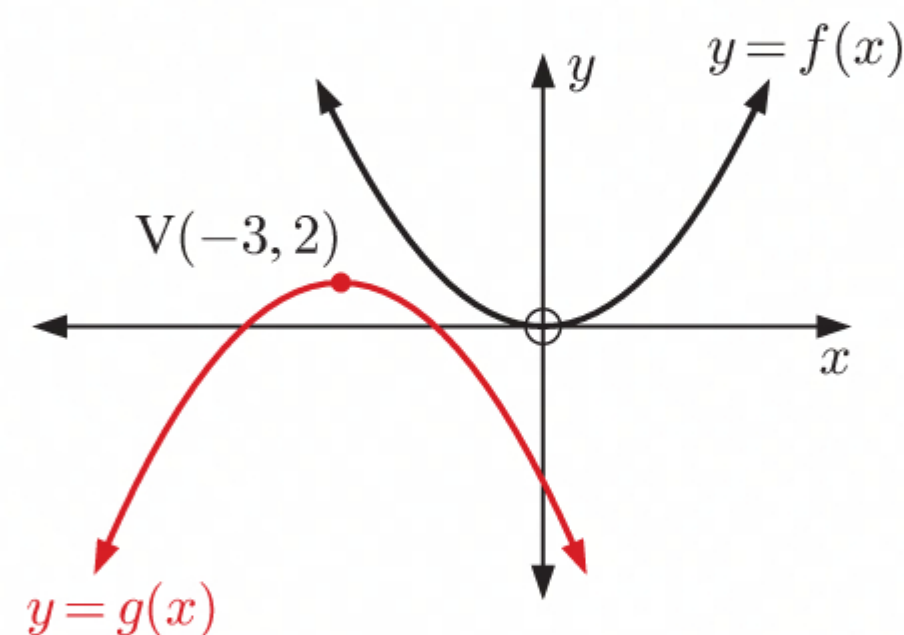
$y = 2f(x+2) - 3$ is a translation of $y = 2f(x)$ through $\begin{pmatrix} -2 \\ -3 \end{pmatrix}$.



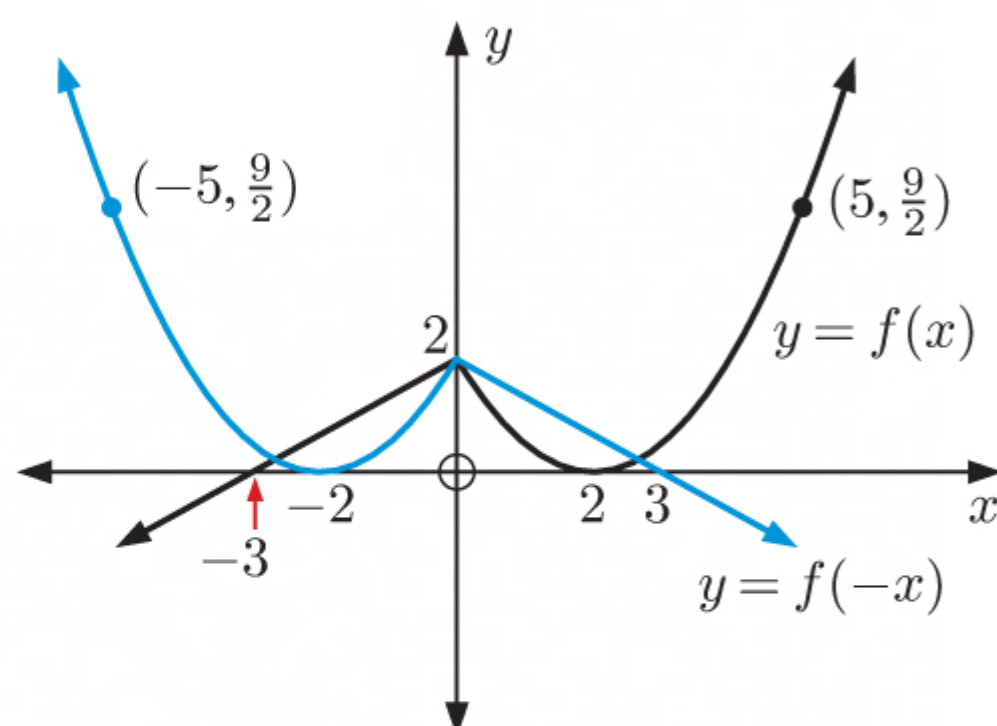
- 4** $y = x^2$ is transformed to $y = -x^2$ by reflecting $y = x^2$ in the x -axis. The vertex is $(0, 0)$.

$y = -x^2$ is transformed to $y = g(x)$ by translating $y = -x^2$ through $\begin{pmatrix} -3 \\ 2 \end{pmatrix}$.

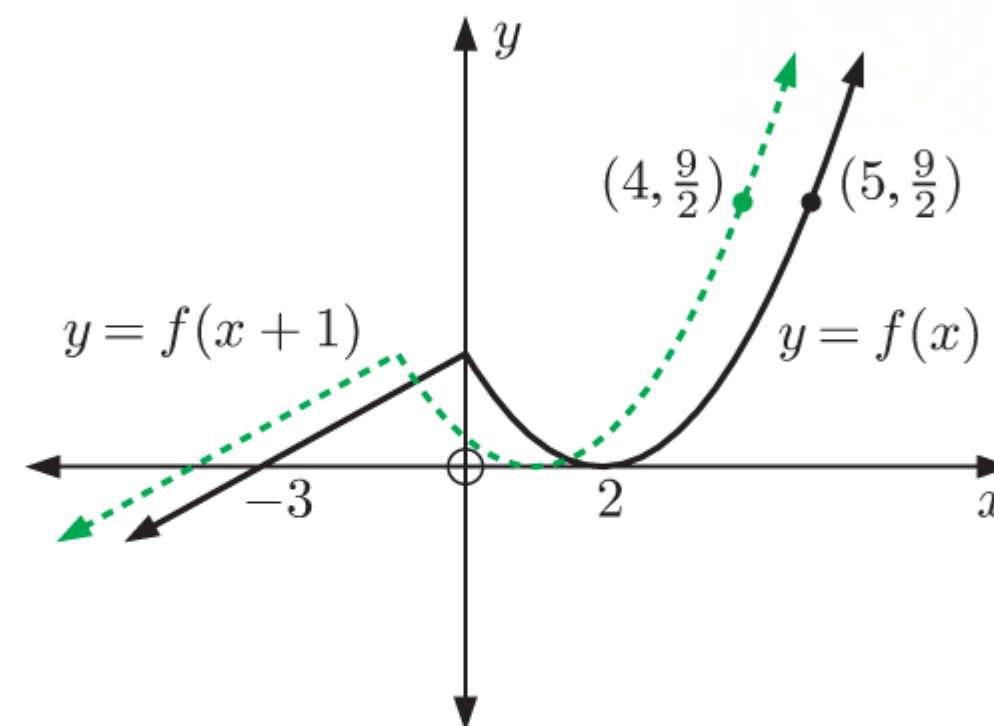
$$\begin{aligned} \therefore g(x) &= -(x+3)^2 + 2 \\ &= -(x^2 + 6x + 9) + 2 \\ &= -x^2 - 6x - 9 + 2 \\ &= -x^2 - 6x - 7 \end{aligned}$$



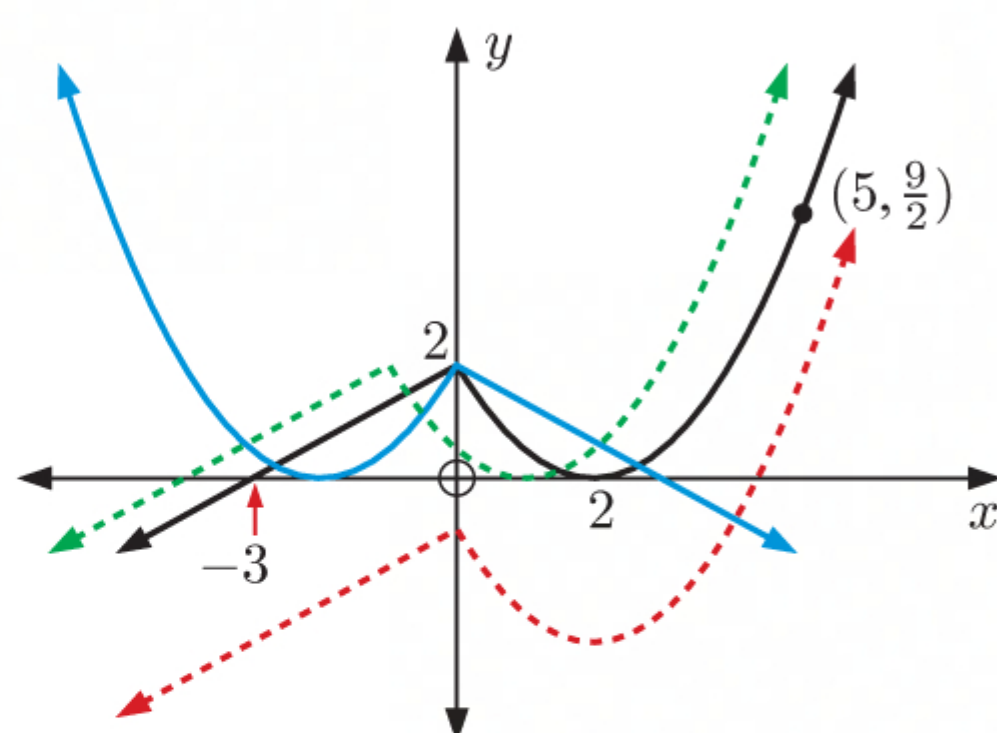
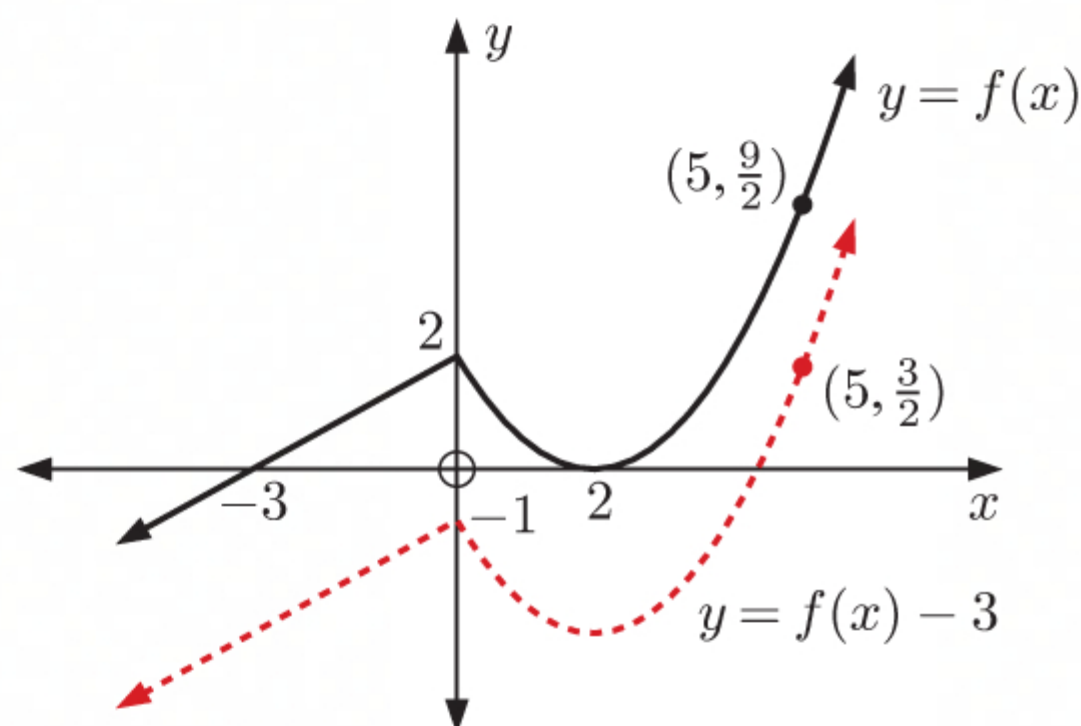
- 5 a** To transform $y = f(x)$ to $y = f(-x)$, we reflect $y = f(x)$ in the y -axis.



- b** To transform $y = f(x)$ to $y = f(x+1)$, we translate $y = f(x)$ through $\begin{pmatrix} -1 \\ 0 \end{pmatrix}$.



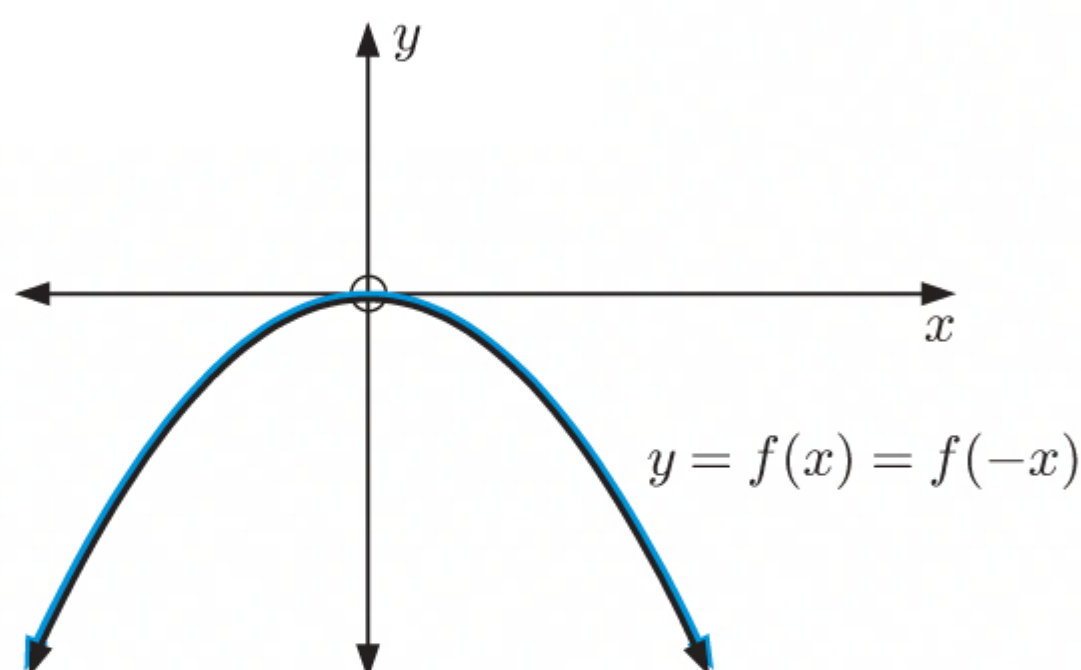
- c** To transform $y = f(x)$ to $y = f(x) - 3$, we translate $y = f(x)$ through $\begin{pmatrix} 0 \\ -3 \end{pmatrix}$.



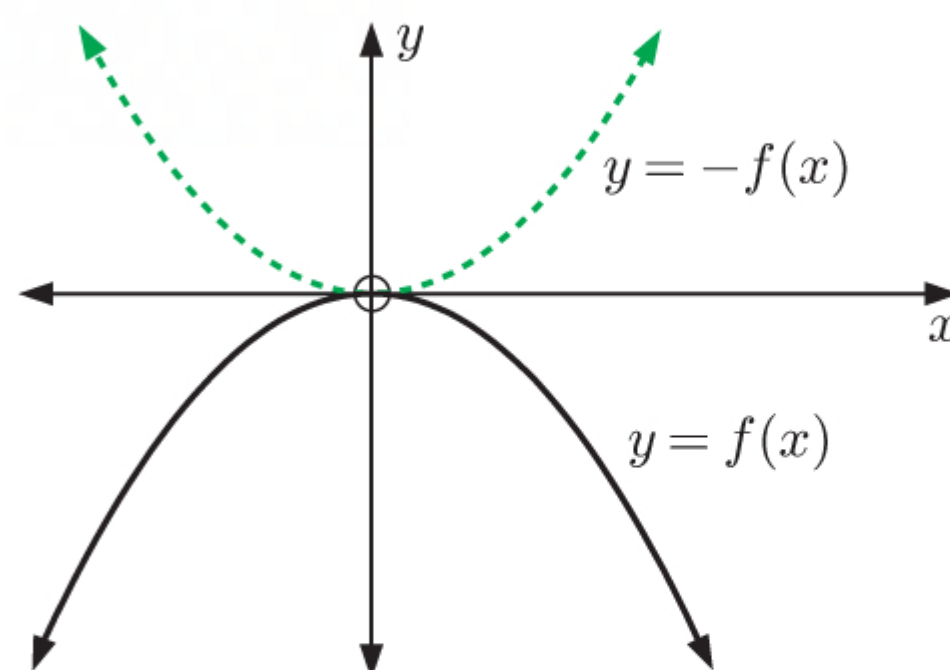
- $\longleftrightarrow y = f(x)$
 $\longleftrightarrow y = f(-x)$
 $\longleftrightarrow y = f(x+1)$
 $\longleftrightarrow y = f(x) - 3$

6 $f(x) = -x^2$

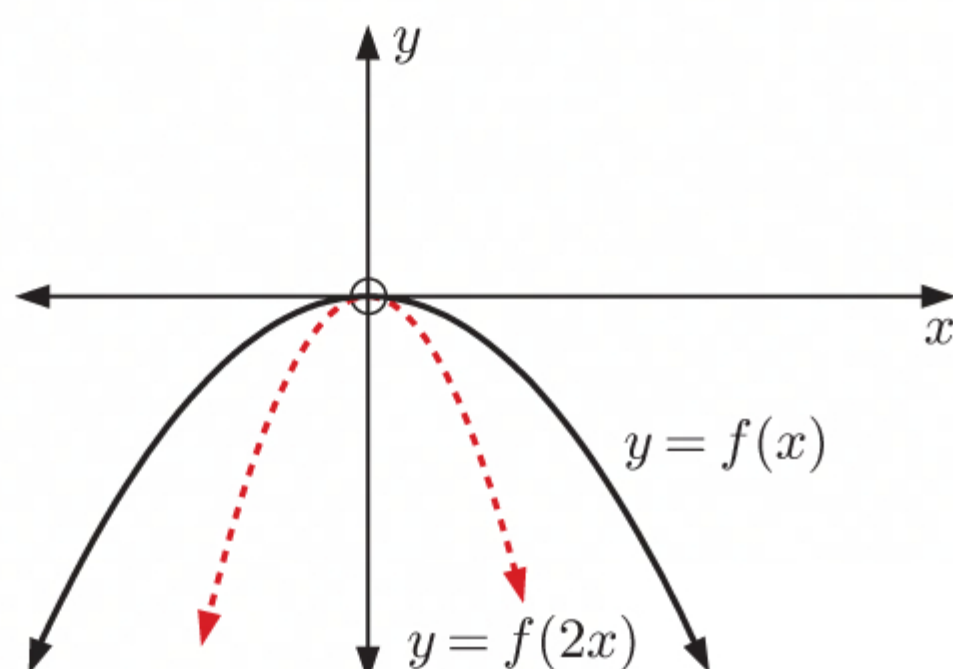
- a** To transform $y = f(x)$ to $y = f(-x)$, we reflect $y = f(x)$ in the y -axis.



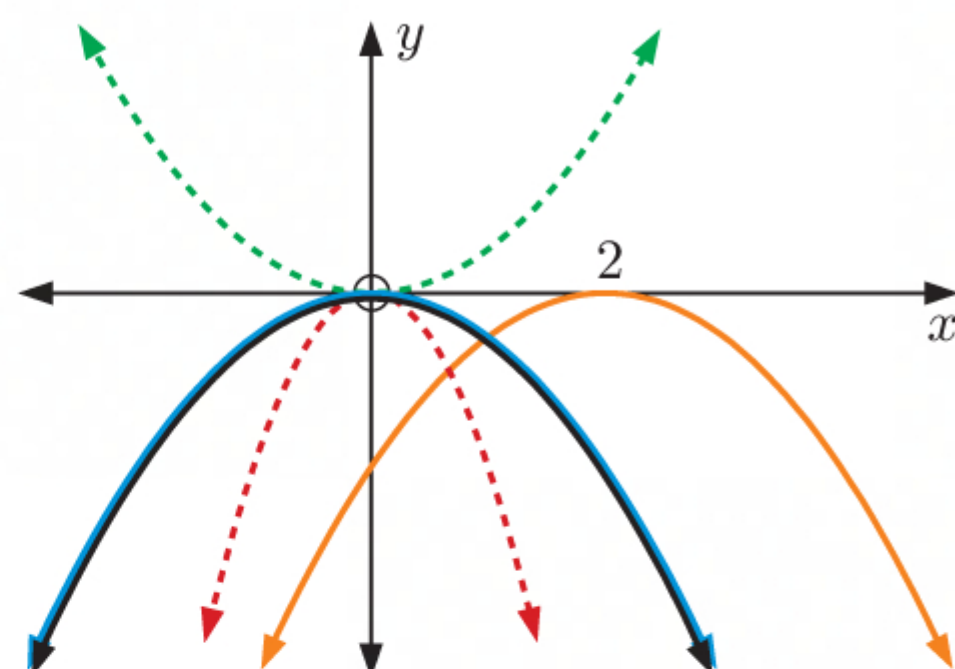
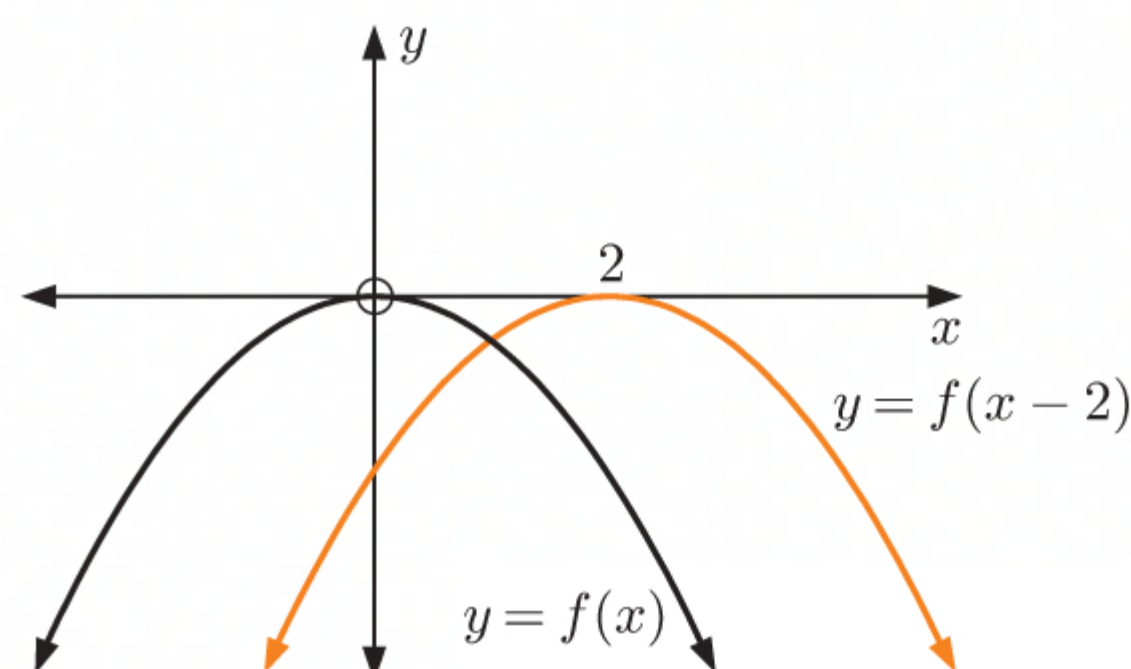
- b** To transform $y = f(x)$ to $y = -f(x)$, we reflect $y = f(x)$ in the x -axis.



- c** To transform $y = f(x)$ to $y = f(2x)$, we horizontally stretch $y = f(x)$ with scale factor $\frac{1}{2}$.

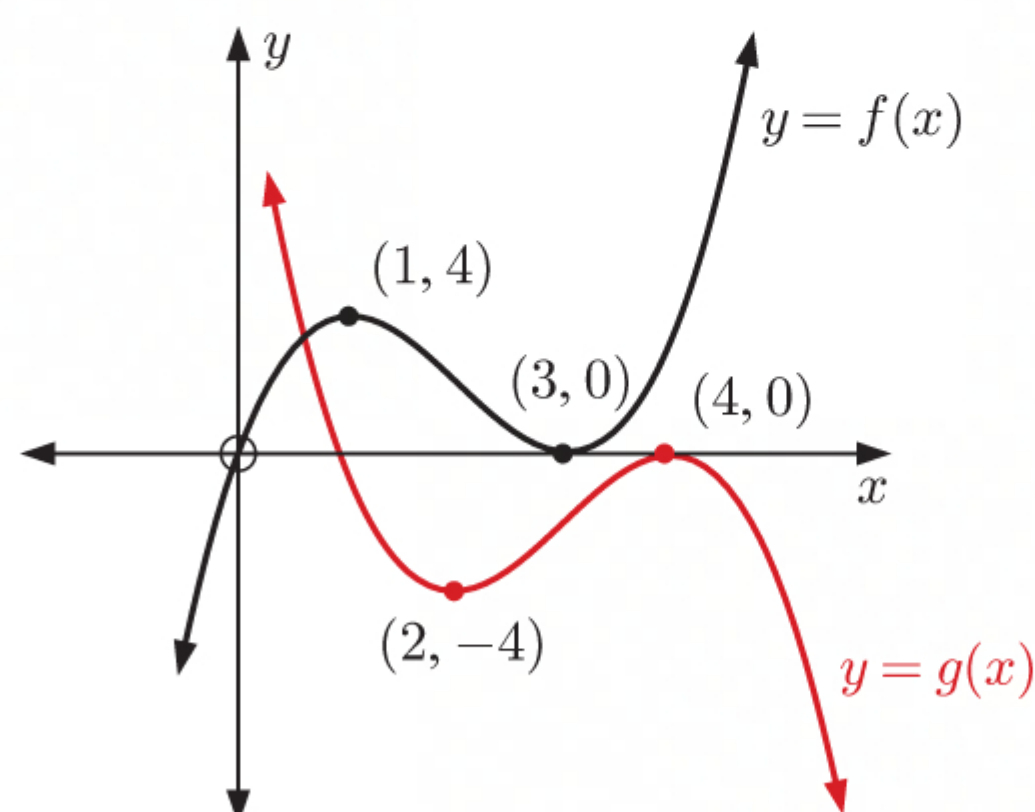


- d** To transform $y = f(x)$ to $y = f(x-2)$, we translate $y = f(x)$ through $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$.



- \longleftrightarrow $y = f(x) = -x^2$
- \longleftrightarrow $y = f(-x)$
- \longleftrightarrow $y = -f(x)$
- \longleftrightarrow $y = f(2x)$
- \longleftrightarrow $y = f(x-2)$

- 7 a** $y = f(x)$ is transformed to $y = -f(x)$ by reflecting $y = f(x)$ in the x -axis.
 $y = -f(x)$ is transformed to $g(x) = -f(x-1)$ by translating $y = -f(x)$ through $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$.



b $(1, 4)$ on $y = f(x)$ is reflected in the x -axis to $(1, -4)$, then translated through $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ to $(2, -4)$ on $y = g(x)$.

$(3, 0)$ on $y = f(x)$ is unchanged by a reflection in the x -axis, then translated through $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ to $(4, 0)$.

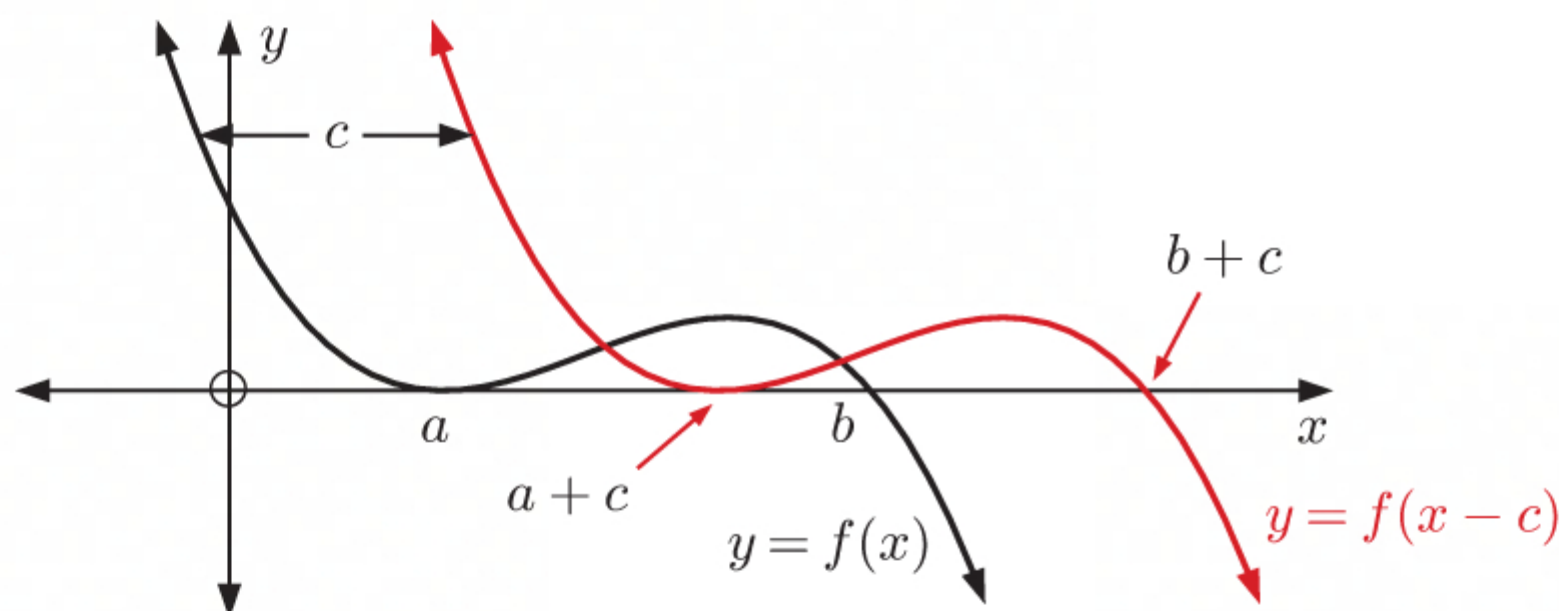
\therefore the turning points of $y = g(x)$ are $(2, -4)$ and $(4, 0)$.

8 $f(x) = -2x^2 + x + 2$ is translated through $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$.

$$\begin{aligned} f(x-1) - 2 &= -2(x-1)^2 + (x-1) + 2 - 2 \\ &= -2(x^2 - 2x + 1) + x - 1 \\ &= -2x^2 + 4x - 2 + x - 1 \\ &= -2x^2 + 5x - 3 \end{aligned}$$

\therefore the image is $y = -2x^2 + 5x - 3$.

9 $y = f(x)$ is transformed to $y = f(x-c)$ by translating $y = f(x)$ through $\begin{pmatrix} c \\ 0 \end{pmatrix}$, $0 < c < b-a$.
The x -intercepts on $y = f(x-c)$ are $a+c$ and $b+c$.



10 $f(x) \xrightarrow{\text{vertical stretch scale factor } \frac{1}{2}} \frac{1}{2}f(x) \xrightarrow{\text{translation } \begin{pmatrix} 2 \\ 3 \end{pmatrix}} \frac{1}{2}f(x-2) + 3$

$\therefore (-1, 6)$ on $y = f(x)$ is transformed to $(-1, 3)$ on $y = \frac{1}{2}f(x)$, then transformed to $(1, 6)$ on $y = \frac{1}{2}f(x-2) + 3$.

11 a $f(x) \xrightarrow{\text{vertical stretch scale factor } 2} 2f(x) \xrightarrow{\text{translation } \begin{pmatrix} -1 \\ 3 \end{pmatrix}} 2f(x+1) + 3$

A vertical stretch with scale factor 2, then a translation through $\begin{pmatrix} -1 \\ 3 \end{pmatrix}$ maps $y = f(x)$ onto $y = 2f(x+1) + 3$.

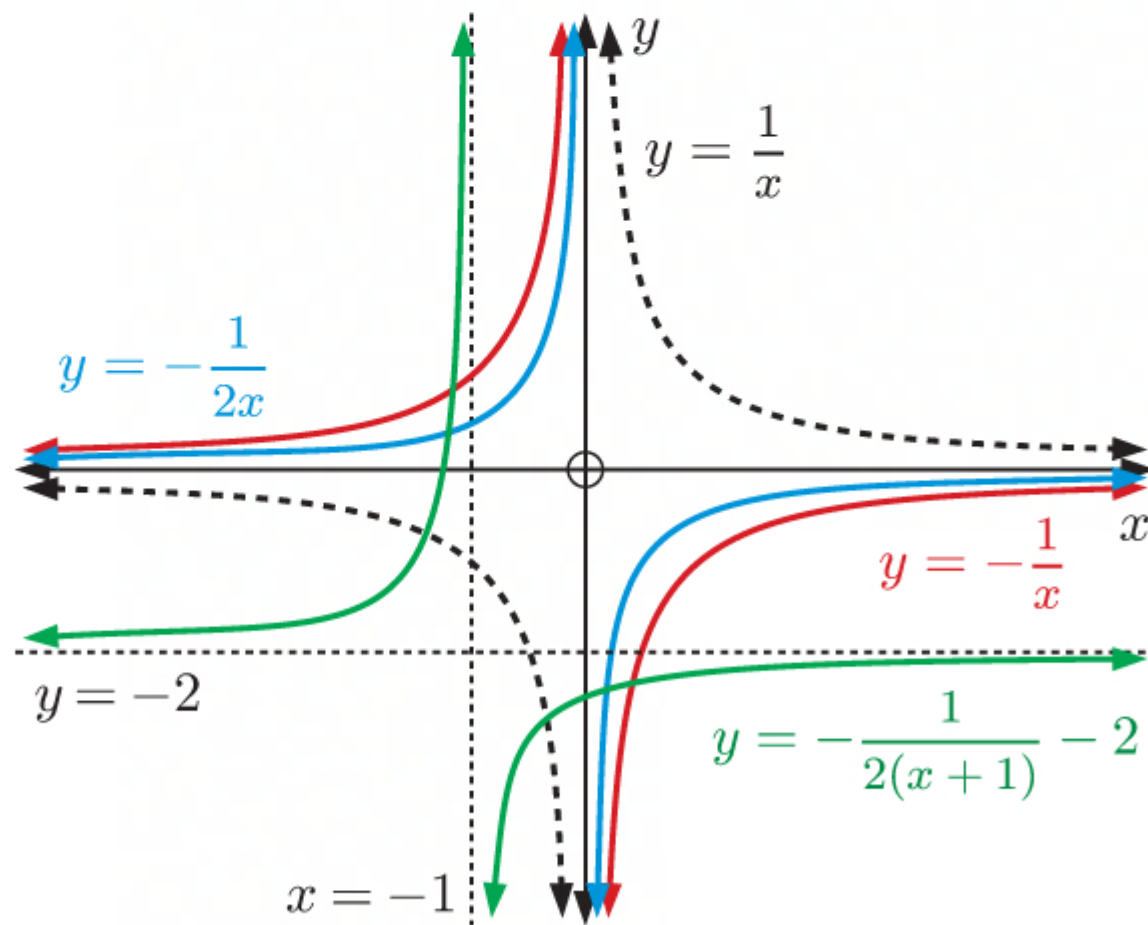
b $f(x) \xrightarrow{\text{horizontal stretch scale factor } \frac{3}{2}} f(\frac{2}{3}x) \xrightarrow{\text{reflection in } x\text{-axis}} -f(\frac{2}{3}x) \xrightarrow{\text{translation } \begin{pmatrix} 0 \\ -6 \end{pmatrix}} -f(\frac{2}{3}x) - 6$

A horizontal stretch with scale factor $\frac{3}{2}$, a reflection in the x -axis, then a translation through $\begin{pmatrix} 0 \\ -6 \end{pmatrix}$ maps $y = f(x)$ onto $y = -f(\frac{2}{3}x) - 6$.

$$\text{c } f(x) \xrightarrow{\text{reflection in } y\text{-axis}} f(-x) \xrightarrow{\text{vertical stretch scale factor } \frac{1}{3}} \frac{1}{3}f(-x) \xrightarrow{\text{translation } \begin{pmatrix} 0 \\ 2 \end{pmatrix}} \frac{1}{3}f(-x) + 2$$

A reflection in the y -axis, a vertical stretch with scale factor $\frac{1}{3}$, then a translation through $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$ maps $y = f(x)$ onto $y = \frac{1}{3}f(-x) + 2$.

12 a



$$\text{b } \frac{1}{x} \xrightarrow{\text{reflection in } x\text{-axis}} -\frac{1}{x} \xrightarrow{\text{vertical stretch scale factor } \frac{1}{2}} -\frac{1}{2x} \xrightarrow{\text{translation } \begin{pmatrix} -1 \\ -2 \end{pmatrix}} -\frac{1}{2(x+1)} - 2$$

A reflection in the x -axis, a vertical stretch with scale factor $\frac{1}{2}$, then a translation through $\begin{pmatrix} -1 \\ -2 \end{pmatrix}$ transforms $y = \frac{1}{x}$ into $y = -\frac{1}{2(x+1)} - 2$.

$$\begin{aligned} \text{c } y &= -\frac{1}{2(x+1)} - 2 \\ \therefore y &= -\frac{1}{2x+2} - \frac{2(2x+2)}{2x+2} \\ \therefore y &= -\frac{1}{2x+2} - \frac{4x+4}{2x+2} \\ \therefore y &= \frac{-1-4x-4}{2x+2} \\ \therefore y &= \frac{-4x-5}{2x+2} \end{aligned}$$

y is undefined when $x = -1$, and as $|x| \rightarrow \infty$, $y \rightarrow -2$.

\therefore the domain is $\{x \mid x \neq -1\}$, and the range is $\{y \mid y \neq -2\}$.

Chapter 5

EXPONENTIAL FUNCTIONS

EXERCISE 5A

1 a $\sqrt[5]{2} = 2^{\frac{1}{5}}$

b $\frac{1}{\sqrt[5]{2}} = \frac{1}{2^{\frac{1}{5}}}$
 $= 2^{-\frac{1}{5}}$

c $2\sqrt{2} = 2^1 \times 2^{\frac{1}{2}}$
 $= 2^{1+\frac{1}{2}}$
 $= 2^{\frac{3}{2}}$

d $4\sqrt{2} = 2^2 \times 2^{\frac{1}{2}}$
 $= 2^{2+\frac{1}{2}}$
 $= 2^{\frac{5}{2}}$

e $\frac{1}{\sqrt[3]{2}} = \frac{1}{2^{\frac{1}{3}}}$
 $= 2^{-\frac{1}{3}}$

f $2 \times \sqrt[3]{2} = 2^1 \times 2^{\frac{1}{3}}$
 $= 2^{1+\frac{1}{3}}$
 $= 2^{\frac{4}{3}}$

g $\frac{4}{\sqrt{2}} = \frac{2^2}{2^{\frac{1}{2}}}$
 $= 2^{2-\frac{1}{2}}$
 $= 2^{\frac{3}{2}}$

h $(\sqrt{2})^3 = (2^{\frac{1}{2}})^3$
 $= 2^{3 \times \frac{1}{2}}$
 $= 2^{\frac{3}{2}}$

i $\frac{1}{\sqrt[3]{16}} = \frac{1}{\sqrt[3]{2^4}}$
 $= \frac{1}{2^{\frac{4}{3}}}$
 $= 2^{-\frac{4}{3}}$

j $\frac{1}{\sqrt{8}} = \frac{1}{\sqrt{2^3}}$
 $= \frac{1}{2^{\frac{3}{2}}}$
 $= 2^{-\frac{3}{2}}$

2 a $\sqrt[3]{3} = 3^{\frac{1}{3}}$

b $\frac{1}{\sqrt[3]{3}} = \frac{1}{3^{\frac{1}{3}}}$
 $= 3^{-\frac{1}{3}}$

c $\sqrt[4]{3} = 3^{\frac{1}{4}}$

d $3\sqrt{3} = 3^1 \times 3^{\frac{1}{2}}$
 $= 3^{\frac{3}{2}}$

e $\frac{1}{9\sqrt{3}} = \frac{1}{3^2 \times 3^{\frac{1}{2}}}$
 $= \frac{1}{3^{\frac{5}{2}}}$
 $= 3^{-\frac{5}{2}}$

3 a $\sqrt[3]{7} = 7^{\frac{1}{3}}$

b $\sqrt[4]{27} = \sqrt[4]{3^3}$
 $= (3^3)^{\frac{1}{4}}$
 $= 3^{\frac{3}{4}}$

c $\sqrt[5]{16} = \sqrt[5]{2^4}$
 $= (2^4)^{\frac{1}{5}}$
 $= 2^{\frac{4}{5}}$

d $\sqrt[3]{32} = \sqrt[3]{2^5}$
 $= (2^5)^{\frac{1}{3}}$
 $= 2^{\frac{5}{3}}$

e $\sqrt[7]{49} = \sqrt[7]{7^2}$
 $= (7^2)^{\frac{1}{7}}$
 $= 7^{\frac{2}{7}}$

f $\frac{1}{\sqrt[3]{7}} = \frac{1}{7^{\frac{1}{3}}}$
 $= 7^{-\frac{1}{3}}$

g $\frac{1}{\sqrt[4]{27}} = \frac{1}{3^{\frac{3}{4}}}$
 $= 3^{-\frac{3}{4}}$

h $\frac{1}{\sqrt[5]{16}} = \frac{1}{2^{\frac{4}{5}}}$
 $= 2^{-\frac{4}{5}}$

$$\begin{aligned} \text{i} \quad \frac{1}{\sqrt[3]{32}} &= \frac{1}{2^{\frac{5}{3}}} \\ &= 2^{-\frac{5}{3}} \end{aligned}$$

$$\begin{aligned} \text{j} \quad \frac{1}{\sqrt[7]{49}} &= \frac{1}{7^{\frac{2}{7}}} \\ &= 7^{-\frac{2}{7}} \end{aligned}$$

$$4 \quad \text{a} \quad \sqrt{x} = x^{\frac{1}{2}}$$

$$\begin{aligned} \text{b} \quad x\sqrt{x} &= x^1 \times x^{\frac{1}{2}} \\ &= x^{\frac{3}{2}} \end{aligned}$$

$$\begin{aligned} \text{c} \quad \frac{1}{\sqrt{x}} &= \frac{1}{x^{\frac{1}{2}}} \\ &= x^{-\frac{1}{2}} \end{aligned}$$

$$\begin{aligned} \text{d} \quad x^2\sqrt{x} &= x^2 \times x^{\frac{1}{2}} \\ &= x^{\frac{5}{2}} \end{aligned}$$

$$\begin{aligned} \text{e} \quad \frac{1}{x\sqrt{x}} &= \frac{1}{x^{\frac{3}{2}}} \\ &= x^{-\frac{3}{2}} \end{aligned}$$

$$5 \quad \text{a} \quad 3^{\frac{3}{4}} \approx 2.28$$

$$\text{b} \quad 4^{-\frac{3}{5}} \approx 0.435$$

$$\text{c} \quad \sqrt[4]{8} \approx 1.68$$

$$\text{d} \quad \sqrt[5]{27} \approx 1.93$$

$$\text{e} \quad \frac{1}{\sqrt[3]{7}} \approx 0.523$$

$$6 \quad \text{a} \quad 5^{\frac{1}{3}} = \sqrt[3]{5}$$

$$\begin{aligned} \text{b} \quad 3^{-\frac{1}{2}} &= \frac{1}{3^{\frac{1}{2}}} \\ &= \frac{1}{\sqrt{3}} \end{aligned}$$

$$\begin{aligned} \text{c} \quad 3^{\frac{5}{2}} &= 3^2 \times 3^{\frac{1}{2}} \\ &= 9\sqrt{3} \end{aligned}$$

$$\begin{aligned} \text{d} \quad m^{\frac{3}{2}} &= m \times m^{\frac{1}{2}} \\ &= m\sqrt{m} \end{aligned}$$

$$\begin{aligned} \text{e} \quad x^{\frac{7}{2}} &= x^3 \times x^{\frac{1}{2}} \\ &= x^3\sqrt{x} \end{aligned}$$

$$\begin{aligned} 7 \quad \text{a} \quad 4^{\frac{3}{2}} &= (2^2)^{\frac{3}{2}} \\ &= 2^3 \\ &= 8 \end{aligned}$$

$$\begin{aligned} \text{b} \quad 8^{\frac{5}{3}} &= (2^3)^{\frac{5}{3}} \\ &= 2^5 \\ &= 32 \end{aligned}$$

$$\begin{aligned} \text{c} \quad 16^{\frac{3}{4}} &= (2^4)^{\frac{3}{4}} \\ &= 2^3 \\ &= 8 \end{aligned}$$

$$\begin{aligned} \text{d} \quad 25^{\frac{3}{2}} &= (5^2)^{\frac{3}{2}} \\ &= 5^3 \\ &= 125 \end{aligned}$$

$$\begin{aligned} \text{e} \quad 32^{\frac{2}{5}} &= (2^5)^{\frac{2}{5}} \\ &= 2^2 \\ &= 4 \end{aligned}$$

$$\begin{aligned} \text{f} \quad 4^{-\frac{1}{2}} &= (2^2)^{-\frac{1}{2}} \\ &= 2^{-1} \\ &= \frac{1}{2} \end{aligned}$$

$$\begin{aligned} \text{g} \quad 9^{-\frac{3}{2}} &= (3^2)^{-\frac{3}{2}} \\ &= 3^{-3} \\ &= \frac{1}{27} \end{aligned}$$

$$\begin{aligned} \text{h} \quad 8^{-\frac{4}{3}} &= (2^3)^{-\frac{4}{3}} \\ &= 2^{-4} \\ &= \frac{1}{16} \end{aligned}$$

$$\begin{aligned} \text{i} \quad 27^{-\frac{4}{3}} &= (3^3)^{-\frac{4}{3}} \\ &= 3^{-4} \\ &= \frac{1}{81} \end{aligned}$$

$$\begin{aligned} \text{j} \quad 125^{-\frac{2}{3}} &= (5^3)^{-\frac{2}{3}} \\ &= 5^{-2} \\ &= \frac{1}{25} \end{aligned}$$

EXERCISE 5B

$$\begin{aligned}
 1 \quad a \quad & x^{\frac{1}{2}} \times x^{-\frac{1}{2}} \\
 &= x^0 \\
 &= 1
 \end{aligned}$$

$$\begin{aligned}
 b \quad & x^{\frac{3}{2}} \times x^{-\frac{1}{2}} \\
 &= x^1 \\
 &= x
 \end{aligned}$$

$$\begin{aligned}
 c \quad & x^2 \times x^{-\frac{3}{2}} \\
 &= x^{\frac{1}{2}} \quad \text{or} \quad \sqrt{x}
 \end{aligned}$$

$$\begin{aligned}
 2 \quad a \quad & x^2(x^3 + 2x^2 + 1) \\
 &= x^2 \times x^3 + x^2 \times 2x^2 + x^2 \times 1 \\
 &= x^5 + 2x^4 + x^2
 \end{aligned}$$

$$\begin{aligned}
 b \quad & 2^x(2^x + 1) \\
 &= 2^x \times 2^x + 2^x \times 1 \\
 &= 2^{2x} + 2^x
 \end{aligned}$$

$$\begin{aligned}
 c \quad & x^{\frac{1}{2}}(x^{\frac{1}{2}} + x^{-\frac{1}{2}}) \\
 &= x^{\frac{1}{2}} \times x^{\frac{1}{2}} + x^{\frac{1}{2}} \times x^{-\frac{1}{2}} \\
 &= x^1 + x^0 \\
 &= x + 1
 \end{aligned}$$

$$\begin{aligned}
 d \quad & 7^x(7^x + 2) \\
 &= 7^x \times 7^x + 7^x \times 2 \\
 &= 7^{2x} + 2(7^x)
 \end{aligned}$$

$$\begin{aligned}
 e \quad & 3^x(2 - 3^{-x}) \\
 &= 3^x \times 2 - 3^x \times 3^{-x} \\
 &= 2(3^x) - 3^0 \\
 &= 2(3^x) - 1
 \end{aligned}$$

$$\begin{aligned}
 f \quad & x^{\frac{1}{2}}(x^{\frac{3}{2}} + 2x^{\frac{1}{2}} + 3x^{-\frac{1}{2}}) \\
 &= x^{\frac{1}{2}} \times x^{\frac{3}{2}} + x^{\frac{1}{2}} \times 2x^{\frac{1}{2}} + x^{\frac{1}{2}} \times 3x^{-\frac{1}{2}} \\
 &= x^2 + 2x^1 + 3x^0 \\
 &= x^2 + 2x + 3
 \end{aligned}$$

$$\begin{aligned}
 g \quad & 2^{-x}(2^x + 5) \\
 &= 2^{-x} \times 2^x + 2^{-x} \times 5 \\
 &= 2^0 + 5(2^{-x}) \\
 &= 1 + 5(2^{-x})
 \end{aligned}$$

$$\begin{aligned}
 h \quad & 5^{-x}(5^{2x} + 5^x) \\
 &= 5^{-x} \times 5^{2x} + 5^{-x} \times 5^x \\
 &= 5^x + 5^0 \\
 &= 5^x + 1
 \end{aligned}$$

$$\begin{aligned}
 i \quad & x^{-\frac{1}{2}}(x^2 + x + x^{\frac{1}{2}}) \\
 &= x^{-\frac{1}{2}} \times x^2 + x^{-\frac{1}{2}} \times x^1 + x^{-\frac{1}{2}} \times x^{\frac{1}{2}} \\
 &= x^{\frac{3}{2}} + x^{\frac{1}{2}} + x^0 \\
 &= x^{\frac{3}{2}} + x^{\frac{1}{2}} + 1
 \end{aligned}$$

$$\begin{aligned}
 j \quad & 3^x(3^x + 5 + 3^{-x}) \\
 &= 3^x \times 3^x + 3^x \times 5 + 3^x \times 3^{-x} \\
 &= 3^{2x} + 5(3^x) + 3^0 \\
 &= 3^{2x} + 5(3^x) + 1
 \end{aligned}$$

$$\begin{aligned}
 k \quad & x^{-\frac{1}{2}}(2x^2 - x + 5x^{\frac{1}{2}}) \\
 &= x^{-\frac{1}{2}} \times 2x^2 - x^{-\frac{1}{2}} \times x^1 + x^{-\frac{1}{2}} \times 5x^{\frac{1}{2}} \\
 &= 2x^{\frac{3}{2}} - x^{\frac{1}{2}} + 5x^0 \\
 &= 2x^{\frac{3}{2}} - x^{\frac{1}{2}} + 5
 \end{aligned}$$

$$\begin{aligned}
 l \quad & 2^{2x}(2^x - 3 - 2^{-2x}) \\
 &= 2^{2x} \times 2^x - 2^{2x} \times 3 - 2^{2x} \times 2^{-2x} \\
 &= 2^{3x} - 3(2^{2x}) - 2^0 \\
 &= 2^{3x} - 3(2^{2x}) - 1
 \end{aligned}$$

$$\begin{aligned}
 3 \quad a \quad & (2^x - 1)(2^x + 3) \\
 &= 2^x \times 2^x + 2^x \times 3 - 1 \times 2^x - 3 \\
 &= 2^{2x} + 2(2^x) - 3 \\
 &= 2^{2x} + 2^{x+1} - 3
 \end{aligned}$$

$$\begin{aligned}
 b \quad & (3^x + 2)(3^x + 5) \\
 &= 3^x \times 3^x + 3^x \times 5 + 2 \times 3^x + 10 \\
 &= 3^{2x} + 7(3^x) + 10
 \end{aligned}$$

$$\begin{aligned}
 \text{c} \quad & (5^x - 2)(5^x - 4) \\
 &= 5^x \times 5^x - 5^x \times 4 - 2 \times 5^x + 8 \\
 &= 5^{2x} - 6(5^x) + 8
 \end{aligned}$$

$$\begin{aligned}
 \text{e} \quad & (3^x - 1)^2 \\
 &= (3^x)^2 - 2 \times 3^x \times 1 + 1^2 \\
 &= 3^{2x} - 2(3^x) + 1
 \end{aligned}$$

$$\begin{aligned}
 \text{g} \quad & (x^{\frac{1}{2}} + 2)(x^{\frac{1}{2}} - 2) \\
 &= (x^{\frac{1}{2}})^2 - 2^2 \\
 &= x - 4
 \end{aligned}$$

$$\begin{aligned}
 \text{i} \quad & (x^{\frac{1}{2}} + x^{-\frac{1}{2}})(x^{\frac{1}{2}} - x^{-\frac{1}{2}}) \\
 &= (x^{\frac{1}{2}})^2 - (x^{-\frac{1}{2}})^2 \\
 &= x^1 - x^{-1} \\
 &= x - \frac{1}{x}
 \end{aligned}$$

$$\begin{aligned}
 \text{k} \quad & (7^x - 7^{-x})^2 \\
 &= (7^x)^2 - 2 \times 7^x \times 7^{-x} + (7^{-x})^2 \\
 &= 7^{2x} - 2 \times 7^0 + 7^{-2x} \\
 &= 7^{2x} - 2 + 7^{-2x}
 \end{aligned}$$

$$\begin{aligned}
 \text{d} \quad & (2^x + 3)^2 \\
 &= (2^x)^2 + 2 \times 2^x \times 3 + 3^2 \\
 &= 2^{2x} + 6(2^x) + 9
 \end{aligned}$$

$$\begin{aligned}
 \text{f} \quad & (4^x + 7)^2 \\
 &= (4^x)^2 + 2 \times 4^x \times 7 + 7^2 \\
 &= 4^{2x} + 14(4^x) + 49
 \end{aligned}$$

$$\begin{aligned}
 \text{h} \quad & (2^x + 3)(2^x - 3) \\
 &= (2^x)^2 - 3^2 \\
 &= 2^{2x} - 9 \\
 &= 4^x - 9
 \end{aligned}$$

$$\begin{aligned}
 \text{j} \quad & \left(x + \frac{2}{x}\right)^2 \\
 &= x^2 + 2 \times x \times \left(\frac{2}{x}\right) + \left(\frac{2}{x}\right)^2 \\
 &= x^2 + 4 + \frac{4}{x^2}
 \end{aligned}$$

$$\begin{aligned}
 \text{l} \quad & (5 - 2^{-x})^2 \\
 &= 5^2 - 2 \times 5 \times 2^{-x} + (2^{-x})^2 \\
 &= 25 - 10(2^{-x}) + 2^{-2x}
 \end{aligned}$$

$$\begin{aligned}
 \text{4 a} \quad & 5^{2x} + 5^x \\
 &= 5^x \times 5^x + 5^x \\
 &= 5^x(5^x + 1)
 \end{aligned}$$

$$\begin{aligned}
 \text{b} \quad & 3^{n+2} + 3^n \\
 &= 3^n \times 3^2 + 3^n \\
 &= 3^n(3^2 + 1) \\
 &= 10(3^n)
 \end{aligned}$$

$$\begin{aligned}
 \text{c} \quad & 7^n + 7^{3n} \\
 &= 7^n + 7^n \times 7^{2n} \\
 &= 7^n(1 + 7^{2n})
 \end{aligned}$$

$$\begin{aligned}
 \text{d} \quad & 5^{n+1} - 5 \\
 &= 5 \times 5^n - 5 \\
 &= 5(5^n - 1)
 \end{aligned}$$

$$\begin{aligned}
 \text{e} \quad & 6^{n+2} - 6 \\
 &= 6 \times 6^{n+1} - 6 \\
 &= 6(6^{n+1} - 1)
 \end{aligned}$$

$$\begin{aligned}
 \text{f} \quad & 4^{n+2} - 16 \\
 &= 4^2 \times 4^n - 16 \\
 &= 16 \times 4^n - 16 \\
 &= 16(4^n - 1)
 \end{aligned}$$

$$\begin{aligned}
 \text{g} \quad & 2^{2n} - 2^{n+3} \\
 &= 2^n \times 2^n - 2^n \times 2^3 \\
 &= 2^n \times 2^n - 2^n \times 8 \\
 &= 2^n(2^n - 8)
 \end{aligned}$$

$$\begin{aligned}
 \text{h} \quad & 2^{n+1} + 2^{n-1} \\
 &= 2^{n-1} \times 2^2 + 2^{n-1} \\
 &= 2^{n-1} \times 4 + 2^{n-1} \\
 &= 5(2^{n-1})
 \end{aligned}$$

$$\begin{aligned}
 \text{i} \quad & 4^{n+1} + 2^{2n-1} \\
 &= (2^2)^{n+1} + 2^{2n-1} \\
 &= 2^{2n+2} + 2^{2n-1} \\
 &= 2^{2n-1} \times 2^3 + 2^{2n-1} \\
 &= 2^{2n-1} \times 8 + 2^{2n-1} \\
 &= 9(2^{2n-1})
 \end{aligned}$$

$$\begin{aligned}
 \text{5 a} \quad & 9^x - 4 \\
 &= (3^x)^2 - 2^2 \\
 &= (3^x + 2)(3^x - 2)
 \end{aligned}$$

$$\begin{aligned}
 \text{b} \quad & 4^x - 25 \\
 &= (2^x)^2 - 5^2 \\
 &= (2^x + 5)(2^x - 5)
 \end{aligned}$$

$$\begin{aligned}
 \text{c} \quad & 16 - 9^x \\
 &= 4^2 - (3^x)^2 \\
 &= (4 + 3^x)(4 - 3^x)
 \end{aligned}$$

$$\begin{aligned} \mathbf{d} \quad & 25 - 4^x \\ &= 5^2 - (2^x)^2 \\ &= (5 + 2^x)(5 - 2^x) \end{aligned}$$

$$\begin{aligned} \mathbf{e} \quad & 9^x - 4^x \\ &= (3^x)^2 - (2^x)^2 \\ &= (3^x + 2^x)(3^x - 2^x) \end{aligned}$$

$$\begin{aligned} \mathbf{f} \quad & 4^x + 6(2^x) + 9 \\ &= (2^x)^2 + 6(2^x) + 9 \\ &= (2^x + 3)^2 \\ &\{a^2 + 6a + 9 = (a + 3)^2\} \end{aligned}$$

$$\begin{aligned} \mathbf{g} \quad & 9^x + 10(3^x) + 25 \\ &= (3^x)^2 + 10(3^x) + 25 \\ &= (3^x + 5)^2 \\ &\{a^2 + 10a + 25 = (a + 5)^2\} \end{aligned}$$

$$\begin{aligned} \mathbf{h} \quad & 4^x - 14(2^x) + 49 \\ &= (2^x)^2 - 14(2^x) + 49 \\ &= (2^x - 7)^2 \\ &\{a^2 - 14a + 49 = (a - 7)^2\} \end{aligned}$$

$$\begin{aligned} \mathbf{i} \quad & 25^x - 4(5^x) + 4 \\ &= (5^x)^2 - 4(5^x) + 4 \\ &= (5^x - 2)^2 \\ &\{a^2 - 4a + 4 = (a - 2)^2\} \end{aligned}$$

$$\begin{aligned} \mathbf{6} \quad \mathbf{a} \quad & (2^x)^2 - 2^x - 2 \\ &= (2^x + 1)(2^x - 2) \\ &\{a^2 - a - 2 = (a + 1)(a - 2)\} \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad & (3^x)^2 + 3^x - 6 \\ &= (3^x + 3)(3^x - 2) \\ &\{a^2 + a - 6 = (a + 3)(a - 2)\} \end{aligned}$$

$$\begin{aligned} \mathbf{c} \quad & 4^x - 7(2^x) + 12 \\ &= (2^x)^2 - 7(2^x) + 12 \\ &= (2^x - 3)(2^x - 4) \\ &\{a^2 - 7a + 12 = (a - 3)(a - 4)\} \end{aligned}$$

$$\begin{aligned} \mathbf{d} \quad & 4^x + 9(2^x) + 18 \\ &= (2^x)^2 + 9(2^x) + 18 \\ &= (2^x + 3)(2^x + 6) \\ &\{a^2 + 9a + 18 = (a + 3)(a + 6)\} \end{aligned}$$

$$\begin{aligned} \mathbf{e} \quad & 4^x - 2^x - 20 \\ &= (2^x)^2 - 2^x - 20 \\ &= (2^x + 4)(2^x - 5) \\ &\{a^2 - a - 20 = (a + 4)(a - 5)\} \end{aligned}$$

$$\begin{aligned} \mathbf{f} \quad & 9^x + 9(3^x) + 14 \\ &= (3^x)^2 + 9(3^x) + 14 \\ &= (3^x + 2)(3^x + 7) \\ &\{a^2 + 9a + 14 = (a + 2)(a + 7)\} \end{aligned}$$

$$\begin{aligned} \mathbf{g} \quad & 9^x + 4(3^x) - 5 \\ &= (3^x)^2 + 4(3^x) - 5 \\ &= (3^x + 5)(3^x - 1) \\ &\{a^2 + 4a - 5 = (a + 5)(a - 1)\} \end{aligned}$$

$$\begin{aligned} \mathbf{h} \quad & 25^x + 5^x - 2 \\ &= (5^x)^2 + 5^x - 2 \\ &= (5^x + 2)(5^x - 1) \\ &\{a^2 + a - 2 = (a + 2)(a - 1)\} \end{aligned}$$

$$\begin{aligned} \mathbf{i} \quad & 49^x - 7^{x+1} + 12 \\ &= (7^x)^2 - 7(7^x) + 12 \\ &= (7^x - 4)(7^x - 3) \\ &\{a^2 - 7a + 12 = (a - 4)(a - 3)\} \end{aligned}$$

$$\begin{aligned} \mathbf{7} \quad \mathbf{a} \quad & \frac{12^n}{6^n} \quad \text{or} \quad \frac{12^n}{6^n} \\ &= \frac{2^n 6^n}{6^n} \quad = \left(\frac{12}{6}\right)^n \\ &= 2^n \quad = 2^n \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad & \frac{20^a}{2^a} \quad \text{or} \quad \frac{20^a}{2^a} \\ &= \frac{2^a 10^a}{2^a} \quad = \left(\frac{20}{2}\right)^a \\ &= 10^a \quad = 10^a \end{aligned}$$

$$\begin{aligned}
 \text{c} \quad & \frac{6^b}{2^b} \quad \text{or} \quad \frac{6^b}{2^b} \\
 & = \frac{2^b 3^b}{2^b} = \left(\frac{6}{2}\right)^b \\
 & = 3^b = 3^b
 \end{aligned}$$

$$\begin{aligned}
 \text{d} \quad & \frac{4^n}{20^n} \quad \text{or} \quad \frac{4^n}{20^n} \\
 & = \frac{4^n}{4^n 5^n} = \left(\frac{4}{20}\right)^n \\
 & = \frac{1}{5^n} = \left(\frac{1}{5}\right)^n
 \end{aligned}$$

$$\begin{aligned}
 \text{e} \quad & \frac{35^x}{7^x} \quad \text{or} \quad \frac{35^x}{7^x} \\
 & = \frac{5^x 7^x}{7^x} = \left(\frac{35}{7}\right)^x \\
 & = 5^x = 5^x
 \end{aligned}$$

$$\begin{aligned}
 \text{f} \quad & \frac{6^a}{8^a} \quad \text{or} \quad \frac{6^a}{8^a} \\
 & = \frac{2^a 3^a}{2^a 4^a} = \left(\frac{6}{8}\right)^a \\
 & = \frac{3^a}{4^a} = \left(\frac{3}{4}\right)^a
 \end{aligned}$$

$$\begin{aligned}
 \text{g} \quad & \frac{24^k}{9^k} \quad \text{or} \quad \frac{24^k}{9^k} \\
 & = \frac{3^k 8^k}{3^k 3^k} = \left(\frac{24}{9}\right)^k \\
 & = \frac{8^k}{3^k} = \left(\frac{8}{3}\right)^k
 \end{aligned}$$

$$\begin{aligned}
 \text{h} \quad & \frac{5^{n+1}}{5^n} \\
 & = \frac{5^n 5^1}{5^n} \\
 & = 5
 \end{aligned}$$

$$\begin{aligned}
 \text{i} \quad & \frac{5^{n+1}}{5} \\
 & = \frac{5^n 5^1}{5} \\
 & = 5^n
 \end{aligned}$$

$$\begin{aligned}
 8 \quad \text{a} \quad & \frac{6^m + 2^m}{2^m} \\
 & = \frac{2^m 3^m + 2^m}{2^m} \\
 & = \frac{2^m (3^m + 1)}{2^m} \\
 & = 3^m + 1
 \end{aligned}$$

$$\begin{aligned}
 \text{b} \quad & \frac{2^n + 12^n}{2^n} \\
 & = \frac{2^n + 2^n 6^n}{2^n} \\
 & = \frac{2^n (1 + 6^n)}{2^n} \\
 & = 1 + 6^n
 \end{aligned}$$

$$\begin{aligned}
 \text{c} \quad & \frac{8^n + 4^n}{2^n} \\
 & = \frac{2^n 4^n + 2^n 2^n}{2^n} \\
 & = \frac{2^n (4^n + 2^n)}{2^n} \\
 & = 4^n + 2^n
 \end{aligned}$$

$$\begin{aligned}
 \text{d} \quad & \frac{12^x - 3^x}{3^x} \\
 & = \frac{3^x 4^x - 3^x}{3^x} \\
 & = \frac{3^x (4^x - 1)}{3^x} \\
 & = 4^x - 1
 \end{aligned}$$

$$\begin{aligned}
 \text{e} \quad & \frac{6^n + 12^n}{1 + 2^n} \\
 & = \frac{6^n + 6^n 2^n}{1 + 2^n} \\
 & = \frac{6^n (1 + 2^n)}{1 + 2^n} \\
 & = 6^n
 \end{aligned}$$

$$\begin{aligned}
 \text{f} \quad & \frac{5^{n+1} - 5^n}{4} \\
 & = \frac{5^n \times 5 - 5^n}{4} \\
 & = \frac{5^n (5 - 1)}{4} \\
 & = 5^n
 \end{aligned}$$

$$\begin{aligned}
 \text{g} \quad & \frac{5^{n+1} - 5^n}{5^n} \\
 & = \frac{5^n \times 5 - 5^n}{5^n} \\
 & = \frac{5^n (5 - 1)}{5^n} \\
 & = 4
 \end{aligned}$$

$$\begin{aligned}
 \text{h} \quad & \frac{4^n - 2^n}{2^n} \\
 & = \frac{2^n 2^n - 2^n}{2^n} \\
 & = \frac{2^n (2^n - 1)}{2^n} \\
 & = 2^n - 1
 \end{aligned}$$

$$\begin{aligned}
 \text{i} \quad & \frac{2^n - 2^{n-1}}{2^n} \\
 & = \frac{2^{n-1} \times 2 - 2^{n-1}}{2^{n-1} \times 2} \\
 & = \frac{2^{n-1} (2 - 1)}{2^{n-1} \times 2} \\
 & = \frac{1}{2}
 \end{aligned}$$

$$\begin{aligned}
 9 \quad a \quad & 2^n(n+1) + 2^n(n-1) \\
 &= 2^n(n+1+n-1) \\
 &= 2^n(2n) \\
 &= n2^{n+1}
 \end{aligned}$$

$$\begin{aligned}
 b \quad & 3^n\left(\frac{n-1}{6}\right) - 3^n\left(\frac{n+1}{6}\right) \\
 &= 3^n\left(\frac{n-1}{6} - \frac{n+1}{6}\right) \\
 &= 3^n\left(-\frac{1}{3}\right) \\
 &= 3^n \times -3^{-1} \\
 &= -3^{n-1}
 \end{aligned}$$

EXERCISE 5C

$$\begin{aligned}
 1 \quad a \quad & 2^x = 32 \\
 \therefore & 2^x = 2^5 \\
 \therefore & x = 5
 \end{aligned}$$

$$\begin{aligned}
 d \quad & 7^x = 1 \\
 \therefore & 7^x = 7^0 \\
 \therefore & x = 0
 \end{aligned}$$

$$\begin{aligned}
 g \quad & 5^x = \frac{1}{125} \\
 \therefore & 5^x = 5^{-3} \\
 \therefore & x = -3
 \end{aligned}$$

$$\begin{aligned}
 j \quad & 3^{x+1} = \frac{1}{27} \\
 \therefore & 3^{x+1} = 3^{-3} \\
 \therefore & x+1 = -3 \\
 \therefore & x = -4
 \end{aligned}$$

$$\begin{aligned}
 2 \quad a \quad & 8^x = 32 \\
 \therefore & (2^3)^x = 2^5 \\
 \therefore & 2^{3x} = 2^5 \\
 \therefore & 3x = 5 \\
 \therefore & x = \frac{5}{3}
 \end{aligned}$$

$$\begin{aligned}
 d \quad & 25^x = \frac{1}{5} \\
 \therefore & (5^2)^x = 5^{-1} \\
 \therefore & 5^{2x} = 5^{-1} \\
 \therefore & 2x = -1 \\
 \therefore & x = -\frac{1}{2}
 \end{aligned}$$

$$\begin{aligned}
 b \quad & 5^x = 25 \\
 \therefore & 5^x = 5^2 \\
 \therefore & x = 2
 \end{aligned}$$

$$\begin{aligned}
 e \quad & 3^x = \frac{1}{3} \\
 \therefore & 3^x = 3^{-1} \\
 \therefore & x = -1
 \end{aligned}$$

$$\begin{aligned}
 h \quad & 4^{x+1} = 64 \\
 \therefore & 4^{x+1} = 4^3 \\
 \therefore & x+1 = 3 \\
 \therefore & x = 2
 \end{aligned}$$

$$\begin{aligned}
 k \quad & 7^{x+1} = 343 \\
 \therefore & 7^{x+1} = 7^3 \\
 \therefore & x+1 = 3 \\
 \therefore & x = 2
 \end{aligned}$$

$$\begin{aligned}
 b \quad & 4^x = \frac{1}{8} \\
 \therefore & (2^2)^x = 2^{-3} \\
 \therefore & 2^{2x} = 2^{-3} \\
 \therefore & 2x = -3 \\
 \therefore & x = -\frac{3}{2}
 \end{aligned}$$

$$\begin{aligned}
 e \quad & 27^x = \frac{1}{9} \\
 \therefore & (3^3)^x = 3^{-2} \\
 \therefore & 3^{3x} = 3^{-2} \\
 \therefore & 3x = -2 \\
 \therefore & x = -\frac{2}{3}
 \end{aligned}$$

$$\begin{aligned}
 c \quad & 3^x = 81 \\
 \therefore & 3^x = 3^4 \\
 \therefore & x = 4
 \end{aligned}$$

$$\begin{aligned}
 f \quad & 2^x = \sqrt{2} \\
 \therefore & 2^x = 2^{\frac{1}{2}} \\
 \therefore & x = \frac{1}{2}
 \end{aligned}$$

$$\begin{aligned}
 i \quad & 2^{x-2} = \frac{1}{32} \\
 \therefore & 2^{x-2} = 2^{-5} \\
 \therefore & x-2 = -5 \\
 \therefore & x = -3
 \end{aligned}$$

$$\begin{aligned}
 l \quad & 5^{1-2x} = \frac{1}{5} \\
 \therefore & 5^{1-2x} = 5^{-1} \\
 \therefore & 1-2x = -1 \\
 \therefore & -2x = -2 \\
 \therefore & x = 1
 \end{aligned}$$

$$\begin{aligned}
 c \quad & 9^x = \frac{1}{27} \\
 \therefore & (3^2)^x = 3^{-3} \\
 \therefore & 3^{2x} = 3^{-3} \\
 \therefore & 2x = -3 \\
 \therefore & x = -\frac{3}{2}
 \end{aligned}$$

$$\begin{aligned}
 f \quad & 16^x = \frac{1}{32} \\
 \therefore & (2^4)^x = 2^{-5} \\
 \therefore & 2^{4x} = 2^{-5} \\
 \therefore & 4x = -5 \\
 \therefore & x = -\frac{5}{4}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{g} \quad & 4^{x+2} = 128 \\
 \therefore & (2^2)^{x+2} = 2^7 \\
 \therefore & 2^{2(x+2)} = 2^7 \\
 \therefore & 2(x+2) = 7 \\
 \therefore & 2x+4 = 7 \\
 \therefore & 2x = 3 \\
 \therefore & x = \frac{3}{2}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{j} \quad & 9^{x-3} = 27 \\
 \therefore & (3^2)^{x-3} = 3^3 \\
 \therefore & 3^{2(x-3)} = 3^3 \\
 \therefore & 2(x-3) = 3 \\
 \therefore & 2x-6 = 3 \\
 \therefore & 2x = 9 \\
 \therefore & x = \frac{9}{2}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{m} \quad & 81^x = 27^{-x} \\
 \therefore & (3^4)^x = (3^3)^{-x} \\
 \therefore & 3^{4x} = 3^{-3x} \\
 \therefore & 4x = -3x \\
 \therefore & 7x = 0 \\
 \therefore & x = 0
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{p} \quad & \left(\frac{1}{3}\right)^{x+1} = 243 \\
 \therefore & (3^{-1})^{x+1} = 3^5 \\
 \therefore & 3^{-(x+1)} = 3^5 \\
 \therefore & -(x+1) = 5 \\
 \therefore & -x-1 = 5 \\
 \therefore & -x = 6 \\
 \therefore & x = -6
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{3} \quad \mathbf{a} \quad & 4^{2x+1} = 8^{1-x} \\
 \therefore & (2^2)^{2x+1} = (2^3)^{1-x} \\
 \therefore & 2^{2(2x+1)} = 2^{3(1-x)} \\
 \therefore & 2(2x+1) = 3(1-x) \\
 \therefore & 4x+2 = 3-3x \\
 \therefore & 7x = 1 \\
 \therefore & x = \frac{1}{7}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{h} \quad & 25^{1-x} = \frac{1}{125} \\
 \therefore & (5^2)^{1-x} = 5^{-3} \\
 \therefore & 5^{2(1-x)} = 5^{-3} \\
 \therefore & 2(1-x) = -3 \\
 \therefore & 2-2x = -3 \\
 \therefore & -2x = -5 \\
 \therefore & x = \frac{5}{2}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{k} \quad & \left(\frac{1}{2}\right)^{x+1} = 8 \\
 \therefore & (2^{-1})^{x+1} = 2^3 \\
 \therefore & 2^{-(x+1)} = 2^3 \\
 \therefore & -(x+1) = 3 \\
 \therefore & -x-1 = 3 \\
 \therefore & -x = 4 \\
 \therefore & x = -4
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{n} \quad & \left(\frac{1}{4}\right)^{1-x} = 32 \\
 \therefore & (2^{-2})^{1-x} = 2^5 \\
 \therefore & 2^{-2(1-x)} = 2^5 \\
 \therefore & -2(1-x) = 5 \\
 \therefore & -2+2x = 5 \\
 \therefore & 2x = 7 \\
 \therefore & x = \frac{7}{2}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{i} \quad & 4^{4x-1} = \frac{1}{2} \\
 \therefore & (2^2)^{4x-1} = 2^{-1} \\
 \therefore & 2^{2(4x-1)} = 2^{-1} \\
 \therefore & 2(4x-1) = -1 \\
 \therefore & 8x-2 = -1 \\
 \therefore & 8x = 1 \\
 \therefore & x = \frac{1}{8}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{l} \quad & \left(\frac{1}{3}\right)^{x+2} = 9 \\
 \therefore & (3^{-1})^{x+2} = 3^2 \\
 \therefore & 3^{-(x+2)} = 3^2 \\
 \therefore & -(x+2) = 2 \\
 \therefore & -x-2 = 2 \\
 \therefore & -x = 4 \\
 \therefore & x = -4
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{o} \quad & \left(\frac{1}{7}\right)^x = 49 \\
 \therefore & (7^{-1})^x = 7^2 \\
 \therefore & 7^{-x} = 7^2 \\
 \therefore & -x = 2 \\
 \therefore & x = -2
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad & 9^{2-x} = \left(\frac{1}{3}\right)^{2x+1} \\
 \therefore & (3^2)^{2-x} = (3^{-1})^{2x+1} \\
 \therefore & 3^{2(2-x)} = 3^{-(2x+1)} \\
 \therefore & 2(2-x) = -(2x+1) \\
 \therefore & 4-2x = -2x-1 \\
 \therefore & 4 = -1
 \end{aligned}$$

which is clearly false, so no solutions exist.

$$\begin{aligned}
 \text{c} \quad & 2^x \times 8^{1-x} = \frac{1}{4} \\
 \therefore & 2^x \times (2^3)^{1-x} = 2^{-2} \\
 \therefore & 2^x \times 2^{3(1-x)} = 2^{-2} \\
 \therefore & 2^{x+3(1-x)} = 2^{-2} \\
 \therefore & x + 3(1-x) = -2 \\
 \therefore & x + 3 - 3x = -2 \\
 \therefore & -2x = -5 \\
 \therefore & x = \frac{5}{2}
 \end{aligned}$$

$$\begin{aligned}
 \text{4 a} \quad & 3 \times 2^x = 24 \\
 \therefore & 2^x = 8 \\
 \therefore & 2^x = 2^3 \\
 \therefore & x = 3
 \end{aligned}$$

$$\begin{aligned}
 \text{b} \quad & 7 \times 2^x = 28 \\
 \therefore & 2^x = 4 \\
 \therefore & 2^x = 2^2 \\
 \therefore & x = 2
 \end{aligned}$$

$$\begin{aligned}
 \text{c} \quad & 4 \times 3^{x+2} = 12 \\
 \therefore & 3^{x+2} = 3 \\
 \therefore & x + 2 = 1 \\
 \therefore & x = -1
 \end{aligned}$$

$$\begin{aligned}
 \text{d} \quad & 12 \times 3^{-x} = \frac{4}{3} \\
 \therefore & 3^{-x} = \frac{4}{3} \times \frac{1}{12} \\
 \therefore & 3^{-x} = \frac{1}{9} \\
 \therefore & 3^{-x} = 3^{-2} \\
 \therefore & -x = -2 \\
 \therefore & x = 2
 \end{aligned}$$

$$\begin{aligned}
 \text{e} \quad & 4 \times \left(\frac{1}{3}\right)^x = 36 \\
 \therefore & \left(\frac{1}{3}\right)^x = 9 \\
 \therefore & (3^{-1})^x = 3^2 \\
 \therefore & 3^{-x} = 3^2 \\
 \therefore & -x = 2 \\
 \therefore & x = -2
 \end{aligned}$$

$$\begin{aligned}
 \text{f} \quad & 5 \times \left(\frac{1}{2}\right)^x = 20 \\
 \therefore & \left(\frac{1}{2}\right)^x = 4 \\
 \therefore & (2^{-1})^x = 2^2 \\
 \therefore & 2^{-x} = 2^2 \\
 \therefore & -x = 2 \\
 \therefore & x = -2
 \end{aligned}$$

$$\begin{aligned}
 \text{5 a} \quad & 4^x - 6(2^x) + 8 = 0 \\
 \therefore & (2^x)^2 - 6(2^x) + 8 = 0 \\
 \therefore & (2^x - 2)(2^x - 4) = 0 \quad \{a^2 - 6a + 8 = (a - 2)(a - 4)\} \\
 \therefore & 2^x = 2 \quad \text{or} \quad 2^x = 4 \\
 \therefore & 2^x = 2^1 \quad \text{or} \quad 2^x = 2^2 \\
 \therefore & x = 1 \quad \text{or} \quad 2
 \end{aligned}$$

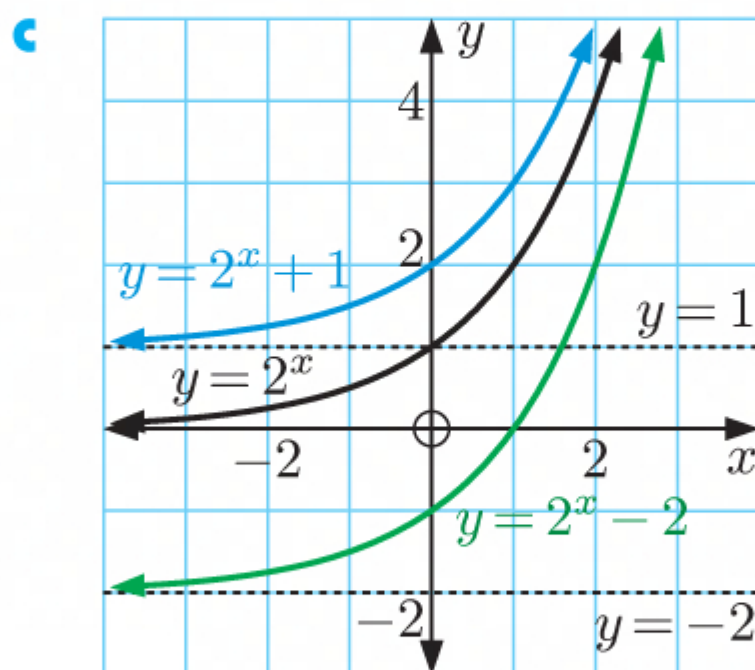
$$\begin{aligned}
 \text{b} \quad & 4^x - 2^x - 2 = 0 \\
 \therefore & (2^x)^2 - 2^x - 2 = 0 \\
 \therefore & (2^x - 2)(2^x + 1) = 0 \quad \{a^2 - a - 2 = (a - 2)(a + 1)\} \\
 \therefore & 2^x = 2 \quad \text{or} \quad 2^x = -1 \\
 \therefore & 2^x = 2^1 \quad \{\text{since } 2^x \text{ cannot be negative}\} \\
 \therefore & x = 1
 \end{aligned}$$

$$\begin{aligned}
 \text{c} \quad & 9^x - 12(3^x) + 27 = 0 \\
 \therefore & (3^x)^2 - 12(3^x) + 27 = 0 \\
 \therefore & (3^x - 3)(3^x - 9) = 0 \quad \{a^2 - 12a + 27 = (a - 3)(a - 9)\} \\
 \therefore & 3^x = 3 \quad \text{or} \quad 3^x = 9 \\
 \therefore & 3^x = 3^1 \quad \text{or} \quad 3^x = 3^2 \\
 \therefore & x = 1 \quad \text{or} \quad 2
 \end{aligned}$$

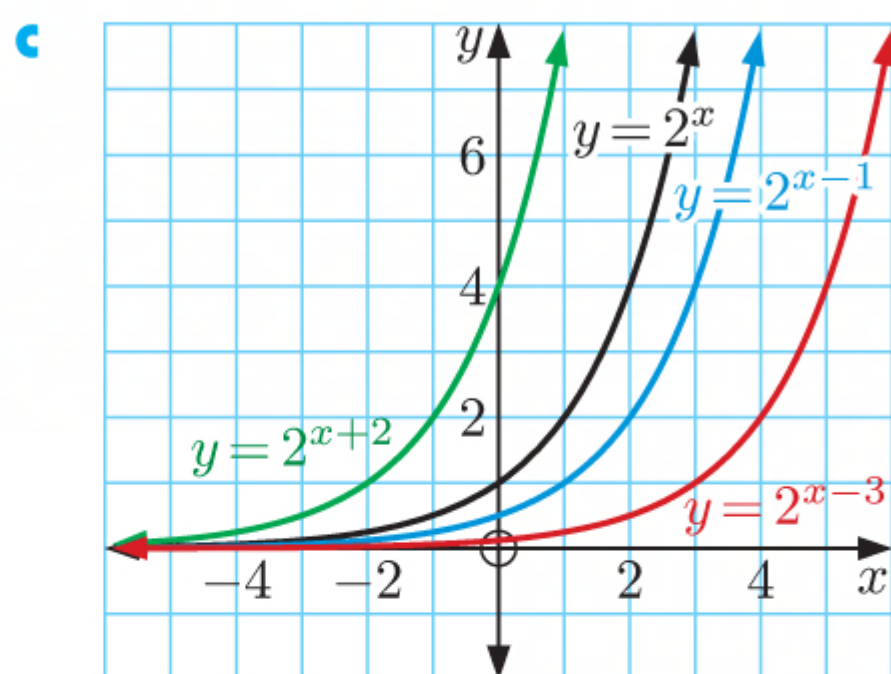
- d** $9^x = 3^x + 6$
 $\therefore (3^x)^2 - 3^x - 6 = 0$
 $\therefore (3^x - 3)(3^x + 2) = 0 \quad \{a^2 - a - 6 = (a - 3)(a + 2)\}$
 $\therefore 3^x = 3 \text{ or } 3^x = -2$
 $\therefore 3^x = 3^1 \quad \{\text{since } 3^x \text{ cannot be negative}\}$
 $\therefore x = 1$
- e** $25^x - 23(5^x) - 50 = 0$
 $\therefore (5^x)^2 - 23(5^x) - 50 = 0$
 $\therefore (5^x - 25)(5^x + 2) = 0 \quad \{a^2 - 23a - 50 = (a - 25)(a + 2)\}$
 $\therefore 5^x = 25 \text{ or } 5^x = -2$
 $\therefore 5^x = 5^2 \quad \{\text{since } 5^x \text{ cannot be negative}\}$
 $\therefore x = 2$
- f** $49^x + 1 = 2(7^x)$
 $\therefore (7^x)^2 - 2(7^x) + 1 = 0$
 $\therefore (7^x - 1)^2 = 0 \quad \{a^2 - 2a + 1 = (a - 1)^2\}$
 $\therefore 7^x = 1$
 $\therefore 7^x = 7^0$
 $\therefore x = 0$

INVESTIGATION 1**GRAPHS OF EXPONENTIAL FUNCTIONS**

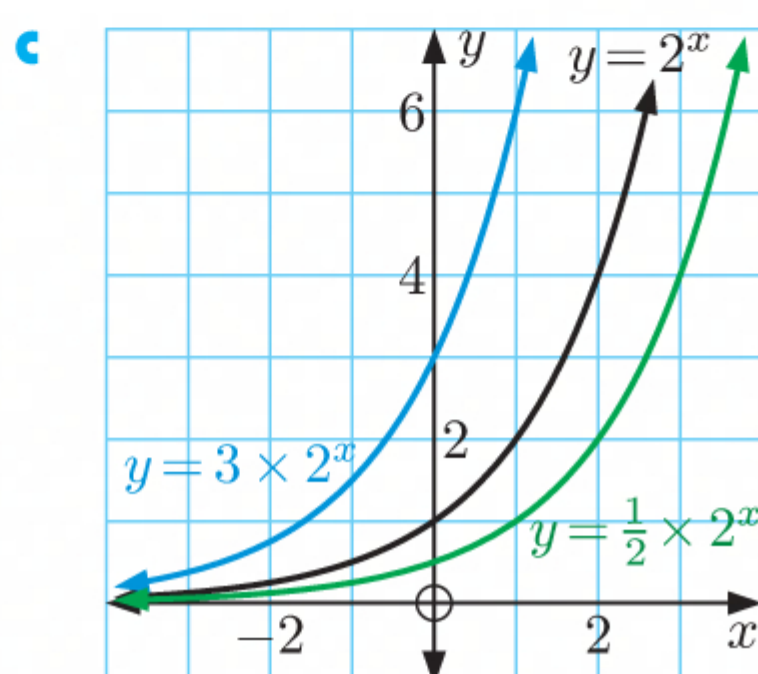
- 1 a** A vertical translation through k units maps $y = a^x$ to $y = a^x + k$.
- b i** The shape of the graph will remain the same.
ii The graph is translated vertically through k units.
iii The horizontal asymptote $y = 0$ is transformed to $y = k$.



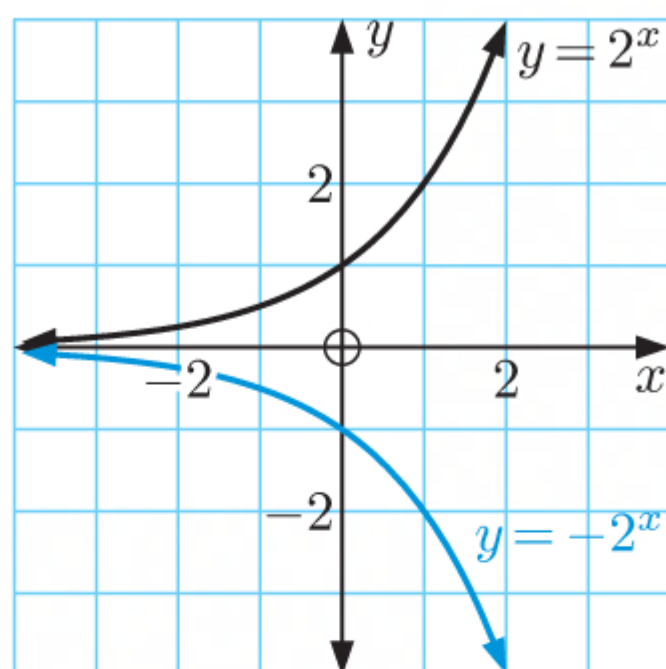
- 2 a** A horizontal translation through h units maps $y = a^x$ to $y = a^{x-h}$.
- b i** The shape of the graph will remain the same.
ii The graph is translated horizontally through h units.
iii The horizontal asymptote $y = 0$ will remain the same.



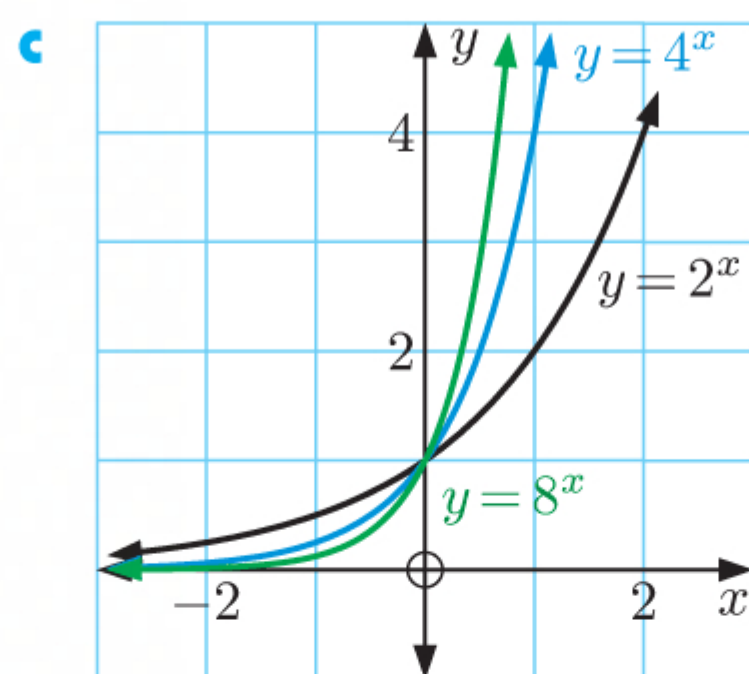
- 3 a** A vertical stretch with scale factor $p > 0$ will map $y = a^x$ to $y = p \times a^x$.
- b**
- i** Each point will become p times its previous distance from the x -axis.
 - ii** The graph is stretched vertically with invariant x -axis and scale factor p .
 - iii** The horizontal asymptote $y = 0$ will remain the same.



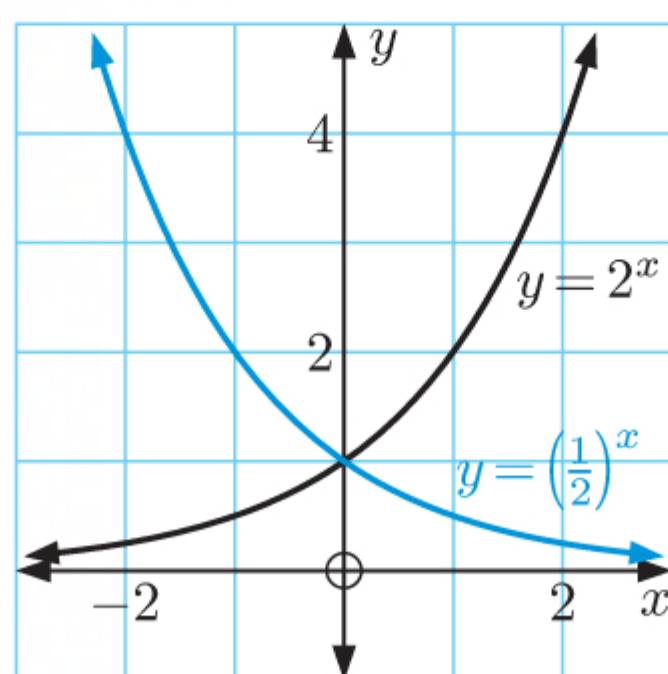
- 4 a** A reflection in the x -axis maps $y = a^x$ to $y = -a^x$.
- b** $y = -2^x$ is a reflection of $y = 2^x$ in the x -axis.



- 5 a** A horizontal stretch with scale factor $\frac{1}{q}$ maps $y = a^x$ to $y = a^{qx}$, $q > 0$.
- b**
- i** Each point will become $\frac{1}{q}$ times its previous distance from the y -axis.
If $0 < q < 1$, the graph becomes flatter, if $q > 1$, the graph becomes steeper, and if $q = 1$, the graph remains unchanged.
 - ii** The graph is stretched horizontally with invariant y -axis and scale factor $\frac{1}{q}$.
 - iii** The horizontal asymptote $y = 0$ will remain the same.

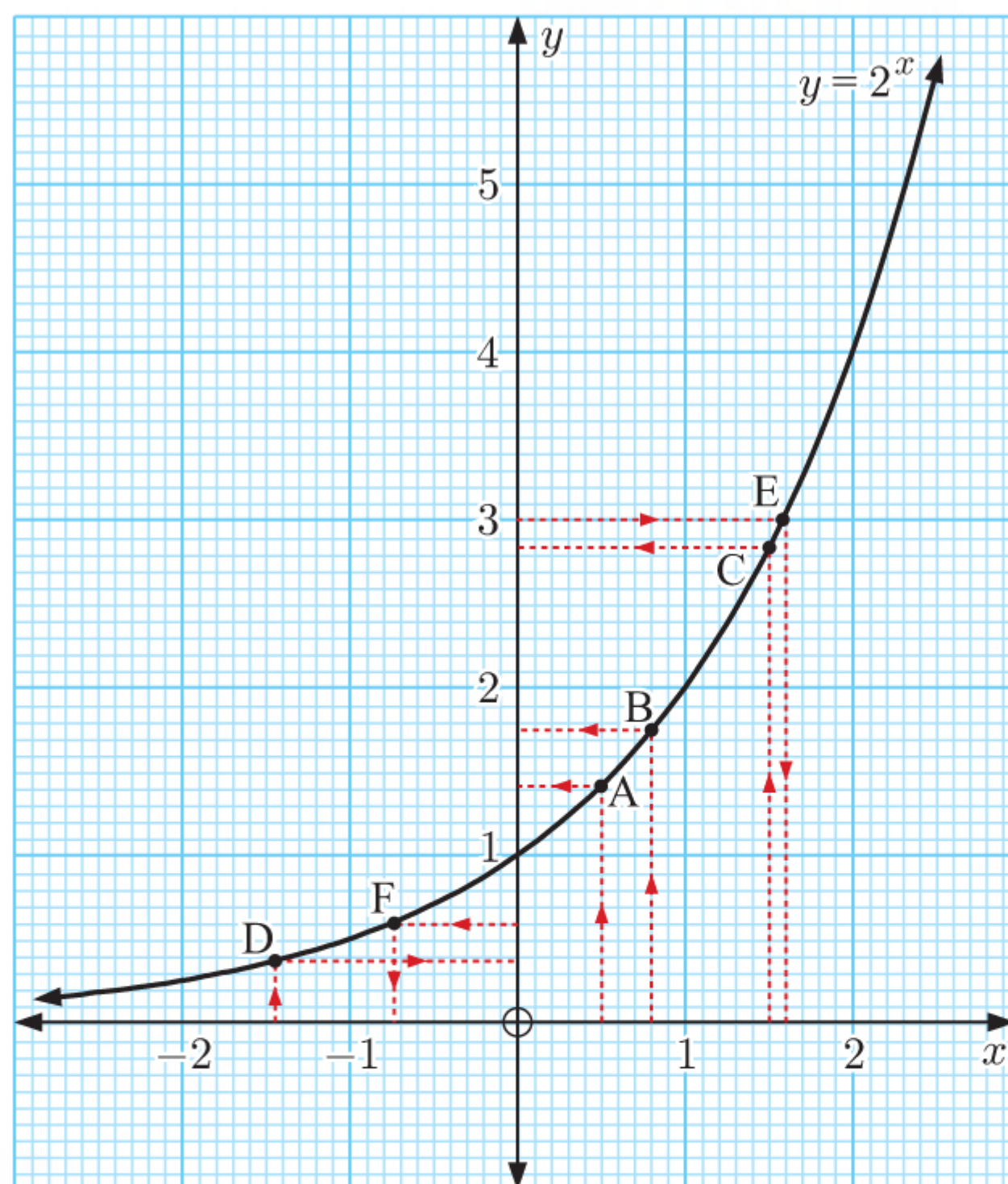


- 6 a** A reflection in the y -axis will map $y = a^x$ to $y = a^{-x}$.
- b** $y = \left(\frac{1}{2}\right)^x = 2^{-x}$ is a reflection of $y = 2^x$ in the y -axis.




EXERCISE 5D


- 1 a i** When $x = \frac{1}{2}$, $y = 2^{\frac{1}{2}}$
From point A, $y \approx 1.4$
 $\therefore 2^{\frac{1}{2}} \approx 1.4$
- ii** When $x = 0.8$, $y = 2^{0.8}$
From point B, $y \approx 1.7$
 $\therefore 2^{0.8} \approx 1.7$
- iii** When $x = 1.5$, $y = 2^{1.5}$
From point C, $y \approx 2.8$
 $\therefore 2^{1.5} \approx 2.8$
- iv** When $x = -\sqrt{2}$, $y = 2^{-\sqrt{2}}$
Using **i** we know $x \approx -1.4$
From point D, $y \approx 0.4$
 $\therefore 2^{-\sqrt{2}} \approx 0.4$





- b i** When $2^x = 3$, $x \approx 1.6$ from point E.
- ii** When $2^x = 0.6$, $x \approx -0.7$ from point F.
- c** The graph of $y = 2^x$ has horizontal asymptote $y = 0$.
 \therefore there is no value of x such that $2^x = 0$.
 $\therefore 2^x = 0$ has no solutions.

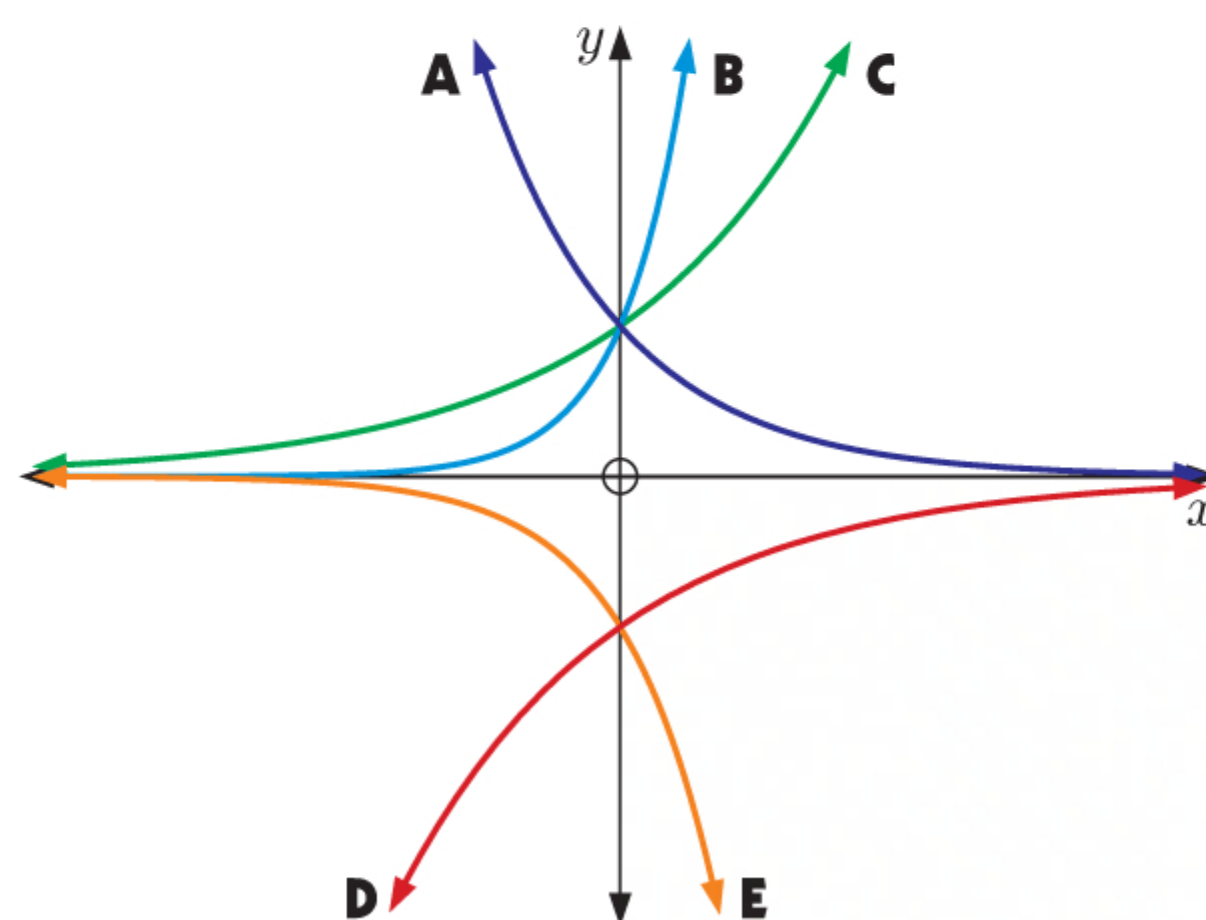
- 2 a, b** Both $y = 2^x$ and $y = 10^x$ have $p > 0$, $a > 1$, and shape 
 $y = 10^x$ is steeper than $y = 2^x$ as $10 > 2$.

$\therefore y = 2^x$ corresponds to **C**, and
 $y = 10^x$ corresponds to **B**.

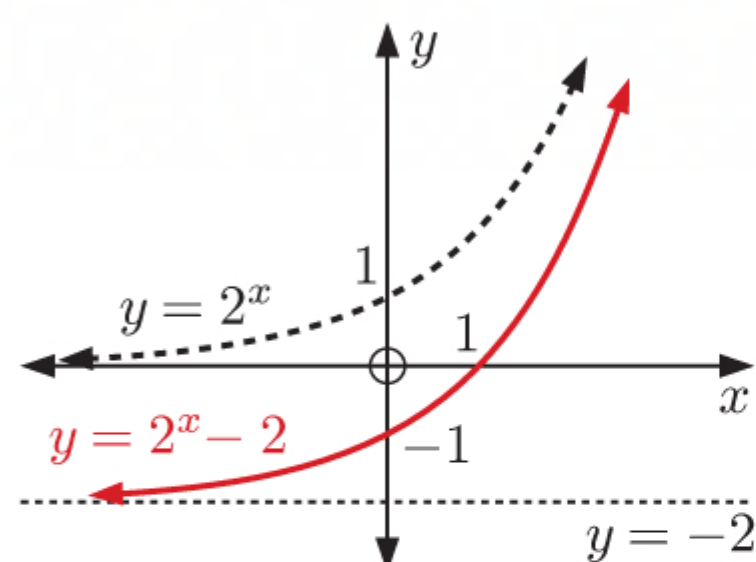
- c** $y = -5^x$ has $p < 0$, $a > 1$, and shape 
 $\therefore y = -5^x$ corresponds to **E**.

- d** $y = \left(\frac{1}{3}\right)^x$ has $p > 0$, $0 < a < 1$, and shape 
 $\therefore y = \left(\frac{1}{3}\right)^x$ corresponds to **A**.

- e** $y = -\left(\frac{1}{2}\right)^x$ has $p < 0$, $0 < a < 1$, and shape 
 $\therefore y = -\left(\frac{1}{2}\right)^x$ corresponds to **D**.

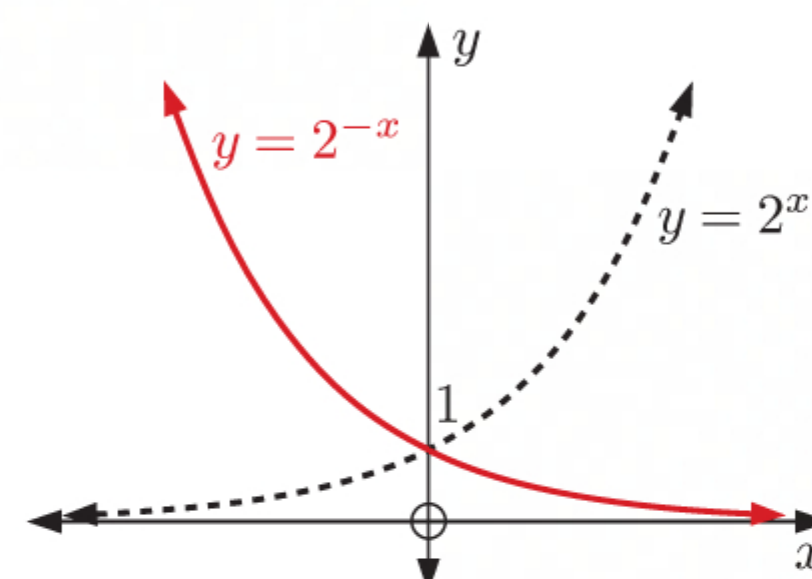


3 a



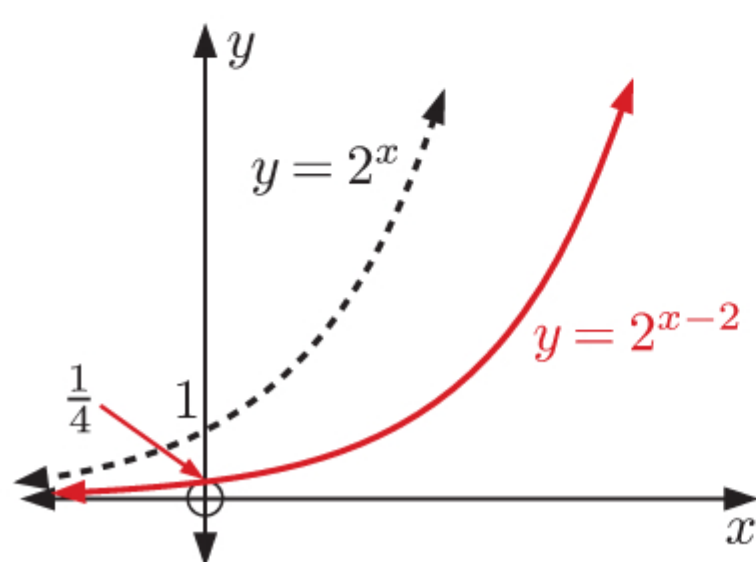
a vertical translation 2 units downwards
 $y = -2$ is the horizontal asymptote

b



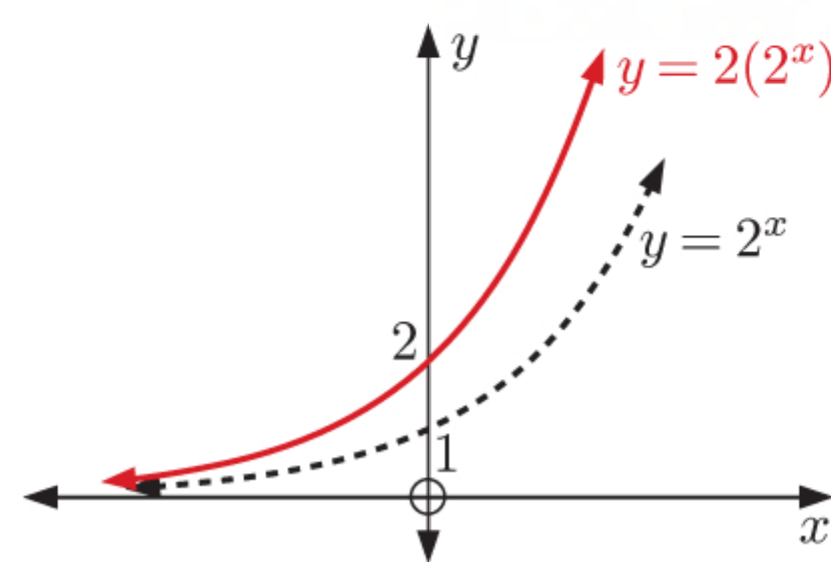
a reflection in the y -axis

c

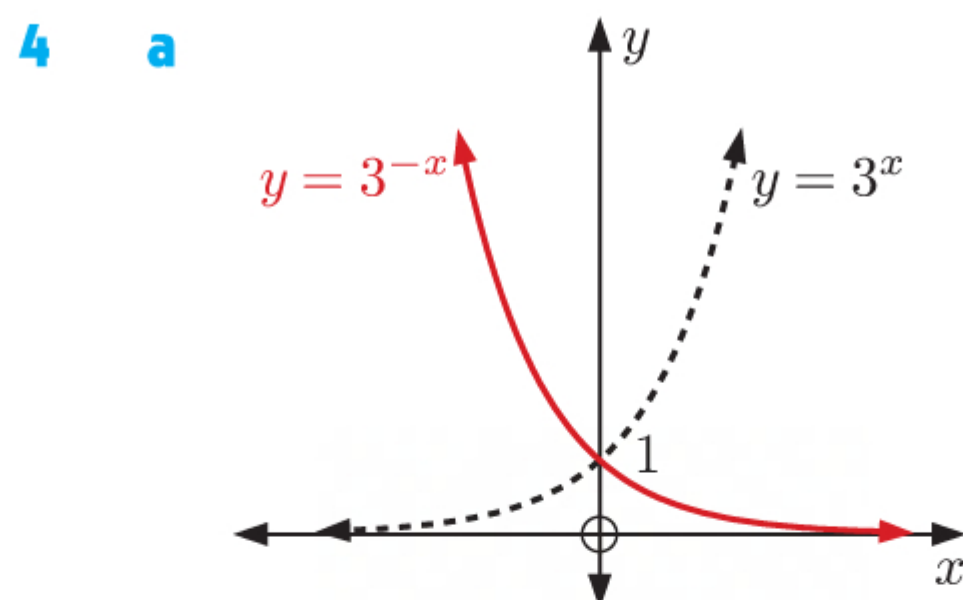


a horizontal translation 2 units right

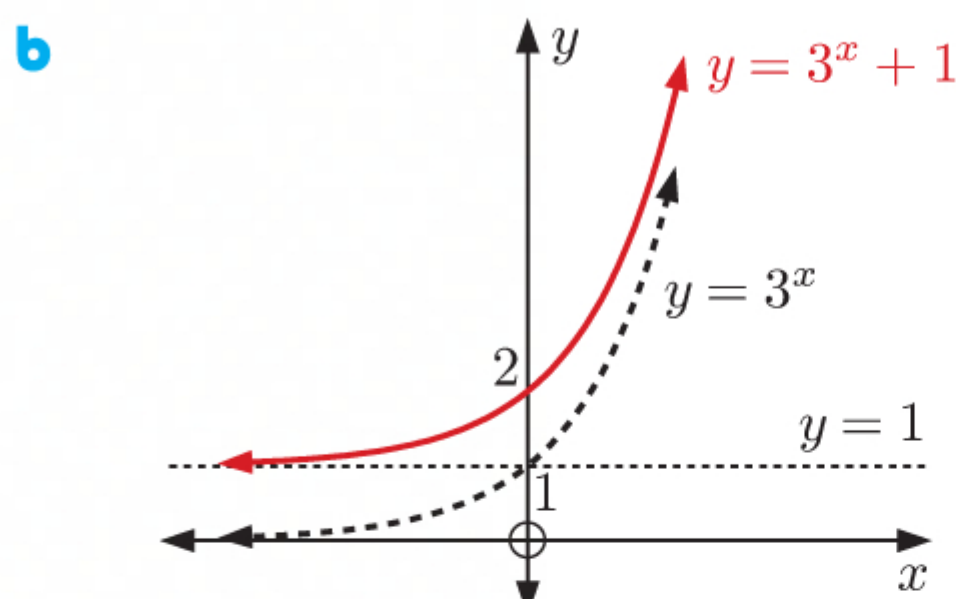
d



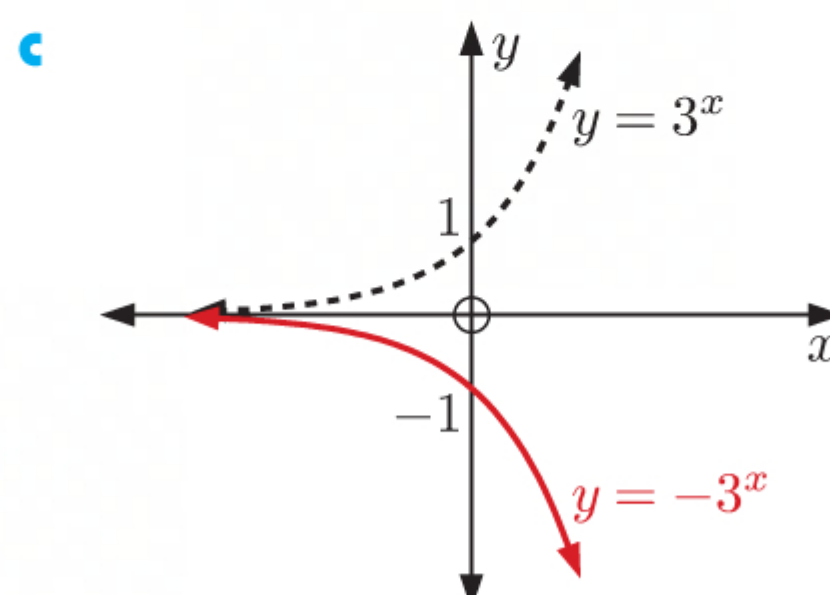
a vertical stretch with scale factor 2



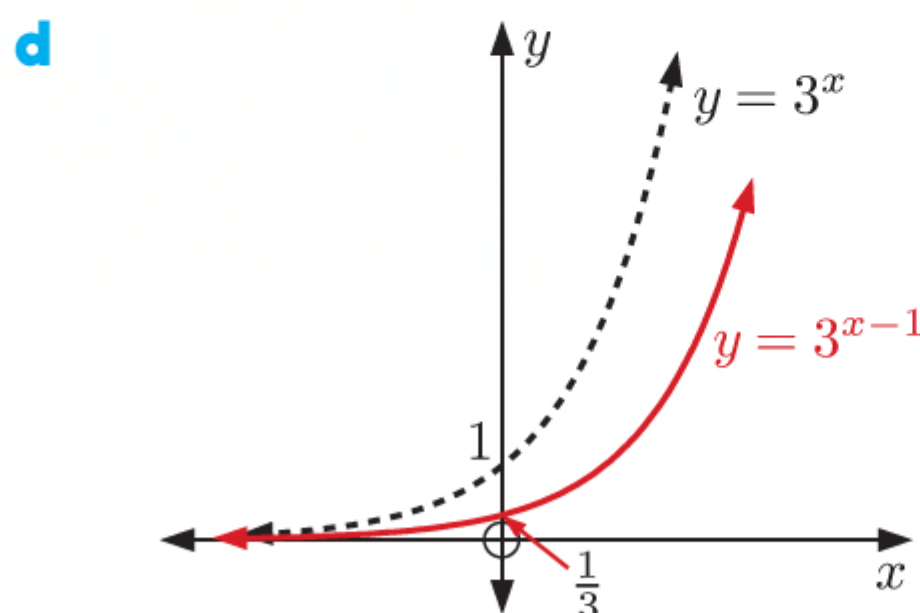
a reflection in the y -axis



a vertical translation 1 unit upwards
 $y = 1$ is the horizontal asymptote



a reflection in the x -axis



a horizontal translation 1 unit right

- 5 a** The graph of $y = 3^{-x}$ has horizontal asymptote $y = 0$.
b The graph of $y = 2^x - 1$ has horizontal asymptote $y = -1$.
c The graph of $y = 3 - 2^{-x}$ has horizontal asymptote $y = 3$.
d The graph of $y = 4 \times 2^x + 2$ has horizontal asymptote $y = 2$.
e The graph of $y = 5 \times 3^{x+2}$ has horizontal asymptote $y = 0$.
f The graph of $y = -2 \times 3^{1-x} - 4$ has horizontal asymptote $y = -4$.

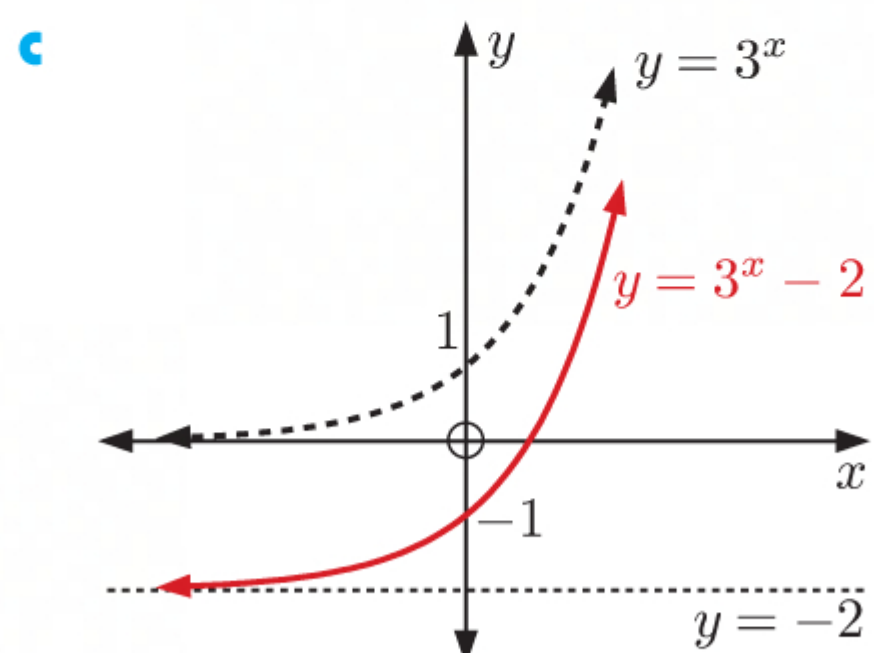
6 $f(x) = 3^x - 2$

a i $f(0) = 3^0 - 2$
 $= 1 - 2$
 $= -1$

ii $f(2) = 3^2 - 2$
 $= 9 - 2$
 $= 7$

iii $f(-2) = 3^{-2} - 2$
 $= \frac{1}{9} - 2$
 $= -\frac{17}{9} = -1\frac{8}{9}$

- b** The graph of $y = 3^x - 2$ has horizontal asymptote $y = -2$.



- d** The domain is $\{x \mid x \in \mathbb{R}\}$.
 The range is $\{y \mid y > -2\}$.

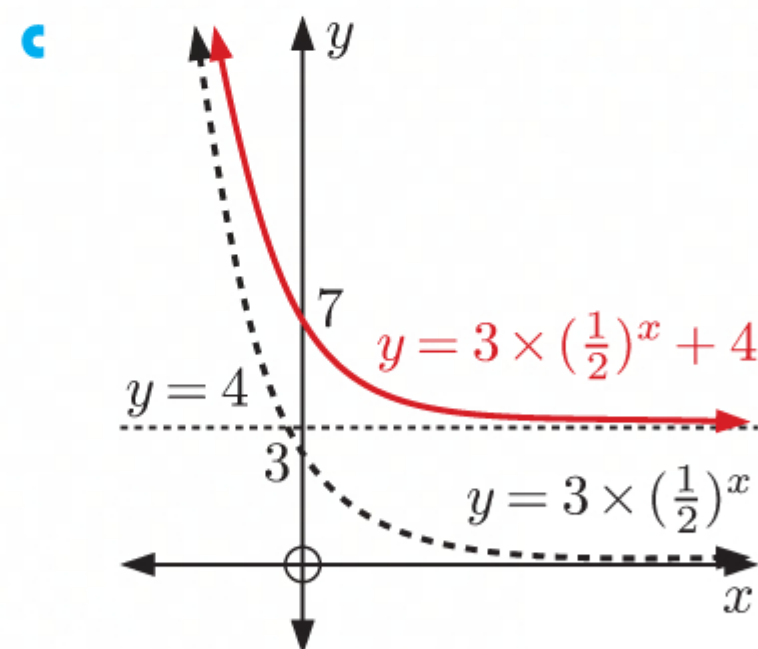
7 $g(x) = 3 \times \left(\frac{1}{2}\right)^x + 4$

a i $g(0) = 3 \times \left(\frac{1}{2}\right)^0 + 4$
 $= 3 \times 1 + 4$
 $= 7$

ii $g(2) = 3 \times \left(\frac{1}{2}\right)^2 + 4$
 $= 3 \times \frac{1}{4} + 4$
 $= \frac{19}{4} = 4\frac{3}{4}$

iii $g(-2) = 3 \times \left(\frac{1}{2}\right)^{-2} + 4$
 $= 3 \times 2^2 + 4$
 $= 3 \times 4 + 4$
 $= 16$

b The graph of $y = 3 \times \left(\frac{1}{2}\right)^x + 4$ has horizontal asymptote $y = 4$.



d The domain is $\{x \mid x \in \mathbb{R}\}$.
 The range is $\{y \mid y > 4\}$.

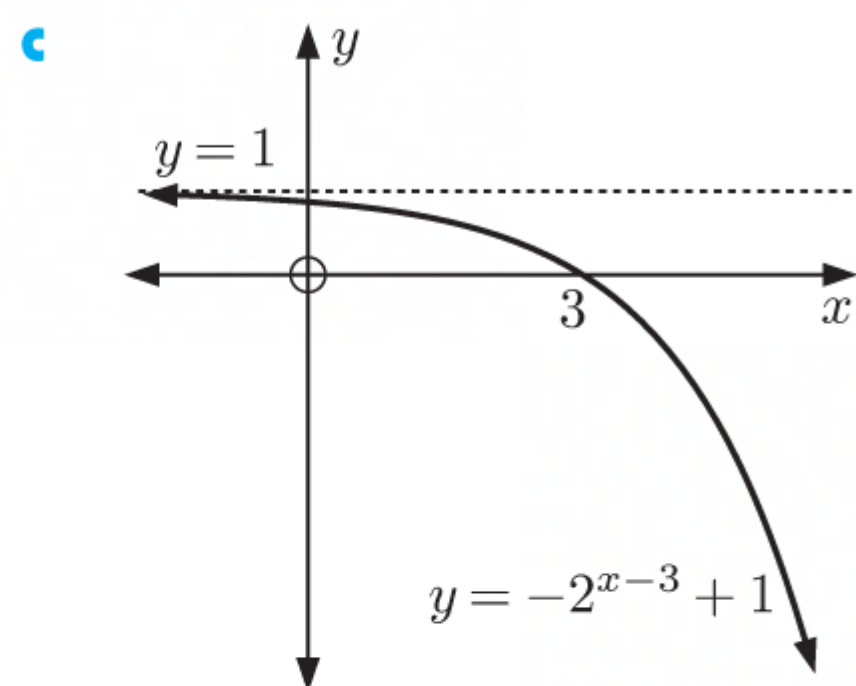
8 $h(x) = -2^{x-3} + 1$

a i $h(0) = -2^{-3} + 1$
 $= -\frac{1}{2^3} + 1$
 $= -\frac{1}{8} + 1$
 $= \frac{7}{8}$

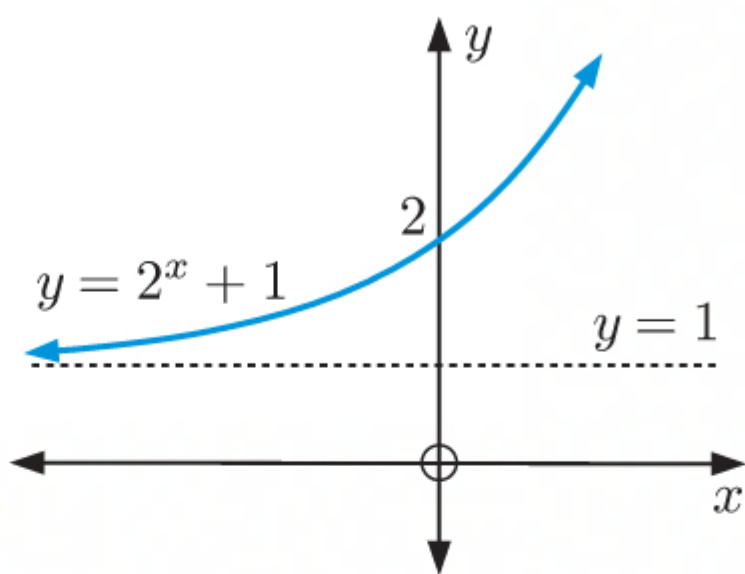
ii $h(3) = -2^0 + 1$
 $= -1 + 1$
 $= 0$

iii $h(6) = -2^3 + 1$
 $= -8 + 1$
 $= -7$

b The graph of $y = -2^{x-3} + 1$ has horizontal asymptote $y = 1$.



d The domain is $\{x \mid x \in \mathbb{R}\}$.
 The range is $\{y \mid y < 1\}$.

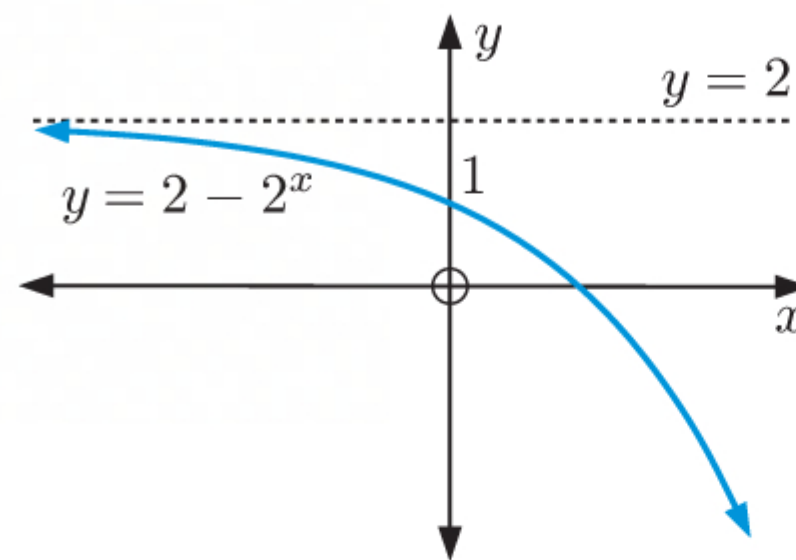
9 a i

When $x = 0$, $y = 1 + 1 = 2$

When $x = 2$, $y = 4 + 1 = 5$

When $x = -2$, $y = \frac{1}{4} + 1 = 1\frac{1}{4}$

- ii** The domain is $\{x \mid x \in \mathbb{R}\}$.
The range is $\{y \mid y > 1\}$.
- iii** Using technology, when $x = \sqrt{2}$, $y \approx 3.67$
- iv** As $x \rightarrow \infty$, $y \rightarrow \infty$
As $x \rightarrow -\infty$, $y \rightarrow 1^+$
- v** The horizontal asymptote is $y = 1$.

b i

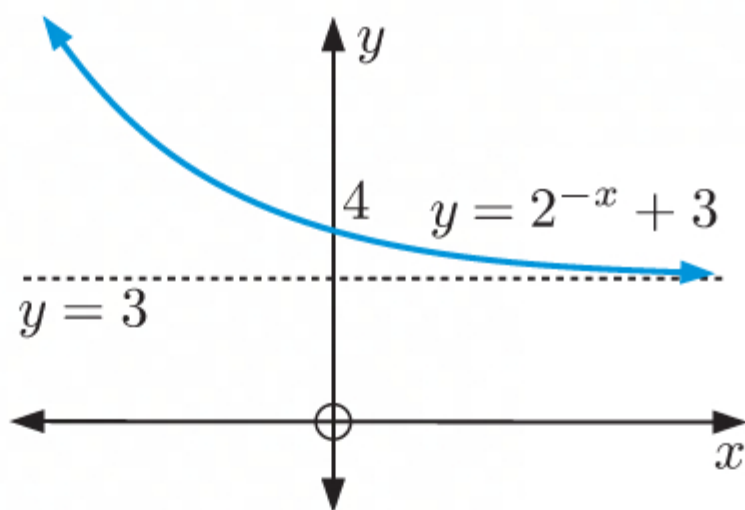
When $x = 0$, $y = 2 - 1 = 1$

When $x = 1$, $y = 2 - 2 = 0$

When $x = 2$, $y = 2 - 4 = -2$

When $x = -2$, $y = 2 - \frac{1}{4} = 1\frac{3}{4}$

- ii** The domain is $\{x \mid x \in \mathbb{R}\}$.
The range is $\{y \mid y < 2\}$.
- iii** Using technology, when $x = \sqrt{2}$, $y \approx -0.665$
- iv** As $x \rightarrow \infty$, $y \rightarrow -\infty$
As $x \rightarrow -\infty$, $y \rightarrow 2^-$
- v** The horizontal asymptote is $y = 2$.

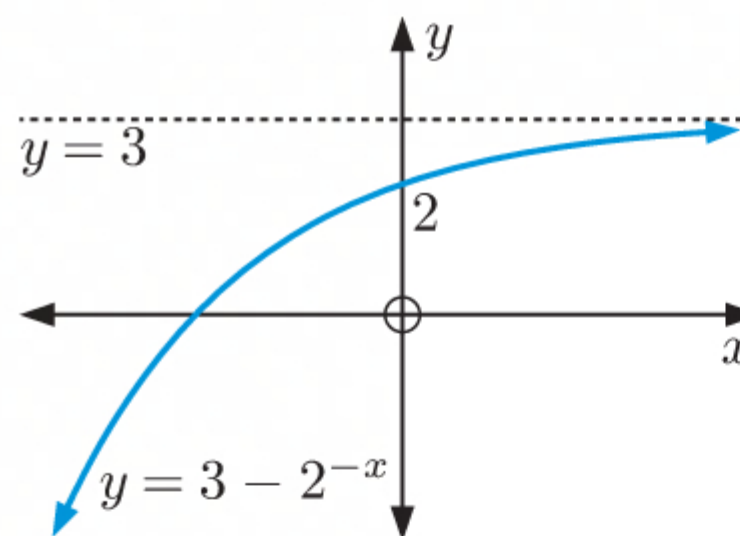
c i

When $x = 0$, $y = 1 + 3 = 4$

When $x = 2$, $y = \frac{1}{4} + 3 = 3\frac{1}{4}$

When $x = -2$, $y = 2^2 + 3 = 7$

- ii** The domain is $\{x \mid x \in \mathbb{R}\}$.
The range is $\{y \mid y > 3\}$.
- iii** Using technology, when $x = \sqrt{2}$, $y \approx 3.38$
- iv** As $x \rightarrow \infty$, $y \rightarrow 3^+$
As $x \rightarrow -\infty$, $y \rightarrow \infty$
- v** The horizontal asymptote is $y = 3$.

d i

When $x = 0$, $y = 3 - 1 = 2$

When $x = 2$, $y = 3 - \frac{1}{4} = 2\frac{3}{4}$

When $x = -2$, $y = 3 - 4 = -1$

- ii** The domain is $\{x \mid x \in \mathbb{R}\}$.
The range is $\{y \mid y < 3\}$.
- iii** Using technology, when $x = \sqrt{2}$, $y \approx 2.62$
- iv** As $x \rightarrow \infty$, $y \rightarrow 3^-$
As $x \rightarrow -\infty$, $y \rightarrow -\infty$
- v** The horizontal asymptote is $y = 3$.

10 a $y = a \times 2^x + b$

When $x = 0$, $y = -5$

$$\therefore a \times 2^0 + b = -5$$

$$\therefore a + b = -5$$

$$\therefore a = -5 - b \quad \dots (*)$$

When $y = 0$, $x = 1$

$$\therefore a \times 2^1 + b = 0$$

$$\therefore 2a + b = 0$$

$$\therefore 2(-5 - b) + b = 0 \quad \{\text{using } (*)\}$$

$$\therefore -10 - 2b + b = 0$$

$$\therefore -10 - b = 0$$

$$\therefore b = -10$$

Substituting $b = -10$ into $(*)$, $a = -5 - (-10)$

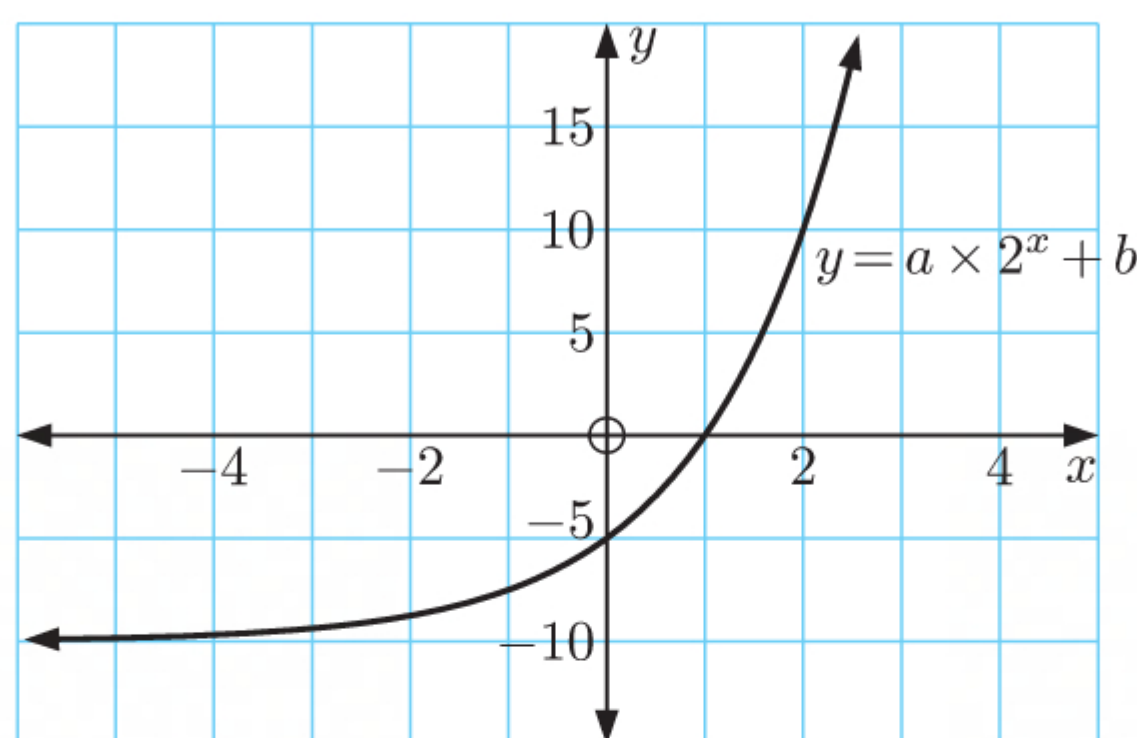
$$\therefore a = 5$$

So, $a = 5$, $b = -10$.

b When $x = 6$, $y = 5 \times 2^6 - 10$

$$\therefore y = 5 \times 64 - 10$$

$$\therefore y = 310$$



11 a $f(0) = 3.5 - a^0$

$$= 3.5 - 1$$

$$= 2.5$$

\therefore the y -intercept is 2.5.

\therefore P is $(0, 2.5)$.

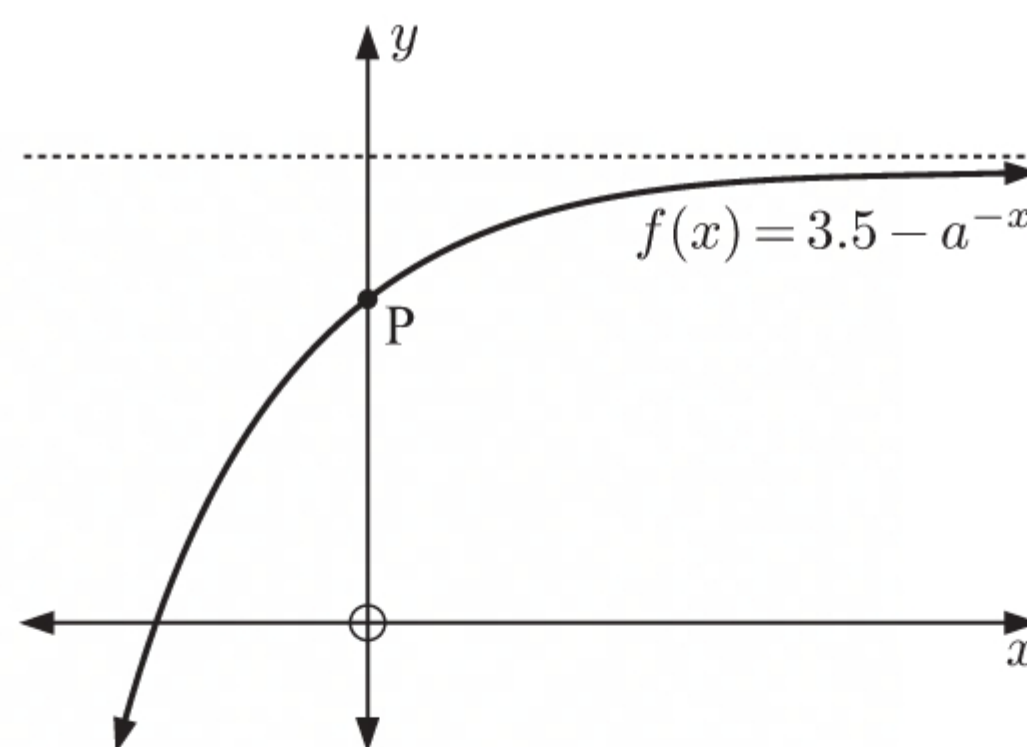
b The point $(-1, 2)$ lies on the graph.

$$\therefore f(-1) = 2$$

$$\therefore 3.5 - a^1 = 2$$

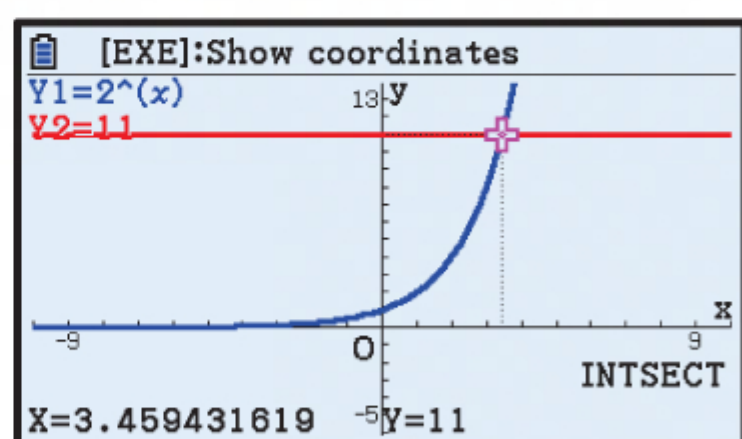
$$\therefore a = 1.5$$

c $f(x) = 3.5 - 1.5^{-x}$ has horizontal asymptote $y = 3.5$.



12 a $2^x = 11$

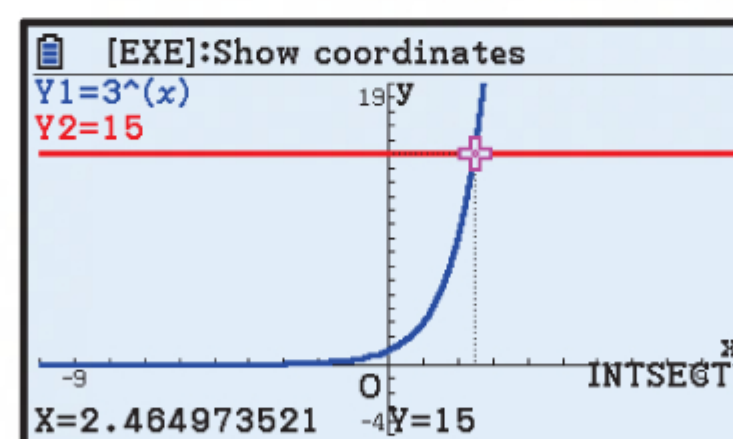
We graph $Y_1 = 2^x$ and $Y_2 = 11$ on the same set of axes and find their point of intersection.



The solution is $x \approx 3.46$.

b $3^x = 15$

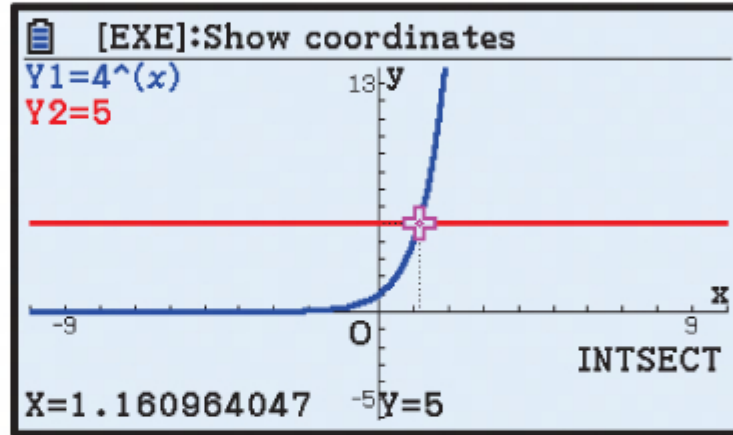
We graph $Y_1 = 3^x$ and $Y_2 = 15$ on the same set of axes and find their point of intersection.



The solution is $x \approx 2.46$.

c $4^x + 5 = 10$
 $\therefore 4^x = 5$

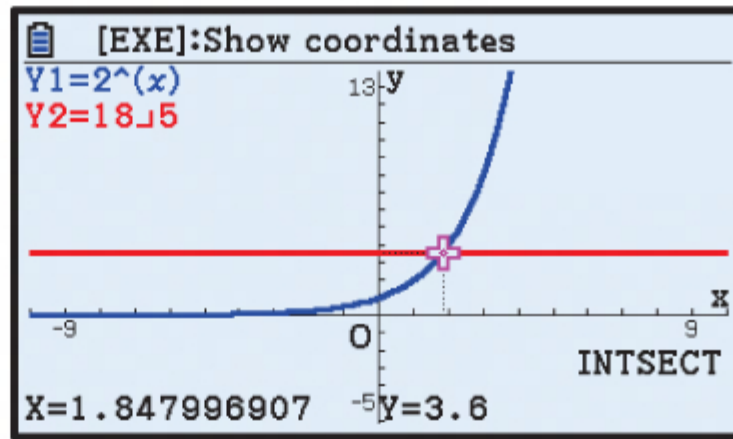
We graph $Y_1 = 4^x$ and $Y_2 = 5$ on the same set of axes and find their point of intersection.



The solution is $x \approx 1.16$.

e $5 \times 2^x = 18$
 $\therefore 2^x = \frac{18}{5}$

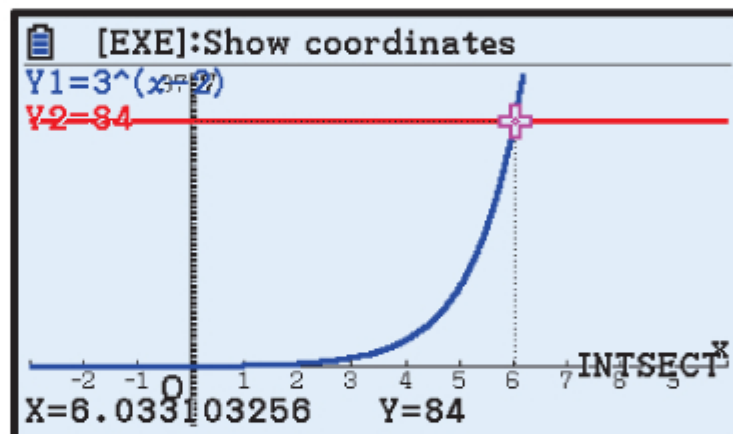
We graph $Y_1 = 2^x$ and $Y_2 = \frac{18}{5}$ on the same set of axes and find their point of intersection.



The solution is $x \approx 1.85$.

g $2 \times 3^{x-2} = 168$
 $\therefore 3^{x-2} = 84$

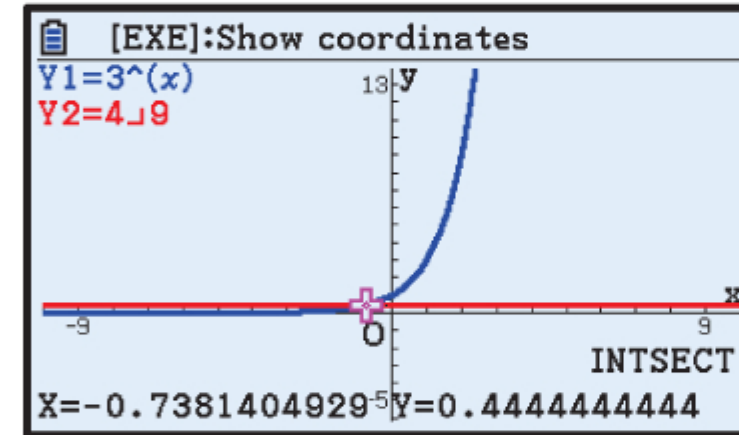
We graph $Y_1 = 3^{x-2}$ and $Y_2 = 84$ on the same set of axes and find their point of intersection.



The solution is $x \approx 6.03$.

d $3^{x+2} = 4$
 $\therefore 3^x \times 3^2 = 4$
 $\therefore 3^x = \frac{4}{9}$

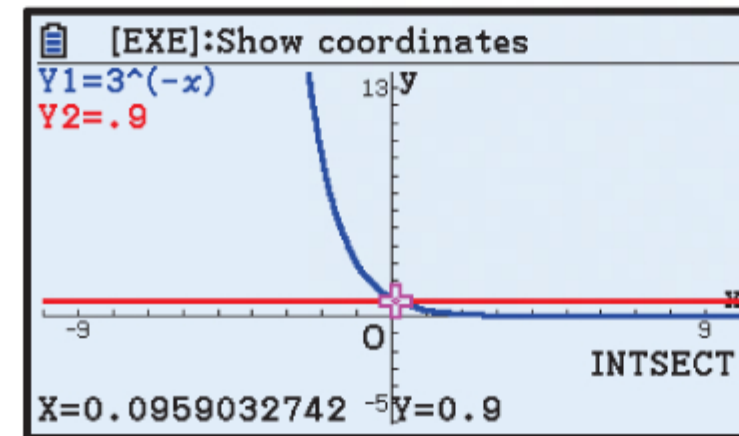
We graph $Y_1 = 3^x$ and $Y_2 = \frac{4}{9}$ on the same set of axes and find their point of intersection.



The solution is $x \approx -0.738$.

f $3^{-x} = 0.9$

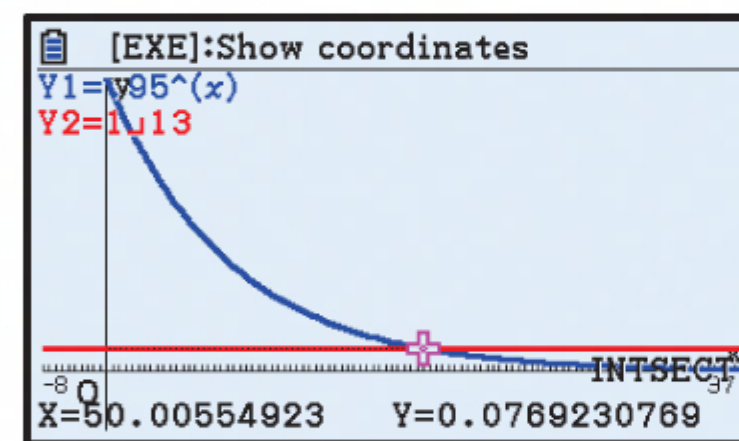
We graph $Y_1 = 3^{-x}$ and $Y_2 = 0.9$ on the same set of axes and find their point of intersection.



The solution is $x \approx 0.0959$.

h $26 \times (0.95)^x = 2$
 $\therefore (0.95)^x = \frac{1}{13}$

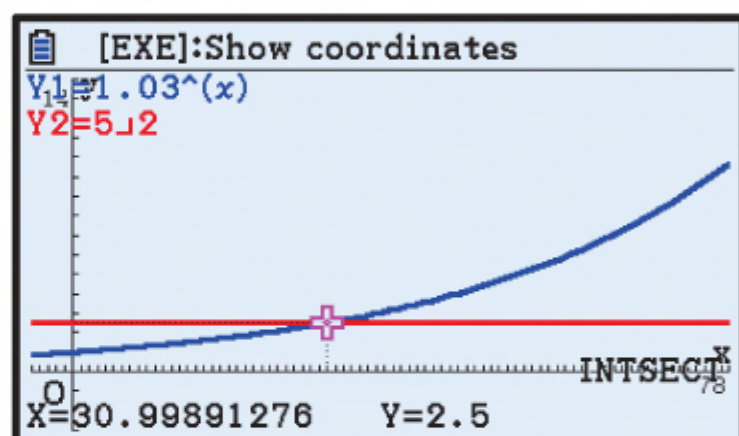
We graph $Y_1 = (0.95)^x$ and $Y_2 = \frac{1}{13}$ on the same set of axes and find their point of intersection.



The solution is $x \approx 50.0$.

$$\begin{aligned} \text{i } 2000 \times (1.03)^x &= 5000 \\ \therefore (1.03)^x &= \frac{5}{2} \end{aligned}$$

We graph $Y_1 = (1.03)^x$ and $Y_2 = \frac{5}{2}$ on the same set of axes and find their point of intersection.



The solution is $x \approx 31.0$.

EXERCISE 5E.1

$$\begin{aligned} \text{1 a } W(0) &= 100 \times (1.07)^0 \\ &= 100 \times 1 \\ &= 100 \end{aligned}$$

\therefore the initial weight was 100 grams.

$$\begin{aligned} \text{b i } W(4) &= 100 \times (1.07)^4 \\ &\approx 131 \end{aligned}$$

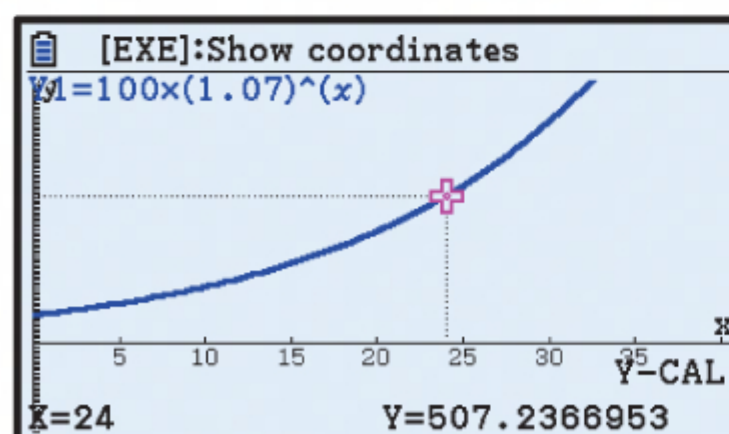
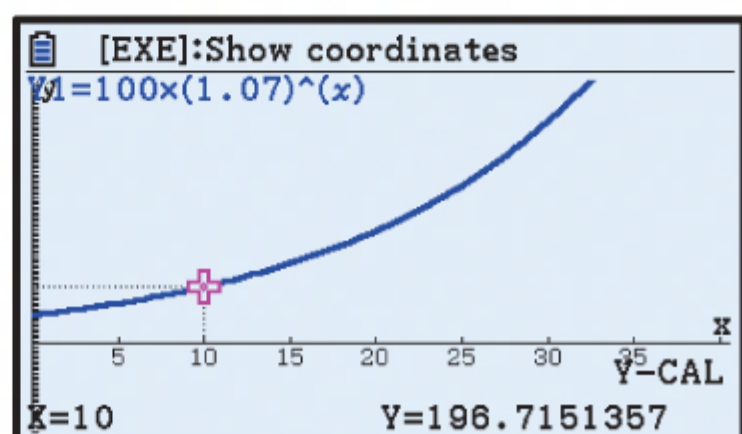
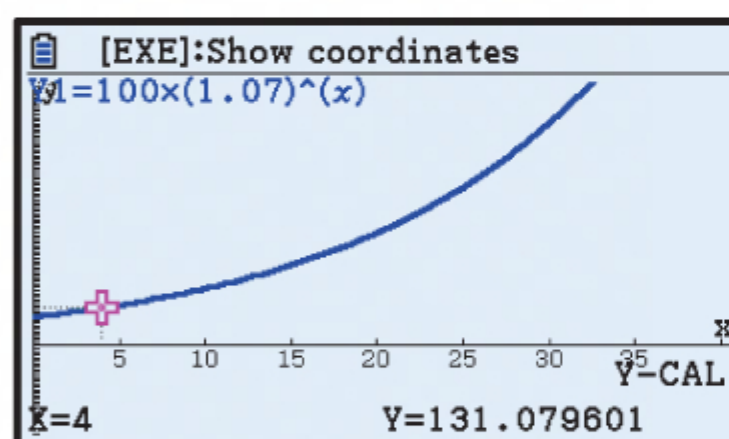
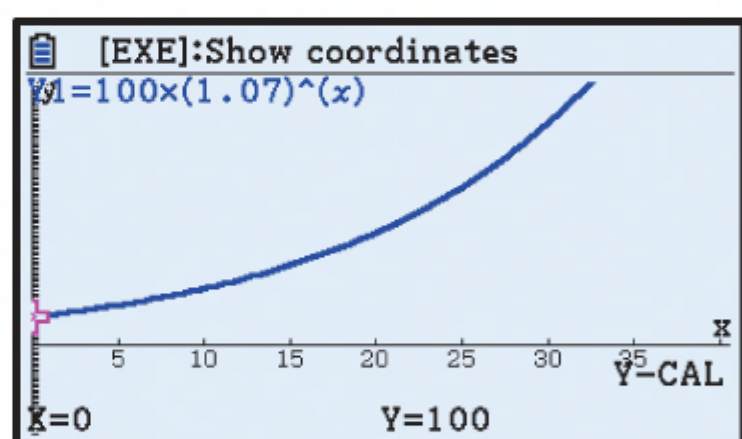
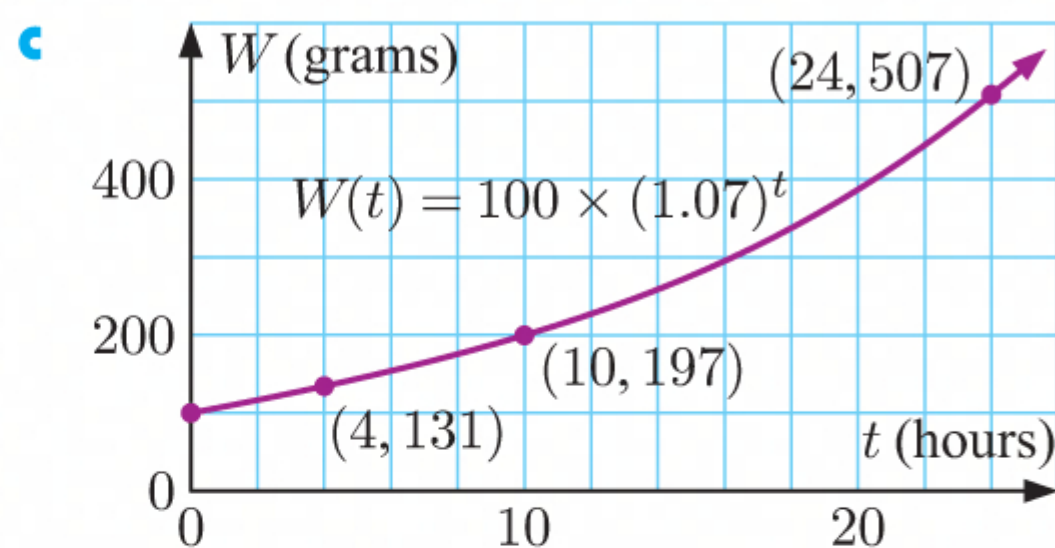
After 4 hours, the weight is about 131 g.

$$\begin{aligned} \text{ii } W(10) &= 100 \times (1.07)^{10} \\ &\approx 197 \end{aligned}$$

After 10 hours, the weight is about 197 g.

$$\begin{aligned} \text{iii } W(24) &= 100 \times (1.07)^{24} \\ &\approx 507 \end{aligned}$$

After 24 hours, the weight is about 507 g.



2 a $P_0 = 50$ possums

b $P(n) = 50 \times (1.23)^n$

i $P(2) = 50 \times (1.23)^2$
 ≈ 75.6

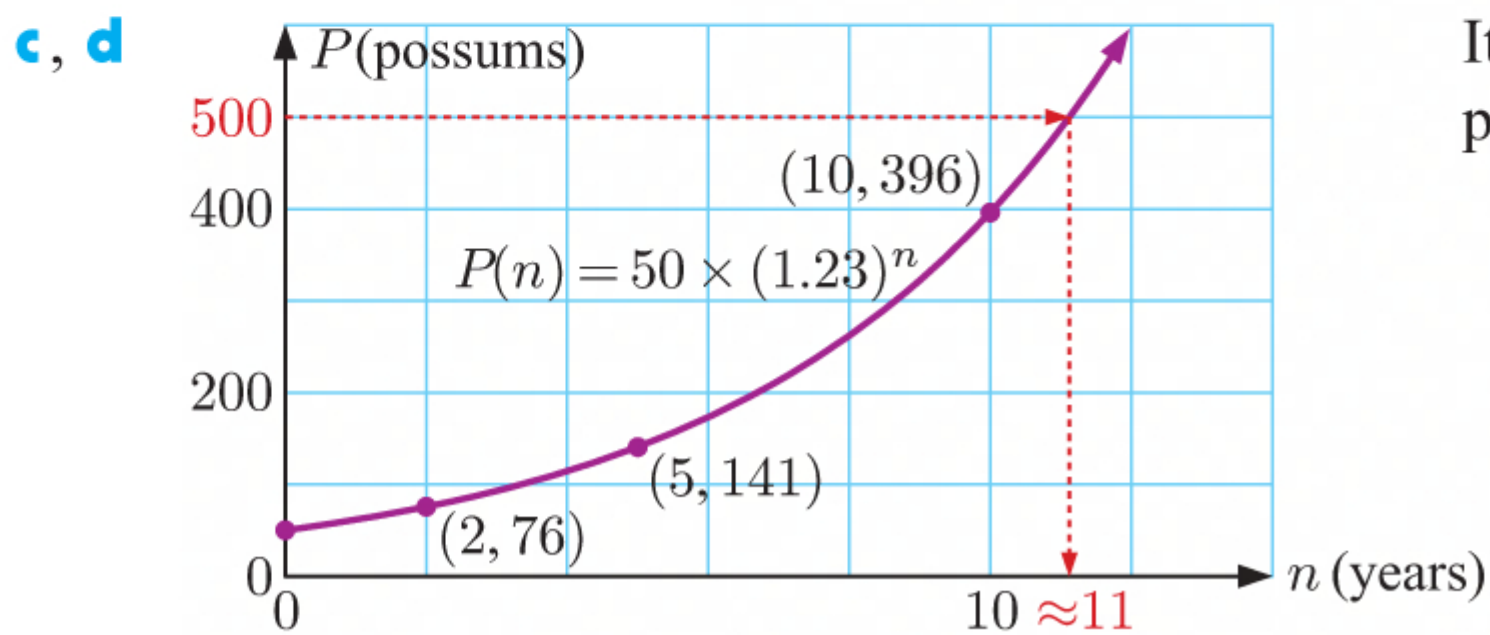
After 2 years, the expected population is about 76 possums.

ii $P(5) = 50 \times (1.23)^5$
 ≈ 141

After 5 years, the expected population is about 141 possums.

iii $P(10) = 50 \times (1.23)^{10}$
 ≈ 396

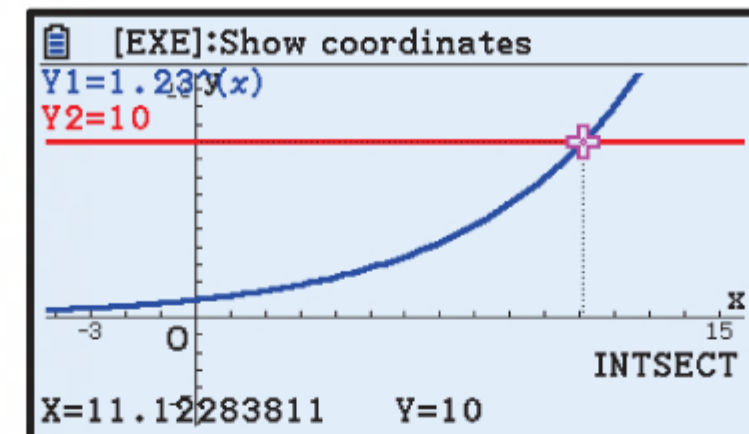
After 10 years, the expected population is about 396 possums.



It will take about 11 years for the population to reach 500.

e $P(n) = 500$
 $\therefore 50 \times (1.23)^n = 500$
 $\therefore (1.23)^n = 10$

We graph $Y_1 = (1.23)^x$ and $Y_2 = 10$ on the same set of axes and find their point of intersection.



The solution is $n \approx 11.1$.

It will take about 11.1 years for the population to reach 500.

3 a $B_0 = 12$ bears

b 2018 is 20 years after 1998, so $t = 20$.

$B(20) = 12 \times (1.13)^{20}$
 ≈ 138

The expected bear population in 2018 is about 138 bears.

c 2008 is 10 years after 1998, so $t = 10$.

$B(10) = 12 \times (1.13)^{10}$

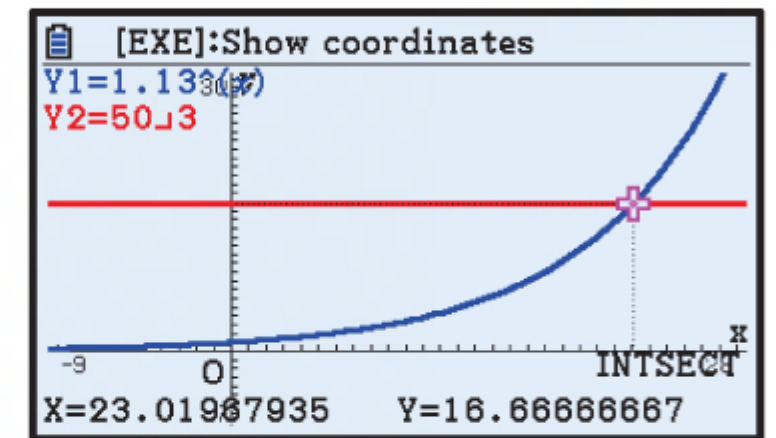
$$\begin{aligned} \text{Percentage increase from 2008 to 2018} &= \left(\frac{B(20) - B(10)}{B(10)} \right) \times 100\% \\ &= \left(\frac{12 \times (1.13)^{20} - 12 \times (1.13)^{10}}{12 \times (1.13)^{10}} \right) \times 100\% \\ &\approx 239\% \end{aligned}$$

d $B(t) = 200$

$$\therefore 12 \times (1.13)^t = 200$$

$$\therefore (1.13)^t = \frac{200}{12} = \frac{50}{3}$$

We graph $Y_1 = (1.13)^x$ and $Y_2 = \frac{50}{3}$ on the same set of axes and find their point of intersection.



The solution is $t \approx 23.0$.

It will take about 23.0 years for the population to reach 200.

4 a $N = 4 \times 1.332^t, t \geq 0$

$$\begin{aligned} \text{When } t = 0, \quad N &= 4 \times 1.332^0 \\ &= 4 \times 1 \\ &= 4 \end{aligned}$$

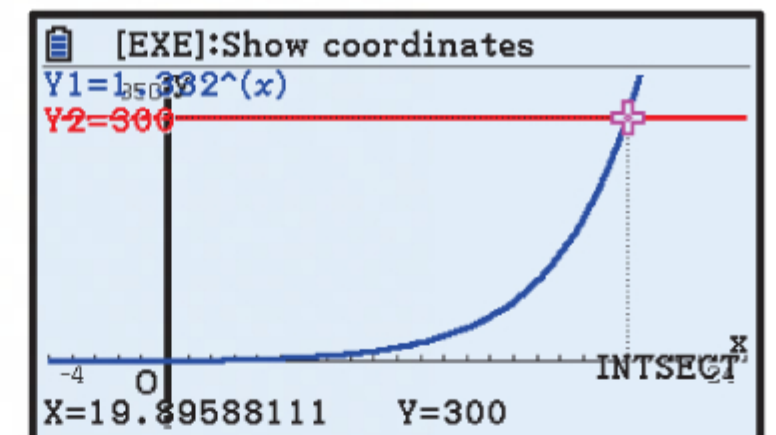
\therefore the number of people initially infected was 4.

c $N = 1200$

$$\therefore 4 \times 1.332^t = 1200$$

$$\therefore 1.332^t = 300$$

We graph $Y_1 = 1.332^x$ and $Y_2 = 300$ on the same set of axes and find their intersection.



The solution is $t \approx 19.9$.

It will take about 19.9 days for everybody in the school to catch the flu.

5 a i
$$\begin{aligned} V(0) &= V_0 \times 2^{0.05(0)} \\ &= V_0 \times 2^0 \\ &= V_0 \times 1 \\ &= V_0 \end{aligned}$$

So, the reaction speed at 0°C is V_0 .

ii
$$\begin{aligned} V(20) &= V_0 \times 2^{0.05(20)} \\ &= V_0 \times 2^1 \\ &= 2V_0 \end{aligned}$$

So, the reaction speed at 20°C is $2V_0$.

b Percentage increase at 20°C compared with $0^\circ\text{C} = \left(\frac{V(20) - V(0)}{V(0)} \right) \times 100\%$

$$\begin{aligned} &= \left(\frac{2V_0 - V_0}{V_0} \right) \times 100\% \\ &= \left(\frac{V_0}{V_0} \right) \times 100\% \\ &= 100\% \end{aligned}$$

So, there is a 100% increase in reaction speed at 20°C compared with 0°C .

$$\begin{aligned}
 \text{c } \left(\frac{V(50) - V(20)}{V(20)} \right) \times 100\% &= \left(\frac{V_0 \times 2^{0.05(50)} - 2V_0}{2V_0} \right) \times 100\% \\
 &= \left(\frac{V_0 \times 2^{2.5} - 2V_0}{2V_0} \right) \times 100\% \\
 &= \left(\frac{V_0(2^{2.5} - 2)}{V_0(2)} \right) \times 100\% \\
 &= \left(\frac{2^{2.5} - 2}{2} \right) \times 100\% \\
 &\approx 183\%
 \end{aligned}$$

This means that there is about a 183% increase in reaction speed at 50°C compared with 20°C.

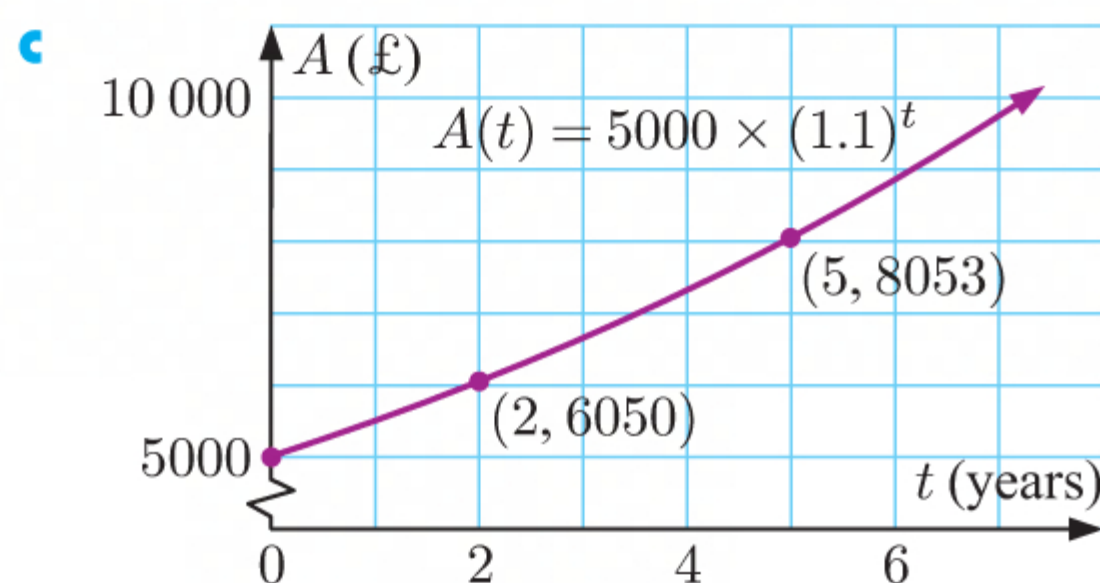
6 a $A(t) = 5000 \times (1.1)^t$, where t is the number of years since Kayla deposited £5000.

b i $A(2) = 5000 \times (1.1)^2$
 $= 6050$

So there was £6050 in the account after 2 years.

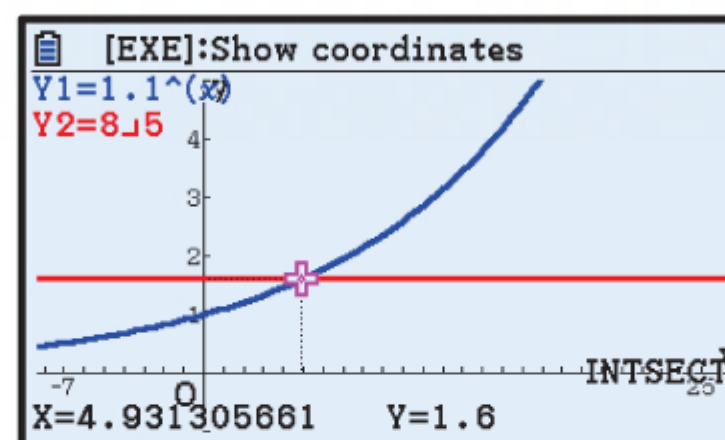
ii $A(5) = 5000 \times (1.1)^5$
 $= 8052.55$

So there was £8052.55 in the account after 5 years.



d $A(t) = 8000$
 $\therefore 5000 \times (1.1)^t = 8000$
 $\therefore (1.1)^t = \frac{8}{5}$

We graph $Y_1 = (1.1)^x$ and $Y_2 = \frac{8}{5}$ on the same set of axes and find their point of intersection.



The solution is $t \approx 4.93$.

It will take about 4.93 years for the amount in the account to reach £8000.

EXERCISE 5E.2

1 $W(t) = 250 \times (0.998)^t$ grams

a $W(0) = 250 \times (0.998)^0$
 $= 250 \times 1$
 $= 250$

\therefore there was initially 250 grams of radioactive substance set aside.

b i $W(400) = 250 \times (0.998)^{400}$
 ≈ 112

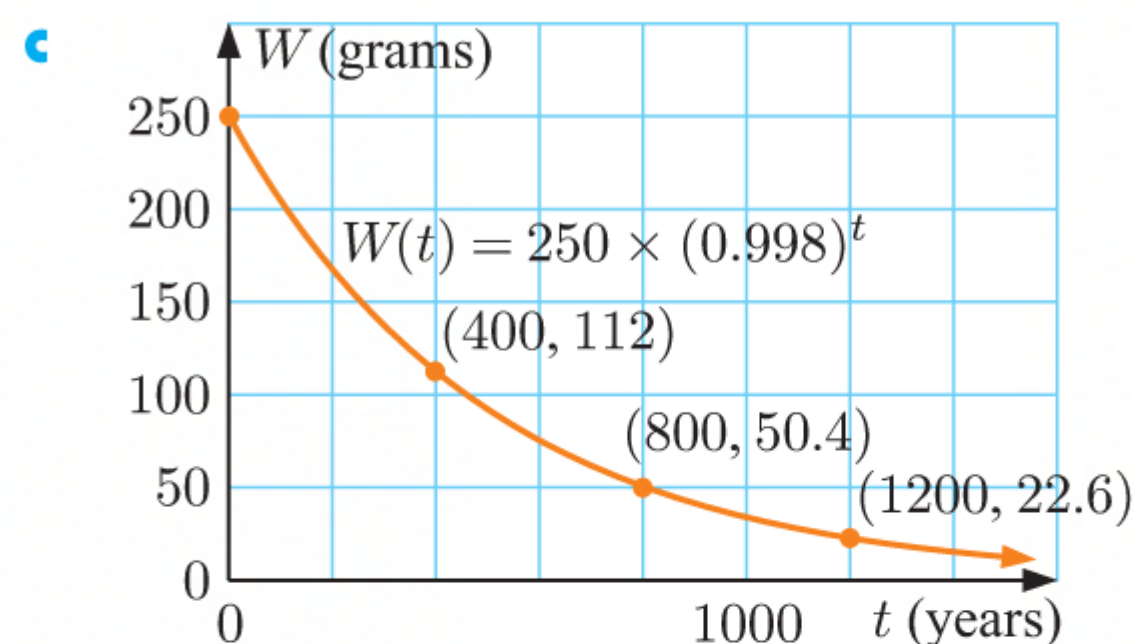
The weight was about 112 grams after 400 years.

ii $W(800) = 250 \times (0.998)^{800}$
 ≈ 50.4

The weight was about 50.4 grams after 800 years.

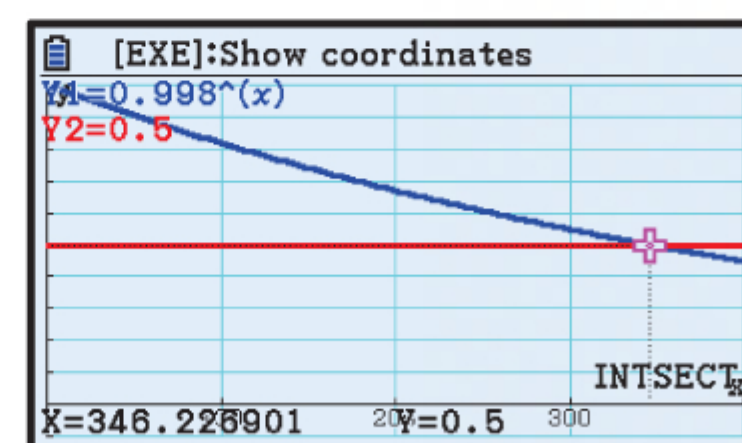
iii $W(1200) = 250 \times (0.998)^{1200}$
 ≈ 22.6

The weight was about 22.6 grams after 1200 years.



d $W(t) = 125$
 $\therefore 250 \times (0.998)^t = 125$
 $\therefore (0.998)^t = 0.5$
 $\therefore t \approx 346.2$ {using technology}

It takes approximately 346 years for the substance to decay to 125 grams.



2 $T(t) = 100 \times (0.986)^t$ °C

a $T(0) = 100 \times (0.986)^0$
 $= 100 \times 1$
 $= 100$

The initial temperature of the liquid was 100°C.

b i $T(15) = 100 \times (0.986)^{15}$
 ≈ 80.9

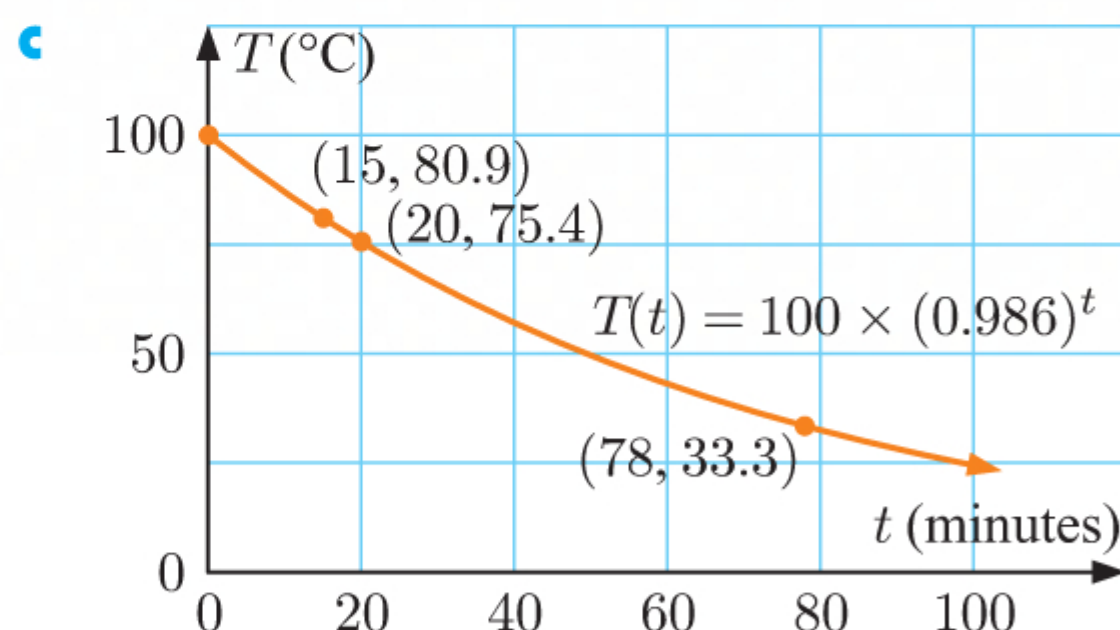
The temperature was about 80.9°C after 15 minutes.

ii $T(20) = 100 \times (0.986)^{20}$
 ≈ 75.4

The temperature was about 75.4°C after 20 minutes.

iii $T(78) = 100 \times (0.986)^{78}$
 ≈ 33.3

The temperature was about 33.3°C after 78 minutes.



3 $W(t) = 1000 \times (0.979)^t$ grams

a $W(0) = 1000 \times (0.979)^0$
 $= 1000 \times 1$
 $= 1000$

The initial weight of the radioactive substance was 1000 grams.

b i $W(10) = 1000 \times (0.979)^{10}$
 ≈ 809

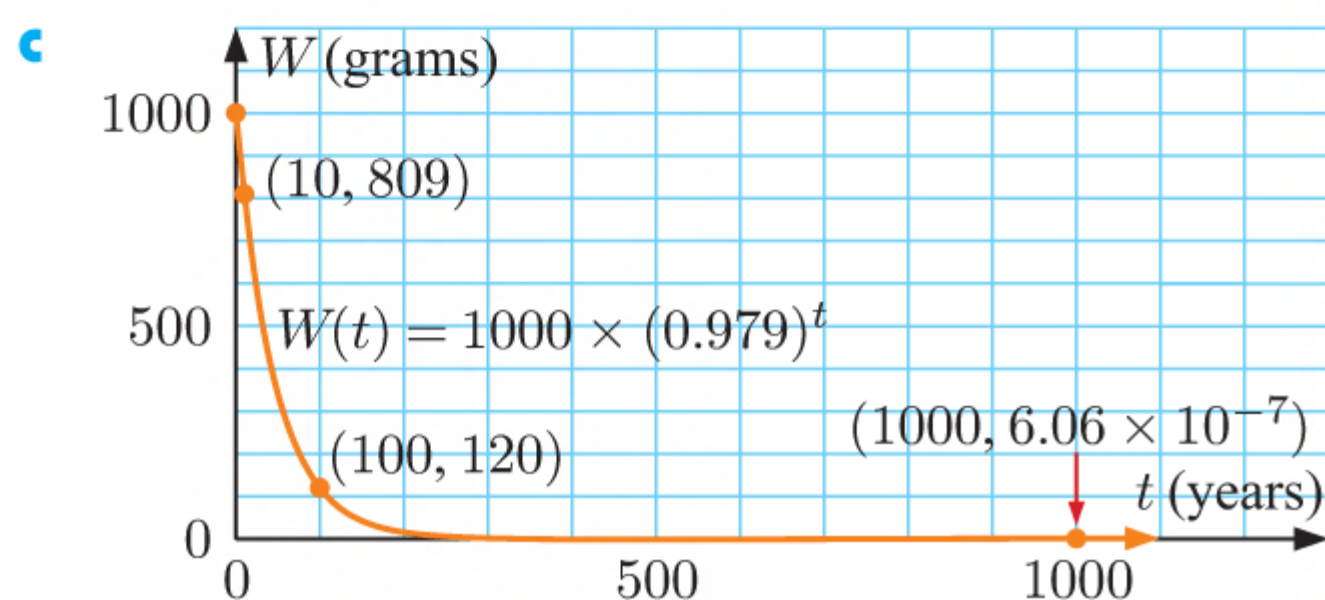
The weight remaining after 10 years was about 809 grams.

ii $W(100) = 1000 \times (0.979)^{100}$
 ≈ 120

The weight remaining after 100 years was about 120 grams.

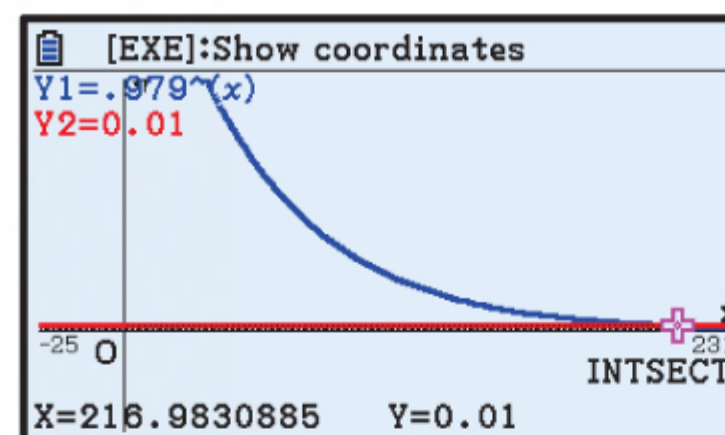
iii $W(1000) = 1000 \times (0.979)^{1000}$
 $\approx 6.06 \times 10^{-7}$

The weight remaining after 1000 years was about 6.06×10^{-7} grams.



d $W(t) = 10$
 $\therefore 1000 \times (0.979)^t = 10$
 $\therefore (0.979)^t = 0.01$
 $\therefore t \approx 217$ {using technology}

10 g of the substance remains after about 217 years.



e Amount remaining after t years $= W(t) = 1000 \times (0.979)^t$ grams.

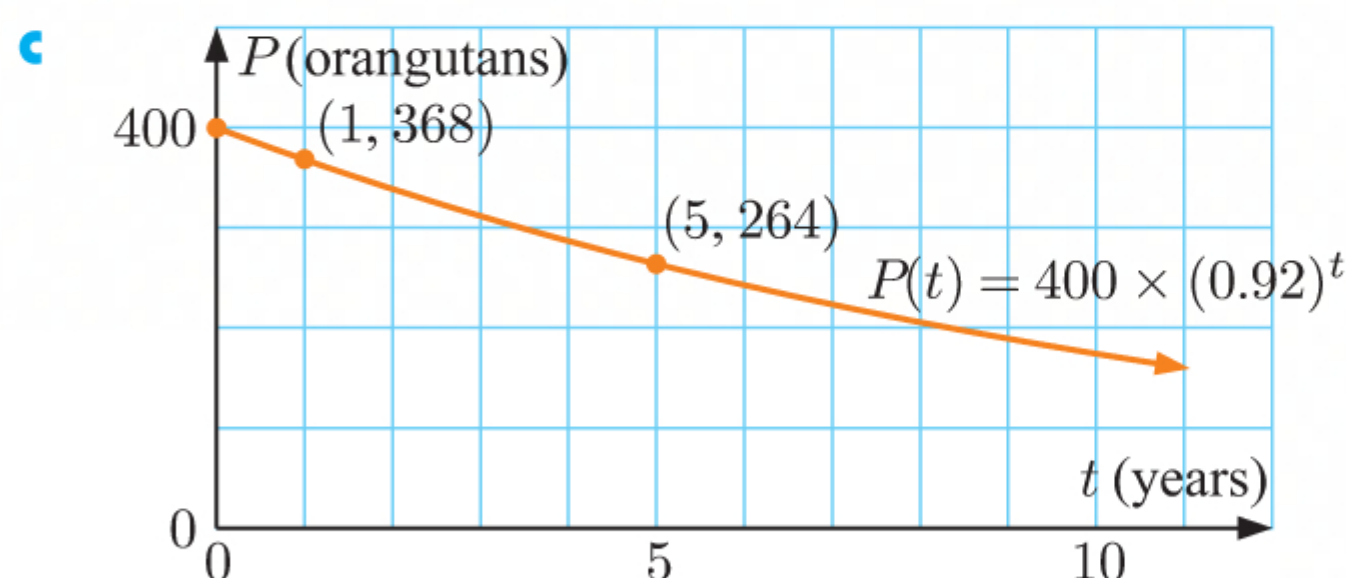
Amount that has decayed after t years $= W(0) - W(t)$
 $= 1000 - 1000 \times (0.979)^t$
 $= 1000(1 - 0.979^t)$ grams

4 a $P(t) = 400 \times (0.92)^t$ orangutans

b i $P(1) = 400 \times (0.92)^1$
 $= 368$
 There were 368 orangutans after 1 year.

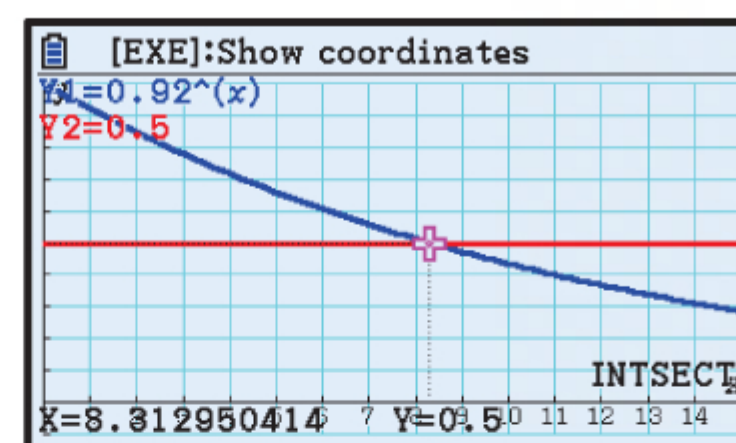
ii $P(5) = 400 \times (0.92)^5$
 ≈ 264

There were about 264 orangutans after 5 years.



$$\begin{aligned}
 \text{d} \quad & P(t) = 200 \\
 \therefore & 400 \times (0.92)^t = 200 \\
 \therefore & (0.92)^t = 0.5 \\
 \therefore & t \approx 8.31 \quad \{\text{using technology}\}
 \end{aligned}$$

The population will fall to 200 after about 8.31 years, or about 8 years and 4 months.



$$5 \quad L = L_0 \times (0.95)^d \text{ units}$$

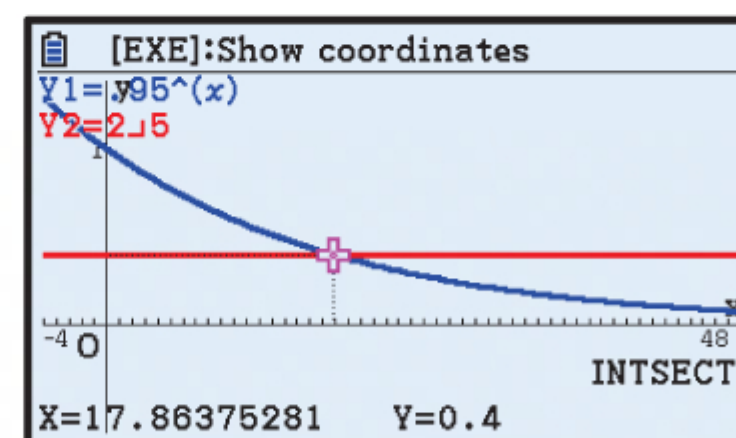
$$\begin{aligned}
 \text{a} \quad & \text{When } d = 0, \quad L = 10 \\
 \therefore & 10 = L_0 \times (0.95)^0 \\
 \therefore & L_0 = 10 \text{ units}
 \end{aligned}$$

$$\begin{aligned}
 \text{b} \quad & \text{When } d = 25, \quad L = 10 \times (0.95)^{25} \\
 & \approx 2.77
 \end{aligned}$$

\therefore the intensity of light 25 m below the surface is about 2.77 units.

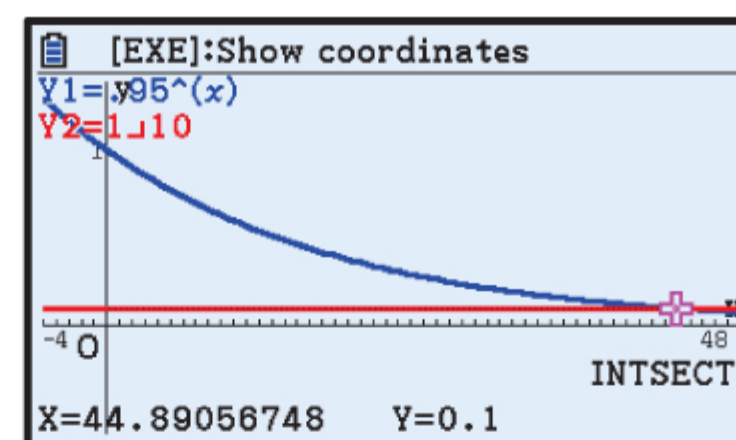
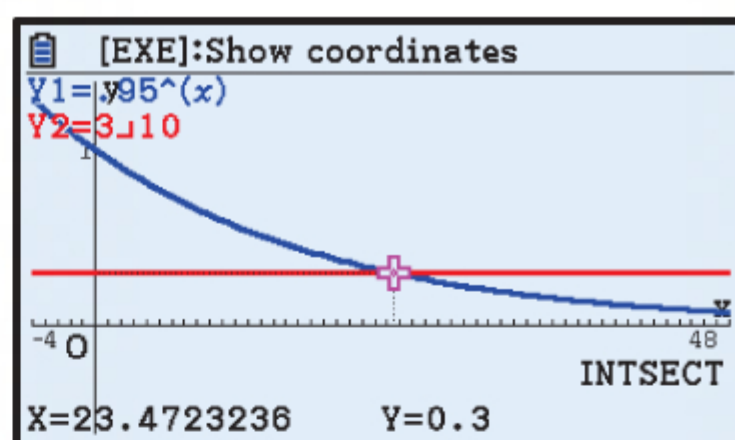
$$\begin{aligned}
 \text{c} \quad & L = 4 \\
 \therefore & 10 \times (0.95)^d = 4 \\
 \therefore & (0.95)^d = \frac{2}{5} \\
 \therefore & d \approx 17.9 \quad \{\text{using technology}\}
 \end{aligned}$$

A depth of about 17.9 m has a light intensity of 4 units.



$$\begin{aligned}
 \text{d} \quad & L = 3 \\
 \therefore & 10 \times (0.95)^d = 3 \\
 \therefore & (0.95)^d = \frac{3}{10} \\
 \therefore & d \approx 23.5 \quad \{\text{using technology}\}
 \end{aligned}$$

$$\begin{aligned}
 & L = 1 \\
 \therefore & 10 \times (0.95)^d = 1 \\
 \therefore & (0.95)^d = \frac{1}{10} \\
 \therefore & d \approx 44.9 \quad \{\text{using technology}\}
 \end{aligned}$$



A depth between approximately 23.5 m and 44.9 m will have a light intensity between 1 and 3 units.

$$6 \quad V = 24\,000 \times r^t \text{ dollars}$$

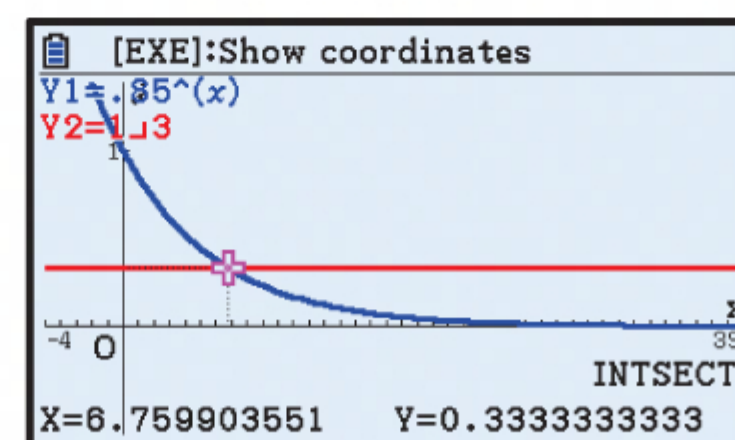
$$\begin{aligned}
 \text{a} \quad & \text{When } t = 0, \quad V = 24\,000 \times r^0 \\
 & = 24\,000 \times 1 \\
 & = 24\,000
 \end{aligned}$$

The value of the car when it was first purchased was \$24 000.

$$\begin{aligned}
 \text{b} \quad & \text{When } t = 1, \quad V = 20\,400 \\
 \therefore & 24\,000 \times r^1 = 20\,400 \\
 \therefore & r = 0.85
 \end{aligned}$$

$$\begin{aligned}
 \text{c} \quad & V = 8000 \\
 \therefore & 24\,000 \times (0.85)^t = 8000 \\
 \therefore & (0.85)^t = \frac{1}{3} \\
 \therefore & t \approx 6.76 \quad \{\text{using technology}\}
 \end{aligned}$$

To the nearest year, it will take about 7 years for the value of the car to reduce to \$8000.



7 $T(t) = -10 + 32 \times 2^{-0.2t} \text{ } ^\circ\text{C}$

a i $T(0) = -10 + 32 \times 2^{-0.2(0)}$
 $= -10 + 32 \times 2^0$
 $= -10 + 32$
 $= 22$

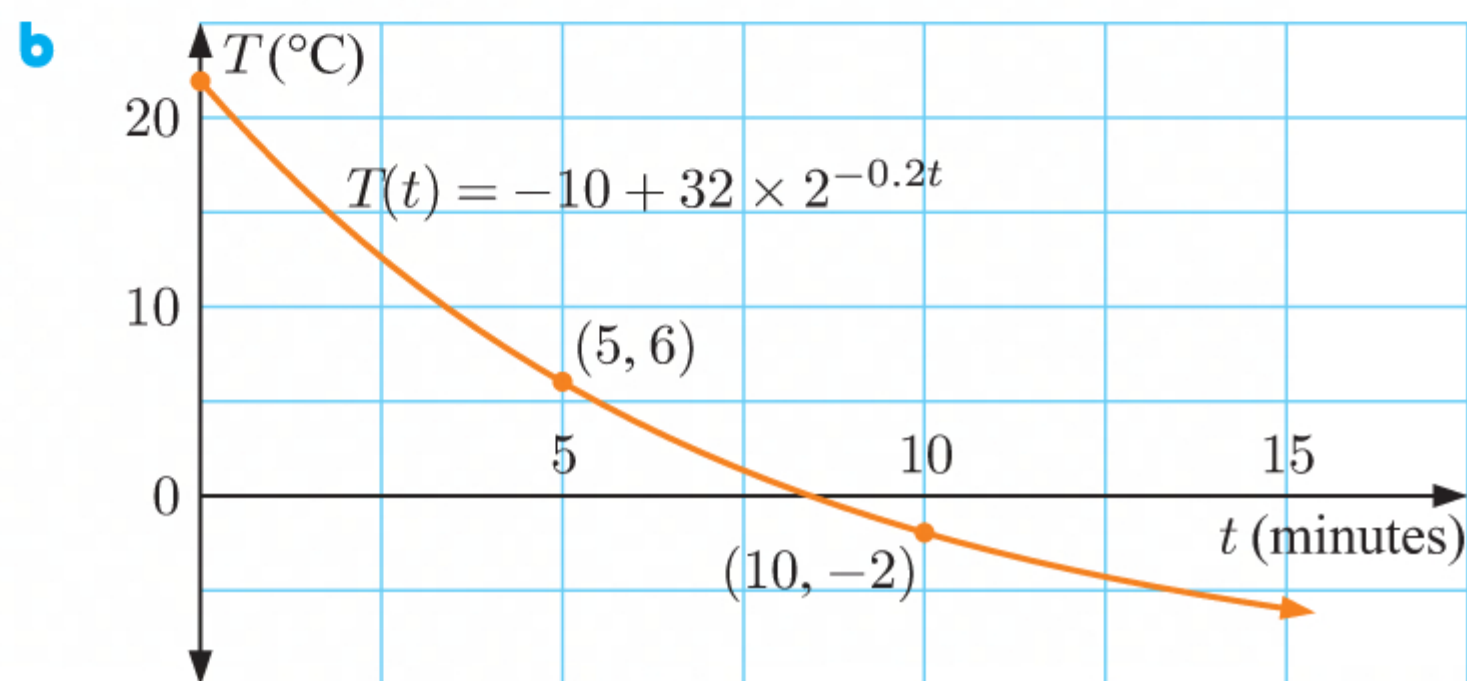
The temperature of the peas was 22°C when placed in the freezer.

iii $T(10) = -10 + 32 \times 2^{-0.2(10)}$
 $= -10 + 32 \times 2^{-2}$
 $= -10 + 8$
 $= -2$

The temperature of the peas was -2°C after 10 minutes.

ii $T(5) = -10 + 32 \times 2^{-0.2(5)}$
 $= -10 + 32 \times 2^{-1}$
 $= -10 + 16$
 $= 6$

The temperature of the peas was 6°C after 5 minutes.



c

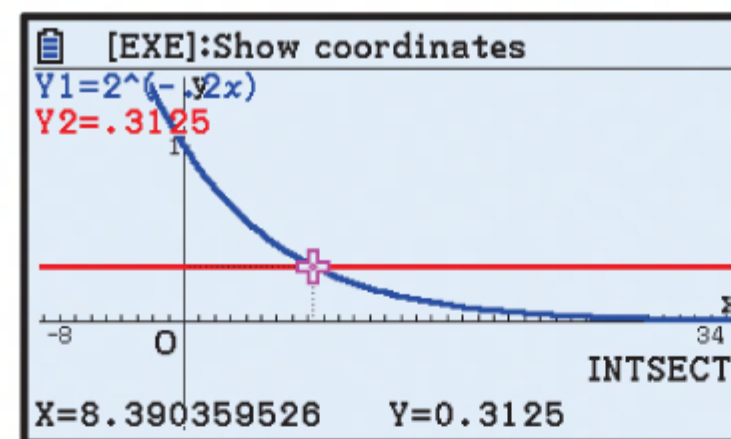
$$T(t) = 0$$

$$\therefore -10 + 32 \times 2^{-0.2t} = 0$$

$$\therefore 32 \times 2^{-0.2t} = 10$$

$$\therefore 2^{-0.2t} = 0.3125$$

$$\therefore t \approx 8.39 \quad \{\text{using technology}\}$$



It takes about 8.39 minutes, or about 8 minutes and 23 seconds, for the temperature of the peas to fall to 0°C .

d $T(t) = -10 + 32 \times 2^{-0.2t}$

Now, $32 \times 2^{-0.2t} > 0$ for all t since $2^{-0.2t} > 0$ for all t .

$$\therefore -10 + 32 \times 2^{-0.2t} > -10 \text{ for all } t.$$

\therefore the temperature of the packet of peas will never reach -10°C .

8 $W_t = W_0 \times 2^{-0.0002t} \text{ grams}$

a When $t = 0$, $W_0 = W_0 \times 2^0$
 $= W_0$

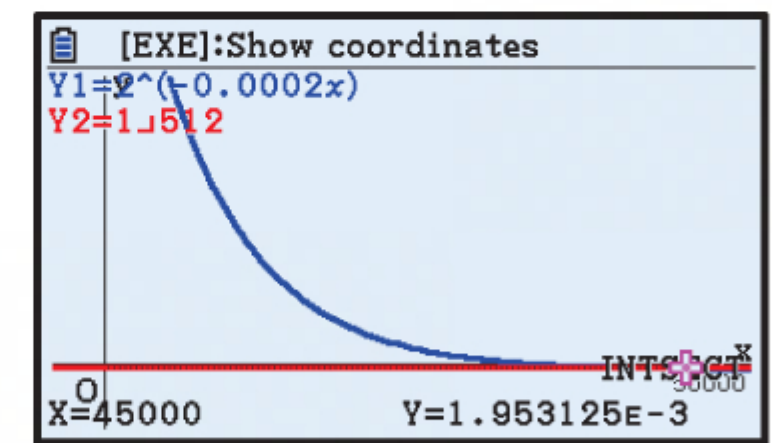
\therefore the original weight was W_0 grams.

$$\begin{aligned}
 \text{b } \left(\frac{W_{1000} - W_0}{W_0} \right) \times 100\% &= \left(\frac{W_0 \times 2^{-0.0002(1000)} - W_0}{W_0} \right) \times 100\% \\
 &= \left(\frac{W_0 \times 2^{-0.2} - W_0}{W_0} \right) \times 100\% \\
 &= \left(\frac{W_0(2^{-0.2} - 1)}{W_0} \right) \times 100\% \\
 &= (2^{-0.2} - 1) \times 100\% \\
 &\approx -12.9\%
 \end{aligned}$$

The percentage weight loss after 1000 years was about 12.9%.

$$\begin{aligned}
 \text{c } W_0 \times 2^{-0.0002t} &= \frac{1}{512} W_0 \\
 \therefore (2^{-0.0002})^t &= \frac{1}{512} \\
 \therefore t &= 45\,000 \quad \{\text{using technology}\}
 \end{aligned}$$

It will take 45 000 years until $\frac{1}{512}$ of the sample remains.



INVESTIGATION 2

CONTINUOUS COMPOUND INTEREST

$$1 \quad u_n = u_0(1+i)^n, \quad u_0 = 1000$$

a Interest paid annually:

$$n = 1, \quad i = 6\% = 0.06$$

$$\begin{aligned}
 \therefore u_1 &= 1000(1 + 0.06)^1 \\
 &= 1060
 \end{aligned}$$

The final amount is \$1060.

c Interest paid monthly:

$$n = 12, \quad i = \frac{6\%}{12} = 0.005$$

$$\begin{aligned}
 \therefore u_{12} &= 1000(1 + 0.005)^{12} \\
 &\approx 1061.68
 \end{aligned}$$

The final amount is \$1061.68.

e Interest paid by the second:

$$n = 365.25 \times 24 \times 60 \times 60 = 31\,557\,600, \quad i = \frac{6\%}{31\,557\,600}$$

$$\begin{aligned}
 \therefore u_{31\,557\,600} &= 1000 \left(1 + \frac{0.06}{31\,557\,600} \right)^{31\,557\,600} \\
 &\approx 1061.84
 \end{aligned}$$

The final amount is \$1061.84.

b Interest paid quarterly:

$$n = 4, \quad i = \frac{6\%}{4} = 0.015$$

$$\begin{aligned}
 \therefore u_4 &= 1000(1 + 0.015)^4 \\
 &\approx 1061.36
 \end{aligned}$$

The final amount is \$1061.36.

d Interest paid daily:

$$n = 365.25, \quad i = \frac{6\%}{365.25}$$

$$\begin{aligned}
 \therefore u_{365.25} &= 1000 \left(1 + \frac{0.06}{365.25} \right)^{365.25} \\
 &\approx 1061.83
 \end{aligned}$$

The final amount is \$1061.83.

f Interest paid by the millisecond:

$$n = 31\,557\,600 \times 1000 = 31\,557\,600\,000, \quad i = \frac{6\%}{31\,557\,600\,000}$$

$$\therefore u_{31\,557\,600\,000} = 1000 \left(1 + \frac{0.06}{31\,557\,600\,000} \right)^{31\,557\,600\,000} \approx 1061.84$$

The final amount is \$1061.84.

Given a fixed interest rate per annum, paying out the interest more frequently results in a higher final amount, but seems to approach a particular value.

$$\begin{aligned} \mathbf{2} \quad u_n &= u_0(1+i)^n \\ &= u_0 \left(1 + \frac{r}{N} \right)^{Nt} \quad \left\{ i = \frac{r}{N}, \quad n = Nt \right\} \\ &= u_0 \left(1 + \frac{1}{a} \right)^{art} \quad \left\{ a = \frac{N}{r}, \quad N = ar \right\} \\ &= u_0 \left[\left(1 + \frac{1}{a} \right)^a \right]^{rt} \end{aligned}$$

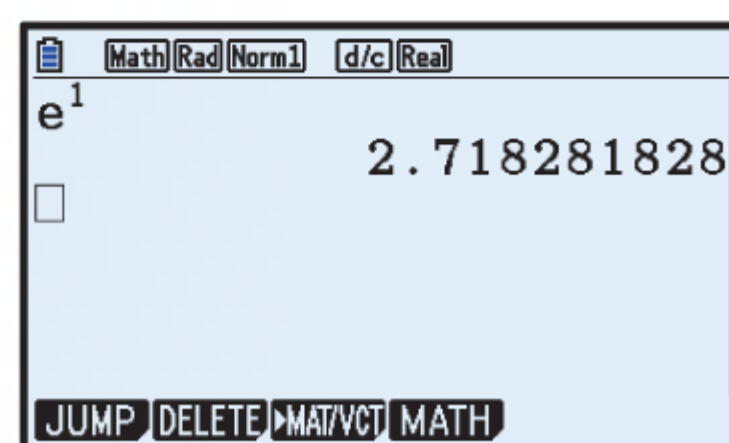
$$\mathbf{3} \quad \mathbf{a} \quad a = \frac{N}{r}, \text{ so as } N \rightarrow \infty, \quad a \rightarrow \infty$$

b

a	$\left(1 + \frac{1}{a}\right)^a$
10	2.593 724 46
100	2.704 813 829
1000	2.716 923 932
10 000	2.718 145 927
100 000	2.718 268 237
1 000 000	2.718 280 469
10 000 000	2.718 281 693

$$\mathbf{4} \quad e^1 \approx 2.718\,281\,828$$

This appears to be the value of $\left(1 + \frac{1}{a}\right)^a$ as $a \rightarrow \infty$.



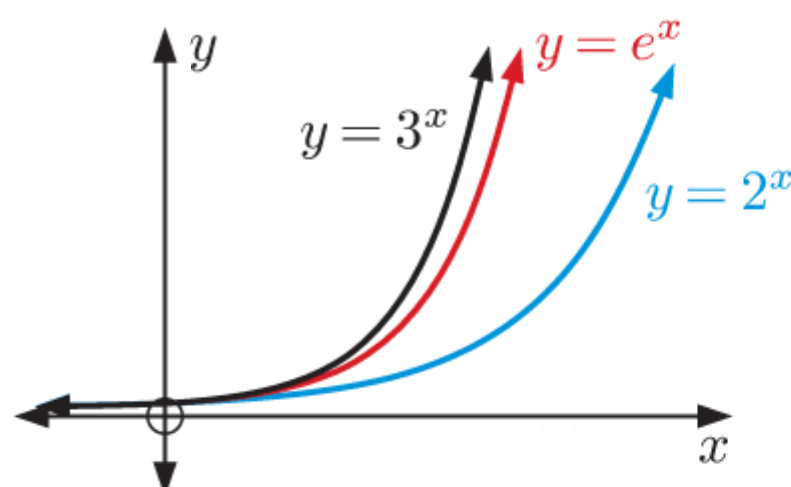
$$\mathbf{5} \quad u_n = u_0 e^{rt}, \quad u_0 = 1000, \quad r = 0.06, \quad t = 1$$

$$\therefore u_n = 1000 \times e^{0.06 \times 1} \approx 1061.84$$

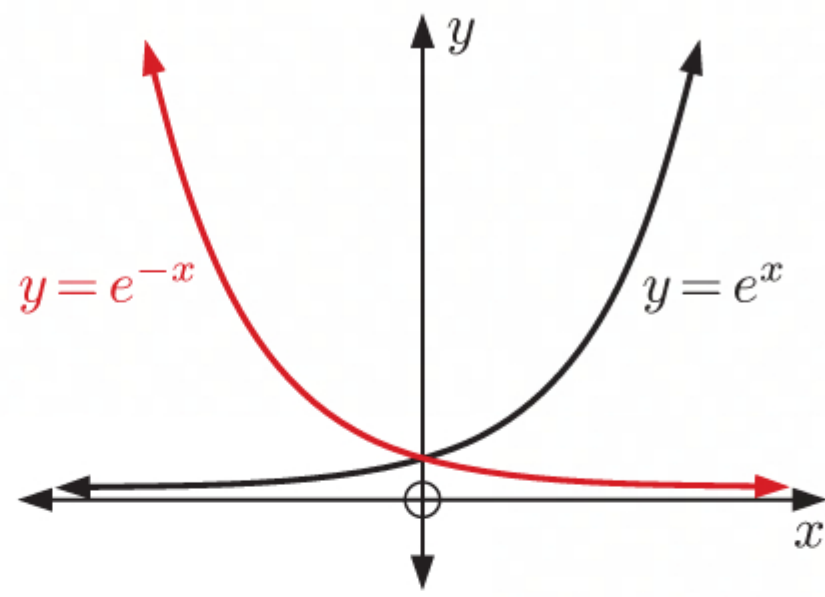
The final amount is \$1061.84.

EXERCISE 5F

1



The graph of $y = e^x$ lies between $y = 2^x$ and $y = 3^x$.

2One is the other reflected in the y -axis.

3 When $x = 0$, $y = pe^0 = p \times 1 = p$
 \therefore the y -intercept is p .

4 a $e^x > 0$ for all x
 $\therefore 2e^x > 0$ for all x
 $\therefore y = 2e^x$ cannot be negative.

b i When $x = -20$, $y = 2e^{-20}$
 $\approx 4.12 \times 10^{-9}$
 $\approx 0.000\,000\,004\,12$

ii When $x = 20$, $y = 2e^{20}$
 $\approx 9.70 \times 10^8$
 $\approx 970\,000\,000$

5 a $e^2 \approx 7.39$ **b** $e^3 \approx 20.1$ **c** $e^{0.7} \approx 2.01$ **d** $\sqrt{e} \approx 1.65$
e $e^{-1} \approx 0.368$

6 a $\sqrt{e} = e^{\frac{1}{2}}$ **b** $\frac{1}{\sqrt{e}} = \frac{1}{e^{\frac{1}{2}}} = e^{-\frac{1}{2}}$ **c** $\frac{1}{e^2} = e^{-2}$ **d** $e\sqrt{e} = e^1 \times e^{\frac{1}{2}} = e^{\frac{3}{2}}$

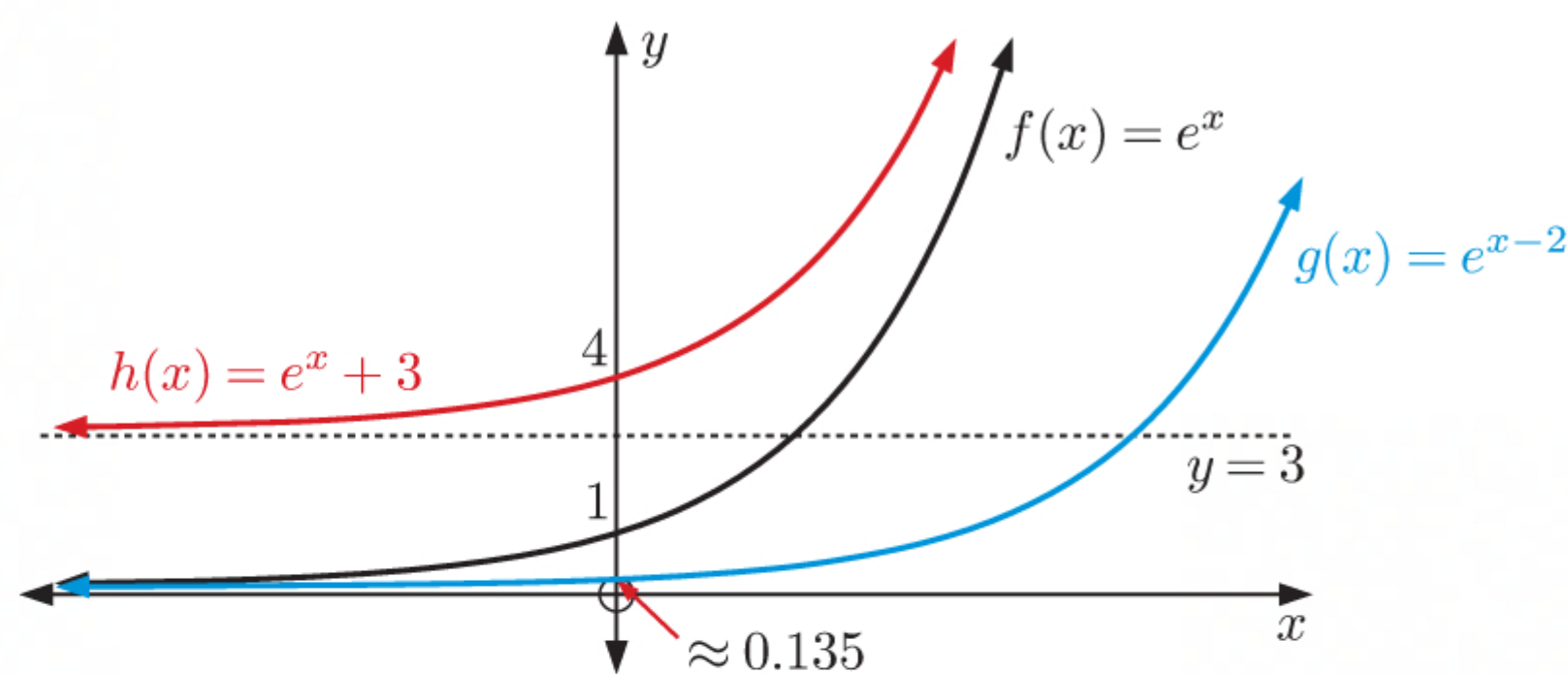
7 a $e^{2.31} \approx 10.074$ **b** $e^{-2.31} \approx 0.099\,261$ **c** $e^{4.829} \approx 125.09$
d $e^{-4.829} \approx 0.007\,994\,5$ **e** $50e^{-0.1764} \approx 41.914$ **f** $80e^{-0.6342} \approx 42.429$
g $1000e^{1.2642} \approx 3540.3$ **h** $0.25e^{-3.6742} \approx 0.006\,342\,4$

8 a $(e^x + 1)^2 = (e^x)^2 + 2 \times e^x \times 1 + 1^2$
 $= e^{2x} + 2e^x + 1$ **b** $(1 + e^x)(1 - e^x) = 1^2 - (e^x)^2$
 $= 1 - e^{2x}$

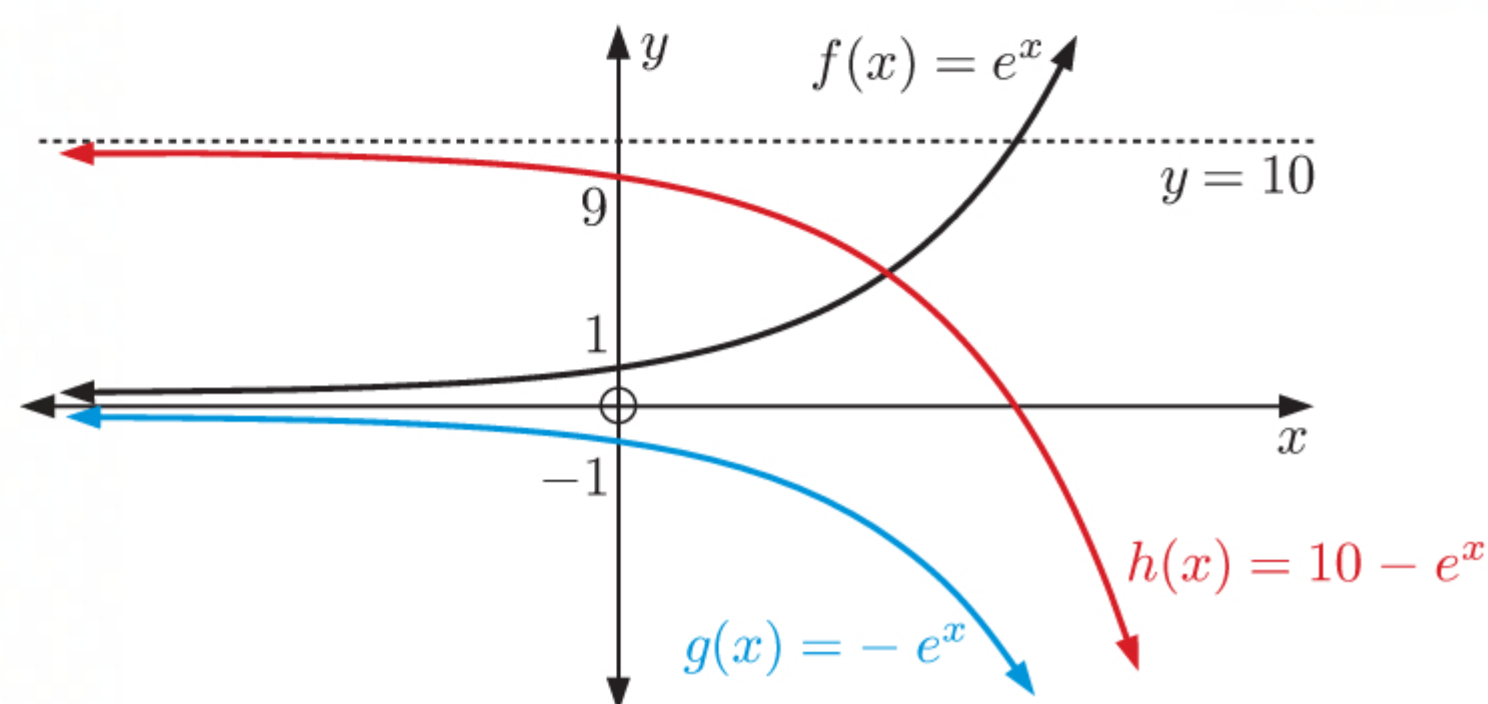
c $e^x(e^{-x} - 3) = e^x \times e^{-x} - e^x \times 3$
 $= e^0 - 3e^x$
 $= 1 - 3e^x$

9 a $e^{2x} + e^x = e^x \times e^x + e^x$
 $= e^x(e^x + 1)$ **b** $e^{2x} - 16 = (e^x)^2 - 4^2$
 $= (e^x + 4)(e^x - 4)$

c $e^{2x} - 8e^x + 12 = (e^x - 2)(e^x - 6)$ $\{a^2 - 8a + 12 = (a - 2)(a - 6)\}$

10 a

- b** The domain of f , g , and h is $\{x \mid x \in \mathbb{R}\}$.
 The range of f is $\{y \mid y > 0\}$. The range of g is $\{y \mid y > 0\}$.
 The range of h is $\{y \mid y > 3\}$.

11 a

- b** The domain of f , g , and h is $\{x \mid x \in \mathbb{R}\}$.
 The range of f is $\{y \mid y > 0\}$. The range of g is $\{y \mid y < 0\}$.
 The range of h is $\{y \mid y < 10\}$.
- c** For f : as $x \rightarrow \infty$, $y \rightarrow \infty$
 as $x \rightarrow -\infty$, $y \rightarrow 0^+$
- For g : as $x \rightarrow \infty$, $y \rightarrow -\infty$
 as $x \rightarrow -\infty$, $y \rightarrow 0^-$
- For h : as $x \rightarrow \infty$, $y \rightarrow -\infty$
 as $x \rightarrow -\infty$, $y \rightarrow 10^-$

12 $W(t) = 2e^{\frac{t}{2}}$ grams

a i $W(0) = 2e^0$
 $= 2 \times 1$
 $= 2$

The weight of the culture is 2 grams initially.

iii $W(1\frac{1}{2}) = 2e^{\frac{3}{4}}$
 ≈ 4.23

The weight of the culture is about 4.23 grams after $1\frac{1}{2}$ hours.

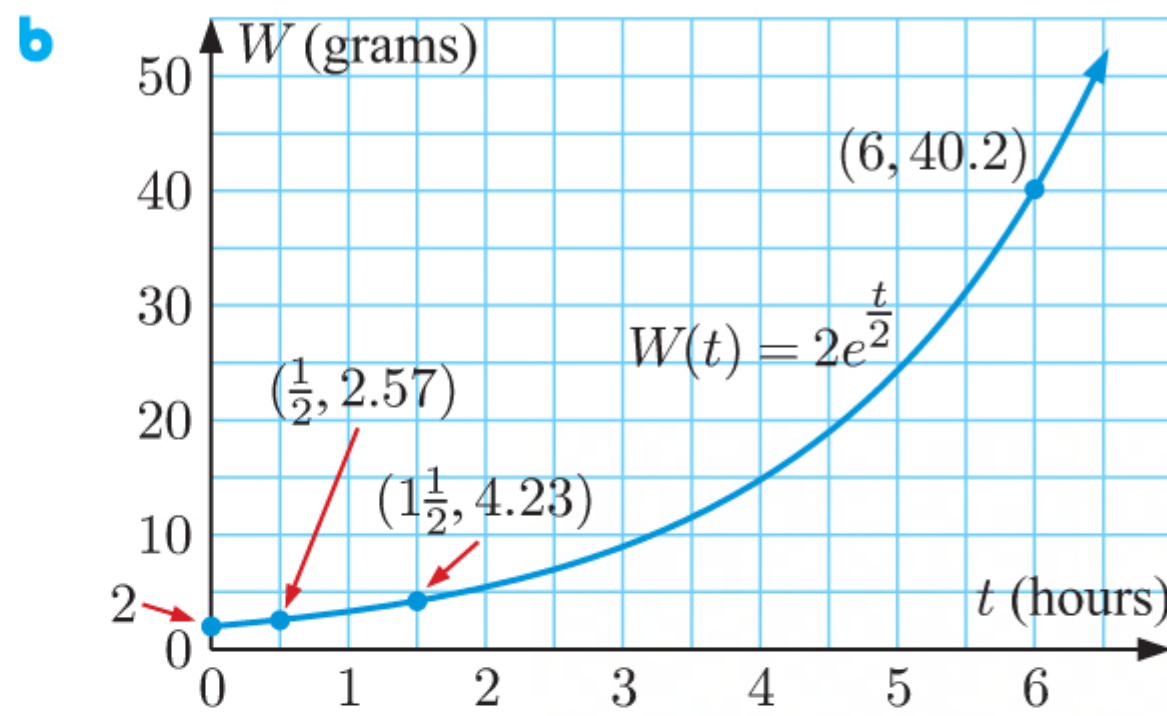
ii $t = 30 \text{ min} = \frac{1}{2} \text{ hour}$

$W(\frac{1}{2}) = 2e^{\frac{1}{4}}$
 ≈ 2.57

The weight of the culture is about 2.57 grams after 30 minutes.

iv $W(6) = 2e^3$
 ≈ 40.2

The weight of the culture is about 40.2 grams after 6 hours.

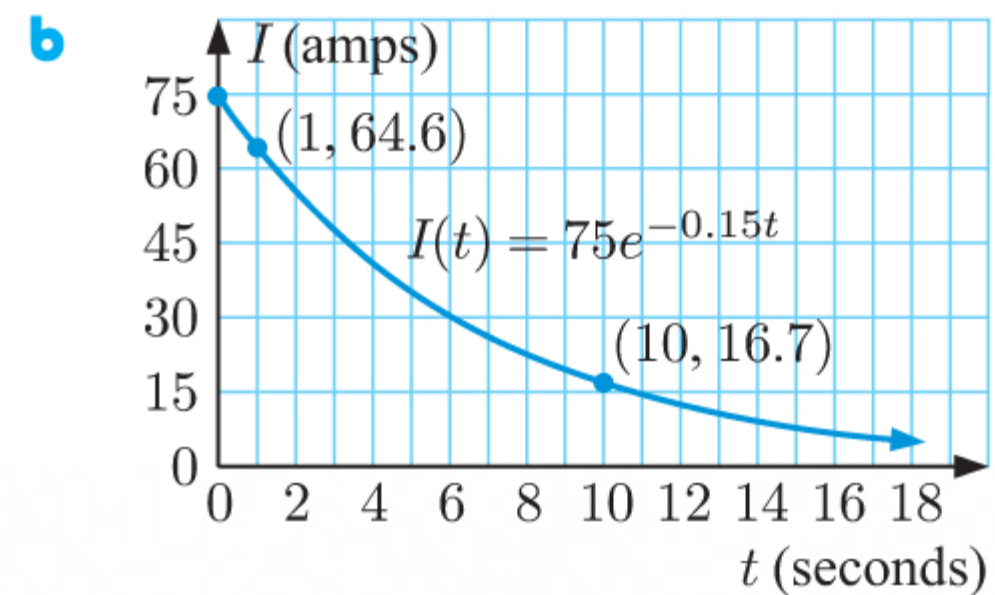


13 a $e^x = \sqrt{e}$
 $\therefore e^x = e^{\frac{1}{2}}$
 $\therefore x = \frac{1}{2}$

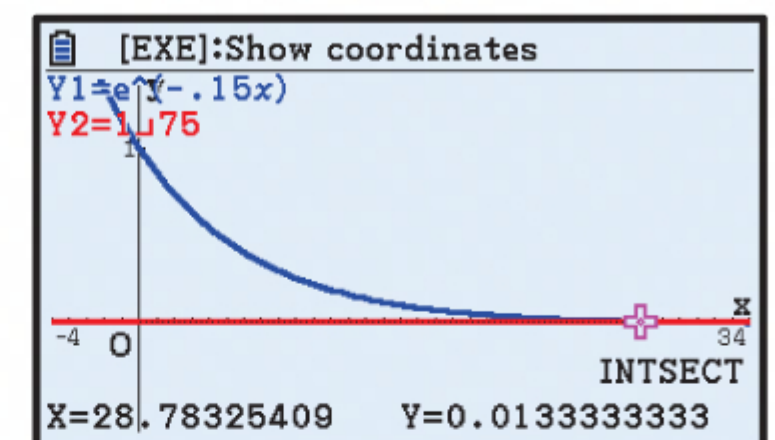
b $e^{\frac{1}{2}x} = \frac{1}{e^2}$
 $\therefore e^{\frac{1}{2}x} = e^{-2}$
 $\therefore \frac{1}{2}x = -2$
 $\therefore x = -4$

14 $I(t) = 75e^{-0.15t}$ amps

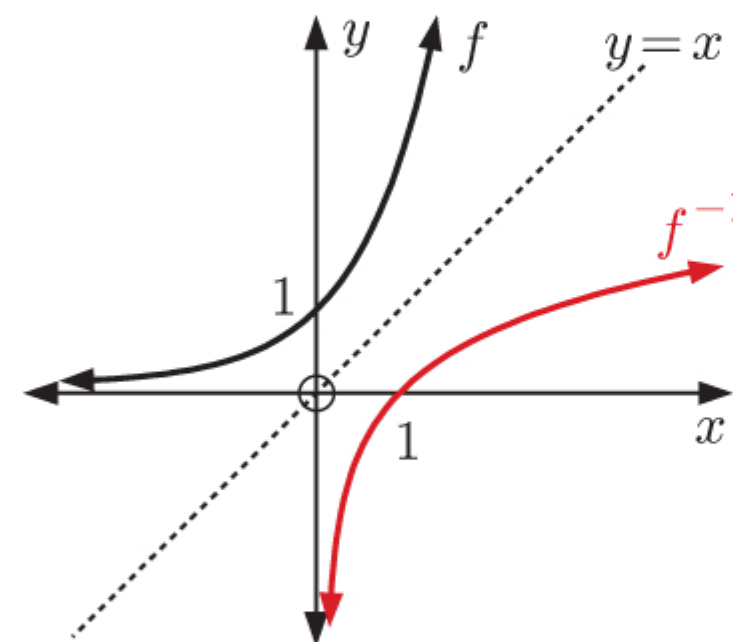
- a i** $I(1) = 75e^{-0.15}$
 ≈ 64.6
 About 64.6 amps of current is still flowing after 1 second.
- ii** $I(10) = 75e^{-1.5}$
 ≈ 16.7
 About 16.7 amps of current is still flowing after 10 seconds.



- c** We need to solve $75e^{-0.15t} = 1$
 $\therefore e^{-0.15t} = \frac{1}{75}$
 $\therefore t \approx 28.8$ {using technology}
 It will take about 28.8 seconds for the current to fall to 1 amp.



- 15 a** f^{-1} is a reflection of f in the line $y = x$



- b** The domain of f^{-1} is $\{x \mid x > 0\}$.
 The range of f^{-1} is $\{y \mid y \in \mathbb{R}\}$.

16 $e^1 \approx \sum_{i=0}^{19} \frac{1}{i!} 1^i \approx 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \dots + \frac{1}{19!} \approx 2.718281828$

REVIEW SET 5A

$$\begin{aligned} 1 \quad a \quad 8^{\frac{2}{3}} &= (2^3)^{\frac{2}{3}} \\ &= 2^2 \\ &= 4 \end{aligned}$$

$$\begin{aligned} b \quad 27^{-\frac{2}{3}} &= (3^3)^{-\frac{2}{3}} \\ &= 3^{-2} \\ &= \frac{1}{3^2} \\ &= \frac{1}{9} \end{aligned}$$

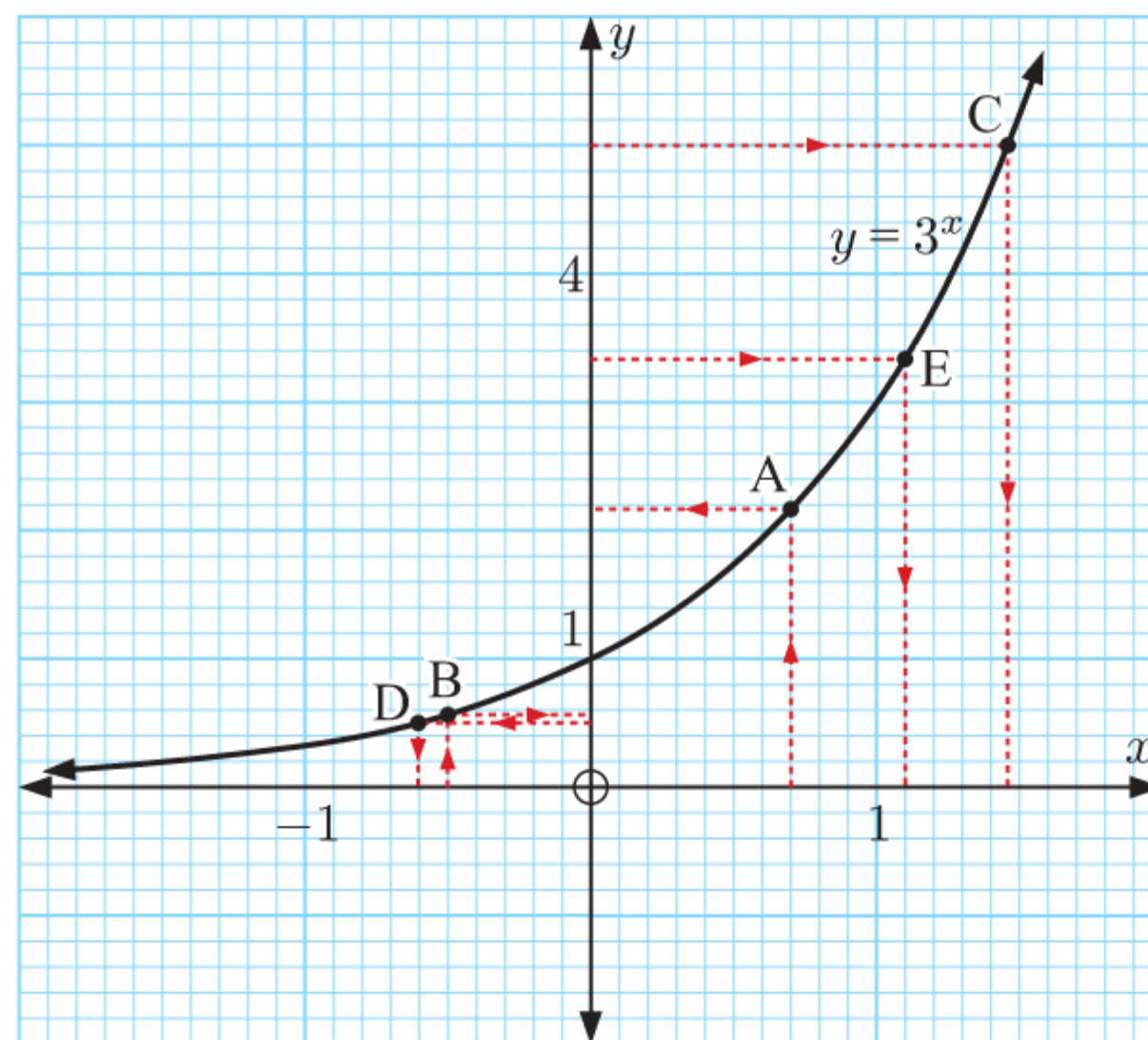
$$\begin{aligned} c \quad 81^{-\frac{1}{4}} &= (3^4)^{-\frac{1}{4}} \\ &= 3^{-1} \\ &= \frac{1}{3} \end{aligned}$$

$$\begin{aligned} 2 \quad a \quad 2^{x-3} &= \frac{1}{32} \\ \therefore 2^{x-3} &= 2^{-5} \\ \therefore x-3 &= -5 \\ \therefore x &= -2 \end{aligned}$$

$$\begin{aligned} b \quad 9^x &= 27^{2-2x} \\ \therefore (3^2)^x &= (3^3)^{2-2x} \\ \therefore 3^{2x} &= 3^{3(2-2x)} \\ \therefore 2x &= 3(2-2x) \\ \therefore 2x &= 6-6x \\ \therefore 8x &= 6 \\ \therefore x &= \frac{6}{8} = \frac{3}{4} \end{aligned}$$

$$\begin{aligned} c \quad e^{2x} &= \frac{1}{\sqrt{e}} \\ \therefore e^{2x} &= e^{-\frac{1}{2}} \\ \therefore 2x &= -\frac{1}{2} \\ \therefore x &= -\frac{1}{4} \end{aligned}$$

- 3 a i When $x = 0.7$, $y = 3^{0.7}$
From point A, $y \approx 2.2$
 $\therefore 3^{0.7} \approx 2.2$
- ii When $x = -0.5$, $y = 3^{-0.5}$
From point B, $y \approx 0.6$
 $\therefore 3^{-0.5} \approx 0.6$
- b i When $3^x = 5$,
 $x \approx 1.45$ from point C.
- ii When $3^x = \frac{1}{2}$,
 $x \approx -0.6$ from point D.
- iii When $6 \times 3^x = 20$,
then $3^x = \frac{20}{6}$
so $x \approx 1.1$ from point E.



$$\begin{aligned} 4 \quad a \quad e^x(e^{-x} + e^x) &= e^x \times e^{-x} + e^x \times e^x \\ &= e^0 + e^{2x} \\ &= 1 + e^{2x} \end{aligned}$$

$$\begin{aligned} b \quad (2^x + 5)^2 &= (2^x)^2 + 2 \times 2^x \times 5 + 5^2 \\ &= 2^{2x} + 10(2^x) + 25 \end{aligned}$$

$$\begin{aligned} c \quad (x^{\frac{1}{2}} - 7)(x^{\frac{1}{2}} + 7) &= (x^{\frac{1}{2}})^2 - 7^2 \\ &= x^1 - 49 \\ &= x - 49 \end{aligned}$$

$$\begin{aligned} 5 \quad a \quad 6 \times 2^x &= 192 \\ \therefore 2^x &= 32 \\ \therefore 2^x &= 2^5 \\ \therefore x &= 5 \end{aligned}$$

$$\begin{aligned} b \quad 4 \times \left(\frac{1}{3}\right)^x &= 324 \\ \therefore (3^{-1})^x &= 81 \\ \therefore 3^{-x} &= 3^4 \\ \therefore -x &= 4 \\ \therefore x &= -4 \end{aligned}$$

6 When $x = 1$, $y = \sqrt{8} \quad \therefore \sqrt{8} = 2^{k(1)}$
 $\therefore (2^3)^{\frac{1}{2}} = 2^k$
 $\therefore 2^{\frac{3}{2}} = 2^k$
 $\therefore k = \frac{3}{2}$

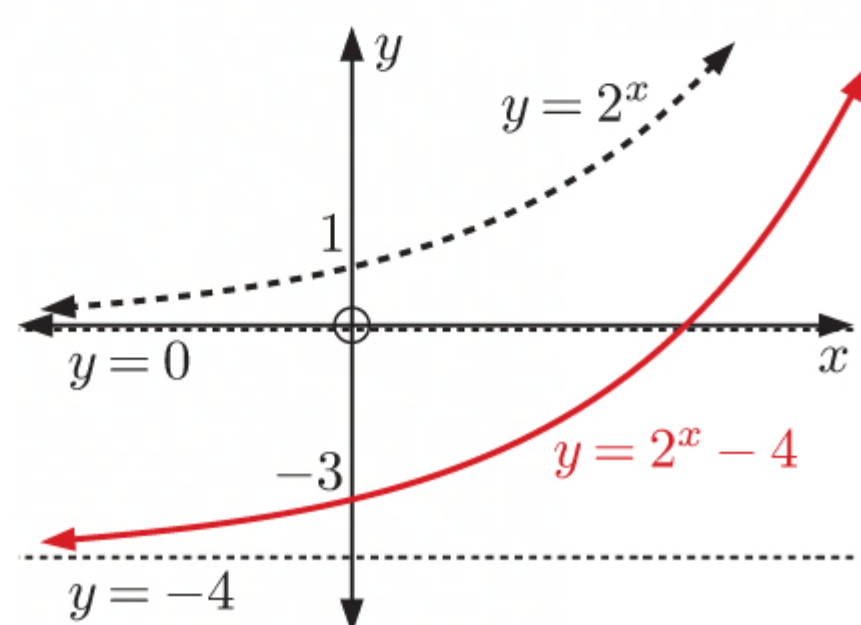
7 $f(x) = 3 \times 2^x$

a $f(0) = 3 \times 2^0$
 $= 3 \times 1$
 $= 3$

b $f(3) = 3 \times 2^3$
 $= 3 \times 8$
 $= 24$

c $f(-2) = 3 \times 2^{-2}$
 $= 3 \times \frac{1}{2^2}$
 $= 3 \times \frac{1}{4}$
 $= \frac{3}{4}$

8

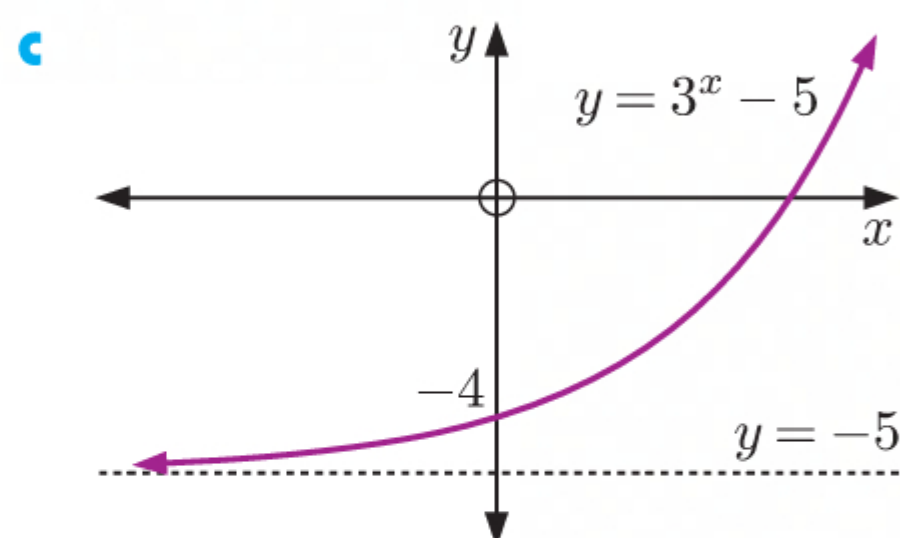


$y = 2^x$ has y -intercept 1 and horizontal asymptote $y = 0$.
 $y = 2^x - 4$ has y -intercept -3 and horizontal asymptote $y = -4$.

9 $y = 3^x - 5$

a When $x = 0$, $y = 3^0 - 5 = 1 - 5 = -4$
When $x = 1$, $y = 3^1 - 5 = 3 - 5 = -2$
When $x = 2$, $y = 3^2 - 5 = 9 - 5 = 4$
When $x = -1$, $y = 3^{-1} - 5 = \frac{1}{3} - 5 = -4\frac{2}{3}$
When $x = -2$, $y = 3^{-2} - 5 = \frac{1}{9} - 5 = -4\frac{8}{9}$

b As $x \rightarrow \infty$, $3^x \rightarrow \infty$
and so $y \rightarrow \infty$
As $x \rightarrow -\infty$, $3^x \rightarrow 0^+$
and so $y \rightarrow -5^+$

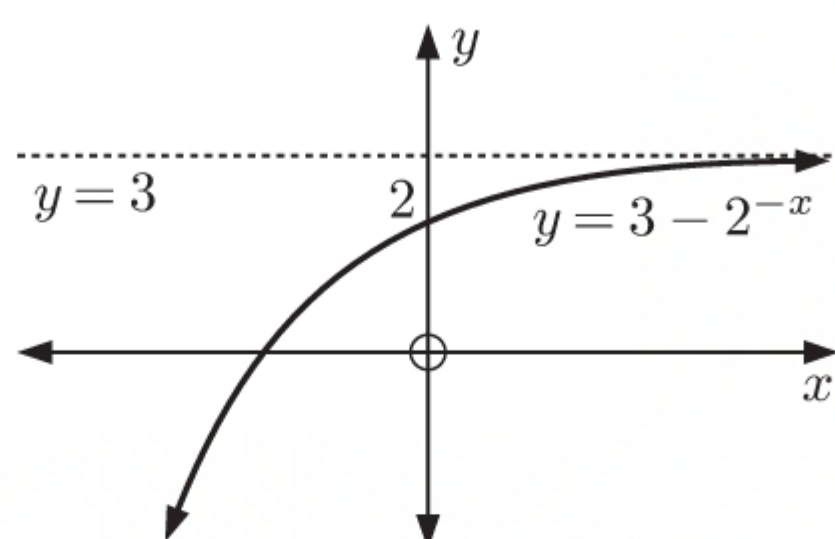
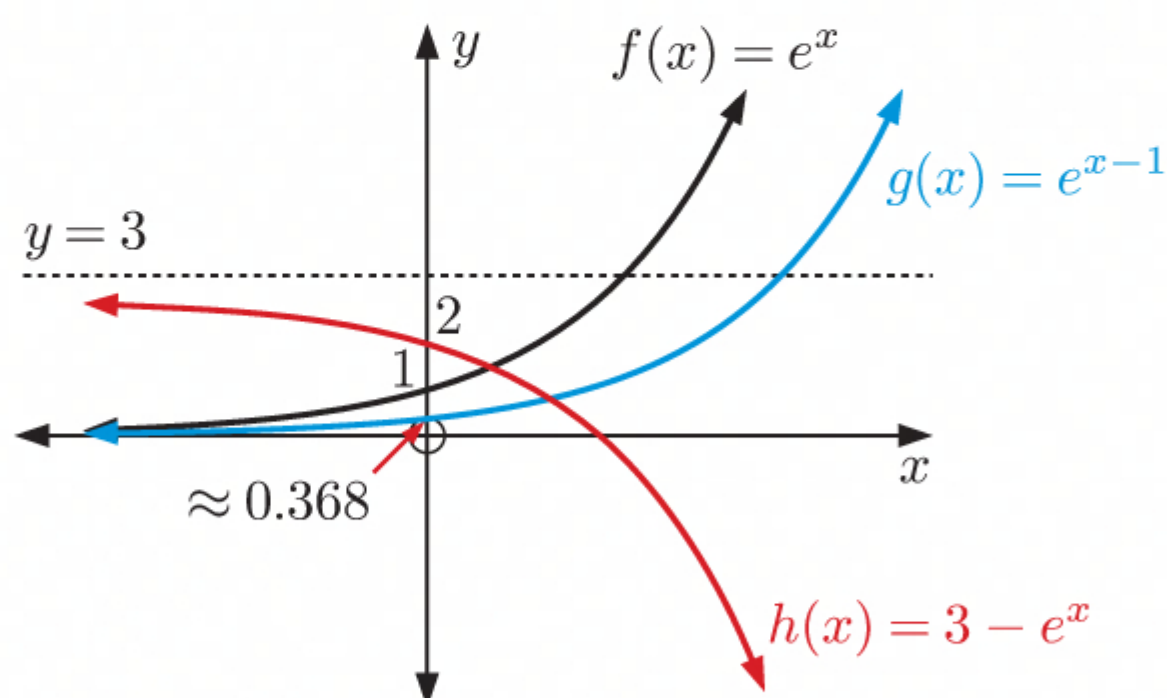


d $y = -5$ is the horizontal asymptote.

10 $y = 3 - 2^{-x}$

a When $x = 0$, $y = 3 - 2^0 = 3 - 1 = 2$
When $x = 1$, $y = 3 - 2^{-1} = 3 - \frac{1}{2} = 2\frac{1}{2}$
When $x = 2$, $y = 3 - 2^{-2} = 3 - \frac{1}{4} = 2\frac{3}{4}$
When $x = -1$, $y = 3 - 2^1 = 3 - 2 = 1$
When $x = -2$, $y = 3 - 2^2 = 3 - 4 = -1$

b As $x \rightarrow \infty$, $2^{-x} \rightarrow 0^+$
and so $y \rightarrow 3^-$
As $x \rightarrow -\infty$, $2^{-x} \rightarrow \infty$
and so $y \rightarrow -\infty$

c**d** $y = 3$ is the horizontal asymptote.**11 a****b** For $f(x)$: the domain is $\{x \mid x \in \mathbb{R}\}$,
the range is $\{y \mid y > 0\}$.For $g(x)$: the domain is $\{x \mid x \in \mathbb{R}\}$,
the range is $\{y \mid y > 0\}$.For $h(x)$: the domain is $\{x \mid x \in \mathbb{R}\}$,
the range is $\{y \mid y < 3\}$.**c** For $f(x)$: as $x \rightarrow \infty$, $f(x) \rightarrow \infty$
as $x \rightarrow -\infty$, $f(x) \rightarrow 0^+$ For $g(x)$: as $x \rightarrow \infty$, $g(x) \rightarrow \infty$
as $x \rightarrow -\infty$, $g(x) \rightarrow 0^+$ For $h(x)$: as $x \rightarrow \infty$, $h(x) \rightarrow -\infty$
as $x \rightarrow -\infty$, $h(x) \rightarrow 3^-$

12 $T(t) = 80 \times (0.913)^t \text{ } ^\circ\text{C}$

a
$$\begin{aligned} T(0) &= 80 \times (0.913)^0 \\ &= 80 \times 1 \\ &= 80 \end{aligned}$$

 \therefore the initial temperature of the dish was 80°C .

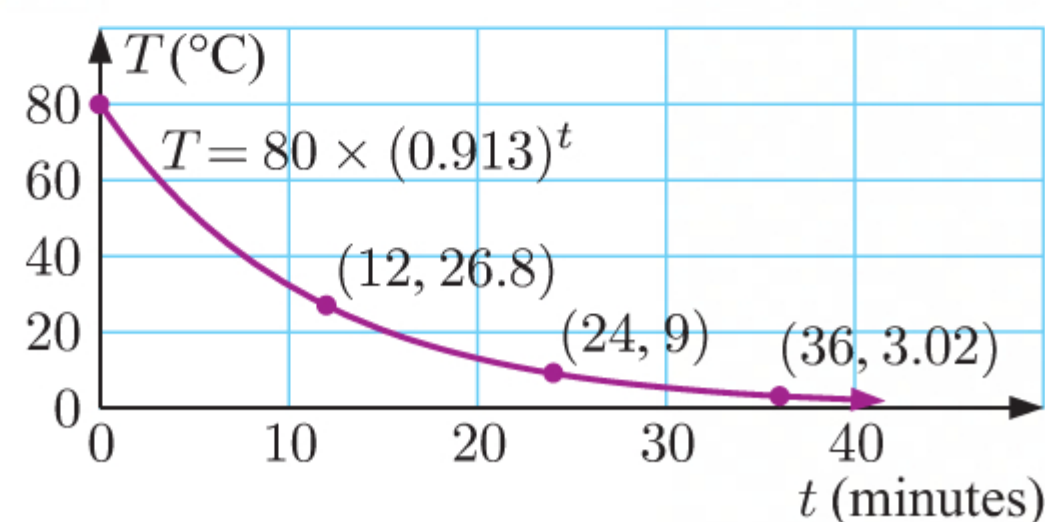
b i
$$\begin{aligned} T(12) &= 80 \times (0.913)^{12} \\ &\approx 26.8 \end{aligned}$$

After 12 minutes, the temperature was about 26.8°C .

iii
$$\begin{aligned} T(36) &= 80 \times (0.913)^{36} \\ &\approx 3.02 \end{aligned}$$

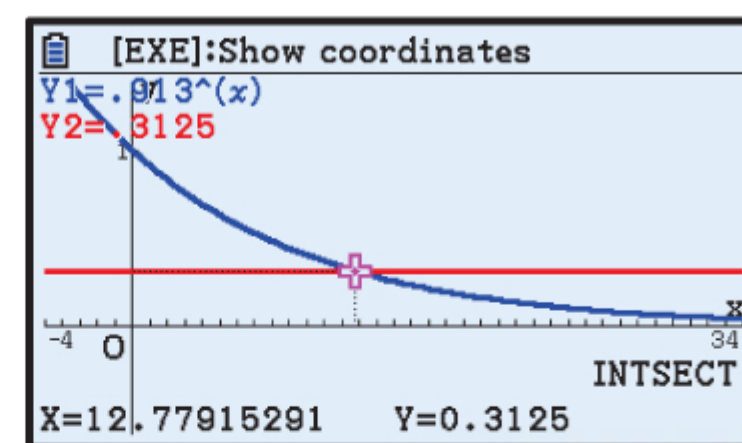
After 36 minutes, the temperature was about 3.02°C .

ii
$$\begin{aligned} T(24) &= 80 \times (0.913)^{24} \\ &\approx 9.00 \end{aligned}$$

After 24 minutes, the temperature was about 9.00°C .**c**

- d** When $T(t) = 25$,
 $80 \times (0.913)^t = 25$
 $\therefore (0.913)^t = 0.3125$
 $\therefore t \approx 12.8$ {using technology}

It takes about 12.8 minutes for the temperature of the dish to fall to 25°C .



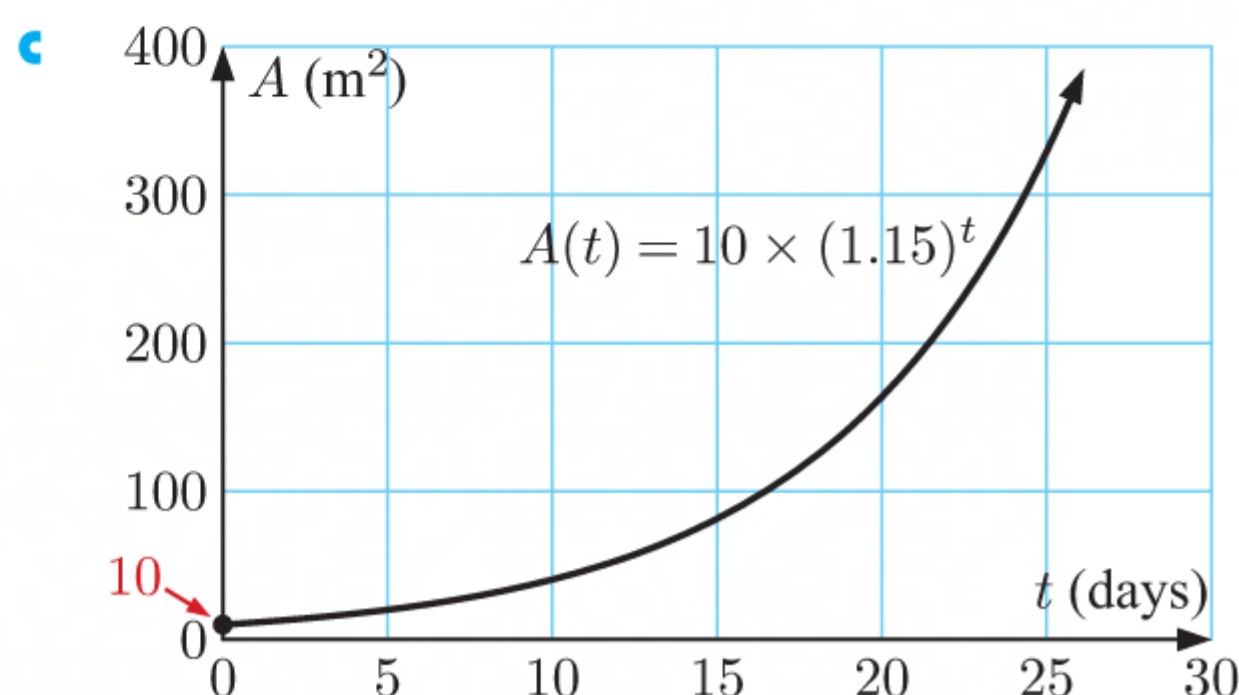
13 a $A(t) = 10 \times (1.15)^t$

b i $A(2) = 10 \times (1.15)^2$
 $= 13.225$

After 2 days, 13.225 m^2 is covered.

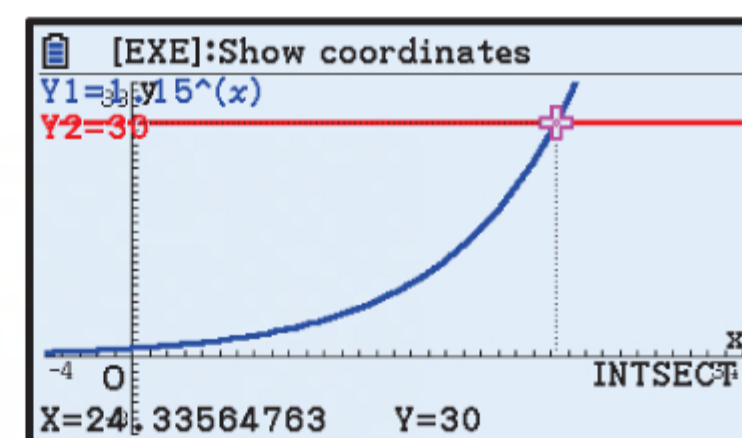
ii $A(5) = 10 \times (1.15)^5$
 ≈ 20.1

After 5 days, about 20.1 m^2 is covered.



d $A(t) = 300$
 $\therefore 10 \times (1.15)^t = 300$
 $\therefore (1.15)^t = 30$
 $\therefore t \approx 24.3$ {using technology}

\therefore it will take about 24.3 days for the affected area to reach 300 m^2 .



REVIEW SET 5B

1 a $3^{\frac{5}{4}} \approx 3.95$

b $27^{-\frac{1}{5}} \approx 0.517$

c $\sqrt[4]{100} \approx 3.16$

2 a $(3 - e^x)^2 = 3^2 - 2 \times 3 \times e^x + (e^x)^2$
 $= 9 - 6e^x + e^{2x}$

b $x^{-\frac{1}{2}}(x^{\frac{3}{2}} - 2x^{\frac{1}{2}} - x^{-\frac{1}{2}})$
 $= x^{-\frac{1}{2}} \times x^{\frac{3}{2}} - x^{-\frac{1}{2}} \times 2x^{\frac{1}{2}} - x^{-\frac{1}{2}} \times x^{-\frac{1}{2}}$
 $= x^1 - 2x^0 - x^{-1}$
 $= x - 2 - x^{-1}$

c $2^{-x}(2^{2x} + 2^x) = 2^{-x} \times 2^{2x} + 2^{-x} \times 2^x$
 $= 2^x + 2^0$
 $= 2^x + 1$

3 a $3^{x+2} - 3^x$
 $= 3^x(3^2 - 1)$
 $= 3^x(9 - 1)$
 $= 8(3^x)$

b $4^x - 2^x - 12$
 $= (2^x)^2 - 2^x - 12$
 $= (2^x + 3)(2^x - 4) \quad \{a^2 - a - 12 = (a + 3)(a - 4)\}$

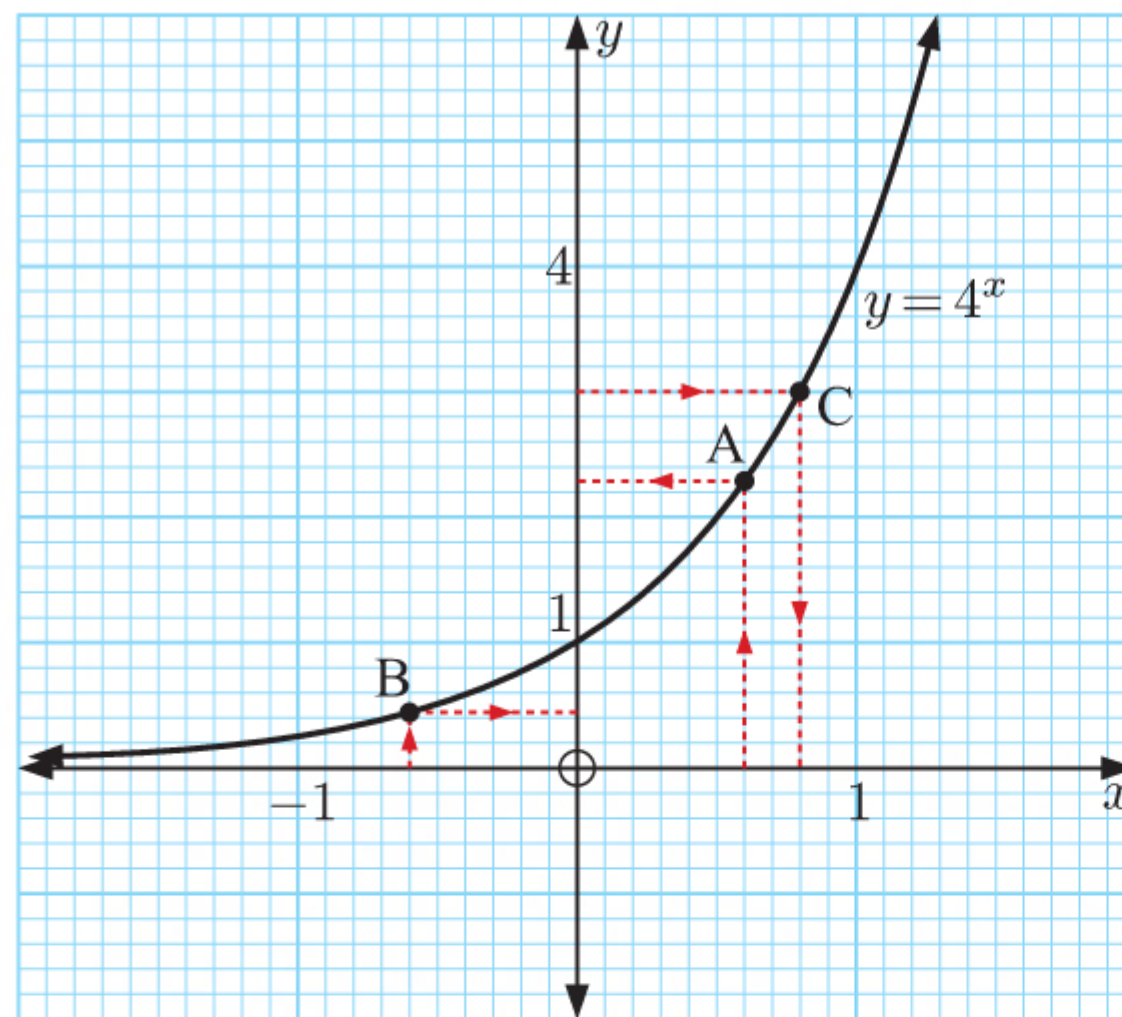
$$\begin{aligned}
 & \mathbf{c} \quad e^{2x} + 2e^x - 15 \\
 & = (e^x)^2 + 2e^x - 15 \\
 & = (e^x + 5)(e^x - 3) \quad \{a^2 + 2a - 15 = (a + 5)(a - 3)\}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{4} \quad \mathbf{a} \quad & 2^{x+1} = 32 \\
 & \therefore 2^{x+1} = 2^5 \\
 & \therefore x + 1 = 5 \\
 & \therefore x = 4
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad & 3 \times \left(\frac{1}{7}\right)^{x+1} = 1029 \\
 & \therefore (7^{-1})^{x+1} = 343 \\
 & \therefore 7^{-(x+1)} = 7^3 \\
 & \therefore -(x+1) = 3 \\
 & \therefore -x - 1 = 3 \\
 & \therefore -x = 4 \\
 & \therefore x = -4
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{c} \quad & 9^x - 10(3^x) + 9 = 0 \\
 & \therefore (3^2)^x - 10(3^x) + 9 = 0 \\
 & \therefore (3^x)^2 - 10(3^x) + 9 = 0 \quad \{a^2 - 10a + 9 = (a - 1)(a - 9)\} \\
 & \therefore (3^x - 1)(3^x - 9) = 0 \\
 & \therefore 3^x = 1 \quad \text{or} \quad 3^x = 9 \\
 & \therefore 3^x = 3^0 \quad \text{or} \quad 3^x = 3^2 \\
 & \therefore x = 0 \quad \text{or} \quad x = 2
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{5} \quad \mathbf{a} \quad \mathbf{i} \quad & \text{When } x = 0.6, \quad y = 4^{0.6} \\
 & \text{From point A, } y \approx 2.3 \\
 & \therefore 4^{0.6} \approx 2.3 \\
 & \mathbf{ii} \quad \text{When } x = -1.1, \quad y = 4^{-1.1} \\
 & \text{From point B, } y \approx 0.2 \\
 & \therefore 4^{-1.1} \approx 0.2 \\
 \mathbf{b} \quad & \text{When } 4^x = 3, \\
 & x \approx 0.8 \quad \text{from point C.}
 \end{aligned}$$



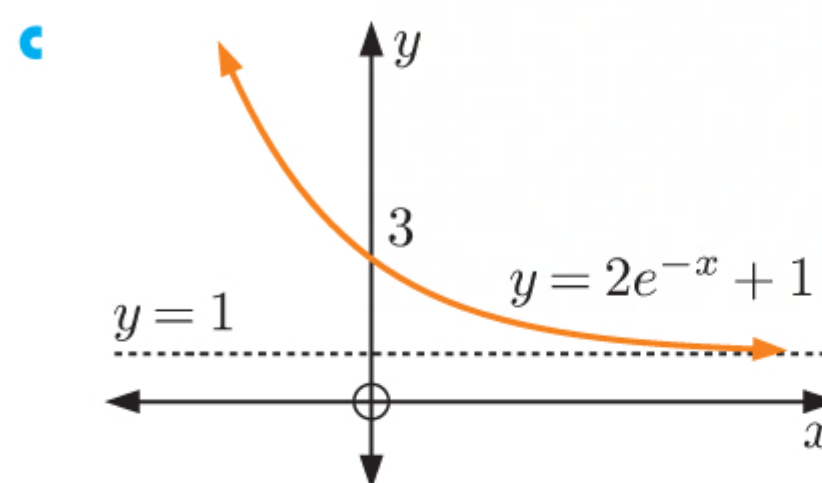
$$\mathbf{6} \quad f(x) = 2^{-x} + 1$$

$$\begin{aligned}
 \mathbf{a} \quad & f\left(\frac{1}{2}\right) = 2^{-\frac{1}{2}} + 1 \\
 & = \frac{1}{\sqrt{2}} + 1 \\
 & \approx 1.71
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad & f(a) = 3 \\
 & \therefore 2^{-a} + 1 = 3 \\
 & \therefore 2^{-a} = 2 \\
 & \therefore 2^{-a} = 2^1 \\
 & \therefore -a = 1 \\
 & \therefore a = -1
 \end{aligned}$$

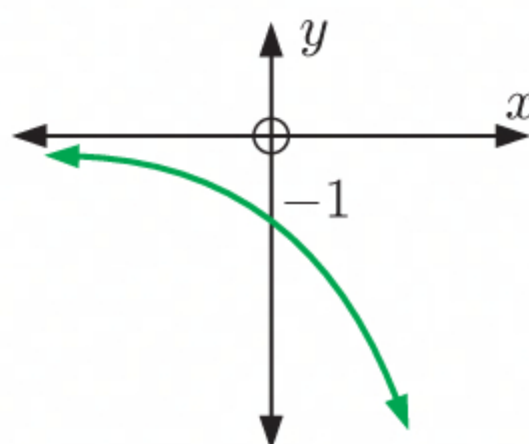
7 $y = 2e^{-x} + 1$

- a** When $x = 0$, $y = 2e^0 + 1 = 3$
 When $x = 1$, $y = 2e^{-1} + 1 \approx 1.74$
 When $x = 2$, $y = 2e^{-2} + 1 \approx 1.27$
 When $x = -1$, $y = 2e^1 + 1 \approx 6.44$
 When $x = -2$, $y = 2e^2 + 1 \approx 15.8$
- b** As $x \rightarrow \infty$, $y \rightarrow 1^+$
 As $x \rightarrow -\infty$, $y \rightarrow \infty$
- d** $y = 1$ is a horizontal asymptote.

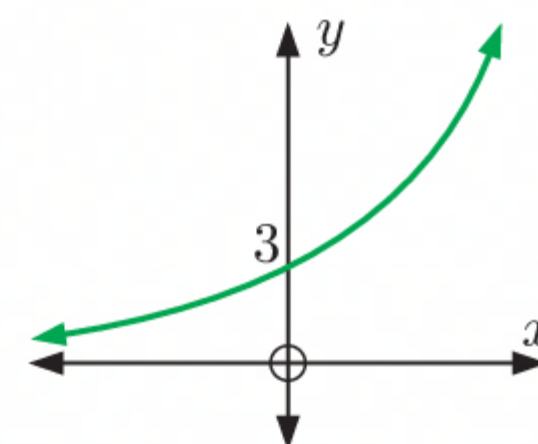


8 Use the general exponential function $y = p \times a^{x-h} + k$.

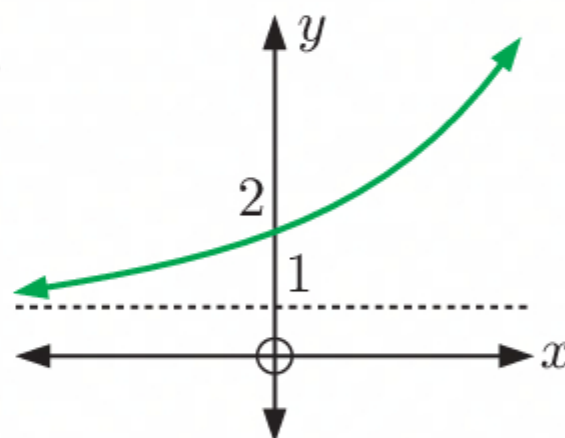
- a** $y = -e^x$
 $p = -1 \quad \therefore p < 0$
 $a = e \quad \therefore a > 1$ } function is decreasing
- When $x = 0$, $y = -e^0 = -1$
 \therefore the y -intercept is -1 .
 \therefore the graph is **C**.



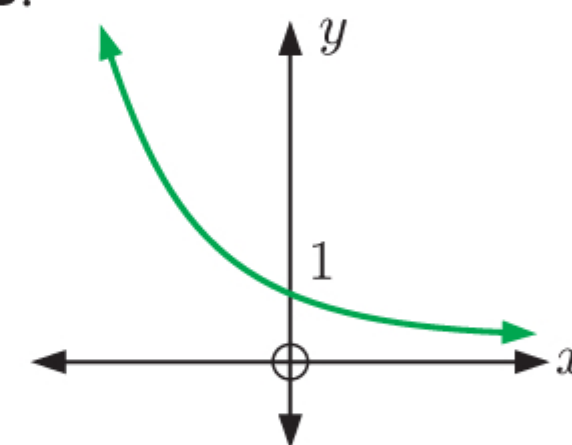
- b** $y = 3 \times 2^x$
 $p = 3 \quad \therefore p > 0$
 $a = 2 \quad \therefore a > 1$ } function is increasing
- When $x = 0$, $y = 3 \times 2^0 = 3$
 \therefore the y -intercept is 3 .
 \therefore the graph is **E**.



- c** $y = e^x + 1$
 $p = 1 \quad \therefore p > 0$
 $a = e \quad \therefore a > 1$ } function is increasing
- When $x = 0$, $y = e^0 + 1 = 2$
 \therefore the y -intercept is 2 .
 $k = 1 \quad \therefore y = 1$ is a horizontal asymptote.
 \therefore the graph is **A**.



- d** $y = 3^{-x} = \frac{1}{3^x} = \left(\frac{1}{3}\right)^x$
 $p = 1 \quad \therefore p > 0$
 $a = \frac{1}{3} \quad \therefore 0 < a < 1$ } function is decreasing
- When $x = 0$, $y = 3^0 = 1$
 \therefore the y -intercept is 1 .
 \therefore the graph is **B**.



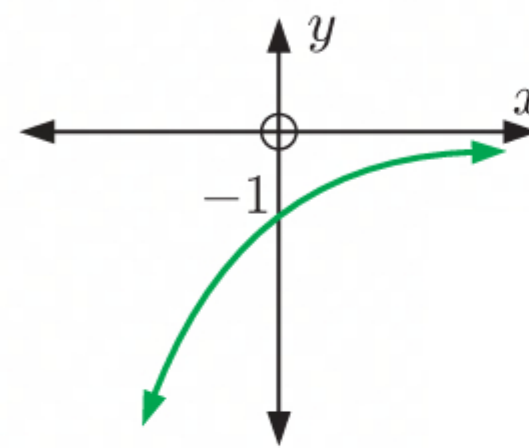
$$\text{e } y = -e^{-x} = -\frac{1}{e^x} = -\left(\frac{1}{e}\right)^x$$

$$\left. \begin{array}{l} p = -1 \quad \therefore p < 0 \\ a = \frac{1}{e} \quad \therefore 0 < a < 1 \end{array} \right\} \text{function is increasing}$$

$$\text{When } x = 0, y = -e^0 = -1$$

\therefore the y -intercept is -1 .

\therefore the graph is **D**.



$$9 \quad f(x) = 3^x$$

$$\text{a i } f(4) = 3^4 \\ = 81$$

$$\text{ii } f(-1) = 3^{-1} \\ = \frac{1}{3}$$

$$\text{b } f(x+2) = kf(x), \quad k \in \mathbb{Z}$$

$$\therefore 3^{x+2} = k \times 3^x$$

$$\therefore 3^2 \times 3^x = k \times 3^x$$

$$\therefore k = 3^2 \quad \{\text{as } 3^x \neq 0\}$$

$$\therefore k = 9$$

$$10 \quad y = a^x$$

$$\text{a } a^{2x} = (a^x)^2 \\ = y^2$$

$$\text{b } a^{-x} = (a^x)^{-1} \\ = y^{-1}$$

$$\text{c } \frac{1}{\sqrt{a^x}} = (a^x)^{-\frac{1}{2}} \\ = y^{-\frac{1}{2}} \quad \text{or} \quad \frac{1}{\sqrt{y}}$$

11 a The clock cost £500 and increases in value by 5% each year.

$$\therefore \text{the value of the clock 1 year after purchase} = £500 \times 1.05 \\ = £525$$

The vase cost £400 and increases in value by 7% each year.

$$\therefore \text{the value of the vase 1 year after purchase} = £400 \times 1.07 \\ = £428$$

b The clock will have value $V(t) = 500 \times (1.05)^t$ pounds, t years after purchase.

The vase will have value $V(t) = 400 \times (1.07)^t$ pounds, t years after purchase.

$$\text{c For the clock, } V(15) = 500 \times (1.05)^{15} \\ \approx £1039.46$$

$$\text{For the vase, } V(15) = 400 \times (1.07)^{15} \\ \approx £1103.61$$

\therefore the vase is more valuable than the clock, 15 years after purchase.

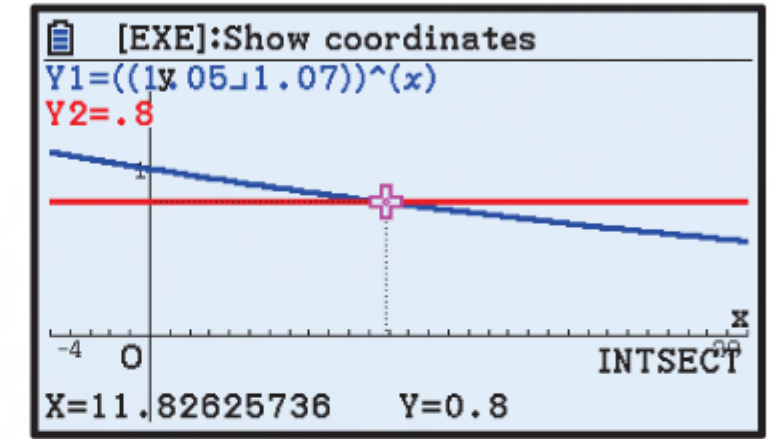
- d** To find when the items are equal in value,
we set $500 \times (1.05)^t = 400 \times (1.07)^t$ and solve for t .

$$\therefore \frac{(1.05)^t}{(1.07)^t} = \frac{400}{500}$$

$$\therefore \left(\frac{1.05}{1.07}\right)^t = 0.8$$

$$\therefore t \approx 11.8 \quad \{\text{using technology}\}$$

So, the items are equal in value after about 11.8 years.



12 $W = 1500 \times (0.993)^t$ grams

- a** When $t = 0$, $W = 1500 \times (0.993)^0$
 $= 1500 \times 1$
 $= 1500$

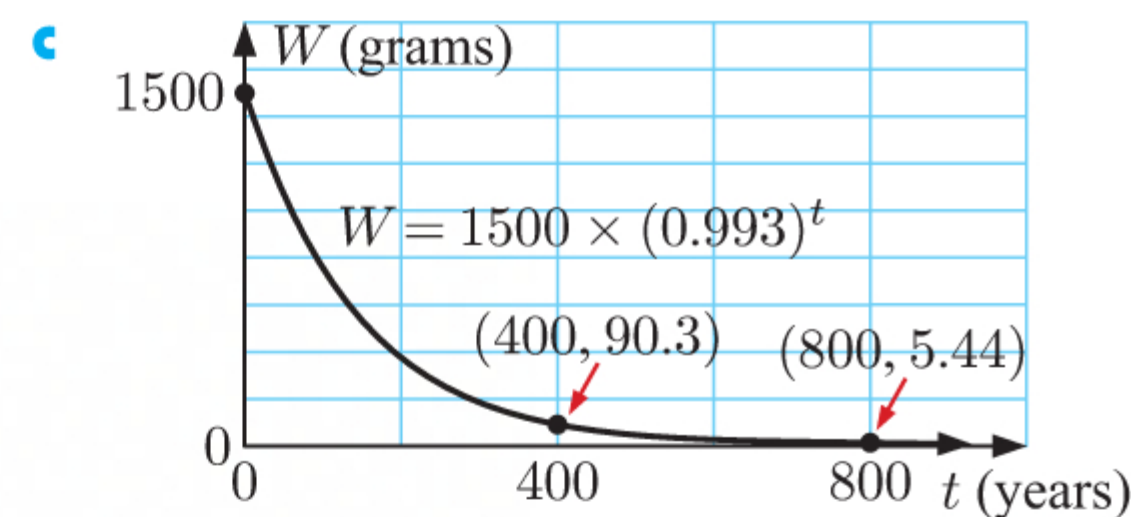
The initial weight of the radioactive substance was 1500 grams.

- b i** When $t = 400$,
 $W = 1500 \times (0.993)^{400}$
 ≈ 90.3

The weight remaining after 400 years was about 90.3 grams.

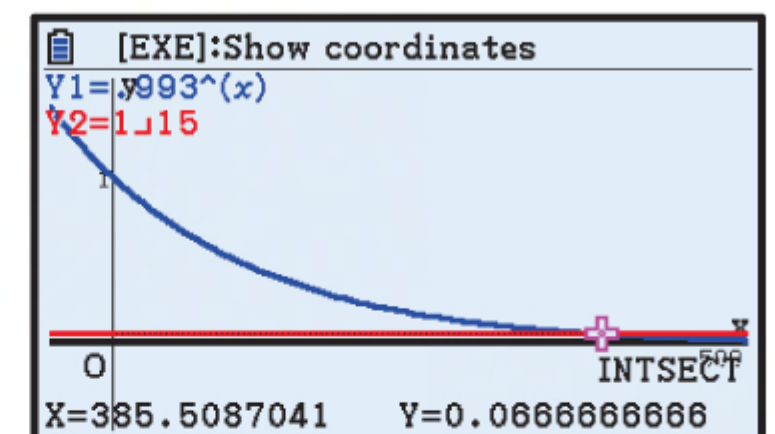
- ii** When $t = 800$,
 $W = 1500 \times (0.993)^{800}$
 ≈ 5.44

The weight remaining after 800 years was about 5.44 grams.



- d** When $W = 100$,
 $1500 \times (0.993)^t = 100$
 $\therefore (0.993)^t = \frac{1}{15}$
 $\therefore t \approx 386 \quad \{\text{using technology}\}$

It takes about 386 years for the weight to reduce to 100 grams.



Chapter 6

LOGARITHMS

EXERCISE 6A

- 1
 - a $\log 10\,000 = \log(10^4)$
 $= 4$
 - b $\log(0.001) = \log(10^{-3})$
 $= -3$
 - c $\log 10 = \log(10^1)$
 $= 1$
 - d $\log 1 = \log(10^0)$
 $= 0$
 - e $\log \sqrt{10} = \log(10^{\frac{1}{2}})$
 $= \frac{1}{2}$
 - f $\log \sqrt[3]{10} = \log(10^{\frac{1}{3}})$
 $= \frac{1}{3}$
 - g $\log\left(\frac{1}{\sqrt[4]{10}}\right) = \log(10^{-\frac{1}{4}})$
 $= -\frac{1}{4}$
 - h $\log(10\sqrt{10}) = \log(10^{\frac{3}{2}})$
 $= \frac{3}{2}$ or $1\frac{1}{2}$
 - i $\log \sqrt[3]{100} = \log((10^2)^{\frac{1}{3}})$
 $= \log(10^{\frac{2}{3}})$
 $= \frac{2}{3}$
 - j $\log\left(\frac{100}{\sqrt{10}}\right) = \log\left(\frac{10^2}{10^{\frac{1}{2}}}\right)$
 $= \log(10^{\frac{3}{2}})$
 $= \frac{3}{2}$ or $1\frac{1}{2}$
 - k $\log(10 \times \sqrt[3]{10}) = \log(10^1 \times 10^{\frac{1}{3}})$
 $= \log(10^{\frac{4}{3}})$
 $= \frac{4}{3}$ or $1\frac{1}{3}$
 - l $\log(1000\sqrt{10}) = \log(10^3 \times 10^{\frac{1}{2}})$
 $= \log(10^{\frac{7}{2}})$
 $= \frac{7}{2}$ or $3\frac{1}{2}$
- 2
 - a $\log(10^n) = n$
 - b $\log(10^a \times 100) = \log(10^a \times 10^2)$
 $= \log(10^{a+2})$
 $= a + 2$
 - c $\log\left(\frac{10}{10^m}\right) = \log(10^{1-m})$
 $= 1 - m$
 - d $\log\left(\frac{10^a}{10^b}\right) = \log(10^{a-b})$
 $= a - b$
- 3
 - a $100 < 237 < 1000$
 $\therefore \log 100 < \log 237 < \log 1000$
 $\therefore \log(10^2) < \log 237 < \log(10^3)$
 $\therefore 2 < \log 237 < 3$
 - b $\log 237 \approx 2.37$
- 4
 - a We know that $\log 1 = \log(10^0) = 0$ and $\log(0.1) = \log(10^{-1}) = -1$.
Also, $0.1 < 0.6 < 1 \therefore \log(0.1) < \log(0.6) < \log 1$
 $\therefore -1 < \log(0.6) < 0$
 - b $\log(0.6) \approx -0.22$ which is between -1 and 0 . ✓

- 5** **a** $\log 76 \approx 1.88$ **b** $\log 114 \approx 2.06$ **c** $\log 3 \approx 0.48$
 d $\log 831 \approx 2.92$ **e** $\log(0.4) \approx -0.40$ **f** $\log 3247 \approx 3.51$
 g $\log(0.008) \approx -2.10$ **h** $\log(-7)$ does not exist

- 6** **a** $\log x > 0$
 $\therefore x > 10^0$
 $\therefore x > 1$
 c $\log x < 0$
 $\therefore x < 10^0$
 $\therefore x < 1$
 but $\log x$ is only defined for $x > 0$
 $\therefore \log x$ is negative when $0 < x < 1$.
- b** $\log x = 0$
 $\therefore x = 10^0$
 $\therefore x = 1$
d $\log x$ is undefined when $x \leq 0$.

- 7** **a** 6
 $= 10^{\log 6}$
 $\approx 10^{0.7782}$ **b** 60
 $= 10^{\log 60}$
 $\approx 10^{1.7782}$ **c** 6000
 $= 10^{\log 6000}$
 $\approx 10^{3.7782}$ **d** 0.6
 $= 10^{\log(0.6)}$
 $\approx 10^{-0.2218}$
 e 0.006
 $= 10^{\log(0.006)}$
 $\approx 10^{-2.2218}$ **f** 15
 $= 10^{\log 15}$
 $\approx 10^{1.1761}$ **g** 1500
 $= 10^{\log 1500}$
 $\approx 10^{3.1761}$ **h** 1.5
 $= 10^{\log(1.5)}$
 $\approx 10^{0.1761}$
 i 0.15
 $= 10^{\log(0.15)}$
 $\approx 10^{-0.8239}$ **j** 0.000 15
 $= 10^{\log(0.000\ 15)}$
 $\approx 10^{-3.8239}$

- 8** **a** **i** $\log 3$
 ≈ 0.477 **ii** $\log 300$
 ≈ 2.477 **b** $300 = 3 \times 10^2$
 $= 10^{\log 3} \times 10^2$
 $= 10^{\log 3 + 2}$
 $\therefore \log 300 = \log(10^{\log 3 + 2})$
 $\therefore \log 300 = \log 3 + 2$

- 9** **a** **i** $\log 5$
 ≈ 0.699 **ii** $\log(0.05)$
 ≈ -1.301 **b** $0.05 = 5 \times 10^{-2}$
 $= 10^{\log 5} \times 10^{-2}$
 $= 10^{\log 5 - 2}$
 $\therefore \log(0.05) = \log(10^{\log 5 - 2})$
 $\therefore \log(0.05) = \log 5 - 2$

- 10** **a** $\log x = 2$
 $\therefore 10^{\log x} = 10^2$
 $\therefore x = 10^2$
 $\therefore x = 100$ **b** $\log x = 1$
 $\therefore 10^{\log x} = 10^1$
 $\therefore x = 10^1$
 $\therefore x = 10$ **c** $\log x = 0$
 $\therefore 10^{\log x} = 10^0$
 $\therefore x = 10^0$
 $\therefore x = 1$

$$\begin{aligned} \text{d} \quad & \log x = -1 \\ \therefore & 10^{\log x} = 10^{-1} \\ \therefore & x = 10^{-1} \\ \therefore & x = \frac{1}{10} \end{aligned}$$

$$\begin{aligned} \text{e} \quad & \log x = \frac{1}{2} \\ \therefore & 10^{\log x} = 10^{\frac{1}{2}} \\ \therefore & x = 10^{\frac{1}{2}} \\ \therefore & x = \sqrt{10} \end{aligned}$$

$$\begin{aligned} \text{f} \quad & \log x = -\frac{1}{2} \\ \therefore & 10^{\log x} = 10^{-\frac{1}{2}} \\ \therefore & x = 10^{-\frac{1}{2}} \\ \therefore & x = \frac{1}{10^{\frac{1}{2}}} \\ \therefore & x = \frac{1}{\sqrt{10}} \end{aligned}$$

$$\begin{aligned} \text{g} \quad & \log x = 4 \\ \therefore & 10^{\log x} = 10^4 \\ \therefore & x = 10^4 \\ \therefore & x = 10\,000 \end{aligned}$$

$$\begin{aligned} \text{h} \quad & \log x = -5 \\ \therefore & 10^{\log x} = 10^{-5} \\ \therefore & x = 10^{-5} \\ \therefore & x = 0.000\,01 \end{aligned}$$

$$\begin{aligned} \text{i} \quad & \log x \approx 0.8351 \\ \therefore & 10^{\log x} \approx 10^{0.8351} \\ \therefore & x \approx 10^{0.8351} \\ \therefore & x \approx 6.84 \end{aligned}$$

$$\begin{aligned} \text{j} \quad & \log x \approx 2.1457 \\ \therefore & 10^{\log x} \approx 10^{2.1457} \\ \therefore & x \approx 10^{2.1457} \\ \therefore & x \approx 140 \end{aligned}$$

$$\begin{aligned} \text{k} \quad & \log x \approx -1.378 \\ \therefore & 10^{\log x} \approx 10^{-1.378} \\ \therefore & x \approx 10^{-1.378} \\ \therefore & x \approx 0.0419 \end{aligned}$$

$$\begin{aligned} \text{l} \quad & \log x \approx -3.1997 \\ \therefore & 10^{\log x} \approx 10^{-3.1997} \\ \therefore & x \approx 10^{-3.1997} \\ \therefore & x \approx 0.000\,631 \end{aligned}$$

EXERCISE 6B

- 1
 - a From $\log_{10} 100 = 2$, we deduce that $10^2 = 100$.
 - b From $\log_{10} 10\,000 = 4$, we deduce that $10^4 = 10\,000$.
 - c From $\log_{10}(0.1) = -1$, we deduce that $10^{-1} = 0.1$.
 - d From $\log_{10} \sqrt{10} = \frac{1}{2}$, we deduce that $10^{\frac{1}{2}} = \sqrt{10}$.
 - e From $\log_2 8 = 3$, we deduce that $2^3 = 8$.
 - f From $\log_3 9 = 2$, we deduce that $3^2 = 9$.
 - g From $\log_2 \left(\frac{1}{4}\right) = -2$, we deduce that $2^{-2} = \frac{1}{4}$.
 - h From $\log_3 \sqrt{27} = 1.5$, we deduce that $3^{1.5} = \sqrt{27}$.
 - i From $\log_5 \left(\frac{1}{\sqrt{5}}\right) = -\frac{1}{2}$, we deduce that $5^{-\frac{1}{2}} = \frac{1}{\sqrt{5}}$.
- 2
 - a From $4^3 = 64$, we deduce that $\log_4 64 = 3$.
 - b From $5^2 = 25$, we deduce that $\log_5 25 = 2$.
 - c From $7^2 = 49$, we deduce that $\log_7 49 = 2$.
 - d From $2^6 = 64$, we deduce that $\log_2 64 = 6$.
 - e From $2^{-3} = \frac{1}{8}$, we deduce that $\log_2 \left(\frac{1}{8}\right) = -3$.
 - f From $10^{-2} = 0.01$, we deduce that $\log_{10}(0.01) = -2$.
 - g From $2^{-1} = \frac{1}{2}$, we deduce that $\log_2 \left(\frac{1}{2}\right) = -1$.
 - h From $3^{-3} = \frac{1}{27}$, we deduce that $\log_3 \left(\frac{1}{27}\right) = -3$.

$$\begin{aligned} \mathbf{3} \quad \mathbf{a} \quad & \log_{10} 100\,000 \\ &= \log_{10}(10^5) \\ &= 5 \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad & \log_{10}(0.01) \\ &= \log_{10}(10^{-2}) \\ &= -2 \end{aligned}$$

$$\begin{aligned} \mathbf{c} \quad & \log_3 \sqrt{3} \\ &= \log_3(3^{\frac{1}{2}}) \\ &= \frac{1}{2} \end{aligned}$$

$$\begin{aligned} \mathbf{d} \quad & \log_2 4 \\ &= \log_2(2^2) \\ &= 2 \end{aligned}$$

$$\begin{aligned} \mathbf{e} \quad & \log_2 64 \\ &= \log_2(2^6) \\ &= 6 \end{aligned}$$

$$\begin{aligned} \mathbf{f} \quad & \log_2 128 \\ &= \log_2(2^7) \\ &= 7 \end{aligned}$$

$$\begin{aligned} \mathbf{g} \quad & \log_5 25 \\ &= \log_5(5^2) \\ &= 2 \end{aligned}$$

$$\begin{aligned} \mathbf{h} \quad & \log_5 125 \\ &= \log_5(5^3) \\ &= 3 \end{aligned}$$

$$\begin{aligned} \mathbf{i} \quad & \log_2(0.125) \\ &= \log_2\left(\frac{1}{8}\right) \\ &= \log_2(2^{-3}) \\ &= -3 \end{aligned}$$

$$\begin{aligned} \mathbf{j} \quad & \log_9 3 \\ &= \log_9(9^{\frac{1}{2}}) \\ &= \frac{1}{2} \end{aligned}$$

$$\begin{aligned} \mathbf{k} \quad & \log_4 16 \\ &= \log_4(4^2) \\ &= 2 \end{aligned}$$

$$\begin{aligned} \mathbf{l} \quad & \log_{36} 6 \\ &= \log_{36}(36^{\frac{1}{2}}) \\ &= \frac{1}{2} \end{aligned}$$

$$\begin{aligned} \mathbf{m} \quad & \log_3 243 \\ &= \log_3(3^5) \\ &= 5 \end{aligned}$$

$$\begin{aligned} \mathbf{n} \quad & \log_2 \sqrt[3]{2} \\ &= \log_2(2^{\frac{1}{3}}) \\ &= \frac{1}{3} \end{aligned}$$

$$\begin{aligned} \mathbf{o} \quad & \log_8 2 \\ &= \log_8(8^{\frac{1}{3}}) \\ &= \frac{1}{3} \end{aligned}$$

$$\begin{aligned} \mathbf{p} \quad & \log_6(6\sqrt{6}) \\ &= \log_6(6^1 \times 6^{\frac{1}{2}}) \\ &= \log_6(6^{\frac{3}{2}}) \\ &= \frac{3}{2} \end{aligned}$$

$$\begin{aligned} \mathbf{q} \quad & \log_4 1 \\ &= \log_4(4^0) \\ &= 0 \end{aligned}$$

$$\begin{aligned} \mathbf{r} \quad & \log_9 9 \\ &= \log_9(9^1) \\ &= 1 \end{aligned}$$

$$\begin{aligned} \mathbf{s} \quad & \log_3\left(\frac{1}{3}\right) \\ &= \log_3(3^{-1}) \\ &= -1 \end{aligned}$$

$$\begin{aligned} \mathbf{t} \quad & \log_{10} \sqrt[4]{1000} \\ &= \log_{10}((10^3)^{\frac{1}{4}}) \\ &= \log_{10}(10^{\frac{3}{4}}) \\ &= \frac{3}{4} \end{aligned}$$

$$\begin{aligned} \mathbf{u} \quad & \log_7\left(\frac{1}{\sqrt{7}}\right) \\ &= \log_7(7^{-\frac{1}{2}}) \\ &= -\frac{1}{2} \end{aligned}$$

$$\begin{aligned} \mathbf{v} \quad & \log_5(25\sqrt{5}) \\ &= \log_5(5^2 \times 5^{\frac{1}{2}}) \\ &= \log_5(5^{\frac{5}{2}}) \\ &= \frac{5}{2} \end{aligned}$$

$$\begin{aligned} \mathbf{w} \quad & \log_3\left(\frac{1}{\sqrt{27}}\right) \\ &= \log_3\left(\frac{1}{(3^3)^{\frac{1}{2}}}\right) \\ &= \log_3(3^{-\frac{3}{2}}) \\ &= -\frac{3}{2} \end{aligned}$$

$$\begin{aligned} \mathbf{x} \quad & \log_4\left(\frac{1}{2\sqrt{2}}\right) \\ &= \log_4(2^{-\frac{3}{2}}) \\ &= \log_4\left((2^2)^{-\frac{3}{4}}\right) \\ &= \log_4(4^{-\frac{3}{4}}) \\ &= -\frac{3}{4} \end{aligned}$$

$$\begin{aligned} \mathbf{4} \quad \mathbf{a} \quad & \log_x(x^2) \\ &= 2, \quad x > 0 \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad & \log_t\left(\frac{1}{t}\right) \\ &= \log_t(t^{-1}) \\ &= -1, \quad t > 0 \end{aligned}$$

$$\begin{aligned} \mathbf{c} \quad & \log_x \sqrt{x} \\ &= \log_x(x^{\frac{1}{2}}) \\ &= \frac{1}{2}, \quad x > 0 \end{aligned}$$

$$\begin{aligned} \mathbf{d} \quad & \log_m(m^3) \\ &= 3, \quad m > 0 \end{aligned}$$

$$\begin{aligned} \mathbf{e} \quad & \log_k \sqrt[4]{k} \\ &= \log_k(k^{\frac{1}{4}}) \\ &= \frac{1}{4}, \quad k > 0 \end{aligned}$$

$$\begin{aligned} \mathbf{f} \quad & \log_x(x\sqrt{x}) \\ &= \log_x(x^1 \times x^{\frac{1}{2}}) \\ &= \log_x(x^{\frac{3}{2}}) \\ &= \frac{3}{2}, \quad x > 0 \end{aligned}$$

$$\begin{aligned} \mathbf{g} \quad & \log_a \left(\frac{1}{a^2} \right) \\ &= \log_a (a^{-2}) \\ &= -2, \quad a > 0 \end{aligned}$$

$$\begin{aligned} \mathbf{h} \quad & \log_a \left(\frac{1}{\sqrt{a}} \right) \\ &= \log_a (a^{-\frac{1}{2}}) \\ &= -\frac{1}{2}, \quad a > 0 \end{aligned}$$

$$\begin{aligned} \mathbf{i} \quad & \log_m \sqrt{m^5} \\ &= \log_m ((m^5)^{\frac{1}{2}}) \\ &= \log_m (m^{\frac{5}{2}}) \\ &= \frac{5}{2}, \quad m > 0 \end{aligned}$$

$$\begin{aligned} \mathbf{5} \quad \mathbf{a} \quad & \log_2 x = 3 \\ & \therefore x = 2^3 \\ & \therefore x = 8 \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad & \log_4 x = \frac{1}{2} \\ & \therefore x = 4^{\frac{1}{2}} \\ & \therefore x = 2 \end{aligned}$$

$$\begin{aligned} \mathbf{c} \quad & \log_x 81 = 4 \\ & \therefore 81 = x^4 \\ & \therefore x = \pm \sqrt[4]{81} \\ & \therefore x = \pm 3 \\ & \therefore x = 3 \quad \{\text{as } x > 0\} \end{aligned}$$

$$\begin{aligned} \mathbf{d} \quad & \log_2 (x - 6) = 3 \\ & \therefore x - 6 = 2^3 \\ & \therefore x - 6 = 8 \\ & \therefore x = 14 \end{aligned}$$

$$\begin{aligned} \mathbf{6} \quad & \log_a b = x \\ & \therefore b = a^x \\ & \therefore \log_b b = \log_b (a^x) \\ & \therefore 1 = x \log_b a \\ & \therefore \log_b a = \frac{1}{x} \end{aligned}$$

INVESTIGATION 1

DISCOVERING THE LAWS OF LOGARITHMS

$$\begin{aligned} \mathbf{1} \quad \mathbf{a} \quad \mathbf{i} \quad & \log 2 + \log 3 \approx 0.778 \\ & \mathbf{iv} \quad \log 6 \approx 0.778 \\ \mathbf{b} \quad & \log m + \log n = \log(mn) \end{aligned}$$

$$\begin{aligned} \mathbf{ii} \quad & \log 3 + \log 7 \approx 1.32 \\ \mathbf{v} \quad & \log 21 \approx 1.32 \end{aligned}$$

$$\begin{aligned} \mathbf{iii} \quad & \log 4 + \log 20 \approx 1.90 \\ \mathbf{vi} \quad & \log 80 \approx 1.90 \end{aligned}$$

$$\begin{aligned} \mathbf{2} \quad \mathbf{a} \quad \mathbf{i} \quad & \log 6 - \log 2 \approx 0.477 \\ & \mathbf{iv} \quad \log 3 \approx 0.477 \\ \mathbf{b} \quad & \log m - \log n = \log \left(\frac{m}{n} \right) \end{aligned}$$

$$\begin{aligned} \mathbf{ii} \quad & \log 12 - \log 3 \approx 0.602 \\ \mathbf{v} \quad & \log 4 \approx 0.602 \end{aligned}$$

$$\begin{aligned} \mathbf{iii} \quad & \log 3 - \log 5 \approx -0.222 \\ \mathbf{vi} \quad & \log(0.6) \approx -0.222 \end{aligned}$$

$$\begin{aligned} \mathbf{3} \quad \mathbf{a} \quad \mathbf{i} \quad & 3 \log 2 \approx 0.903 \\ & \mathbf{iv} \quad \log(2^3) \approx 0.903 \\ \mathbf{b} \quad & m \log b = \log(b^m) \end{aligned}$$

$$\begin{aligned} \mathbf{ii} \quad & 2 \log 5 \approx 1.40 \\ \mathbf{v} \quad & \log(5^2) \approx 1.40 \end{aligned}$$

$$\begin{aligned} \mathbf{iii} \quad & -4 \log 3 \approx -1.91 \\ \mathbf{vi} \quad & \log(3^{-4}) \approx -1.91 \end{aligned}$$

EXERCISE 6C

$$\begin{aligned} \mathbf{1} \quad \mathbf{a} \quad & \log 8 + \log 2 \\ &= \log(8 \times 2) \\ &= \log 16 \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad & \log 4 + \log 5 \\ &= \log(4 \times 5) \\ &= \log 20 \end{aligned}$$

$$\begin{aligned} \mathbf{c} \quad & \log 40 - \log 5 \\ &= \log \left(\frac{40}{5} \right) \\ &= \log 8 \end{aligned}$$

$$\begin{aligned} \mathbf{d} \quad & \log p - \log m \\ &= \log \left(\frac{p}{m} \right) \end{aligned}$$

$$\begin{aligned} \mathbf{e} \quad & \log_4 8 - \log_4 2 \\ &= \log_4 \left(\frac{8}{2} \right) \\ &= \log_4 4 \\ &= 1 \end{aligned}$$

$$\begin{aligned} \mathbf{f} \quad & \log 5 + \log(0.4) \\ &= \log(5 \times 0.4) \\ &= \log 2 \end{aligned}$$

$$\begin{aligned}
 \text{g} \quad & \log 250 + \log 4 \\
 &= \log(250 \times 4) \\
 &= \log 1000 \\
 &= \log(10^3) \\
 &= 3
 \end{aligned}$$

$$\begin{aligned}
 \text{h} \quad & \log_5 100 - \log_5 4 \\
 &= \log_5 \left(\frac{100}{4} \right) \\
 &= \log_5 25 \\
 &= \log_5 (5^2) \\
 &= 2
 \end{aligned}$$

$$\begin{aligned}
 \text{i} \quad & \log 2 + \log 3 + \log 4 \\
 &= \log(2 \times 3 \times 4) \\
 &= \log 24
 \end{aligned}$$

$$\begin{aligned}
 \text{j} \quad & \log 5 + \log 4 - \log 2 \\
 &= \log \left(\frac{5 \times 4}{2} \right) \\
 &= \log 10 \\
 &= 1
 \end{aligned}$$

$$\begin{aligned}
 \text{k} \quad & \log_3 6 - \log_3 2 - \log_3 3 \\
 &= \log_3 (6 \div 2 \div 3) \\
 &= \log_3 1 \\
 &= \log_3 (3^0) \\
 &= 0
 \end{aligned}$$

$$\begin{aligned}
 \text{l} \quad & \log \left(\frac{4}{3} \right) + \log 3 + \log 7 \\
 &= \log \left(\frac{4}{3} \times 3 \times 7 \right) \\
 &= \log 28
 \end{aligned}$$

$$\begin{aligned}
 \text{2 a} \quad & \log 7 + 2 \\
 &= \log 7 + \log(10^2) \\
 &= \log(7 \times 100) \\
 &= \log 700
 \end{aligned}$$

$$\begin{aligned}
 \text{b} \quad & \log 4 - 1 \\
 &= \log 4 - \log(10^1) \\
 &= \log \left(\frac{4}{10} \right) \\
 &= \log \left(\frac{2}{5} \right)
 \end{aligned}$$

$$\begin{aligned}
 \text{c} \quad & 1 + \log_2 3 \\
 &= \log_2 (2^1) + \log_2 3 \\
 &= \log_2 (2 \times 3) \\
 &= \log_2 6
 \end{aligned}$$

$$\begin{aligned}
 \text{d} \quad & \log_3 5 - 2 \\
 &= \log_3 5 - \log_3 (3^2) \\
 &= \log_3 5 - \log_3 9 \\
 &= \log_3 \left(\frac{5}{9} \right)
 \end{aligned}$$

$$\begin{aligned}
 \text{e} \quad & 2 + \log 2 \\
 &= \log(10^2) + \log 2 \\
 &= \log(100 \times 2) \\
 &= \log 200
 \end{aligned}$$

$$\begin{aligned}
 \text{f} \quad & \log 50 - 4 \\
 &= \log 50 - \log(10^4) \\
 &= \log 50 - \log 10\,000 \\
 &= \log \left(\frac{50}{10\,000} \right) \\
 &= \log(0.005)
 \end{aligned}$$

$$\begin{aligned}
 \text{g} \quad & t + \log w \\
 &= \log(10^t) + \log w \\
 &= \log(10^t \times w)
 \end{aligned}$$

$$\begin{aligned}
 \text{h} \quad & \log_m 40 - 2 \\
 &= \log_m 40 - \log_m (m^2) \\
 &= \log_m \left(\frac{40}{m^2} \right)
 \end{aligned}$$

$$\begin{aligned}
 \text{i} \quad & 3 - \log_5 50 \\
 &= \log_5 (5^3) - \log_5 50 \\
 &= \log_5 125 - \log_5 50 \\
 &= \log_5 \left(\frac{125}{50} \right) \\
 &= \log_5 \left(\frac{5}{2} \right)
 \end{aligned}$$

$$\begin{aligned}
 \text{3 a} \quad & 5 \log 2 + \log 3 \\
 &= \log(2^5) + \log 3 \\
 &= \log 32 + \log 3 \\
 &= \log(32 \times 3) \\
 &= \log 96
 \end{aligned}$$

$$\begin{aligned}
 \text{b} \quad & 2 \log 3 + 3 \log 2 \\
 &= \log(3^2) + \log(2^3) \\
 &= \log 9 + \log 8 \\
 &= \log(9 \times 8) \\
 &= \log 72
 \end{aligned}$$

$$\begin{aligned}
 \text{c} \quad & 3 \log 4 - \log 8 \\
 &= \log(4^3) - \log 8 \\
 &= \log 64 - \log 8 \\
 &= \log \left(\frac{64}{8} \right) \\
 &= \log 8
 \end{aligned}$$

$$\begin{aligned}
 \text{d} \quad & 2 \log_3 5 - 3 \log_3 2 \\
 &= \log_3 (5^2) - \log_3 (2^3) \\
 &= \log_3 25 - \log_3 8 \\
 &= \log_3 \left(\frac{25}{8} \right)
 \end{aligned}$$

$$\begin{aligned}
 \text{e} \quad & \frac{1}{2} \log_6 4 + \log_6 3 \\
 &= \log_6 (4^{\frac{1}{2}}) + \log_6 3 \\
 &= \log_6 2 + \log_6 3 \\
 &= \log_6 (2 \times 3) \\
 &= \log_6 6 \\
 &= 1
 \end{aligned}$$

$$\begin{aligned}
 \text{f} \quad & \frac{1}{3} \log \left(\frac{1}{8} \right) \\
 &= \log \left(\left(\frac{1}{8} \right)^{\frac{1}{3}} \right) \\
 &= \log \left((2^{-3})^{\frac{1}{3}} \right) \\
 &= \log(2^{-1}) \\
 &= \log \left(\frac{1}{2} \right)
 \end{aligned}$$

$$\begin{aligned}
 \text{g} \quad & 3 - \log 2 - 2 \log 5 \\
 &= \log(10^3) - \log 2 - \log(5^2) \\
 &= \log 1000 - \log 2 - \log 25 \\
 &= \log(1000 \div 2 \div 25) \\
 &= \log 20
 \end{aligned}$$

$$\begin{aligned}
 \text{h} \quad & 1 - 3 \log 2 + \log 20 \\
 &= \log(10^1) - \log(2^3) + \log 20 \\
 &= \log 10 - \log 8 + \log 20 \\
 &= \log(10 \div 8 \times 20) \\
 &= \log 25
 \end{aligned}$$

$$\begin{aligned}
 \text{i} \quad & 2 - \frac{1}{2} \log_n 4 - \log_n 5 \\
 &= \log_n(n^2) - \log_n(4^{\frac{1}{2}}) - \log_n 5 \\
 &= \log_n(n^2) - \log_n 2 - \log_n 5 \\
 &= \log_n(n^2 \div 2 \div 5) \\
 &= \log_n\left(\frac{n^2}{10}\right)
 \end{aligned}$$

$$\begin{aligned}
 \text{4 a} \quad & \frac{\log 4}{\log 2} \\
 &= \frac{\log(2^2)}{\log 2} \\
 &= \frac{2 \log 2}{\log 2} \\
 &= 2
 \end{aligned}$$

$$\begin{aligned}
 \text{b} \quad & \frac{\log_5 27}{\log_5 9} \\
 &= \frac{\log_5(3^3)}{\log_5(3^2)} \\
 &= \frac{3 \log_5 3}{2 \log_5 3} \\
 &= \frac{3}{2}
 \end{aligned}$$

$$\begin{aligned}
 \text{c} \quad & \frac{\log 8}{\log 2} \\
 &= \frac{\log(2^3)}{\log 2} \\
 &= \frac{3 \log 2}{\log 2} \\
 &= 3
 \end{aligned}$$

$$\begin{aligned}
 \text{d} \quad & \frac{\log 3}{\log 9} \\
 &= \frac{\log 3}{\log(3^2)} \\
 &= \frac{\log 3}{2 \log 3} \\
 &= \frac{1}{2}
 \end{aligned}$$

$$\begin{aligned}
 \text{e} \quad & \frac{\log_3 25}{\log_3(0.2)} \\
 &= \frac{\log_3(5^2)}{\log_3(5^{-1})} \\
 &= \frac{2 \log_3 5}{-1 \log_3 5} \\
 &= -2
 \end{aligned}$$

$$\begin{aligned}
 \text{f} \quad & \frac{\log_4 8}{\log_4(0.25)} \\
 &= \frac{\log_4(2^3)}{\log_4(2^{-2})} \quad \{0.25 = \frac{1}{4} = \frac{1}{2^2}\} \\
 &= \frac{3 \log_4 2}{-2 \log_4 2} \\
 &= -\frac{3}{2}
 \end{aligned}$$

$$\begin{aligned}
 \text{5 a} \quad & \log 9 = \log(3^2) \\
 &= 2 \log 3 \quad \checkmark
 \end{aligned}$$

$$\begin{aligned}
 \text{b} \quad & \log \sqrt{2} = \log(2^{\frac{1}{2}}) \\
 &= \frac{1}{2} \log 2 \quad \checkmark
 \end{aligned}$$

$$\begin{aligned}
 \text{c} \quad & \log\left(\frac{1}{8}\right) = \log\left(\frac{1}{2^3}\right) \\
 &= \log(2^{-3}) \\
 &= -3 \log 2 \quad \checkmark
 \end{aligned}$$

$$\begin{aligned}
 \text{d} \quad & \log\left(\frac{1}{5}\right) = \log(5^{-1}) \\
 &= -1 \times \log 5 \\
 &= -\log 5 \quad \checkmark
 \end{aligned}$$

$$\begin{aligned}
 \text{e} \quad & \log 5 = \log\left(\frac{10}{2}\right) \\
 &= \log(10^1) - \log 2 \\
 &= 1 - \log 2 \quad \checkmark
 \end{aligned}$$

$$\begin{aligned}
 \text{f} \quad & \log 5000 \\
 &= \log\left(\frac{10\,000}{2}\right) \\
 &= \log(10^4) - \log 2 \\
 &= 4 - \log 2 \quad \checkmark
 \end{aligned}$$

$$\begin{aligned}
 \text{6} \quad & \log(a \times 10^k) \\
 &= \log a + \log 10^k \\
 &= \log a + k
 \end{aligned}$$

$$\begin{aligned}
 7 \quad a \quad & \log_b 6 \\
 &= \log_b (2 \times 3) \\
 &= \log_b 2 + \log_b 3 \\
 &= p + q
 \end{aligned}$$

$$\begin{aligned}
 b \quad & \log_b 45 \\
 &= \log_b (9 \times 5) \\
 &= \log_b (3^2 \times 5) \\
 &= \log_b (3^2) + \log_b 5 \\
 &= 2 \log_b 3 + \log_b 5 \\
 &= 2q + r
 \end{aligned}$$

$$\begin{aligned}
 c \quad & \log_b 108 \\
 &= \log_b (4 \times 27) \\
 &= \log_b (2^2 \times 3^3) \\
 &= \log_b (2^2) + \log_b (3^3) \\
 &= 2 \log_b 2 + 3 \log_b 3 \\
 &= 2p + 3q
 \end{aligned}$$

$$\begin{aligned}
 d \quad & \log_b \left(\frac{5\sqrt{3}}{2} \right) \\
 &= \log_b (5 \times 3^{\frac{1}{2}}) - \log_b 2 \\
 &= \log_b 5 + \log_b (3^{\frac{1}{2}}) - \log_b 2 \\
 &= \log_b 5 + \frac{1}{2} \log_b 3 - \log_b 2 \\
 &= r + \frac{1}{2}q - p
 \end{aligned}$$

$$\begin{aligned}
 e \quad & \log_b \left(\frac{5}{32} \right) \\
 &= \log_b 5 - \log_b 32 \\
 &= \log_b 5 - \log_b (2^5) \\
 &= \log_b 5 - 5 \log_b 2 \\
 &= r - 5p
 \end{aligned}$$

$$\begin{aligned}
 f \quad & \log_b \left(\frac{2}{9} \right) \\
 &= \log_b 2 - \log_b 9 \\
 &= \log_b 2 - \log_b (3^2) \\
 &= \log_b 2 - 2 \log_b 3 \\
 &= p - 2q
 \end{aligned}$$

$$\begin{aligned}
 8 \quad a \quad & \log_2 (PR) \\
 &= \log_2 P + \log_2 R \\
 &= x + z
 \end{aligned}$$

$$\begin{aligned}
 b \quad & \log_2 (RQ^2) \\
 &= \log_2 R + \log_2 (Q^2) \\
 &= \log_2 R + 2 \log_2 Q \\
 &= z + 2y
 \end{aligned}$$

$$\begin{aligned}
 c \quad & \log_2 \left(\frac{PR}{Q} \right) \\
 &= \log_2 (PR) - \log_2 Q \\
 &= \log_2 P + \log_2 R - \log_2 Q \\
 &= x + z - y
 \end{aligned}$$

$$\begin{aligned}
 d \quad & \log_2 (P^2 \sqrt{Q}) \\
 &= \log_2 (P^2) + \log_2 (Q^{\frac{1}{2}}) \\
 &= 2 \log_2 P + \frac{1}{2} \log_2 Q \\
 &= 2x + \frac{1}{2}y
 \end{aligned}$$

$$\begin{aligned}
 e \quad & \log_2 \left(\frac{Q^3}{\sqrt{R}} \right) \\
 &= \log_2 (Q^3) - \log_2 (R^{\frac{1}{2}}) \\
 &= 3 \log_2 Q - \frac{1}{2} \log_2 R \\
 &= 3y - \frac{1}{2}z
 \end{aligned}$$

$$\begin{aligned}
 f \quad & \log_2 \left(\frac{R^2 \sqrt{Q}}{P^3} \right) \\
 &= \log_2 (R^2) + \log_2 (Q^{\frac{1}{2}}) - \log_2 (P^3) \\
 &= 2 \log_2 R + \frac{1}{2} \log_2 Q - 3 \log_2 P \\
 &= 2z + \frac{1}{2}y - 3x
 \end{aligned}$$

$$\begin{aligned}
 9 \quad a \quad & \log_t (N^2) = 1.72 \\
 \therefore 2 \log_t N &= 1.72 \\
 \therefore \log_t N &= 0.86
 \end{aligned}$$

$$\begin{aligned}
 b \quad & \log_t (MN) \\
 &= \log_t M + \log_t N \\
 &= 1.29 + 0.86 \\
 &= 2.15
 \end{aligned}$$

$$\begin{aligned}
 c \quad & \log_t \left(\frac{N^2}{\sqrt{M}} \right) \\
 &= \log_t (N^2) - \log_t (M^{\frac{1}{2}}) \\
 &= 1.72 - \frac{1}{2} \log_t M \\
 &= 1.72 - \frac{1}{2}(1.29) \\
 &= 1.075
 \end{aligned}$$

$$\begin{aligned}
10 \quad & \log(8!) - \log(7!) + \log(6!) - \log(5!) + \log(4!) - \log(3!) + \log(2!) - \log(1!) \\
&= \log\left(\frac{8!}{7!}\right) + \log\left(\frac{6!}{5!}\right) + \log\left(\frac{4!}{3!}\right) + \log\left(\frac{2!}{1!}\right) \\
&= \log\left(\frac{8 \times 7!}{7!}\right) + \log\left(\frac{6 \times 5!}{5!}\right) + \log\left(\frac{4 \times 3!}{3!}\right) + \log\left(\frac{2 \times 1!}{1!}\right) \\
&= \log 8 + \log 6 + \log 4 + \log 2 \\
&= \log(8 \times 6 \times 4 \times 2) \\
&= \log 384
\end{aligned}$$

11 We consider all of the possible ways of writing $\log_2(6!)$ in the form $a + \log_2 b$, where $a, b \in \mathbb{Z}$.

$$\begin{aligned}
&\log_2(6!) = \log_2 720 & \text{or} & \log_2(6!) = \log_2 720 \\
&= \log_2(2 \times 360) & &= \log_2(4 \times 180) \\
&= \log_2 2 + \log_2 360 & &= \log_2 4 + \log_2 180 \\
&= 1 + \log_2 360 & &= \log_2(2^2) + \log_2 180 \\
& & &= 2 + \log_2 180
\end{aligned}$$

$$\begin{aligned}
\text{or} \quad \log_2(6!) &= \log_2 720 & \text{or} \quad \log_2(6!) &= 720 \\
&= \log_2(8 \times 90) & &= \log_2(16 \times 45) \\
&= \log_2 8 + \log_2 90 & &= \log_2 16 + \log_2 45 \\
&= \log_2(2^3) + \log_2 90 & &= \log_2(2^4) + \log_2 45 \\
&= 3 + \log_2 90 & &= 4 + \log_2 45
\end{aligned}$$

$\log_2(6!) = 4 + \log_2 45$ where $a = 4$ and $b = 45$ gives the smallest possible value of b .

EXERCISE 6D

$$\begin{aligned}
1 \quad a \quad & \ln(e^2) \\
&= 2
\end{aligned}$$

$$\begin{aligned}
b \quad & \ln(e^4) \\
&= 4
\end{aligned}$$

$$\begin{aligned}
c \quad & \ln((\sqrt{e})^3) \\
&= \ln((e^{\frac{1}{2}})^3) \\
&= \ln(e^{\frac{3}{2}}) \\
&= \frac{3}{2}
\end{aligned}$$

$$\begin{aligned}
d \quad & \ln 1 \\
&= \ln(e^0) \\
&= 0
\end{aligned}$$

$$\begin{aligned}
e \quad & \ln\left(\frac{1}{e}\right) \\
&= \ln(e^{-1}) \\
&= -1
\end{aligned}$$

$$\begin{aligned}
f \quad & \ln \sqrt[3]{e} \\
&= \ln(e^{\frac{1}{3}}) \\
&= \frac{1}{3}
\end{aligned}$$

$$\begin{aligned}
g \quad & \ln\left(\frac{1}{e^2}\right) \\
&= \ln(e^{-2}) \\
&= -2
\end{aligned}$$

$$\begin{aligned}
h \quad & \ln\left(\frac{1}{\sqrt{e}}\right) \\
&= \ln(e^{-\frac{1}{2}}) \\
&= -\frac{1}{2}
\end{aligned}$$

$$\begin{aligned}
2 \quad a \quad & e^{\ln 3} \\
&= 3
\end{aligned}$$

$$\begin{aligned}
b \quad & e^{2 \ln 3} \\
&= (e^{\ln 3})^2 \\
&= 3^2 \\
&= 9
\end{aligned}$$

$$\begin{aligned}
c \quad & e^{-\ln 5} \\
&= (e^{\ln 5})^{-1} \\
&= 5^{-1} \\
&= \frac{1}{5}
\end{aligned}$$

$$\begin{aligned}
d \quad & e^{-2 \ln 2} \\
&= (e^{\ln 2})^{-2} \\
&= 2^{-2} \\
&= \frac{1}{2^2} \\
&= \frac{1}{4}
\end{aligned}$$

$$\begin{aligned} \mathbf{e} \quad \ln(e^a) \\ = a \end{aligned}$$

$$\begin{aligned} \mathbf{f} \quad \ln(e \times e^a) \\ = \ln(e^{1+a}) \\ = 1 + a \end{aligned}$$

$$\begin{aligned} \mathbf{g} \quad \ln(e^a \times e^b) \\ = \ln(e^{a+b}) \\ = a + b \end{aligned}$$

$$\begin{aligned} \mathbf{h} \quad \ln((e^a)^b) \\ = \ln(e^{ab}) \\ = ab \end{aligned}$$

$$\mathbf{3} \quad \mathbf{a} \quad \ln 12 \approx 2.485$$

$$\mathbf{b} \quad \ln 68 \approx 4.220$$

$$\mathbf{c} \quad \ln(1.4) \approx 0.336$$

$$\mathbf{d} \quad \ln(0.7) \approx -0.357$$

$$\mathbf{e} \quad \ln 500 \approx 6.215$$

$\mathbf{4} \quad x$ does not exist such that $e^x = -2$ or 0 since $e^x > 0$ for all $x \in \mathbb{R}$.
 $\therefore \ln(-2)$ and $\ln 0$ do not exist.

$$\begin{aligned} \mathbf{5} \quad \mathbf{a} \quad 6 &= e^{\ln 6} \\ &\approx e^{1.7918} \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad 60 &= e^{\ln 60} \\ &\approx e^{4.0943} \end{aligned}$$

$$\begin{aligned} \mathbf{c} \quad 6000 &= e^{\ln 6000} \\ &\approx e^{8.6995} \end{aligned}$$

$$\begin{aligned} \mathbf{d} \quad 0.6 &= e^{\ln(0.6)} \\ &\approx e^{-0.5108} \end{aligned}$$

$$\begin{aligned} \mathbf{e} \quad 0.006 &= e^{\ln(0.006)} \\ &\approx e^{-5.1160} \end{aligned}$$

$$\begin{aligned} \mathbf{f} \quad 15 &= e^{\ln 15} \\ &\approx e^{2.7081} \end{aligned}$$

$$\begin{aligned} \mathbf{g} \quad 1500 &= e^{\ln 1500} \\ &\approx e^{7.3132} \end{aligned}$$

$$\begin{aligned} \mathbf{h} \quad 1.5 &= e^{\ln(1.5)} \\ &\approx e^{0.4055} \end{aligned}$$

$$\begin{aligned} \mathbf{i} \quad 0.15 &= e^{\ln(0.15)} \\ &\approx e^{-1.8971} \end{aligned}$$

$$\begin{aligned} \mathbf{j} \quad 0.000\,15 &= e^{\ln(0.000\,15)} \\ &\approx e^{-8.8049} \end{aligned}$$

$$\begin{aligned} \mathbf{6} \quad \mathbf{a} \quad \ln x &= 3 \\ \therefore x &= e^3 \\ \therefore x &\approx 20.1 \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad \ln x &= 1 \\ \therefore x &= e^1 \\ \therefore x &= e \end{aligned}$$

$$\begin{aligned} \mathbf{c} \quad \ln x &= 0 \\ \therefore x &= e^0 \\ \therefore x &= 1 \end{aligned}$$

$$\begin{aligned} \mathbf{d} \quad \ln x &= -1 \\ \therefore x &= e^{-1} \\ \therefore x &\approx 0.368 \end{aligned}$$

$$\begin{aligned} \mathbf{e} \quad \ln x &= -5 \\ \therefore x &= e^{-5} \\ \therefore x &\approx 0.006\,74 \end{aligned}$$

$$\begin{aligned} \mathbf{f} \quad \ln x &\approx 0.835 \\ \therefore x &\approx e^{0.835} \\ \therefore x &\approx 2.30 \end{aligned}$$

$$\begin{aligned} \mathbf{g} \quad \ln x &\approx 2.145 \\ \therefore x &\approx e^{2.145} \\ \therefore x &\approx 8.54 \end{aligned}$$

$$\begin{aligned} \mathbf{h} \quad \ln x &\approx -3.2971 \\ \therefore x &\approx e^{-3.2971} \\ \therefore x &\approx 0.0370 \end{aligned}$$

$$\mathbf{7} \quad \mathbf{a} \quad \mathbf{i} \quad \ln(e^x) = x$$

$$\mathbf{ii} \quad e^{\ln x} = x$$

$\mathbf{b} \quad y = e^x$ and $y = \ln x$ are inverses of each other.

$$\begin{aligned} \mathbf{8} \quad \mathbf{a} \quad \ln 15 + \ln 3 \\ = \ln(15 \times 3) \\ = \ln 45 \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad \ln 15 - \ln 3 \\ = \ln\left(\frac{15}{3}\right) \\ = \ln 5 \end{aligned}$$

$$\begin{aligned} \mathbf{c} \quad \ln 20 - \ln 5 \\ = \ln\left(\frac{20}{5}\right) \\ = \ln 4 \end{aligned}$$

$$\begin{aligned} \mathbf{d} \quad \ln 4 + \ln 6 \\ = \ln(4 \times 6) \\ = \ln 24 \end{aligned}$$

$$\begin{aligned} \mathbf{e} \quad \ln 5 + \ln(0.2) \\ = \ln(5 \times 0.2) \\ = \ln 1 \\ = 0 \end{aligned}$$

$$\begin{aligned} \mathbf{f} \quad \ln 2 + \ln 3 + \ln 5 \\ = \ln(2 \times 3 \times 5) \\ = \ln 30 \end{aligned}$$

$$\begin{aligned} \mathbf{g} \quad 1 + \ln 4 \\ = \ln(e^1) + \ln 4 \\ = \ln(e \times 4) \\ = \ln(4e) \end{aligned}$$

$$\begin{aligned} \mathbf{h} \quad \ln 6 - 1 \\ = \ln 6 - \ln(e^1) \\ = \ln\left(\frac{6}{e}\right) \end{aligned}$$

$$\begin{aligned} \mathbf{i} \quad \ln 5 + \ln 8 - \ln 2 \\ = \ln(5 \times 8 \div 2) \\ = \ln 20 \end{aligned}$$

$$\begin{aligned}
 \text{j} \quad & 2 + \ln 4 \\
 &= \ln(e^2) + \ln 4 \\
 &= \ln(e^2 \times 4) \\
 &= \ln(4e^2)
 \end{aligned}$$

$$\begin{aligned}
 \text{k} \quad & \ln 20 - 2 \\
 &= \ln 20 - \ln(e^2) \\
 &= \ln\left(\frac{20}{e^2}\right)
 \end{aligned}$$

$$\begin{aligned}
 \text{l} \quad & \ln 12 - \ln 4 - \ln 3 \\
 &= \ln(12 \div 4 \div 3) \\
 &= \ln 1 \\
 &= 0
 \end{aligned}$$

$$\begin{aligned}
 \text{9 a} \quad & 5 \ln 3 + \ln 4 \\
 &= \ln(3^5) + \ln 4 \\
 &= \ln(243 \times 4) \\
 &= \ln 972
 \end{aligned}$$

$$\begin{aligned}
 \text{b} \quad & 3 \ln 2 + 2 \ln 5 \\
 &= \ln(2^3) + \ln(5^2) \\
 &= \ln(8 \times 25) \\
 &= \ln 200
 \end{aligned}$$

$$\begin{aligned}
 \text{c} \quad & 3 \ln 2 - \ln 8 \\
 &= \ln(2^3) - \ln 8 \\
 &= \ln\left(\frac{8}{8}\right) \\
 &= \ln 1
 \end{aligned}$$

$$\begin{aligned}
 \text{d} \quad & 3 \ln 4 - 2 \ln 2 \\
 &= \ln(4^3) - \ln(2^2) \\
 &= \ln\left(\frac{64}{4}\right) \\
 &= \ln 16
 \end{aligned}$$

$$\begin{aligned}
 \text{e} \quad & \frac{1}{3} \ln 8 + \ln 3 \\
 &= \ln(8^{\frac{1}{3}}) + \ln 3 \\
 &= \ln(2 \times 3) \\
 &= \ln 6
 \end{aligned}$$

$$\begin{aligned}
 \text{f} \quad & \frac{1}{3} \ln\left(\frac{1}{27}\right) \\
 &= \ln\left(\left(\frac{1}{27}\right)^{\frac{1}{3}}\right) \\
 &= \ln\left(\frac{1}{27^{\frac{1}{3}}}\right) \\
 &= \ln\left(\frac{1}{3}\right)
 \end{aligned}$$

$$\begin{aligned}
 \text{g} \quad & -\ln 2 \\
 &= \ln(2^{-1}) \\
 &= \ln\left(\frac{1}{2}\right)
 \end{aligned}$$

$$\begin{aligned}
 \text{h} \quad & -\ln\left(\frac{1}{2}\right) \\
 &= \ln\left(\left(\frac{1}{2}\right)^{-1}\right) \\
 &= \ln 2
 \end{aligned}$$

$$\begin{aligned}
 \text{i} \quad & -2 \ln\left(\frac{1}{4}\right) \\
 &= \ln\left(\left(\frac{1}{4}\right)^{-2}\right) \\
 &= \ln(4^2) \\
 &= \ln 16
 \end{aligned}$$

$$\begin{aligned}
 \text{j} \quad & 4 \ln 2 + 2 \\
 &= \ln(2^4) + \ln(e^2) \\
 &= \ln 16 + \ln(e^2) \\
 &= \ln(16e^2)
 \end{aligned}$$

$$\begin{aligned}
 \text{k} \quad & \frac{1}{2} \ln 9 - 1 \\
 &= \ln(9^{\frac{1}{2}}) - \ln(e^1) \\
 &= \ln 3 - \ln e \\
 &= \ln\left(\frac{3}{e}\right)
 \end{aligned}$$

$$\begin{aligned}
 \text{l} \quad & -3 \ln 2 + \frac{1}{2} \\
 &= \ln(2^{-3}) + \ln(e^{\frac{1}{2}}) \\
 &= \ln\left(\frac{1}{8}\right) + \ln \sqrt{e} \\
 &= \ln\left(\frac{\sqrt{e}}{8}\right)
 \end{aligned}$$

$$\begin{aligned}
 \text{10 a} \quad & \ln 27 \\
 &= \ln(3^3) \\
 &= 3 \ln 3
 \end{aligned}$$

$$\begin{aligned}
 \text{b} \quad & \ln \sqrt{3} \\
 &= \ln(3^{\frac{1}{2}}) \\
 &= \frac{1}{2} \ln 3
 \end{aligned}$$

$$\begin{aligned}
 \text{c} \quad & \ln\left(\frac{1}{16}\right) \\
 &= \ln\left(\frac{1}{2^4}\right) \\
 &= \ln(2^{-4}) \\
 &= -4 \ln 2
 \end{aligned}$$

$$\begin{aligned}
 \text{d} \quad & \ln\left(\frac{1}{6}\right) \\
 &= \ln(6^{-1}) \\
 &= -1 \times \ln 6 \\
 &= -\ln 6
 \end{aligned}$$

$$\begin{aligned}
 \text{e} \quad & \ln\left(\frac{1}{\sqrt{2}}\right) \\
 &= \ln(2^{-\frac{1}{2}}) \\
 &= -\frac{1}{2} \ln 2
 \end{aligned}$$

$$\begin{aligned}
 \text{f} \quad & \ln\left(\frac{e}{5}\right) \\
 &= \ln(e^1) - \ln 5 \\
 &= 1 - \ln 5
 \end{aligned}$$

$$\begin{aligned}
 \text{g} \quad & \ln(6e) \\
 &= \ln 6 + \ln e \\
 &= \ln 6 + 1
 \end{aligned}$$

$$\begin{aligned}
 \text{h} \quad & \ln \sqrt[3]{5} \\
 &= \ln(5^{\frac{1}{3}}) \\
 &= \frac{1}{3} \ln 5
 \end{aligned}$$

$$\begin{aligned}
 \text{i} \quad & \ln\left(\frac{1}{\sqrt[5]{2}}\right) \\
 &= \ln(2^{-\frac{1}{5}}) \\
 &= -\frac{1}{5} \ln 2
 \end{aligned}$$

$$\begin{aligned}
 \text{j} \quad & \ln\left(\frac{e^2}{8}\right) \\
 &= \ln(e^2) - \ln 8 \\
 &= \ln(e^2) - \ln(2^3) \\
 &= 2 - 3\ln 2
 \end{aligned}$$

$$\begin{aligned}
 \text{k} \quad & \ln\left(\frac{\sqrt{3}}{e^4}\right) \\
 &= \ln(3^{\frac{1}{2}}) - \ln(e^4) \\
 &= \frac{1}{2}\ln 3 - 4
 \end{aligned}$$

$$\begin{aligned}
 \text{l} \quad & \ln\left(\frac{1}{16 \times \sqrt[3]{e}}\right) \\
 &= \ln 1 - \ln(16 \times \sqrt[3]{e}) \\
 &= 0 - (\ln 16 + \ln(e^{\frac{1}{3}})) \\
 &= -(\ln(2^4) + \frac{1}{3}) \\
 &= -(4\ln 2 + \frac{1}{3}) \\
 &= -4\ln 2 - \frac{1}{3}
 \end{aligned}$$

EXERCISE 6E

$$\begin{aligned}
 \text{1 a} \quad & y = 2^x \\
 \therefore \log y &= \log(2^x) \\
 \therefore \log y &= x \log 2
 \end{aligned}$$

$$\begin{aligned}
 \text{c} \quad & M = ad^4 \\
 \therefore \log M &= \log(ad^4) \\
 \therefore \log M &= \log a + \log(d^4) \\
 \therefore \log M &= \log a + 4\log d
 \end{aligned}$$

$$\begin{aligned}
 \text{e} \quad & R = b\sqrt{l} \\
 \therefore \log R &= \log(bl^{\frac{1}{2}}) \\
 \therefore \log R &= \log b + \log(l^{\frac{1}{2}}) \\
 \therefore \log R &= \log b + \frac{1}{2}\log l
 \end{aligned}$$

$$\begin{aligned}
 \text{g} \quad & y = ab^x \\
 \therefore \log y &= \log(ab^x) \\
 \therefore \log y &= \log a + \log(b^x) \\
 \therefore \log y &= \log a + x \log b
 \end{aligned}$$

$$\begin{aligned}
 \text{i} \quad & L = \frac{ab}{c} \\
 \therefore \log L &= \log\left(\frac{ab}{c}\right) \\
 \therefore \log L &= \log(ab) - \log c \\
 \therefore \log L &= \log a + \log b - \log c
 \end{aligned}$$

$$\begin{aligned}
 \text{b} \quad & y = 20b^3 \\
 \therefore \log y &= \log(20b^3) \\
 \therefore \log y &= \log 20 + \log(b^3) \\
 \therefore \log y &= \log 20 + 3\log b
 \end{aligned}$$

$$\begin{aligned}
 \text{d} \quad & T = 5\sqrt{d} \\
 \therefore \log T &= \log(5d^{\frac{1}{2}}) \\
 \therefore \log T &= \log 5 + \log(d^{\frac{1}{2}}) \\
 \therefore \log T &= \log 5 + \frac{1}{2}\log d
 \end{aligned}$$

$$\begin{aligned}
 \text{f} \quad & Q = \frac{a}{b^n} \\
 \therefore \log Q &= \log\left(\frac{a}{b^n}\right) \\
 \therefore \log Q &= \log a - \log(b^n) \\
 \therefore \log Q &= \log a - n \log b
 \end{aligned}$$

$$\begin{aligned}
 \text{h} \quad & F = \frac{20}{\sqrt{n}} \\
 \therefore \log F &= \log\left(\frac{20}{n^{\frac{1}{2}}}\right) \\
 \therefore \log F &= \log 20 - \log(n^{\frac{1}{2}}) \\
 \therefore \log F &= \log 20 - \frac{1}{2}\log n
 \end{aligned}$$

$$\begin{aligned}
 \text{j} \quad & N = \sqrt{\frac{a}{b}} \\
 \therefore N &= \left(\frac{a}{b}\right)^{\frac{1}{2}} \\
 \therefore \log N &= \log\left(\left(\frac{a}{b}\right)^{\frac{1}{2}}\right) \\
 \therefore \log N &= \frac{1}{2}\log\left(\frac{a}{b}\right) \\
 \therefore \log N &= \frac{1}{2}\log a - \frac{1}{2}\log b
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{k} \quad S &= 200 \times 2^t \\
 \therefore \log S &= \log(200 \times 2^t) \\
 \therefore \log S &= \log 200 + \log(2^t) \\
 \therefore \log S &= \log 200 + t \log 2
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{2} \quad \mathbf{a} \quad \log D &= \log e + \log 2 \\
 \therefore \log D &= \log(2e) \\
 \therefore D &= 2e
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{c} \quad \log P &= \frac{1}{2} \log x \\
 \therefore \log P &= \log(x^{\frac{1}{2}}) \\
 \therefore P &= \sqrt{x}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{e} \quad \log B &= 3 \log m - 2 \log n \\
 \therefore \log B &= \log(m^3) - \log(n^2) \\
 \therefore \log B &= \log\left(\frac{m^3}{n^2}\right) \\
 \therefore B &= \frac{m^3}{n^2}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{g} \quad \log P &= 3 \log x + 1 \\
 \therefore \log P &= \log(x^3) + \log(10^1) \\
 \therefore \log P &= \log(10x^3) \\
 \therefore P &= 10x^3
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{3} \quad \mathbf{a} \quad \ln D &= \ln x + 1 \\
 \therefore \ln D - \ln x &= 1 \\
 \therefore \ln\left(\frac{D}{x}\right) &= 1 \\
 \therefore \frac{D}{x} &= e^1 \\
 \therefore D &= ex
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{c} \quad \ln P &= \frac{1}{2} \ln x \\
 \therefore \ln P &= \ln(x^{\frac{1}{2}}) \\
 \therefore P &= \sqrt{x}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{l} \quad y &= \frac{a^m}{b^n} \\
 \therefore \log y &= \log\left(\frac{a^m}{b^n}\right) \\
 \therefore \log y &= \log(a^m) - \log(b^n) \\
 \therefore \log y &= m \log a - n \log b
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad \log_a F &= \log_a 5 - \log_a t \\
 \therefore \log_a F &= \log_a\left(\frac{5}{t}\right) \\
 \therefore F &= \frac{5}{t}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{d} \quad \log_n M &= 2 \log_n b + \log_n c \\
 \therefore \log_n M &= \log_n(b^2) + \log_n c \\
 \therefore \log_n M &= \log_n(b^2 c) \\
 \therefore M &= b^2 c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{f} \quad \log N &= -\frac{1}{3} \log p \\
 \therefore \log N &= \log(p^{-\frac{1}{3}}) \\
 \therefore \log N &= \log\left(\frac{1}{\sqrt[3]{p}}\right) \\
 \therefore N &= \frac{1}{\sqrt[3]{p}}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{h} \quad \log_a Q &= 2 - \log_a x \\
 \therefore \log_a Q &= \log_a(a^2) - \log_a x \\
 \therefore \log_a Q &= \log_a\left(\frac{a^2}{x}\right) \\
 \therefore Q &= \frac{a^2}{x}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad \ln F &= -\ln p + 2 \\
 \therefore \ln F + \ln p &= 2 \\
 \therefore \ln(Fp) &= 2 \\
 \therefore Fp &= e^2 \\
 \therefore F &= \frac{e^2}{p}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{d} \quad \ln M &= 2 \ln y + 3 \\
 \therefore \ln M - 2 \ln y &= 3 \\
 \therefore \ln M - \ln(y^2) &= 3 \\
 \therefore \ln\left(\frac{M}{y^2}\right) &= 3 \\
 \therefore \frac{M}{y^2} &= e^3 \\
 \therefore M &= e^3 y^2
 \end{aligned}$$

$$\begin{aligned}
 \text{e} \quad & \ln B = 3 \ln t - 1 \\
 & \therefore \ln B - 3 \ln t = -1 \\
 & \therefore \ln B - \ln(t^3) = -1 \\
 & \therefore \ln\left(\frac{B}{t^3}\right) = -1 \\
 & \therefore \frac{B}{t^3} = e^{-1} \\
 & \therefore B = \frac{t^3}{e}
 \end{aligned}$$

$$\begin{aligned}
 \text{g} \quad & \ln Q \approx 3 \ln x + 2.159 \\
 & \therefore \ln Q - 3 \ln x \approx 2.159 \\
 & \therefore \ln Q - \ln(x^3) \approx 2.159 \\
 & \therefore \ln\left(\frac{Q}{x^3}\right) \approx 2.159 \\
 & \therefore \frac{Q}{x^3} \approx e^{2.159} \\
 & \therefore \frac{Q}{x^3} \approx 8.66 \\
 & \therefore Q \approx 8.66x^3
 \end{aligned}$$

$$\begin{aligned}
 4 \quad \text{a} \quad & y = 3 \times 2^x \\
 & \therefore \log_2 y = \log_2(3 \times 2^x) \\
 & \therefore \log_2 y = \log_2 3 + \log_2(2^x) \\
 & \therefore \log_2 y = \log_2 3 + x
 \end{aligned}$$

$$\begin{aligned}
 \text{c} \quad \text{i} \quad & \text{When } y = 3, \\
 & x = \log_2\left(\frac{3}{3}\right) \\
 & \therefore x = \log_2 1 \\
 & \therefore x = 0
 \end{aligned}$$

$$\begin{aligned}
 5 \quad \text{a} \quad & \log_3 27 + \log_3\left(\frac{1}{3}\right) = \log_3 x \\
 & \therefore \log_3\left(27 \times \frac{1}{3}\right) = \log_3 x \\
 & \therefore \log_3 9 = \log_3 x \\
 & \therefore x = 9
 \end{aligned}$$

$$\begin{aligned}
 \text{c} \quad & \log_5 125 - \log_5 \sqrt{5} = \log_5 x \\
 & \therefore \log_5\left(\frac{125}{\sqrt{5}}\right) = \log_5 x \\
 & \therefore x = \frac{125}{\sqrt{5}} = 25\sqrt{5}
 \end{aligned}$$

$$\begin{aligned}
 \text{f} \quad & \ln N = -\frac{1}{3} \ln g \\
 & \therefore \ln N = \ln(g^{-\frac{1}{3}}) \\
 & \therefore N = g^{-\frac{1}{3}} \\
 & \therefore N = \frac{1}{\sqrt[3]{g}}
 \end{aligned}$$

$$\begin{aligned}
 \text{h} \quad & \ln D \approx 0.4 \ln n - 0.6582 \\
 & \therefore \ln D - 0.4 \ln n \approx -0.6582 \\
 & \therefore \ln D - \ln(n^{0.4}) \approx -0.6582 \\
 & \therefore \ln\left(\frac{D}{n^{0.4}}\right) \approx -0.6582 \\
 & \therefore \frac{D}{n^{0.4}} \approx e^{-0.6582} \\
 & \therefore \frac{D}{n^{0.4}} \approx 0.518 \\
 & \therefore D \approx 0.518n^{0.4}
 \end{aligned}$$

$$\begin{aligned}
 \text{b} \quad & \log_2 y = \log_2 3 + x \quad \{\text{from a}\} \\
 & \therefore x = \log_2 y - \log_2 3 \\
 & \therefore x = \log_2\left(\frac{y}{3}\right)
 \end{aligned}$$

$$\begin{aligned}
 \text{ii} \quad & \text{When } y = 12, \\
 & x = \log_2\left(\frac{12}{3}\right) \\
 & \therefore x = \log_2 4 \\
 & \therefore x = \log_2(2^2) \\
 & \therefore x = 2
 \end{aligned}$$

$$\begin{aligned}
 \text{iii} \quad & \text{When } y = 30, \\
 & x = \log_2\left(\frac{30}{3}\right) \\
 & \therefore x = \log_2 10 \\
 & \therefore x \approx 3.32
 \end{aligned}$$

$$\begin{aligned}
 \text{b} \quad & \log_5 x = \log_5 8 - \log_5(6 - x) \\
 & \therefore \log_5 x = \log_5\left(\frac{8}{6 - x}\right) \\
 & \therefore x = \frac{8}{6 - x} \quad \text{Note: } x > 0 \\
 & \therefore 6x - x^2 = 8 \quad \text{and } 6 - x > 0 \\
 & \therefore x^2 - 6x + 8 = 0 \quad \text{so } 0 < x < 6 \\
 & \therefore (x - 2)(x - 4) = 0 \\
 & \therefore x = 2 \text{ or } 4
 \end{aligned}$$

$$\begin{aligned}
 \text{d} \quad & \log_{20} x = 1 + \log_{20} 10 \\
 & \therefore \log_{20} x = \log_{20}(20^1) + \log_{20} 10 \\
 & \quad = \log_{20} 200 \\
 & \therefore x = 200
 \end{aligned}$$

$$\begin{aligned}
 \text{e } \log x + \log(x+1) &= \log 30 \\
 \therefore \log[x(x+1)] &= \log 30 \\
 \therefore x^2 + x &= 30 \\
 \therefore x^2 + x - 30 &= 0 \\
 \therefore (x+6)(x-5) &= 0 \\
 \therefore x &= -6 \text{ or } 5 \\
 \therefore x &= 5 \quad \{x > 0\}
 \end{aligned}$$

$$\begin{aligned}
 \text{f } \log(x+2) - \log(x-2) &= \log 5 \\
 \therefore \log\left(\frac{x+2}{x-2}\right) &= \log 5 \\
 \therefore \frac{x+2}{x-2} &= 5 \\
 \therefore x+2 &= 5x-10 \\
 \therefore -4x &= -12 \\
 \therefore x &= 3
 \end{aligned}$$

Note: $x+2 > 0$ and $x-2 > 0$
 $\therefore x > 2$ ✓

$$\begin{aligned}
 \text{6 a } x &= \log_2 7 \\
 \therefore 2^x &= 2^{\log_2 7} \\
 \therefore 2^x &= 7
 \end{aligned}$$

$$\begin{aligned}
 \text{b } \log(2^x) &= \log 7 \\
 \therefore x \log 2 &= \log 7 \\
 \therefore x &= \frac{\log 7}{\log 2} \\
 \text{But } x &= \log_2 7 \quad \{\text{from a}\} \\
 \therefore \log_2 7 &= \frac{\log 7}{\log 2} \approx 2.81
 \end{aligned}$$

$$\begin{aligned}
 \text{7 a } a^x &= b, \quad a, b > 0 \\
 \therefore \log_a(a^x) &= \log_a b \\
 \therefore x &= \log_a b
 \end{aligned}$$

$$\begin{aligned}
 \text{b } a^x &= b \\
 \therefore \log(a^x) &= \log b
 \end{aligned}$$

$$\begin{aligned}
 \text{c Using b, } x \log a &= \log b \\
 \therefore x &= \frac{\log b}{\log a}
 \end{aligned}$$

and using part a, $x = \log_a b = \frac{\log b}{\log a}$

EXERCISE 6F

$$\begin{aligned}
 \text{1 a } \log_3 7 \\
 &= \frac{\log 7}{\log 3} \\
 &\approx 1.77 \\
 \text{Check: } \frac{\ln 7}{\ln 3} &\approx 1.77 \quad \checkmark
 \end{aligned}$$

$$\begin{aligned}
 \text{c } \log_5 180 \\
 &= \frac{\log 180}{\log 5} \\
 &\approx 3.23 \\
 \text{Check: } \frac{\ln 180}{\ln 5} &\approx 3.23 \quad \checkmark
 \end{aligned}$$

$$\begin{aligned}
 \text{b } \log_2 40 \\
 &= \frac{\log 40}{\log 2} \\
 &\approx 5.32 \\
 \text{Check: } \frac{\ln 40}{\ln 2} &\approx 5.32 \quad \checkmark
 \end{aligned}$$

$$\begin{aligned}
 \text{d } \log_{\frac{1}{2}} 1250 \\
 &= \frac{\log 1250}{\log \frac{1}{2}} \\
 &\approx -10.3 \\
 \text{Check: } \frac{\ln 1250}{\ln \frac{1}{2}} &\approx -10.3 \quad \checkmark
 \end{aligned}$$

$$\begin{aligned}
 \text{e} \quad & \log_3(0.067) \\
 &= \frac{\log(0.067)}{\log 3} \\
 &\approx -2.46 \\
 \text{Check: } & \frac{\ln(0.067)}{\ln 3} \approx -2.46 \quad \checkmark
 \end{aligned}$$

$$\begin{aligned}
 \text{f} \quad & \log_{0.4}(0.006\,984) \\
 &= \frac{\log(0.006\,984)}{\log(0.4)} \\
 &\approx 5.42 \\
 \text{Check: } & \frac{\ln(0.006\,984)}{\ln(0.4)} \approx 5.42 \quad \checkmark
 \end{aligned}$$

$$\begin{aligned}
 \text{2} \quad & \log_m n \times \log_n(m^2) \\
 &= \frac{\log_n n}{\log_n m} \times \log_n(m^2) \quad \left\{ \log_b a = \frac{\log_c a}{\log_c b} \right\} \\
 &= \frac{1}{\log_n m} \times 2 \log_n m \\
 &= 1 \times 2 \\
 &= 2
 \end{aligned}$$

$$\begin{aligned}
 \text{3} \quad & \frac{4}{\log_5 4} + \frac{3}{\log_7 8} = \frac{4}{\frac{\log 4}{\log 5}} + \frac{3}{\frac{\log 8}{\log 7}} \quad \left\{ \log_b a = \frac{\log_c a}{\log_c b} \right\} \\
 &= \frac{4 \log 5}{\log 4} + \frac{3 \log 7}{\log 8} \\
 &= \frac{4 \log 5}{2 \log 2} + \frac{3 \log 7}{3 \log 2} \\
 &= \frac{2 \log 5}{\log 2} + \frac{\log 7}{\log 2} \\
 &= \frac{\log(5^2)}{\log 2} + \frac{\log 7}{\log 2} \\
 &= \log_2 25 + \log_2 7 \quad \left\{ \frac{\log_c a}{\log_c b} = \log_b a \right\} \\
 &= \log_2(25 \times 7) \quad \{ \log_c(ab) = \log_c a + \log_c b \} \\
 &= \log_2 175 \\
 \therefore & 2^{\frac{4}{\log_5 4} + \frac{3}{\log_7 8}} = 2^{\log_2 175} = 175
 \end{aligned}$$

$$\begin{aligned}
 \text{4} \quad \text{a} \quad & \log_4(x^3) + \log_2 \sqrt{x} = 8 \\
 \therefore & \frac{\log_2(x^3)}{\log_2 4} + \log_2(x^{\frac{1}{2}}) = 8 \quad \left\{ \log_b a = \frac{\log_c a}{\log_c b} \right\} \\
 \therefore & \frac{1}{2} \times 3 \log_2 x + \frac{1}{2} \log_2 x = 8 \quad \{ m \log_a b = \log_a(b^m) \} \\
 \therefore & 2 \log_2 x = 8 \\
 \therefore & \log_2 x = 4 \\
 \therefore & x = 2^4 \\
 \therefore & x = 16
 \end{aligned}$$

$$\begin{aligned}
 \text{b} \quad & \log_{16}(x^5) = \log_{64} 125 - \log_4 \sqrt{x} \\
 \therefore & \frac{\log_4(x^5)}{\log_4 16} = \frac{\log_4 125}{\log_4 64} - \log_4(x^{\frac{1}{2}}) \quad \left\{ \log_b a = \frac{\log_c a}{\log_c b} \right\} \\
 \therefore & \frac{1}{2} \times 5 \log_4 x = \frac{1}{3} \log_4 125 - \frac{1}{2} \log_4 x \\
 \therefore & 3 \log_4 x = \log_4(125^{\frac{1}{3}}) \\
 \therefore & \log_4 x = \frac{1}{3} \log_4 5 \\
 \therefore & \log_4 x = \log_4(5^{\frac{1}{3}}) \\
 \therefore & x = \sqrt[3]{5} \approx 1.71
 \end{aligned}$$

$$\begin{aligned}
 \text{5} \quad & x = 2 \log_3 y \\
 & = \frac{2 \log_y y}{\log_y 3} \quad \left\{ \log_a b = \frac{\log_c a}{\log_c b} \right\} \\
 \therefore & \log_y 3 = \frac{2}{x} \quad \dots (*) \\
 \text{Thus} \quad & \log_y 81 = \log_y(3^4) \\
 & = 4 \log_y 3 \\
 & = 4 \left(\frac{2}{x} \right) \quad \{\text{using } (*)\} \\
 & = \frac{8}{x}
 \end{aligned}$$

EXERCISE 6G

$$\begin{aligned}
 \text{1} \quad \text{a} \quad & 2^4 = 16 \text{ and } 2^5 = 32 \\
 & \text{Since } 16 < 20 < 32, \text{ then } 2^4 < 20 < 2^5, \text{ and the} \\
 & \text{solution to } 2^x = 20 \text{ lies between } x = 4 \text{ and } x = 5.
 \end{aligned}$$

$$\text{c} \quad x = \frac{\log 20}{\log 2} \approx 4.32$$

$$\begin{aligned}
 \text{2} \quad \text{a} \quad & 3^3 = 27 \text{ and } 3^4 = 81 \\
 & \text{Since } 27 < 40 < 81, \text{ then } 3^3 < 40 < 3^4, \text{ and the} \\
 & \text{solution to } 3^x = 40 \text{ lies between } x = 3 \text{ and } x = 4.
 \end{aligned}$$

$$\text{c} \quad x = \frac{\log 40}{\log 3} \approx 3.36$$

$$\begin{aligned}
 \text{b} \quad & 2^x = 20 \\
 \therefore & \log(2^x) = \log 20 \\
 \therefore & x \log 2 = \log 20 \\
 \therefore & x = \frac{\log 20}{\log 2}
 \end{aligned}$$

$$\begin{aligned}
 \text{b} \quad & 3^x = 40 \\
 \therefore & \log(3^x) = \log 40 \\
 \therefore & x \log 3 = \log 40 \\
 \therefore & x = \frac{\log 40}{\log 3}
 \end{aligned}$$

$$\begin{aligned}
 \text{3} \quad \text{a} \quad \text{i} \quad & 2^x = 10 \\
 \therefore & \log(2^x) = \log 10 \\
 \therefore & x \log 2 = \log 10^1 \\
 \therefore & x = \frac{1}{\log 2}
 \end{aligned}$$

$$\text{ii} \quad x = \frac{1}{\log 2} \approx 3.32$$

$$\begin{aligned}
 \text{b} \quad \text{i} \quad & 3^x = 20 \\
 \therefore & \log(3^x) = \log 20 \\
 \therefore & x \log 3 = \log 20 \\
 \therefore & x = \frac{\log 20}{\log 3}
 \end{aligned}$$

$$\text{ii} \quad x = \frac{\log 20}{\log 3} \approx 2.73$$

$$\begin{aligned} \text{c i} \quad & 4^x = 50 \\ \therefore \log(4^x) &= \log 50 \\ \therefore x \log 4 &= \log 50 \\ \therefore x &= \frac{\log 50}{\log 4} \end{aligned}$$

$$\text{ii} \quad x = \frac{\log 50}{\log 4} \approx 2.82$$

$$\begin{aligned} \text{e i} \quad & \left(\frac{3}{4}\right)^x = 0.1 \\ \therefore \log\left(\frac{3}{4}\right)^x &= \log(10^{-1}) \\ \therefore x \log\left(\frac{3}{4}\right) &= -1 \\ \therefore x &= -\frac{1}{\log\left(\frac{3}{4}\right)} \end{aligned}$$

$$\text{ii} \quad x = -\frac{1}{\log\left(\frac{3}{4}\right)} \approx 8.00$$

$$\begin{aligned} \text{4 a} \quad & 5^x = 40 \\ \therefore \log(5^x) &= \log 40 \\ \therefore x \log 5 &= \log 40 \\ \therefore x &= \frac{\log 40}{\log 5} \approx 2.29 \end{aligned}$$

$$\begin{aligned} \text{c} \quad & 2^{x+4} = 5^{2-x} \\ \therefore \log(2^x \times 2^4) &= \log(5^2 \times 5^{-x}) \\ \therefore \log(2^x) + \log(2^4) &= \log(5^2) + \log(5^{-x}) \\ \therefore x \log 2 + 4 \log 2 &= 2 \log 5 - x \log 5 \\ \therefore x(\log 2 + \log 5) &= 2 \log 5 - 4 \log 2 \\ \therefore x &= \frac{2 \log 5 - 4 \log 2}{\log 2 + \log 5} \approx 0.194 \end{aligned}$$

$$\begin{aligned} \text{5 a} \quad & e^x = 10 \\ \therefore x &= \ln 10 \end{aligned}$$

$$\begin{aligned} \text{d} \quad & e^{\frac{x}{2}} = 5 \\ \therefore \frac{x}{2} &= \ln 5 \\ \therefore x &= 2 \ln 5 \end{aligned}$$

$$\begin{aligned} \text{b} \quad & e^x = 1000 \\ \therefore x &= \ln 1000 \end{aligned}$$

$$\begin{aligned} \text{e} \quad & e^{2x} = 18 \\ \therefore 2x &= \ln 18 \\ \therefore x &= \frac{1}{2} \ln 18 \end{aligned}$$

$$\begin{aligned} \text{d i} \quad & \left(\frac{1}{2}\right)^x = 0.0625 \\ \therefore \log\left(\frac{1}{2}\right)^x &= \log\left(\frac{1}{16}\right) \\ \therefore x \log(2^{-1}) &= \log(2^{-4}) \\ \therefore x &= \frac{-4 \log 2}{-\log 2} \\ \therefore x &= 4 \end{aligned}$$

$$\text{ii} \quad x = 4$$

$$\begin{aligned} \text{f i} \quad & 10^x = 0.000\,015 \\ \therefore \log 10^x &= \log(0.000\,015) \\ \therefore x \log 10 &= \log(0.000\,015) \\ \therefore x &= \log(0.000\,015) \end{aligned}$$

$$\begin{aligned} \text{ii} \quad & x = \log(0.000\,015) \\ & \approx -4.82 \end{aligned}$$

$$\begin{aligned} \text{b} \quad & 3^x = 2^{x+3} \\ \therefore \log(3^x) &= \log(2^x \times 2^3) \\ \therefore \log(3^x) &= \log(2^x) + \log(2^3) \\ \therefore x \log 3 &= x \log 2 + 3 \log 2 \\ \therefore x(\log 3 - \log 2) &= 3 \log 2 \\ \therefore x &= \frac{3 \log 2}{\log 3 - \log 2} \approx 5.13 \end{aligned}$$

$$\begin{aligned} \text{c} \quad & 2e^x = 0.3 \\ \therefore e^x &= 0.15 \\ \therefore x &= \ln(0.15) \end{aligned}$$

$$\begin{aligned} \text{f} \quad & e^{-\frac{x}{2}} = 1 \\ \therefore -\frac{x}{2} &= \ln 1 \\ \therefore -\frac{x}{2} &= 0 \\ \therefore x &= 0 \end{aligned}$$

$$\begin{aligned}
 \text{6 a} \quad & 3 \times 2^x = 75 \\
 & \therefore 2^x = 25 \\
 & \therefore \log(2^x) = \log 25 \\
 & \therefore x \log 2 = \log 25 \\
 & \therefore x = \frac{\log 25}{\log 2}
 \end{aligned}$$

$$\begin{aligned}
 \text{c} \quad & 5 \times (0.8)^x = 3 \\
 & \therefore (0.8)^x = 0.6 \\
 & \therefore \log((0.8)^x) = \log(0.6) \\
 & \therefore x \log(0.8) = \log(0.6) \\
 & \therefore x = \frac{\log(0.6)}{\log(0.8)}
 \end{aligned}$$

$$\begin{aligned}
 \text{e} \quad & 300 \times 5^{0.1x} = 1000 \\
 & \therefore 5^{0.1x} = \frac{10}{3} \\
 & \therefore \log(5^{0.1x}) = \log\left(\frac{10}{3}\right) \\
 & \therefore 0.1x \log 5 = \log\left(\frac{10}{3}\right) \\
 & \therefore x \log 5 = 10 \log\left(\frac{10}{3}\right) \\
 & \therefore x = \frac{10 \log\left(\frac{10}{3}\right)}{\log 5}
 \end{aligned}$$

$$\begin{aligned}
 \text{7 a} \quad & 25^x - 3 \times 5^x = 0 \\
 & \therefore 5^x(5^x - 3) = 0 \\
 & \therefore 5^x = 0 \text{ or } 5^x - 3 = 0 \\
 & \quad \therefore 5^x = 3 \quad \{5^x \neq 0\} \\
 & \therefore \log(5^x) = \log 3 \\
 & \therefore x \log 5 = \log 3 \\
 & \therefore x = \frac{\log 3}{\log 5}
 \end{aligned}$$

$$\begin{aligned}
 \text{c} \quad & 2^x - 2 \times 4^x = 0 \\
 & \therefore 2^x(1 - 2 \times 2^x) = 0 \\
 & \therefore 2^x = 0 \text{ or } 1 - 2 \times 2^x = 0 \\
 & \quad \therefore 1 = 2 \times 2^x \quad \{2^x \neq 0\} \\
 & \quad \therefore 2^x = \frac{1}{2} \\
 & \quad \therefore 2^x = 2^{-1} \\
 & \quad \therefore x = -1
 \end{aligned}$$

$$\begin{aligned}
 \text{b} \quad & 7 \times (1.5)^x = 20 \\
 & \therefore (1.5)^x = \frac{20}{7} \\
 & \therefore \log((1.5)^x) = \log\left(\frac{20}{7}\right) \\
 & \therefore x \log(1.5) = \log\left(\frac{20}{7}\right) \\
 & \therefore x = \frac{\log\left(\frac{20}{7}\right)}{\log(1.5)}
 \end{aligned}$$

$$\begin{aligned}
 \text{d} \quad & 4 \times 2^{-x} = 0.12 \\
 & \therefore 2^{-x} = 0.03 \\
 & \therefore \log(2^{-x}) = \log(0.03) \\
 & \therefore -x \log 2 = \log(0.03) \\
 & \therefore x = -\frac{\log(0.03)}{\log 2}
 \end{aligned}$$

$$\begin{aligned}
 \text{f} \quad & 32 \times e^{-0.25x} = 4 \\
 & \therefore e^{-0.25x} = \frac{1}{8} \\
 & \therefore \ln(e^{-0.25x}) = \ln\left(\frac{1}{8}\right) \\
 & \therefore -0.25x \ln e = \ln 1 - \ln 8 \\
 & \therefore -0.25x = 0 - \ln 8 \\
 & \therefore x = 4 \ln 8
 \end{aligned}$$

$$\begin{aligned}
 \text{b} \quad & 8 \times 9^x - 3^x = 0 \\
 & \therefore 3^x(8 \times 3^x - 1) = 0 \\
 & \therefore 3^x = 0 \text{ or } 8 \times 3^x - 1 = 0 \\
 & \quad \therefore 8 \times 3^x = 1 \quad \{3^x \neq 0\} \\
 & \quad \therefore 3^x = \frac{1}{8} \\
 & \therefore \log(3^x) = \log\left(\frac{1}{8}\right) \\
 & \therefore x \log 3 = \log\left(\frac{1}{8}\right) \\
 & \therefore x = \frac{\log(8^{-1})}{\log 3} \\
 & \therefore x = -\frac{\log 8}{\log 3}
 \end{aligned}$$

$$\begin{aligned}
 \text{8 a} \quad & e^{2x} = 2e^x \\
 \therefore & e^{2x} - 2e^x = 0 \\
 \therefore & e^x(e^x - 2) = 0 \\
 \therefore & e^x = 0 \text{ or } 2 \\
 \therefore & x = \ln 2 \quad \{e^x > 0\}
 \end{aligned}$$

$$\begin{aligned}
 \text{b} \quad & e^x = e^{-x} \\
 \therefore & e^{2x} = 1 \\
 \therefore & 2x = 0 \\
 \therefore & x = 0
 \end{aligned}$$

$$\begin{aligned}
 \text{c} \quad & e^{2x} - 5e^x + 6 = 0 \\
 \therefore & (e^x - 2)(e^x - 3) = 0 \\
 \therefore & e^x = 2 \text{ or } 3 \\
 \therefore & x = \ln 2 \text{ or } \ln 3
 \end{aligned}$$

$$\begin{aligned}
 \text{d} \quad & e^x + 2 = 3e^{-x} \\
 \therefore & e^{2x} + 2e^x = 3 \\
 \therefore & e^{2x} + 2e^x - 3 = 0 \\
 \therefore & (e^x + 3)(e^x - 1) = 0 \\
 \therefore & e^x = -3 \text{ or } 1 \\
 \therefore & e^x = 1 \quad \{e^x > 0\} \\
 \therefore & x = 0
 \end{aligned}$$

$$\begin{aligned}
 \text{e} \quad & 1 + 12e^{-x} = e^x \\
 \therefore & e^x + 12 = e^{2x} \\
 \therefore & e^{2x} - e^x - 12 = 0 \\
 \therefore & (e^x - 4)(e^x + 3) = 0 \\
 \therefore & e^x = 4 \text{ or } -3 \\
 \therefore & e^x = 4 \quad \{e^x > 0\} \\
 \therefore & x = \ln 4
 \end{aligned}$$

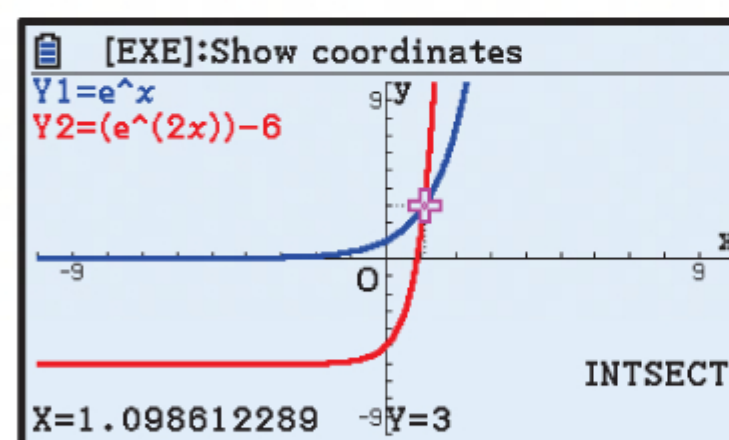
$$\begin{aligned}
 \text{f} \quad & e^x + e^{-x} = 3 \\
 \therefore & e^{2x} + 1 = 3e^x \\
 \therefore & e^{2x} - 3e^x + 1 = 0 \\
 \therefore & e^x = \frac{-(-3) \pm \sqrt{(-3)^2 - 4(1)(1)}}{2(1)} \\
 & = \frac{3 \pm \sqrt{5}}{2} \\
 \therefore & x = \ln\left(\frac{3 + \sqrt{5}}{2}\right) \text{ or } \ln\left(\frac{3 - \sqrt{5}}{2}\right)
 \end{aligned}$$

9 a The functions $y = e^x$ and $y = e^{2x} - 6$ meet where

$$\begin{aligned}
 & e^x = e^{2x} - 6 \\
 \therefore & e^{2x} - e^x - 6 = 0 \\
 \therefore & (e^x + 2)(e^x - 3) = 0 \\
 \therefore & e^x = -2 \text{ or } 3 \\
 \therefore & e^x = 3 \quad \{e^x > 0\} \\
 \therefore & x = \ln 3
 \end{aligned}$$

When $x = \ln 3$, $y = e^{\ln 3} = 3$

\therefore the functions meet at $(\ln 3, 3)$.



- b** The functions $y = 2e^x + 1$ and $y = 7 - e^x$ meet where

$$2e^x + 1 = 7 - e^x$$

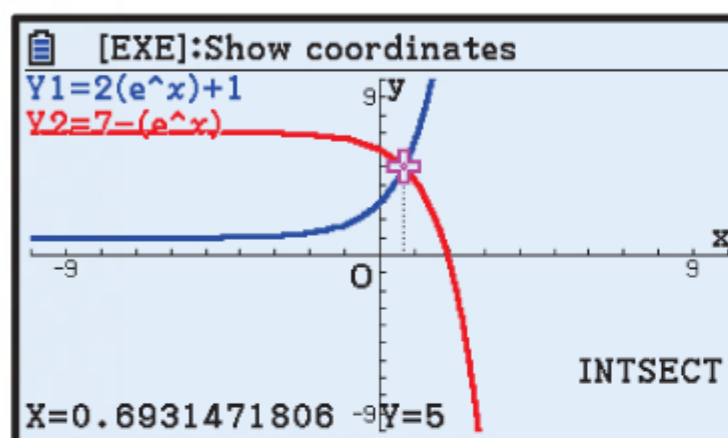
$$\therefore 3e^x = 6$$

$$\therefore e^x = 2$$

$$\therefore x = \ln 2$$

$$\text{When } x = \ln 2, y = 2e^{\ln 2} + 1 = 5$$

\therefore the functions meet at $(\ln 2, 5)$.



- c** The functions $y = 3 - e^x$ and $y = 5e^{-x} - 3$ meet where

$$3 - e^x = 5e^{-x} - 3$$

$$\therefore e^x - 6 + 5e^{-x} = 0$$

$$\therefore e^{2x} - 6e^x + 5 = 0$$

$$\therefore (e^x - 1)(e^x - 5) = 0$$

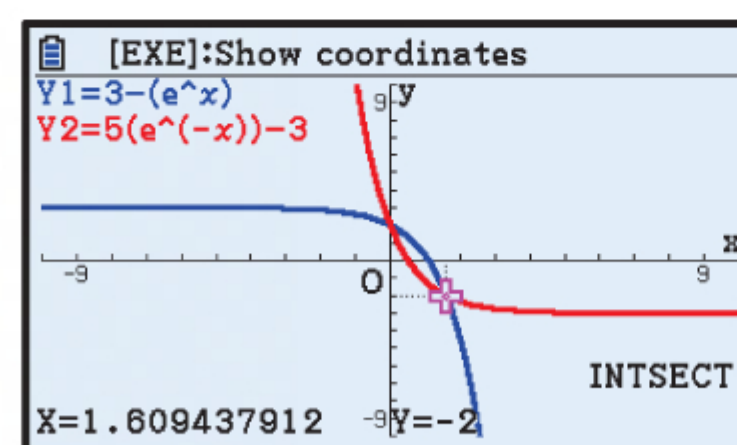
$$\therefore e^x = 1 \text{ or } 5$$

$$\therefore x = 0 \text{ or } \ln 5$$

$$\text{When } x = 0, y = 3 - e^0 = 2$$

$$\text{When } x = \ln 5, y = 3 - e^{\ln 5} = -2$$

\therefore the functions meet at $(0, 2)$ and at $(\ln 5, -2)$.



- 10** $P(t) = 852 \times (1.07)^t$ turtles

- a** When $P(t) = 1000$,

$$852 \times (1.07)^t = 1000$$

$$\therefore (1.07)^t = \frac{1000}{852}$$

$$\therefore t \log(1.07) = \log\left(\frac{1000}{852}\right)$$

$$\therefore t = \frac{\log\left(\frac{1000}{852}\right)}{\log(1.07)} \approx 2.37$$

The population will reach 1000 turtles in about 2.37 years.

- b** When $P(t) = 1500$,

$$852 \times (1.07)^t = 1500$$

$$\therefore (1.07)^t = \frac{1500}{852}$$

$$\therefore t \log(1.07) = \log\left(\frac{1500}{852}\right)$$

$$\therefore t = \frac{\log\left(\frac{1500}{852}\right)}{\log(1.07)} \approx 8.36$$

The population will reach 1500 turtles in about 8.36 years.

- 11** $W(t) = 20 \times 2^{0.15t}$ grams

- a** When $W(t) = 30$,

$$20 \times 2^{0.15t} = 30$$

$$\therefore 2^{0.15t} = 1.5$$

$$\therefore \log(2^{0.15t}) = \log(1.5)$$

$$\therefore 0.15t \log 2 = \log(1.5)$$

$$\therefore t = \frac{\log(1.5)}{0.15 \times \log 2}$$

$$\therefore t \approx 3.90$$

It takes about 3.90 hours for the weight to reach 30 g.

- b** When $W(t) = 100$,

$$20 \times 2^{0.15t} = 100$$

$$\therefore 2^{0.15t} = 5$$

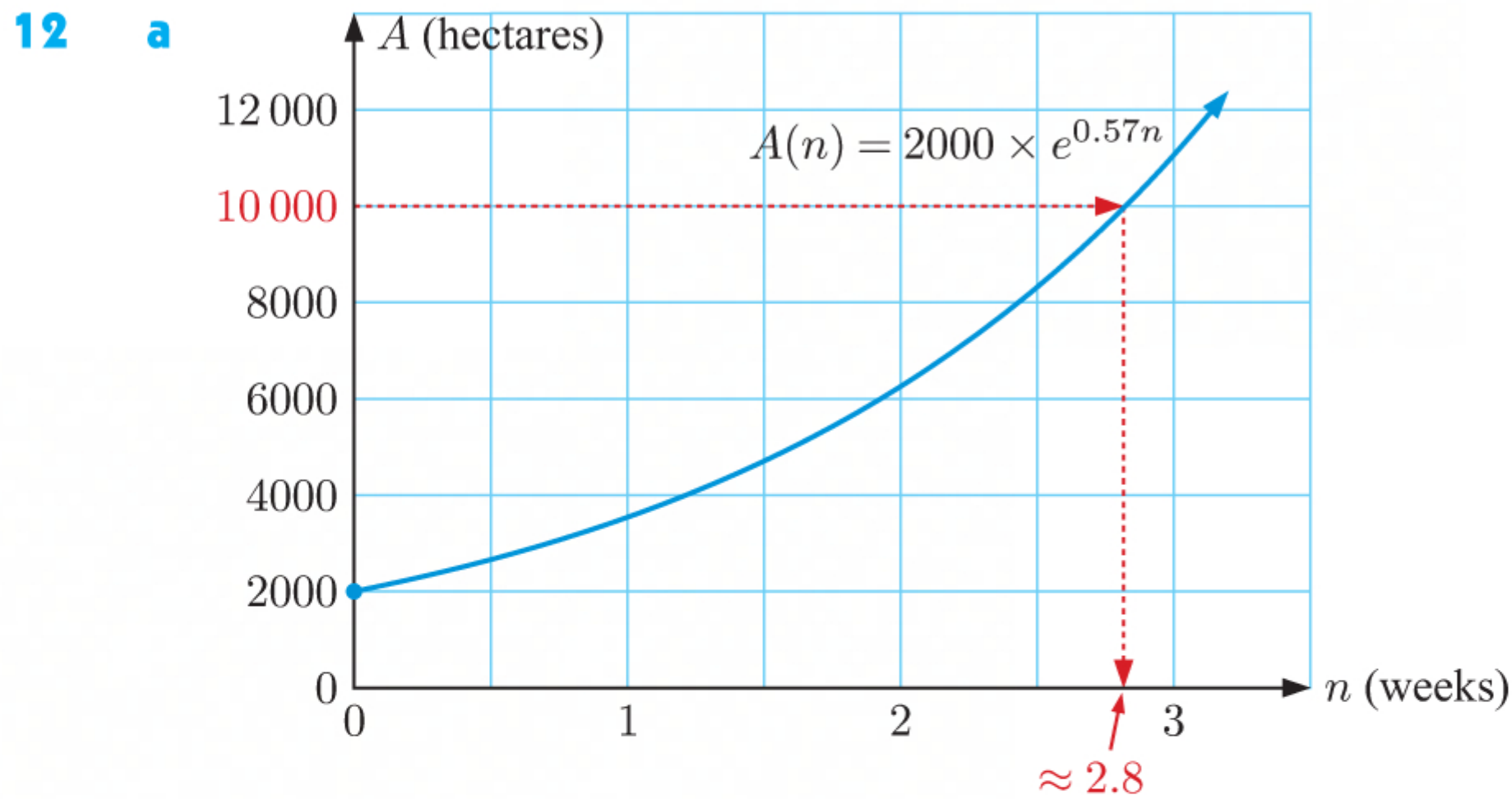
$$\therefore \log(2^{0.15t}) = \log 5$$

$$\therefore 0.15t \log 2 = \log 5$$

$$\therefore t = \frac{\log 5}{0.15 \times \log 2}$$

$$\therefore t \approx 15.5$$

It takes about 15.5 hours for the weight to reach 100 g.



b When $A(n) = 10\,000$, $t \approx 2.8$
 \therefore we estimate that it will take about 2.8 weeks for the infested area to reach 10 000 hectares.

c When $A(n) = 10\,000$, $2000 \times e^{0.57n} = 10\,000$
 $\therefore e^{0.57n} = 5$
 $\therefore \ln(e^{0.57n}) = \ln 5$
 $\therefore 0.57n = \ln 5$
 $\therefore n = \frac{\ln 5}{0.57}$
 $\therefore n \approx 2.82$

\therefore it takes about 2.82 weeks for the infested area to reach 10 000 hectares.

13 $u_0 = 360\,000$, $u_n = 550\,000$, $i = 7.5\% = 0.075$

$$u_n = u_0(1 + i)^n$$

$$\therefore 550\,000 = 360\,000 \times (1 + 0.075)^n$$

$$\therefore (1.075)^n = \frac{55}{36}$$

$$\therefore \log(1.075)^n = \log\left(\frac{55}{36}\right)$$

$$\therefore n \log(1.075) = \log\left(\frac{55}{36}\right)$$

$$\therefore n = \frac{\log\left(\frac{55}{36}\right)}{\log(1.075)} \approx 5.86$$

\therefore we expect the house to be worth £550 000 in about 5.86 years or about 5 years and 10 months.

14 $u_0 = 10\,000$, $u_n = 15\,000$, $i = 4.8\% = 0.048$

$$u_n = u_0(1 + i)^n$$

$$\therefore 15\,000 = 10\,000 \times (1 + 0.048)^n$$

$$\therefore (1.048)^n = 1.5$$

$$\therefore \log(1.048)^n = \log(1.5)$$

$$\therefore n \log(1.048) = \log(1.5)$$

$$\therefore n = \frac{\log(1.5)}{\log(1.048)}$$

$$\therefore n \approx 8.648$$

\therefore it would take 9 years for Thabo's investment to grow to \$15 000.
 {interest compounded annually}

15 a 8.4% p.a. compounded monthly

$$\text{is } \frac{8.4\%}{12} = 0.7\% \text{ a month}$$

$$= 0.007$$

$$\text{So } r = 1 + 0.007$$

$$\therefore r = 1.007$$

b $t_0 = 15\,000$, $t_n = 25\,000$, and $r = 1.007$

$$t_n = t_0 \times r^n$$

$$\therefore 25\,000 = 15\,000 \times (1.007)^n$$

$$\therefore (1.007)^n = \frac{25}{15} = \frac{5}{3}$$

$$\therefore \log(1.007)^n = \log\left(\frac{5}{3}\right)$$

$$\therefore n \log(1.007) = \log\left(\frac{5}{3}\right)$$

$$\therefore n = \frac{\log\left(\frac{5}{3}\right)}{\log(1.007)} \approx 73.23$$

\therefore Dien will withdraw the money after 74 months.

16 $M_t = 1000 \times e^{-0.04t}$

$$M_0 = 1000 \times e^0 = 1000$$

\therefore the initial mass of the substance is 1000 grams.

a For the mass to halve, $M_t = 500$

$$\therefore 1000e^{-0.04t} = 500$$

$$\therefore e^{-0.04t} = 0.5$$

$$\therefore -0.04t = \ln(0.5)$$

$$\therefore t = \frac{\ln(0.5)}{-0.04}$$

$$\therefore t \approx 17.3$$

\therefore it will take about 17.3 years for the mass to halve.

c When $M_t = 1\%$ of original value,

$$1000e^{-0.04t} = 0.01 \times 1000$$

$$\therefore e^{-0.04t} = 0.01$$

$$\therefore -0.04t = \ln(0.01)$$

$$\therefore t = \frac{\ln(0.01)}{-0.04}$$

$$\therefore t \approx 115$$

\therefore it will take about 115 years for the mass to reach 1% of its original value.

b When $M_t = 25$ g,

$$1000e^{-0.04t} = 25$$

$$\therefore e^{-0.04t} = 0.025$$

$$\therefore -0.04t = \ln(0.025)$$

$$\therefore t = \frac{\ln(0.025)}{-0.04}$$

$$\therefore t \approx 92.2$$

\therefore it will take about 92.2 years for the mass to reach 25 grams.

17 $I = I_0 \times 2^{-0.02t}$ amps

When I is 10% of its original value,

$$I = 10\% \text{ of } I_0$$

$$\therefore I_0 \times 2^{-0.02t} = 0.1 \times I_0$$

$$\therefore 2^{-0.02t} = 0.1$$

$$\therefore \log(2^{-0.02t}) = \log(0.1)$$

$$\therefore -0.02t \log 2 = \log(10^{-1})$$

$$\therefore -\frac{1}{50}t \log 2 = -1$$

$$\therefore t = \frac{50}{\log 2}$$

\therefore it takes $\frac{50}{\log 2}$ seconds for the current to drop to 10% of its original value.

18 $V(t) = 50(1 - e^{-0.2t}) \text{ m s}^{-1}$

So, when $V(t) = 40$, $50(1 - e^{-0.2t}) = 40$

$$\therefore 1 - e^{-0.2t} = \frac{4}{5}$$

$$\therefore e^{-0.2t} = \frac{1}{5}$$

$$\therefore \ln(e^{-0.2t}) = \ln\left(\frac{1}{5}\right)$$

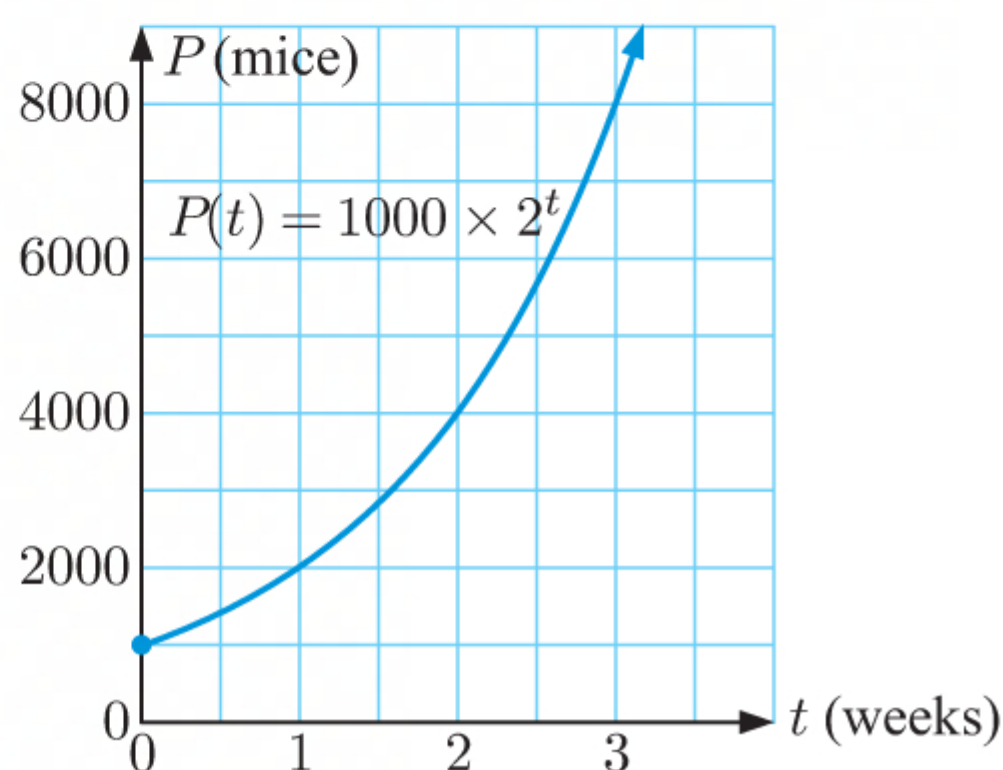
$$\therefore -0.2t = \ln(5^{-1})$$

$$\therefore -\frac{1}{5}t = -\ln 5$$

$$\therefore t = 5 \ln 5$$

\therefore it will take $5 \ln 5$ seconds for the sky diver's speed to reach 40 m s^{-1} .

19 a



b When $P(t) = 20\,000$,

$$1000 \times 2^t = 20\,000$$

$$\therefore 2^t = 20$$

$$\therefore \log(2^t) = \log 20$$

$$\therefore t \log 2 = \log 20$$

$$\therefore t = \frac{\log 20}{\log 2} \approx 4.32$$

\therefore it will take about 4.32 weeks for the population to reach 20 000 mice.

c

$$P = 1000 \times 2^t$$

$$\therefore 2^t = \frac{P}{1000}$$

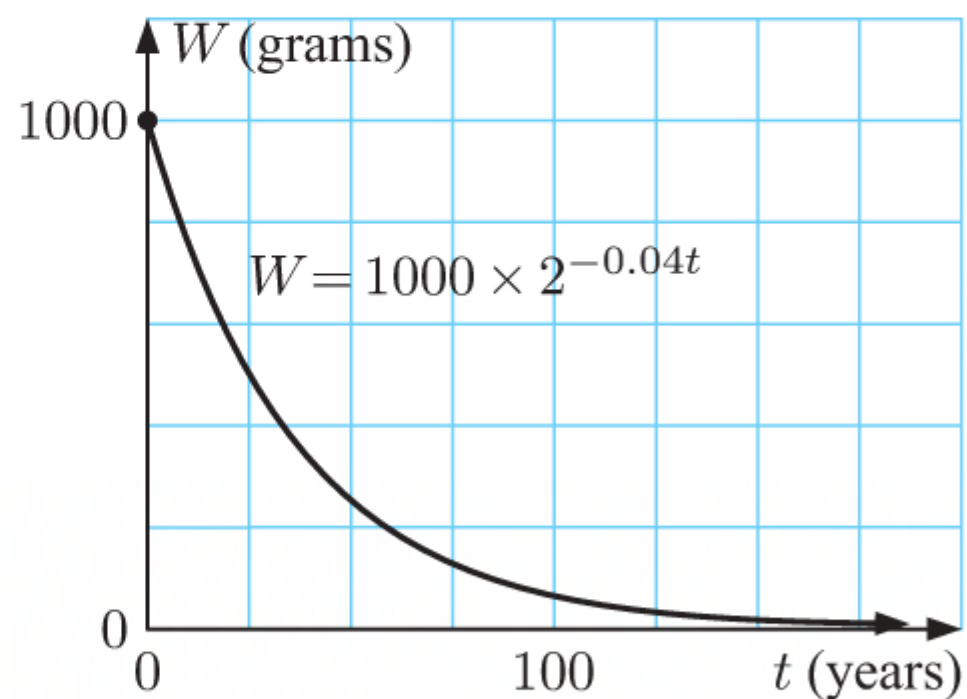
$$\therefore \log(2^t) = \log\left(\frac{P}{1000}\right)$$

$$\therefore t \log 2 = \log P - \log 1000$$

$$\therefore t = \frac{\log P - \log(10^3)}{\log 2}$$

$$\therefore t = \frac{\log P - 3}{\log 2}$$

20 a



b

$$W = 1000 \times 2^{-0.04t}$$

$$\therefore 2^{-0.04t} = \frac{W}{1000}$$

$$\therefore \log(2^{-0.04t}) = \log\left(\frac{W}{1000}\right)$$

$$\therefore -0.04t \log 2 = \log W - \log 1000$$

$$\therefore 0.04t \log 2 = \log(10^3) - \log W$$

$$\therefore t = \frac{3 - \log W}{0.04 \log 2}$$

c i When $W = 20$,

$$t = \frac{3 - \log 20}{0.04 \log 2} \\ \approx 141$$

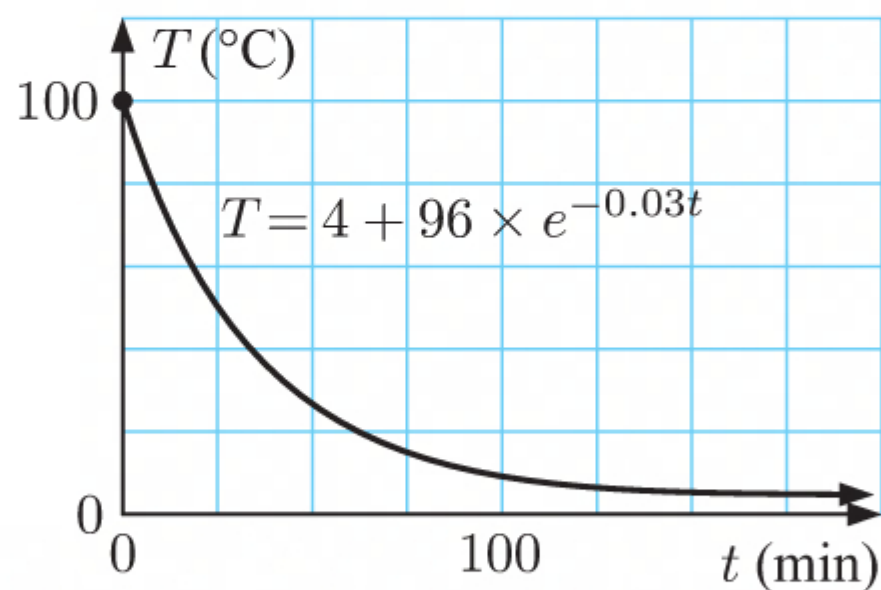
\therefore it will take about 141 years for the weight to reach 20 grams.

ii When $W = 0.001$,

$$t = \frac{3 - \log(0.001)}{0.04 \log 2} \\ \approx 498$$

\therefore it will take about 498 years for the weight to reach 0.001 grams.

21 a



c i When $T = 25$,

$$t = \frac{\ln 96 - \ln 21}{0.03} \\ \approx 50.7$$

\therefore it will take about 50.7 minutes for the temperature to reach 25°C .

b

$$T = 4 + 96 \times e^{-0.03t} \\ \therefore 96 \times e^{-0.03t} = T - 4 \\ \therefore e^{-0.03t} = \frac{T - 4}{96} \\ \therefore -0.03t = \ln\left(\frac{T - 4}{96}\right) \\ \therefore -0.03t = \ln(T - 4) - \ln 96 \\ \therefore t = \frac{\ln 96 - \ln(T - 4)}{0.03}$$

ii When $T = 5$,

$$t = \frac{\ln 96 - \ln 1}{0.03} \\ \approx 152$$

\therefore it will take about 152 minutes for the temperature to reach 5°C .

22 $V = 60(1 - 2^{-0.2t}) \text{ m s}^{-1}$

When $V = v$, $60(1 - 2^{-0.2t}) = v$

$$\therefore 1 - 2^{-0.2t} = \frac{v}{60}$$

$$\therefore 2^{-0.2t} = 1 - \frac{v}{60}$$

$$\therefore \log(2^{-0.2t}) = \log\left(1 - \frac{v}{60}\right)$$

$$\therefore -0.2t \log 2 = \log\left(1 - \frac{v}{60}\right)$$

$$\therefore t = \frac{\log\left(1 - \frac{v}{60}\right)}{-0.2 \times \log 2}$$

$$\therefore t = \frac{-5 \log\left(1 - \frac{v}{60}\right)}{\log 2} \text{ seconds}$$

23 $V(t) = 650(4 + 2 \times e^{-0.1t})$

a as $t \rightarrow \infty$, $e^{-0.1t} \rightarrow 0^+$

\therefore the speed of the meteor is decreasing.

$$\begin{aligned}
 \text{b i } V(0) &= 650(4 + 2 \times e^{-0.1 \times 0}) \\
 &= 650(4 + 2 \times 1) \\
 &= 650(6) \\
 &= 3900
 \end{aligned}$$

The speed of the meteor when it was first sighted was 3900 m s^{-1} .

$$\begin{aligned}
 \text{ii } V(120) &= 650(4 + 2 \times e^{-0.1 \times 120}) \\
 &= 650(4 + 2 \times e^{-12}) \\
 &\approx 2600 \text{ m s}^{-1}
 \end{aligned}$$

The speed of the meteor after 2 minutes was about 2600 m s^{-1} .

$$\begin{aligned}
 \text{c } \text{When } V(t) &= 3000, \\
 650(4 + 2 \times e^{-0.1t}) &= 3000 \\
 \therefore 4 + 2 \times e^{-0.1t} &= \frac{60}{13} \\
 \therefore 2 \times e^{-0.1t} &= \frac{8}{13} \\
 \therefore e^{-0.1t} &= \frac{4}{13} \\
 \therefore -0.1t &= \ln\left(\frac{4}{13}\right) \\
 \therefore t &= \frac{\ln\left(\frac{4}{13}\right)}{-0.1} \\
 &\approx 11.8
 \end{aligned}$$

It will take about 11.8 seconds for the meteor's speed to reach 3000 m s^{-1} .

INVESTIGATION 2

THE “RULE OF 72”

- 1 a A population growing at 2% per year will double in approximately $\frac{72}{2} = 36$ years.
- b A population growing at 8% per year will double in approximately $\frac{72}{8} = 9$ years.
- c A population growing at 12% per year will double in approximately $\frac{72}{12} = 6$ years.

- 2 Let T be the doubling time.

$$\begin{aligned}
 \text{a For a population growing at 2\% per year, we require } (1.02)^T &= 2, \\
 \therefore \log((1.02)^T) &= \log 2 \\
 \therefore T \log(1.02) &= \log 2 \\
 \therefore T &= \frac{\log 2}{\log(1.02)} \\
 &\approx 35.003
 \end{aligned}$$

$$\begin{aligned}
 \text{For a population growing at 8\% per year, we require } (1.08)^T &= 2 \\
 \therefore \log((1.08)^T) &= \log 2 \\
 \therefore T \log(1.08) &= \log 2 \\
 \therefore T &= \frac{\log 2}{\log(1.08)} \\
 &\approx 9.006
 \end{aligned}$$

$$\begin{aligned}
 \text{For a population growing at 12\% per year, we require } (1.12)^T &= 2 \\
 \therefore \log((1.12)^T) &= \log 2 \\
 \therefore T \log(1.12) &= \log 2 \\
 \therefore T &= \frac{\log 2}{\log(1.12)} \\
 &\approx 6.116
 \end{aligned}$$

b Our estimates in **1** using the “rule of 72” were very close to the values obtained in **2 a**.

3 a Let the percentage increase each year be $r\%$, and the doubling time be T .

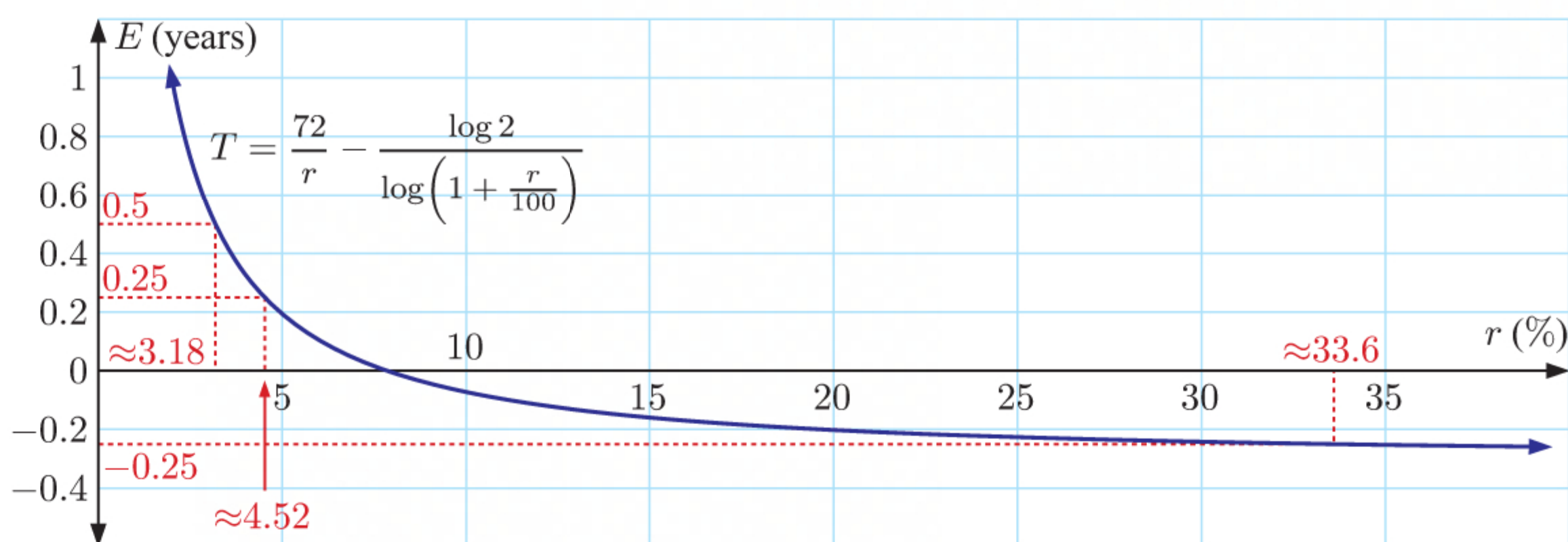
$$\text{We require } \left(1 + \frac{r}{100}\right)^T = 2$$

$$\therefore \log\left(\left(1 + \frac{r}{100}\right)^T\right) = \log 2$$

$$\therefore T \log\left(1 + \frac{r}{100}\right) = \log 2$$

$$\therefore T = \frac{\log 2}{\log\left(1 + \frac{r}{100}\right)}$$

b The error in estimating the exact doubling time is $E = \frac{72}{r} - \frac{\log 2}{\log\left(1 + \frac{r}{100}\right)}$.



c i 6 months is 0.5 years.

From the graph, $E = 0.5$ when $r \approx 3.18$.

\therefore the estimate is accurate to within 6 months for growth rates larger than about 3.18%.

ii 3 months is 0.25 years.

From the graph, $E = 0.25$ when $r \approx 4.52$
and $E = -0.25$ when $r \approx 33.6$

\therefore the estimate is accurate to within 3 months for growth rates between about 4.52% and 33.6%.

EXERCISE 6H

1 a $f : x \mapsto \log_2 x - 2$

i $\log_2 x$ is defined when $x > 0$

So, the domain is $\{x \mid x > 0\}$ and the range is $\{y \mid y \in \mathbb{R}\}$.

ii As $x \rightarrow 0^+$, $y \rightarrow -\infty$, so the vertical asymptote is $x = 0$.

As $x \rightarrow \infty$, $y \rightarrow \infty$, so there is no horizontal asymptote.

When $x = 0$, y is undefined, so there is no y -intercept.

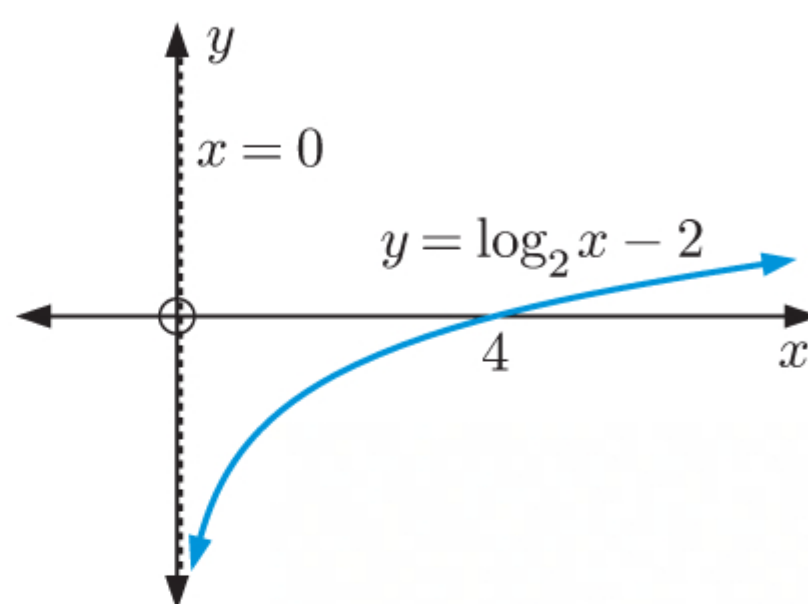
When $y = 0$, $\log_2 x = 2$

$$\therefore x = 2^2$$

$$\therefore x = 4$$

So, the x -intercept is 4.

iii



iv

$$\begin{aligned}
 f(x) &= -1 \\
 \therefore \log_2 x - 2 &= -1 \\
 \therefore \log_2 x &= 1 \\
 \therefore x &= 2^1 \\
 \therefore x &= 2
 \end{aligned}$$

$$\begin{aligned}
 \text{v } f \text{ is defined by } y &= \log_2 x - 2 \\
 \therefore f^{-1} \text{ is defined by } x &= \log_2 y - 2 \\
 \therefore x + 2 &= \log_2 y \\
 \therefore y &= 2^{x+2} \\
 \therefore f^{-1}(x) &= 2^{x+2}
 \end{aligned}$$

$$\text{b } f : x \mapsto \log_3(x + 1)$$

i $\log_3(x + 1)$ is defined when $x + 1 > 0$, that is, when $x > -1$.
So, the domain is $\{x \mid x > -1\}$ and the range is $\{y \mid y \in \mathbb{R}\}$.

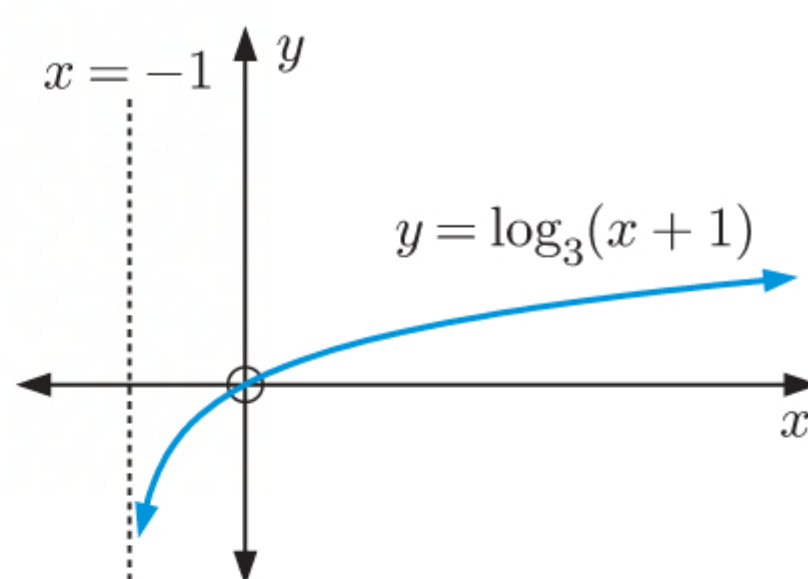
ii As $x \rightarrow -1^+$, $y \rightarrow -\infty$, so the vertical asymptote is $x = -1$.
As $x \rightarrow \infty$, $y \rightarrow \infty$, so there is no horizontal asymptote.

When $x = 0$, $\log_3 1 = 0$, so the y -intercept is 0.

$$\begin{aligned}
 \text{When } y = 0, \quad \log_3(x + 1) &= 0 \\
 \therefore x + 1 &= 3^0 \\
 \therefore x &= 0
 \end{aligned}$$

So, the x -intercept is 0.

iii



iv

$$\begin{aligned}
 f(x) &= -1 \\
 \therefore \log_3(x + 1) &= -1 \\
 \therefore x + 1 &= 3^{-1} \\
 \therefore x &= -\frac{2}{3}
 \end{aligned}$$

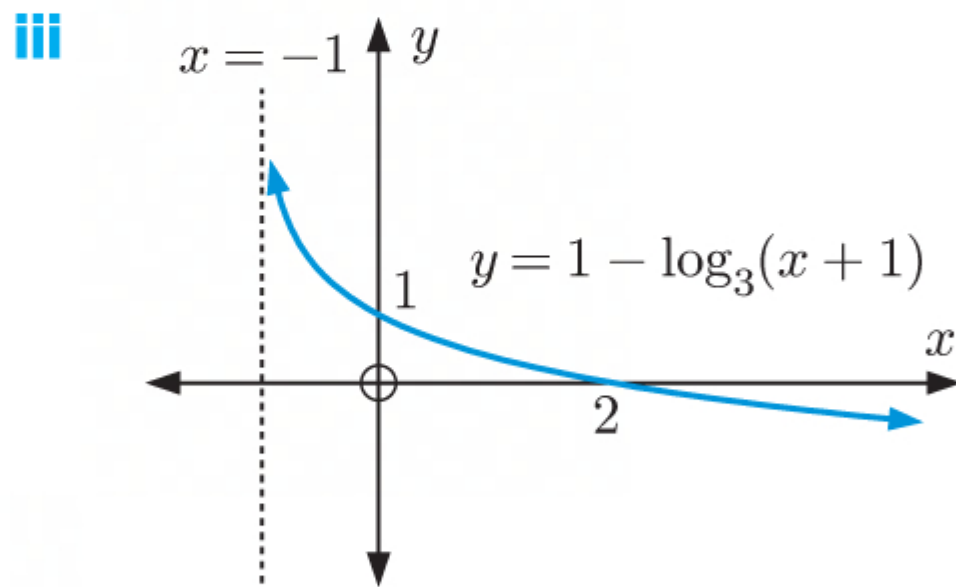
$$\begin{aligned}
 \text{v } f \text{ is defined by } y &= \log_3(x + 1) \\
 \therefore f^{-1} \text{ is defined by } x &= \log_3(y + 1) \\
 \therefore y + 1 &= 3^x \\
 \therefore y &= 3^x - 1 \\
 \therefore f^{-1}(x) &= 3^x - 1
 \end{aligned}$$

$$\text{c } f : x \mapsto 1 - \log_3(x + 1)$$

i $\log_3(x + 1)$ is defined when $x + 1 > 0$, that is, when $x > -1$.
So, the domain is $\{x \mid x > -1\}$ and the range is $\{y \mid y \in \mathbb{R}\}$.

- ii As $x \rightarrow -1^+$, $y \rightarrow \infty$, so the vertical asymptote is $x = -1$.
 As $x \rightarrow \infty$, $y \rightarrow -\infty$, so there is no horizontal asymptote.
 When $x = 0$, $1 - \log_3 1 = 1 - 0 = 1$, so the y -intercept is 1.
 When $y = 0$, $1 = \log_3(x + 1)$
 $\therefore 3^1 = x + 1$
 $\therefore x = 2$

So, the x -intercept is 2.



iv

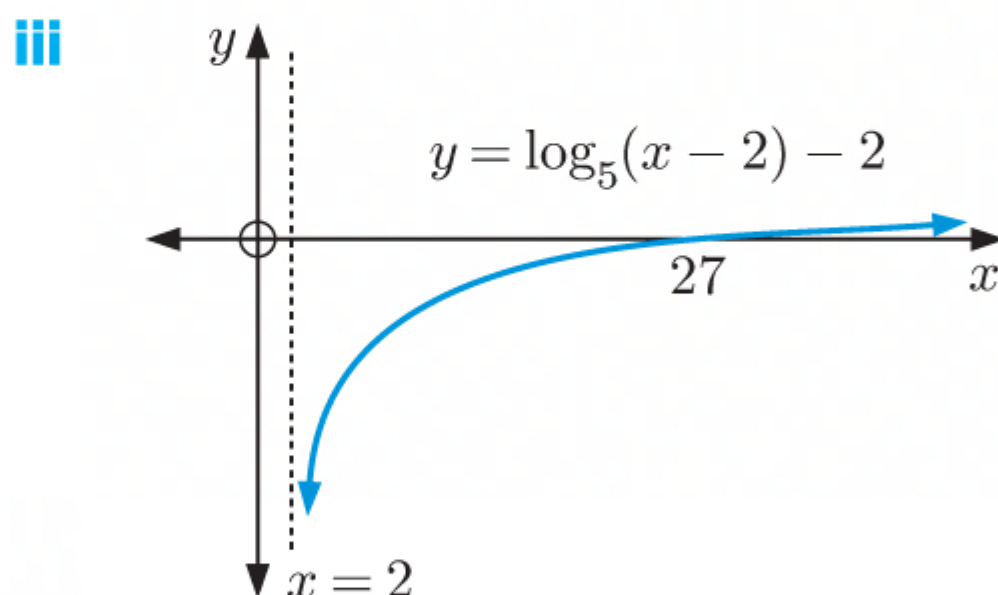
$$\begin{aligned} f(x) &= -1 \\ \therefore 1 - \log_3(x + 1) &= -1 \\ \therefore -\log_3(x + 1) &= -2 \\ \therefore \log_3(x + 1) &= 2 \\ \therefore x + 1 &= 3^2 \\ \therefore x &= 8 \end{aligned}$$

- v f is defined by $y = 1 - \log_3(x + 1)$
 $\therefore f^{-1}$ is defined by $x = 1 - \log_3(y + 1)$
 $\therefore \log_3(y + 1) = 1 - x$
 $\therefore y + 1 = 3^{1-x}$
 $\therefore y = 3^{1-x} - 1$
 $\therefore f^{-1}(x) = 3^{1-x} - 1$

d $f : x \mapsto \log_5(x - 2) - 2$

- i $\log_5(x - 2)$ is defined when $x - 2 > 0$, that is, when $x > 2$.
 So, the domain is $\{x \mid x > 2\}$ and the range is $\{y \mid y \in \mathbb{R}\}$.
 ii As $x \rightarrow 2^+$, $y \rightarrow -\infty$, so the vertical asymptote is $x = 2$.
 As $x \rightarrow \infty$, $y \rightarrow \infty$, so there is no horizontal asymptote.
 When $x = 0$, y is undefined, so there is no y -intercept.
 When $y = 0$, $\log_5(x - 2) = 2$
 $\therefore x - 2 = 5^2$
 $\therefore x = 27$

So, the x -intercept is 27.



iv

$$\begin{aligned} f(x) &= -1 \\ \therefore \log_5(x - 2) - 2 &= -1 \\ \therefore \log_5(x - 2) &= 1 \\ \therefore x - 2 &= 5^1 \\ \therefore x &= 7 \end{aligned}$$

$$\begin{aligned}
 \text{v} \quad & f \text{ is defined by } y = \log_5(x-2) - 2 \\
 \therefore f^{-1} \text{ is defined by } & x = \log_5(y-2) - 2 \\
 & \therefore x+2 = \log_5(y-2) \\
 & \therefore y-2 = 5^{x+2} \\
 & \therefore y = 5^{x+2} + 2 \\
 \therefore f^{-1}(x) &= 5^{x+2} + 2
 \end{aligned}$$

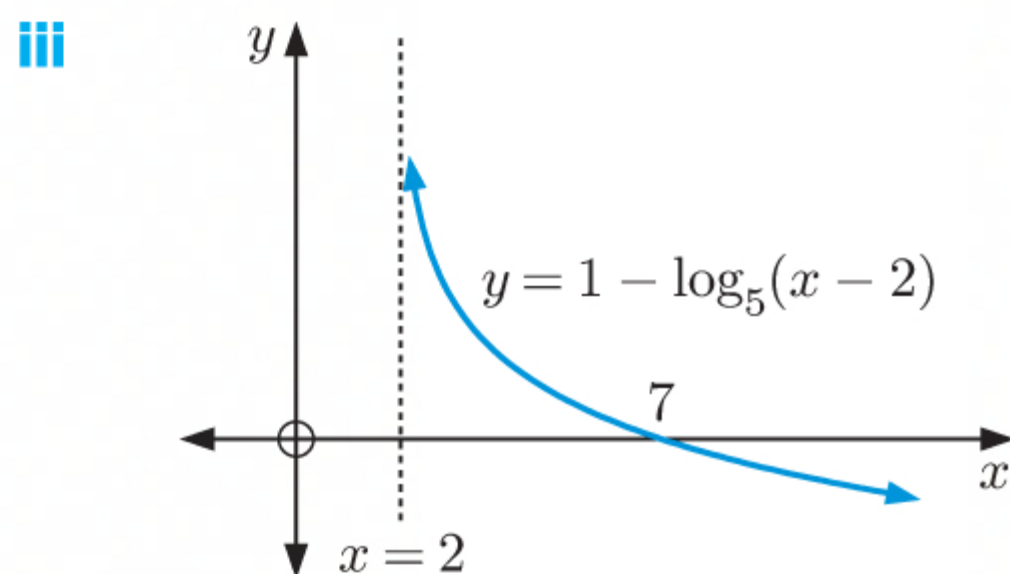
$$\text{e} \quad f : x \mapsto 1 - \log_5(x-2)$$

i $\log_5(x-2)$ is defined when $x-2 > 0$, that is, when $x > 2$
So, the domain is $\{x \mid x > 2\}$ and the range is $\{y \mid y \in \mathbb{R}\}$.

ii As $x \rightarrow 2^+$, $y \rightarrow \infty$, so the vertical asymptote is $x = 2$.
As $x \rightarrow \infty$, $y \rightarrow -\infty$, so there is no horizontal asymptote.
When $x = 0$, y is undefined, so there is no y -intercept.

$$\begin{aligned}
 \text{When } y = 0, \quad & \log_5(x-2) = 1 \\
 & \therefore x-2 = 5^1 \\
 & \therefore x = 7
 \end{aligned}$$

So, the x -intercept is 7.



$$\begin{aligned}
 \text{iv} \quad & f(x) = -1 \\
 \therefore 1 - \log_5(x-2) &= -1 \\
 \therefore -\log_5(x-2) &= -2 \\
 \therefore \log_5(x-2) &= 2 \\
 \therefore x-2 &= 5^2 \\
 \therefore x &= 27
 \end{aligned}$$

$$\begin{aligned}
 \text{v} \quad & f \text{ is defined by } y = 1 - \log_5(x-2) \\
 \therefore f^{-1} \text{ is defined by } & x = 1 - \log_5(y-2) \\
 & \therefore \log_5(y-2) = 1-x \\
 & \therefore y-2 = 5^{1-x} \\
 & \therefore y = 5^{1-x} + 2 \\
 \therefore f^{-1}(x) &= 5^{1-x} + 2
 \end{aligned}$$

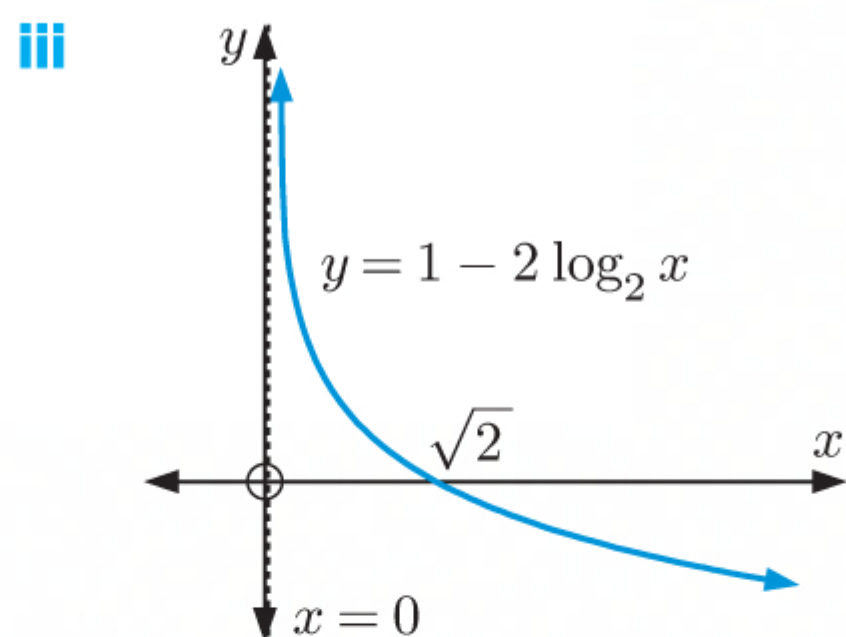
$$\text{f} \quad f : x \mapsto 1 - 2\log_2 x$$

i $\log_2 x$ is defined when $x > 0$
So, the domain is $\{x \mid x > 0\}$ and the range is $\{y \mid y \in \mathbb{R}\}$.

ii As $x \rightarrow 0^+$, $y \rightarrow \infty$, so the vertical asymptote is $x = 0$.
As $x \rightarrow \infty$, $y \rightarrow -\infty$, so there is no horizontal asymptote.
When $x = 0$, y is undefined, so there is no y -intercept.

$$\begin{aligned}
 \text{When } y = 0, \quad & 2\log_2 x = 1 \\
 \therefore \log_2 x &= \frac{1}{2} \\
 \therefore x &= 2^{\frac{1}{2}} \\
 \therefore x &= \sqrt{2}
 \end{aligned}$$

So, the x -intercept is $\sqrt{2}$.



iv

$$\begin{aligned}
 f(x) &= -1 \\
 \therefore 1 - 2 \log_2 x &= -1 \\
 \therefore -2 \log_2 x &= -2 \\
 \therefore 2 \log_2 x &= 2 \\
 \therefore \log_2 x &= 1 \\
 \therefore x &= 2^1 \\
 \therefore x &= 2
 \end{aligned}$$

v

$$\begin{aligned}
 f \text{ is defined by } y &= 1 - 2 \log_2 x \\
 \therefore f^{-1} \text{ is defined by } x &= 1 - 2 \log_2 y \\
 \therefore 2 \log_2 y &= 1 - x \\
 \therefore \log_2 y &= \frac{1-x}{2} \\
 \therefore y &= 2^{\frac{1-x}{2}} \\
 \therefore f^{-1}(x) &= 2^{\frac{1-x}{2}}
 \end{aligned}$$

2 a $f(x) = \ln x - 4$

i $f(x)$ is a translation of $y = \ln x$ through $\begin{pmatrix} 0 \\ -4 \end{pmatrix}$.

ii Domain = $\{x \mid x > 0\}$, Range = $\{y \mid y \in \mathbb{R}\}$

iii As $x \rightarrow 0^+$, $y \rightarrow -\infty$, so the vertical asymptote is $x = 0$.

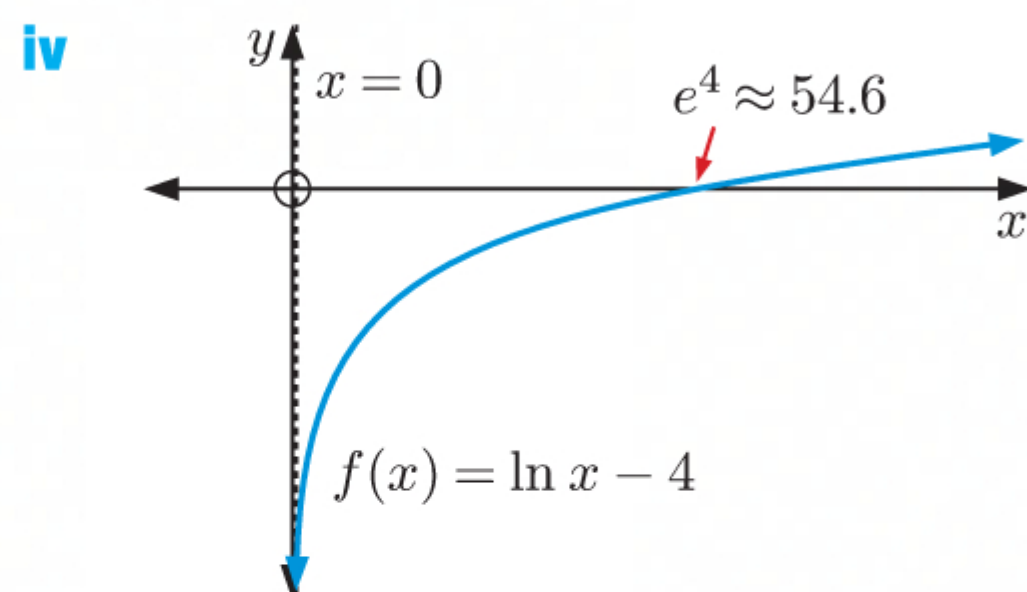
As $x \rightarrow \infty$, $y \rightarrow \infty$, so there is no horizontal asymptote.

When $x = 0$, y is undefined, so there is no y -intercept.

When $y = 0$, $\ln x = 4$

$$\therefore x = e^4 \approx 54.6$$

So, the x -intercept is e^4 .



v

$$\begin{aligned}
 f \text{ is defined by } y &= \ln x - 4 \\
 \therefore f^{-1} \text{ is defined by } x &= \ln y - 4 \\
 \therefore x + 4 &= \ln y \\
 \therefore y &= e^{x+4} \\
 \therefore f^{-1}(x) &= e^{x+4}
 \end{aligned}$$

b $f(x) = \ln(x-1) + 2$

i $f(x)$ is a translation of $y = \ln x$ through $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

ii Domain = $\{x \mid x > 1\}$, Range = $\{y \mid y \in \mathbb{R}\}$

iii As $x \rightarrow 1^+$, $y \rightarrow -\infty$, so the vertical asymptote is $x = 1$.

As $x \rightarrow \infty$, $y \rightarrow \infty$, so there is no horizontal asymptote.

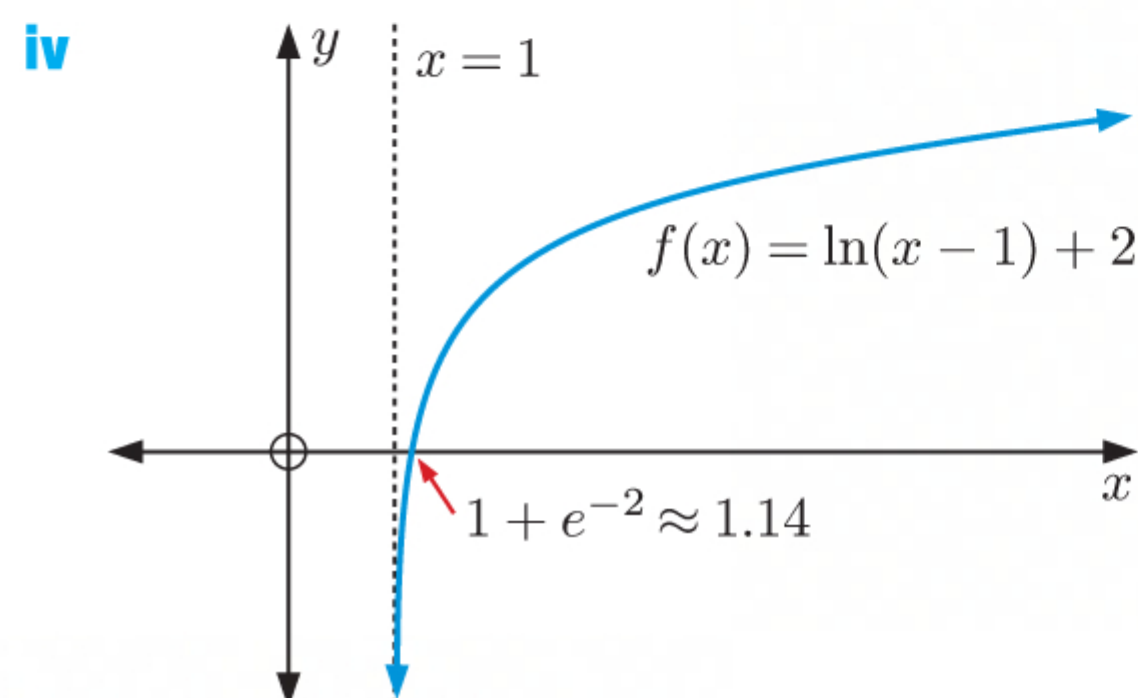
When $x = 0$, y is undefined, so there is no y -intercept.

When $y = 0$, $\ln(x - 1) = -2$

$$\therefore x - 1 = e^{-2}$$

$$\therefore x = 1 + e^{-2} \approx 1.14$$

So, the x -intercept is $1 + e^{-2}$.



v f is defined by $y = \ln(x - 1) + 2$

$\therefore f^{-1}$ is defined by $x = \ln(y - 1) + 2$

$$\therefore x - 2 = \ln(y - 1)$$

$$\therefore y - 1 = e^{x-2}$$

$$\therefore y = e^{x-2} + 1$$

$$\therefore f^{-1}(x) = e^{x-2} + 1$$

c $f(x) = 3 \ln x - 1$

i $f(x)$ is a vertical stretch of $y = \ln x$ with scale factor 3, then a translation through

$$\begin{pmatrix} 0 \\ -1 \end{pmatrix}.$$

ii Domain = $\{x \mid x > 0\}$, Range = $\{y \mid y \in \mathbb{R}\}$

iii As $x \rightarrow 0^+$, $y \rightarrow -\infty$, so the vertical asymptote is $x = 0$.

As $x \rightarrow \infty$, $y \rightarrow \infty$, so there is no horizontal asymptote.

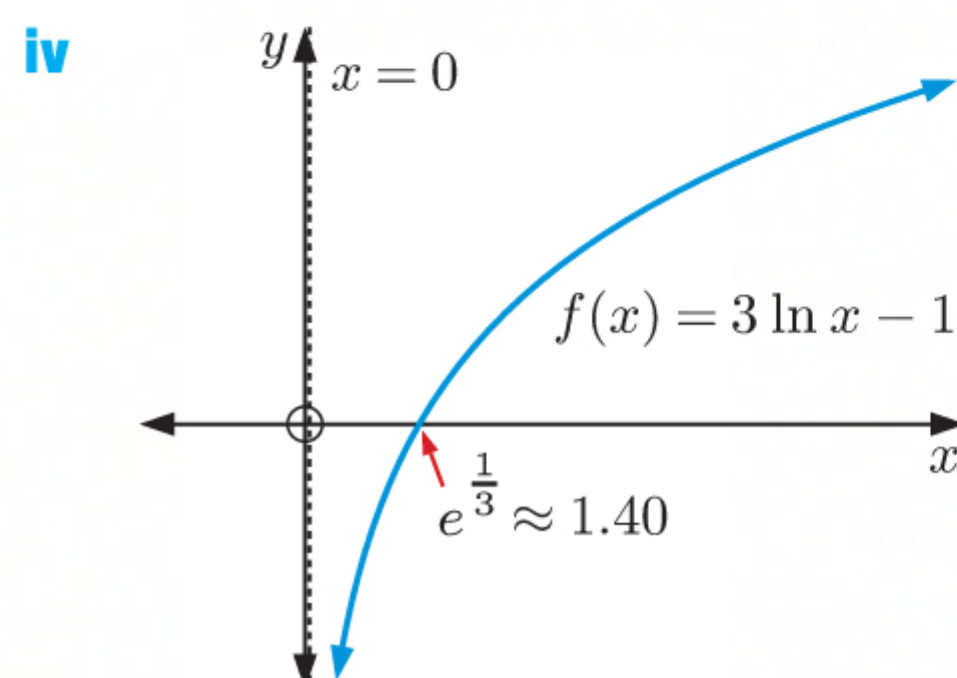
When $x = 0$, y is undefined, so there is no y -intercept.

When $y = 0$, $3 \ln x = 1$

$$\therefore \ln x = \frac{1}{3}$$

$$\therefore x = e^{\frac{1}{3}} \approx 1.40$$

So, the x -intercept is $e^{\frac{1}{3}}$.



v f is defined by $y = 3 \ln x - 1$

$\therefore f^{-1}$ is defined by $x = 3 \ln y - 1$

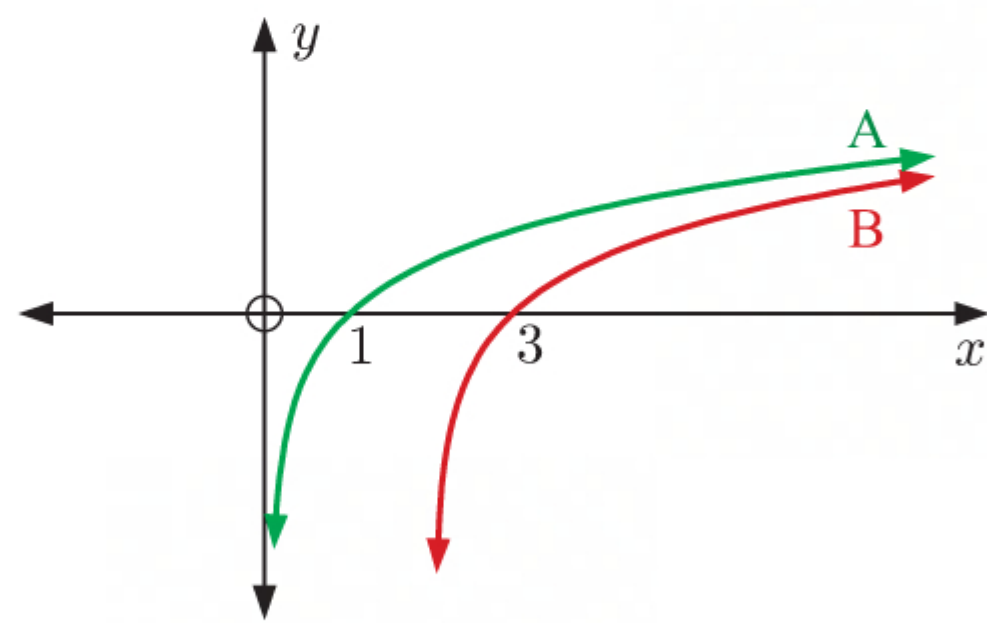
$$\therefore x + 1 = 3 \ln y$$

$$\therefore \ln y = \frac{x+1}{3}$$

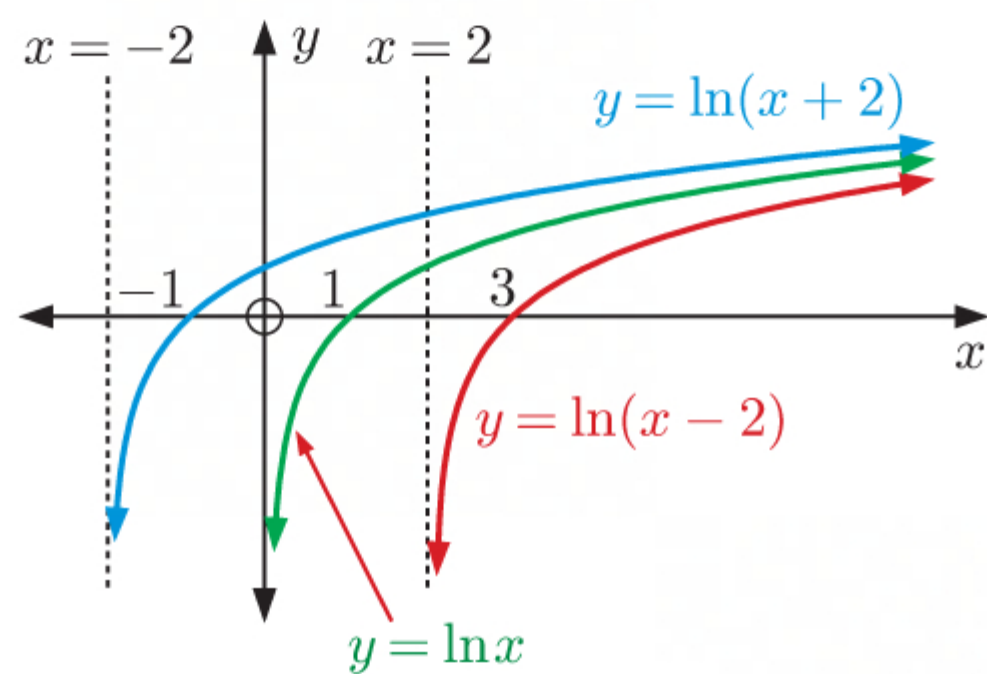
$$\therefore y = e^{\frac{x+1}{3}}$$

$$\therefore f^{-1}(x) = e^{\frac{x+1}{3}}$$

- 3 a** For $y = \ln x$, when $y = 0$, $\ln x = 0$
 $\therefore x = e^0$
 $\therefore x = 1$
 \therefore A is $y = \ln x$ as its x -intercept is 1,
 so B must be $y = \ln(x - 2)$.

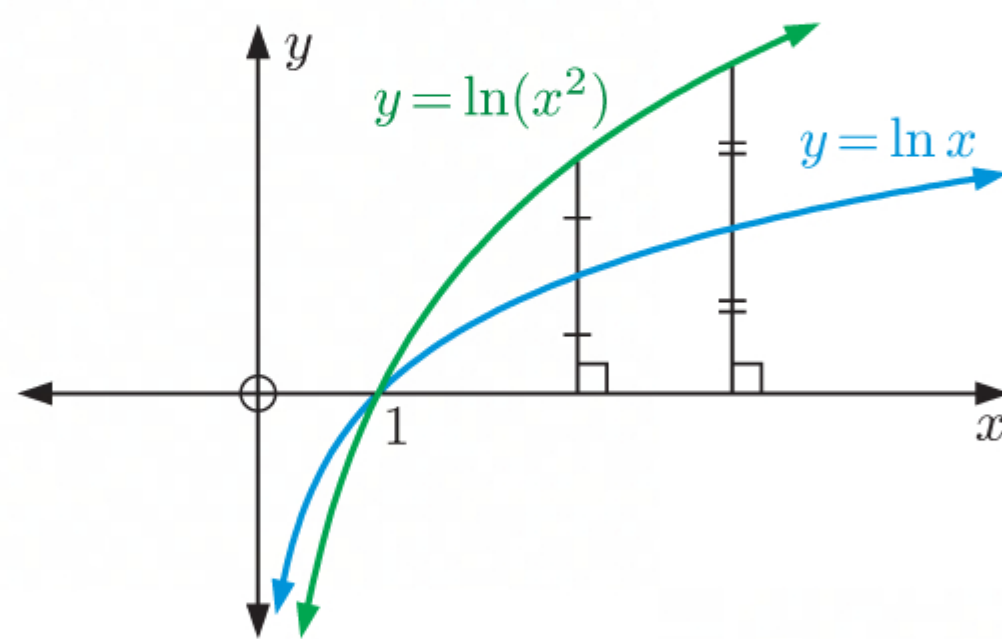


- b** $y = \ln(x - 2)$ is a horizontal translation of $y = \ln x$, 2 units to the right.
 $y = \ln(x + 2)$ is a horizontal translation of $y = \ln x$, 2 units to the left.

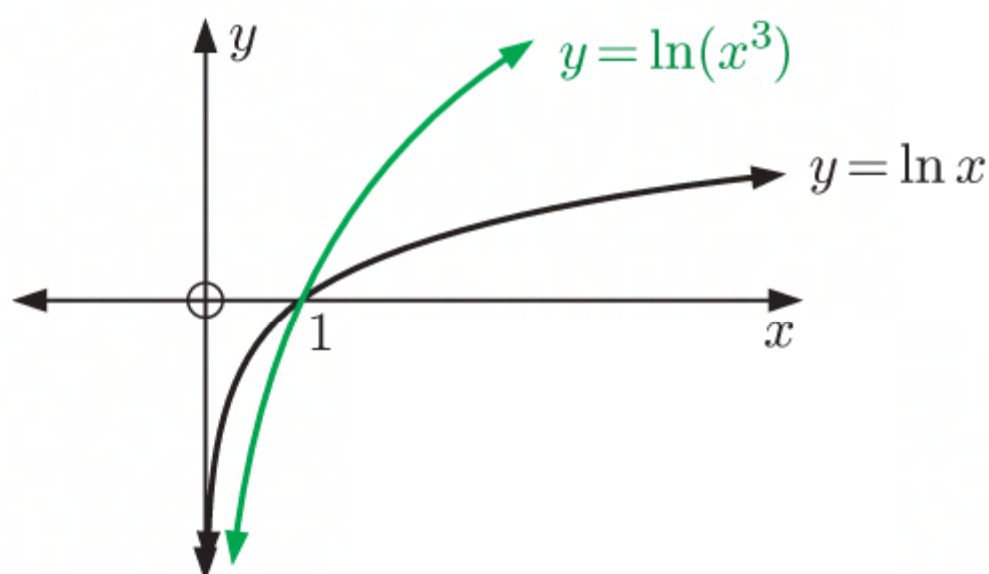


- c** $y = \ln x$ has domain $\{x \mid x > 0\}$.
 As $x \rightarrow 0^+$, $y \rightarrow -\infty$, so $y = \ln x$ has vertical asymptote $x = 0$.
 $y = \ln(x - 2)$ has domain $\{x \mid x > 2\}$.
 As $x \rightarrow 2^+$, $y \rightarrow -\infty$, so $y = \ln(x - 2)$ has vertical asymptote $x = 2$.
 $y = \ln(x + 2)$ has domain $\{x \mid x > -2\}$.
 As $x \rightarrow -2^+$, $y \rightarrow -\infty$, so $y = \ln(x + 2)$ has vertical asymptote $x = -2$.

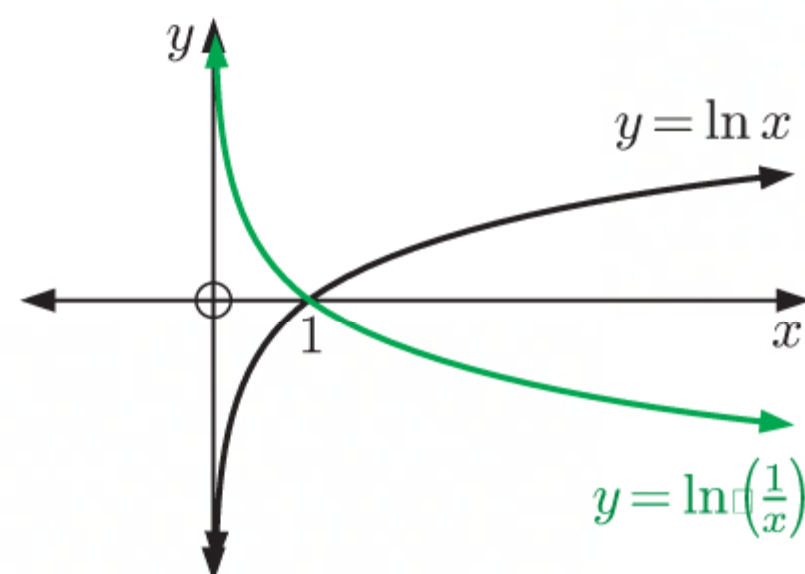
- 4** $y = \ln(x^2)$, $x > 0$
 $= 2 \ln x \quad \{m \ln b = \ln(b^m)\}$
 So, the y -values are twice as large for $y = \ln(x^2)$ as they are for $y = \ln x$. Therefore, yes, Kelly is correct.



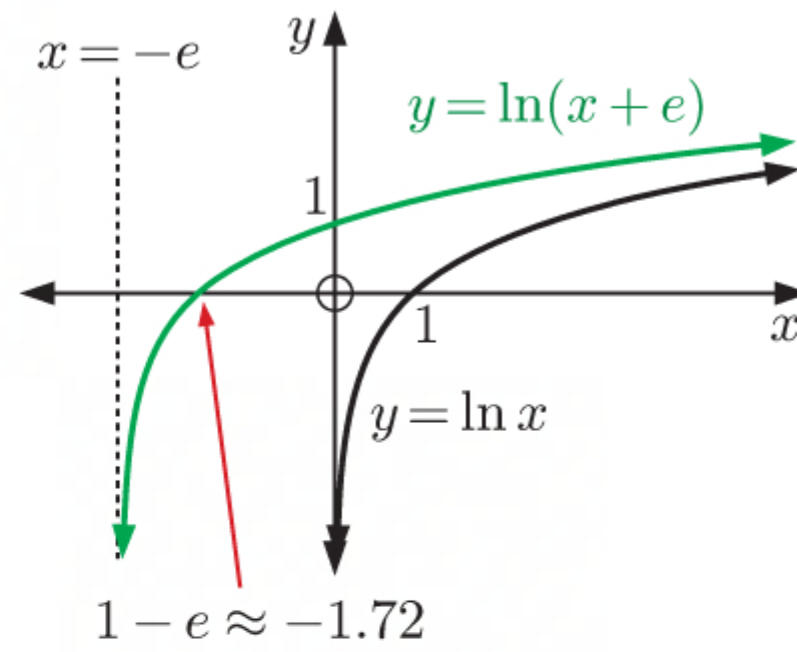
- 5 a** $y = \ln(x^3) = 3 \ln x$ is a vertical stretch of $y = \ln x$ with scale factor 3.



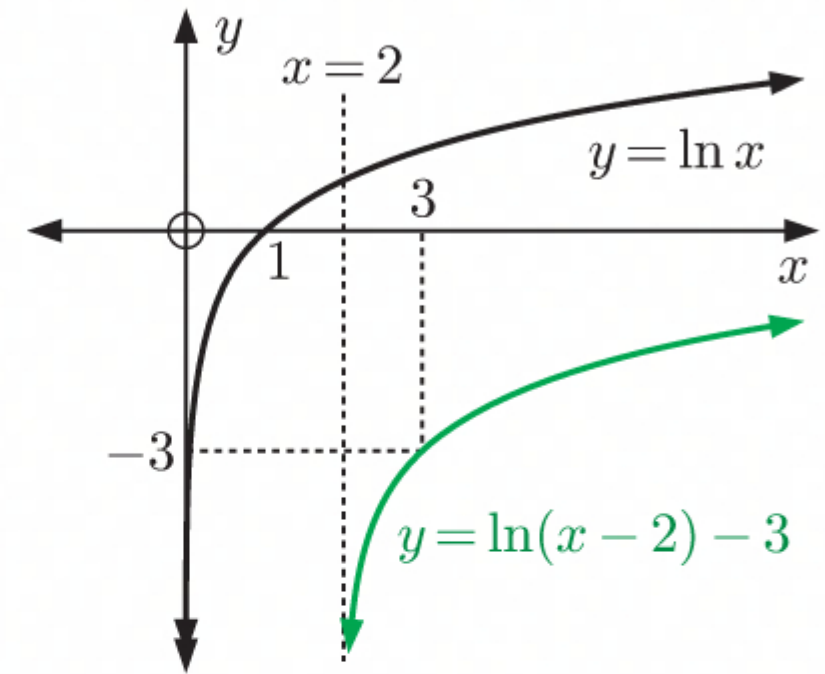
- b** $y = \ln\left(\frac{1}{x}\right) = \ln(x^{-1}) = -\ln x$ is a reflection of $y = \ln x$ in the x -axis.



- c** $y = \ln(x + e)$ is a horizontal translation of $y = \ln x$, e units to the left.

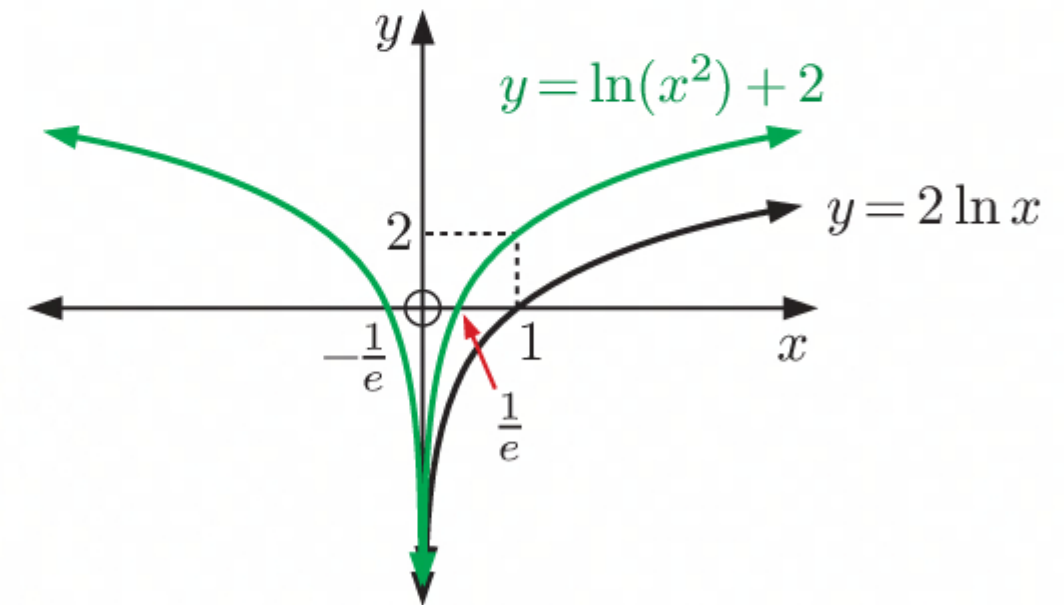


- d** $y = \ln(x - 2) - 3$ is a translation of $y = \ln x$ through $\begin{pmatrix} 2 \\ -3 \end{pmatrix}$.



- e** For $x > 0$, $y = \ln(x^2) + 2 = 2 \ln x + 2$ is a vertical translation of $y = 2 \ln x$, 2 units upwards.

For $x < 0$, $y = \ln(x^2) + 2 = \ln((-x)^2) + 2$ is a reflection of $y = \ln(x^2) + 2$, $x > 0$, in the y -axis.



6 $f(x) = be^x$, $g(x) = \ln(bx)$

a $(f \circ g)(x) = f(g(x))$
 $= f(\ln(bx))$
 $= be^{\ln(bx)}$
 $= b(bx)$
 $= b^2x$

b $(g \circ f)(x) = g(f(x))$
 $= g(be^x)$
 $= \ln(b \times be^x)$
 $= \ln(b^2e^x)$
 $= \ln(b^2) + \ln(e^x)$
 $= 2 \ln b + x$

c $(f \circ g)(x) = (g \circ f)(x)$
 $\therefore b^2x = 2 \ln b + x$
 $\therefore b^2x - x = 2 \ln b$
 $\therefore x(b^2 - 1) = 2 \ln b$
 $\therefore x = \frac{2 \ln b}{b^2 - 1}$

7 $f: x \mapsto e^{2x}$, $g: x \mapsto 2x - 1$

a f is defined by $y = e^{2x}$
 $\therefore f^{-1}$ is defined by $x = e^{2y}$
 $\therefore 2y = \ln x$
 $\therefore y = \frac{1}{2} \ln x$
 $\therefore f^{-1}(x) = \frac{1}{2} \ln x$

$(f^{-1} \circ g)(x) = f^{-1}(g(x))$
 $= f^{-1}(2x - 1)$
 $= \frac{1}{2} \ln(2x - 1)$

b $(g \circ f)(x) = g(f(x))$
 $= g(e^{2x})$
 $= 2e^{2x} - 1$

$(g \circ f)(x)$ is defined by $y = 2e^{2x} - 1$
 $\therefore (g \circ f)^{-1}(x)$ is defined by $x = 2e^{2y} - 1$

$\therefore x + 1 = 2e^{2y}$
 $\therefore e^{2y} = \frac{x + 1}{2}$

$\therefore 2y = \ln\left(\frac{x + 1}{2}\right)$

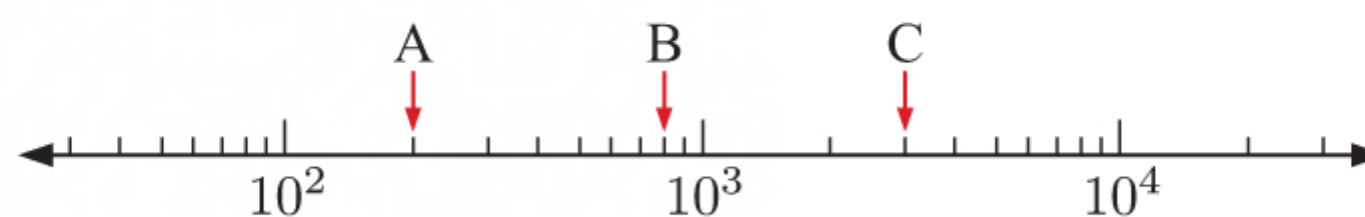
$\therefore y = \frac{1}{2} \ln\left(\frac{x + 1}{2}\right)$

$\therefore (g \circ f)^{-1}(x) = \frac{1}{2} \ln\left(\frac{x + 1}{2}\right)$

INVESTIGATION 3

LOGARITHMIC SCALES

- 1 a A - 200,
B - 800,
C - 3000



- b The minor ticks correspond to integer multiples of each power of 10. For example, the minor ticks between 10^1 and 10^2 are:

$$20 = 2 \times 10^1 = 10^{\log 2} \times 10^1 = 10^{1+\log 2} \approx 10^{1.301}$$

$$30 = 3 \times 10^1 = 10^{\log 3} \times 10^1 = 10^{1+\log 3} \approx 10^{1.477}$$

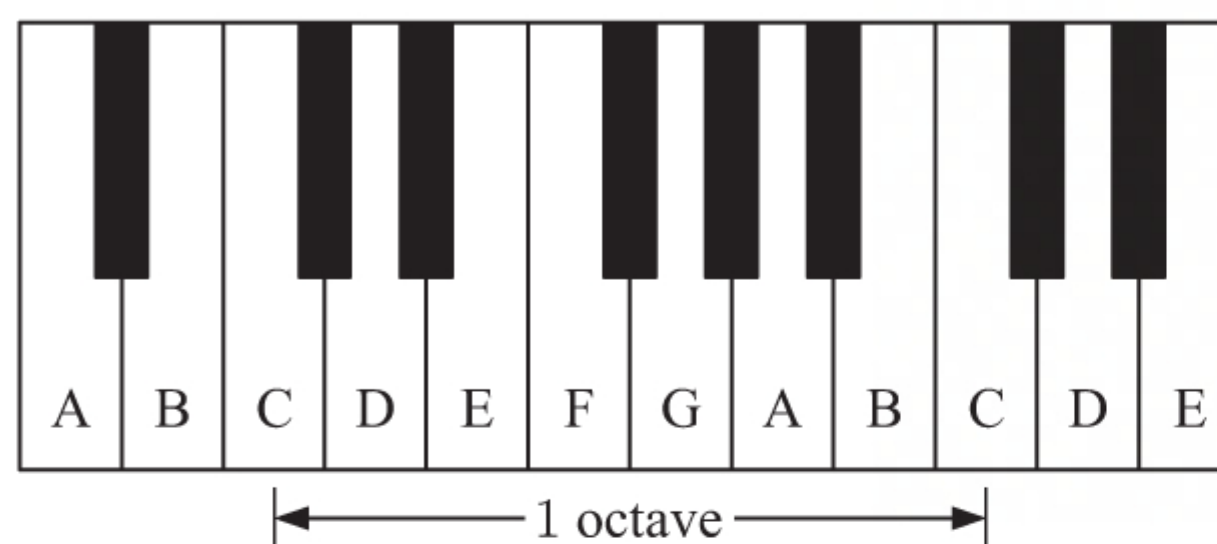
$$40 = 4 \times 10^1 = 10^{\log 4} \times 10^1 = 10^{1+\log 4} \approx 10^{1.602}$$

$$50 = 5 \times 10^1 = 10^{\log 5} \times 10^1 = 10^{1+\log 5} \approx 10^{1.699} \text{ and so on.}$$

The minor ticks are therefore placed between the major ticks in the positions $\log 2$, $\log 3$, $\log 4$, $\log 5$, and these values are not evenly spaced.

- c 0 is not on the logarithmic scale since $\log 0$ is undefined.

- 2 a Two notes separated by 3 octaves are 3 orders of magnitude apart.
b $f = 261.6 \times 2^n$ Hz



- c From b, $f = 261.6 \times 2^n$
 $\therefore n = \log_2 \left(\frac{f}{261.6} \right)$

Let f_P and f_Q be the frequencies of notes P and Q respectively, where Q is one note above P. There are 12 notes in an octave, and they are evenly spaced on the logarithmic scale.

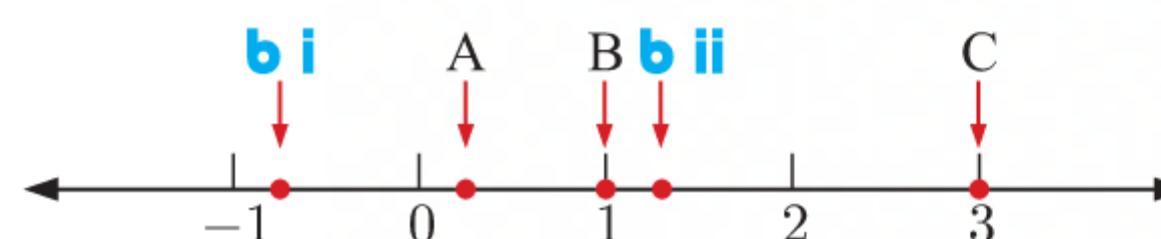
$$\therefore \log_2 \left(\frac{f_Q}{261.6} \right) - \log_2 \left(\frac{f_P}{261.6} \right) = \frac{1}{12}$$

$$\therefore \log_2 \left(\frac{f_Q}{f_P} \right) = \frac{1}{12}$$

$$\therefore \frac{f_Q}{f_P} = 2^{\frac{1}{12}}$$

So, the ratio between the frequencies of two consecutive notes is $2^{\frac{1}{12}} : 1 \approx 1.06 : 1$.

- 3 a C is $10^{3-1} = 100$ times larger than B.



- 4 a i If $M = 0$, $I = I_0$, the earthquake intensity is equal to the reference intensity.
ii If $M = 1$, $I = 10I_0$, the earthquake intensity is 10 times greater than the reference intensity.
b The magnitude of an earthquake follows a logarithmic scale. A magnitude 6 earthquake is $10^{6-3} = 1000$ times more intense than a magnitude 3 earthquake.

- c** The intensity I_4 of an earthquake with magnitude 4 obeys $4 = \log\left(\frac{I_4}{I_0}\right)$
- $$\therefore \frac{I_4}{I_0} = 10^4$$
- $$\therefore I_4 = 10^4 I_0$$

The magnitude of an earthquake with half this intensity is $M = \log\left(\frac{\frac{1}{2} \times 10^4 I_0}{I_0}\right)$

$$= \log \frac{1}{2} + \log 10^4$$

$$= 4 - \log 2$$

$$\approx 3.70$$

- 5 a** The possible concentrations of H_3O^+ take extremely small values that would otherwise be impossible to compare.
- b** $\text{pH} = -\log C$
- i** If $C = 0.000\,234$, $\text{pH} = -\log(0.000\,234) \approx 3.63$
- ii** If $\text{pH} = 7$, $7 = -\log C$
 $\therefore \log C = -7$
 $\therefore C = 10^{-7}$ units
- c** If $C > 1$, then $\log C > 0$
 $\therefore \text{pH} = -\log C < 0$

So, it is possible for a solution to have negative pH. In this case, the concentration of H_3O^+ is greater than 1 unit.

REVIEW SET 6A

- 1 a** $\log \sqrt{10}$
 $= \log(10^{\frac{1}{2}})$
 $= \frac{1}{2}$
- b** $\log\left(\frac{1}{\sqrt[3]{10}}\right)$
 $= \log(10^{-\frac{1}{3}})$
 $= -\frac{1}{3}$
- c** $\log(10^a \times 10^{b+1})$
 $= \log(10^{a+b+1})$
 $= a + b + 1$
- 2 a** $\log_4 64$
 $= \log_4(4^3)$
 $= 3$
- b** $\log_2 256$
 $= \log_2(2^8)$
 $= 8$
- c** $\log_2(0.25)$
 $= \log_2\left(\frac{1}{4}\right)$
 $= \log_2(2^{-2})$
 $= -2$
- d** $\log_{25} 5$
 $= \log_{25}(25^{\frac{1}{2}})$
 $= \frac{1}{2}$
- e** $\log_8 1$
 $= \log_8(8^0)$
 $= 0$
- f** $\log_{81} 3$
 $= \log_{81}(81^{\frac{1}{4}})$
 $= \frac{1}{4}$
- g** $\log_9\left(\frac{1}{9}\right)$
 $= \log_9(9^{-1})$
 $= -1$
- h** $\log_k \sqrt{k}$
 $= \log_k(k^{\frac{1}{2}})$
 $= \frac{1}{2}$
 provided $k > 0$,
 $k \neq 1$
- 3 a** $\log 27$
 ≈ 1.431
- b** $\log(0.58)$
 ≈ -0.237
- c** $\log 400$
 ≈ 2.602
- d** $\ln 40$
 ≈ 3.689

- 4 a** $4 \ln 2 + 2 \ln 3$
 $= \ln(2^4) + \ln(3^2)$
 $= \ln(16 \times 9)$
 $= \ln 144$
- b** $\frac{1}{2} \ln 9 - \ln 2$
 $= \ln(9^{\frac{1}{2}}) - \ln 2$
 $= \ln 3 - \ln 2$
 $= \ln(\frac{3}{2})$
- c** $2 \ln 5 - 1$
 $= \ln(5^2) - \ln(e^1)$
 $= \ln\left(\frac{25}{e}\right)$
- d** $\frac{1}{4} \ln 81$
 $= \ln(3^4)^{\frac{1}{4}}$
 $= \ln(3^1)$
 $= \ln 3$
- 5 a** $\log 16 + 2 \log 3$
 $= \log 16 + \log(3^2)$
 $= \log(16 \times 9)$
 $= \log 144$
- b** $\log_2 16 - 2 \log_2 3$
 $= \log_2 16 - \log_2(3^2)$
 $= \log_2(\frac{16}{9})$
- c** $2 + \log_4 5$
 $= \log_4(4^2) + \log_4 5$
 $= \log_4(16 \times 5)$
 $= \log_4 80$
- 6 a** $\log_5 36$
 $= \log_5(4 \times 9)$
 $= \log_5(2^2 \times 3^2)$
 $= \log_5(2^2) + \log_5(3^2)$
 $= 2 \log_5 2 + 2 \log_5 3$
 $= 2A + 2B$
- b** $\log_5 54$
 $= \log_5(2 \times 27)$
 $= \log_5(2 \times 3^3)$
 $= \log_5 2 + \log_5(3^3)$
 $= \log_5 2 + 3 \log_5 3$
 $= A + 3B$
- c** $\log_5(8\sqrt{3})$
 $= \log_5(2^3 \times 3^{\frac{1}{2}})$
 $= \log_5(2^3) + \log_5(3^{\frac{1}{2}})$
 $= 3 \log_5 2 + \frac{1}{2} \log_5 3$
 $= 3A + \frac{1}{2}B$
- d** $\log_5(\sqrt{6})$
 $= \log_5(6^{\frac{1}{2}})$
 $= \frac{1}{2} \log_5 6$
 $= \frac{1}{2} \log_5(2 \times 3)$
 $= \frac{1}{2}(\log_5 2 + \log_5 3)$
 $= \frac{1}{2}(A + B)$
- e** $\log_5(20.25)$
 $= \log_5(\frac{81}{4})$
 $= \log_5\left(\frac{3^4}{2^2}\right)$
 $= \log_5(3^4) - \log_5(2^2)$
 $= 4 \log_5 3 - 2 \log_5 2$
 $= 4B - 2A$
- f** $\log_5(\frac{8}{9})$
 $= \log_5\left(\frac{2^3}{3^2}\right)$
 $= \log_5(2^3) - \log_5(3^2)$
 $= 3 \log_5 2 - 2 \log_5 3$
 $= 3A - 2B$
- 7 a** $M = ab^n$
 $\therefore \log M = \log(ab^n)$
 $\therefore \log M = \log a + \log(b^n)$
 $\therefore \log M = \log a + n \log b$
- c** $G = \frac{a^2b}{c}$
 $\therefore \log G = \log\left(\frac{a^2b}{c}\right)$
 $\therefore \log G = \log(a^2b) - \log c$
 $\therefore \log G = \log(a^2) + \log b - \log c$
 $\therefore \log G = 2 \log a + \log b - \log c$
- 8 a** $3^x = 300$
 $\therefore \log(3^x) = \log 300$
 $\therefore x \log 3 = \log 300$
 $\therefore x = \frac{\log 300}{\log 3}$
 $\therefore x \approx 5.19$
- b** $30 \times 5^{1-x} = 0.15$
 $\therefore 5^{1-x} = 0.005$
 $\therefore \log(5^{1-x}) = \log(0.005)$
 $\therefore (1-x) \log 5 = \log(0.005)$
 $\therefore 1-x = \frac{\log(0.005)}{\log 5}$
 $\therefore x = 1 - \frac{\log(0.005)}{\log 5}$
 $\therefore x \approx 4.29$

$$\begin{aligned}
 \text{c} \quad & 3^{x+2} = 2^{1-x} \\
 & \therefore \log(3^{x+2}) = \log(2^{1-x}) \\
 & \therefore (x+2)\log 3 = (1-x)\log 2 \\
 & \therefore x\log 3 + 2\log 3 = \log 2 - x\log 2 \\
 & \therefore x(\log 3 + \log 2) = \log 2 - 2\log 3 \\
 & \therefore x\log(3 \times 2) = \log 2 - \log(3^2) \\
 & \therefore x\log 6 = \log\left(\frac{2}{9}\right) \\
 & \therefore x = \frac{\log(\frac{2}{9})}{\log 6} \\
 & \therefore x \approx -0.839
 \end{aligned}$$

$$\begin{aligned}
 \text{9 a} \quad & e^{2x} = 3e^x \\
 & \therefore e^{2x} - 3e^x = 0 \\
 & \therefore e^x(e^x - 3) = 0 \\
 & \therefore e^x - 3 = 0 \quad \{e^x > 0 \text{ for all } x\} \\
 & \therefore e^x = 3 \\
 & \therefore x = \ln 3
 \end{aligned}$$

$$\begin{aligned}
 \text{b} \quad & e^{2x} - 7e^x + 12 = 0 \\
 & \therefore (e^x - 3)(e^x - 4) = 0 \\
 & \therefore e^x - 3 = 0 \quad \text{or} \quad e^x - 4 = 0 \\
 & \therefore e^x = 3 \quad \text{or} \quad e^x = 4 \\
 & \therefore x = \ln 3 \quad \text{or} \quad \ln 4
 \end{aligned}$$

$$\begin{aligned}
 \text{10 a} \quad & \ln P = 1.5 \ln Q + \ln T \\
 & \therefore \ln P = \ln(Q^{1.5}) + \ln T \\
 & \quad = \ln(TQ^{1.5}) \\
 & \therefore P = TQ^{1.5}
 \end{aligned}$$

$$\begin{aligned}
 \text{b} \quad & \ln M = 1.2 - 0.5 \ln N \\
 & \therefore \ln M + \ln(N^{\frac{1}{2}}) = 1.2 \\
 & \therefore \ln(M\sqrt{N}) = 1.2 \\
 & \therefore M\sqrt{N} = e^{1.2} \\
 & \therefore M = \frac{e^{1.2}}{\sqrt{N}}
 \end{aligned}$$

$$\begin{aligned}
 \text{11 a} \quad & 3e^x - 5 = -2e^{-x} \\
 & \therefore 3e^{2x} - 5e^x = -2 \\
 & \therefore 3e^{2x} - 5e^x + 2 = 0 \\
 & \therefore (3e^x - 2)(e^x - 1) = 0 \\
 & \therefore 3e^x - 2 = 0 \quad \text{or} \quad e^x - 1 = 0 \\
 & \therefore e^x = \frac{2}{3} \quad \text{or} \quad e^x = 1 \\
 & \therefore x = \ln\left(\frac{2}{3}\right) \quad \text{or} \quad x = 0
 \end{aligned}$$

$$\begin{aligned}
 \text{b} \quad & 2\ln x - 3\ln\left(\frac{1}{x}\right) = 10 \\
 & \therefore \ln(x^2) - 3\ln(x^{-1}) = 10 \\
 & \therefore \ln(x^2) + 3\ln x = 10 \\
 & \therefore \ln(x^2) + \ln(x^3) = 10 \\
 & \therefore \ln(x^2 \times x^3) = 10 \\
 & \therefore \ln(x^5) = 10 \\
 & \therefore x^5 = e^{10} \\
 & \therefore x = (e^{10})^{\frac{1}{5}} \\
 & \therefore x = e^2
 \end{aligned}$$

$$\begin{aligned}
 \text{12 a} \quad & \log_2 x = -3 \\
 & \therefore x = 2^{-3} \\
 & \therefore x = \frac{1}{8}
 \end{aligned}$$

$$\begin{aligned}
 \text{b} \quad & \log_5 x \approx 2.743 \\
 & \therefore x \approx 5^{2.743} \\
 & \therefore x \approx 82.7
 \end{aligned}$$

$$\begin{aligned}
 \text{c} \quad & \log_3 x \approx -3.145 \\
 & \therefore x \approx 3^{-3.145} \\
 & \therefore x \approx 0.0316
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{13} \quad \mathbf{a} \quad \mathbf{i} \quad & 2^x = 50 \\
 & \therefore \log(2^x) = \log 50 \\
 & \therefore x \log 2 = \log 50 \\
 & \therefore x = \frac{\log 50}{\log 2}
 \end{aligned}$$

$$\mathbf{ii} \quad x = \frac{\log 50}{\log 2} \approx 5.64$$

$$\begin{aligned}
 \mathbf{c} \quad \mathbf{i} \quad & (0.6)^x = 0.01 \\
 & \therefore \log(0.6)^x = \log(10^{-2}) \\
 & \therefore x \log(0.6) = -2 \\
 & \therefore x = \frac{-2}{\log(0.6)}
 \end{aligned}$$

$$\mathbf{ii} \quad x = \frac{-2}{\log(0.6)} \approx 9.02$$

$$\mathbf{14} \quad \log_a b = x$$

$$\begin{aligned}
 \text{Now, } \log_a \left(\frac{1}{b} \right) &= \log_a (b^{-1}) \\
 &= -\log_a b \\
 &= -x
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{15} \quad & 16^x - 5 \times 8^x = 0 \\
 \therefore (2^4)^x - 5 \times (2^3)^x &= 0 \\
 \therefore 2^{4x} - 5 \times 2^{3x} &= 0 \\
 \therefore 2^{3x}(2^x - 5) &= 0 \\
 \therefore 2^x - 5 = 0 \quad \{2^{3x} \neq 0\} \\
 \therefore 2^x &= 5 \\
 \therefore x &= \log_2 5
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{16} \quad \mathbf{a} \quad \ln x &= 5 \\
 \therefore x &= e^5
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad 3 \ln x + 2 &= 0 \\
 \therefore 3 \ln x &= -2 \\
 \therefore \ln x &= -\frac{2}{3} \\
 \therefore x &= e^{-\frac{2}{3}}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{c} \quad e^x &= 400 \\
 \therefore x &= \ln 400
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{d} \quad e^{2x+1} &= 11 \\
 \therefore 2x + 1 &= \ln 11 \\
 \therefore 2x &= \ln 11 - 1 \\
 \therefore x &= \frac{\ln 11 - 1}{2}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{e} \quad 25e^{\frac{x}{2}} &= 750 \\
 \therefore e^{\frac{x}{2}} &= 30 \\
 \therefore \frac{x}{2} &= \ln 30 \\
 \therefore x &= 2 \ln 30
 \end{aligned}$$

17 $f(x) = e^{3x-4} + 1$

a f is defined by $y = e^{3x-4} + 1$
 $\therefore f^{-1}$ is defined by $x = e^{3y-4} + 1$
 $\therefore e^{3y-4} = x - 1$
 $\therefore 3y - 4 = \ln(x - 1)$
 $\therefore 3y = \ln(x - 1) + 4$
 $\therefore y = \frac{\ln(x - 1) + 4}{3}$
 $\therefore f^{-1}(x) = \frac{\ln(x - 1) + 4}{3}$

b $f^{-1}(8) - f^{-1}(3)$
 $= \frac{\ln 7 + 4}{3} - \frac{\ln 2 + 4}{3}$
 $= \frac{\ln 7 + 4 - \ln 2 - 4}{3}$
 $= \frac{1}{3} \ln\left(\frac{7}{2}\right)$

18 $g: x \mapsto \log_3(x + 2) - 2$

a $g(x)$ is a translation of $y = \log_3 x$ through $\begin{pmatrix} -2 \\ -2 \end{pmatrix}$.

b Domain = $\{x \mid x > -2\}$, Range = $\{y \mid y \in \mathbb{R}\}$

c As $x \rightarrow -2^+$, $y \rightarrow -\infty$, so the vertical asymptote is $x = -2$.

As $x \rightarrow \infty$, $y \rightarrow \infty$, so there is no horizontal asymptote.

When $x = 0$, $y = \log_3 2 - 2 \approx -1.37$.

So, the y -intercept is ≈ -1.37 .

When $y = 0$, $\log_3(x + 2) = 2$

$\therefore x + 2 = 3^2$

$\therefore x = 7$

So, the x -intercept is 7.

d g is defined by $y = \log_3(x + 2) - 2$

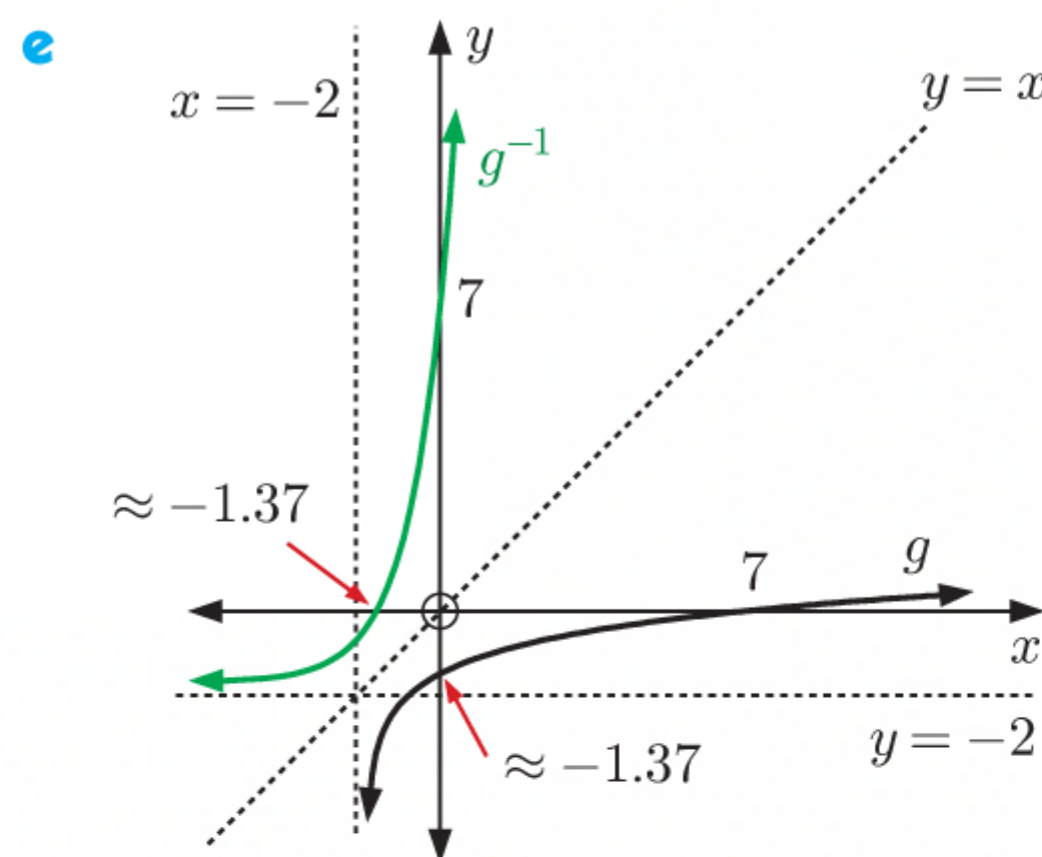
$\therefore g^{-1}$ is defined by $x = \log_3(y + 2) - 2$

$\therefore x + 2 = \log_3(y + 2)$

$\therefore y + 2 = 3^{x+2}$

$\therefore y = 3^{x+2} - 2$

$\therefore g^{-1}(x) = 3^{x+2} - 2$



19 $W_t = 8000 \times e^{-\frac{t}{20}}$

$$W_0 = 8000 \times e^0 = 8000$$

\therefore the initial weight is 8000 grams.

a For W_t to halve, $W_t = 4000$

$$\therefore 8000 \times e^{-\frac{t}{20}} = 4000$$

$$\therefore e^{-\frac{t}{20}} = \frac{1}{2}$$

$$\therefore -\frac{t}{20} = \ln\left(\frac{1}{2}\right)$$

$$\therefore t = -20 \ln\left(\frac{1}{2}\right)$$

$$\therefore t \approx 13.9$$

\therefore it will take about 13.9 weeks for the weight to halve.

c When $W_t = 0.1\%$ of W_0 ,

$$8000 \times e^{-\frac{t}{20}} = 0.001 \times 8000$$

$$\therefore e^{-\frac{t}{20}} = 0.001$$

$$\therefore -\frac{t}{20} = \ln(0.001)$$

$$\therefore t = -20 \ln(0.001)$$

$$\therefore t \approx 138$$

\therefore it will take about 138 weeks for the weight to reach 0.1% of its original value.

b When $W_t = 1000$,

$$8000 \times e^{-\frac{t}{20}} = 1000$$

$$\therefore e^{-\frac{t}{20}} = \frac{1}{8}$$

$$\therefore -\frac{t}{20} = \ln\left(\frac{1}{8}\right)$$

$$\therefore t = -20 \ln\left(\frac{1}{8}\right)$$

$$\therefore t \approx 41.6$$

\therefore it will take about 41.6 weeks for the weight to reach 1000 grams.

20 $P(t) = 80 \times (1.15)^t$ seals

a $P(0) = 80 \times (1.15)^0$
 $= 80 \times 1$
 $= 80$

So the initial population was 80 seals.

We need to find t when $P(t) = 2 \times 80 = 160$

$$\therefore 80 \times (1.15)^t = 160$$

$$\therefore (1.15)^t = 2$$

$$\therefore \log(1.15)^t = \log 2$$

$$\therefore t \log(1.15) = \log 2$$

$$\therefore t = \frac{\log 2}{\log(1.15)}$$

$$\approx 4.96$$

\therefore it took about 4.96 years or about 4 years and $11\frac{1}{2}$ months for the population to double in size.

$$\begin{aligned}
 \text{b Percentage increase in first 4 years} &= \left(\frac{P(4) - P(0)}{P(0)} \right) \times 100\% \\
 &= \left(\frac{80 \times (1.15)^4 - 80}{80} \right) \times 100\% \\
 &\approx 74.9\%
 \end{aligned}$$

21 a $f(x) = \log_2(x + 4) - 1$

i Domain = $\{x \mid x > -4\}$, Range = $\{y \mid y \in \mathbb{R}\}$

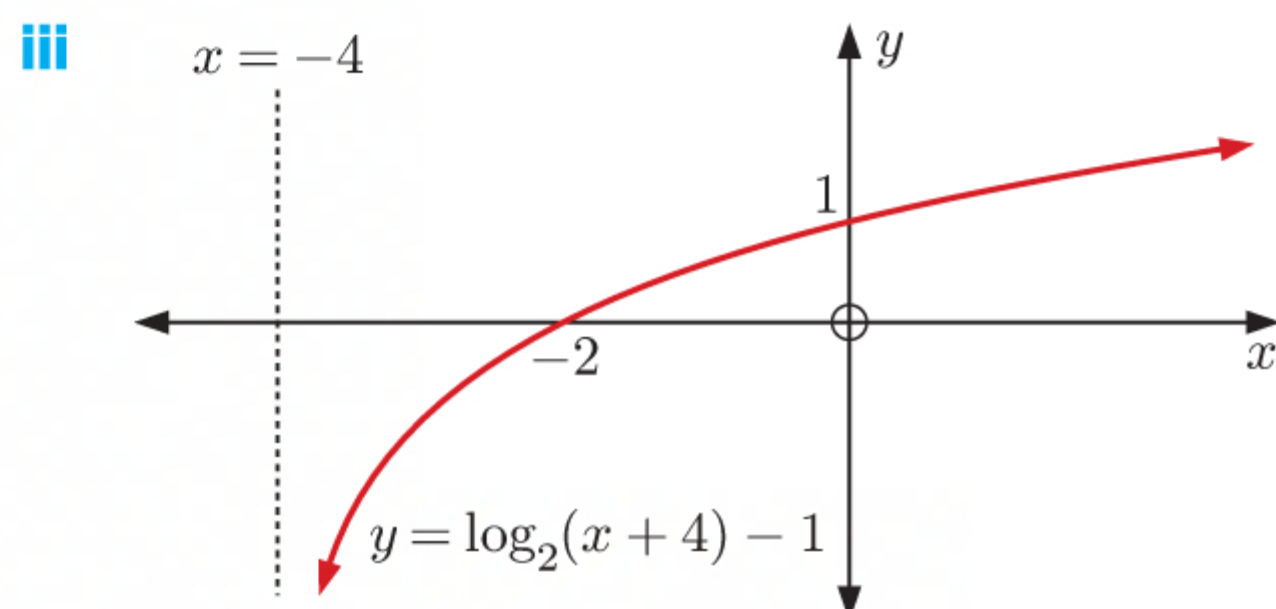
ii As $x \rightarrow -4^+$, $y \rightarrow -\infty$, so the vertical asymptote is $x = -4$.

As $x \rightarrow \infty$, $y \rightarrow \infty$, so there is no horizontal asymptote.

When $x = 0$, $\log_2 4 - 1 = 1$, so the y -intercept is 1.

$$\begin{aligned}
 \text{When } y = 0, \quad \log_2(x + 4) &= 1 \\
 \therefore x + 4 &= 2^1 \\
 \therefore x &= -2
 \end{aligned}$$

So, the x -intercept is -2 .



b $f(x) = \ln x + 2$

i Domain = $\{x \mid x > 0\}$, Range = $\{y \mid y \in \mathbb{R}\}$

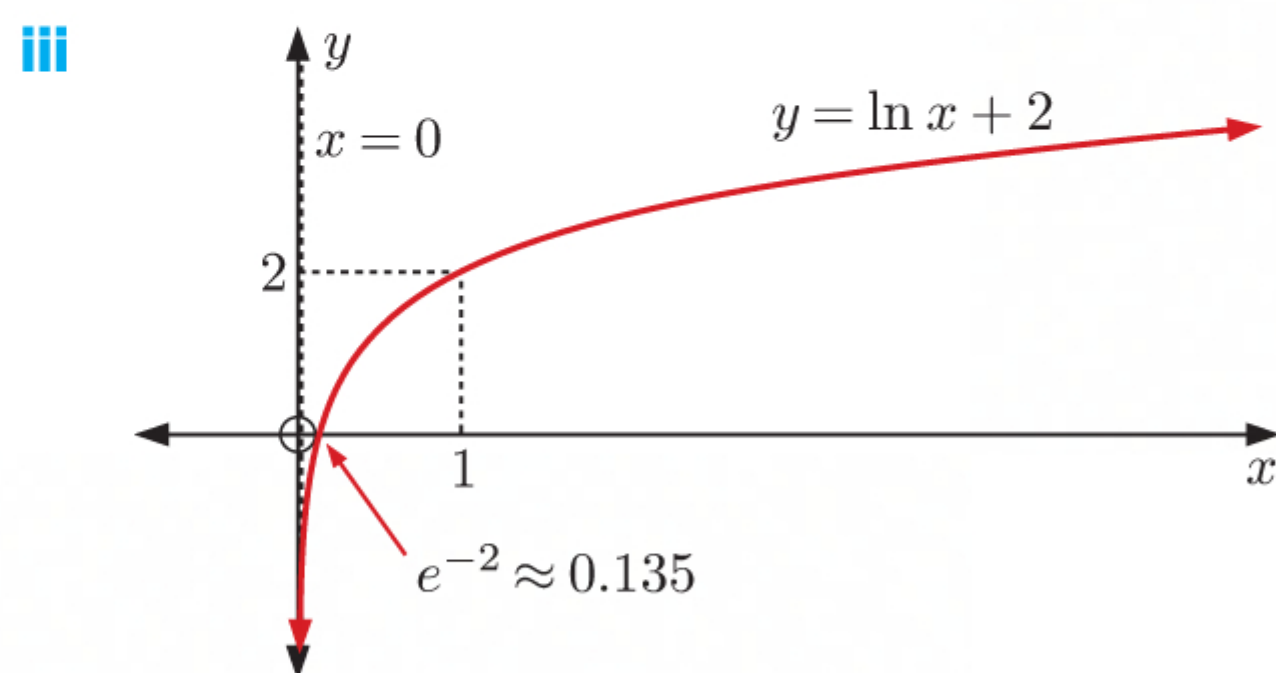
ii As $x \rightarrow 0^+$, $y \rightarrow -\infty$, so the vertical asymptote is $x = 0$.

As $x \rightarrow \infty$, $y \rightarrow \infty$, so there is no horizontal asymptote.

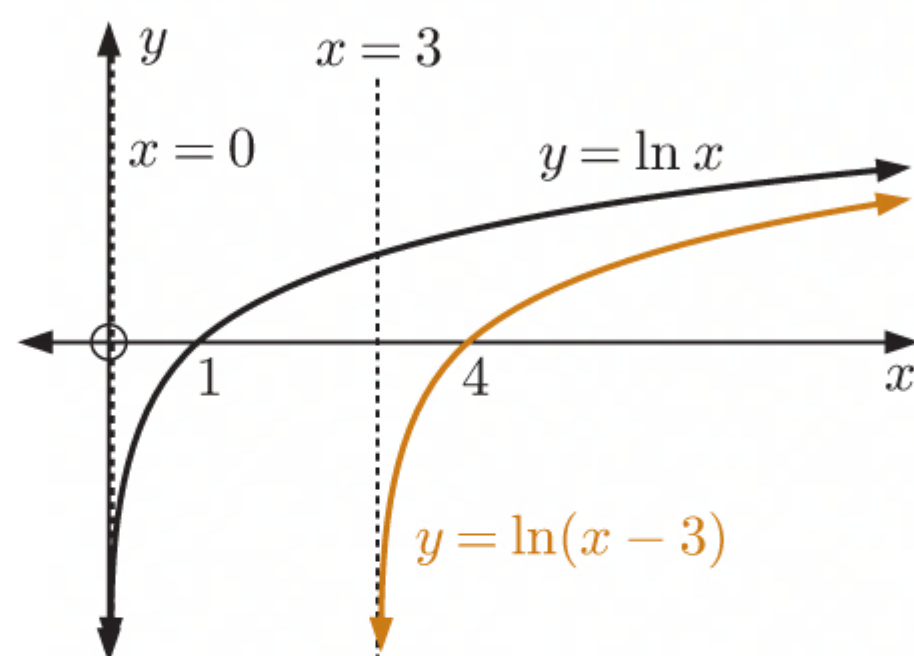
When $x = 0$, y is undefined, so there is no y -intercept.

$$\begin{aligned}
 \text{When } y = 0, \quad \ln x &= -2 \\
 \therefore x &= e^{-2} \approx 0.135
 \end{aligned}$$

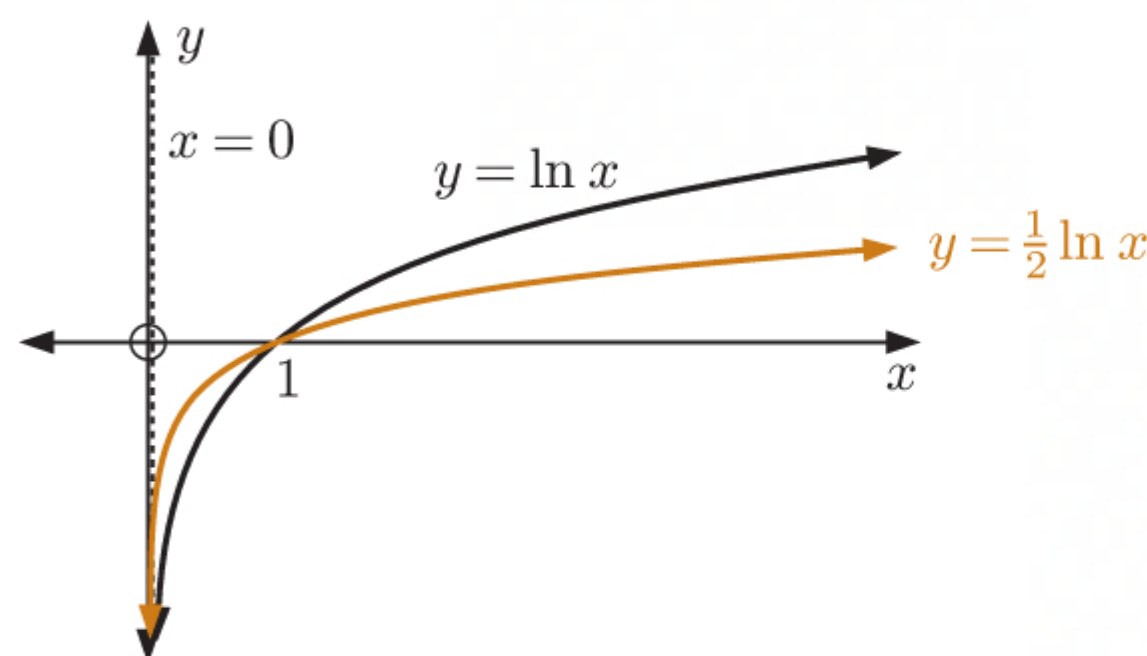
So, the x -intercept is e^{-2} .



- 22 a** $y = \ln(x - 3)$ is a horizontal translation of $y = \ln x$, 3 units to the right.



- b** $y = \frac{1}{2} \ln x$ is a vertical stretch of $y = \ln x$ with scale factor $\frac{1}{2}$.



- 23** $f(x) = e^x$, $g(x) = \ln(x + 4)$, $x > -4$

a $(f \circ g)(5) = f(g(5))$
 $= f(\ln 9)$
 $= e^{\ln 9}$
 $= 9$

b $(g \circ f)(0) = g(f(0))$
 $= g(e^0)$
 $= g(1)$
 $= \ln 5$

REVIEW SET 6B

1 a $\log \sqrt{1000}$
 $= \log((10^3)^{\frac{1}{2}})$
 $= \log(10^{\frac{3}{2}})$
 $= \frac{3}{2}$

b $\log\left(\frac{10}{\sqrt[3]{10}}\right)$
 $= \log\left(\frac{10^1}{10^{\frac{1}{3}}}\right)$
 $= \log(10^{\frac{2}{3}})$
 $= \frac{2}{3}$

c $\log\left(\frac{10^a}{10^{-b}}\right)$
 $= \log(10^{a-(-b)})$
 $= \log(10^{a+b})$
 $= a + b$

2 a $\log_2 128$
 $= \log_2(2^7)$
 $= 7$

b $\log_3\left(\frac{1}{27}\right)$
 $= \log_3(3^{-3})$
 $= -3$

c $\log_5\left(\frac{1}{\sqrt{5}}\right)$
 $= \log_5(5^{-\frac{1}{2}})$
 $= -\frac{1}{2}$

3 a $32 = 10^{\log 32}$
 $\approx 10^{1.5051}$

b 0.0013
 $= 10^{\log(0.0013)}$
 $\approx 10^{-2.8861}$

c 8.963×10^{-5}
 $= 10^{\log(8.963)} \times 10^{-5}$
 $= 10^{\log(8.963) - 5}$
 $\approx 10^{-4.0475}$

4 a $\ln(e\sqrt{e})$
 $= \ln(e^1 e^{\frac{1}{2}})$
 $= \ln(e^{\frac{3}{2}})$
 $= \frac{3}{2}$

b $\ln\left(\frac{1}{e^3}\right)$
 $= \ln(e^{-3})$
 $= -3$

c $\ln(e^{2x}) = 2x$

d $\ln\left(\frac{e}{e^x}\right)$
 $= \ln(e^{1-x})$
 $= 1 - x$

$$\begin{aligned}
 \mathbf{5} \quad \mathbf{a} \quad \frac{\log_2 25}{\log_2 125} &= \frac{\log_2(5^2)}{\log_2(5^3)} \\
 &= \frac{2 \log_2 5}{3 \log_2 5} \\
 &= \frac{2}{3}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad \frac{\log 64}{\log 32} &= \frac{\log(2^6)}{\log(2^5)} \\
 &= \frac{6 \log 2}{5 \log 2} \\
 &= \frac{6}{5}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{c} \quad \frac{\log_5 81}{\log_5 \sqrt{3}} &= \frac{\log_5(3^4)}{\log_5(3^{\frac{1}{2}})} \\
 &= \frac{4 \log_5 3}{\frac{1}{2} \log_5 3} \\
 &= \frac{4}{\frac{1}{2}} \\
 &= 8
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{6} \quad \mathbf{a} \quad e^{4 \ln x} \\
 &= (e^{\ln x})^4 \\
 &= x^4
 \end{aligned}$$

$$\mathbf{b} \quad \ln(e^5) = 5$$

$$\begin{aligned}
 \mathbf{c} \quad \ln(\sqrt{e}) &= \ln(e^{\frac{1}{2}}) \\
 &= \frac{1}{2}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{d} \quad 10^{\log x + \log 3} \\
 &= 10^{\log x} \times 10^{\log 3} \\
 &= x \times 3 \\
 &= 3x
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{e} \quad \ln\left(\frac{1}{e^x}\right) &= \ln(e^{-x}) \\
 &= -x
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{f} \quad \frac{\log(x^2)}{\log_3 9} \\
 &= \frac{\log(x^2)}{\log_3(3^2)} \\
 &= \frac{\log(x^2)}{2} \\
 &= \frac{1}{2} \log(x^2) \\
 &= \log x
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{7} \quad \mathbf{a} \quad 20 &= e^{\ln 20} \\
 &\approx e^{2.9957}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad 3000 &= e^{\ln 3000} \\
 &\approx e^{8.0064}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{c} \quad 0.075 &= e^{\ln(0.075)} \\
 &\approx e^{-2.5903}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{8} \quad \mathbf{a} \quad \mathbf{i} \quad 5^x &= 7 \\
 \therefore \log(5^x) &= \log 7 \\
 \therefore x \log 5 &= \log 7 \\
 \therefore x &= \frac{\log 7}{\log 5} \\
 \mathbf{ii} \quad x &= \frac{\log 7}{\log 5} \approx 1.21
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad \mathbf{i} \quad 2^x &= 0.1 \\
 \therefore \log(2^x) &= \log(0.1) \\
 \therefore x \log 2 &= \log(10^{-1}) \\
 \therefore x &= -\frac{1}{\log 2} \\
 \mathbf{ii} \quad x &= -\frac{1}{\log 2} \approx -3.32
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{9} \quad \mathbf{a} \quad \ln 60 - \ln 20 \\
 &= \ln\left(\frac{60}{20}\right) \\
 &= \ln 3
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad \ln 4 + \ln 1 \\
 &= \ln 4 + 0 \\
 &= \ln 4
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{c} \quad \ln 200 - \ln 8 + \ln 5 \\
 &= \ln\left(\frac{200}{8}\right) + \ln 5 \\
 &= \ln\left(\frac{200}{8} \times 5\right) \\
 &= \ln 125
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{10} \quad \mathbf{a} \quad e^{2x} &= 70 \\
 \therefore \ln(e^{2x}) &= \ln 70 \\
 \therefore 2x &= \ln 70 \\
 \therefore x &= \frac{\ln 70}{2}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad 3 \times (1.3)^x &= 11 \\
 \therefore (1.3)^x &= \frac{11}{3} \\
 \therefore \log(1.3)^x &= \log\left(\frac{11}{3}\right) \\
 \therefore x \log(1.3) &= \log\left(\frac{11}{3}\right) \\
 \therefore x &= \frac{\log(\frac{11}{3})}{\log(1.3)}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{c} \quad 5 \times 2^{0.3x} &= 16 \\
 \therefore 2^{0.3x} &= \frac{16}{5} \\
 \therefore \log(2^{0.3x}) &= \log\left(\frac{16}{5}\right) \\
 \therefore 0.3x \log 2 &= \log\left(\frac{16}{5}\right) \\
 \therefore x &= \frac{\log(\frac{16}{5})}{0.3 \log 2} \\
 \therefore x &= \frac{10 \log(\frac{16}{5})}{3 \log 2}
 \end{aligned}$$

11 $\log x = \ln x$
 $\therefore \log x = \frac{\log x}{\log e} \quad \left\{ \log_a b = \frac{\log_c b}{\log_c a} \right\}$
 $\therefore \log e \log x = \log x$
 $\therefore \log x (\log e - 1) = 0$
 $\therefore \log x = 0 \quad \{\log e \neq 1\}$
 $\therefore x = 10^0$
 $\therefore x = 1$
 $\therefore x = 1$ is the only value of x for which $\log x = \ln x$.

12 a $P = 3 \times b^x$
 $\therefore \log P = \log(3 \times b^x)$
 $\therefore \log P = \log 3 + \log(b^x)$
 $\therefore \log P = \log 3 + x \log b$

b $m = \frac{n^3}{p^2}$
 $\therefore \log m = \log \left(\frac{n^3}{p^2} \right)$
 $\therefore \log m = \log(n^3) - \log(p^2)$
 $\therefore \log m = 3 \log n - 2 \log p$

13 $\log_3 7 \times 2 \log_7 x = \log_3 7 \times 2 \times \frac{\log_3 x}{\log_3 7} \quad \left\{ \log_b a = \frac{\log_c a}{\log_c b} \right\}$
 $= 2 \log_3 x$

14 a $\log T = 2 \log x - \log y$
 $\therefore \log T = \log(x^2) - \log y$
 $\therefore \log T = \log \left(\frac{x^2}{y} \right)$
 $\therefore T = \frac{x^2}{y}$

b $\log_2 K = \log_2 n + \frac{1}{2} \log_2 t$
 $\therefore \log_2 K = \log_2 n + \log_2(t^{\frac{1}{2}})$
 $\therefore \log_2 K = \log_2(n \times \sqrt{t})$
 $\therefore K = n\sqrt{t}$

15 a $\ln 32 = \ln(2^5)$
 $= 5 \ln 2$

b $\ln 125 = \ln(5^3)$
 $= 3 \ln 5$

c $\ln 729 = \ln(3^6)$
 $= 6 \ln 3$

16 $\log_2 x$ is defined for all $x > 0$
 \therefore the domain is $\{x \mid x > 0\}$
and the range is $y \in \mathbb{R}$.
 $\ln(x+5)$ is defined for all $x > -5$
 \therefore the domain is $\{x \mid x > -5\}$
and the range is $y \in \mathbb{R}$.

So, the completed table is:

	$y = \log_2 x$	$y = \ln(x+5)$
Domain	$\{x \mid x > 0\}$	$\{x \mid x > -5\}$
Range	$\{y \mid y \in \mathbb{R}\}$	$\{y \mid y \in \mathbb{R}\}$

17 a $4^x - 2^x - 20$
 $= (2^x)^2 - 2^x - 20$
 $= (2^x + 4)(2^x - 5) \quad \{a^2 - a - 20 = (a+4)(a-5)\}$

b

$$\begin{aligned}
 2^x (2^x - 1) &= 20 \\
 \therefore (2^x)^2 - 2^x - 20 &= 0 \\
 \therefore (2^x + 4)(2^x - 5) &= 0 \quad \{\text{using a}\} \\
 \therefore 2^x &= -4 \text{ or } 5 \\
 \text{But } 2^x \text{ is never negative} \quad \therefore 2^x &= 5 \\
 \therefore \log(2^x) &= \log 5 \\
 \therefore x \log 2 &= \log 5 \\
 \therefore x &= \frac{\log 5}{\log 2} \text{ or } \log_2 5
 \end{aligned}$$

c i If $p = \log_5 2$ then

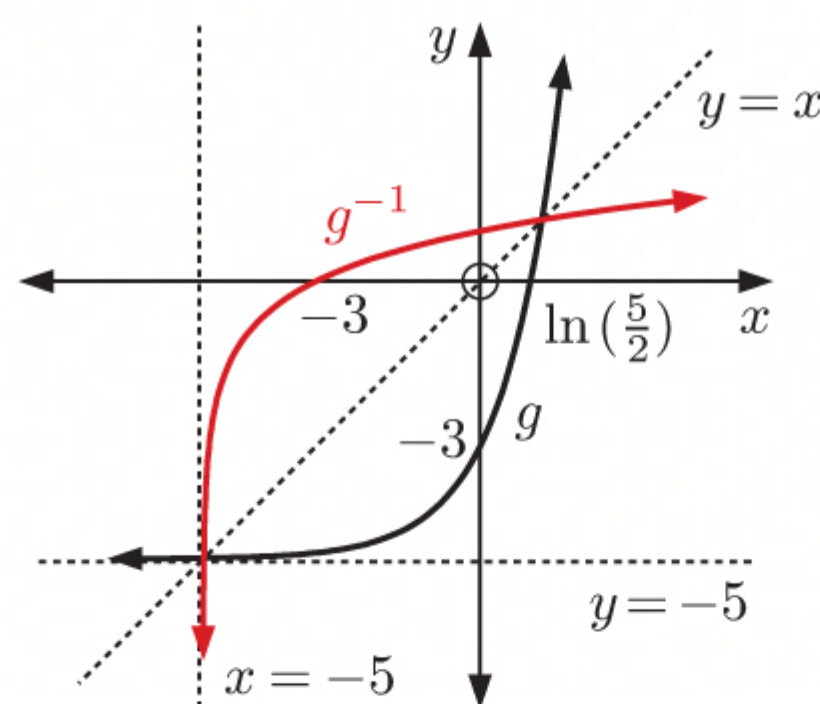
$$\begin{aligned}
 p &= \frac{\log 2}{\log 5} \\
 \therefore x &= \frac{1}{p}
 \end{aligned}$$

ii

$$\begin{aligned}
 8^x &= 5^{1-x} \\
 \therefore 2^{3x} &= 5^{1-x} \\
 \therefore \log(2^{3x}) &= \log(5^{1-x}) \\
 \therefore 3x \log 2 &= (1-x) \log 5 \\
 \therefore \frac{1-x}{3x} &= \frac{\log 2}{\log 5} = p \\
 \therefore 1-x &= 3px \\
 \therefore x(3p+1) &= 1 \\
 \therefore x &= \frac{1}{3p+1}
 \end{aligned}$$

18 $g: x \mapsto 2e^x - 5$

$$\begin{aligned}
 \text{a } g \text{ is defined by } y &= 2e^x - 5 \\
 \therefore g^{-1} \text{ is defined by } x &= 2e^y - 5 \\
 \therefore x+5 &= 2e^y \\
 \therefore e^y &= \frac{x+5}{2} \\
 \therefore y &= \ln\left(\frac{x+5}{2}\right) \\
 \therefore g^{-1}(x) &= \ln\left(\frac{x+5}{2}\right)
 \end{aligned}$$

b

c g : Domain = $\{x \mid x \in \mathbb{R}\}$, Range = $\{y \mid y > -5\}$
 g^{-1} : Domain = $\{x \mid x > -5\}$, Range = $\{y \mid y \in \mathbb{R}\}$

d Consider $g(x) = 2e^x - 5$:As $x \rightarrow -\infty$, $y \rightarrow -5^+$, so the horizontal asymptote is $y = -5$.When $x = 0$, $y = 2e^0 - 5$

$$\therefore y = -3$$

So, the y -intercept is -3 .When $y = 0$, $2e^x - 5 = 0$

$$\therefore 2e^x = 5$$

$$\therefore e^x = \frac{5}{2}$$

$$\therefore x = \ln\left(\frac{5}{2}\right)$$

So, the x -intercept is $\ln\left(\frac{5}{2}\right) \approx 0.916$.

Since $g^{-1}(x)$ is the reflection of $g(x)$ in the line $y = x$, the x and y -intercepts of $g(x)$ are the y and x -intercepts of $g^{-1}(x)$ respectively.

The horizontal asymptote of $g(x)$ corresponds to a vertical asymptote of $g^{-1}(x)$.

So, $g^{-1}(x)$ has vertical asymptote $x = -5$, x -intercept -3 , and y -intercept $\ln\left(\frac{5}{2}\right) \approx 0.916$.

19 $T = 60e^{-0.1t} + 20$ °C

When $T = 40$, $60e^{-0.1t} + 20 = 40$

$$\therefore 60e^{-0.1t} = 20$$

$$\therefore e^{-0.1t} = \frac{1}{3}$$

$$\therefore -0.1t = \ln\left(\frac{1}{3}\right)$$

$$\therefore t = -10 \times -\ln 3$$

$$= 10 \ln 3 \text{ minutes as required}$$

20 $W(t) = 2500 \times 3^{-\frac{t}{3000}}$ grams

a $W(0) = 2500 \times 3^0$
 $= 2500 \times 1$
 $= 2500$

\therefore the initial weight was 2500 grams.

b When $W(t) = 30\%$ of original weight,

$$2500 \times 3^{-\frac{t}{3000}} = 0.3 \times 2500$$

$$\therefore 3^{-\frac{t}{3000}} = 0.3$$

$$\therefore \log(3^{-\frac{t}{3000}}) = \log(0.3)$$

$$\therefore -\frac{t}{3000} \times \log 3 = \log(0.3)$$

$$\therefore t = \frac{-\log(0.3) \times 3000}{\log 3}$$

$$\therefore t \approx 3287.7$$

\therefore it takes about 3288 years for the isotope to reduce to 30% of its original weight.

c Percentage change after 1500 years $= \left(\frac{W(1500) - W(0)}{W(0)} \right) \times 100\%$
 $= \left(\frac{2500 \times 3^{-\frac{1}{2}} - 2500}{2500} \right) \times 100\%$
 $\approx -42.3\%$

\therefore the percentage of weight lost after 1500 years is about 42.3%.

21 a $5^{\frac{x}{2}} = 9$

$$\therefore \log(5^{\frac{x}{2}}) = \log 9$$

$$\therefore \frac{x}{2} \log 5 = \log 9$$

$$\therefore x = \frac{2 \log 9}{\log 5}$$

b $e^x = 30$

$$\therefore x = \ln 30$$

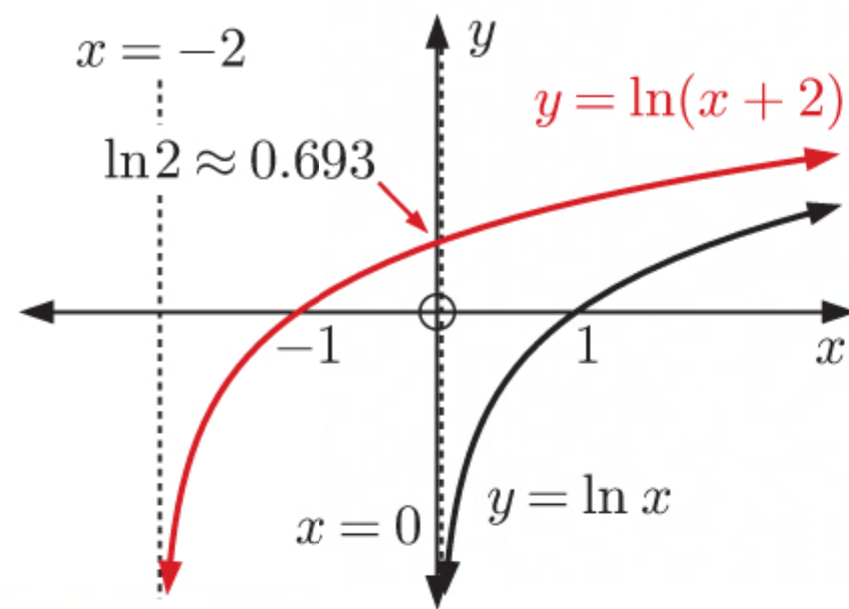
c $e^{1-3x} = 2$

$$\therefore 1 - 3x = \ln 2$$

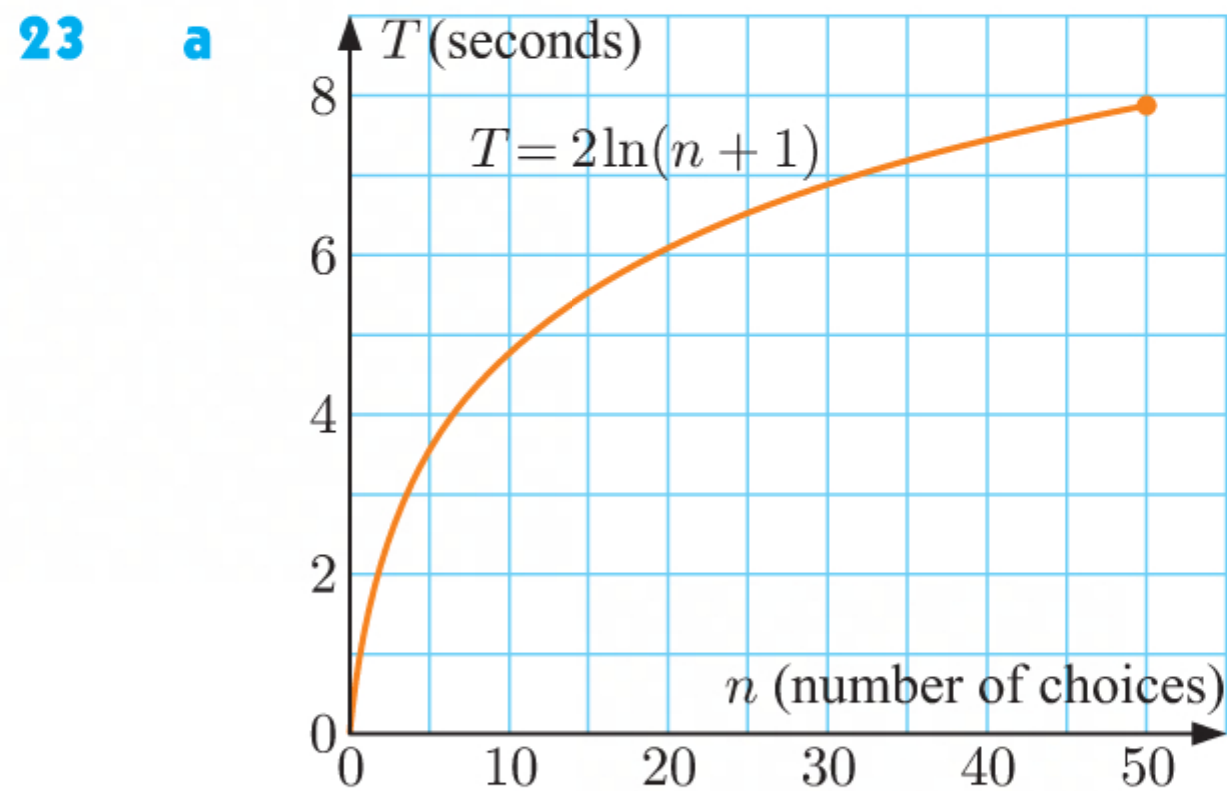
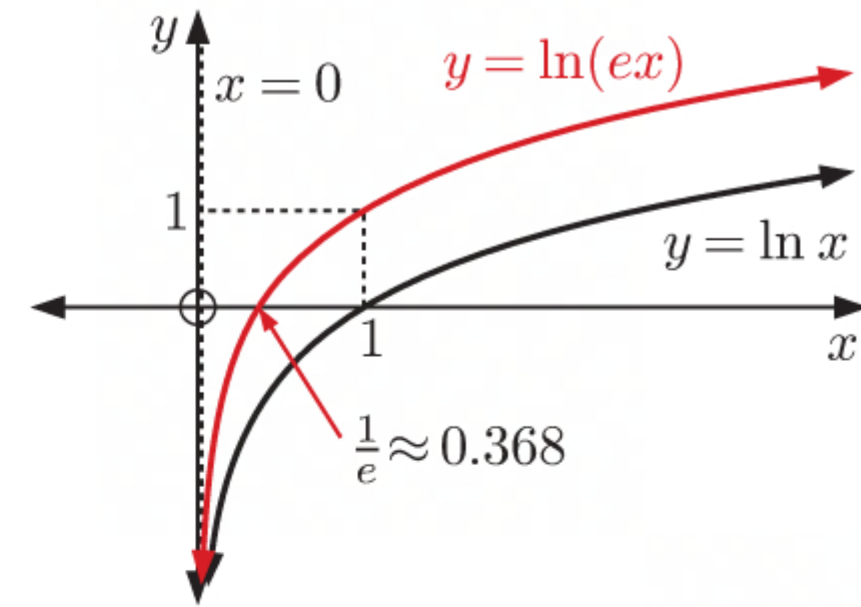
$$\therefore 3x = 1 - \ln 2$$

$$\therefore x = \frac{1 - \ln 2}{3}$$

- 22 a** $y = \ln(x + 2)$ is a translation of $y = \ln x$, 2 units to the left.



- b** $y = \ln(ex)$ is a horizontal stretch of $y = \ln x$ with scale factor $\frac{1}{e}$.



- b** $T = 2 \ln(n + 1)$ seconds
- i** When $n = 5$, $T = 2 \ln 6 \approx 3.58$ seconds
 - ii** When $n = 15$, $T = 2 \ln 16 \approx 5.55$ seconds
- c** When $n = 20$, $T = 2 \ln 21 \approx 6.09$ seconds
 When $n = 40$, $T = 2 \ln 41 \approx 7.43$ seconds
 $2 \ln 41 - 2 \ln 21 \approx 1.34$ seconds longer

Chapter 7

THE UNIT CIRCLE AND RADIAN MEASURE

EXERCISE 7A

1 a $90^\circ = \left(90 \times \frac{\pi}{180}\right)$ radians
 $= \frac{\pi}{2}$ radians

c $30^\circ = \left(30 \times \frac{\pi}{180}\right)$ radians
 $= \frac{\pi}{6}$ radians

e $9^\circ = \left(9 \times \frac{\pi}{180}\right)$ radians
 $= \frac{\pi}{20}$ radians

g $225^\circ = \left(225 \times \frac{\pi}{180}\right)$ radians
 $= \frac{5\pi}{4}$ radians

i $360^\circ = \left(360 \times \frac{\pi}{180}\right)$ radians
 $= 2\pi$ radians

k $315^\circ = \left(315 \times \frac{\pi}{180}\right)$ radians
 $= \frac{7\pi}{4}$ radians

m $36^\circ = \left(36 \times \frac{\pi}{180}\right)$ radians
 $= \frac{\pi}{5}$ radians

o $230^\circ = \left(230 \times \frac{\pi}{180}\right)$ radians
 $= \frac{23\pi}{18}$ radians

2 a $36.7^\circ = \left(36.7 \times \frac{\pi}{180}\right)$ radians
 ≈ 0.641 radians

c $317.9^\circ = \left(317.9 \times \frac{\pi}{180}\right)$ radians
 ≈ 5.55 radians

e $396.7^\circ = \left(396.7 \times \frac{\pi}{180}\right)$ radians
 ≈ 6.92 radians

3 a $\frac{\pi}{5} = \left(\frac{\pi}{5} \times \frac{180}{\pi}\right)^\circ$
 $= 36^\circ$

c $\frac{3\pi}{4} = \left(\frac{3\pi}{4} \times \frac{180}{\pi}\right)^\circ$
 $= 135^\circ$

b $60^\circ = \left(60 \times \frac{\pi}{180}\right)$ radians
 $= \frac{\pi}{3}$ radians

d $18^\circ = \left(18 \times \frac{\pi}{180}\right)$ radians
 $= \frac{\pi}{10}$ radians

f $135^\circ = \left(135 \times \frac{\pi}{180}\right)$ radians
 $= \frac{3\pi}{4}$ radians

h $270^\circ = \left(270 \times \frac{\pi}{180}\right)$ radians
 $= \frac{3\pi}{2}$ radians

j $720^\circ = \left(720 \times \frac{\pi}{180}\right)$ radians
 $= 4\pi$ radians

l $540^\circ = \left(540 \times \frac{\pi}{180}\right)$ radians
 $= 3\pi$ radians

n $80^\circ = \left(80 \times \frac{\pi}{180}\right)$ radians
 $= \frac{4\pi}{9}$ radians

b $137.2^\circ = \left(137.2 \times \frac{\pi}{180}\right)$ radians
 ≈ 2.39 radians

d $219.6^\circ = \left(219.6 \times \frac{\pi}{180}\right)$ radians
 ≈ 3.83 radians

b $\frac{3\pi}{5} = \left(\frac{3\pi}{5} \times \frac{180}{\pi}\right)^\circ$
 $= 108^\circ$

d $\frac{\pi}{18} = \left(\frac{\pi}{18} \times \frac{180}{\pi}\right)^\circ$
 $= 10^\circ$

$$\text{e } \frac{\pi}{9} = \left(\frac{\pi}{9} \times \frac{180}{\pi}\right)^\circ = 20^\circ$$

$$\text{f } \frac{7\pi}{9} = \left(\frac{7\pi}{9} \times \frac{180}{\pi}\right)^\circ = 140^\circ$$

$$\text{g } \frac{\pi}{10} = \left(\frac{\pi}{10} \times \frac{180}{\pi}\right)^\circ = 18^\circ$$

$$\text{h } \frac{3\pi}{20} = \left(\frac{3\pi}{20} \times \frac{180}{\pi}\right)^\circ = 27^\circ$$

$$\text{i } \frac{7\pi}{6} = \left(\frac{7\pi}{6} \times \frac{180}{\pi}\right)^\circ = 210^\circ$$

$$\text{j } \frac{\pi}{8} = \left(\frac{\pi}{8} \times \frac{180}{\pi}\right)^\circ = 22.5^\circ$$

$$\text{4 a } 2 \text{ radians} = \left(2 \times \frac{180}{\pi}\right)^\circ \approx 114.59^\circ$$

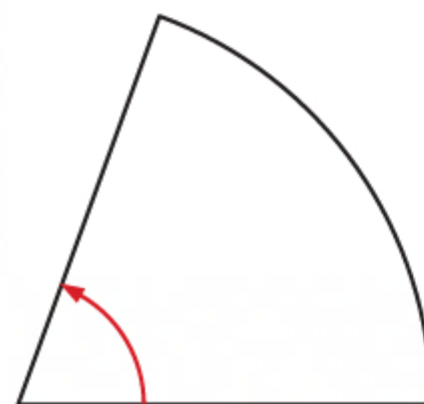
$$\text{b } 1.53 \text{ radians} = \left(1.53 \times \frac{180}{\pi}\right)^\circ \approx 87.66^\circ$$

$$\text{c } 0.867 \text{ radians} = \left(0.867 \times \frac{180}{\pi}\right)^\circ \approx 49.68^\circ$$

$$\text{d } 3.179 \text{ radians} = \left(3.179 \times \frac{180}{\pi}\right)^\circ \approx 182.14^\circ$$

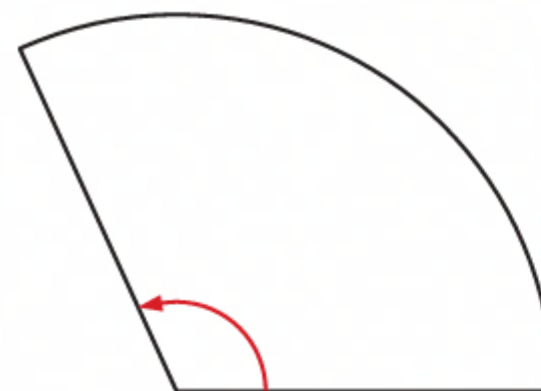
$$\text{e } 5.267 \text{ radians} = \left(5.267 \times \frac{180}{\pi}\right)^\circ \approx 301.78^\circ$$

5 a 70° corresponds with diagram **F**.



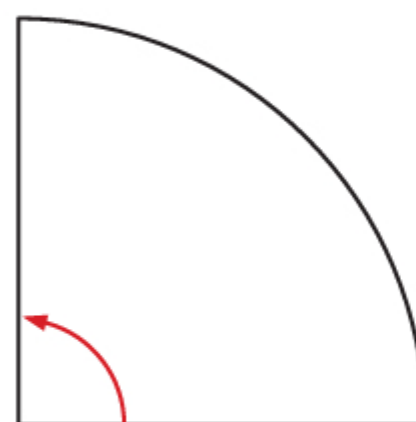
$$\text{b } 2^c = \left(2 \times \frac{180}{\pi}\right)^\circ \approx 115^\circ$$

So, 2^c corresponds with diagram **B**.

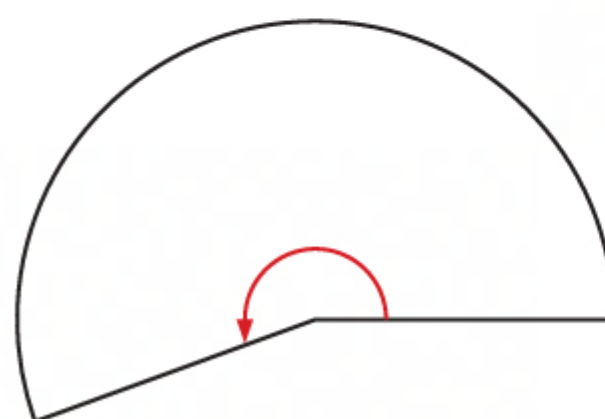


$$\text{c } \frac{\pi}{2} = \left(\frac{\pi}{2} \times \frac{180}{\pi}\right)^\circ = 90^\circ$$

So, $\frac{\pi}{2}$ corresponds with diagram **D**.

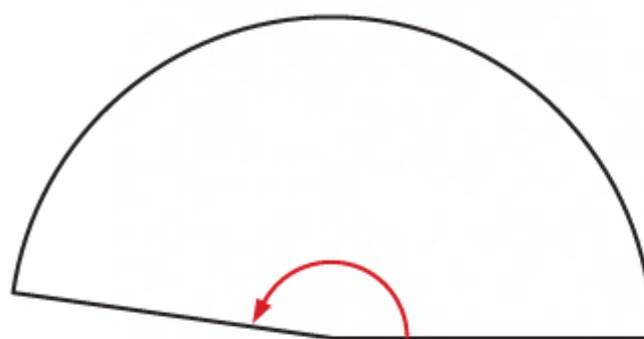


d 200° corresponds with diagram **A**.



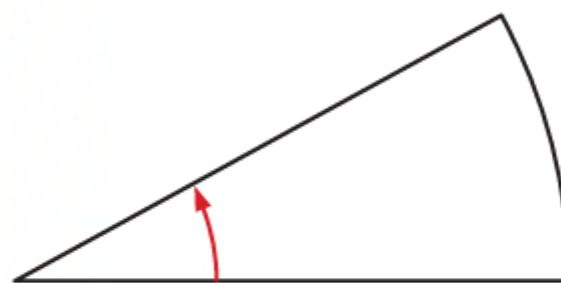
e $3^c = \left(3 \times \frac{180}{\pi}\right)^\circ$
 $\approx 172^\circ$

So, 3^c corresponds with diagram **E**.



f $0.5 \text{ radians} = \left(0.5 \times \frac{180}{\pi}\right)^\circ$
 $\approx 28.6^\circ$

So, 0.5 corresponds with diagram **C**.



6 a

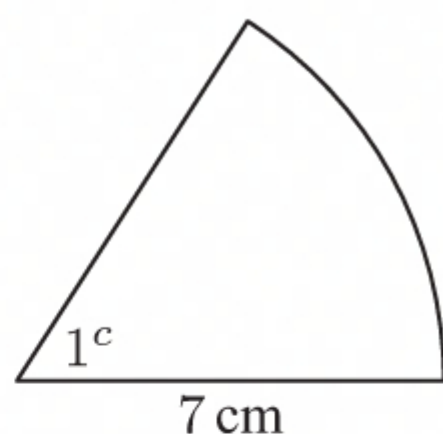
Degrees	0	45	90	135	180	225	270	315	360
Radians	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$	2π

b

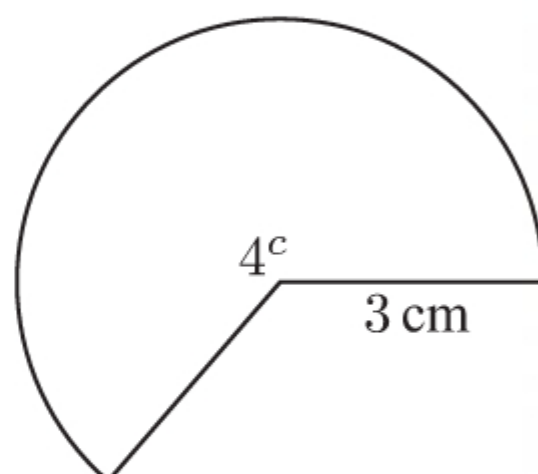
Degrees	0	30	60	90	120	150	180	210	240	270	300	330	360
Radians	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$	2π

EXERCISE 7B

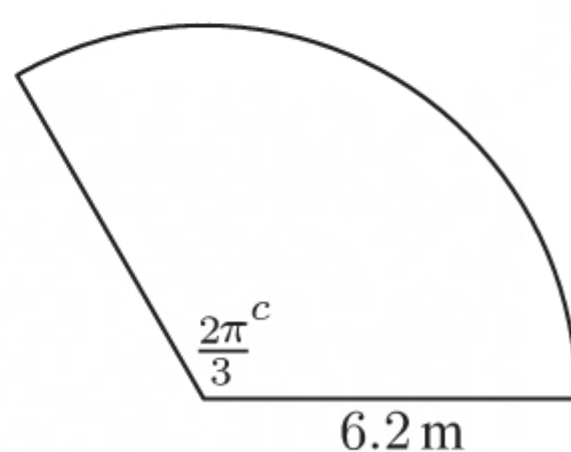
1 a arc length $= \theta r$
 $= 1 \times 7$
 $= 7 \text{ cm}$



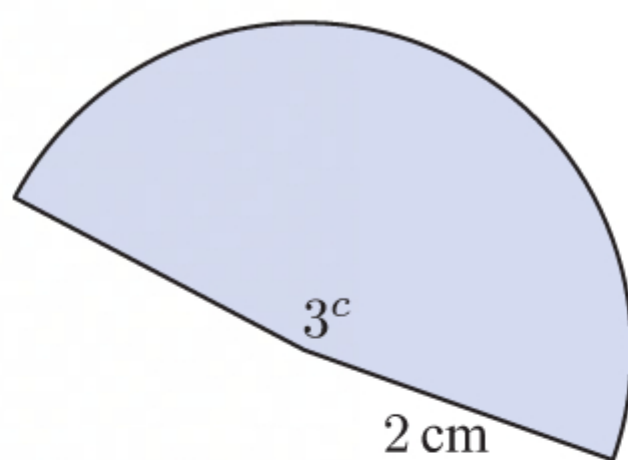
b arc length $= \theta r$
 $= 4 \times 3$
 $= 12 \text{ cm}$



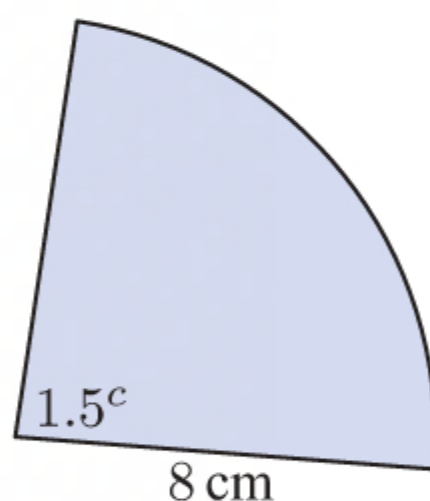
c arc length $= \theta r$
 $= \frac{2\pi}{3} \times 6.2$
 $\approx 13.0 \text{ m}$



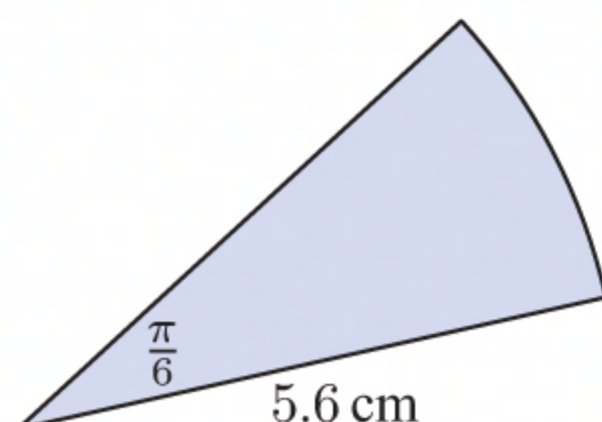
2 a $\text{area} = \frac{1}{2}\theta r^2$
 $= \frac{1}{2} \times 3 \times 2^2$
 $= 6 \text{ cm}^2$



b $\text{area} = \frac{1}{2}\theta r^2$
 $= \frac{1}{2} \times 1.5 \times 8^2$
 $= 48 \text{ cm}^2$



c $\text{area} = \frac{1}{2}\theta r^2$
 $= \frac{1}{2} \times \frac{\pi}{6} \times 5.6^2$
 $\approx 8.21 \text{ cm}^2$

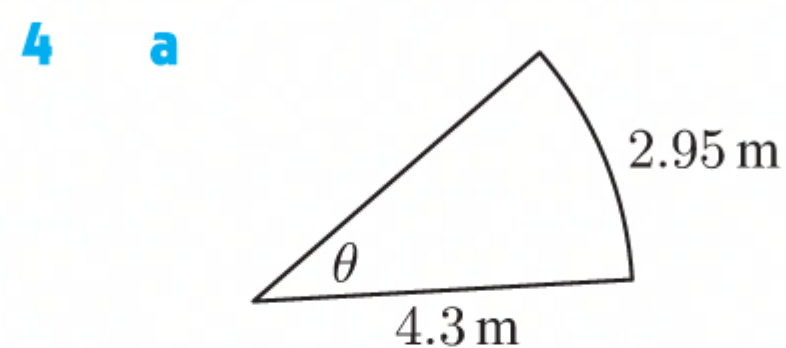


3 a $\text{arc length} = \theta r$
 $= \frac{7\pi}{4} \times 9$
 $\approx 49.5 \text{ cm}$

$\text{area} = \frac{1}{2}\theta r^2$
 $= \frac{1}{2} \times \frac{7\pi}{4} \times 9^2$
 $\approx 223 \text{ cm}^2$

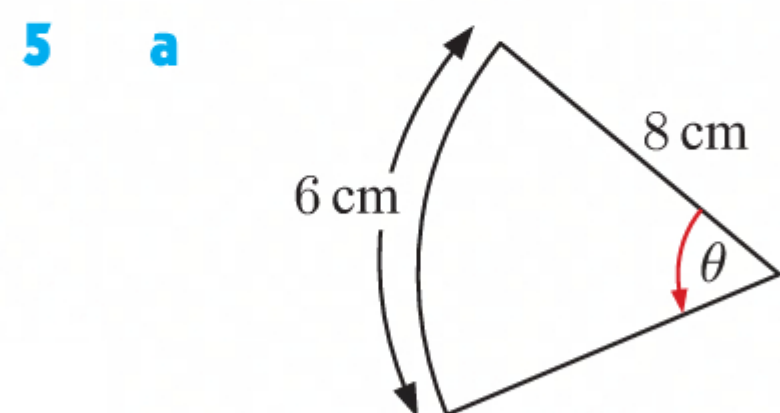
b $\text{arc length} = \theta r$
 $= 4.67 \times 4.93$
 $\approx 23.0 \text{ cm}$

$\text{area} = \frac{1}{2}\theta r^2$
 $= \frac{1}{2} \times 4.67 \times 4.93^2$
 $\approx 56.8 \text{ cm}^2$



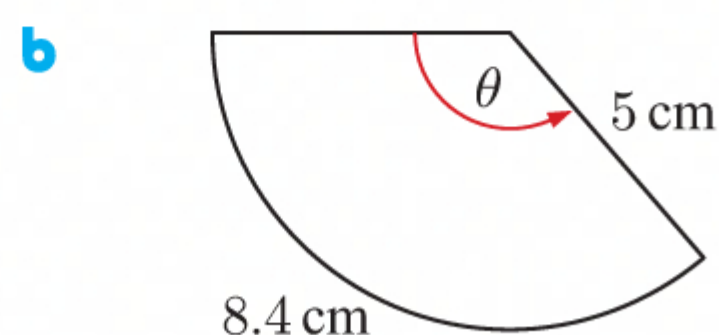
$l = \theta r$
 $\therefore 2.95 = \theta \times 4.3$
 $\therefore \theta = \frac{2.95}{4.3}$
 $\therefore \theta \approx 0.686^\circ$

b $\text{area} = \frac{1}{2}\theta r^2$
 $\therefore 30 = \frac{1}{2} \times \theta \times 10^2$
 $\therefore \theta = \frac{30 \times 2}{100}$
 $\therefore \theta = 0.6^\circ$



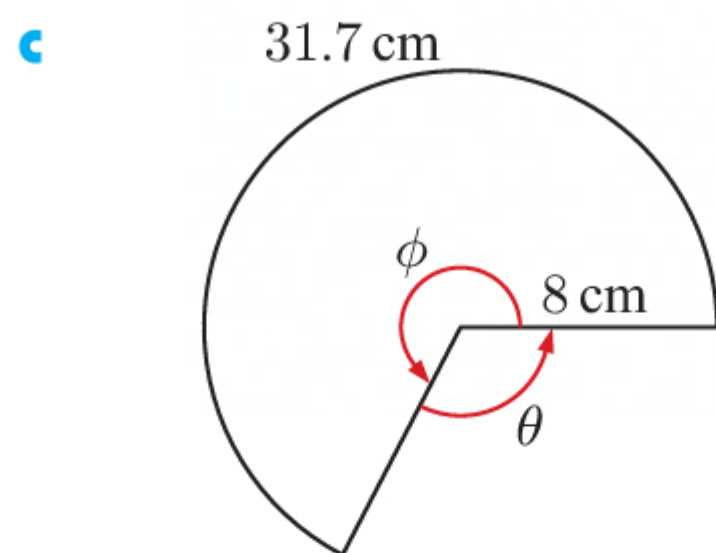
$l = \theta r$
 $\therefore 6 = \theta \times 8$
 $\therefore \theta = \frac{6}{8}$
 $\therefore \theta = 0.75^\circ$

$\text{area} = \frac{1}{2}\theta r^2$
 $= \frac{1}{2} \times 0.75 \times 8^2$
 $= 24 \text{ cm}^2$



$l = \theta r$
 $\therefore 8.4 = \theta \times 5$
 $\therefore \theta = \frac{8.4}{5}$
 $\therefore \theta = 1.68^\circ$

$\text{area} = \frac{1}{2}\theta r^2$
 $= \frac{1}{2} \times 1.68 \times 5^2$
 $= 21 \text{ cm}^2$



$$l = \phi r$$

$$\therefore 31.7 = \phi \times 8$$

$$\therefore \phi = \frac{31.7}{8}$$

$$\therefore \phi \approx 3.96^c$$

But $\theta = 2\pi - \phi$

$$\therefore \theta \approx 2.32^c$$

$$\text{area} = \frac{1}{2}\phi r^2$$

$$= \frac{1}{2} \times \frac{31.7}{8} \times 8^2$$

$$= 126.8 \text{ cm}^2$$

6 a $l = \theta r$

$$\therefore r = \frac{l}{\theta}$$

$$\therefore r = \frac{5.92}{1.88}$$

$$\therefore r \approx 3.15$$

The radius is about 3.15 m.

b $\text{area} = \frac{1}{2}\theta r^2$

$$= \frac{1}{2} \times 1.88 \times \left(\frac{5.92}{1.88}\right)^2$$

$$\approx 9.32 \text{ m}^2$$

7 a $\text{area} = \frac{1}{2}\theta r^2$

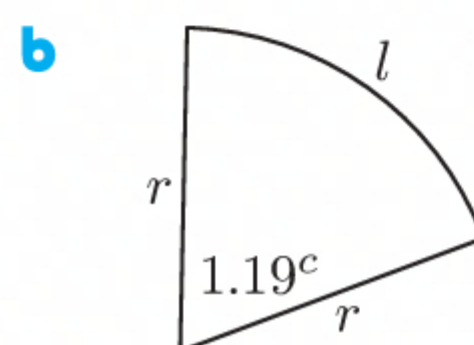
$$\therefore 20.8 = \frac{1}{2} \times 1.19 \times r^2$$

$$\therefore \frac{20.8 \times 2}{1.19} = r^2$$

$$\therefore r = \sqrt{\frac{20.8 \times 2}{1.19}} \quad \{r > 0\}$$

$$\therefore r \approx 5.91$$

The radius is about 5.91 cm.



perimeter

$$= l + 2r$$

$$\approx 1.19 \times 5.912 + 2 \times 5.912$$

$$\approx 18.9 \text{ cm}$$

8 a $\tan \alpha = \frac{5}{15}$

$$\therefore \alpha = \tan^{-1}\left(\frac{1}{3}\right)$$

$$\therefore \alpha \approx 0.3218^c$$

b $\theta + 2\alpha = \pi$ {angles on a line}

$$\therefore \theta \approx \pi - 2 \times 0.3218$$

$$\therefore \theta \approx 2.498^c$$

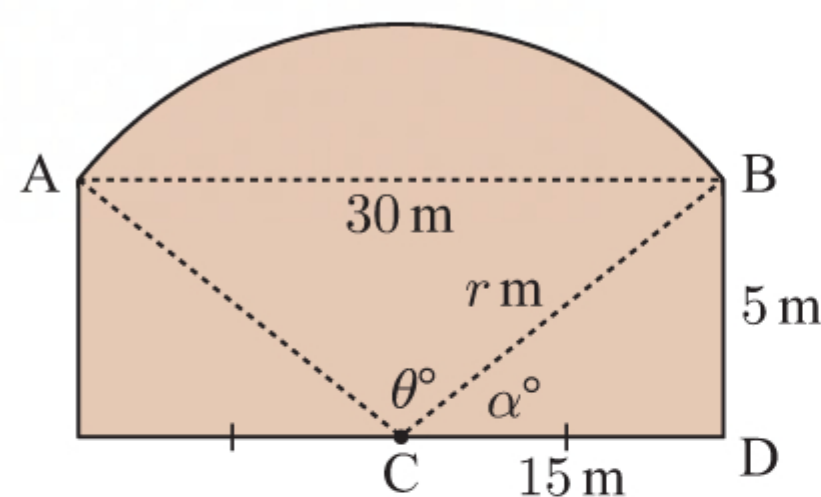
c $\text{area} = 2 \times \text{area of } \triangle CDB + \text{area of sector}$

$$= 2 \times \frac{1}{2} \times CD \times BD + \left(\frac{\theta}{2\pi}\right) \times \pi \times r^2$$

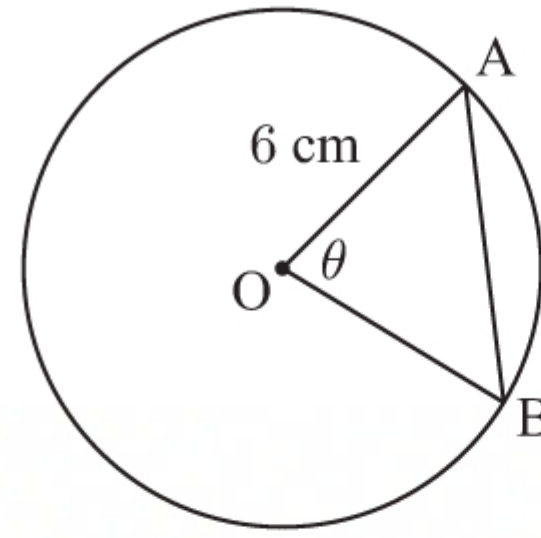
Now $r^2 = 5^2 + 15^2 = 250$

$$\therefore \text{area} \approx 2 \times \frac{1}{2} \times 15 \times 5 + \left(\frac{2.498}{2\pi}\right) \times \pi \times 250$$

$$\approx 387 \text{ m}^2$$

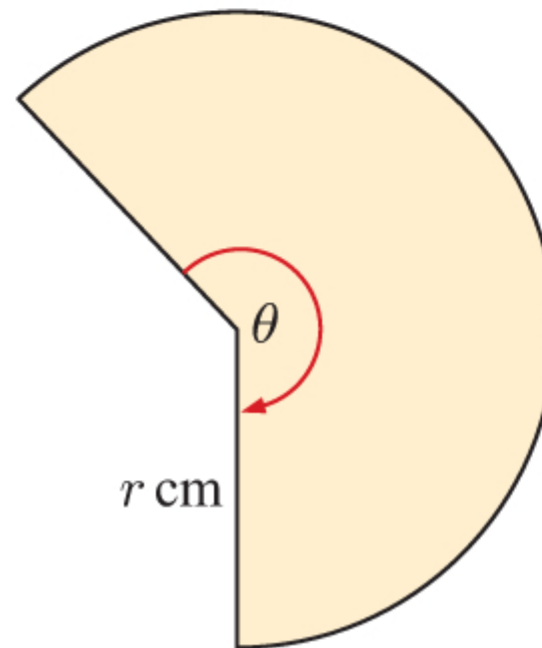
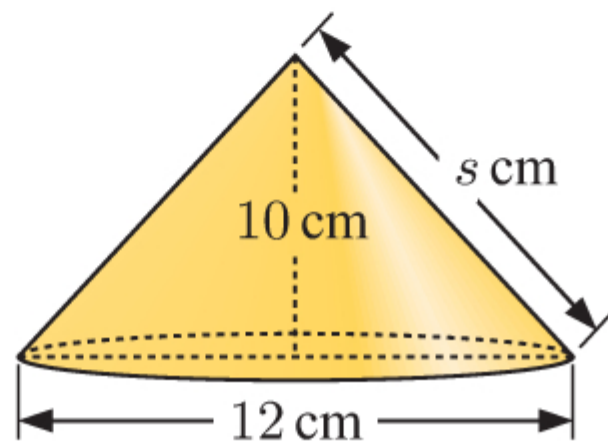


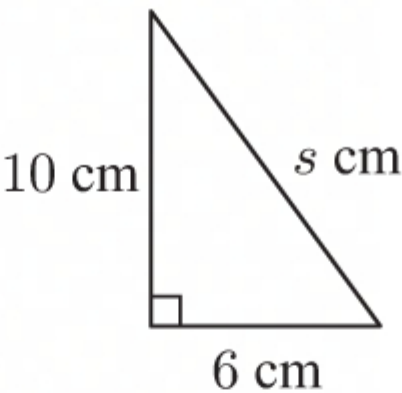
- 9 a Perimeter of sector OAB = $r + r + \theta r$
 $\therefore 12 + 2\pi = 6 + 6 + (\theta \times 6)$
 $\therefore 12 + 2\pi = 12 + 6\theta$
 $\therefore 2\pi = 6\theta$
 $\therefore \theta = \frac{\pi}{3}$



- b OA = OB {equal radii of circle}
 $\therefore \triangle OAB$ is isosceles with $\widehat{OAB} = \widehat{OBA}$.
 Since $\theta = \frac{\pi}{3} = 60^\circ$,
 then $\widehat{OAB} + \widehat{OBA} = 180^\circ - 60^\circ = 120^\circ$ {angles in a triangle}
 $\therefore \widehat{OAB} = \widehat{OBA} = 60^\circ$
 $\therefore \triangle OAB$ is equilateral.
 \therefore the length of the chord [AB] is 6 cm.

10



- a  $s^2 = 6^2 + 10^2$ {Pythagoras}
 $\therefore s = \sqrt{6^2 + 10^2}$
 $\therefore s = \sqrt{136}$
 $\therefore s \approx 11.7$
 \therefore slant length is about 11.7 cm.

- b $r = s \approx 11.7$

- c arc length = circumference of cone base
 $= 2\pi \times 6$
 $= 12\pi$
 ≈ 37.7 cm

- d arc length = θr
 $\therefore 12\pi = \theta \times \sqrt{136}$
 $\therefore \theta = \frac{12\pi}{\sqrt{136}}$
 $\therefore \theta \approx 3.23^c$

- 11 Since [AT] is a tangent, \widehat{OTA} is a right angle.

We let $\widehat{TOA} = \theta$.

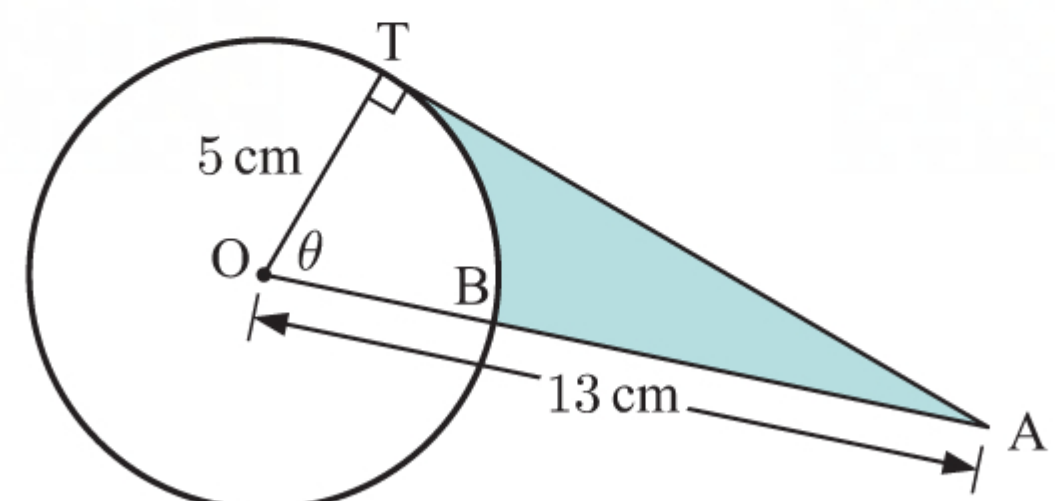
$$\therefore \cos \theta = \frac{5}{13}$$

$$\therefore \theta \approx 1.176^c$$

$$\text{arc length BT} = \theta r$$

$$\approx 1.176 \times 5$$

$$\approx 5.88 \text{ cm}$$

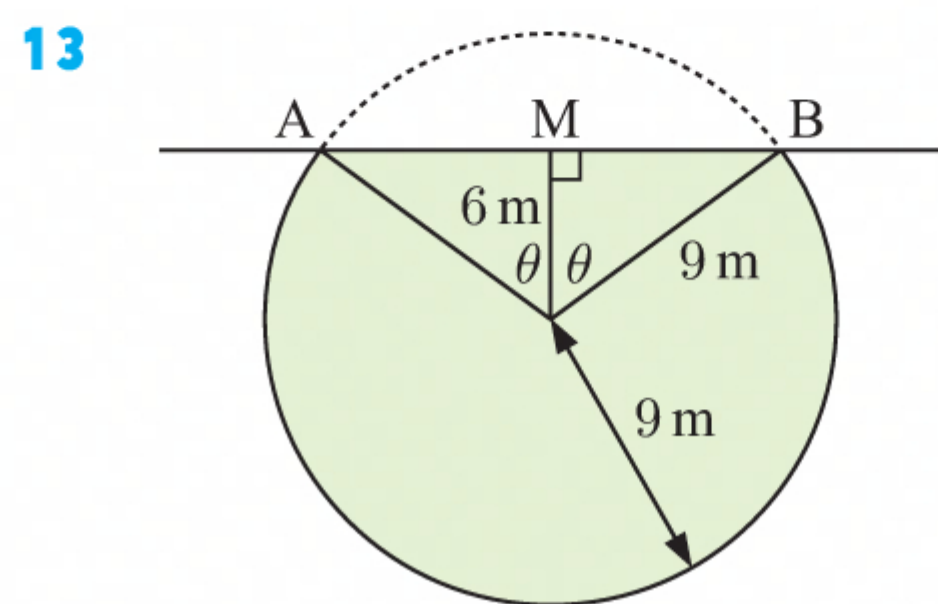


$$\begin{aligned} AT^2 + OT^2 &= OA^2 \quad \{\text{Pythagoras}\} \\ \therefore AT^2 &= 13^2 - 5^2 \\ \therefore AT &= 12 \text{ cm} \end{aligned}$$

$$\begin{aligned} \text{perimeter} &= AT + \text{arc length } BT + AB \\ &\approx 12 + 5.88 + (13 - 5) \\ &\approx 25.9 \text{ cm} \end{aligned}$$

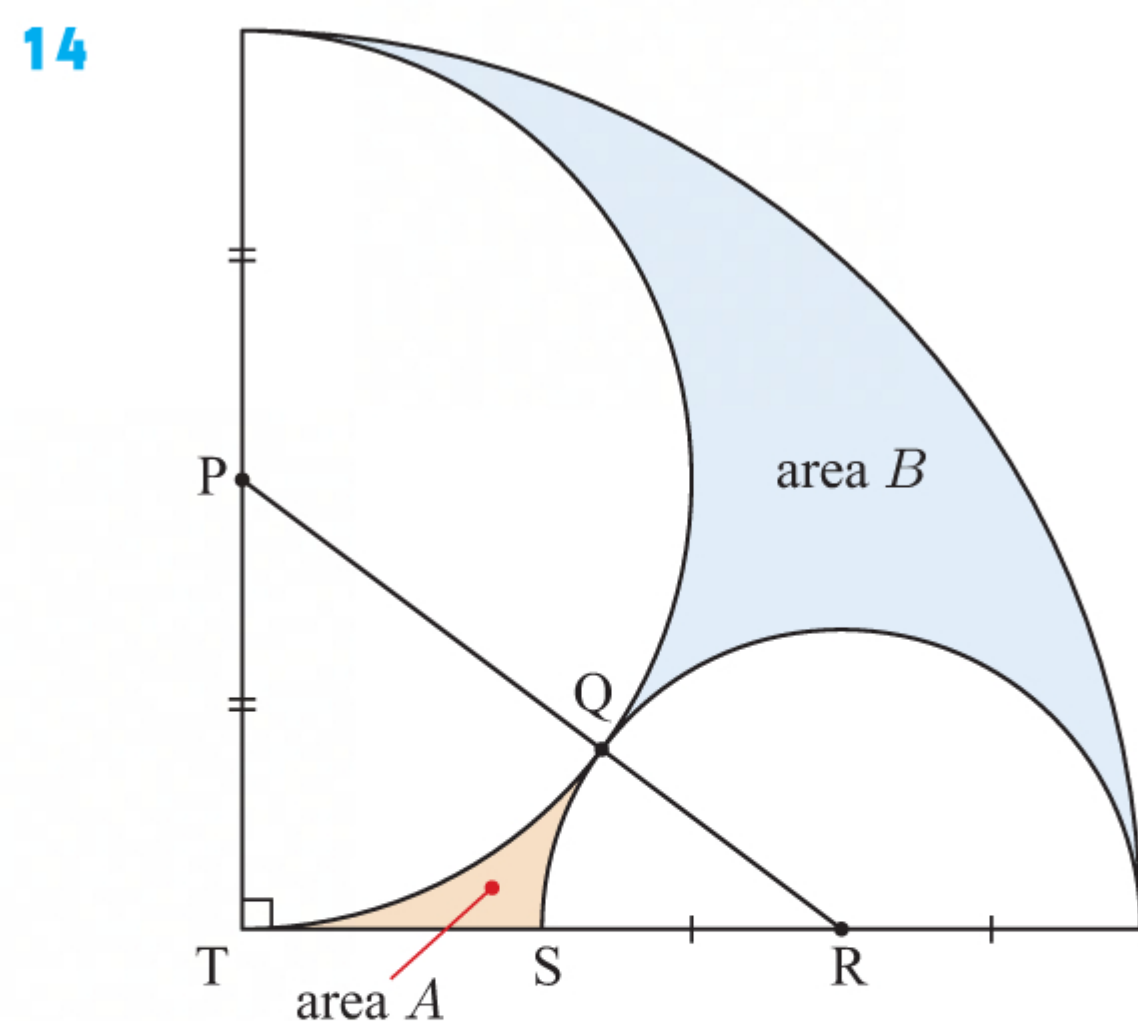
$$\begin{aligned} \mathbf{12} \quad \mathbf{a} \quad l &= \left(\frac{\theta}{360}\right) \times 2\pi r \\ &= \frac{1}{60} \times 2 \times \pi \times 6370 \text{ km} \\ &\approx 1.853 \text{ km} \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad \text{time} &= \frac{\text{distance}}{\text{speed}} \\ &= \frac{2130 \text{ km}}{480 \text{ n miles h}^{-1}} \\ &\approx \frac{2130 \text{ km}}{480 \times 1.853 \text{ km h}^{-1}} \\ &\approx 2.395 \text{ hours} \\ &\approx 2 \text{ hours } 24 \text{ min} \end{aligned}$$

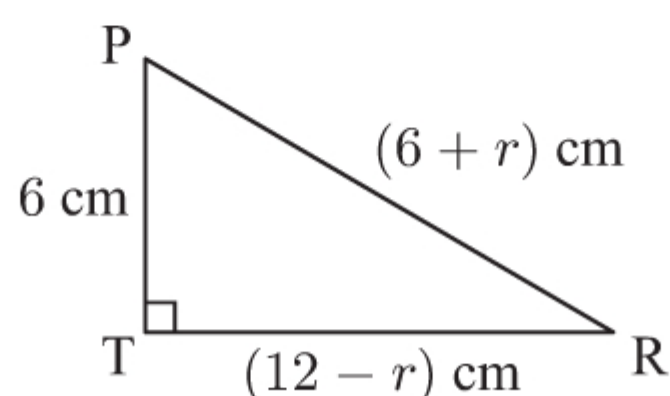


$$\begin{aligned} \cos \theta &= \frac{6}{9} = \frac{2}{3} \\ \therefore \theta &= \cos^{-1}\left(\frac{2}{3}\right) \\ \therefore \theta &\approx 0.841^c \\ \text{So, } 2\pi - 2\theta &\approx 4.601^c \\ \text{Now } MB &= \sqrt{9^2 - 6^2} \quad \{\text{Pythagoras}\} \\ &= \sqrt{45} \end{aligned}$$

$$\begin{aligned} \therefore \text{available feeding area} &= \text{area of } \triangle + \text{area of sector} \\ &\approx \frac{1}{2} \times 2 \times \sqrt{45} \times 6 + \frac{1}{2} \times 4.601 \times 9^2 \\ &\approx 227 \text{ m}^2 \end{aligned}$$



- a** Let the smaller semi-circle have radius r cm.
In $\triangle PTR$, we have:



$$\begin{aligned} \text{Thus } (6 + r)^2 &= 6^2 + (12 - r)^2 \\ \therefore 36 + 12r + r^2 &= 36 + 144 - 24r + r^2 \\ \therefore 36r &= 144 \\ \therefore r &= 4 \end{aligned}$$

$$\begin{aligned} \text{b } \cos(\widehat{\text{TPR}}) &= \frac{6}{10} = 0.6 & \widehat{\text{PRT}} &= \frac{\pi}{2} - 0.927^c \\ \therefore \widehat{\text{TPR}} &= \cos^{-1}(0.6) \approx 0.927^c & & \approx 0.644^c \end{aligned}$$

$$\begin{aligned} \text{i area } A &= \text{area } \triangle \text{PTR} - (\text{area sector PQT} + \text{area sector RQS}) \\ &\approx \frac{1}{2}(8 \times 6) - \left(\frac{1}{2}(0.927)(6^2) + \frac{1}{2}(0.644)(4^2) \right) \\ &\approx 24 - 16.69 - 5.15 \\ &\approx 2.16 \text{ cm}^2 \end{aligned}$$

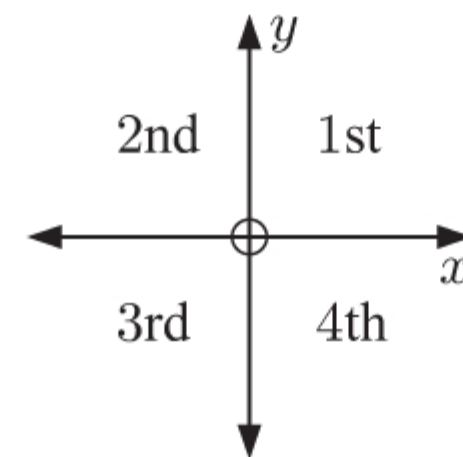
$$\begin{aligned} \text{ii area } B &= \text{area of quarter circle} - \text{area of semi-circles} - \text{area } A \\ &\approx \frac{1}{4}\pi(12^2) - \frac{1}{2}\pi(6^2) - \frac{1}{2}\pi(4^2) - 2.16 \\ &\approx 36\pi - 18\pi - 8\pi - 2.16 \\ &\approx 10\pi - 2.16 \\ &\approx 29.3 \text{ cm}^2 \end{aligned}$$

INVESTIGATION

THE TRIGONOMETRIC RATIOS

1

Quadrant	$\cos \theta$	$\sin \theta$	$\tan \theta$
1	positive	positive	positive
2	negative	positive	negative
3	negative	negative	positive
4	positive	negative	negative

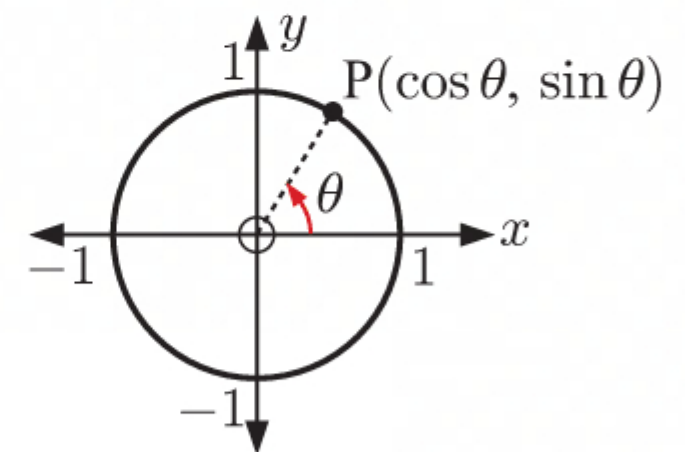


- 2**
- $\sin \theta$, $\cos \theta$, and $\tan \theta$ are all positive in quadrant 1
 - only $\sin \theta$ is positive in quadrant 2
 - only $\tan \theta$ is positive in quadrant 3
 - only $\cos \theta$ is positive in quadrant 4.

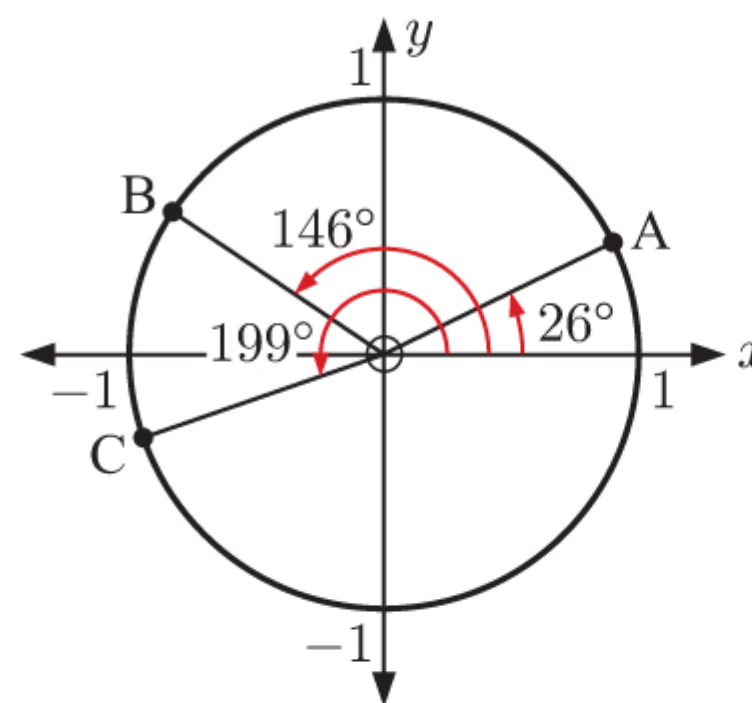
EXERCISE 7C

- 1** Since any point P on the unit circle has coordinates $(\cos \theta, \sin \theta)$, then:

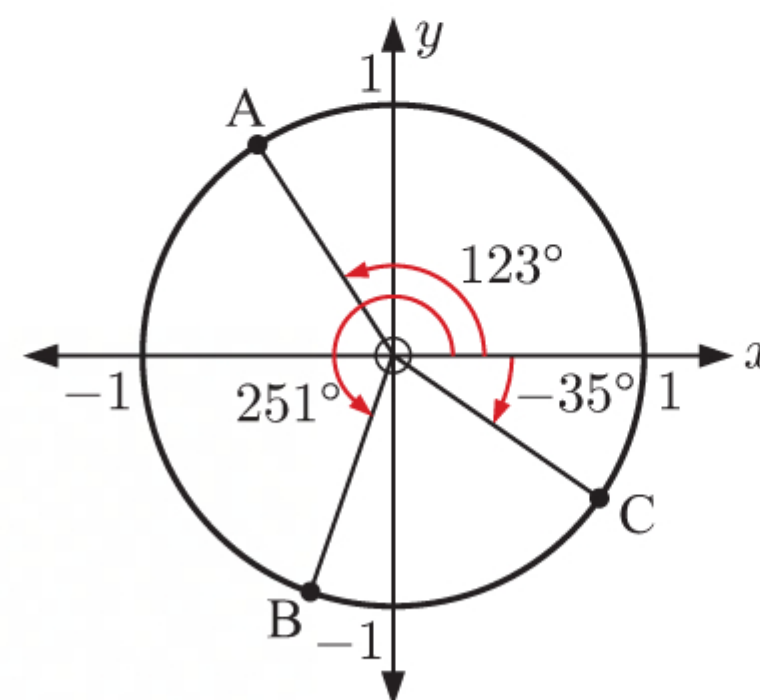
θ (degrees)	0°	90°	180°	270°	360°	450°
θ (radians)	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π	$\frac{5\pi}{2}$
sine	0	1	0	-1	0	1
cosine	1	0	-1	0	1	0
tangent	0	undefined	0	undefined	0	undefined



- 2 a i** $A(\cos 26^\circ, \sin 26^\circ)$, $B(\cos 146^\circ, \sin 146^\circ)$,
 $C(\cos 199^\circ, \sin 199^\circ)$
- ii** $A(0.899, 0.438)$, $B(-0.829, 0.559)$,
 $C(-0.946, -0.326)$



- b i** $A(\cos 123^\circ, \sin 123^\circ)$, $B(\cos 251^\circ, \sin 251^\circ)$,
 $C(\cos(-35^\circ), \sin(-35^\circ))$
- ii** $A(-0.545, 0.839)$, $B(-0.326, -0.946)$,
 $C(0.819, -0.574)$



- 3 a i** $\frac{1}{\sqrt{2}} \approx 0.707$ **ii** $\frac{\sqrt{3}}{2} \approx 0.866$

b

θ (degrees)	30°	45°	60°	135°	150°	240°	315°
θ (radians)	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	$\frac{4\pi}{3}$	$\frac{7\pi}{4}$
sine	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{\sqrt{2}}$
cosine	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	$-\frac{1}{\sqrt{2}}$	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$\frac{1}{\sqrt{2}}$
tangent	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	-1	$-\frac{1}{\sqrt{3}}$	$\sqrt{3}$	-1

4 a

Quadrant	Degree measure	Radian measure	$\cos \theta$	$\sin \theta$	$\tan \theta$
1	$0^\circ < \theta < 90^\circ$	$0 < \theta < \frac{\pi}{2}$	positive	positive	positive
2	$90^\circ < \theta < 180^\circ$	$\frac{\pi}{2} < \theta < \pi$	negative	positive	negative
3	$180^\circ < \theta < 270^\circ$	$\pi < \theta < \frac{3\pi}{2}$	negative	negative	positive
4	$270^\circ < \theta < 360^\circ$	$\frac{3\pi}{2} < \theta < 2\pi$	positive	negative	negative

- b i** $\cos \theta$ is positive in quadrants 1 and 4.
ii $\cos \theta$ is negative in quadrants 2 and 3.
iii $\cos \theta$ and $\sin \theta$ are both negative in quadrant 3.
iv $\cos \theta$ is negative and $\sin \theta$ is positive in quadrant 2.

$$\begin{aligned} \mathbf{5} \quad \mathbf{a} \quad & \cos 400^\circ \\ &= \cos(360^\circ + 40^\circ) \\ &= \cos 40^\circ \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad & \sin \frac{5\pi}{7} \\ &= \sin\left(\frac{5\pi}{7} + 2\pi\right) \\ &= \sin \frac{19\pi}{7} \end{aligned}$$

$$\begin{aligned} \mathbf{c} \quad & \tan \frac{13\pi}{8} \\ &= \tan\left(\frac{13\pi}{8} - 3\pi\right) \\ &= \tan\left(-\frac{11\pi}{8}\right) \end{aligned}$$

$$\begin{aligned} \mathbf{6} \quad \tan 230^\circ &= \tan(180^\circ + 50^\circ) \\ &= \tan 50^\circ \end{aligned}$$

\therefore **B** and **D** have the same value.

$$\begin{aligned} \mathbf{7} \quad \sin 40^\circ &= \sin \frac{40 \times \pi}{180} \\ &= \sin \frac{2\pi}{9} \end{aligned}$$

\therefore **B** and **E** have the same value.

$$\mathbf{8} \quad \mathbf{a} \quad \mathbf{i} \quad \sin 100^\circ \approx 0.985$$

$$\mathbf{ii} \quad \sin 80^\circ \approx 0.985$$

$$\mathbf{iii} \quad \sin 120^\circ \approx 0.866$$

$$\mathbf{iv} \quad \sin 60^\circ \approx 0.866$$

$$\mathbf{v} \quad \sin 150^\circ = 0.5$$

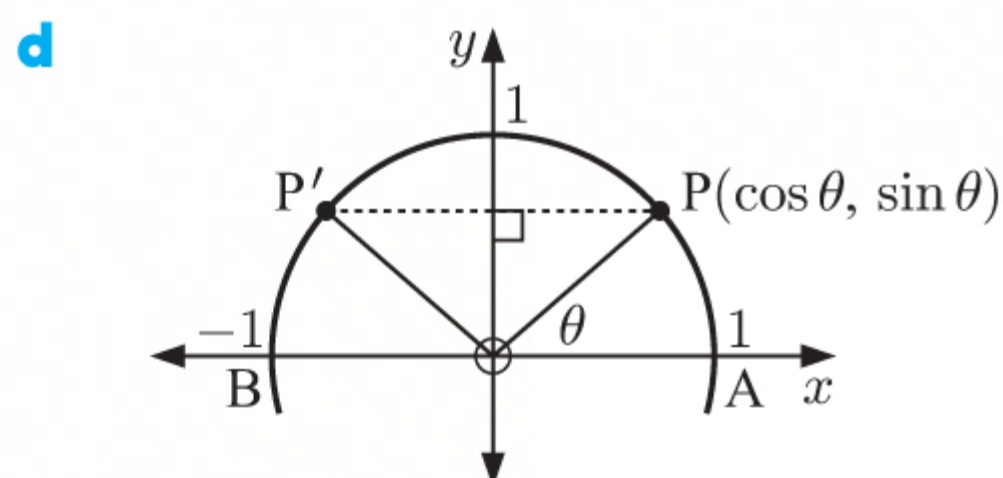
$$\mathbf{vi} \quad \sin 30^\circ = 0.5$$

$$\mathbf{vii} \quad \sin 45^\circ \approx 0.707$$

$$\mathbf{viii} \quad \sin 135^\circ \approx 0.707$$

$$\mathbf{b} \quad \sin(180^\circ - \theta) = \sin \theta \quad \text{as the points have the same } y\text{-coordinate.}$$

$$\mathbf{c} \quad \sin(\pi - \theta) = \sin \theta$$



The diagram shows P reflected in the y -axis to P' , so $\widehat{P'OB} = \widehat{POA} = \theta$, and P' has coordinates $(-\cos \theta, \sin \theta)$.

But $\widehat{AOP'} = 180^\circ - \theta$
 $\{\widehat{AOP'} + \widehat{P'OB} = 180^\circ\}$, so P' has coordinates $(\cos(180^\circ - \theta), \sin(180^\circ - \theta))$.

$$\begin{aligned} \therefore \sin(180^\circ - \theta) &= \sin \theta \\ &\quad \{\text{equating } y\text{-coordinates of } P'\} \end{aligned}$$

$$\mathbf{e} \quad \mathbf{i} \quad 180^\circ - 45^\circ = 135^\circ$$

$$\mathbf{ii} \quad 180^\circ - 51^\circ = 129^\circ$$

$$\mathbf{iii} \quad \pi - \frac{\pi}{3} = \frac{2\pi}{3}$$

$$\mathbf{iv} \quad \pi - \frac{\pi}{6} = \frac{5\pi}{6} \quad \{\text{using } \sin(\pi - \theta) = \sin \theta\}$$

$$\mathbf{9} \quad \mathbf{a} \quad \mathbf{i} \quad \cos 70^\circ \approx 0.342$$

$$\mathbf{ii} \quad \cos 110^\circ \approx -0.342$$

$$\mathbf{iii} \quad \cos 60^\circ = 0.5$$

$$\mathbf{iv} \quad \cos 120^\circ = -0.5$$

$$\mathbf{v} \quad \cos 25^\circ \approx 0.906$$

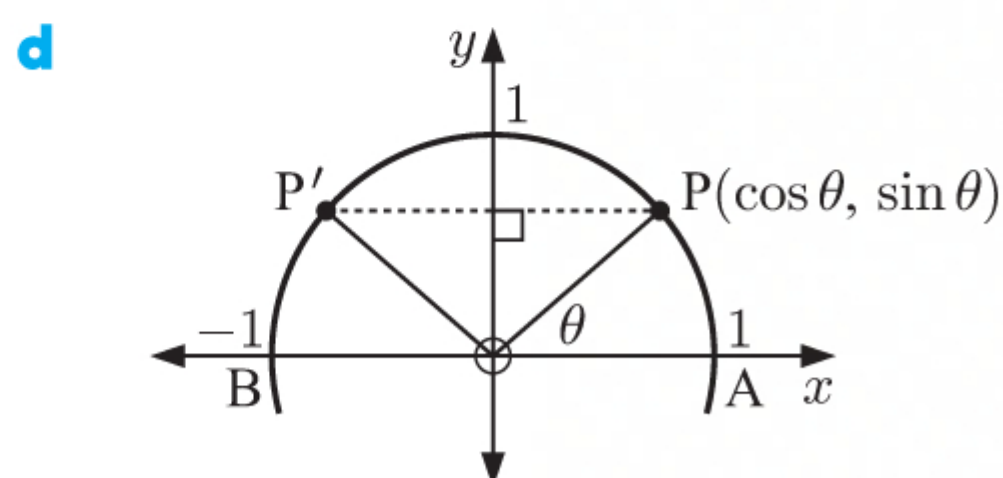
$$\mathbf{vi} \quad \cos 155^\circ \approx -0.906$$

$$\mathbf{vii} \quad \cos 80^\circ \approx 0.174$$

$$\mathbf{viii} \quad \cos 100^\circ \approx -0.174$$

$$\mathbf{b} \quad \cos(180^\circ - \theta) = -\cos \theta$$

$$\mathbf{c} \quad \cos(\pi - \theta) = -\cos \theta$$



The diagram shows P reflected in the y -axis to P' , so $\widehat{P'OB} = \widehat{POA} = \theta$, and P' has coordinates $(-\cos \theta, \sin \theta)$.

But $\widehat{AOP'} = 180^\circ - \theta$, so P' has coordinates $(\cos(180^\circ - \theta), \sin(180^\circ - \theta))$.

$$\begin{aligned} \therefore \cos(180^\circ - \theta) &= -\cos \theta \\ &\quad \{\text{equating } x\text{-coordinates of } P'\} \end{aligned}$$

e i $180^\circ - 40^\circ = 140^\circ$

ii $180^\circ - 19^\circ = 161^\circ$

iii $\pi - \frac{\pi}{5} = \frac{4\pi}{5}$

iv $\pi - \frac{2\pi}{5} = \frac{3\pi}{5}$ {using $\cos(\pi - \theta) = -\cos \theta$ }

10 $\tan \theta = \frac{\sin \theta}{\cos \theta}$

$$\begin{aligned} \therefore \tan(\pi - \theta) &= \frac{\sin(\pi - \theta)}{\cos(\pi - \theta)} \\ &= \frac{\sin \theta}{-\cos \theta} \\ &= -\frac{\sin \theta}{\cos \theta} \\ &= -\tan \theta \end{aligned}$$

11 a $\sin 137^\circ$
 $= \sin(180 - 137)^\circ$
 $= \sin 43^\circ$
 ≈ 0.6820

b $\sin 59^\circ$
 $= \sin(180 - 59)^\circ$
 $= \sin 121^\circ$
 ≈ 0.8572

c $\cos 143^\circ$
 $= -\cos(180 - 143)^\circ$
 $= -\cos 37^\circ$
 ≈ -0.7986

d $\cos 24^\circ$
 $= -\cos(180 - 24)^\circ$
 $= -\cos 156^\circ$
 ≈ 0.9135

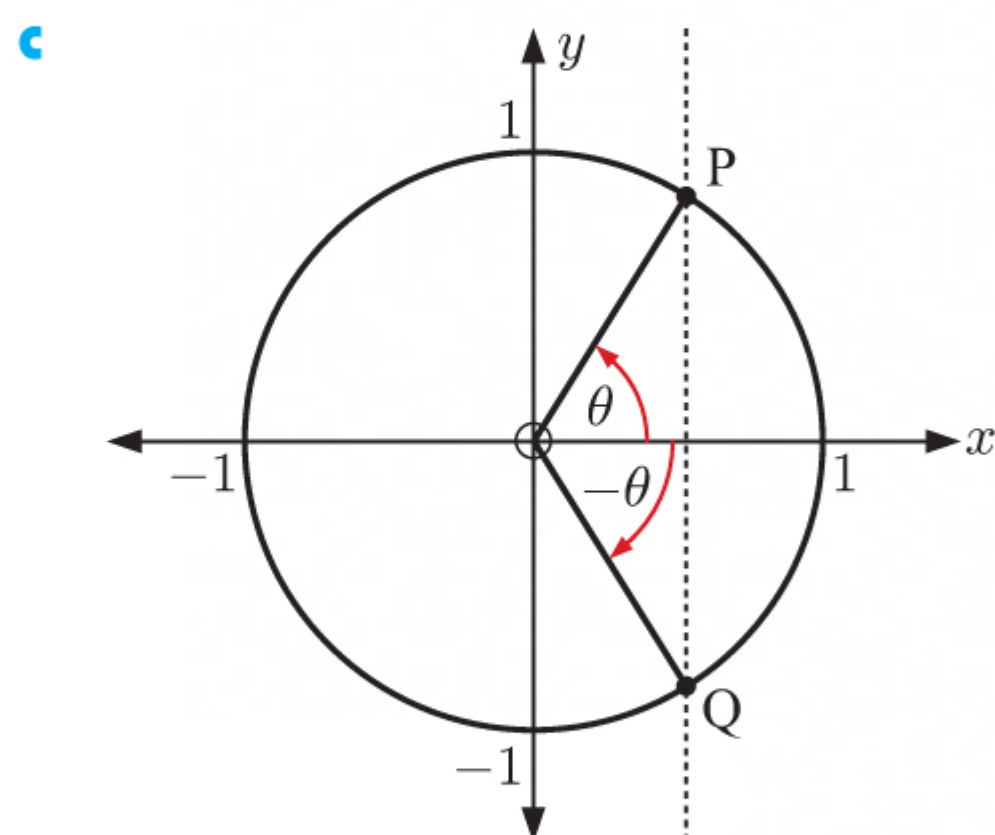
e $\sin 115^\circ$
 $= \sin(180 - 115)^\circ$
 $= \sin 65^\circ$
 ≈ 0.9063

f $\cos 132^\circ$
 $= -\cos(180 - 132)^\circ$
 $= -\cos 48^\circ$
 ≈ -0.6691

12 a

θ (radians)	$\sin \theta$	$\sin(-\theta)$	$\cos \theta$	$\cos(-\theta)$
0.75	≈ 0.682	≈ -0.682	≈ 0.732	≈ 0.732
1.772	≈ 0.980	≈ -0.980	≈ -0.200	≈ -0.200
3.414	≈ -0.269	≈ 0.269	≈ -0.963	≈ -0.963
6.25	≈ -0.0332	≈ 0.0332	≈ 0.999	≈ 0.999
-1.17	≈ -0.921	≈ 0.921	≈ 0.390	≈ 0.390

b We deduce that $\sin(-\theta) = -\sin \theta$ and $\cos(-\theta) = \cos \theta$.



P is reflected in the x -axis to Q, so Q has coordinates $(\cos \theta, -\sin \theta)$.

But Q has coordinates $(\cos(-\theta), \sin(-\theta))$.

$$\therefore Q(\cos(-\theta), \sin(-\theta)) = Q(\cos \theta, -\sin \theta).$$

$$\therefore \cos(-\theta) = \cos \theta \quad \text{and} \quad \sin(-\theta) = -\sin \theta$$

So the deduction is correct.

d There are 2π radians in a circle.

$$\therefore \text{we could write Q with coordinates } (\cos(2\pi - \theta), \sin(2\pi - \theta)).$$

From **c** we know that Q has coordinates $(\cos \theta, -\sin \theta)$

$$\therefore \cos(2\pi - \theta) = \cos \theta \quad \text{and} \quad \sin(2\pi - \theta) = -\sin \theta$$

$$\begin{aligned}
 \text{e } \tan(2\pi - \theta) &= \frac{\sin(2\pi - \theta)}{\cos(2\pi - \theta)} \\
 &= \frac{-\sin \theta}{\cos \theta} \\
 &= -\frac{\sin \theta}{\cos \theta} \\
 &= -\tan \theta
 \end{aligned}$$

- 13 a** The angle between [OP] and the positive x -axis is $(\frac{\pi}{2} - \theta)$.
 \therefore P is $(\cos(\frac{\pi}{2} - \theta), \sin(\frac{\pi}{2} - \theta))$

b i In $\triangle OXP$, $\sin \theta = \frac{XP}{OP} = \frac{XP}{1}$
 $\therefore XP = \sin \theta$

ii In $\triangle OXP$, $\cos \theta = \frac{OX}{OP} = \frac{OX}{1}$
 $\therefore OX = \cos \theta$

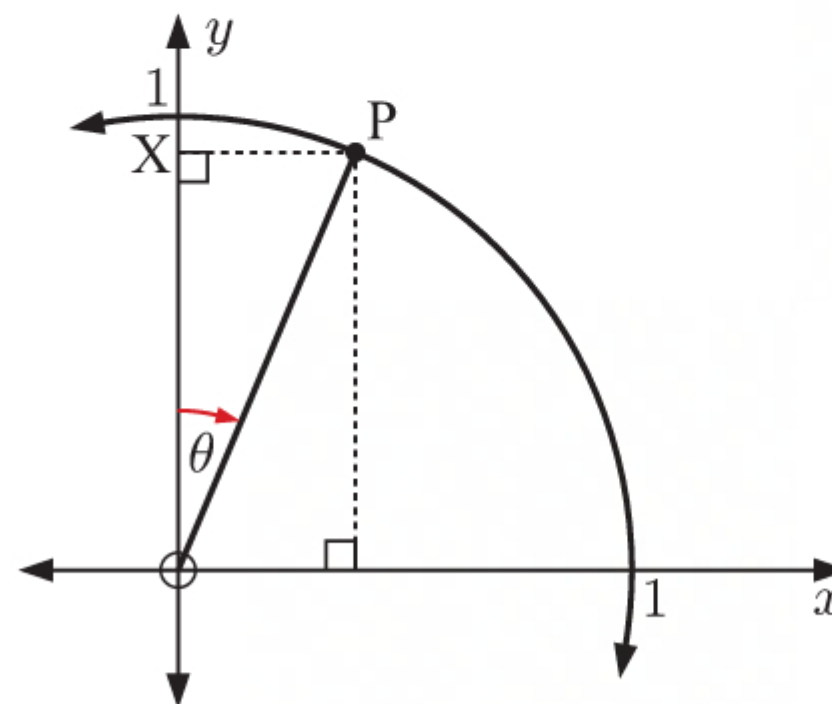
c i $\cos(\frac{\pi}{2} - \theta) = XP = \sin \theta$

ii $\sin(\frac{\pi}{2} - \theta) = OX = \cos \theta$

d i $\cos \frac{\pi}{5} = \sin(\frac{\pi}{2} - \frac{\pi}{5}) = \sin \frac{3\pi}{10} \approx 0.809$

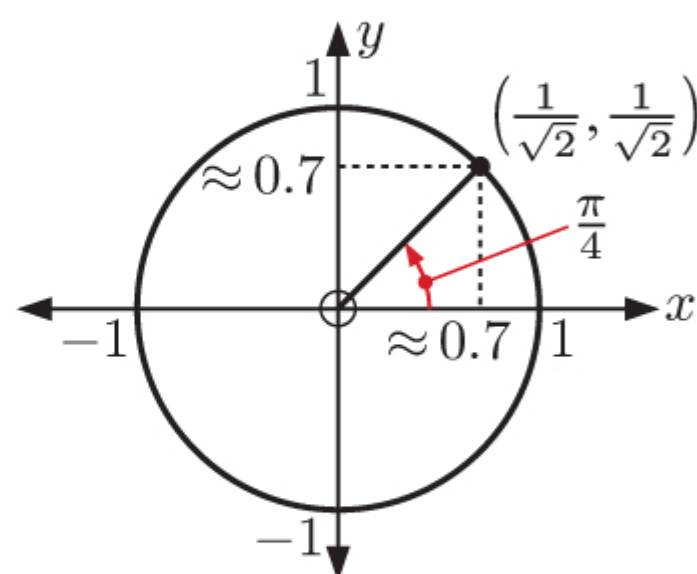
ii $\sin \frac{\pi}{8} = \cos(\frac{\pi}{2} - \frac{\pi}{8}) = \cos \frac{3\pi}{8} \approx 0.383$

$$\begin{aligned}
 \text{e } \tan(\frac{\pi}{2} - \theta) &= \frac{\sin(\frac{\pi}{2} - \theta)}{\cos(\frac{\pi}{2} - \theta)} \\
 &= \frac{\cos \theta}{\sin \theta} \\
 &= \frac{1}{\tan \theta}
 \end{aligned}$$



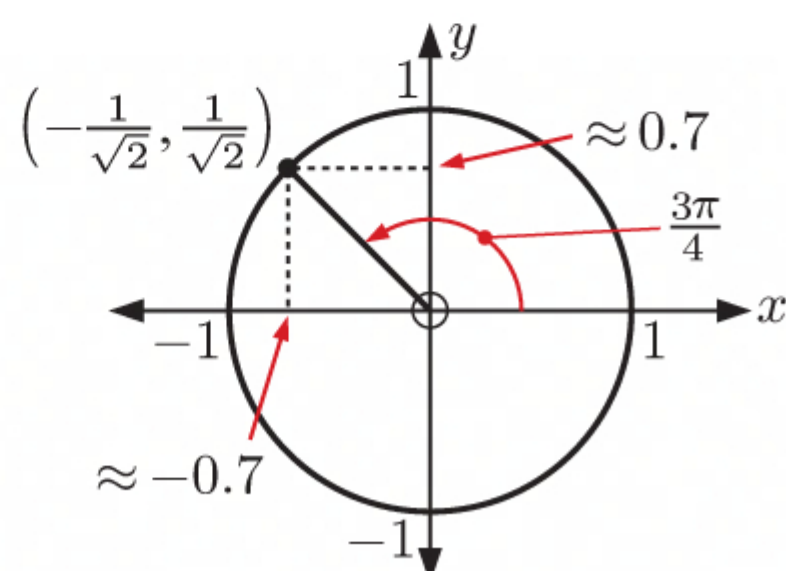
EXERCISE 7D

1 a



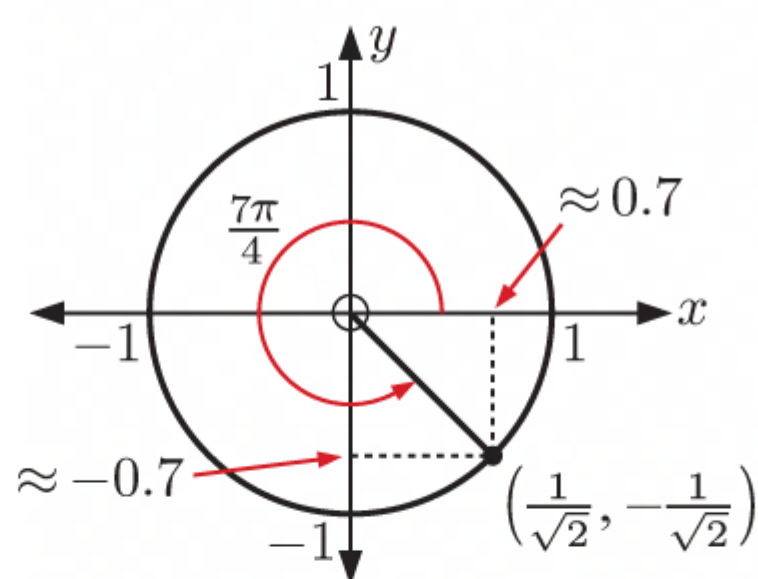
$$\begin{aligned}
 \sin \frac{\pi}{4} &= \frac{1}{\sqrt{2}} \\
 \cos \frac{\pi}{4} &= \frac{1}{\sqrt{2}} \\
 \tan \frac{\pi}{4} &= \frac{\frac{1}{\sqrt{2}}}{\frac{1}{\sqrt{2}}} = 1
 \end{aligned}$$

b



$$\begin{aligned}
 \sin \frac{3\pi}{4} &= \frac{1}{\sqrt{2}} \\
 \cos \frac{3\pi}{4} &= -\frac{1}{\sqrt{2}} \\
 \tan \frac{3\pi}{4} &= \frac{\frac{1}{\sqrt{2}}}{-\frac{1}{\sqrt{2}}} = -1
 \end{aligned}$$

c

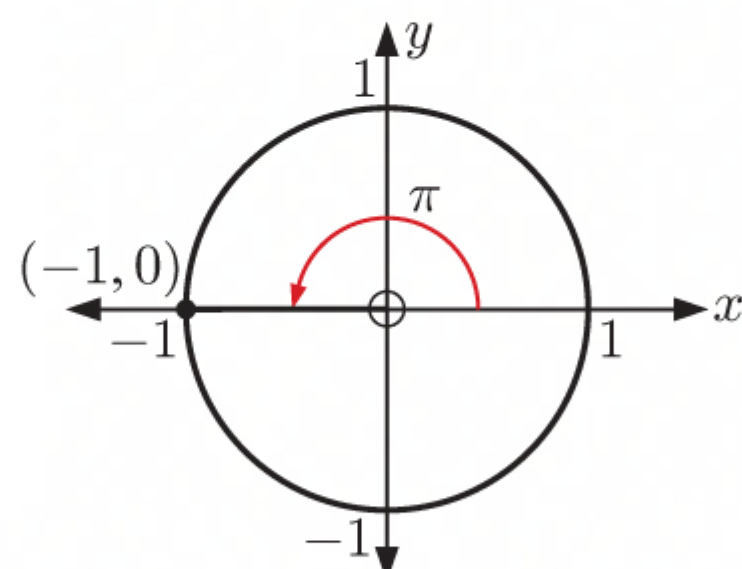


$$\sin \frac{7\pi}{4} = -\frac{1}{\sqrt{2}}$$

$$\cos \frac{7\pi}{4} = \frac{1}{\sqrt{2}}$$

$$\tan \frac{7\pi}{4} = \frac{-\frac{1}{\sqrt{2}}}{\frac{1}{\sqrt{2}}} = -1$$

d

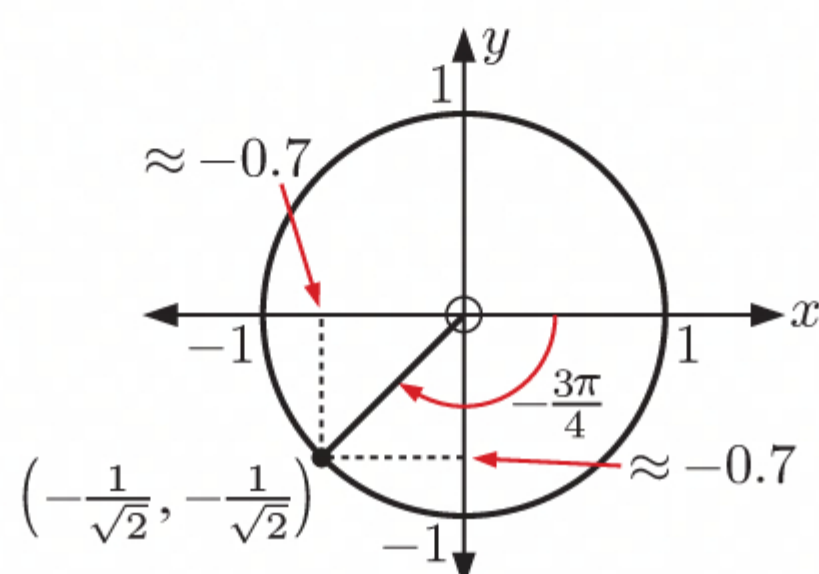


$$\sin \pi = 0$$

$$\cos \pi = -1$$

$$\tan \pi = \frac{0}{-1} = 0$$

e

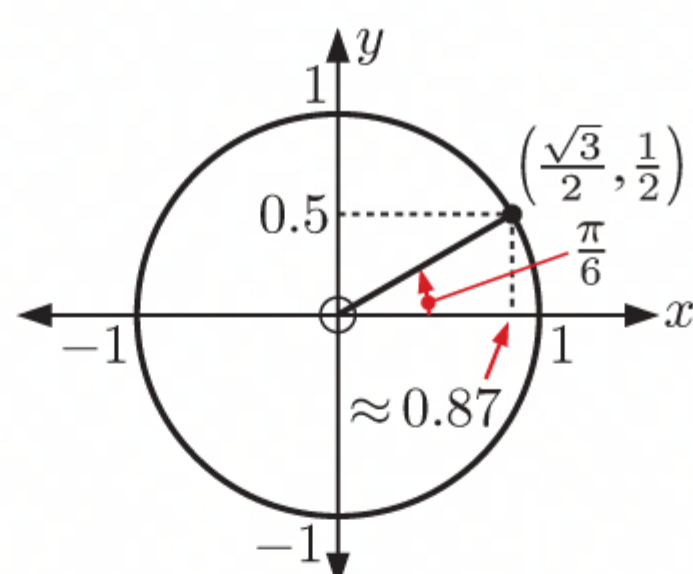


$$\sin\left(-\frac{3\pi}{4}\right) = -\frac{1}{\sqrt{2}}$$

$$\cos\left(-\frac{3\pi}{4}\right) = -\frac{1}{\sqrt{2}}$$

$$\tan\left(-\frac{3\pi}{4}\right) = \frac{-\frac{1}{\sqrt{2}}}{-\frac{1}{\sqrt{2}}} = 1$$

2 a

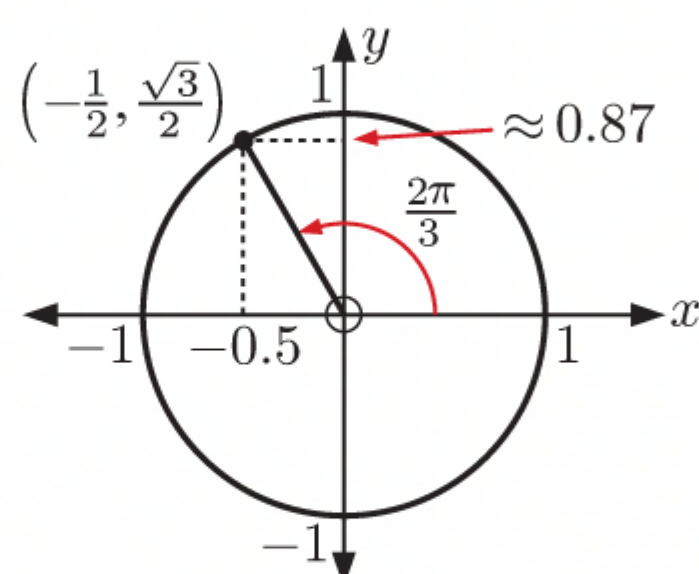


$$\sin \frac{\pi}{6} = \frac{1}{2}$$

$$\cos \frac{\pi}{6} = \frac{\sqrt{3}}{2}$$

$$\tan \frac{\pi}{6} = \frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}} = \frac{1}{\sqrt{3}}$$

b

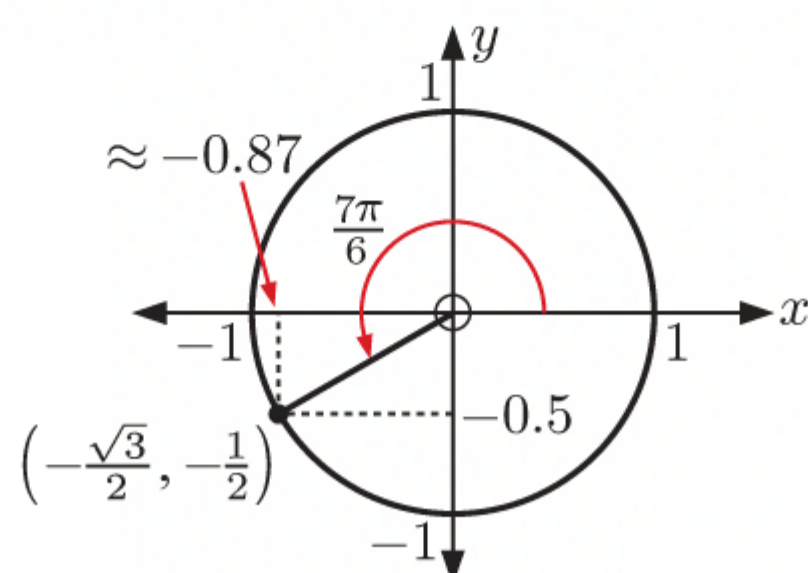


$$\sin \frac{2\pi}{3} = \frac{\sqrt{3}}{2}$$

$$\cos \frac{2\pi}{3} = -\frac{1}{2}$$

$$\tan \frac{2\pi}{3} = \frac{\frac{\sqrt{3}}{2}}{-\frac{1}{2}} = -\sqrt{3}$$

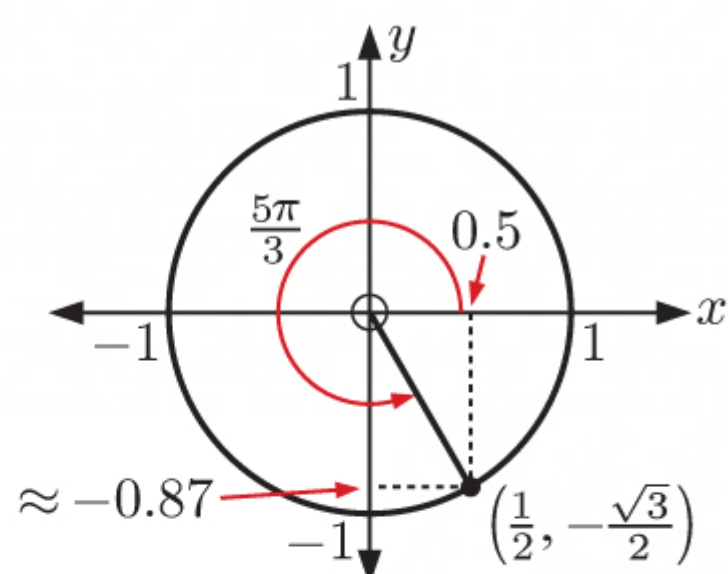
c



$$\sin \frac{7\pi}{6} = -\frac{1}{2}$$

$$\cos \frac{7\pi}{6} = -\frac{\sqrt{3}}{2}$$

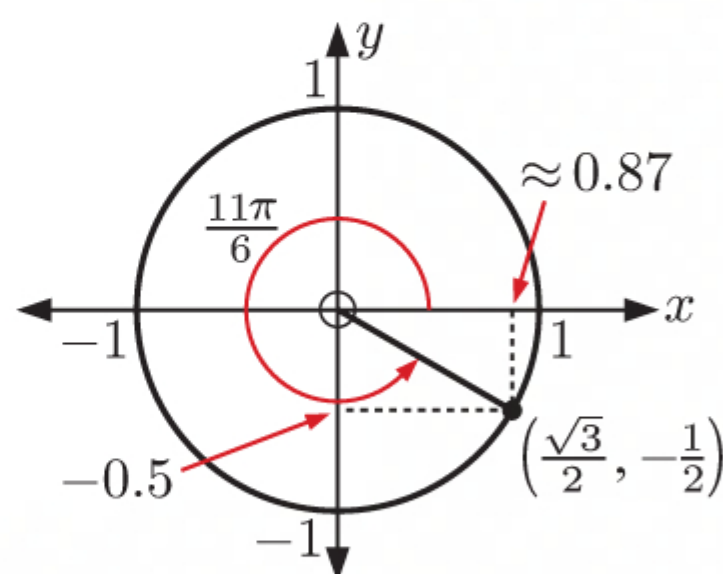
$$\tan \frac{7\pi}{6} = \frac{-\frac{1}{2}}{-\frac{\sqrt{3}}{2}} = \frac{1}{\sqrt{3}}$$

d

$$\sin \frac{5\pi}{3} = -\frac{\sqrt{3}}{2}$$

$$\cos \frac{5\pi}{3} = \frac{1}{2}$$

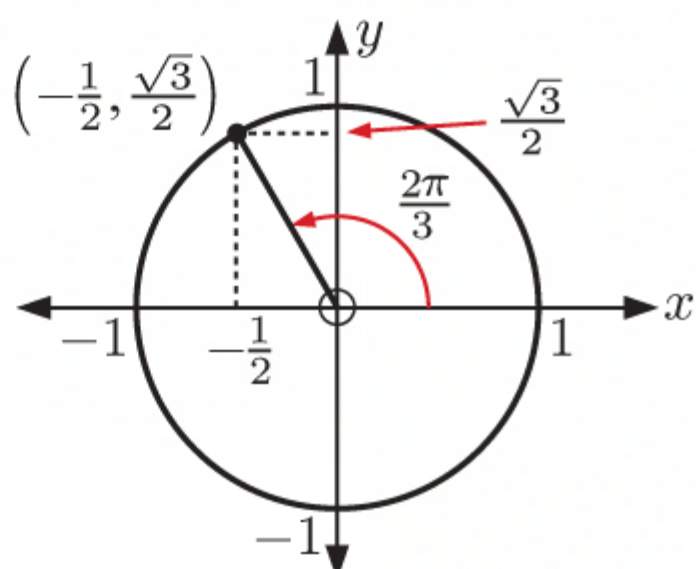
$$\tan \frac{5\pi}{3} = \frac{-\frac{\sqrt{3}}{2}}{\frac{1}{2}} = -\sqrt{3}$$

e

$$\sin \frac{11\pi}{6} = -\frac{1}{2}$$

$$\cos \frac{11\pi}{6} = \frac{\sqrt{3}}{2}$$

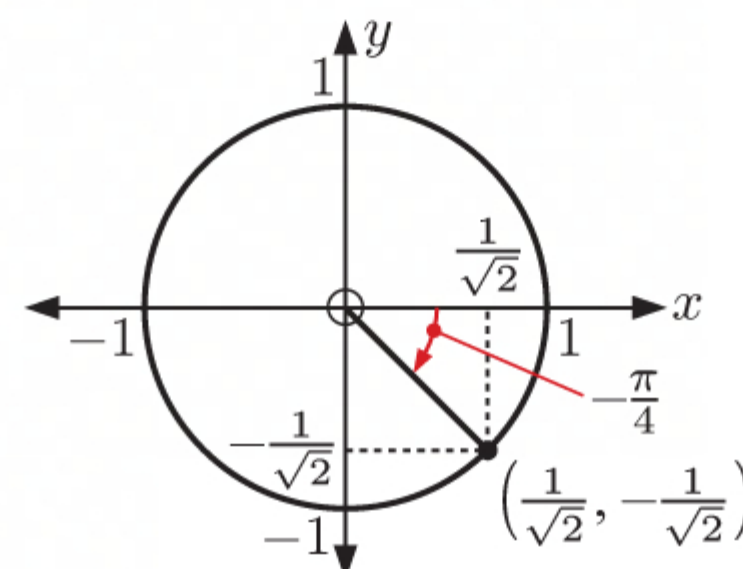
$$\tan \frac{11\pi}{6} = \frac{-\frac{1}{2}}{\frac{\sqrt{3}}{2}} = -\frac{1}{\sqrt{3}}$$

3 a $\frac{2\pi}{3}$ is a multiple of $\frac{\pi}{6}$ 

$$\text{So, } \cos \frac{2\pi}{3} = -\frac{1}{2}$$

$$\sin \frac{2\pi}{3} = \frac{\sqrt{3}}{2}$$

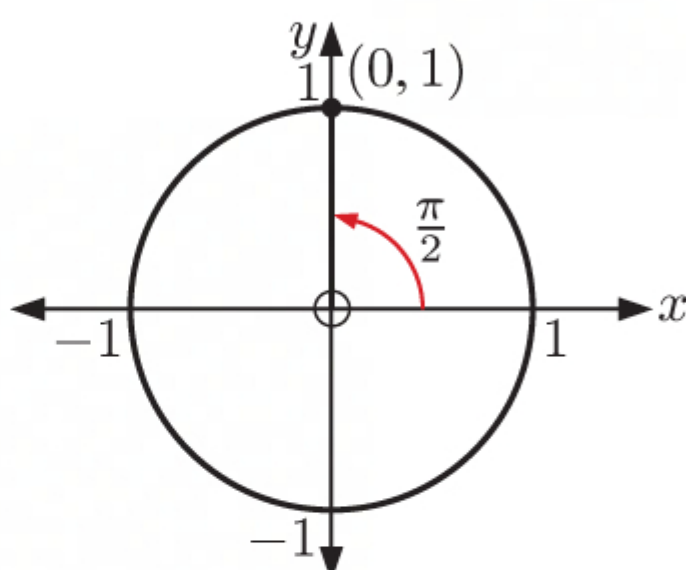
$$\tan \frac{2\pi}{3} = -\sqrt{3}$$

b $-\frac{\pi}{4}$ is a multiple of $\frac{\pi}{4}$ 

$$\text{So, } \cos\left(-\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$$

$$\sin\left(-\frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}}$$

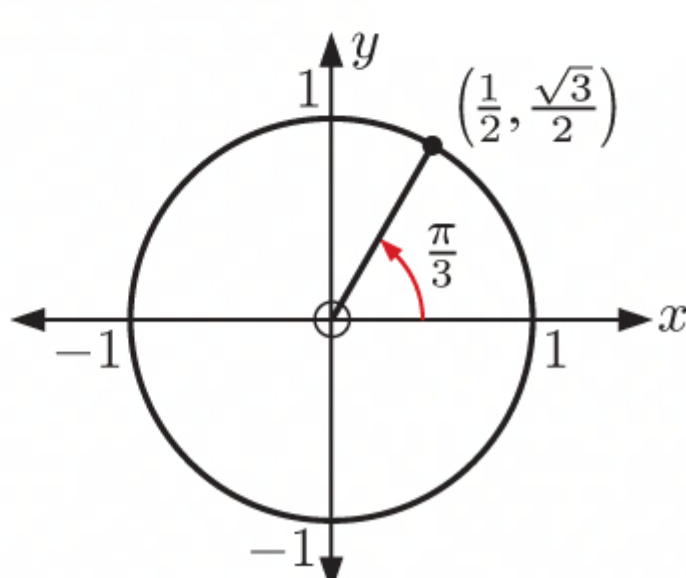
$$\tan\left(-\frac{\pi}{4}\right) = -1$$

4 a

$$\cos \frac{\pi}{2} = 0, \quad \sin \frac{\pi}{2} = 1$$

$$\text{b } \tan \frac{\pi}{2} = \frac{\sin \frac{\pi}{2}}{\cos \frac{\pi}{2}} = \frac{1}{0}$$

$\therefore \tan \frac{\pi}{2}$ is undefined.

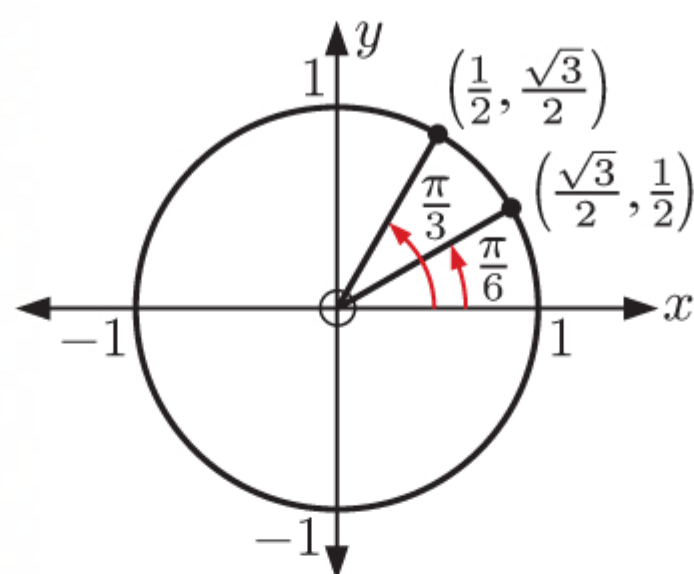
5 a

$$\sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}$$

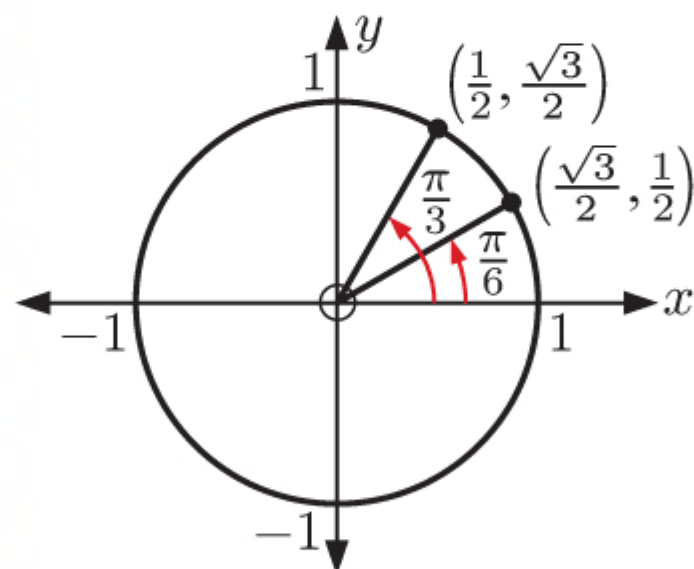
$$\therefore \sin^2\left(\frac{\pi}{3}\right) = \sin \frac{\pi}{3} \times \sin \frac{\pi}{3}$$

$$= \frac{\sqrt{3}}{2} \times \frac{\sqrt{3}}{2}$$

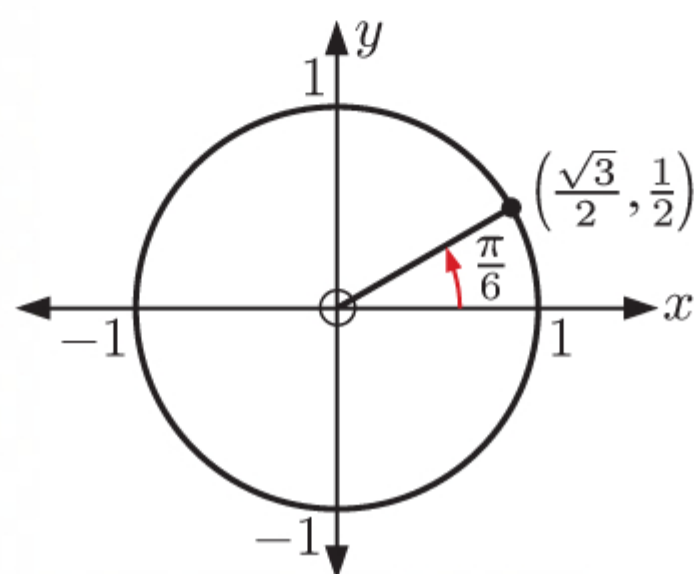
$$= \frac{3}{4}$$

b

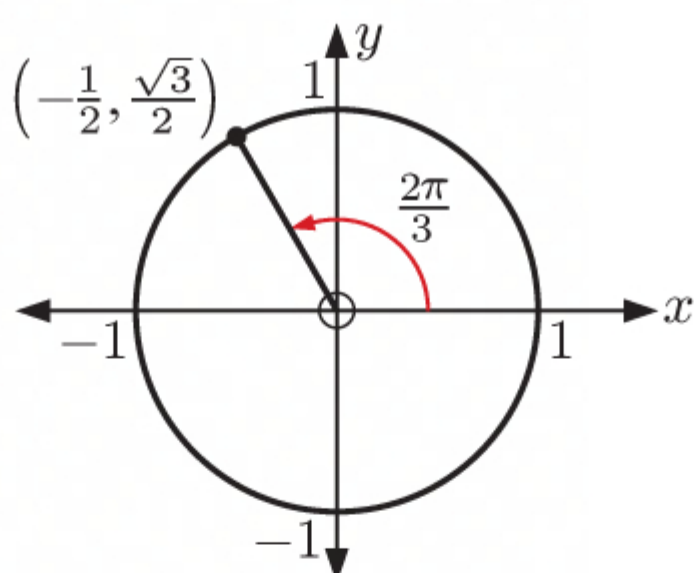
$$\begin{aligned}\sin \frac{\pi}{6} &= \frac{1}{2} \quad \text{and} \quad \cos \frac{\pi}{3} = \frac{1}{2} \\ \therefore \sin \frac{\pi}{6} \cos \frac{\pi}{3} &= \frac{1}{2} \times \frac{1}{2} \\ &= \frac{1}{4}\end{aligned}$$

c

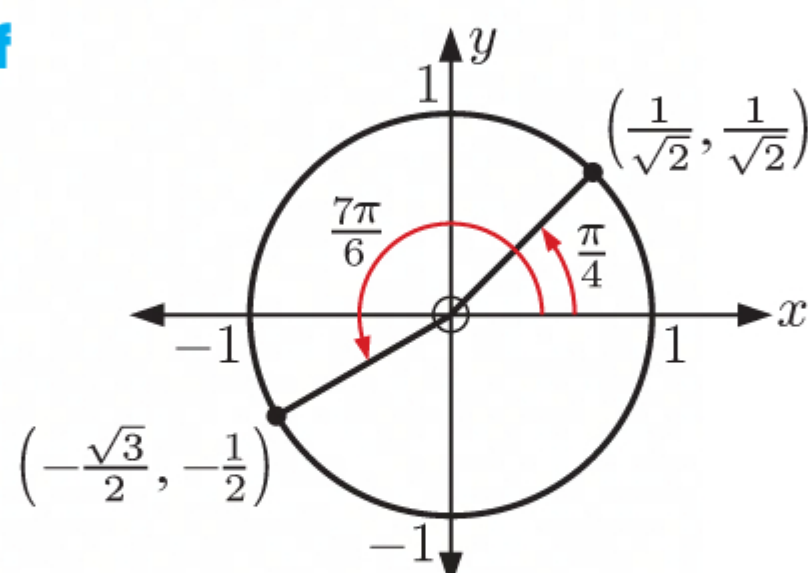
$$\begin{aligned}\sin \frac{\pi}{3} &= \frac{\sqrt{3}}{2} \quad \text{and} \quad \cos \frac{\pi}{6} = \frac{\sqrt{3}}{2} \\ \therefore 4 \sin \frac{\pi}{3} \cos \frac{\pi}{6} &= 4 \left(\frac{\sqrt{3}}{2} \right) \left(\frac{\sqrt{3}}{2} \right) \\ &= 3\end{aligned}$$

d

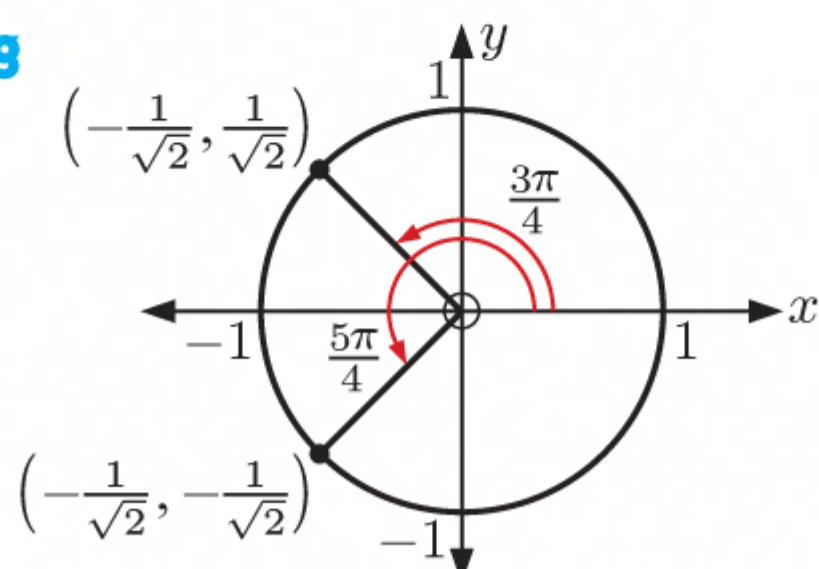
$$\begin{aligned}\cos \frac{\pi}{6} &= \frac{\sqrt{3}}{2} \\ \therefore 1 - \cos^2 \left(\frac{\pi}{6} \right) &= 1 - \left(\frac{\sqrt{3}}{2} \right)^2 \\ &= 1 - \frac{3}{4} \\ &= \frac{1}{4}\end{aligned}$$

e

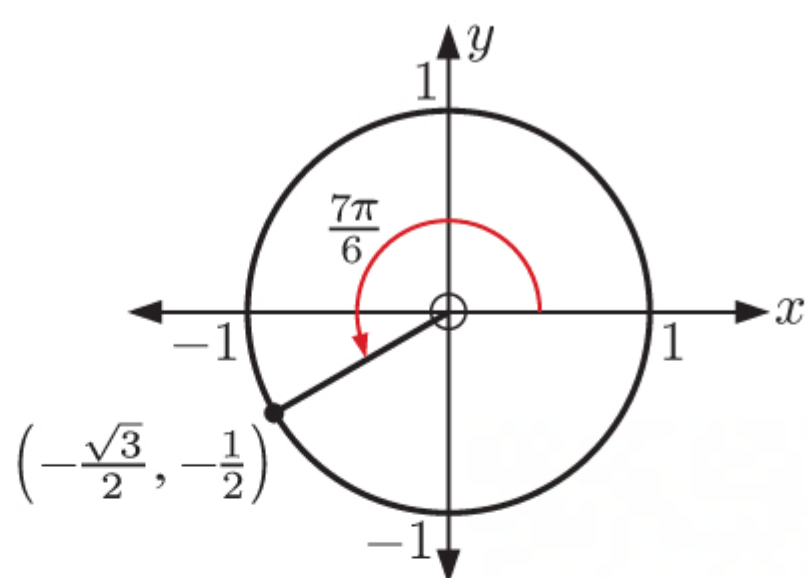
$$\begin{aligned}\sin \frac{2\pi}{3} &= \frac{\sqrt{3}}{2} \\ \therefore \sin^2 \left(\frac{2\pi}{3} \right) - 1 &= \left(\frac{\sqrt{3}}{2} \right)^2 - 1 \\ &= \frac{3}{4} - 1 \\ &= -\frac{1}{4}\end{aligned}$$

f

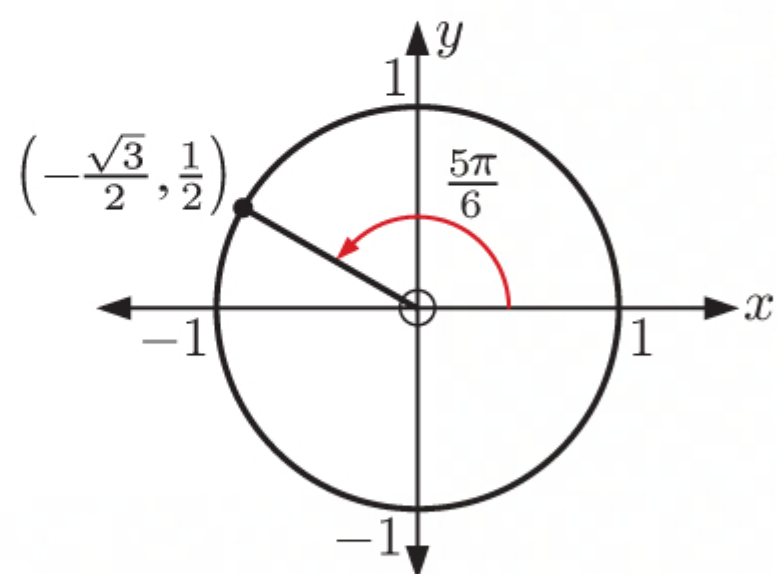
$$\begin{aligned}\cos \frac{\pi}{4} &= \frac{1}{\sqrt{2}} \quad \text{and} \quad \sin \frac{7\pi}{6} = -\frac{1}{2} \\ \therefore \cos^2 \left(\frac{\pi}{4} \right) - \sin \frac{7\pi}{6} &= \left(\frac{1}{\sqrt{2}} \right)^2 - \left(-\frac{1}{2} \right) \\ &= \frac{1}{2} + \frac{1}{2} \\ &= 1\end{aligned}$$

g

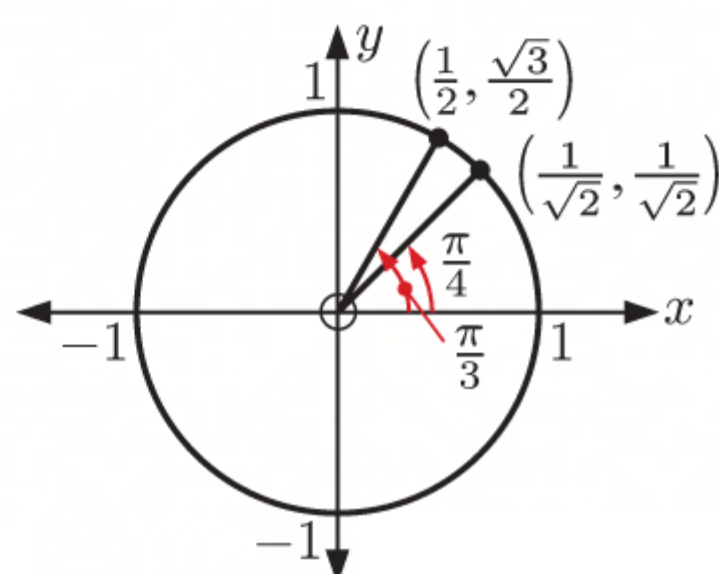
$$\begin{aligned}\sin \frac{3\pi}{4} &= \frac{1}{\sqrt{2}} \quad \text{and} \quad \cos \frac{5\pi}{4} = -\frac{1}{\sqrt{2}} \\ \therefore \sin \frac{3\pi}{4} - \cos \frac{5\pi}{4} &= \frac{1}{\sqrt{2}} - \left(-\frac{1}{\sqrt{2}} \right) \\ &= \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} \\ &= \frac{2}{\sqrt{2}} \quad \text{or} \quad \sqrt{2}\end{aligned}$$

h

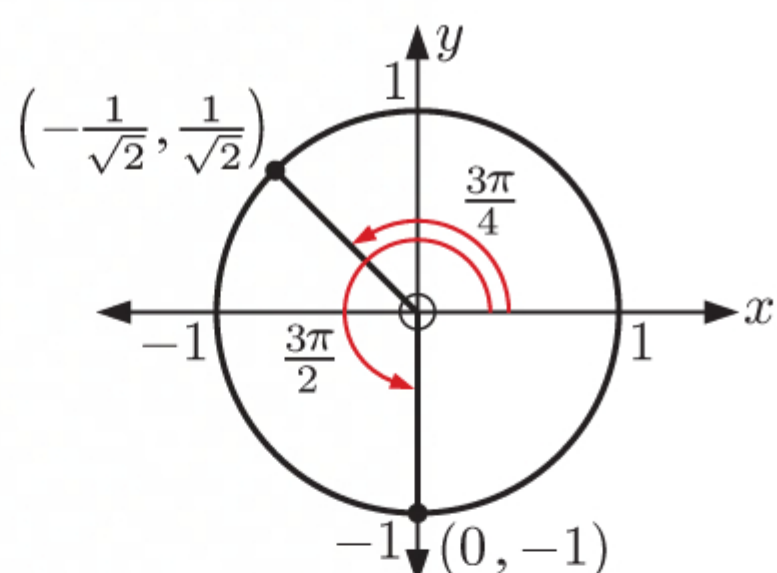
$$\begin{aligned}\sin \frac{7\pi}{6} &= -\frac{1}{2} \\ \therefore 1 - 2 \sin^2 \left(\frac{7\pi}{6} \right) &= 1 - 2 \left(-\frac{1}{2} \right)^2 \\ &= 1 - 2 \times \frac{1}{4} \\ &= \frac{1}{2}\end{aligned}$$

i

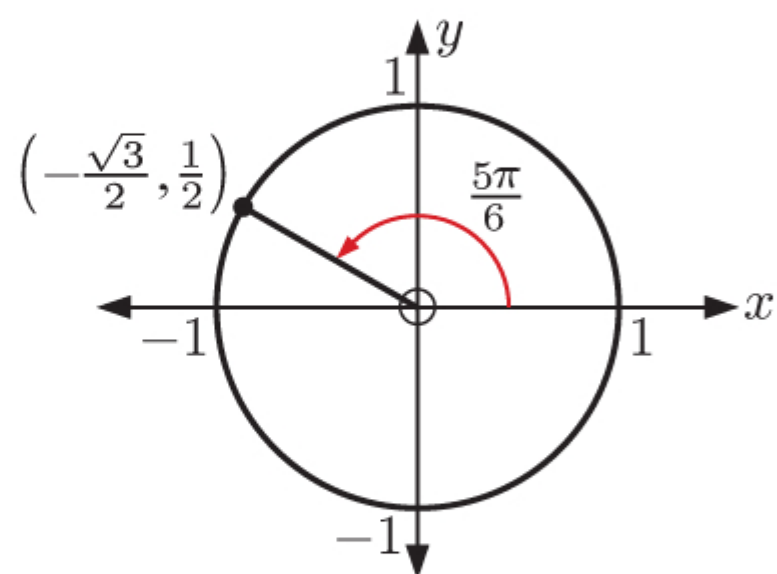
$$\begin{aligned}\cos \frac{5\pi}{6} &= -\frac{\sqrt{3}}{2} \quad \text{and} \quad \sin \frac{5\pi}{6} = \frac{1}{2} \\ \therefore \cos^2 \left(\frac{5\pi}{6} \right) - \sin^2 \left(\frac{5\pi}{6} \right) &= \left(-\frac{\sqrt{3}}{2} \right)^2 - \left(\frac{1}{2} \right)^2 \\ &= \frac{3}{4} - \frac{1}{4} \\ &= \frac{1}{2}\end{aligned}$$

j

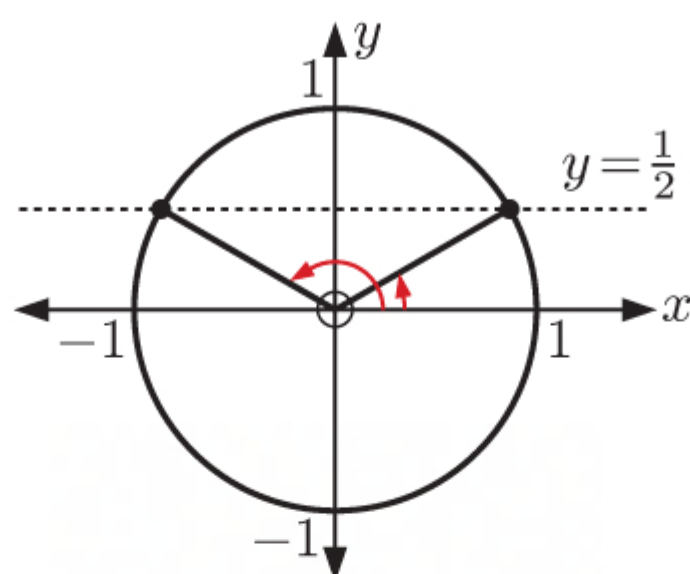
$$\begin{aligned}\tan \frac{\pi}{3} &= \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = \sqrt{3} \quad \text{and} \quad \sin \frac{\pi}{4} = \frac{1}{\sqrt{2}} \\ \therefore \tan^2 \left(\frac{\pi}{3} \right) - 2 \sin^2 \left(\frac{\pi}{4} \right) &= \left(\sqrt{3} \right)^2 - 2 \left(\frac{1}{\sqrt{2}} \right)^2 \\ &= 3 - 2 \left(\frac{1}{2} \right) \\ &= 2\end{aligned}$$

k

$$\begin{aligned}\tan \left(-\frac{5\pi}{4} \right) &= \frac{\frac{1}{\sqrt{2}}}{-\frac{1}{\sqrt{2}}} = -1 \quad \text{and} \quad \sin \frac{3\pi}{2} = -1 \\ \therefore 2 \tan \left(-\frac{5\pi}{4} \right) - \sin \frac{3\pi}{2} &= 2(-1) - (-1) \\ &= -1\end{aligned}$$

l

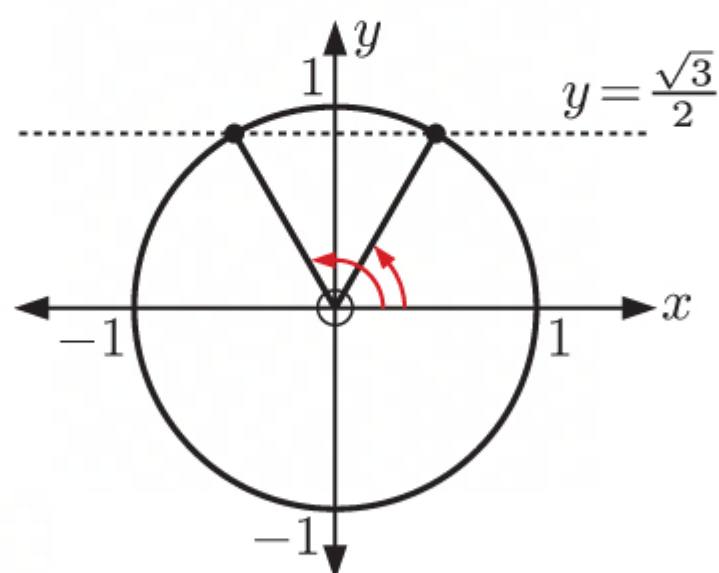
$$\begin{aligned}\tan \frac{5\pi}{6} &= \frac{\frac{1}{2}}{-\frac{\sqrt{3}}{2}} = -\frac{1}{\sqrt{3}} \\ \therefore \frac{2 \tan \frac{5\pi}{6}}{1 - \tan^2 \left(\frac{5\pi}{6} \right)} &= \frac{2 \left(-\frac{1}{\sqrt{3}} \right)}{1 - \left(-\frac{1}{\sqrt{3}} \right)^2} \\ &= \frac{-\frac{2}{\sqrt{3}}}{1 - \frac{1}{3}} \\ &= \frac{-\frac{2}{\sqrt{3}}}{\frac{2}{3}} \\ &= -\frac{3}{\sqrt{3}} = -\sqrt{3}\end{aligned}$$

6 a

Since the sine is $\frac{1}{2}$, we draw the horizontal line $y = \frac{1}{2}$.

Because $\frac{1}{2}$ is involved, we know the required angles are multiples of $\frac{\pi}{6}$.

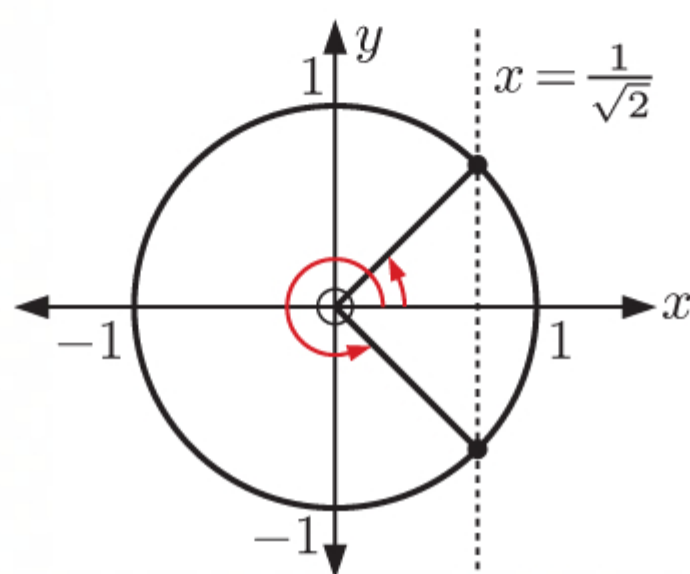
They are $\frac{\pi}{6}$ and $\frac{5\pi}{6}$.

b

Since the sine is $\frac{\sqrt{3}}{2}$, we draw the horizontal line $y = \frac{\sqrt{3}}{2}$.

Because $\frac{\sqrt{3}}{2}$ is involved, we know the required angles are multiples of $\frac{\pi}{6}$.

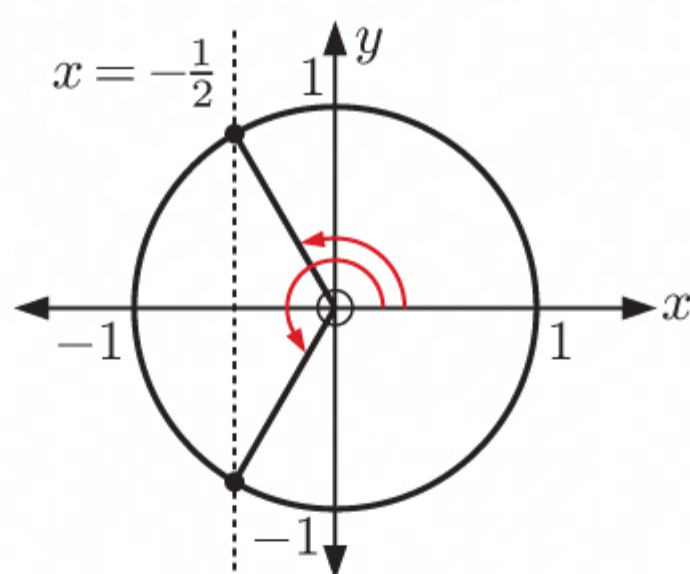
They are $\frac{\pi}{3}$ and $\frac{2\pi}{3}$.

c

Since the cosine is $\frac{1}{\sqrt{2}}$, we draw the vertical line $x = \frac{1}{\sqrt{2}}$.

Because $\frac{1}{\sqrt{2}}$ is involved, we know the required angles are multiples of $\frac{\pi}{4}$.

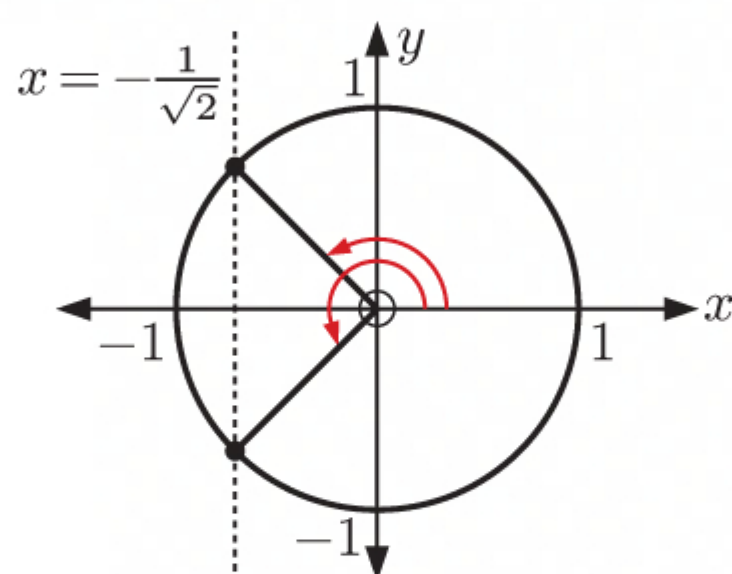
They are $\frac{\pi}{4}$ and $\frac{7\pi}{4}$.

d

Since the cosine is $-\frac{1}{2}$, we draw the vertical line $x = -\frac{1}{2}$.

Because $\frac{1}{2}$ is involved, we know the required angles are multiples of $\frac{\pi}{6}$.

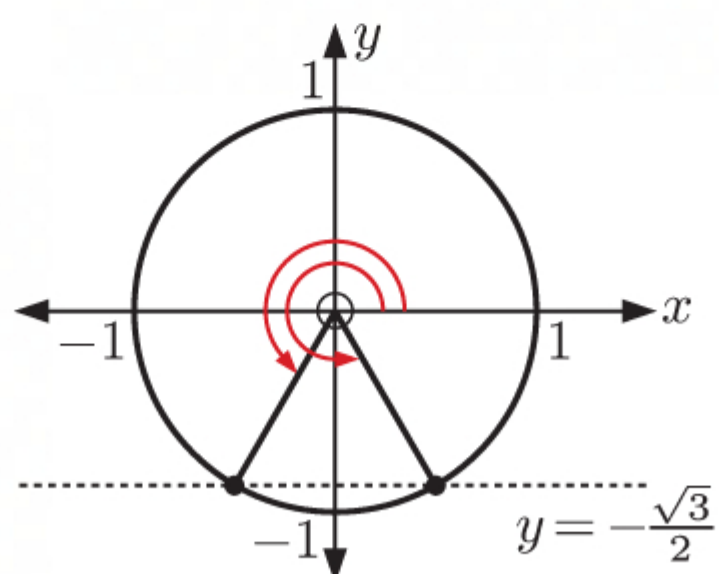
They are $\frac{2\pi}{3}$ and $\frac{4\pi}{3}$.

e

Since the cosine is $-\frac{1}{\sqrt{2}}$, we draw the vertical line $x = -\frac{1}{\sqrt{2}}$.

Because $\frac{1}{\sqrt{2}}$ is involved, we know the required angles are multiples of $\frac{\pi}{4}$.

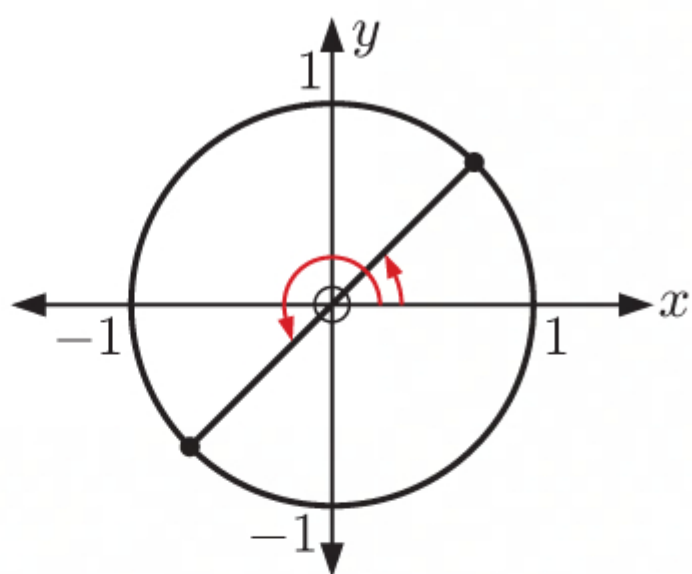
They are $\frac{3\pi}{4}$ and $\frac{5\pi}{4}$.

f

Since the sine is $-\frac{\sqrt{3}}{2}$, we draw the horizontal line $y = -\frac{\sqrt{3}}{2}$.

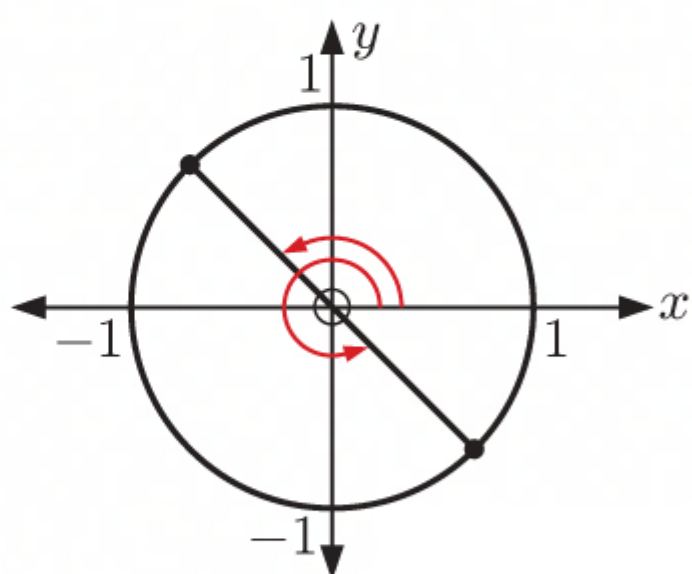
Because $\frac{\sqrt{3}}{2}$ is involved, we know the required angles are multiples of $\frac{\pi}{6}$.

They are $\frac{4\pi}{3}$ and $\frac{5\pi}{3}$.

7 a

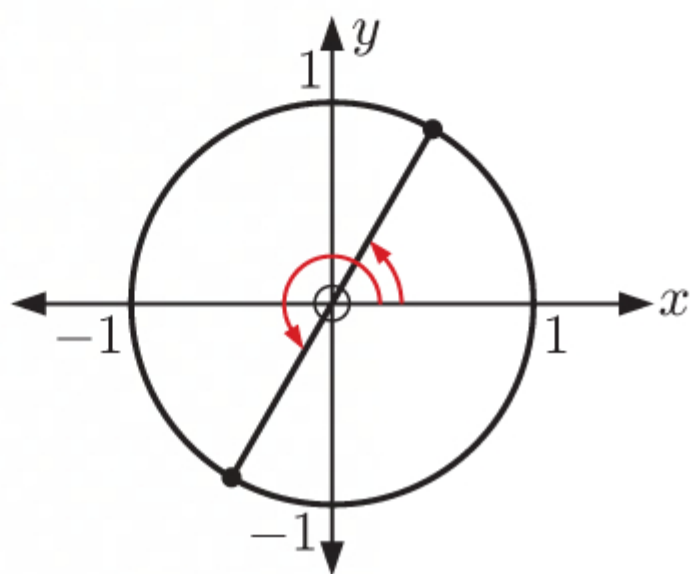
Since the tangent is 1, the sine and cosine must be identical (since $\tan \theta = \frac{\sin \theta}{\cos \theta}$).

This is only true when the angles are $\frac{\pi}{4}$ and $\frac{5\pi}{4}$.

b

Since the tangent is -1 , the sine and cosine must be equal in value but opposite in sign (since $\tan \theta = \frac{\sin \theta}{\cos \theta}$).

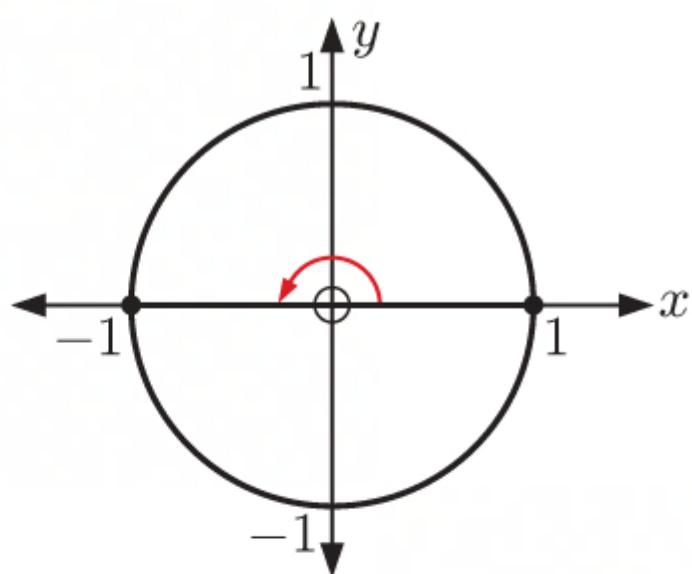
This is only true when the angles are $\frac{3\pi}{4}$ and $\frac{7\pi}{4}$.

c

Since the tangent is $\sqrt{3} = \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = \frac{-\frac{\sqrt{3}}{2}}{-\frac{1}{2}}$, the sine must be $\pm \frac{\sqrt{3}}{2}$, and the cosine must be $\pm \frac{1}{2}$ (since $\tan \theta = \frac{\sin \theta}{\cos \theta}$).

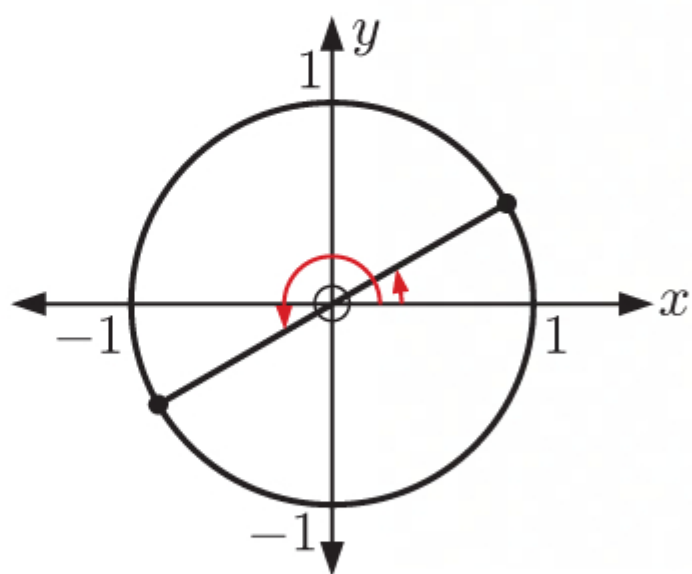
Because $\frac{\sqrt{3}}{2}$ is involved, we know the required angles are multiples of $\frac{\pi}{6}$.

They are $\frac{\pi}{3}$ and $\frac{4\pi}{3}$.

d

Since the tangent is 0, the sine must be 0 (since $\tan \theta = \frac{\sin \theta}{\cos \theta}$).

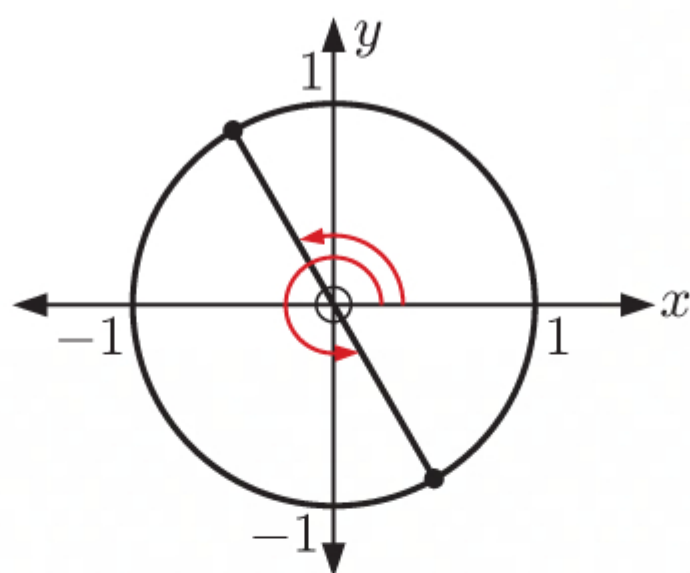
This is only true when the cosine is ± 1 , and the angles are 0, π , and 2π .

e

Since the tangent is $\frac{1}{\sqrt{3}} = \frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}} = \frac{-\frac{1}{2}}{-\frac{\sqrt{3}}{2}}$, the sine must be $\pm \frac{1}{2}$, and the cosine must be $\pm \frac{\sqrt{3}}{2}$ (since $\tan \theta = \frac{\sin \theta}{\cos \theta}$).

Because $\frac{1}{2}$ and $\frac{\sqrt{3}}{2}$ are both involved, we know the required angles are multiples of $\frac{\pi}{6}$.

They are $\frac{\pi}{6}$ and $\frac{7\pi}{6}$.

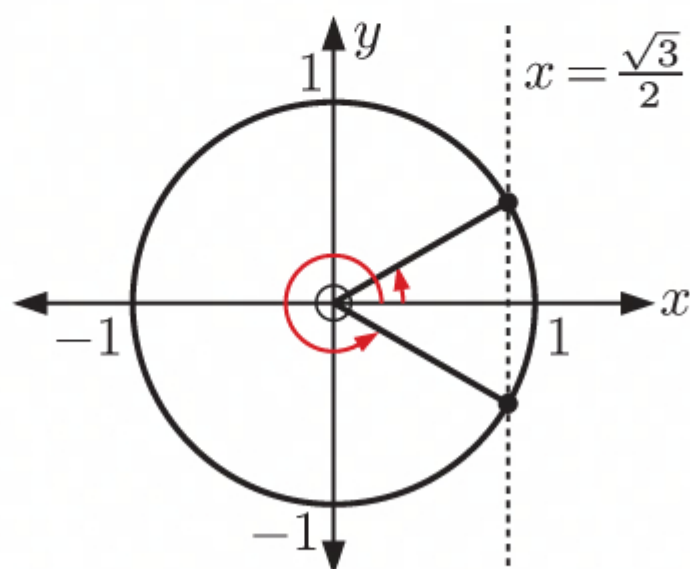
f

Since the tangent is $-\sqrt{3} = \frac{\frac{\sqrt{3}}{2}}{-\frac{1}{2}} = \frac{-\frac{\sqrt{3}}{2}}{\frac{1}{2}}$, the sine

must be $\pm\frac{\sqrt{3}}{2}$, and the cosine must be $\mp\frac{1}{2}$ (since $\tan \theta = \frac{\sin \theta}{\cos \theta}$).

Because $\frac{1}{2}$ and $\frac{\sqrt{3}}{2}$ are both involved, we know the required angles are multiples of $\frac{\pi}{6}$.

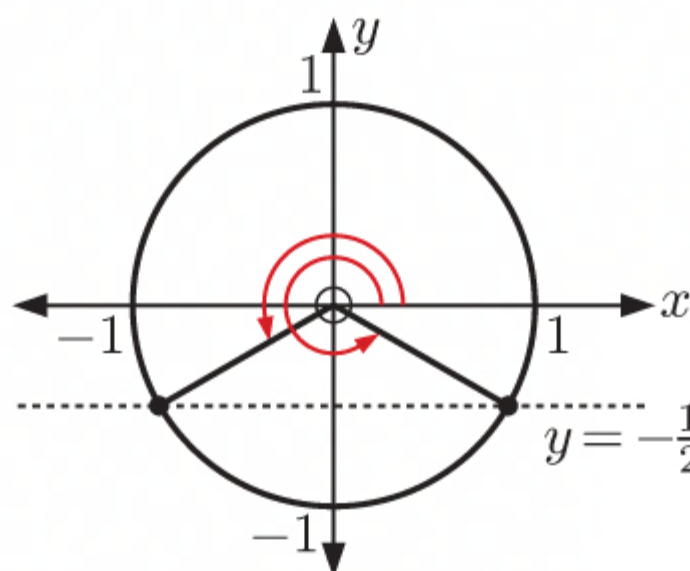
They are $\frac{2\pi}{3}$ and $\frac{5\pi}{3}$.

8 a

Since the cosine is $\frac{\sqrt{3}}{2}$, we draw the vertical line $x = \frac{\sqrt{3}}{2}$.

Because $\frac{\sqrt{3}}{2}$ is involved, we know the required angles are multiples of $\frac{\pi}{6}$.

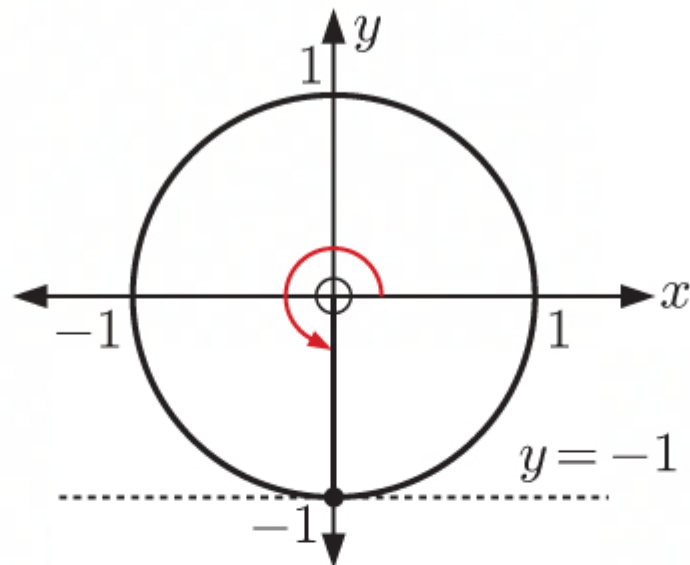
They are $\frac{\pi}{6}$, $\frac{11\pi}{6}$, $\frac{13\pi}{6}$, and $\frac{23\pi}{6}$.

b

Since the sine is $-\frac{1}{2}$, we draw the horizontal line $y = -\frac{1}{2}$.

Because $\frac{1}{2}$ is involved, we know the required angles are multiples of $\frac{\pi}{6}$.

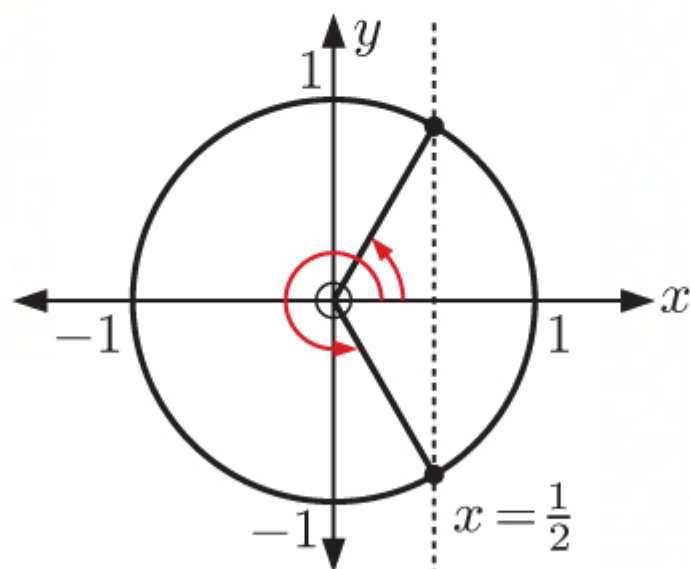
They are $\frac{7\pi}{6}$, $\frac{11\pi}{6}$, $\frac{19\pi}{6}$, and $\frac{23\pi}{6}$.

c

Since the sine is -1 , we draw the horizontal line $y = -1$.

Because 1 is involved, we know the required angles are multiples of $\frac{\pi}{2}$.

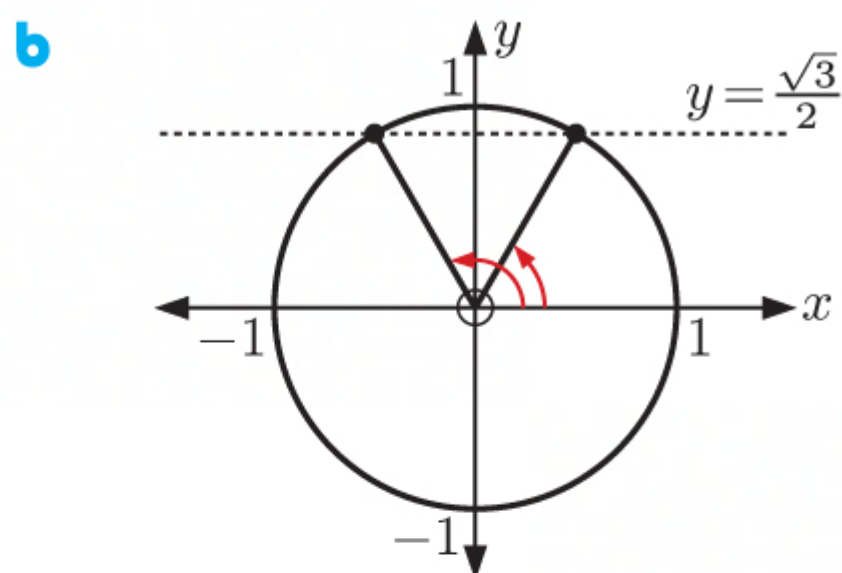
They are $\frac{3\pi}{2}$ and $\frac{7\pi}{2}$.

9 a

Since $\cos \theta = \frac{1}{2}$, we draw the vertical line $x = \frac{1}{2}$.

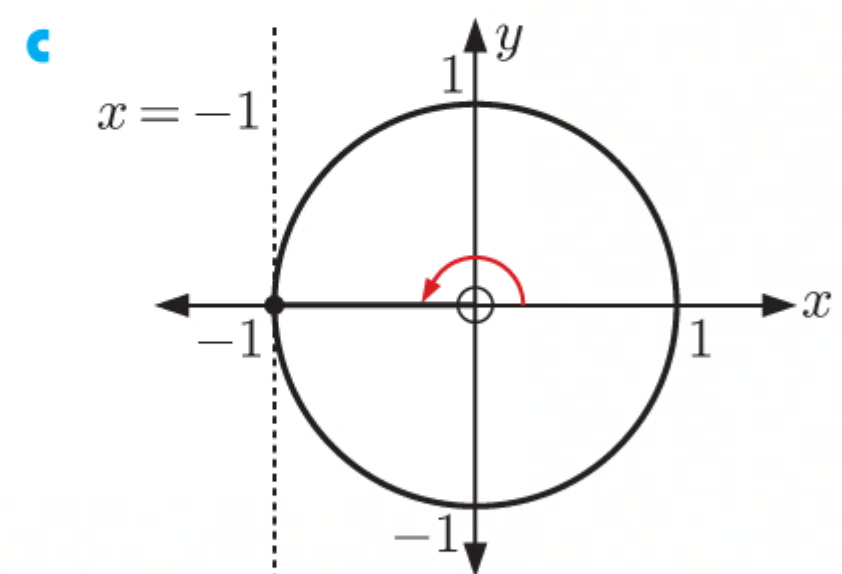
Because $\frac{1}{2}$ is involved, we know the required angles are multiples of $\frac{\pi}{3}$.

$\therefore \theta = \frac{\pi}{3}, \frac{5\pi}{3}$



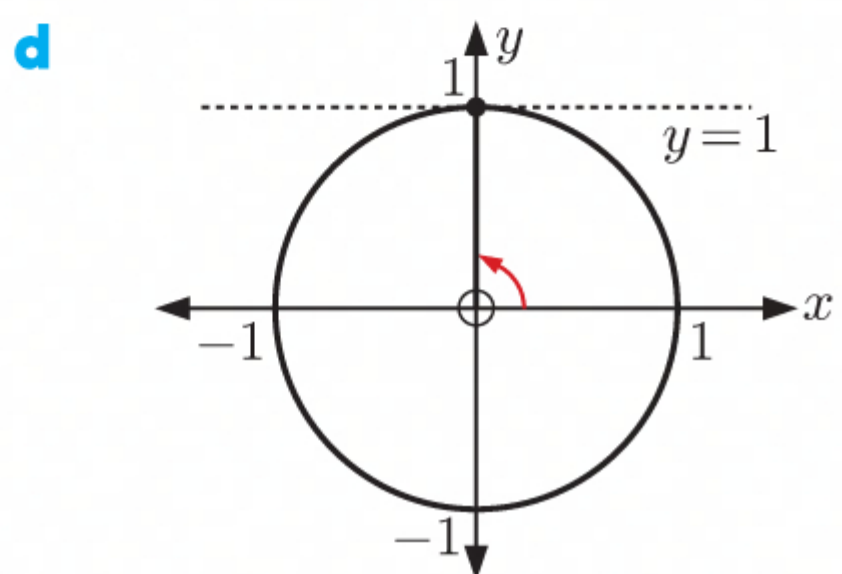
Since $\sin \theta = \frac{\sqrt{3}}{2}$, we draw the horizontal line $y = \frac{\sqrt{3}}{2}$.
Because $\frac{\sqrt{3}}{2}$ is involved, we know the required angles are multiples of $\frac{\pi}{6}$.

$$\therefore \theta = \frac{\pi}{3}, \frac{2\pi}{3}$$



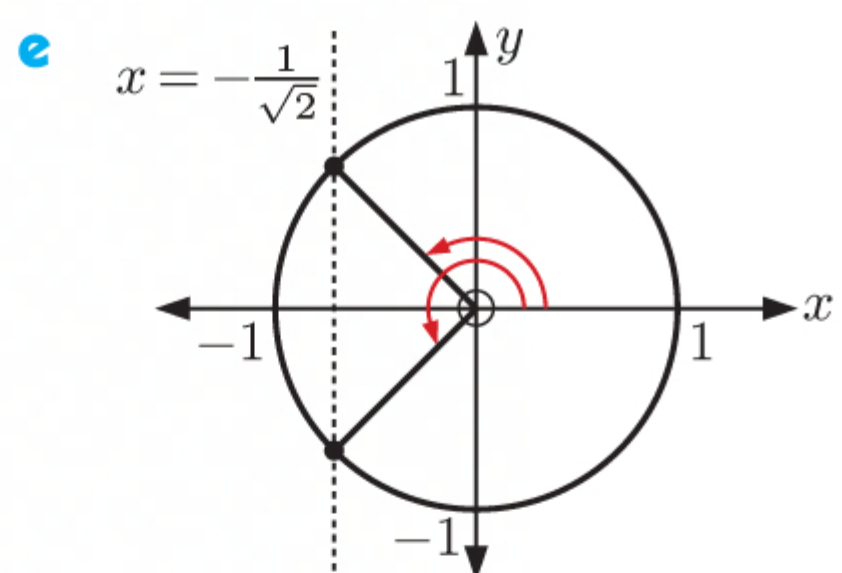
Since $\cos \theta = -1$, we draw the vertical line $x = -1$.
Because 1 is involved, we know the required angles are multiples of $\frac{\pi}{2}$.

$$\therefore \theta = \pi$$



Since $\sin \theta = 1$, we draw the horizontal line $y = 1$.
Because 1 is involved, we know the required angles are multiples of $\frac{\pi}{2}$.

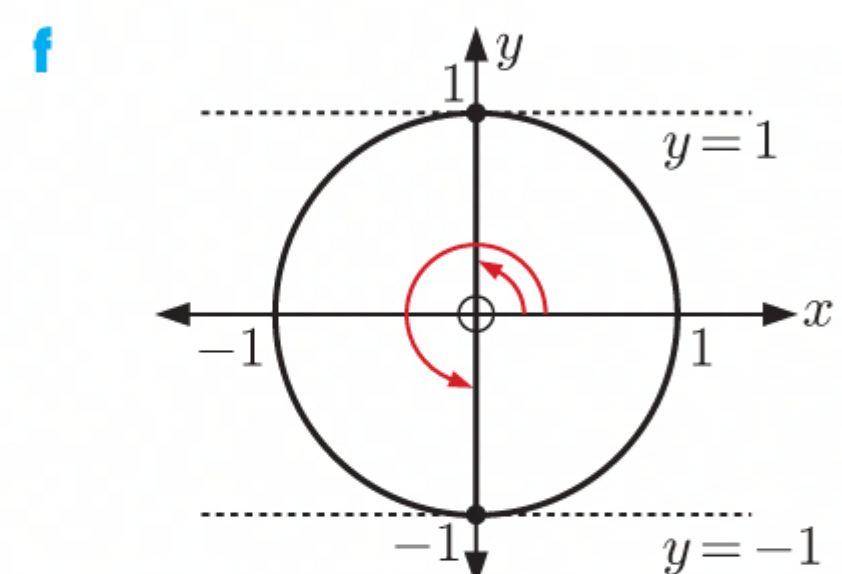
$$\therefore \theta = \frac{\pi}{2}$$



Since $\cos \theta = -\frac{1}{\sqrt{2}}$, we draw the vertical line $x = -\frac{1}{\sqrt{2}}$.

Because $\frac{1}{\sqrt{2}}$ is involved, we know the required angles are multiples of $\frac{\pi}{4}$.

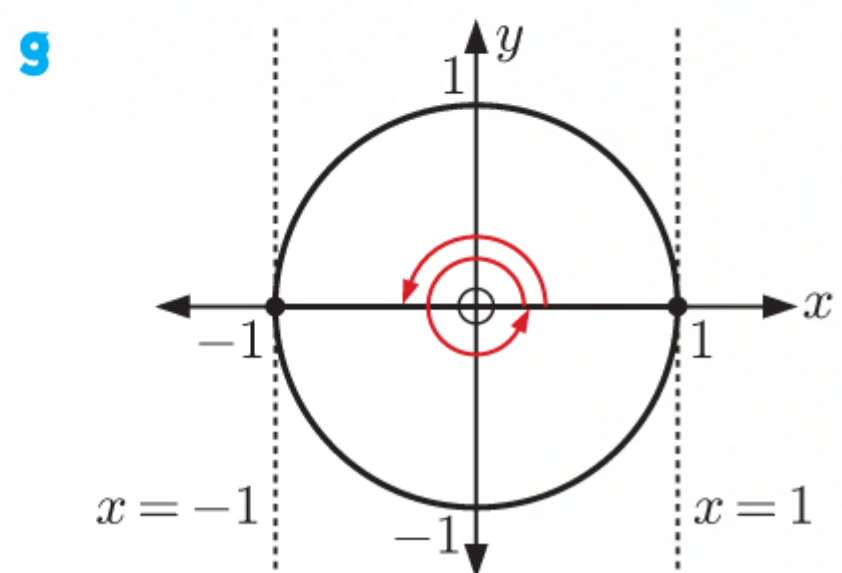
$$\therefore \theta = \frac{3\pi}{4}, \frac{5\pi}{4}$$



Since $\sin^2 \theta = 1$, then $\sin \theta = \pm 1$, so we draw the horizontal lines $y = 1$ and $y = -1$.

Because 1 is involved, we know the required angles are multiples of $\frac{\pi}{2}$.

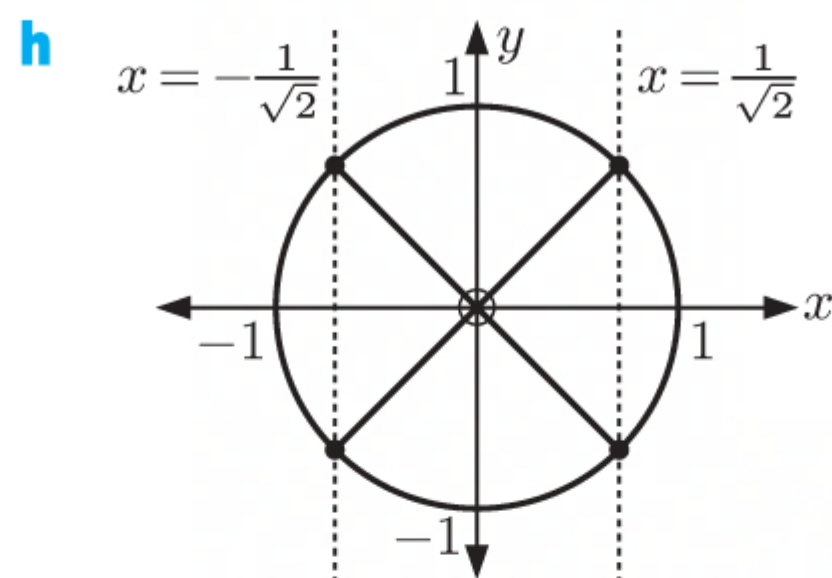
$$\therefore \theta = \frac{\pi}{2}, \frac{3\pi}{2}$$



Since $\cos^2 \theta = 1$, then $\cos \theta = \pm 1$, so we draw the vertical lines $x = -1$ and $x = 1$.

Because 1 is involved, we know the required angles are multiples of $\frac{\pi}{2}$.

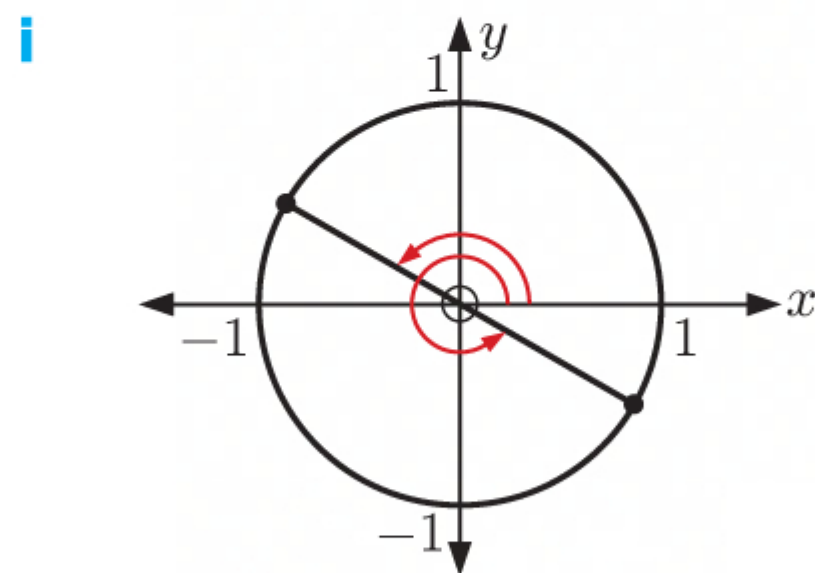
$$\therefore \theta = 0, \pi, 2\pi$$



Since $\cos^2 \theta = \frac{1}{2}$, then $\cos \theta = \pm \frac{1}{\sqrt{2}}$, so we draw the vertical lines $x = -\frac{1}{\sqrt{2}}$ and $x = \frac{1}{\sqrt{2}}$.

Because $\frac{1}{\sqrt{2}}$ is involved, we know the required angles are multiples of $\frac{\pi}{4}$.

$$\therefore \theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$$

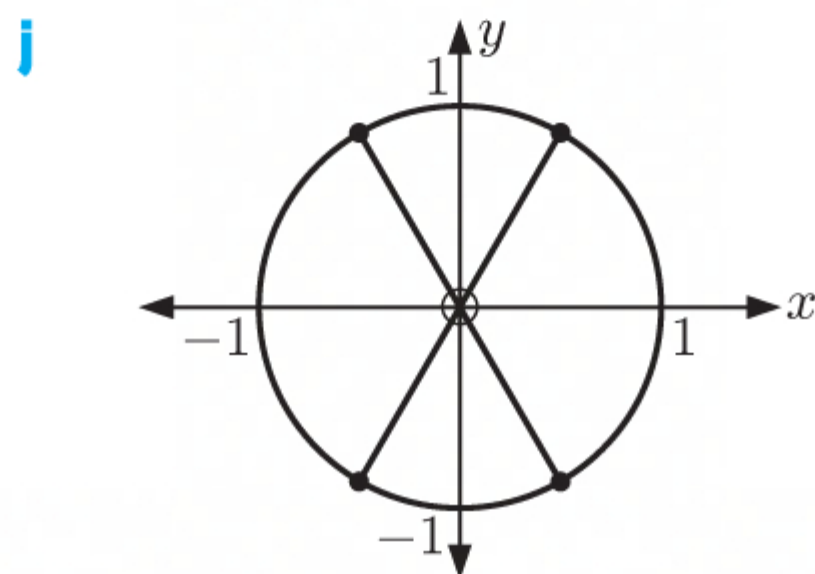


Since $\tan \theta = -\frac{1}{\sqrt{3}} = \frac{-\frac{1}{2}}{\frac{\sqrt{3}}{2}} = \frac{\frac{1}{2}}{-\frac{\sqrt{3}}{2}}$, then

$$\sin \theta = \mp \frac{1}{2} \text{ and } \cos \theta = \pm \frac{\sqrt{3}}{2} \text{ (since } \tan \theta = \frac{\sin \theta}{\cos \theta} \text{)}.$$

Because $\frac{1}{2}$ and $\frac{\sqrt{3}}{2}$ are both involved, we know the required angles are multiples of $\frac{\pi}{6}$.

$$\therefore \theta = \frac{5\pi}{6}, \frac{11\pi}{6}$$



Since $\tan^2 \theta = 3$, then $\tan \theta = \pm \sqrt{3} = \pm \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}}$.

So, $\sin \theta = \pm \frac{\sqrt{3}}{2}$ and $\cos \theta = \pm \frac{1}{2}$

(since $\tan \theta = \frac{\sin \theta}{\cos \theta}$).

Because $\frac{\sqrt{3}}{2}$ and $\frac{1}{2}$ are both involved, we know the required angles are multiples of $\frac{\pi}{6}$.

$$\therefore \theta = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$$

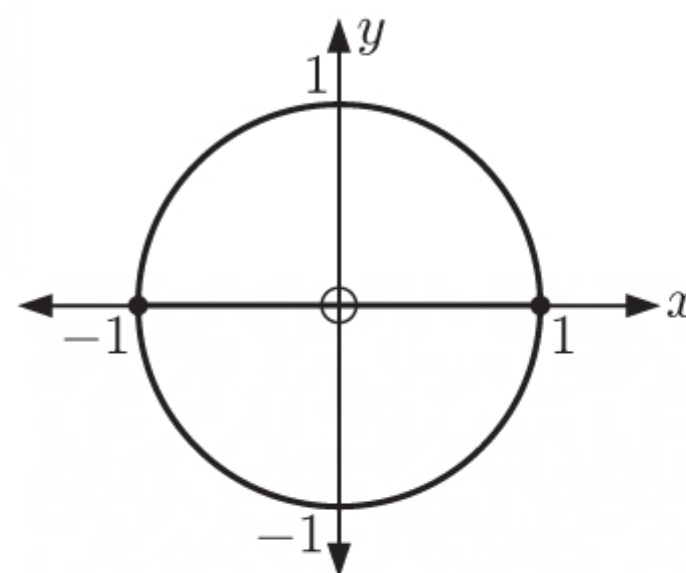
10 a $\tan \theta$ is zero when

$$\tan \theta = \frac{\sin \theta}{\cos \theta} = \frac{0}{\cos \theta}$$

$$\therefore \text{ when } \sin \theta = 0$$

$$\therefore \theta = \dots, -\pi, 0, \pi, 2\pi, \dots$$

$$\therefore \theta = k\pi, \text{ for } k \in \mathbb{Z}$$



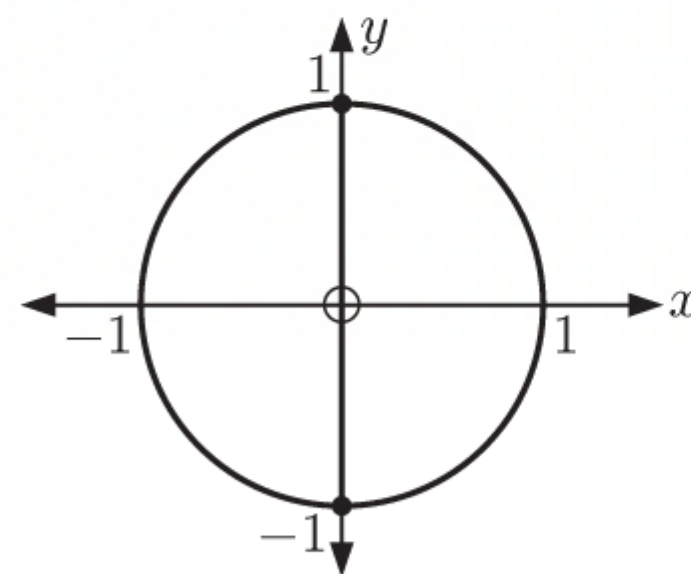
b $\tan \theta$ is undefined when

$$\tan \theta = \frac{\sin \theta}{\cos \theta} = \frac{\sin \theta}{0}$$

$$\therefore \text{ when } \cos \theta = 0$$

$$\therefore \theta = \dots, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \dots$$

$$\therefore \theta = \frac{\pi}{2} + k\pi, \text{ for } k \in \mathbb{Z}$$



EXERCISE 7E

1 a $\cos^2 \theta + \sin^2 \theta = 1$

$$\therefore \cos^2 \theta + \left(\frac{1}{2}\right)^2 = 1$$

$$\therefore \cos^2 \theta = \frac{3}{4}$$

$$\therefore \cos \theta = \pm \frac{\sqrt{3}}{2}$$

c $\cos^2 \theta + \sin^2 \theta = 1$

$$\therefore \cos^2 \theta + 0^2 = 1$$

$$\therefore \cos \theta = \pm 1$$

2 a $\cos^2 \theta + \sin^2 \theta = 1$

$$\therefore \left(\frac{4}{5}\right)^2 + \sin^2 \theta = 1$$

$$\therefore \sin^2 \theta = \frac{9}{25}$$

$$\therefore \sin \theta = \pm \frac{3}{5}$$

c $\cos^2 \theta + \sin^2 \theta = 1$

$$\therefore 1^2 + \sin^2 \theta = 1$$

$$\therefore \sin^2 \theta = 0$$

$$\therefore \sin \theta = 0$$

3 a $\cos^2 \theta + \sin^2 \theta = 1$

$$\therefore \frac{4}{9} + \sin^2 \theta = 1$$

$$\therefore \sin^2 \theta = \frac{5}{9}$$

$$\therefore \sin \theta = \pm \frac{\sqrt{5}}{3}$$

But θ is in quadrant 1

where $\sin \theta > 0$

$$\therefore \sin \theta = \frac{\sqrt{5}}{3}$$

c $\cos^2 \theta + \sin^2 \theta = 1$

$$\therefore \cos^2 \theta + \frac{9}{25} = 1$$

$$\therefore \cos^2 \theta = \frac{16}{25}$$

$$\therefore \cos \theta = \pm \frac{4}{5}$$

But θ is in quadrant 4

where $\cos \theta > 0$

$$\therefore \cos \theta = \frac{4}{5}$$

b $\cos^2 \theta + \sin^2 \theta = 1$

$$\therefore \cos^2 \theta + \left(-\frac{1}{3}\right)^2 = 1$$

$$\therefore \cos^2 \theta = \frac{8}{9}$$

$$\therefore \cos \theta = \pm \frac{\sqrt{8}}{3}$$

$$\therefore \cos \theta = \pm \frac{2\sqrt{2}}{3}$$

d $\cos^2 \theta + \sin^2 \theta = 1$

$$\therefore \cos^2 \theta + (-1)^2 = 1$$

$$\therefore \cos \theta = 0$$

b $\cos^2 \theta + \sin^2 \theta = 1$

$$\therefore \left(-\frac{3}{4}\right)^2 + \sin^2 \theta = 1$$

$$\therefore \sin^2 \theta = \frac{7}{16}$$

$$\therefore \sin \theta = \pm \frac{\sqrt{7}}{4}$$

d $\cos^2 \theta + \sin^2 \theta = 1$

$$\therefore 0^2 + \sin^2 \theta = 1$$

$$\therefore \sin \theta = \pm 1$$

b $\cos^2 \theta + \sin^2 \theta = 1$

$$\therefore \cos^2 \theta + \frac{4}{25} = 1$$

$$\therefore \cos^2 \theta = \frac{21}{25}$$

$$\therefore \cos \theta = \pm \frac{\sqrt{21}}{5}$$

But θ is in quadrant 2

where $\cos \theta < 0$

$$\therefore \cos \theta = -\frac{\sqrt{21}}{5}$$

d $\cos^2 \theta + \sin^2 \theta = 1$

$$\therefore \frac{25}{169} + \sin^2 \theta = 1$$

$$\therefore \sin^2 \theta = \frac{144}{169}$$

$$\therefore \sin \theta = \pm \frac{12}{13}$$

But θ is in quadrant 3

where $\sin \theta < 0$

$$\therefore \sin \theta = -\frac{12}{13}$$

4 a $\cos^2 \theta + \sin^2 \theta = 1$

$$\therefore \cos^2 \theta + \frac{1}{9} = 1$$

$$\therefore \cos^2 \theta = \frac{8}{9}$$

$$\therefore \cos \theta = \pm \frac{2\sqrt{2}}{3}$$

But θ is in quadrant 2

where $\cos \theta < 0$

$$\therefore \cos \theta = -\frac{2\sqrt{2}}{3}$$

$$\begin{aligned} \text{and so } \tan \theta &= \frac{\sin \theta}{\cos \theta} \\ &= \frac{\frac{1}{3}}{-\frac{2\sqrt{2}}{3}} \\ &= -\frac{1}{2\sqrt{2}} \end{aligned}$$

c $\cos^2 \theta + \sin^2 \theta = 1$

$$\therefore \cos^2 \theta + \frac{1}{3} = 1$$

$$\therefore \cos^2 \theta = \frac{2}{3}$$

$$\therefore \cos \theta = \pm \frac{\sqrt{2}}{\sqrt{3}}$$

But θ is in quadrant 3

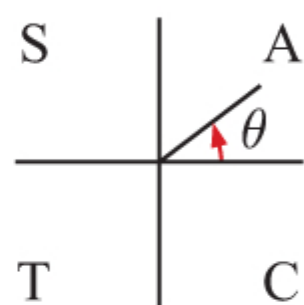
where $\cos \theta < 0$

$$\therefore \cos \theta = -\frac{\sqrt{2}}{\sqrt{3}}$$

$$\begin{aligned} \text{and so } \tan \theta &= \frac{\sin \theta}{\cos \theta} \\ &= \frac{-\frac{1}{\sqrt{3}}}{-\frac{\sqrt{2}}{\sqrt{3}}} \\ &= \frac{1}{\sqrt{2}} \end{aligned}$$

5 a $\tan \theta = \frac{\sin \theta}{\cos \theta} = \frac{2}{3}$

$$\therefore \sin \theta = \frac{2}{3} \cos \theta$$



Now $\cos^2 \theta + \sin^2 \theta = 1$

$$\therefore \cos^2 \theta + \frac{4}{9} \cos^2 \theta = 1$$

$$\therefore \frac{13}{9} \cos^2 \theta = 1$$

$$\therefore \cos^2 \theta = \frac{9}{13}$$

$$\therefore \cos \theta = \pm \frac{3}{\sqrt{13}}$$

But θ is in quadrant 1 where $\cos \theta$ and $\sin \theta$ are positive.

$$\therefore \cos \theta = \frac{3}{\sqrt{13}}, \quad \sin \theta = \frac{2}{\sqrt{13}}$$

b $\cos^2 \theta + \sin^2 \theta = 1$

$$\therefore \frac{1}{25} + \sin^2 \theta = 1$$

$$\therefore \sin^2 \theta = \frac{24}{25}$$

$$\therefore \sin \theta = \pm \frac{2\sqrt{6}}{5}$$

But θ is in quadrant 4

where $\sin \theta < 0$

$$\therefore \sin \theta = -\frac{2\sqrt{6}}{5}$$

$$\begin{aligned} \text{and so } \tan \theta &= \frac{\sin \theta}{\cos \theta} \\ &= \frac{-\frac{2\sqrt{6}}{5}}{\frac{1}{5}} \\ &= -2\sqrt{6} \end{aligned}$$

d $\cos^2 \theta + \sin^2 \theta = 1$

$$\therefore \frac{9}{16} + \sin^2 \theta = 1$$

$$\therefore \sin^2 \theta = \frac{7}{16}$$

$$\therefore \sin \theta = \pm \frac{\sqrt{7}}{4}$$

But θ is in quadrant 2

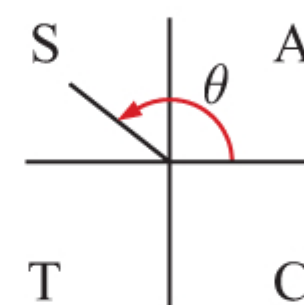
where $\sin \theta > 0$

$$\therefore \sin \theta = \frac{\sqrt{7}}{4}$$

$$\begin{aligned} \text{and so } \tan \theta &= \frac{\sin \theta}{\cos \theta} \\ &= \frac{\frac{\sqrt{7}}{4}}{-\frac{3}{4}} \\ &= -\frac{\sqrt{7}}{3} \end{aligned}$$

b $\tan \theta = \frac{\sin \theta}{\cos \theta} = -\frac{4}{3}$

$$\therefore \sin \theta = -\frac{4}{3} \cos \theta$$



Now $\cos^2 \theta + \sin^2 \theta = 1$

$$\therefore \cos^2 \theta + \frac{16}{9} \cos^2 \theta = 1$$

$$\therefore \frac{25}{9} \cos^2 \theta = 1$$

$$\therefore \cos^2 \theta = \frac{9}{25}$$

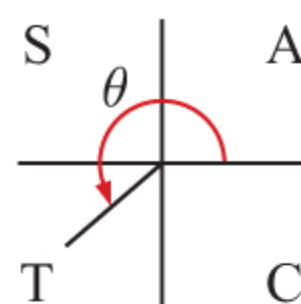
$$\therefore \cos \theta = \pm \frac{3}{5}$$

But θ is in quadrant 2 where $\cos \theta$ is negative and $\sin \theta$ is positive.

$$\therefore \cos \theta = -\frac{3}{5}, \quad \sin \theta = \frac{4}{5}$$

$$\text{c } \tan \theta = \frac{\sin \theta}{\cos \theta} = \frac{\sqrt{5}}{3}$$

$$\therefore \sin \theta = \frac{\sqrt{5}}{3} \cos \theta$$



$$\text{Now } \cos^2 \theta + \sin^2 \theta = 1$$

$$\therefore \cos^2 \theta + \frac{5}{9} \cos^2 \theta = 1$$

$$\therefore \frac{14}{9} \cos^2 \theta = 1$$

$$\therefore \cos^2 \theta = \frac{9}{14}$$

$$\therefore \cos \theta = \pm \frac{3}{\sqrt{14}}$$

But θ is in quadrant 3 where $\cos \theta$ and $\sin \theta$ are both negative.

$$\therefore \cos \theta = -\frac{3}{\sqrt{14}}, \quad \sin \theta = -\frac{\sqrt{5}}{\sqrt{14}}$$

6

$$\tan \theta = \frac{\sin \theta}{\cos \theta} = k$$

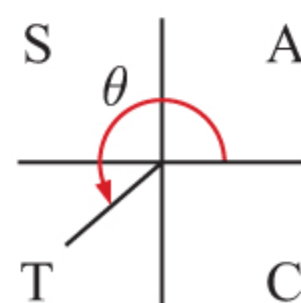
$$\therefore \sin \theta = k \cos \theta$$

$$\text{Now } \cos^2 \theta + \sin^2 \theta = 1$$

$$\therefore \cos^2 \theta + k^2 \cos^2 \theta = 1$$

$$\therefore (k^2 + 1) \cos^2 \theta = 1$$

$$\therefore \cos \theta = \frac{\pm 1}{\sqrt{k^2 + 1}}$$

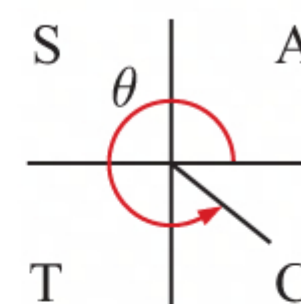


But θ is in quadrant 3 where $\cos \theta$ and $\sin \theta$ are both negative, and $\tan \theta$ is positive
 $\therefore k$ is positive.

$$\therefore \cos \theta = \frac{-1}{\sqrt{k^2 + 1}}, \quad \sin \theta = \frac{-k}{\sqrt{k^2 + 1}}$$

$$\text{d } \tan \theta = \frac{\sin \theta}{\cos \theta} = -\frac{12}{5}$$

$$\therefore \sin \theta = -\frac{12}{5} \cos \theta$$



$$\text{Now } \cos^2 \theta + \sin^2 \theta = 1$$

$$\therefore \cos^2 \theta + \frac{144}{25} \cos^2 \theta = 1$$

$$\therefore \frac{169}{25} \cos^2 \theta = 1$$

$$\therefore \cos^2 \theta = \frac{25}{169}$$

$$\therefore \cos \theta = \pm \frac{5}{13}$$

But θ is in quadrant 4 where $\cos \theta$ is positive and $\sin \theta$ is negative.

$$\therefore \cos \theta = \frac{5}{13}, \quad \sin \theta = -\frac{12}{13}$$

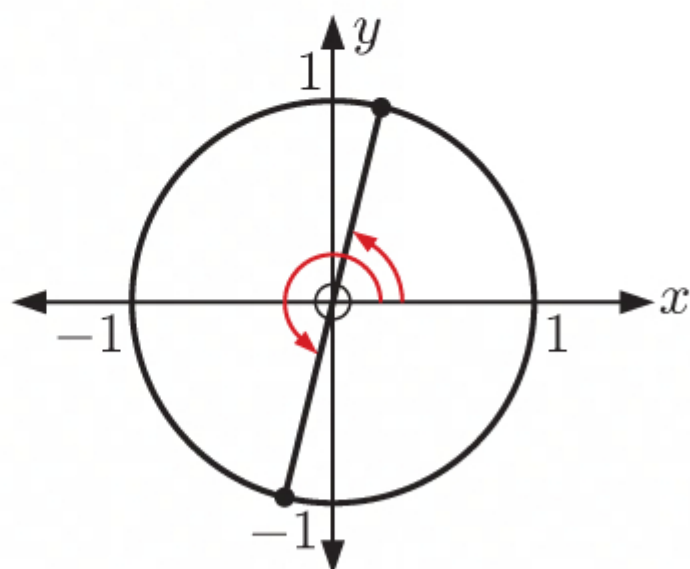
EXERCISE 7F

1

$$\text{a } \tan \theta = 4$$

Using technology,

$$\tan^{-1}(4) \approx 75.96^\circ$$



$$\therefore \theta \approx 75.96^\circ \text{ or } 180^\circ + 75.96^\circ$$

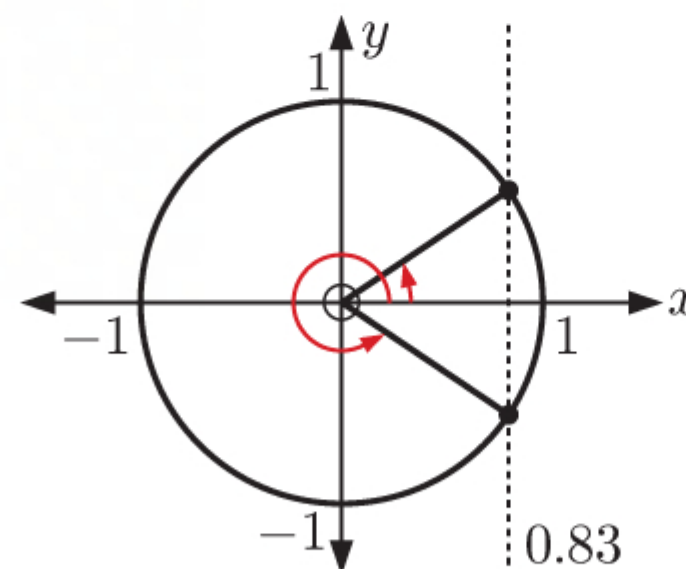
$$\therefore \theta \approx 76.0^\circ \text{ or } 256^\circ$$

b

$$\cos \theta = 0.83$$

Using technology,

$$\cos^{-1}(0.83) \approx 33.90^\circ$$



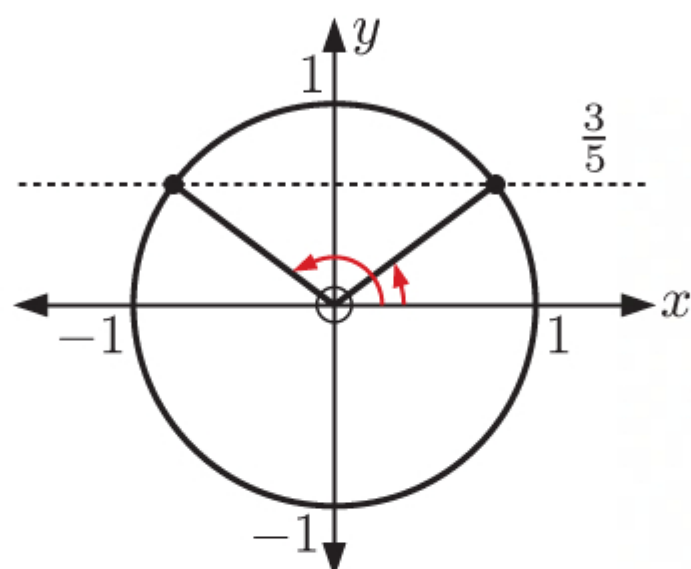
$$\therefore \theta \approx 33.90^\circ \text{ or } 360^\circ - 33.90^\circ$$

$$\therefore \theta \approx 33.9^\circ \text{ or } 326.1^\circ$$

c $\sin \theta = \frac{3}{5}$

Using technology,

$$\sin^{-1}\left(\frac{3}{5}\right) \approx 36.87^\circ$$



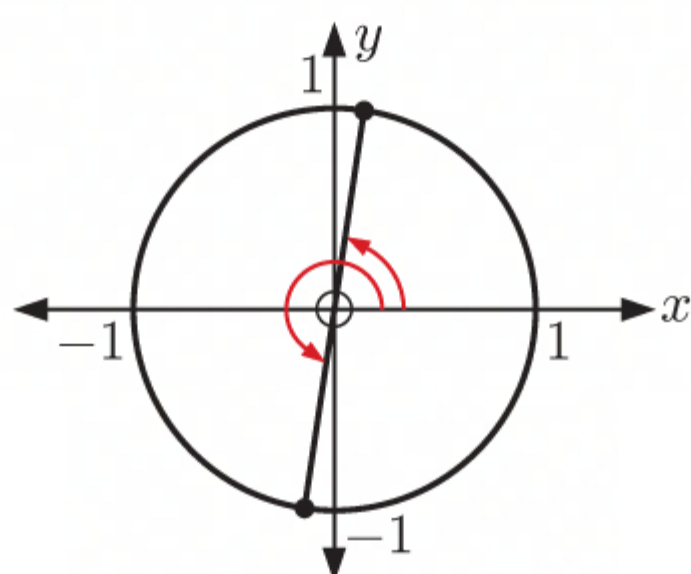
$$\therefore \theta \approx 36.87^\circ \text{ or } 180^\circ - 36.87^\circ$$

$$\therefore \theta \approx 36.9^\circ \text{ or } 143.1^\circ$$

e $\tan \theta = 6.67$

Using technology,

$$\tan^{-1}(6.67) \approx 81.47^\circ$$



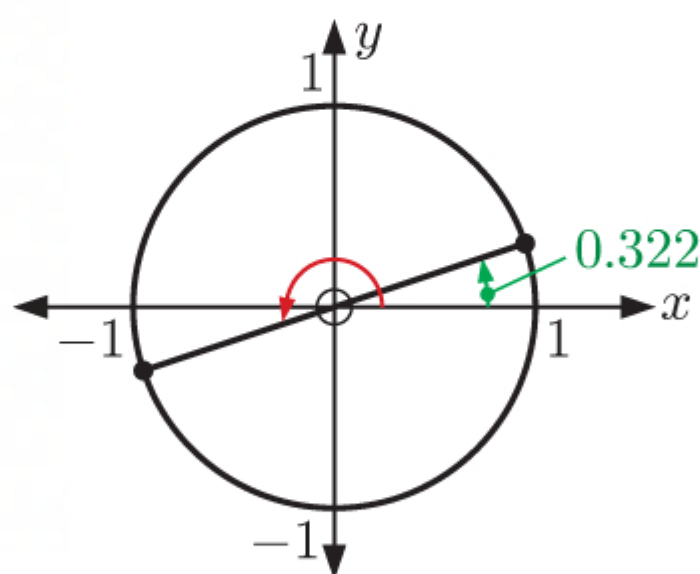
$$\therefore \theta \approx 81.47^\circ \text{ or } 180^\circ + 81.47^\circ$$

$$\therefore \theta \approx 81.5^\circ \text{ or } 261.5^\circ$$

2 a $\tan \theta = \frac{1}{3}$

Using technology,

$$\tan^{-1}\left(\frac{1}{3}\right) \approx 0.322$$



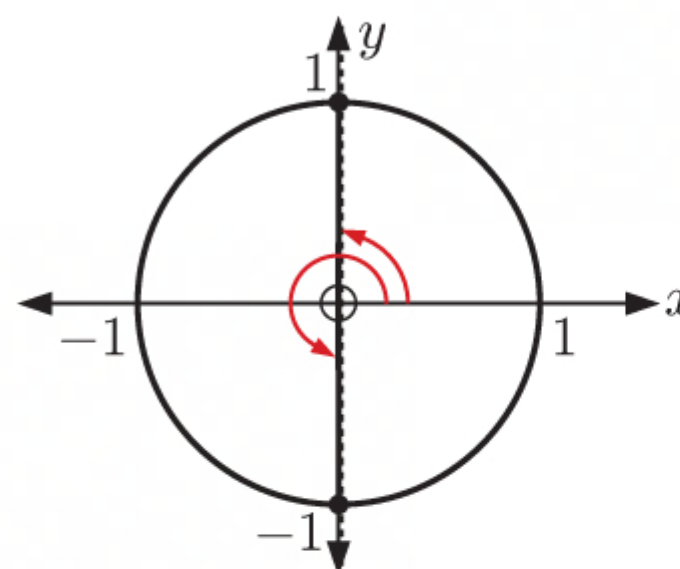
But $0 \leq \theta \leq 2\pi$

$$\therefore \theta \approx 0.322 \text{ or } \pi + 0.322$$

$$\therefore \theta \approx 0.322 \text{ or } 3.46$$

d $\cos \theta = 0$

$$\cos^{-1}(0) = 90^\circ$$



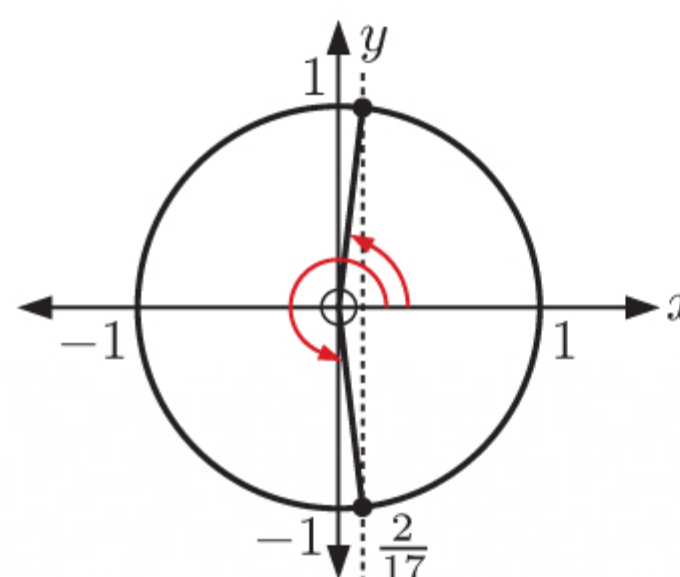
$$\therefore \theta = 90^\circ \text{ or } 360^\circ - 90^\circ$$

$$\therefore \theta = 90^\circ \text{ or } 270^\circ$$

f $\cos \theta = \frac{2}{17}$

Using technology,

$$\cos^{-1}\left(\frac{2}{17}\right) \approx 83.24^\circ$$



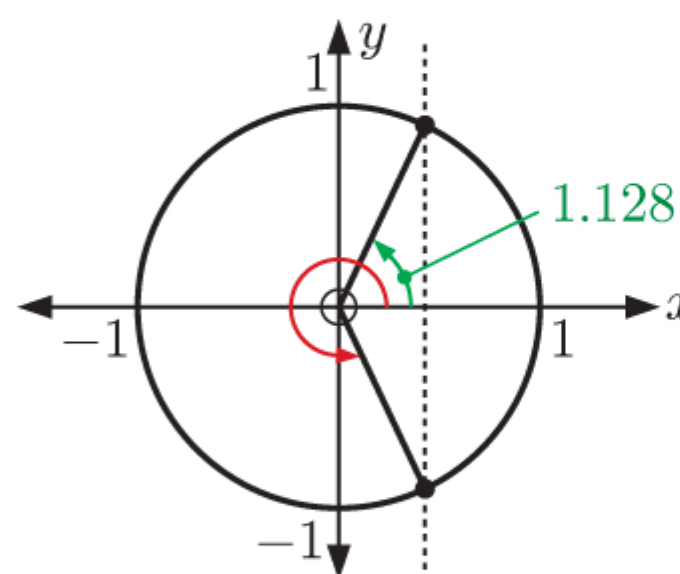
$$\therefore \theta \approx 83.24^\circ \text{ or } 360^\circ - 83.24^\circ$$

$$\therefore \theta \approx 83.2^\circ \text{ or } 276.8^\circ$$

b $\cos \theta = \frac{3}{7}$

Using technology,

$$\cos^{-1}\left(\frac{3}{7}\right) \approx 1.128$$



But $0 \leq \theta \leq 2\pi$

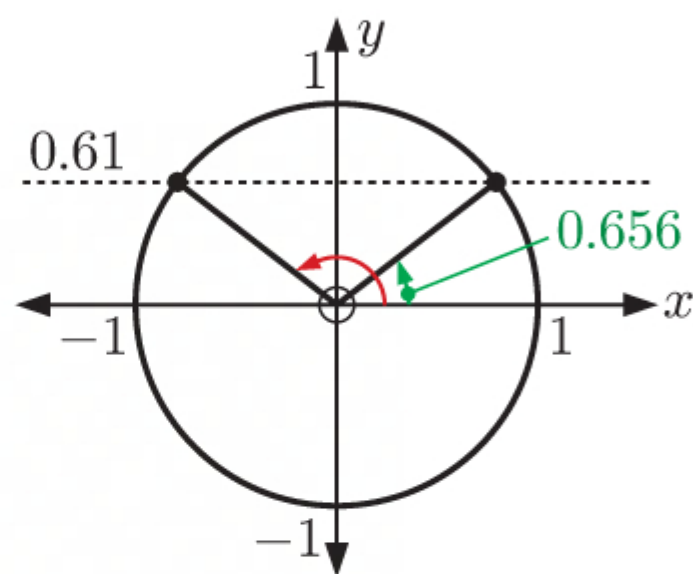
$$\therefore \theta \approx 1.128 \text{ or } 2\pi - 1.128$$

$$\therefore \theta \approx 1.13 \text{ or } 5.16$$

c $\sin \theta = 0.61$

Using technology,

$$\sin^{-1}(0.61) \approx 0.656$$



But $0 \leq \theta \leq 2\pi$

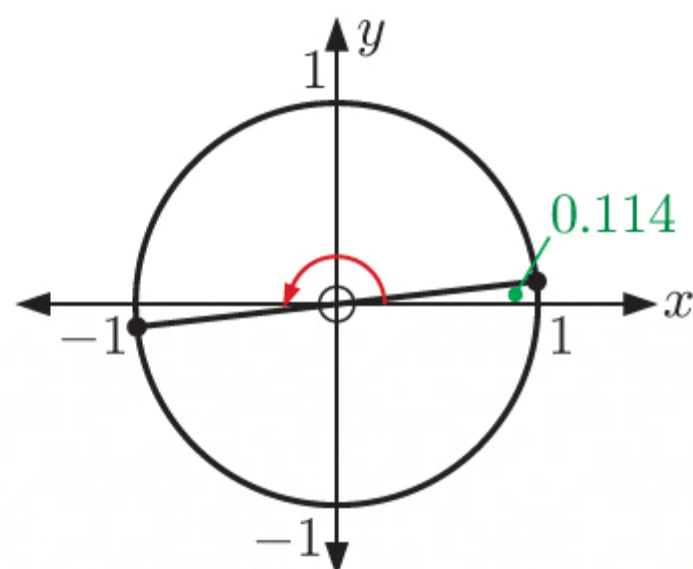
$$\therefore \theta \approx 0.656 \text{ or } \pi - 0.656$$

$$\therefore \theta \approx 0.656 \text{ or } 2.49$$

e $\tan \theta = 0.114$

Using technology,

$$\tan^{-1}(0.114) \approx 0.114$$



But $0 \leq \theta \leq 2\pi$

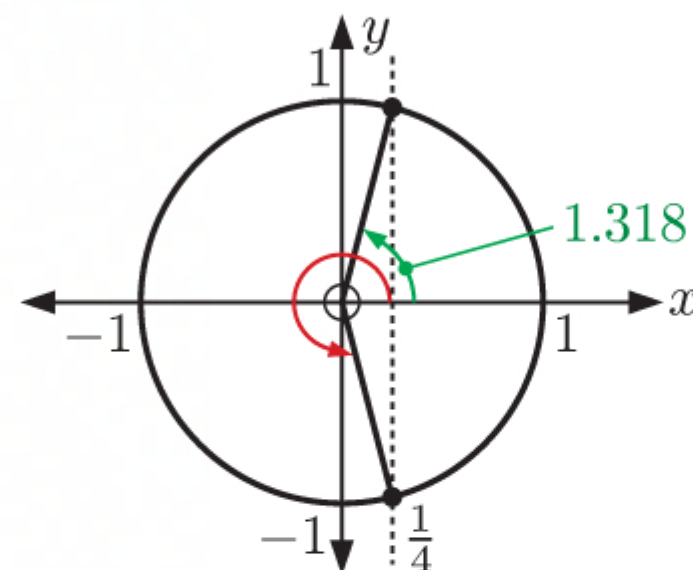
$$\therefore \theta \approx 0.114 \text{ or } \pi + 0.114$$

$$\therefore \theta \approx 0.114 \text{ or } 3.26$$

d $\cos \theta = \frac{1}{4}$

Using technology,

$$\cos^{-1}\left(\frac{1}{4}\right) \approx 1.318$$



But $0 \leq \theta \leq 2\pi$

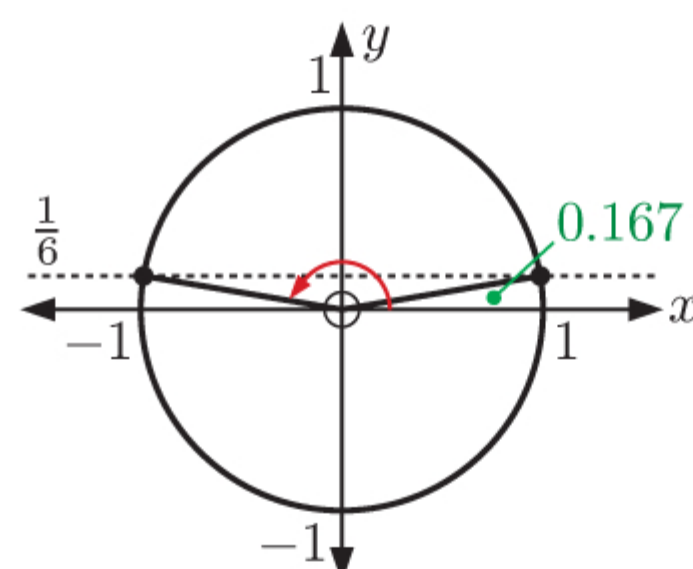
$$\therefore \theta \approx 1.318 \text{ or } 2\pi - 1.318$$

$$\therefore \theta \approx 1.32 \text{ or } 4.97$$

f $\sin \theta = \frac{1}{6}$

Using technology,

$$\sin^{-1}\left(\frac{1}{6}\right) \approx 0.167$$



But $0 \leq \theta \leq 2\pi$

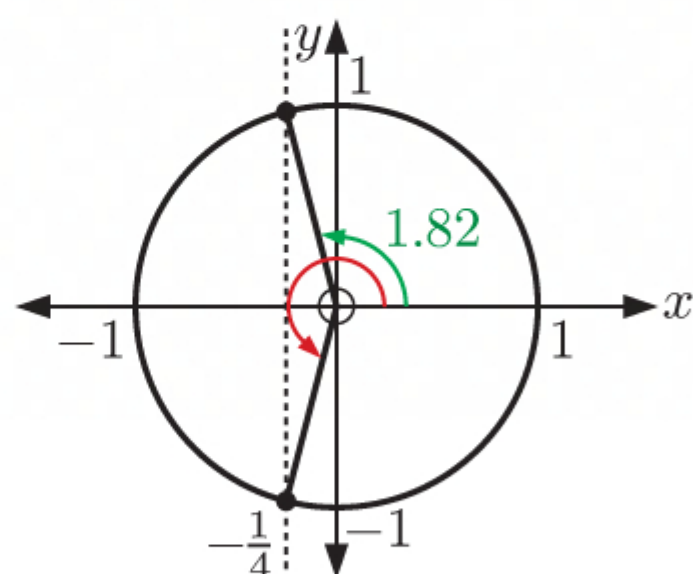
$$\therefore \theta \approx 0.167 \text{ or } \pi - 0.167$$

$$\therefore \theta \approx 0.167 \text{ or } 2.97$$

3 a $\cos \theta = -\frac{1}{4}$

Using technology,

$$\cos^{-1}\left(-\frac{1}{4}\right) \approx 1.82$$



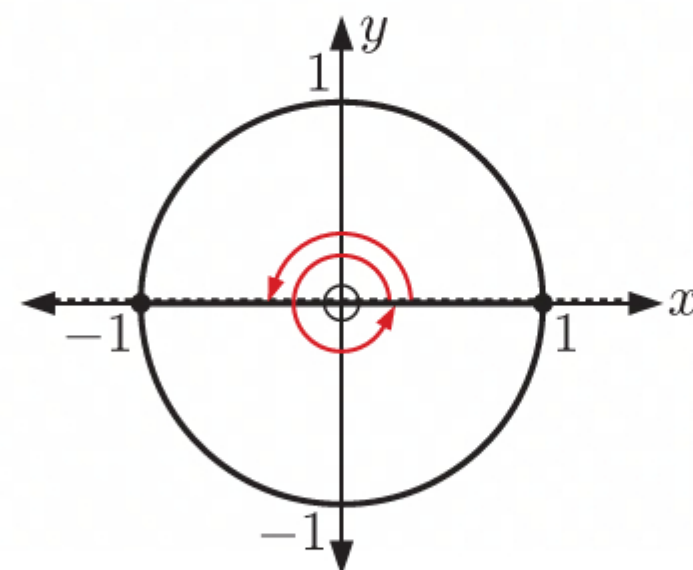
But $0 \leq \theta \leq 2\pi$

$$\therefore \theta \approx 1.82 \text{ or } 2\pi - 1.82$$

$$\therefore \theta \approx 1.82 \text{ or } 4.46$$

b $\sin \theta = 0$

$$\therefore \sin^{-1}(0) = 0$$



But $0 \leq \theta \leq 2\pi$

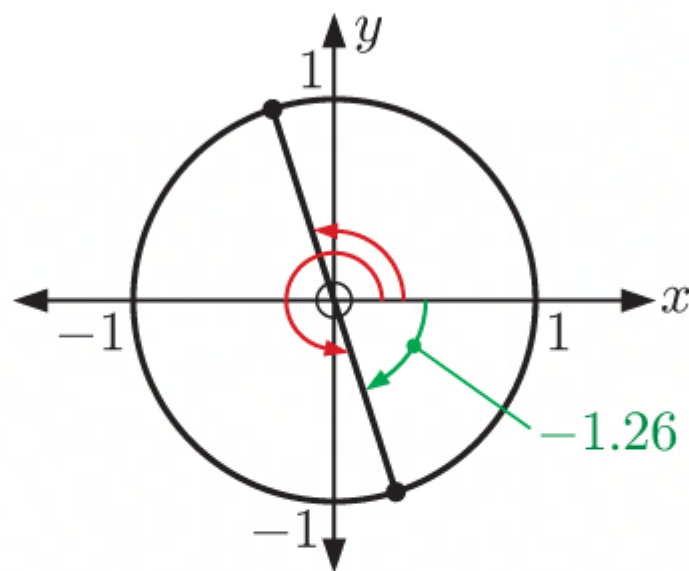
$$\therefore \theta = 0 \text{ or } \pi - 0 \text{ or } 2\pi$$

$$\therefore \theta = 0, \pi, \text{ or } 2\pi$$

c $\tan \theta = -3.1$

Using technology,

$$\tan^{-1}(-3.1) \approx -1.26$$



But $0 \leq \theta \leq 2\pi$

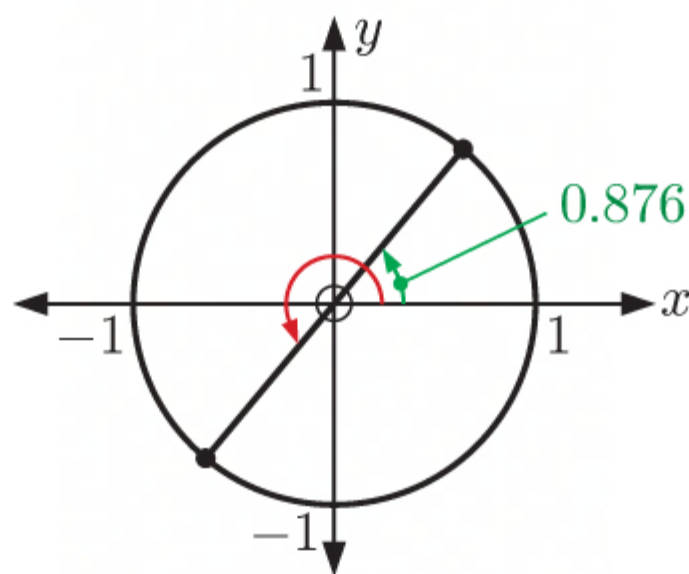
$$\therefore \theta \approx \pi - 1.26 \text{ or } 2\pi - 1.26$$

$$\therefore \theta \approx 1.88 \text{ or } 5.02$$

e $\tan \theta = 1.2$

Using technology,

$$\tan^{-1}(1.2) \approx 0.876$$



But $0 \leq \theta \leq 2\pi$

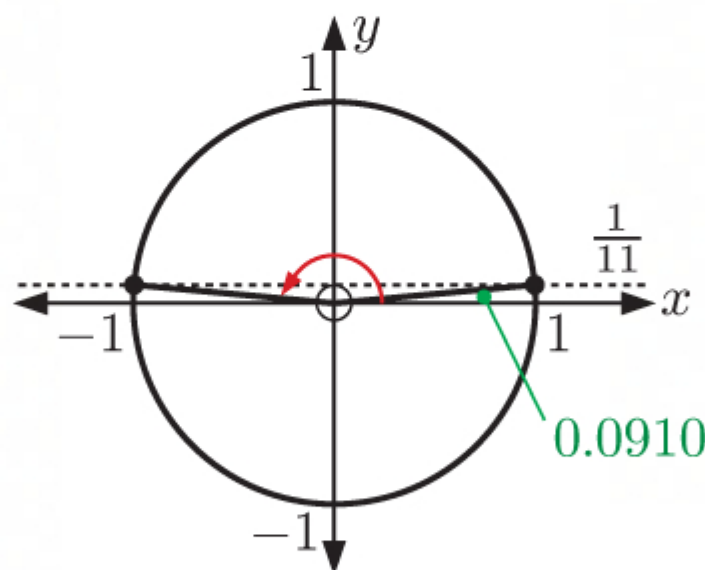
$$\therefore \theta \approx 0.876 \text{ or } \pi + 0.876$$

$$\therefore \theta \approx 0.876 \text{ or } 4.02$$

g $\sin \theta = \frac{1}{11}$

Using technology,

$$\sin^{-1}\left(\frac{1}{11}\right) \approx 0.0910$$



But $0 \leq \theta \leq 2\pi$

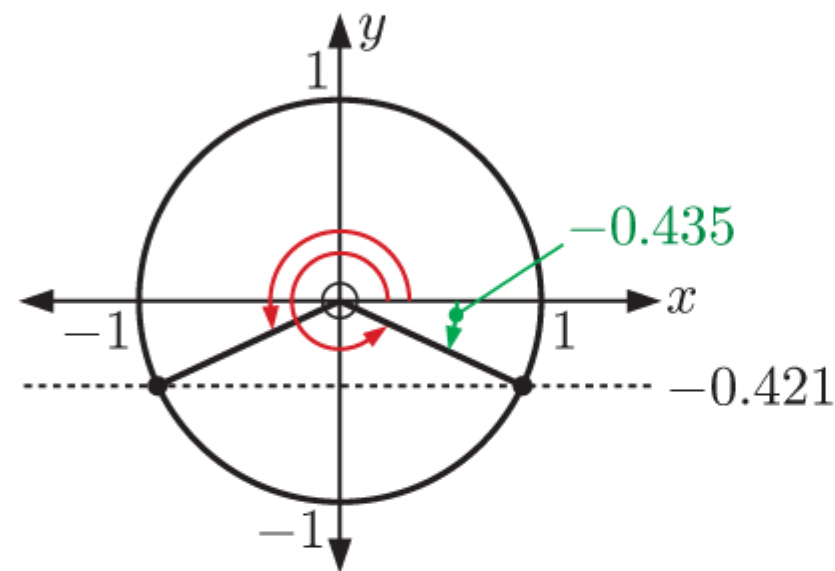
$$\therefore \theta \approx 0.0910 \text{ or } \pi - 0.0910$$

$$\therefore \theta \approx 0.0910 \text{ or } 3.05$$

d $\sin \theta = -0.421$

Using technology,

$$\sin^{-1}(-0.421) \approx -0.435$$



But $0 \leq \theta \leq 2\pi$

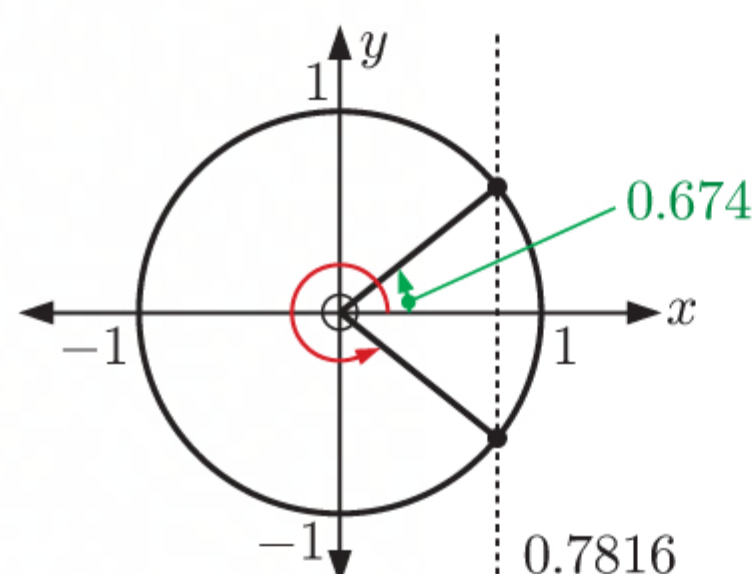
$$\therefore \theta \approx \pi + 0.435 \text{ or } 2\pi - 0.435$$

$$\therefore \theta \approx 3.58 \text{ or } 5.85$$

f $\cos \theta = 0.7816$

Using technology,

$$\cos^{-1}(0.7816) \approx 0.674$$



But $0 \leq \theta \leq 2\pi$

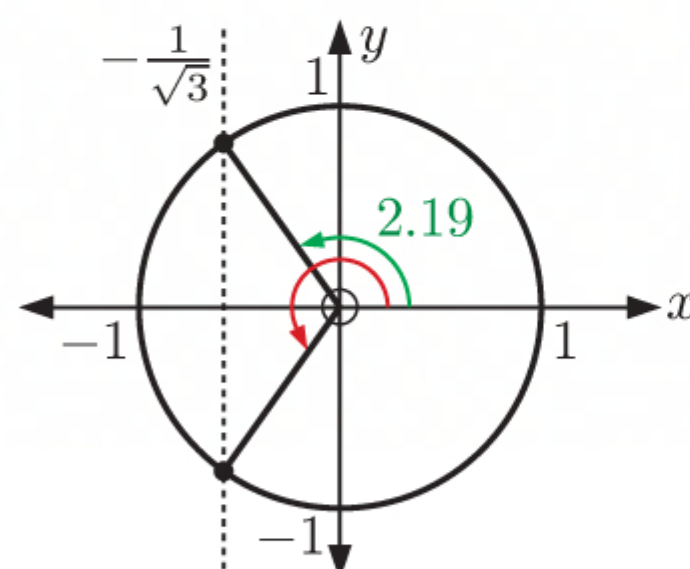
$$\therefore \theta \approx 0.674 \text{ or } 2\pi - 0.674$$

$$\therefore \theta \approx 0.674 \text{ or } 5.61$$

h $\cos \theta = -\frac{1}{\sqrt{3}}$

Using technology,

$$\cos^{-1}\left(-\frac{1}{\sqrt{3}}\right) \approx 2.19$$



But $0 \leq \theta \leq 2\pi$

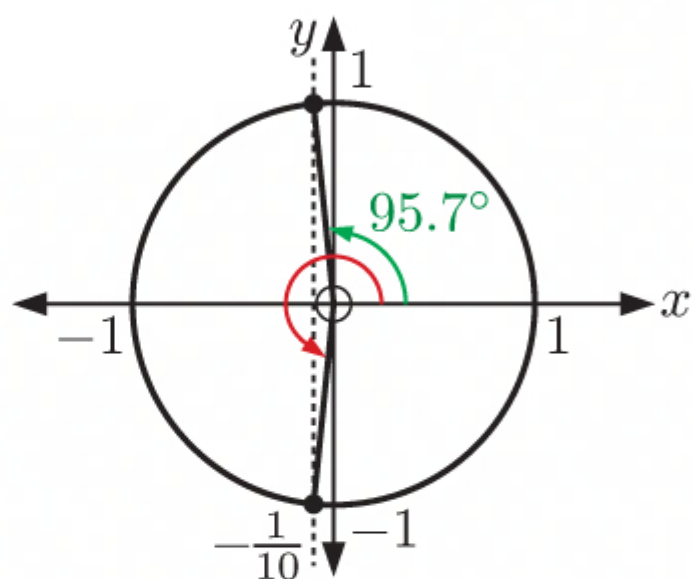
$$\therefore \theta \approx 2.19 \text{ or } 2\pi - 2.19$$

$$\therefore \theta \approx 2.19 \text{ or } 4.10$$

4 a $\cos \theta = -\frac{1}{10}$

Using technology,

$$\cos^{-1}\left(-\frac{1}{10}\right) \approx 95.7^\circ$$

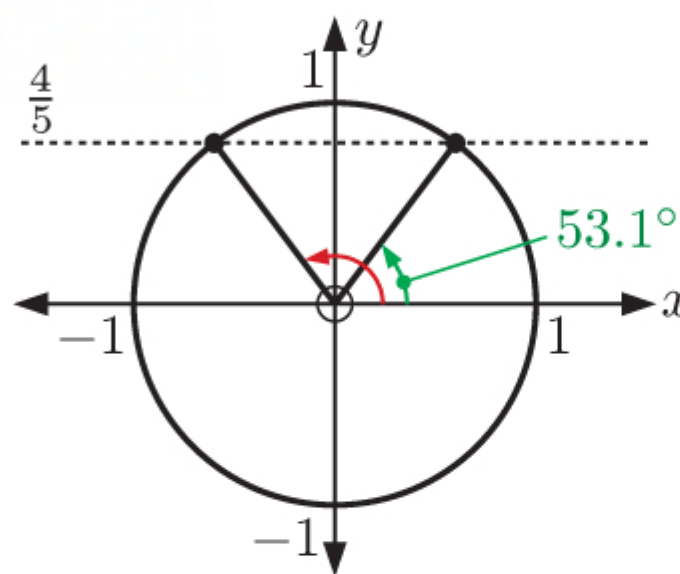


But $-180^\circ \leq \theta \leq 180^\circ$
 $\therefore \theta \approx -95.7^\circ$ or 95.7°

b $\sin \theta = \frac{4}{5}$

Using technology,

$$\sin^{-1}\left(\frac{4}{5}\right) \approx 53.1^\circ$$

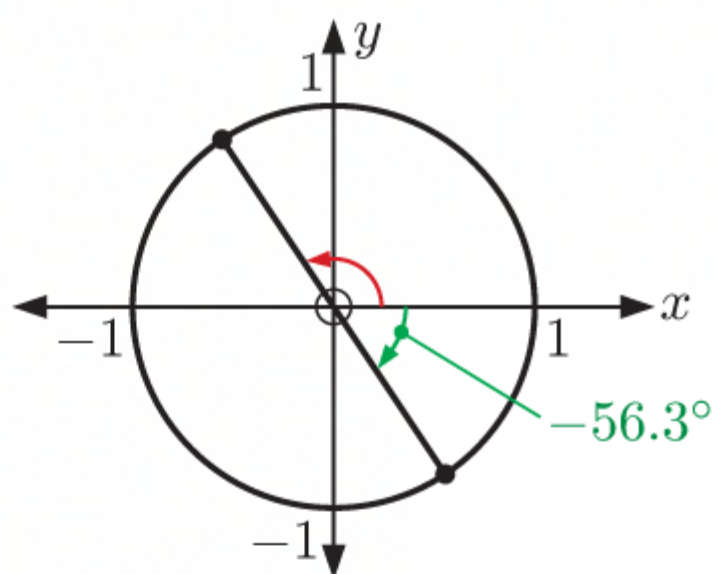


But $-180^\circ \leq \theta \leq 180^\circ$
 $\therefore \theta \approx 53.1^\circ$ or $180^\circ - 53.1^\circ$
 $\therefore \theta \approx 53.1^\circ$ or 126.9°

c $\tan \theta = -\frac{3}{2}$

Using technology,

$$\tan^{-1}\left(-\frac{3}{2}\right) \approx -56.3^\circ$$

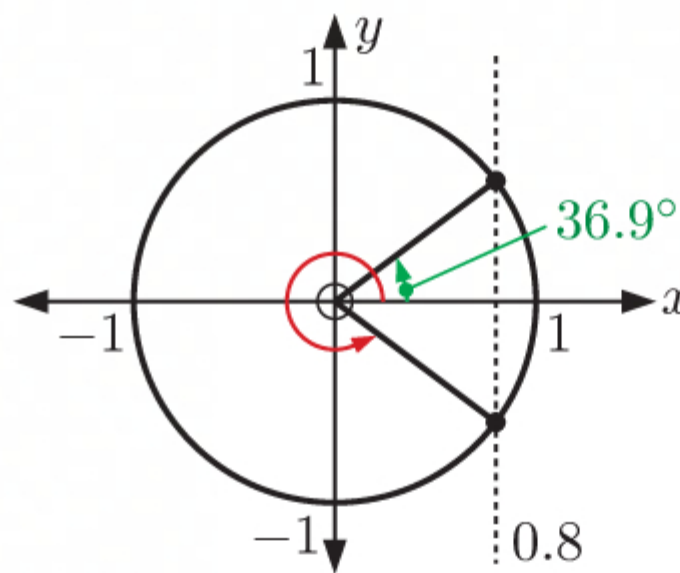


But $-180^\circ \leq \theta \leq 180^\circ$
 $\therefore \theta \approx -56.3^\circ$ or $180^\circ - 56.3^\circ$
 $\therefore \theta \approx -56.3^\circ$ or 123.7°

d $\cos \theta = 0.8$

Using technology,

$$\cos^{-1}(0.8) \approx 36.9^\circ$$

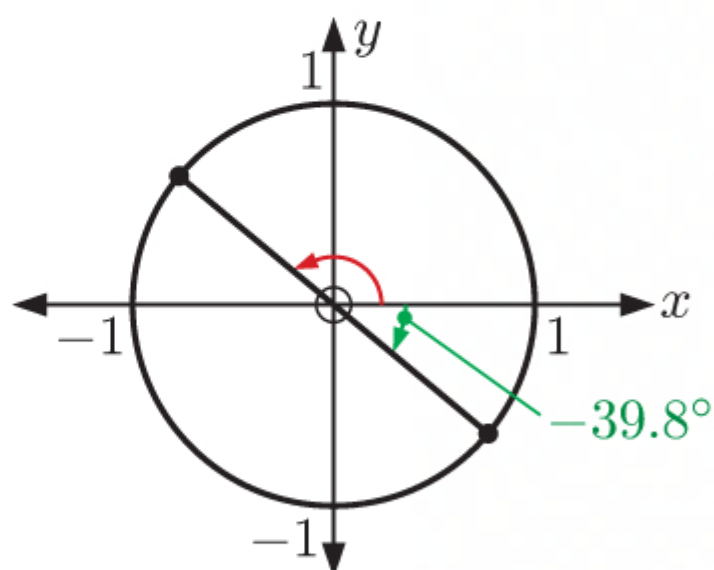


But $-180^\circ \leq \theta \leq 180^\circ$
 $\therefore \theta \approx -36.9^\circ$ or 36.9°

e $\tan \theta = -\frac{5}{6}$

Using technology,

$$\tan^{-1}\left(-\frac{5}{6}\right) \approx -39.8^\circ$$

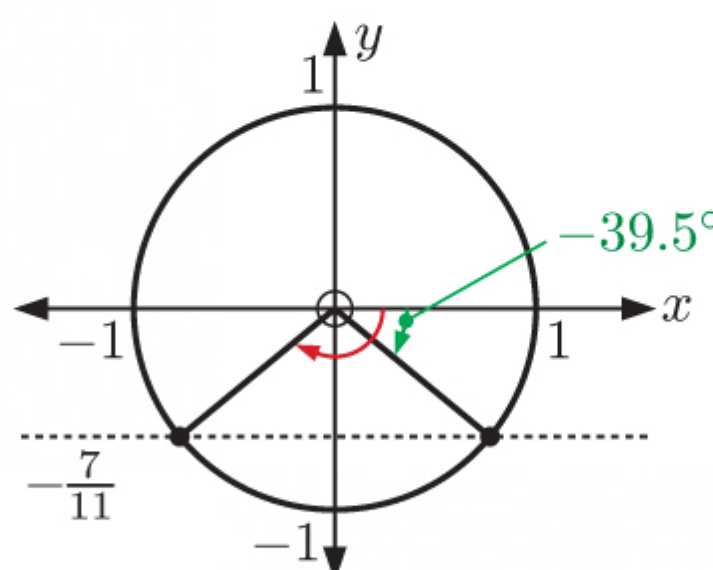


But $-180^\circ \leq \theta \leq 180^\circ$
 $\therefore \theta \approx -39.8^\circ$ or $180^\circ - 39.8^\circ$
 $\therefore \theta \approx -39.8^\circ$ or 140.2°

f $\sin \theta = -\frac{7}{11}$

Using technology,

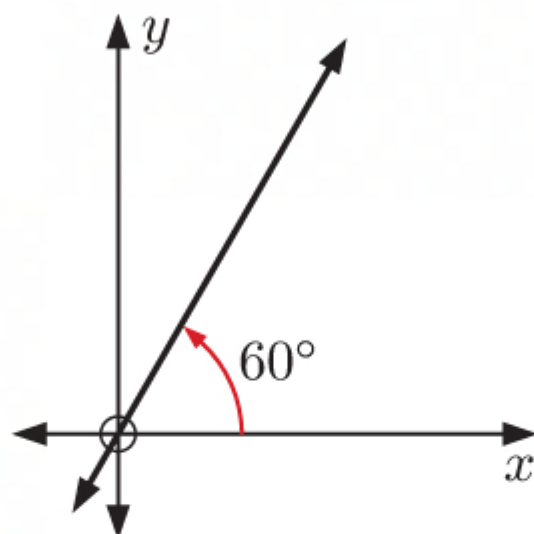
$$\sin^{-1}\left(-\frac{7}{11}\right) \approx -39.5^\circ$$



But $-180^\circ \leq \theta \leq 180^\circ$
 $\therefore \theta \approx 39.5^\circ - 180^\circ$ or -39.5°
 $\therefore \theta \approx -140.5^\circ$ or -39.5°

EXERCISE 7G

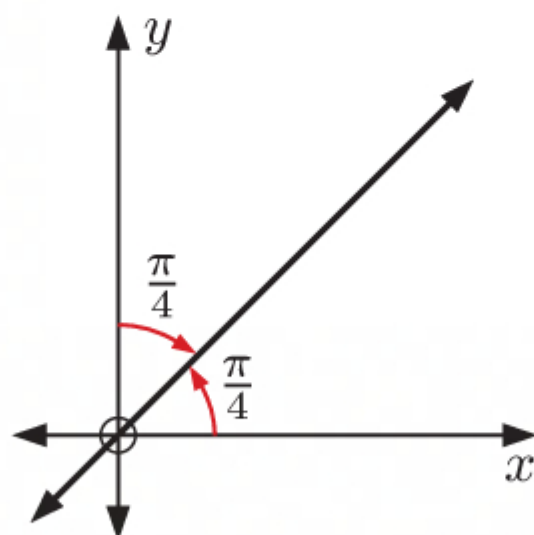
1 a



The line has gradient $m = \tan 60^\circ = \sqrt{3}$ and y -intercept 0.

\therefore the line has equation $y = \sqrt{3}x$.

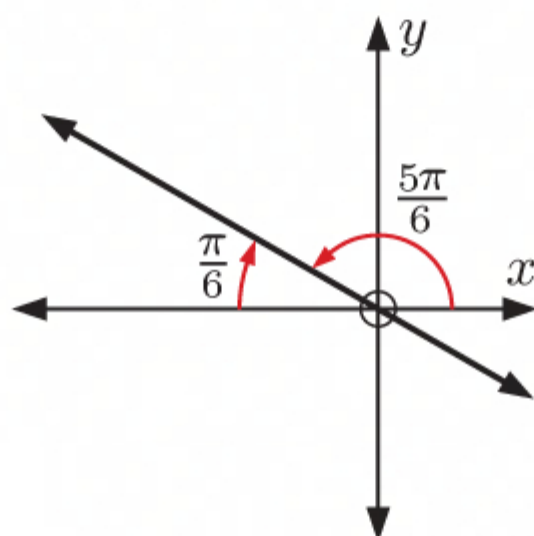
b



The line has gradient $m = \tan \frac{\pi}{4} = 1$ and y -intercept 0.

\therefore the line has equation $y = x$.

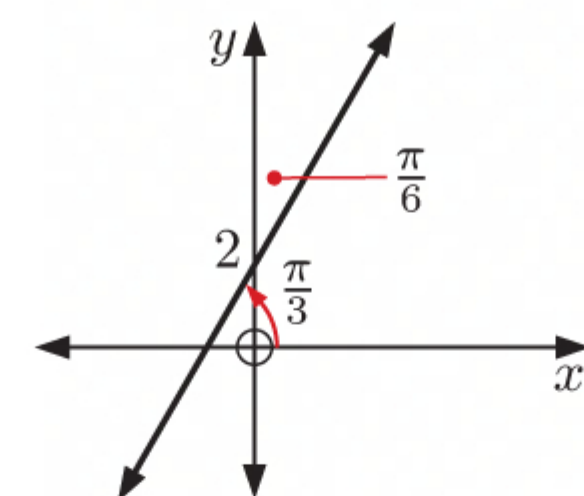
c



The line has gradient $m = \tan \frac{5\pi}{6} = -\frac{1}{\sqrt{3}}$ and y -intercept 0.

\therefore the line has equation $y = -\frac{1}{\sqrt{3}}x$.

2 a

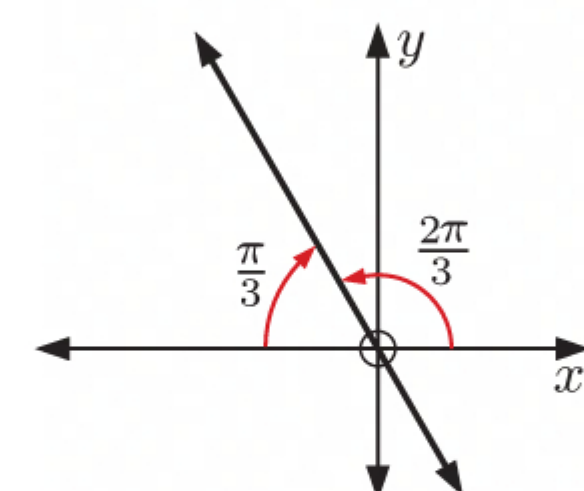


The line makes an angle of $\pi - \frac{\pi}{2} - \frac{\pi}{6} = \frac{\pi}{3}$ with the positive x -axis.

\therefore the line has gradient $m = \tan \frac{\pi}{3} = \sqrt{3}$ and y -intercept 2.

\therefore the line has equation $y = \sqrt{3}x + 2$.

b

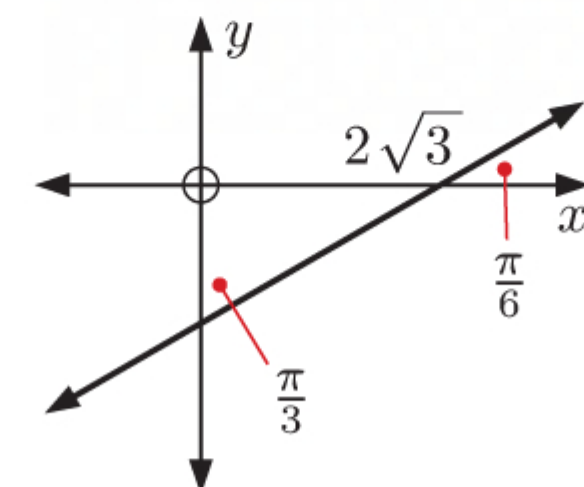


The line makes an angle of $\pi - \frac{\pi}{3} = \frac{2\pi}{3}$ with the positive x -axis.

\therefore the line has gradient $m = \tan \frac{2\pi}{3} = -\sqrt{3}$ and y -intercept 0.

\therefore the line has equation $y = -\sqrt{3}x$.

c



The line makes an angle of $\pi - \frac{\pi}{2} - \frac{\pi}{3} = \frac{\pi}{6}$ with the positive x -axis.

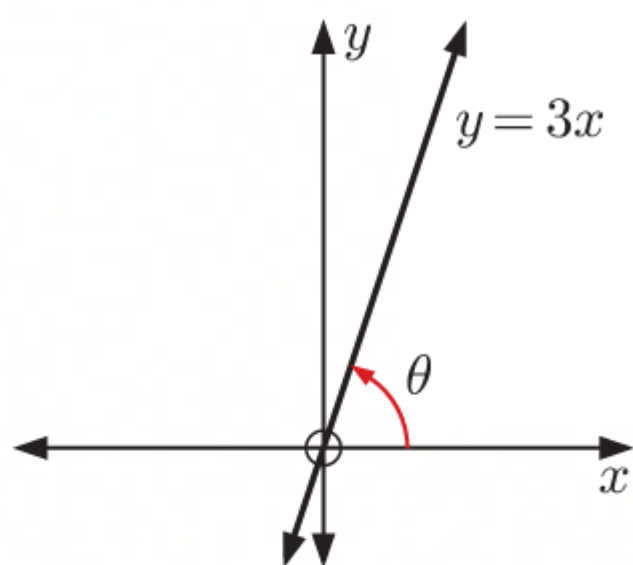
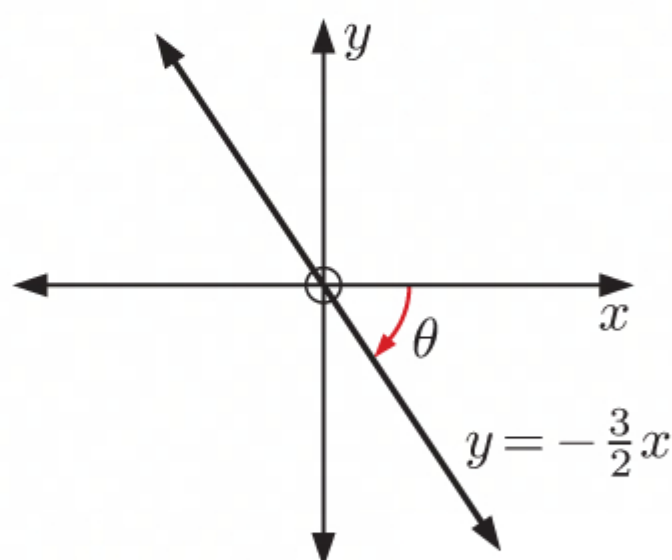
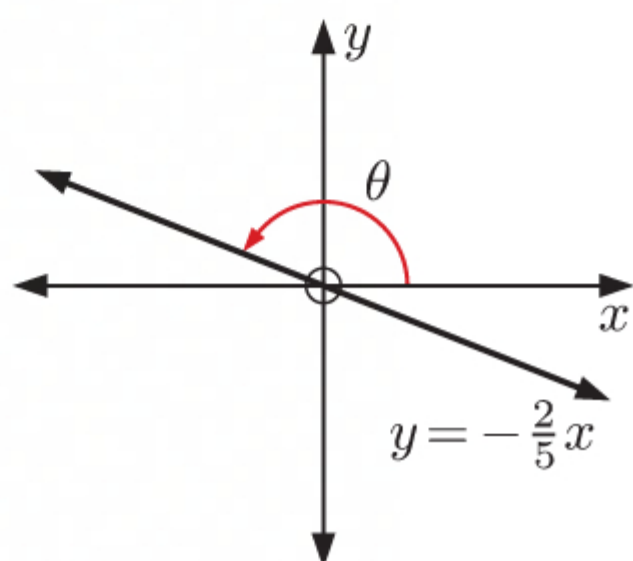
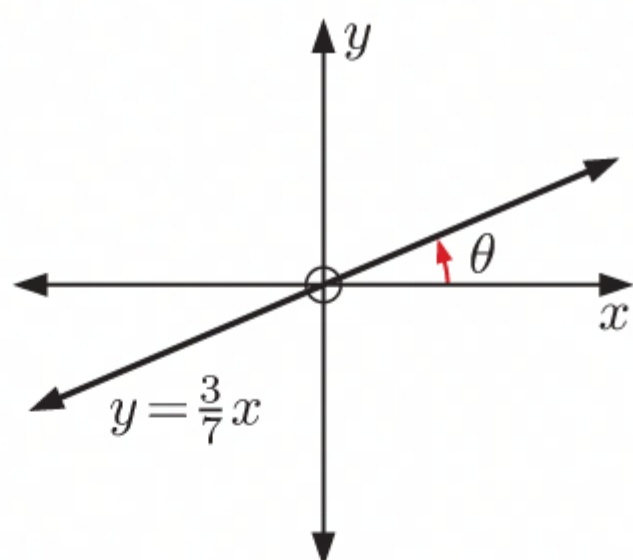
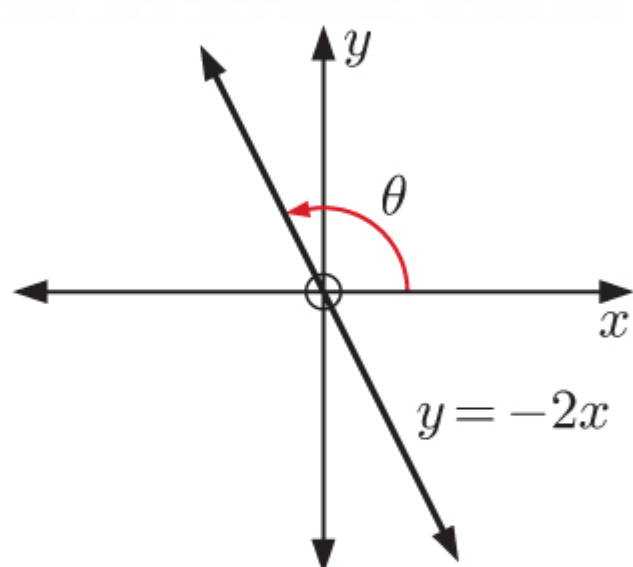
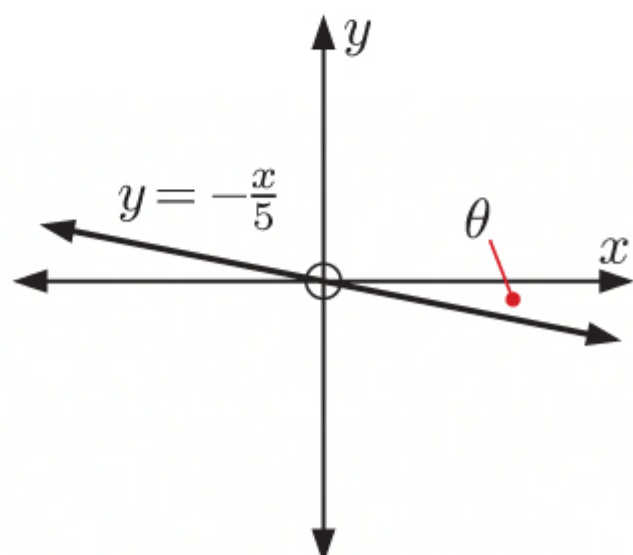
\therefore the line has gradient $m = \tan \frac{\pi}{6} = \frac{1}{\sqrt{3}}$.

\therefore the line has equation $y = \frac{1}{\sqrt{3}}x + c$.

When $x = 2\sqrt{3}$, $y = 0$, $\therefore 0 = \frac{1}{\sqrt{3}}(2\sqrt{3}) + c$

$$\therefore 0 = 2 + c$$

$$\therefore c = -2$$

3 aThe line has gradient 3, so $\tan \theta = 3$.Using technology, $\tan^{-1}(3) \approx 1.25$ $\therefore \theta \approx 1.25$ **b**The line has gradient $-\frac{3}{2}$, so $\tan \theta = -\frac{3}{2}$.Using technology, $\tan^{-1}\left(-\frac{3}{2}\right) \approx -0.983$ $\therefore \theta \approx -0.983$ **c**The line has gradient $-\frac{2}{5}$, so $\tan \theta = -\frac{2}{5}$.Using technology, $\tan^{-1}\left(-\frac{2}{5}\right) \approx -0.381$ But $0 < \theta < \pi$, so $\theta \approx \pi - 0.381 \approx 2.76$ **4 a**The line has gradient $\frac{3}{7}$, so $\tan \theta = \frac{3}{7}$.Using technology, $\tan^{-1}\left(\frac{3}{7}\right) \approx 23.2^\circ$ $\therefore \theta \approx 23.2^\circ$ **b**The line has gradient -2 , so $\tan \theta = -2$.Using technology, $\tan^{-1}(-2) \approx -63.4^\circ$ But $0^\circ < \theta < 180^\circ$, so $\theta \approx 180^\circ - 63.4^\circ \approx 117^\circ$ **c**The line has gradient $-\frac{1}{5}$, so $\tan \theta = -\frac{1}{5}$.Using technology, $\tan^{-1}\left(-\frac{1}{5}\right) \approx -11.3^\circ$ $\therefore \theta \approx -11.3^\circ$

REVIEW SET 7A

$$1 \quad a \quad 120^\circ = \left(120 \times \frac{\pi}{180}\right) \text{ radians} \\ = \frac{2\pi}{3} \text{ radians}$$

$$c \quad 150^\circ = \left(150 \times \frac{\pi}{180}\right) \text{ radians} \\ = \frac{5\pi}{6} \text{ radians}$$

$$2 \quad a \quad \frac{2\pi}{5} = \left(\frac{2\pi}{5} \times \frac{180}{\pi}\right)^\circ \\ = 72^\circ$$

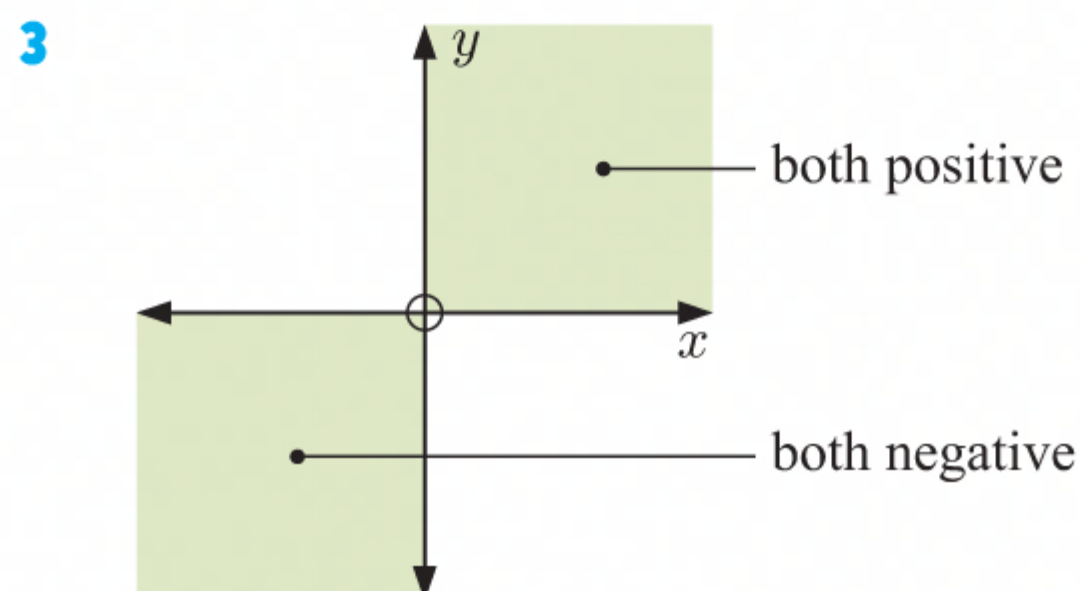
$$c \quad \frac{7\pi}{9} = \left(\frac{7\pi}{9} \times \frac{180}{\pi}\right)^\circ \\ = 140^\circ$$

$$b \quad 225^\circ = \left(225 \times \frac{\pi}{180}\right) \text{ radians} \\ = \frac{5\pi}{4} \text{ radians}$$

$$d \quad 540^\circ = \left(540 \times \frac{\pi}{180}\right) \text{ radians} \\ = 3\pi \text{ radians}$$

$$b \quad \frac{5\pi}{4} = \left(\frac{5\pi}{4} \times \frac{180}{\pi}\right)^\circ \\ = 225^\circ$$

$$d \quad \frac{11\pi}{6} = \left(\frac{11\pi}{6} \times \frac{180}{\pi}\right)^\circ \\ = 330^\circ$$



$$4 \quad a \quad \text{The point is } (\cos 320^\circ, \sin 320^\circ) \approx (0.766, -0.643).$$

$$b \quad \text{The point is } (\cos 163^\circ, \sin 163^\circ) \approx (-0.956, 0.292).$$

$$c \quad \text{The point is } (\cos 0.68^c, \sin 0.68^c) \approx (0.778, 0.629).$$

$$5 \quad \text{arc length} = \theta r \\ = 1.5 \times 8 \\ = 12 \text{ cm}$$

$$6 \quad a \quad \sin \frac{2\pi}{3} = \sin\left(\pi - \frac{2\pi}{3}\right) \\ = \sin \frac{\pi}{3} \\ \therefore \theta = \frac{\pi}{3}$$

$$c \quad \cos 276^\circ = \cos(360 - 276)^\circ \\ = \cos 84^\circ \\ \therefore \theta = 84^\circ$$

$$b \quad \sin 165^\circ = \sin(180 - 165)^\circ \\ = \sin 15^\circ \\ \therefore \theta = 15^\circ$$

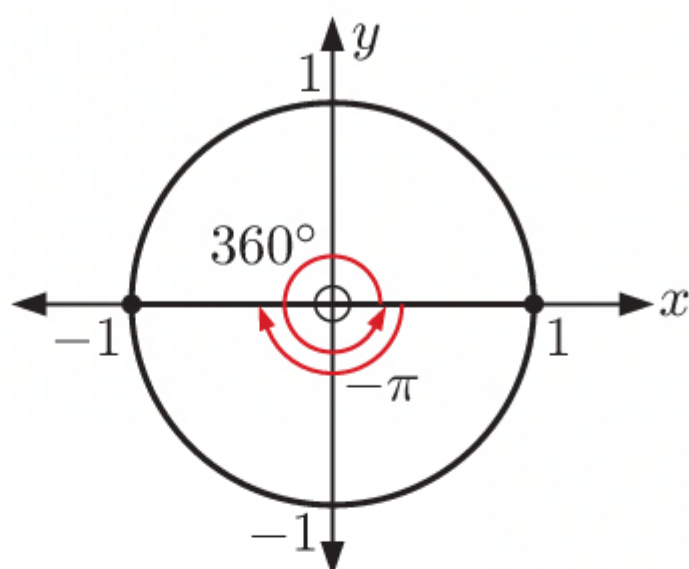
$$7 \quad a \quad \sin 159^\circ = \sin(180 - 159)^\circ \\ = \sin 21^\circ \\ \approx 0.358$$

$$c \quad \cos 75^\circ = -\cos(180 - 75)^\circ \\ = -\cos 105^\circ \\ \approx 0.259$$

$$b \quad \cos 92^\circ = -\cos(180 - 92)^\circ \\ = -\cos 88^\circ \\ \approx -0.035$$

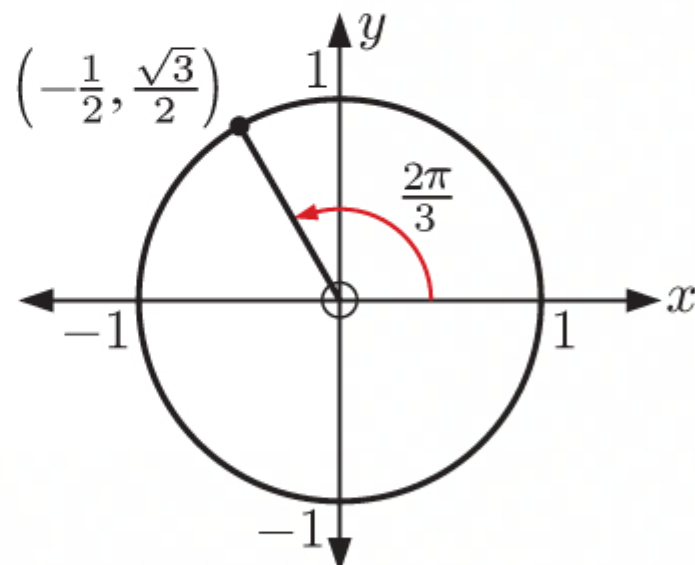
$$d \quad \tan(-133^\circ) = \tan(180 - 133)^\circ \\ = \tan 47^\circ \\ \approx 1.072$$

8



a $\cos 360^\circ = 1, \sin 360^\circ = 0$

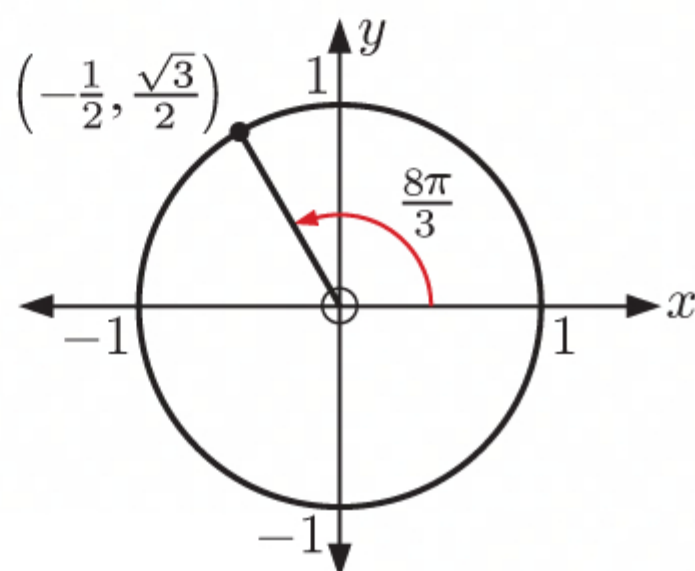
b $\cos(-\pi) = -1, \sin(-\pi) = 0$

9 **a**

$$\sin \frac{2\pi}{3} = \frac{\sqrt{3}}{2}$$

$$\cos \frac{2\pi}{3} = -\frac{1}{2}$$

$$\begin{aligned} \tan \frac{2\pi}{3} &= \frac{\frac{\sqrt{3}}{2}}{-\frac{1}{2}} \\ &= -\sqrt{3} \end{aligned}$$

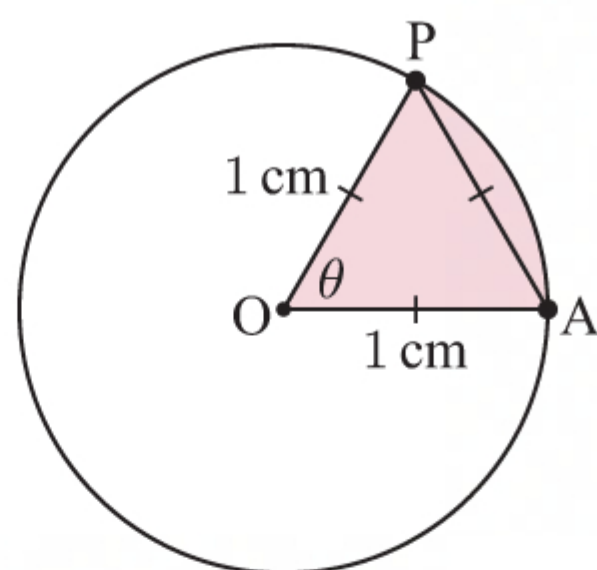
b

$$\sin \frac{8\pi}{3} = \frac{\sqrt{3}}{2}$$

$$\cos \frac{8\pi}{3} = -\frac{1}{2}$$

$$\begin{aligned} \tan \frac{8\pi}{3} &= \frac{\frac{\sqrt{3}}{2}}{-\frac{1}{2}} \\ &= -\sqrt{3} \end{aligned}$$

10



a i $\theta = 60^\circ$ {equilateral triangle}

ii $\theta = \frac{\pi}{3}$ radians

b arc length $AP = \theta r$

$$= \frac{\pi}{3} \times 1$$

$$= \frac{\pi}{3} \text{ cm}$$

c sector area $= \frac{1}{2}\theta r^2$

$$= \frac{1}{2} \times \frac{\pi}{3} \times 1^2$$

$$= \frac{\pi}{6} \text{ cm}^2$$

11 $\cos^2 x + \sin^2 x = 1$

$$\therefore \cos^2 x + \frac{1}{16} = 1$$

$$\therefore \cos^2 x = \frac{15}{16}$$

$$\therefore \cos x = \pm \frac{\sqrt{15}}{4}$$

But x is in quadrant 3 where $\cos x < 0$

$$\therefore \cos x = -\frac{\sqrt{15}}{4}$$

and so $\tan x = \frac{\sin x}{\cos x} = \frac{-\frac{1}{4}}{-\frac{\sqrt{15}}{4}} = \frac{1}{\sqrt{15}}$

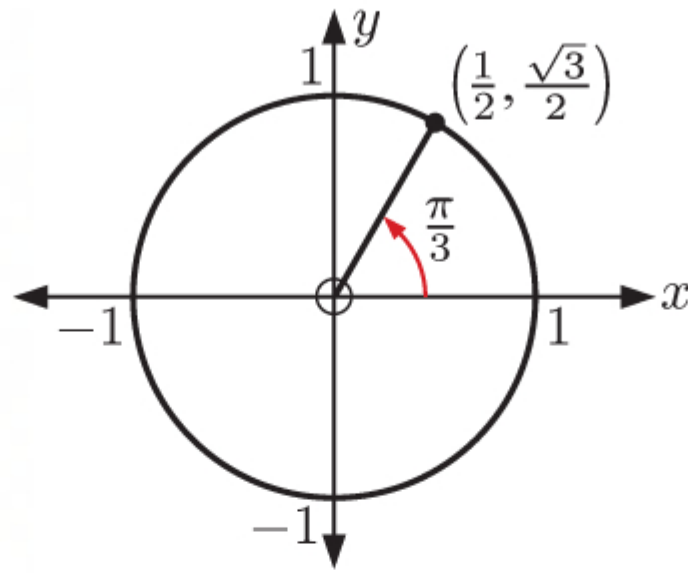
12 $\cos^2 \theta + \sin^2 \theta = 1$

$$\therefore \frac{9}{16} + \sin^2 \theta = 1$$

$$\therefore \sin^2 \theta = \frac{7}{16}$$

$$\therefore \sin \theta = \pm \frac{\sqrt{7}}{4}$$

13 a

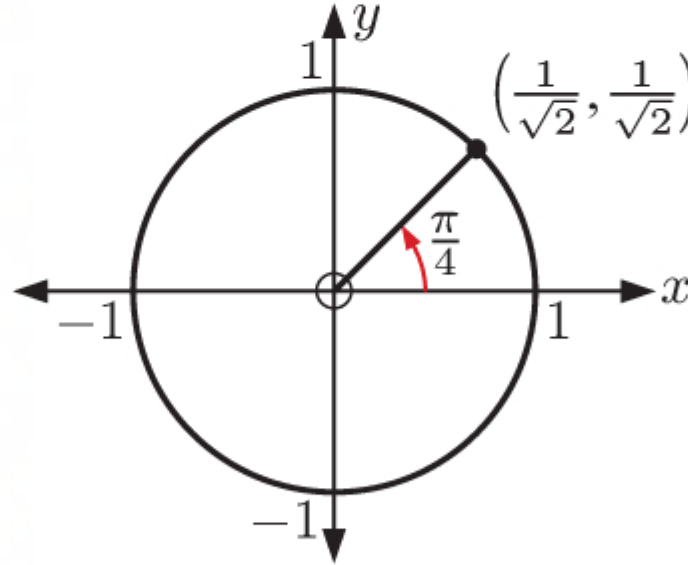


$$\sin \frac{\pi}{3} = \frac{\sqrt{3}}{2} \quad \text{and} \quad \cos \frac{\pi}{3} = \frac{1}{2}$$

$$\therefore 2 \sin \frac{\pi}{3} \cos \frac{\pi}{3} = 2 \times \frac{\sqrt{3}}{2} \times \frac{1}{2}$$

$$= \frac{\sqrt{3}}{2}$$

b

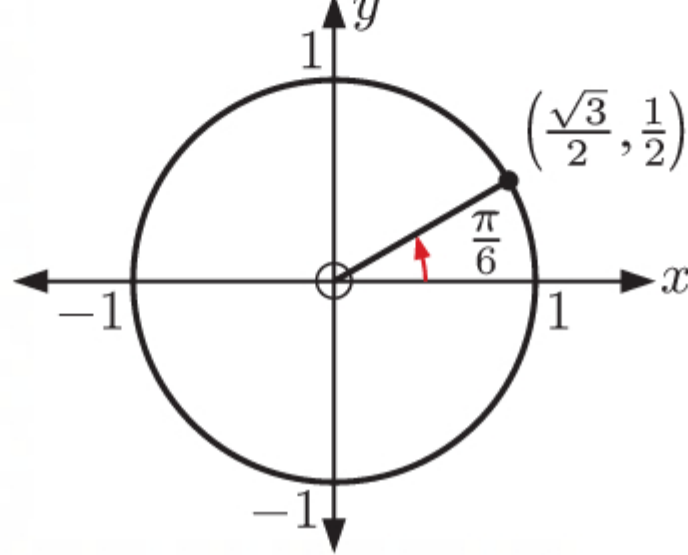


$$\tan \frac{\pi}{4} = \frac{\frac{1}{\sqrt{2}}}{\frac{1}{\sqrt{2}}} = 1$$

$$\therefore \tan^2\left(\frac{\pi}{4}\right) - 1 = 1^2 - 1$$

$$= 0$$

c



$$\cos \frac{\pi}{6} = \frac{\sqrt{3}}{2} \quad \text{and} \quad \sin \frac{\pi}{6} = \frac{1}{2}$$

$$\therefore \cos^2\left(\frac{\pi}{6}\right) - \sin^2\left(\frac{\pi}{6}\right) = \left(\frac{\sqrt{3}}{2}\right)^2 - \left(\frac{1}{2}\right)^2$$

$$= \frac{3}{4} - \frac{1}{4}$$

$$= \frac{1}{2}$$

14 $\tan x = \frac{\sin x}{\cos x} = -\frac{3}{2}$

$$\therefore \sin x = -\frac{3}{2} \cos x$$

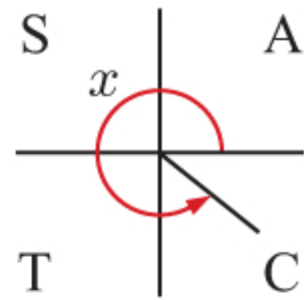
Now $\cos^2 x + \sin^2 x = 1$

$$\therefore \cos^2 x + \frac{9}{4} \cos^2 x = 1$$

$$\therefore \frac{13}{4} \cos^2 x = 1$$

$$\therefore \cos^2 x = \frac{4}{13}$$

$$\therefore \cos x = \pm \frac{2}{\sqrt{13}}$$


 But x is in quadrant 4, so $\cos x$ is positive and $\sin x$ is negative.

$$\therefore \cos x = \frac{2}{\sqrt{13}}, \quad \sin x = -\frac{3}{\sqrt{13}}$$

a $\cos x = \frac{2}{\sqrt{13}}$

b $\sin x = -\frac{3}{\sqrt{13}}$

15 Perimeter of sector $= 2r + \theta r$

$$= 2 \times 4 + 1 \times 4$$

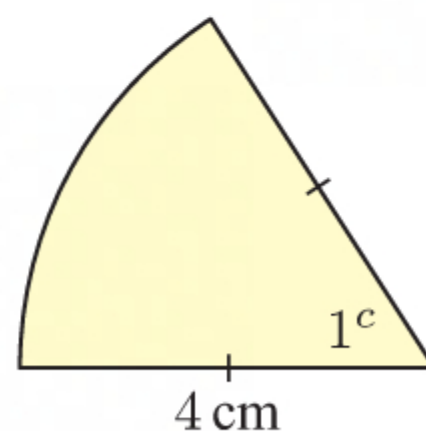
$$= 8 + 4$$

$$= 12 \text{ cm}$$

Area of sector $= \frac{1}{2} \theta r^2$

$$= \frac{1}{2} \times 1 \times 4^2$$

$$= 8 \text{ cm}^2$$



16

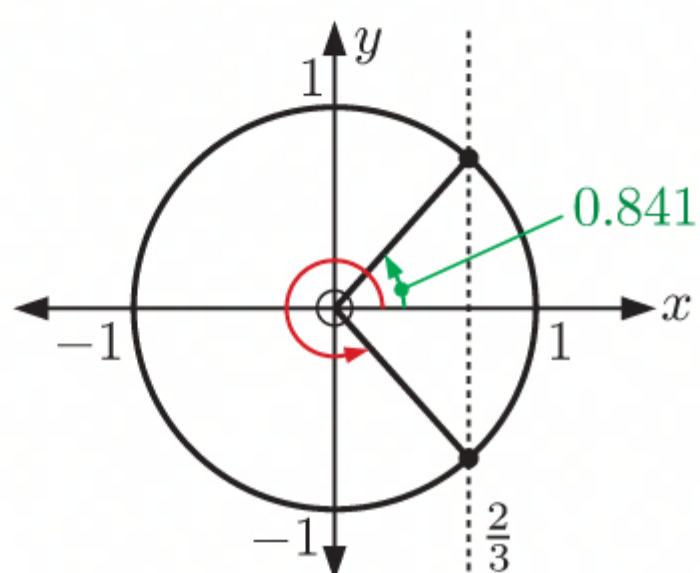
$$\begin{aligned}\cos^2 \theta + \sin^2 \theta &= 1 \\ \therefore \left(\frac{\sqrt{11}}{\sqrt{17}}\right)^2 + \sin^2 \theta &= 1 \\ \therefore \sin^2 \theta &= \frac{6}{17} \\ \therefore \sin \theta &= \pm \frac{\sqrt{6}}{\sqrt{17}}\end{aligned}$$

But θ is acute, $\therefore \sin \theta = \frac{\sqrt{6}}{\sqrt{17}}$

$$\tan \theta = \frac{\frac{\sqrt{6}}{\sqrt{17}}}{\frac{\sqrt{11}}{\sqrt{17}}} = \frac{\sqrt{6}}{\sqrt{11}}$$

17 a $\cos \theta = \frac{2}{3}$

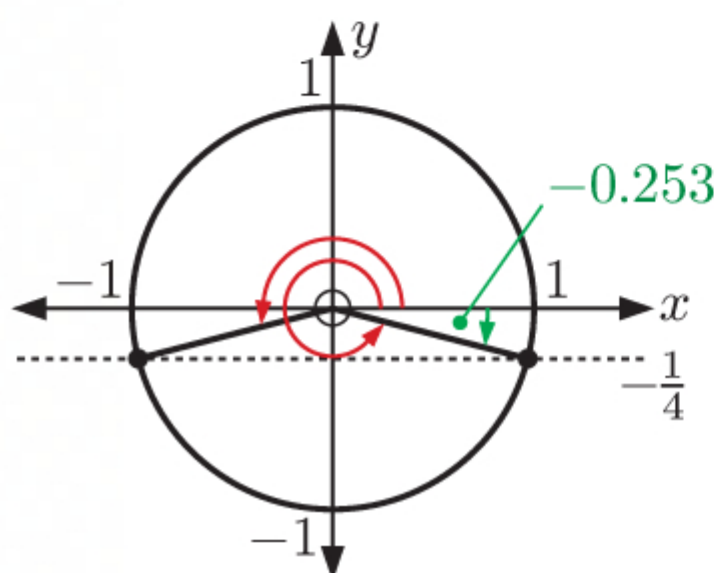
Using technology,
 $\cos^{-1}\left(\frac{2}{3}\right) \approx 0.841$



$$\begin{aligned}\therefore \theta &\approx 0.841 \text{ or } 2\pi - 0.841 \\ \therefore \theta &\approx 0.841 \text{ or } 5.44\end{aligned}$$

b $\sin \theta = -\frac{1}{4}$

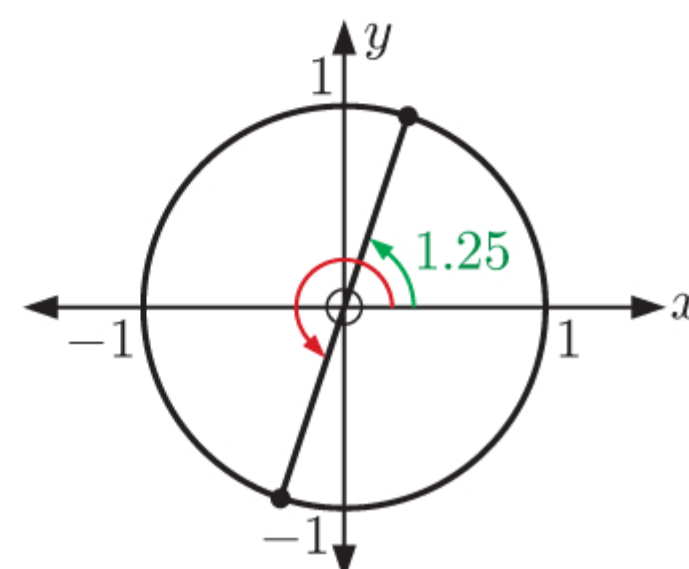
Using technology,
 $\sin^{-1}\left(-\frac{1}{4}\right) \approx -0.253$



$$\begin{aligned}\text{But } 0 \leq \theta < 2\pi \\ \therefore \theta &\approx \pi + 0.253 \text{ or } 2\pi - 0.253 \\ \therefore \theta &\approx 3.39 \text{ or } 6.03\end{aligned}$$

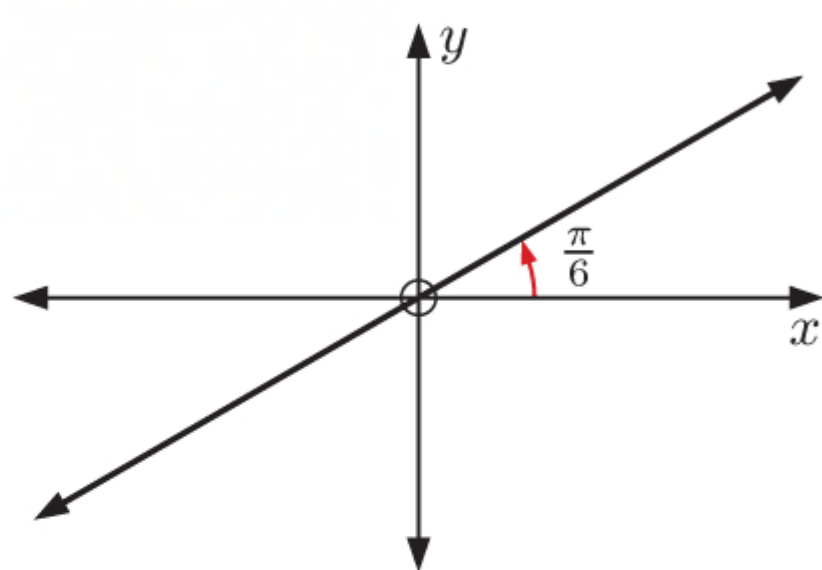
c $\tan \theta = 3$

Using technology,
 $\tan^{-1}(3) \approx 1.25$



$$\begin{aligned}\therefore \theta &\approx 1.25 \text{ or } \pi + 1.25 \\ \therefore \theta &\approx 1.25 \text{ or } 4.39\end{aligned}$$

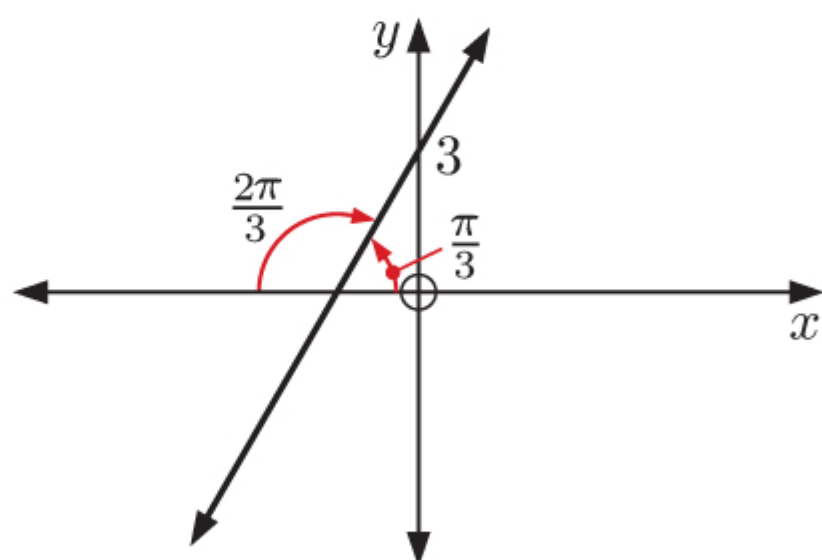
18 a



The line has gradient $m = \tan \frac{\pi}{6} = \frac{1}{\sqrt{3}}$ and y -intercept 0.

$$\therefore \text{the line has equation } y = \frac{1}{\sqrt{3}}x.$$

b



The graph makes an angle of $\pi - \frac{2\pi}{3} = \frac{\pi}{3}$ with the positive x -axis.

\therefore the line has gradient $m = \tan \frac{\pi}{3} = \sqrt{3}$ and y -intercept 3.

$$\therefore \text{the line has equation } y = \sqrt{3}x + 3.$$

REVIEW SET 7B

$$1 \quad \mathbf{a} \quad 71^\circ = \left(71 \times \frac{\pi}{180}\right) \text{ radians} \\ \approx 1.239 \text{ radians}$$

$$\mathbf{c} \quad -142^\circ = \left(-142 \times \frac{\pi}{180}\right) \text{ radians} \\ \approx -2.478 \text{ radians}$$

$$2 \quad \mathbf{a} \quad 3^c = \left(3 \times \frac{180}{\pi}\right)^\circ \\ \approx 171.89^\circ$$

$$\mathbf{c} \quad 0.435^c = \left(0.435 \times \frac{180}{\pi}\right)^\circ \\ \approx 24.92^\circ$$

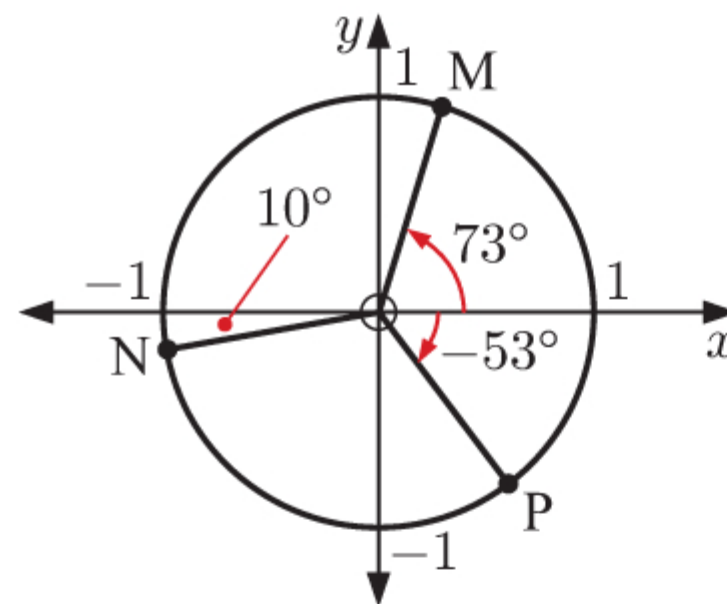
$$\mathbf{b} \quad 124.6^\circ = \left(124.6 \times \frac{\pi}{180}\right) \text{ radians} \\ \approx 2.175 \text{ radians}$$

$$\mathbf{b} \quad 1.46^c = \left(1.46 \times \frac{180}{\pi}\right)^\circ \\ \approx 83.65^\circ$$

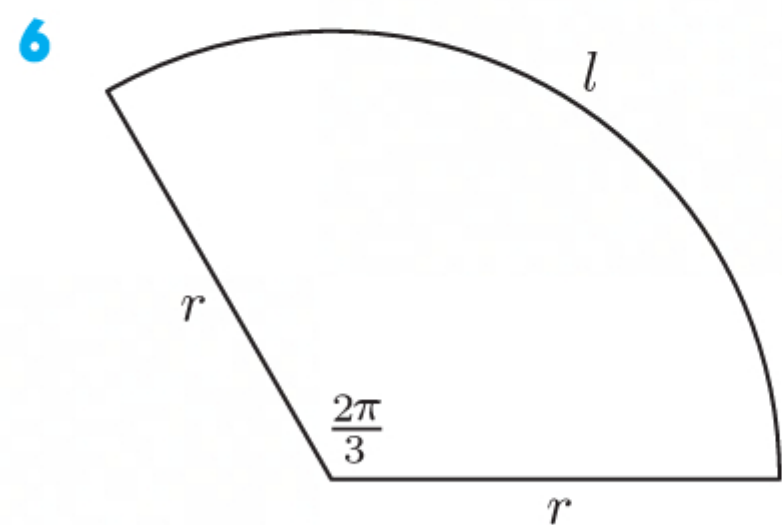
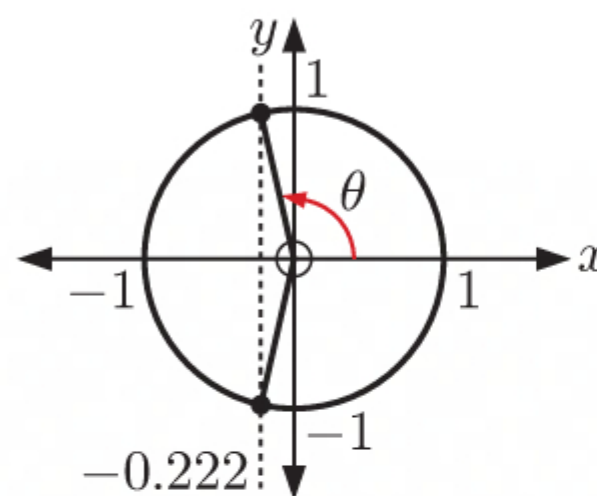
$$\mathbf{d} \quad -5.271^c = \left(-5.271 \times \frac{180}{\pi}\right)^\circ \\ \approx -302.01^\circ$$

$$3 \quad \text{area} = \frac{1}{2}\theta r^2 \\ = \frac{1}{2} \times \frac{5\pi}{12} \times 13^2 \\ \approx 111 \text{ cm}^2$$

$$4 \quad \begin{aligned} M(\cos 73^\circ, \sin 73^\circ) &\approx M(0.292, 0.956), \\ N(\cos 190^\circ, \sin 190^\circ) &\approx N(-0.985, -0.174), \\ P(\cos(-53^\circ), \sin(-53^\circ)) &= P(\cos 307^\circ, \sin 307^\circ) \\ &\approx P(0.602, -0.799) \end{aligned}$$



$$5 \quad \begin{aligned} \text{The } x\text{-coordinate of A} &= -0.222 \\ \therefore \cos \theta &= -0.222 \\ \therefore \theta &= \cos^{-1}(-0.222) \\ \therefore \theta &\approx 103^\circ \end{aligned}$$



$$\text{perimeter} = l + 2r$$

$$\therefore 36 = \frac{2\pi r}{3} + 2r$$

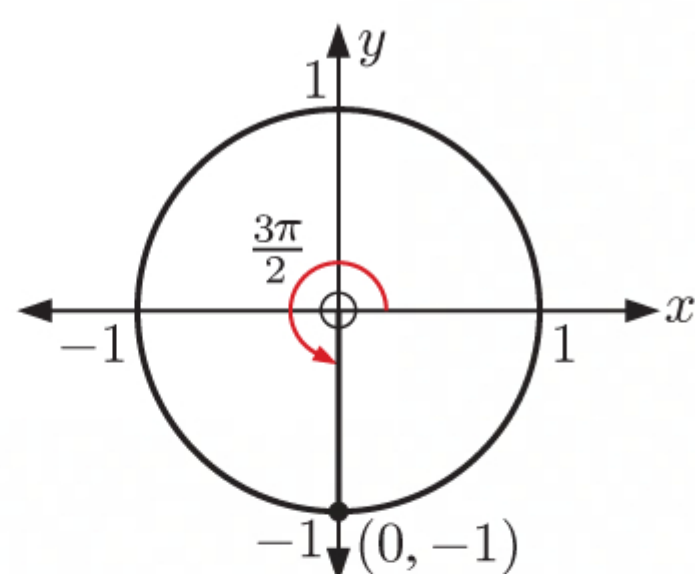
$$\therefore 36 = r\left(\frac{2\pi}{3} + 2\right)$$

$$\therefore r = \frac{36}{\frac{2\pi}{3} + 2}$$

$$\therefore r \approx 8.7925$$

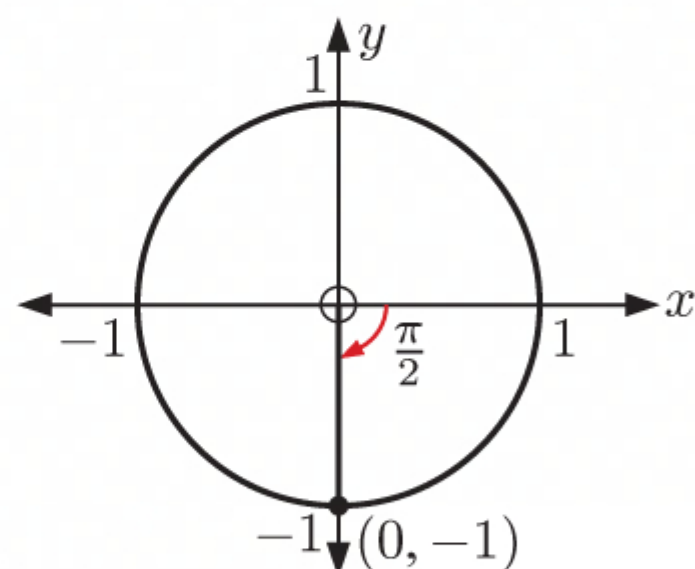
$$\therefore \text{the radius is } \approx 8.79 \text{ cm.}$$

$$\begin{aligned} \text{area} &\approx \frac{1}{2} \times \frac{2\pi}{3} \times 8.7925^2 \\ &\approx 81.0 \text{ cm}^2 \end{aligned}$$

7 a

$$\cos \frac{3\pi}{2} = 0$$

$$\sin \frac{3\pi}{2} = -1$$

b

$$\cos\left(-\frac{\pi}{2}\right) = 0$$

$$\sin\left(-\frac{\pi}{2}\right) = -1$$

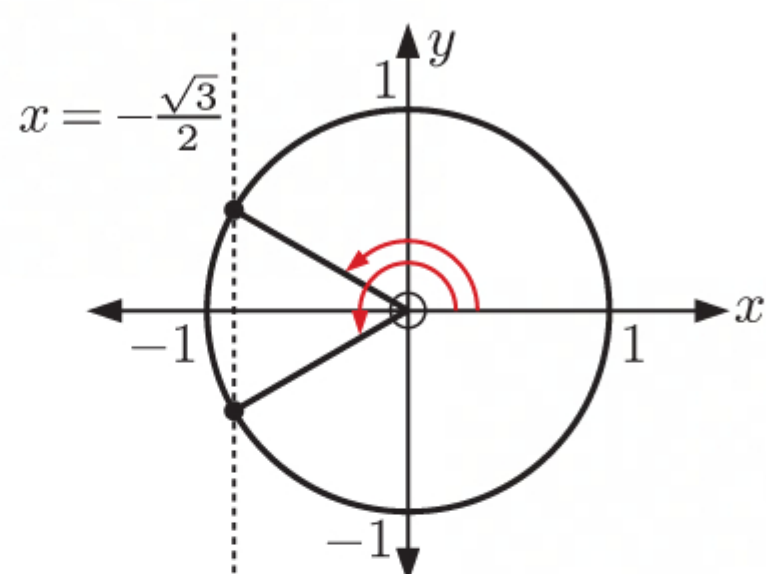
8 a $\sin(\pi - \theta) = \sin \theta$
 $\therefore \sin(\pi - p) = \sin p$
 $= m$

b $\sin(\theta + 2\pi) = \sin \theta$
 $\therefore \sin(p + 2\pi) = \sin p$
 $= m$

c $\cos^2 p + \sin^2 p = 1$
 $\therefore \cos^2 p + m^2 = 1$
 $\therefore \cos^2 p = 1 - m^2$
 $\therefore \cos p = \pm \sqrt{1 - m^2}$

d $\tan p = \frac{\sin p}{\cos p}$
 $= \frac{m}{\sqrt{1 - m^2}}$

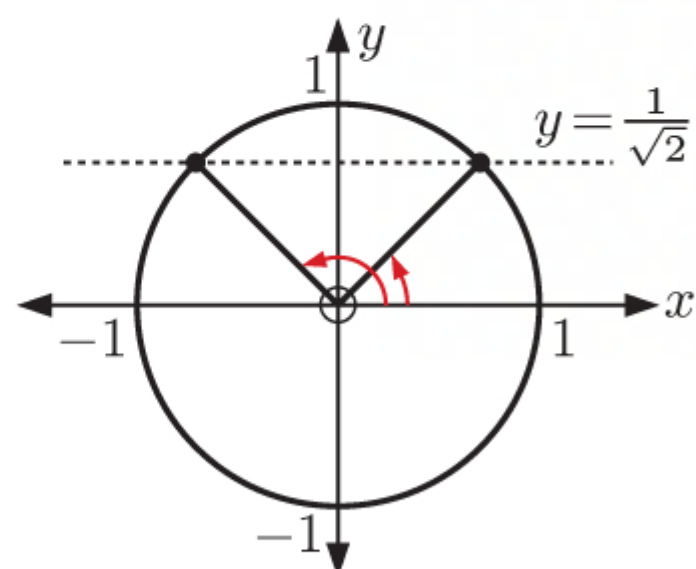
But p is acute, $\therefore \cos p = \sqrt{1 - m^2}$

9 a

Since the cosine is $-\frac{\sqrt{3}}{2}$, we draw the vertical line $x = -\frac{\sqrt{3}}{2}$.

Because $\frac{\sqrt{3}}{2}$ is involved, we know the required angles are multiples of 30° .

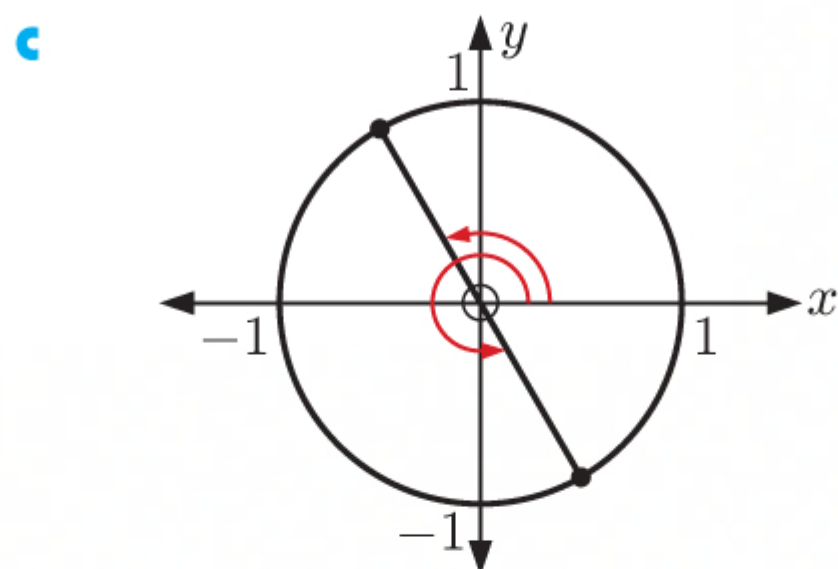
They are 150° and 210° .

b

Since the sine is $\frac{1}{\sqrt{2}}$, we draw the horizontal line $y = \frac{1}{\sqrt{2}}$.

Because $\frac{1}{\sqrt{2}}$ is involved, we know the required angles are multiples of 45° .

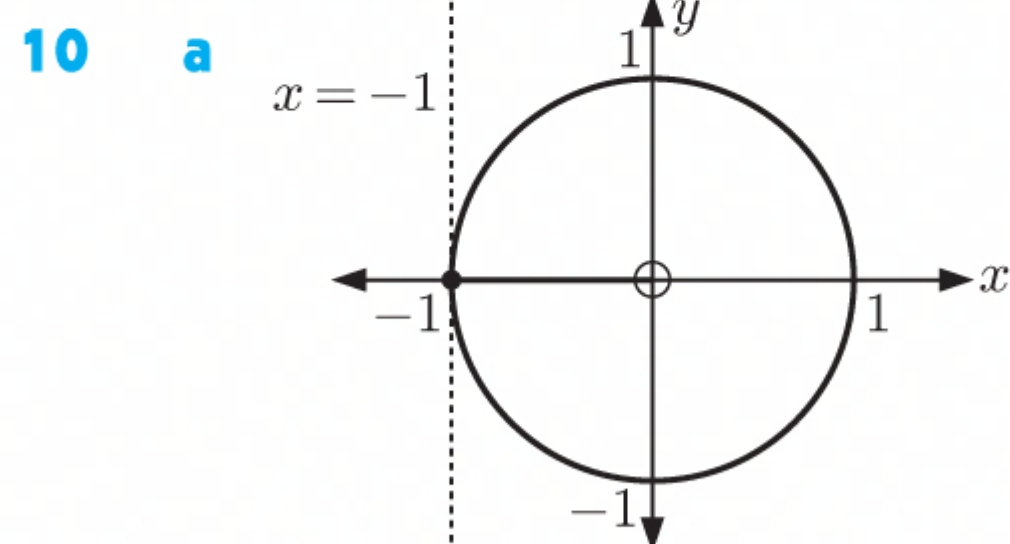
They are 45° and 135° .



Since the tangent is $-\sqrt{3} = \frac{\frac{\sqrt{3}}{2}}{-\frac{1}{2}} = \frac{-\frac{\sqrt{3}}{2}}{\frac{1}{2}}$, the sine must be $\pm\frac{\sqrt{3}}{2}$, and the cosine must be $\mp\frac{1}{2}$ (since $\tan \theta = \frac{\sin \theta}{\cos \theta}$).

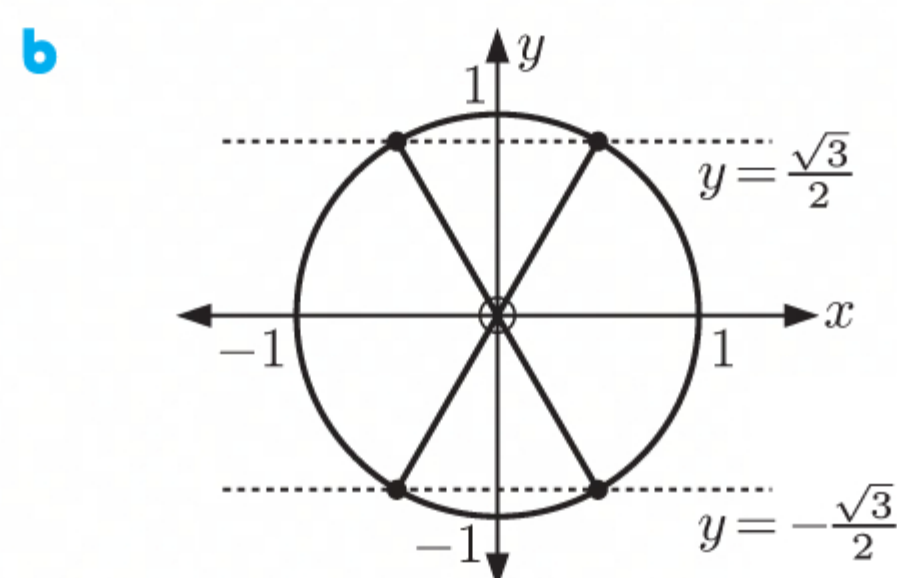
Because $\frac{\sqrt{3}}{2}$ and $\frac{1}{2}$ are both involved, we know the required angles are multiples of 30° .

They are 120° and 300° .



Since $\cos \theta = -1$, we draw the vertical line $x = -1$. Because 1 is involved, we know the required angles are multiples of $\frac{\pi}{2}$.

$$\therefore \theta = \pi$$



Since $\sin^2 \theta = \frac{3}{4}$, then $\sin \theta = \pm\frac{\sqrt{3}}{2}$, so we draw the horizontal lines $y = \frac{\sqrt{3}}{2}$ and $y = -\frac{\sqrt{3}}{2}$.

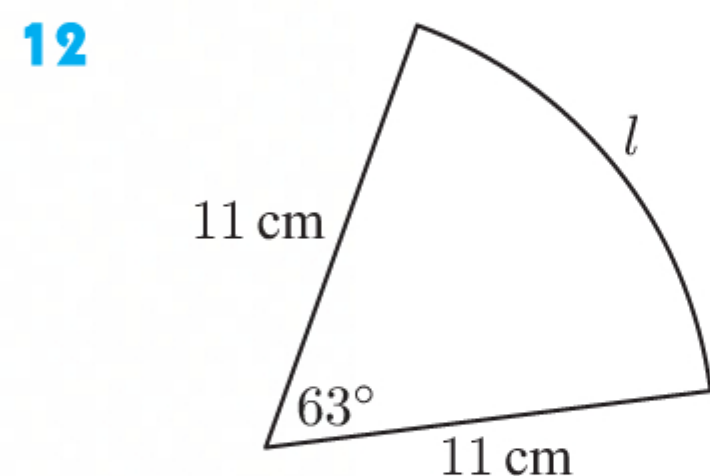
Because $\frac{\sqrt{3}}{2}$ is involved, we know the required angles are multiples of $\frac{\pi}{6}$.

$$\therefore \theta = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$$

11 a $\sin 47^\circ = \sin(180 - 47)^\circ$
 $= \sin 133^\circ$

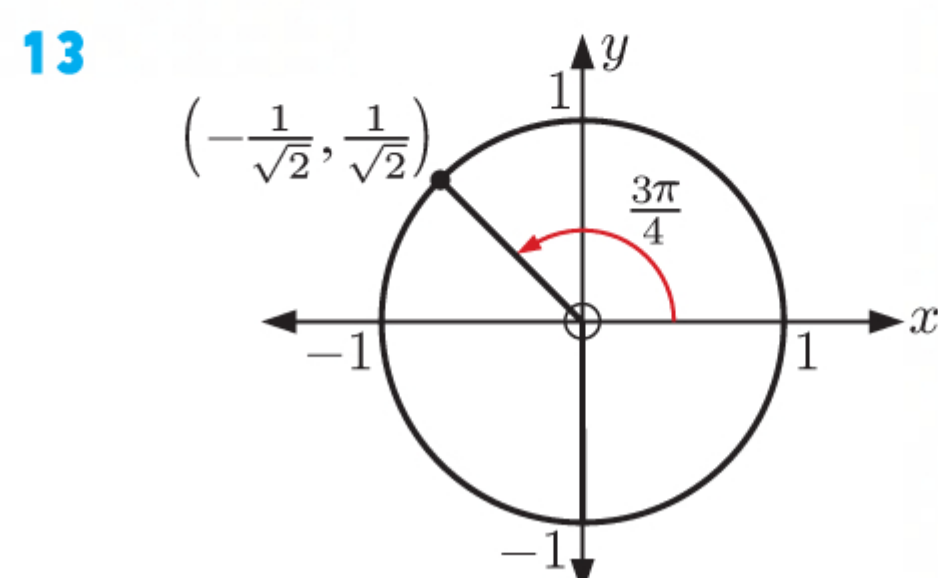
b $\sin \frac{\pi}{15} = \sin(\pi - \frac{\pi}{15})$
 $= \sin \frac{14\pi}{15}$

c $\cos 186^\circ = \cos(360 - 186)^\circ$
 $= \cos 174^\circ$



$$\begin{aligned} \text{perimeter} &= l + 2r \\ &= \left(\frac{63}{360}\right) \times 2\pi \times 11 + 2 \times 11 \\ &\approx 34.1 \text{ cm} \end{aligned}$$

$$\begin{aligned} \text{area} &= \left(\frac{63}{360}\right) \times \pi \times 11^2 \\ &\approx 66.5 \text{ cm}^2 \end{aligned}$$



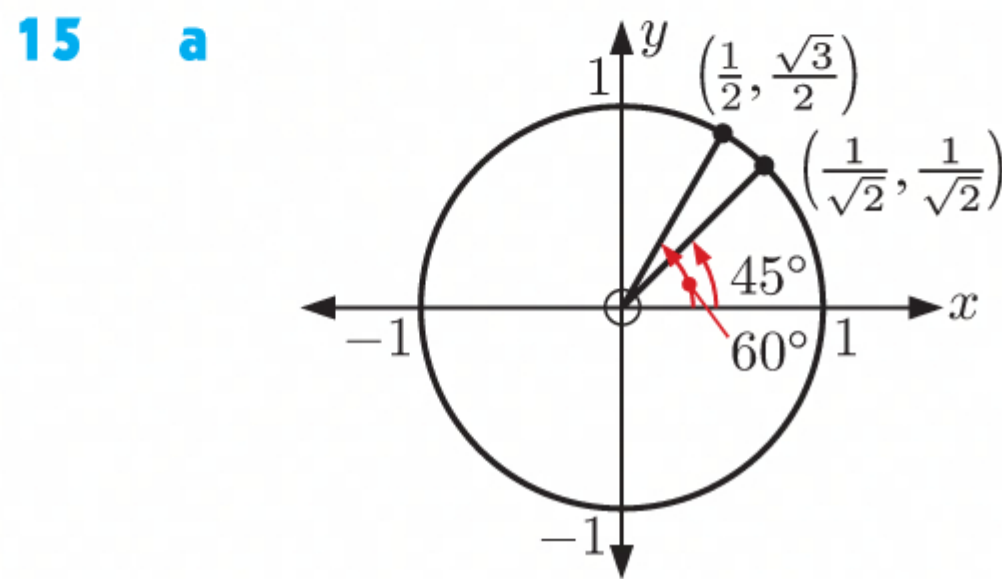
$$\begin{aligned} &\cos \frac{3\pi}{4} - \sin \frac{3\pi}{4} \\ &= -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} \\ &= -\frac{2}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} \\ &= -\sqrt{2} \end{aligned}$$

$$\begin{aligned}
 14 \quad a \quad & \cos^2 \theta + \sin^2 \theta = 1 \\
 & \therefore \frac{9}{16} + \sin^2 \theta = 1 \\
 & \therefore \sin^2 \theta = \frac{7}{16} \\
 & \therefore \sin \theta = \pm \frac{\sqrt{7}}{4}
 \end{aligned}$$

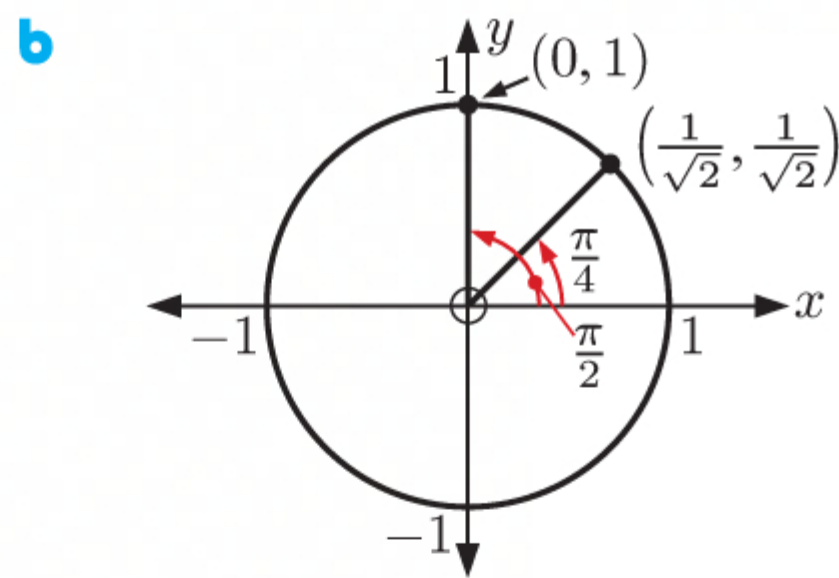
But θ is in quadrant 2, where $\sin \theta > 0$
 $\therefore \sin \theta = \frac{\sqrt{7}}{4}$

$$b \quad \tan \theta = \frac{\sin \theta}{\cos \theta} = \frac{\frac{\sqrt{7}}{4}}{-\frac{3}{4}} = -\frac{\sqrt{7}}{3}$$

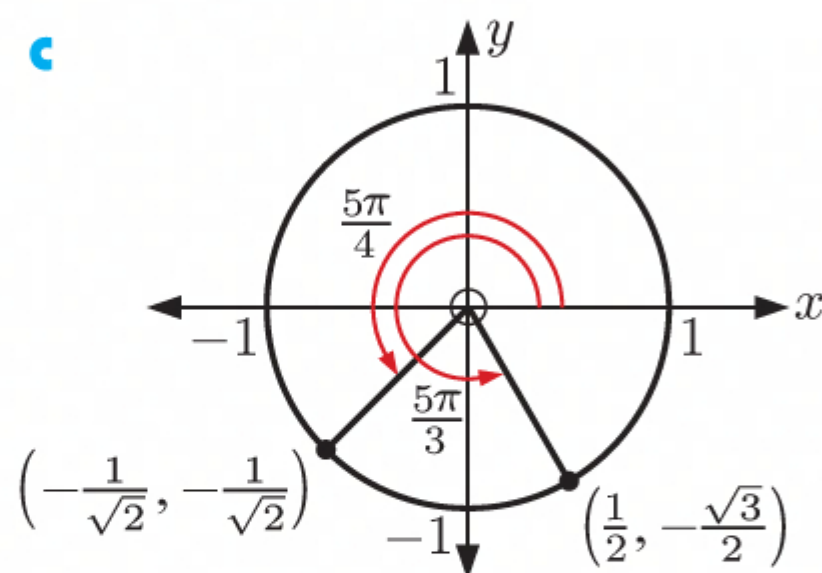
$$\begin{aligned}
 c \quad \cos(\pi - \theta) &= -\cos \theta \\
 &= \frac{3}{4}
 \end{aligned}$$



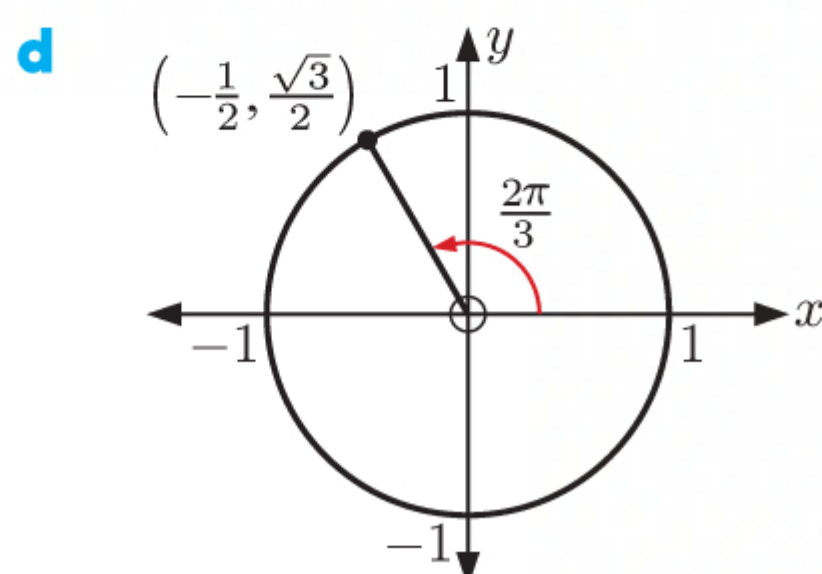
$$\begin{aligned}
 \tan 60^\circ &= \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = \sqrt{3} \quad \text{and} \quad \sin 45^\circ = \frac{1}{\sqrt{2}} \\
 \therefore \tan^2 60^\circ - \sin^2 45^\circ &= (\sqrt{3})^2 - \left(\frac{1}{\sqrt{2}}\right)^2 \\
 &= 3 - \frac{1}{2} \\
 &= 2\frac{1}{2}
 \end{aligned}$$



$$\begin{aligned}
 \cos \frac{\pi}{4} &= \frac{1}{\sqrt{2}} \quad \text{and} \quad \sin \frac{\pi}{2} = 1 \\
 \therefore \cos^2 \left(\frac{\pi}{4}\right) + \sin^2 \frac{\pi}{2} &= \left(\frac{1}{\sqrt{2}}\right)^2 + 1 \\
 &= \frac{1}{2} + 1 \\
 &= 1\frac{1}{2}
 \end{aligned}$$

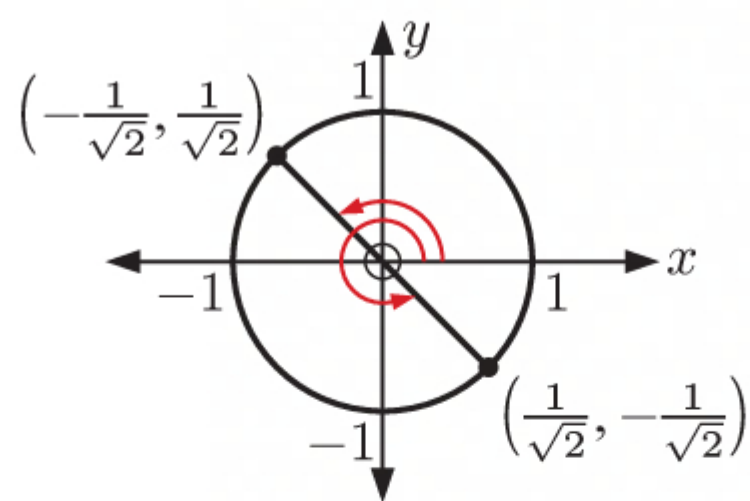


$$\begin{aligned}
 \cos \frac{5\pi}{3} &= \frac{1}{2} \quad \text{and} \quad \tan \frac{5\pi}{4} = \frac{-\frac{1}{\sqrt{2}}}{-\frac{1}{\sqrt{2}}} = 1 \\
 \therefore \cos \frac{5\pi}{3} - \tan \frac{5\pi}{4} &= \frac{1}{2} - 1 \\
 &= -\frac{1}{2}
 \end{aligned}$$



$$\begin{aligned}
 \tan \frac{2\pi}{3} &= \frac{\frac{\sqrt{3}}{2}}{-\frac{1}{2}} = -\sqrt{3} \\
 \therefore \tan^2 \left(\frac{2\pi}{3}\right) &= (-\sqrt{3})^2 \\
 &= 3
 \end{aligned}$$

16


 When $\cos \theta = -\sin \theta$,

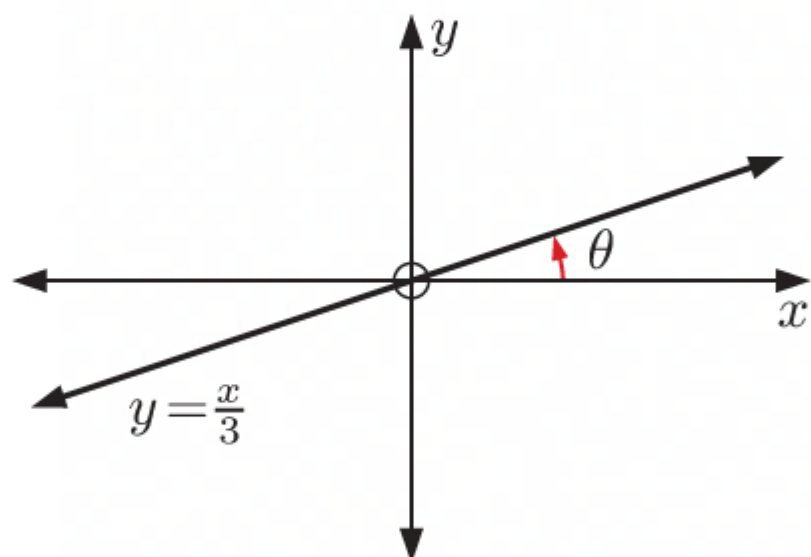
$$\frac{\sin \theta}{\cos \theta} = -1$$

$$\therefore \tan \theta = -1$$

which only occurs at the two points shown.

 So, $\theta = \frac{3\pi}{4}$ or $\frac{7\pi}{4}$.

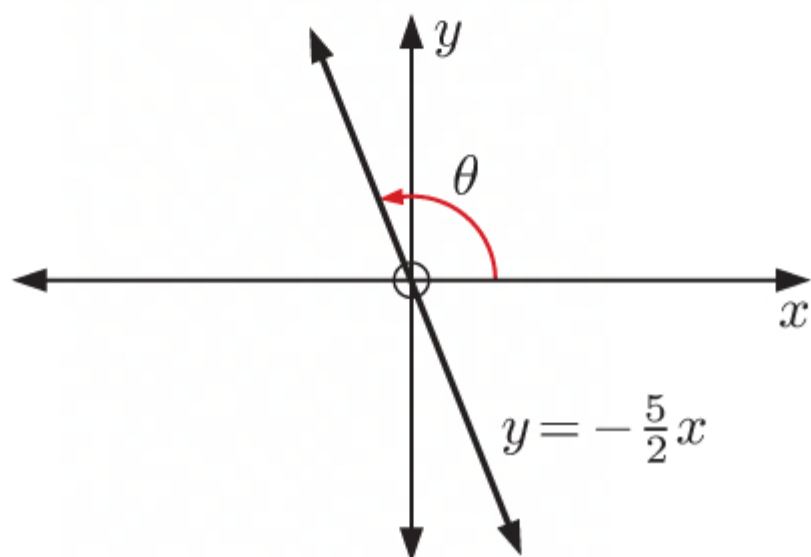
17 a


 The line has gradient $\frac{1}{3}$, so $\tan \theta = \frac{1}{3}$.

 Using technology, $\tan^{-1}\left(\frac{1}{3}\right) \approx 0.322$

$$\therefore \theta \approx 0.322$$

b


 The line has gradient $-\frac{5}{2}$, so $\tan \theta = -\frac{5}{2}$.

 Using technology, $\tan^{-1}\left(-\frac{5}{2}\right) \approx -1.19$

 But $0 < \theta < \pi$, so $\theta \approx \pi - 1.19 \approx 1.95$

18 [AB], [AC], and [BC] are all radii,

 so $AB = AC = BC = r$.

 Hence $\triangle ABC$ is equilateral

 and so $\widehat{CAB} = \frac{\pi}{3}$.

$$\sin \frac{\pi}{3} = \frac{CD}{AC}$$

$$\therefore CD = \sin \frac{\pi}{3} \times AC = \frac{\sqrt{3}}{2}r$$

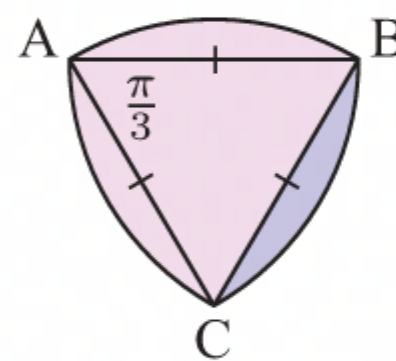
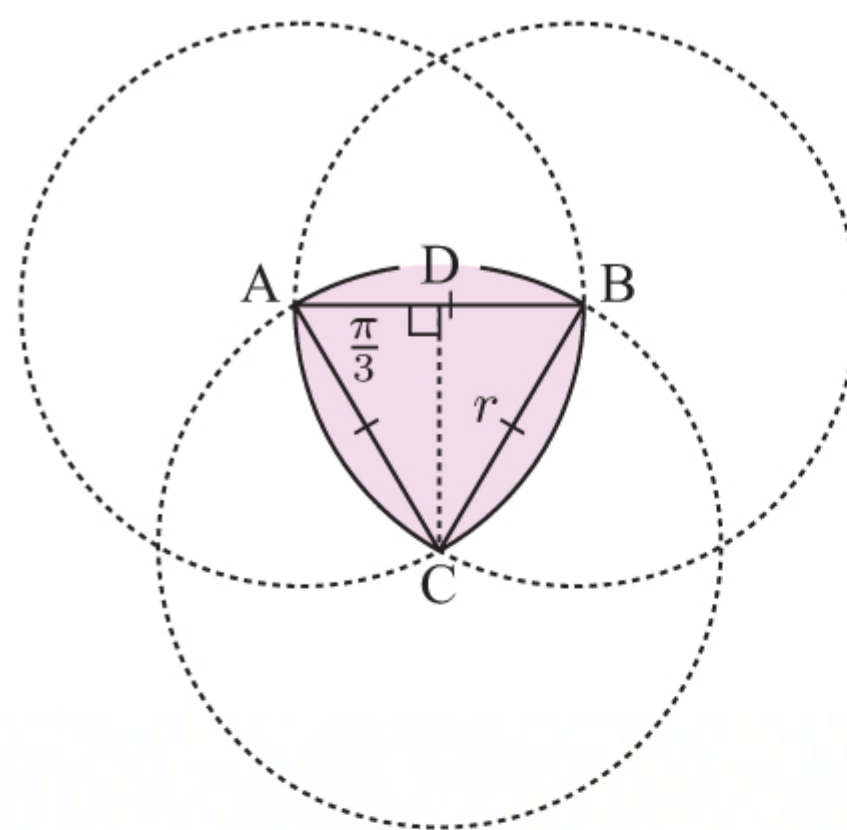
$$\therefore \text{area of } \triangle ABC = \frac{1}{2} \times r \times \frac{\sqrt{3}}{2}r$$

$$= \frac{\sqrt{3}}{4}r^2$$

 purple shaded area = area of sector - area of \triangle

$$= \frac{1}{2} \times \frac{\pi}{3} \times r^2 - \frac{\sqrt{3}}{4}r^2$$

$$= \frac{\pi}{6}r^2 - \frac{\sqrt{3}}{4}r^2$$


 \therefore area of shaded region = $3 \times$ purple shaded area + area of \triangle

$$= 3 \left[\frac{\pi}{6}r^2 - \frac{\sqrt{3}}{4}r^2 \right] + \frac{\sqrt{3}}{4}r^2$$

$$= \frac{\pi}{2}r^2 - \frac{3\sqrt{3}}{4}r^2 + \frac{\sqrt{3}}{4}r^2$$

$$= \frac{\pi}{2}r^2 - \frac{2\sqrt{3}}{4}r^2$$

$$\therefore A = \frac{r^2}{2}(\pi - \sqrt{3})$$

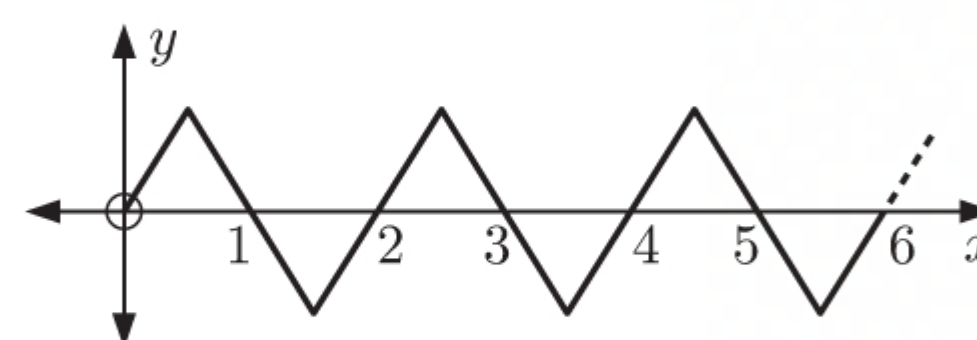
Chapter 8

TRIGONOMETRIC FUNCTIONS

EXERCISE 8A

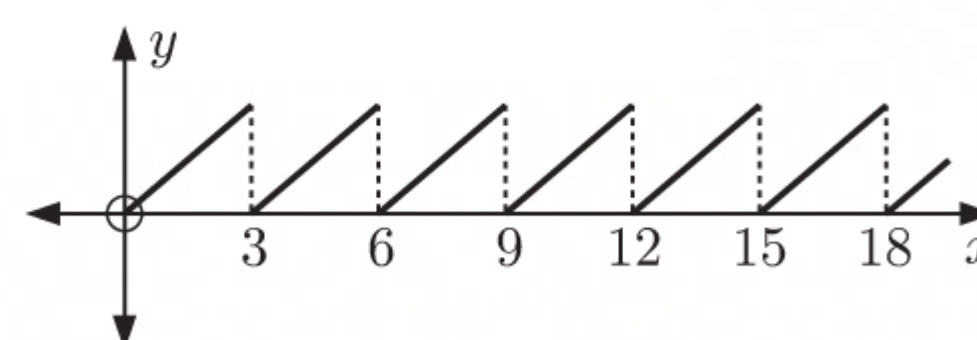
- 1 a This graph repeats itself over and over in a horizontal direction, in intervals of the same length.

\therefore this graph shows periodic behaviour.



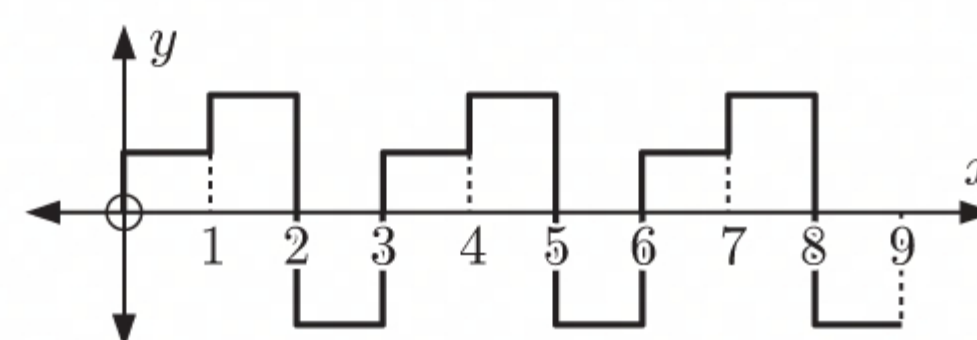
- b This graph repeats itself over and over in a horizontal direction, in intervals of the same length.

\therefore this graph shows periodic behaviour.



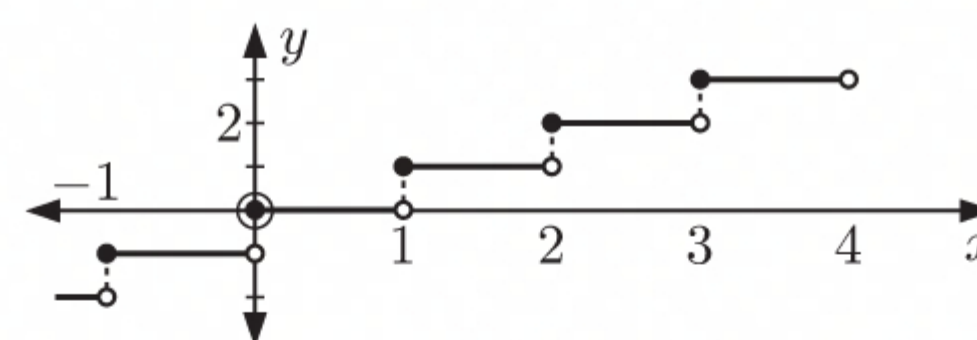
- c This graph repeats itself over and over in a horizontal direction, in intervals of the same length.

\therefore this graph shows periodic behaviour.



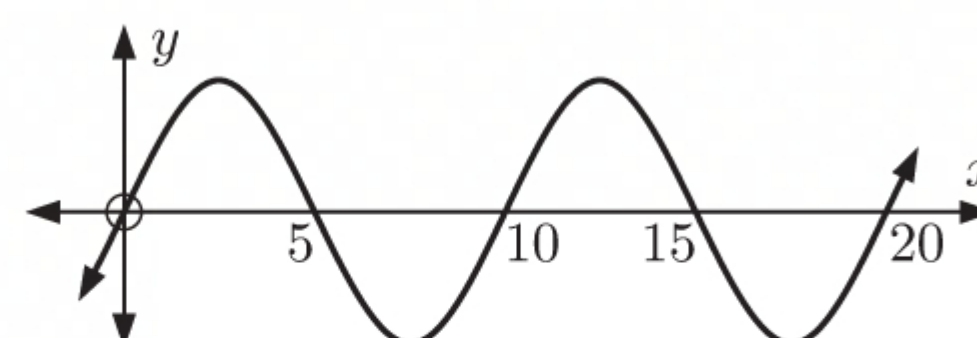
- d This graph repeats itself over and over in intervals of the same length, but not in a horizontal direction.

\therefore this graph does not show periodic behaviour.



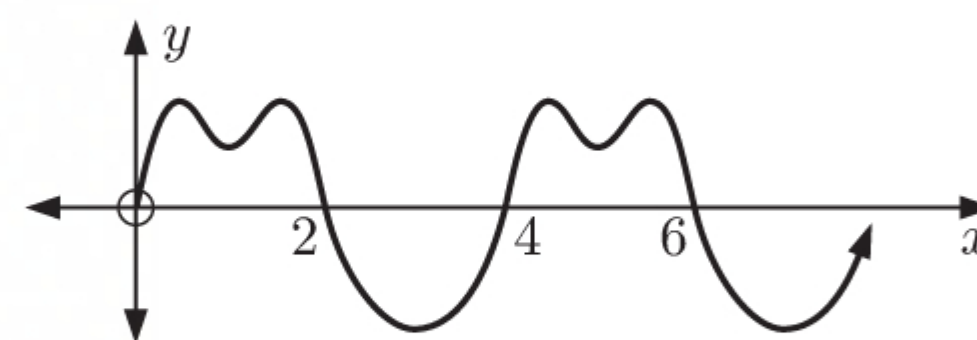
- e This graph repeats itself over and over in a horizontal direction, in intervals of the same length.

\therefore this graph shows periodic behaviour.



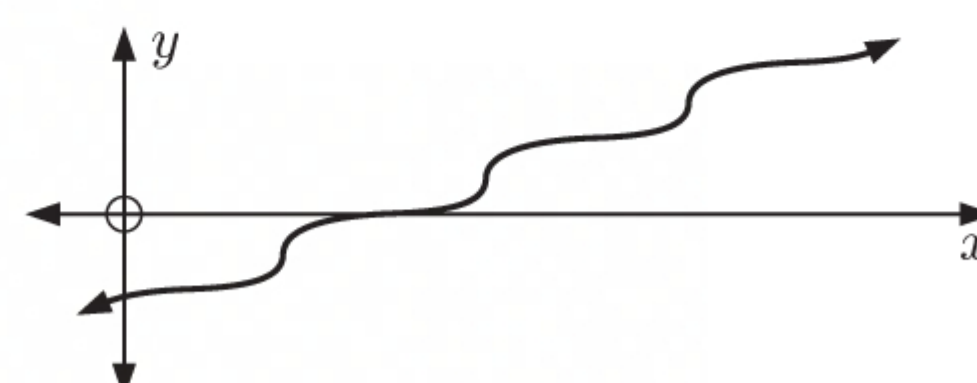
- f This graph repeats itself over and over in a horizontal direction, in intervals of the same length.

\therefore this graph shows periodic behaviour.

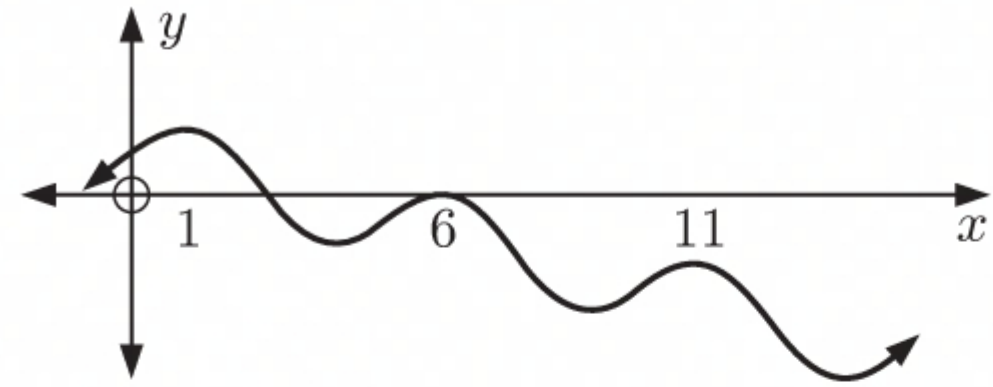


- g This graph repeats itself over and over in intervals of the same length, but not in a horizontal direction.

\therefore this graph does not show periodic behaviour.

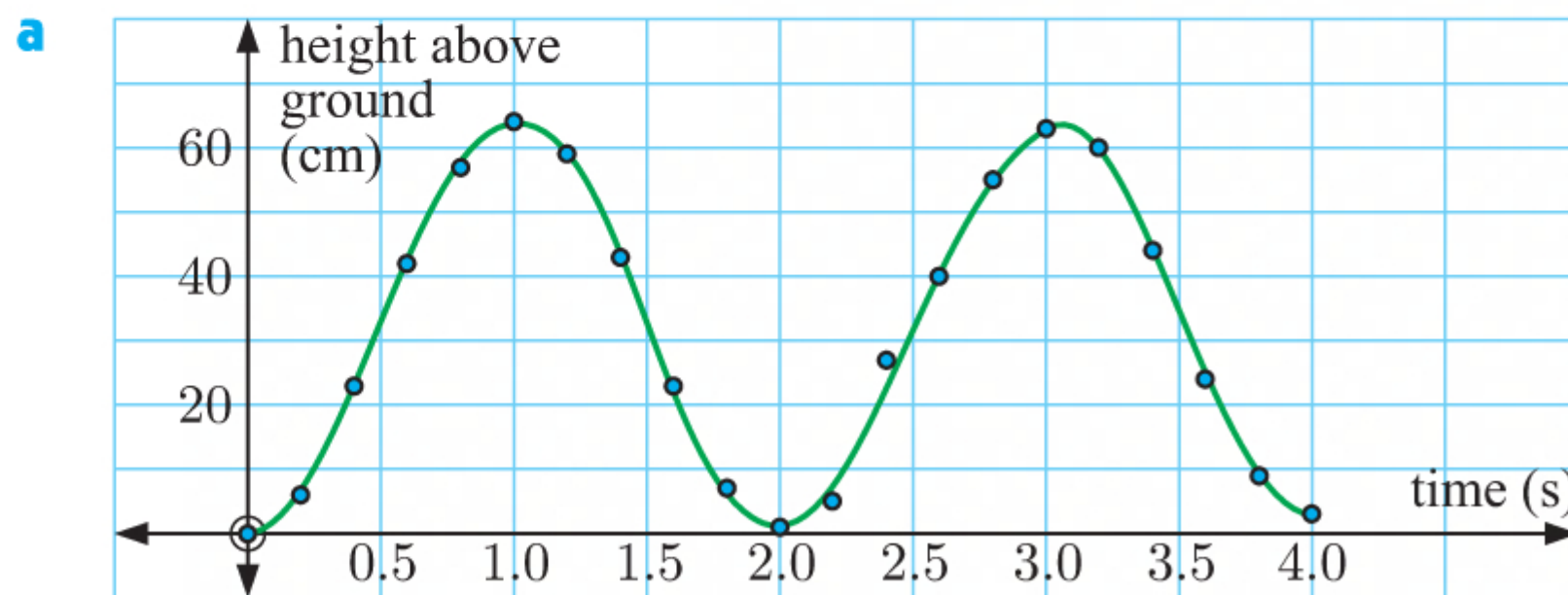


- h** This graph repeats itself over and over in intervals of the same length, but not in a horizontal direction.
 \therefore this graph does not show periodic behaviour.



2	Time (seconds)	0	0.2	0.4	0.6	0.8	1	1.2	1.4	1.6	1.8	2
	Height above ground (cm)	0	6	23	42	57	64	59	43	23	7	1

Time (seconds)	2.2	2.4	2.6	2.8	3	3.2	3.4	3.6	3.8	4
Height above ground (cm)	5	27	40	55	63	60	44	24	9	3



- b** A curve can be fitted to the data, as time is continuous.
c The graph repeats itself in a horizontal direction, in intervals of the same length.
 \therefore this data shows periodic behaviour.

- i** The minimum data value is 0 cm and the maximum data value is 64 cm.

The principal axis is $y = \frac{\text{max} + \text{min}}{2}$

$$\approx \frac{64 + 0}{2}$$

$$\therefore y \approx 32$$

- ii** The maximum value is ≈ 64 cm.

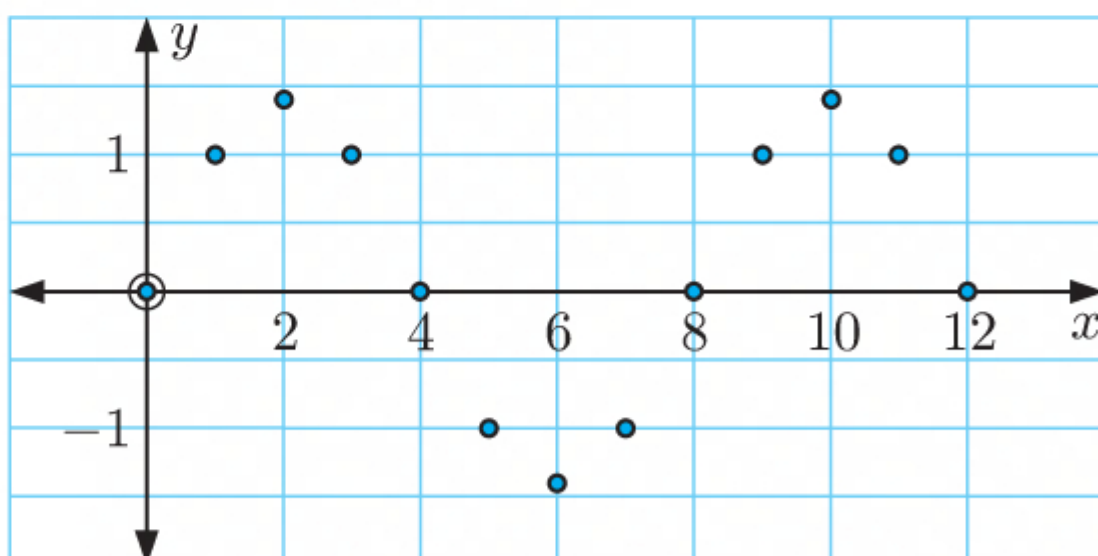
- iii** The period $\approx 3.0 - 1.0$
 ≈ 2 seconds

iv The amplitude $= \frac{\text{max} - \text{min}}{2}$

$$\approx \frac{64 - 0}{2}$$

$$\approx 32 \text{ cm}$$

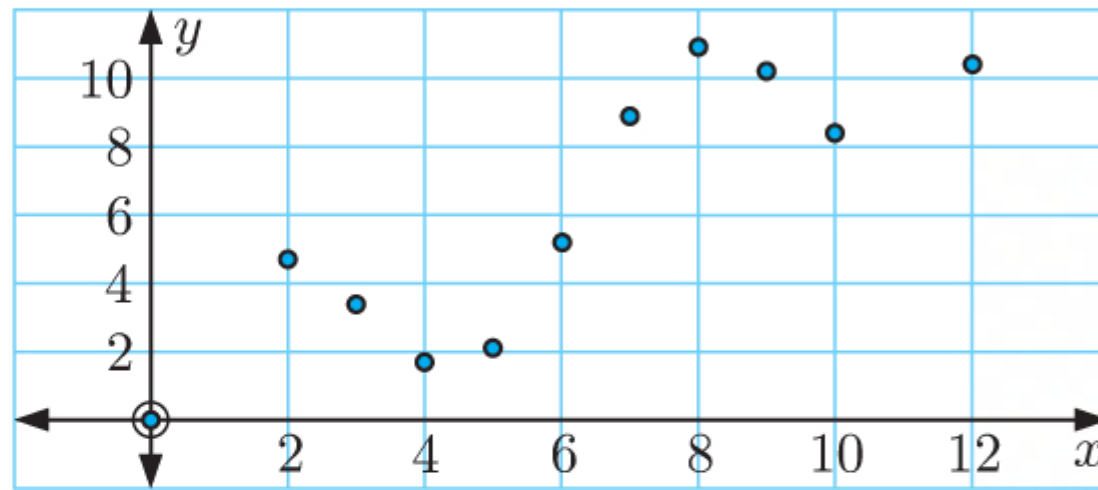
3	a	x	0	1	2	3	4	5	6	7	8	9	10	11	12
		y	0	1	1.4	1	0	-1	-1.4	-1	0	1	1.4	1	0



This data exhibits periodic behaviour, as the graph repeats itself in intervals of the same length in a horizontal direction.

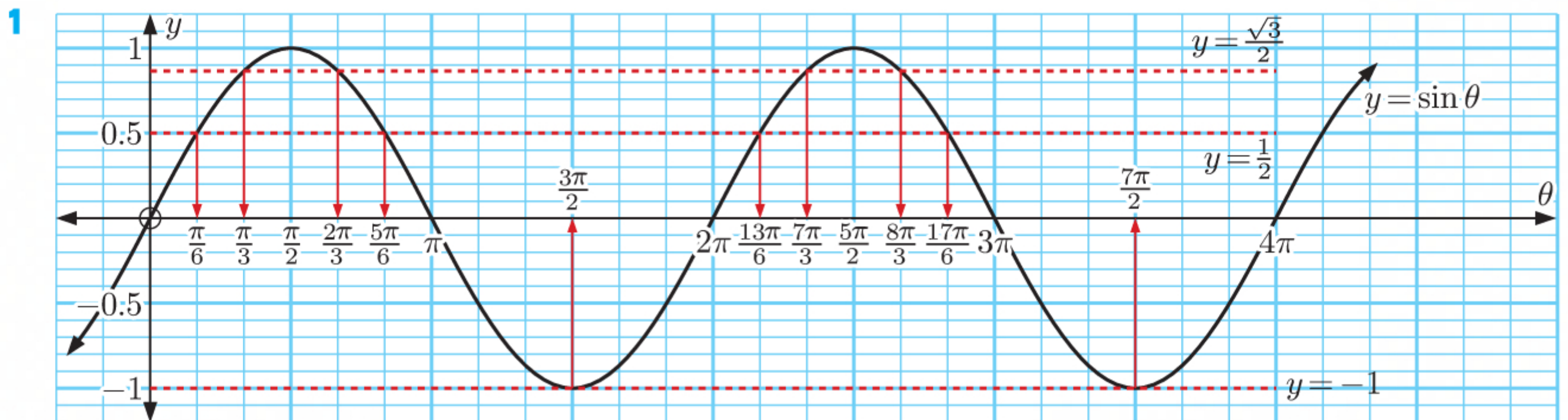
b

x	0	2	3	4	5	6	7	8	9	10	12
y	0	4.7	3.4	1.7	2.1	5.2	8.9	10.9	10.2	8.4	10.4

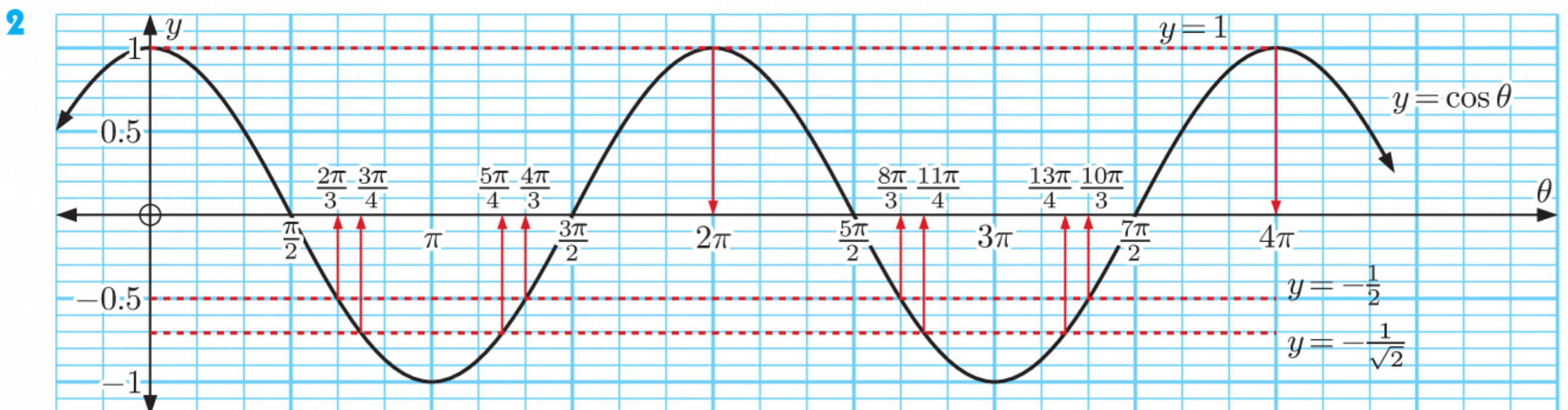


There is not enough information to say this data is periodic.

EXERCISE 8B



- a** The y -intercept is 0.
- b**
- i** When $\sin \theta = 0$, $0 \leq \theta \leq 4\pi$, $\theta = 0, \pi, 2\pi, 3\pi$, or 4π .
 - ii** When $\sin \theta = -1$, $0 \leq \theta \leq 4\pi$, $\theta = \frac{3\pi}{2}$ or $\frac{7\pi}{2}$.
 - iii** When $\sin \theta = \frac{1}{2}$, $0 \leq \theta \leq 4\pi$, $\theta = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}$, or $\frac{17\pi}{6}$.
 - iv** When $\sin \theta = \frac{\sqrt{3}}{2}$, $0 \leq \theta \leq 4\pi$, $\theta = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{7\pi}{3}$, or $\frac{8\pi}{3}$.
- c**
- i** On $0 \leq \theta \leq 4\pi$, $\sin \theta$ is positive for $0 < \theta < \pi$, $2\pi < \theta < 3\pi$.
 - ii** On $0 \leq \theta \leq 4\pi$, $\sin \theta$ is negative for $\pi < \theta < 2\pi$, $3\pi < \theta < 4\pi$.
- d** The range is $\{y \mid -1 \leq y \leq 1\}$.

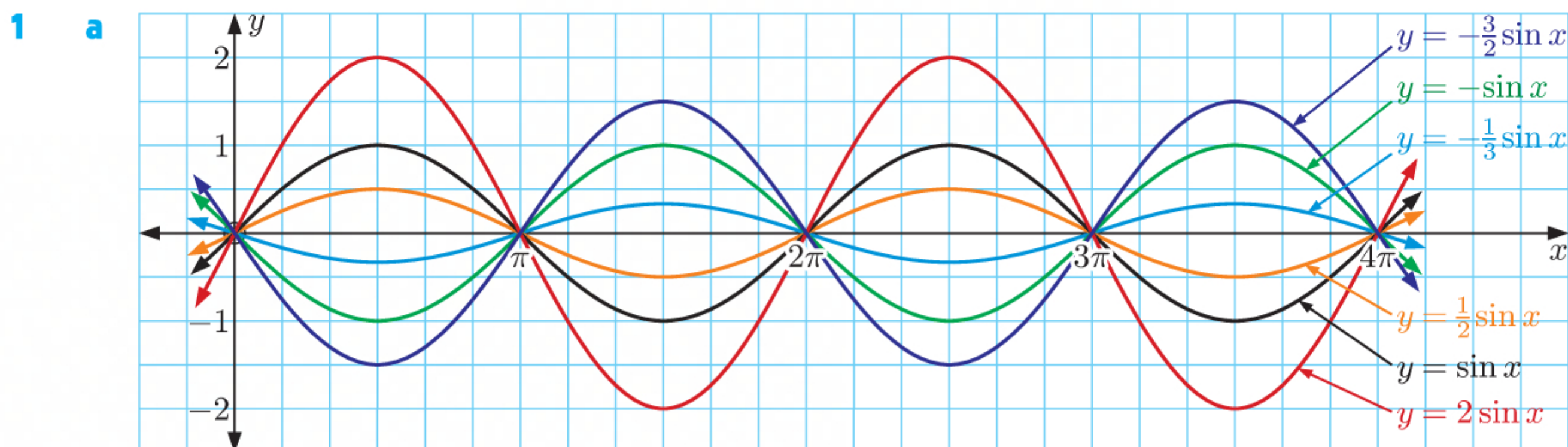


- a** The y -intercept is 1.
- b**
- i** When $\cos \theta = 0$, $0 \leq \theta \leq 4\pi$, $\theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}$, or $\frac{7\pi}{2}$.
 - ii** When $\cos \theta = 1$, $0 \leq \theta \leq 4\pi$, $\theta = 0, 2\pi$, or 4π .

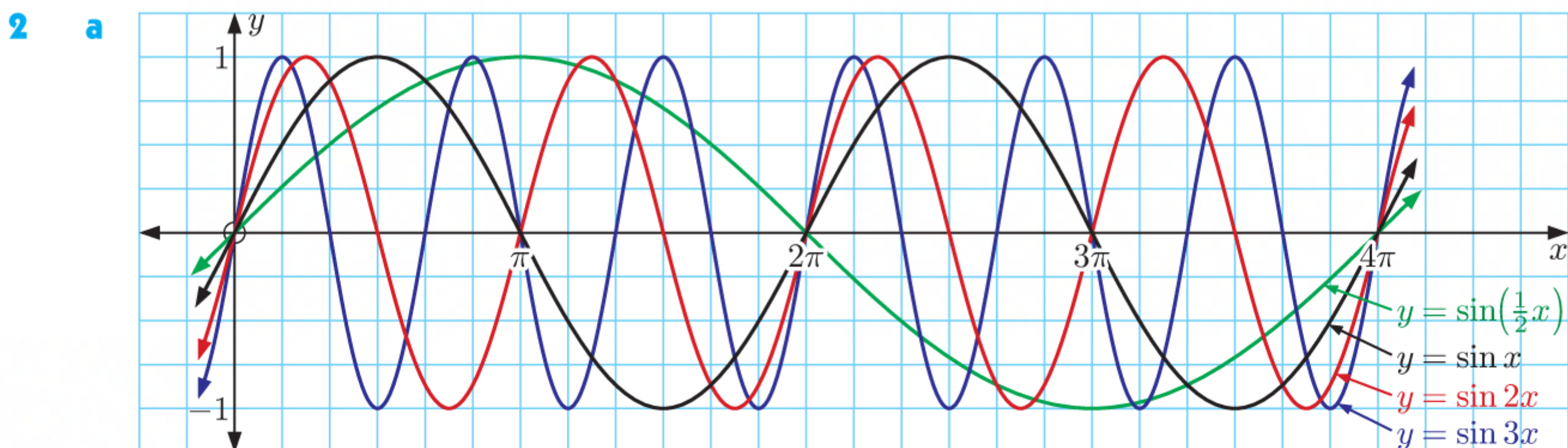
- iii When $\cos \theta = -\frac{1}{2}$, $0 \leq \theta \leq 4\pi$, $\theta = \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{8\pi}{3}$, or $\frac{10\pi}{3}$.
- iv When $\cos \theta = -\frac{1}{\sqrt{2}}$, $0 \leq \theta \leq 4\pi$, $\theta = \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{11\pi}{4}$, or $\frac{13\pi}{4}$.
- c
 - i On $0 \leq \theta \leq 4\pi$, $\cos \theta$ is positive for $0 \leq \theta < \frac{\pi}{2}$, $\frac{3\pi}{2} < \theta < \frac{5\pi}{2}$, $\frac{7\pi}{2} < \theta \leq 4\pi$.
 - ii On $0 \leq \theta \leq 4\pi$, $\cos \theta$ is negative for $\frac{\pi}{2} < \theta < \frac{3\pi}{2}$, $\frac{5\pi}{2} < \theta < \frac{7\pi}{2}$.
- d The range is $\{y \mid -1 \leq y \leq 1\}$.

INVESTIGATION

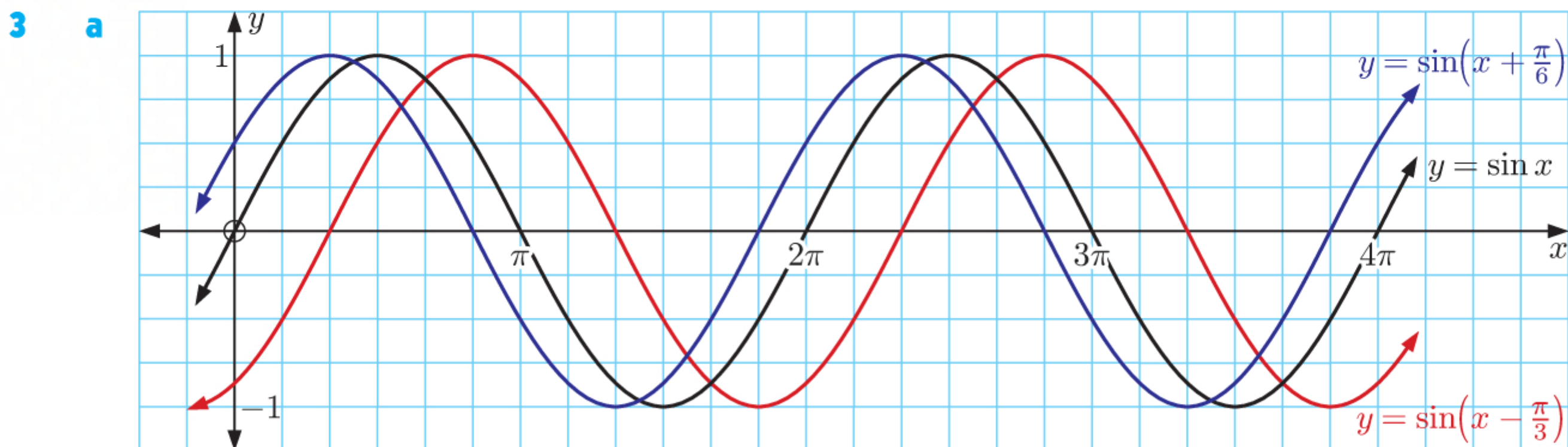
FAMILIES OF TRIGONOMETRIC FUNCTIONS



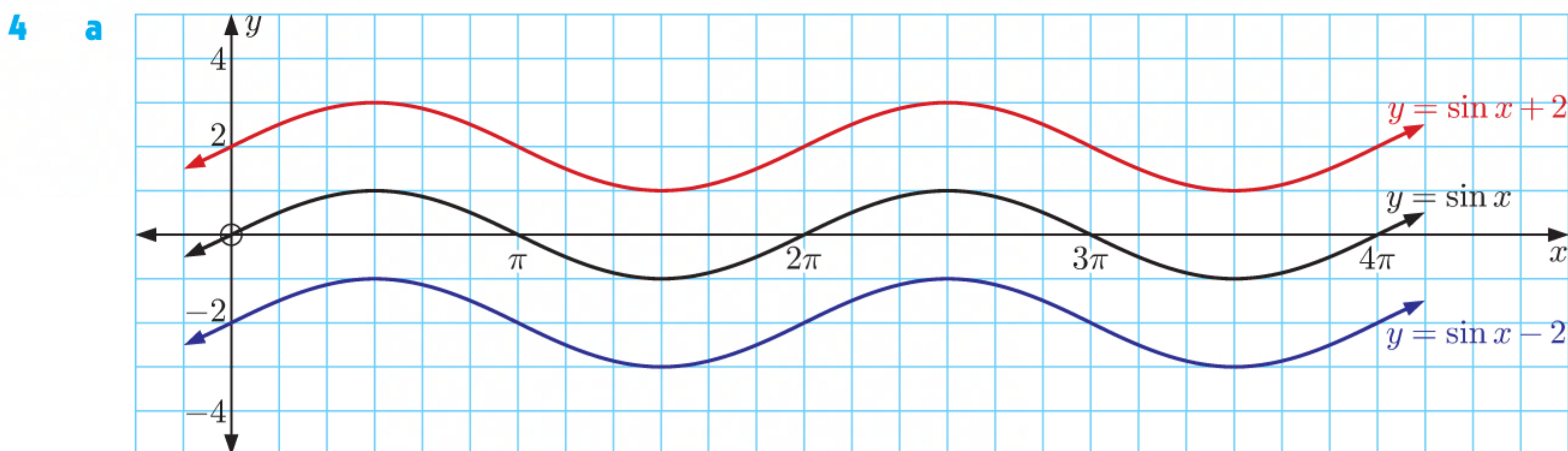
- b For graphs of the form $y = a \sin x$:
- i the sign of a affects where the graph is positive or negative
 - ii the size of a affects the **amplitude** of the graph.



- b For graphs of the form $y = \sin bx$, $b > 0$, the period is $\frac{2\pi}{b}$.



- b A horizontal translation of c units moves $y = \sin x$ to $y = \sin(x - c)$.



- b** A vertical translation of d units moves $y = \sin x$ to $y = \sin x + d$.
- c** The principal axis of $y = \sin x + d$ is $y = d$.
- 5** $y = a \sin(b(x - c)) + d$ is obtained from $y = \sin x$ by a vertical stretch with scale factor $|a|$ and a horizontal stretch with scale factor $\frac{1}{b}$, a reflection in the x -axis if $a < 0$, and a translation through $\begin{pmatrix} c \\ d \end{pmatrix}$.

EXERCISE 8C

- 1 a** $y = \sin x - 1$ is a vertical translation of $y = \sin x$ downwards by 1 unit.
So, a vertical translation 1 unit downwards will map $y = \sin x$ onto $y = \sin x - 1$.
- b** $y = \sin\left(x - \frac{\pi}{4}\right)$ is a horizontal translation of $y = \sin x$ to the right by $\frac{\pi}{4}$ units.
So, a horizontal translation of $\frac{\pi}{4}$ units to the right will map $y = \sin x$ onto $y = \sin\left(x - \frac{\pi}{4}\right)$.
- c** $y = 2 \sin x$ is a vertical stretch of $y = \sin x$ with scale factor 2.
So, a vertical stretch with scale factor 2 will map $y = \sin x$ onto $y = 2 \sin x$.
- d** $y = \sin 4x$ is a horizontal stretch of $y = \sin x$ with scale factor $\frac{1}{4}$.
So, a horizontal stretch with scale factor $\frac{1}{4}$ will map $y = \sin x$ onto $y = \sin 4x$.
- e** $y = \sin \frac{x}{4}$ is a horizontal stretch of $y = \sin x$ with scale factor $\frac{1}{\frac{1}{4}} = 4$.
So, a horizontal stretch with scale factor 4 will map $y = \sin x$ onto $y = \sin \frac{x}{4}$.
- f** $y = \sin\left(x - \frac{\pi}{3}\right) + 2$ is a horizontal translation of $y = \sin x$ to the right by $\frac{\pi}{3}$ units followed by a vertical translation upwards by 2 units.
So, a translation of $\begin{pmatrix} \frac{\pi}{3} \\ 2 \end{pmatrix}$ will map $y = \sin x$ onto $y = \sin\left(x - \frac{\pi}{3}\right) + 2$.
- 2 a** $y = \frac{1}{2} \cos x$ is a vertical stretch of $y = \cos x$ with scale factor $\frac{1}{2}$.
So, a vertical stretch with scale factor $\frac{1}{2}$ will map $y = \cos x$ on $y = \frac{1}{2} \cos x$.
- b** $y = -\cos x$ is a reflection of $y = \cos x$ in the x -axis.
So, a reflection of $y = \cos x$ in the x -axis will map $y = \cos x$ onto $y = -\cos x$.
- c** $y = \cos\left(x + \frac{\pi}{6}\right) - 2$ is a horizontal translation of $y = \cos x$ to the left $\frac{\pi}{6}$ units followed by a vertical translation downwards by 2 units.

3 a $y = \sin 5x$ has period $\frac{2\pi}{b} = \frac{2\pi}{5}$

b $y = \sin(0.6x)$ has period $\frac{2\pi}{b} = \frac{2\pi}{0.6}$
 $= \frac{2\pi}{\frac{3}{5}}$
 $= \frac{10\pi}{3}$

c $y = \sin \pi x$ has period $\frac{2\pi}{b} = \frac{2\pi}{\pi}$
 $= 2$

d $y = \cos 3x$ has period $\frac{2\pi}{b} = \frac{2\pi}{3}$

e $y = \cos \frac{x}{3}$ has period $\frac{2\pi}{b} = \frac{2\pi}{\frac{1}{3}}$
 $= 6\pi$

f $y = \cos \frac{\pi x}{50}$ has period $\frac{2\pi}{b} = \frac{2\pi}{\frac{\pi}{50}}$
 $= 100$

4 $y = \sin bx, \quad b > 0$

a period $= \frac{2\pi}{b}$
 $\therefore 5\pi = \frac{2\pi}{b}$
 $\therefore b = \frac{2}{5}$

b period $= \frac{2\pi}{b}$
 $\therefore \frac{2\pi}{3} = \frac{2\pi}{b}$
 $\therefore b = 3$

c period $= \frac{2\pi}{b}$
 $\therefore 12\pi = \frac{2\pi}{b}$
 $\therefore b = \frac{1}{6}$

d period $= \frac{2\pi}{b}$
 $\therefore 4 = \frac{2\pi}{b}$
 $\therefore 4b = 2\pi$
 $\therefore b = \frac{\pi}{2}$

e period $= \frac{2\pi}{b}$
 $\therefore 100 = \frac{2\pi}{b}$
 $\therefore 100b = 2\pi$
 $\therefore b = \frac{\pi}{50}$

5 a $y = 4 \cos 2x$ has maximum value $4(1) = 4$ {when $\cos 2x = 1$ }
 and minimum value $4(-1) = -4$ {when $\cos 2x = -1$ }

b $y = 3 \cos x + 5$ has maximum value $3(1) + 5 = 8$ {when $\cos x = 1$ }
 and minimum value $3(-1) + 5 = 2$ {when $\cos x = -1$ }

c $y = -2 \cos(x - 3) - 4$ has maximum value $-2(-1) - 4 = -2$
 {when $\cos(x - 3) = -1$ }
 and minimum value $-2(1) - 4 = -6$
 {when $\cos(x - 3) = 1$ }

6 a The amplitude is $a = 4$.

b The period is $\frac{2\pi}{b} = \frac{2\pi}{3}$.

c $y = 4 \sin 3x + 2$ has maximum value $4(1) + 2 = 6$ {when $\sin 3x = 1$ }
 and minimum value $4(-1) + 2 = -2$ {when $\sin 3x = -1$ }
 \therefore the range is $\{y \mid -2 \leq y \leq 6\}$.

7 $y = a \cos(b(x - c)) + d$

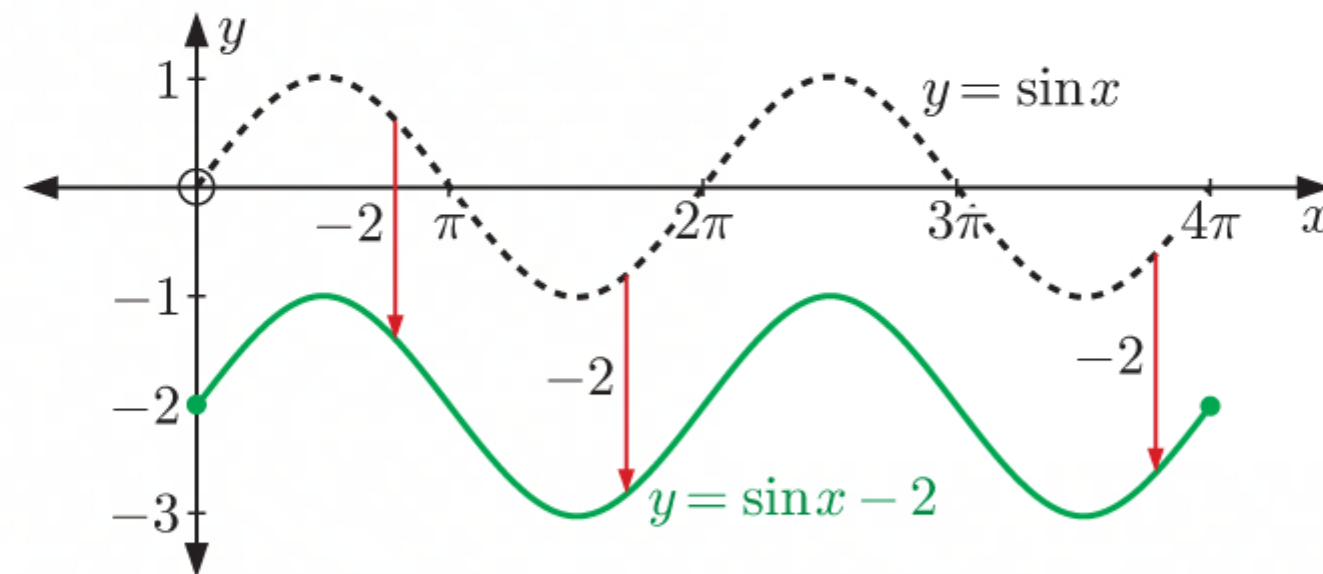
a controls the amplitude {amplitude = $|a|$ }.

b controls the period {period = $\frac{2\pi}{b}$ }.

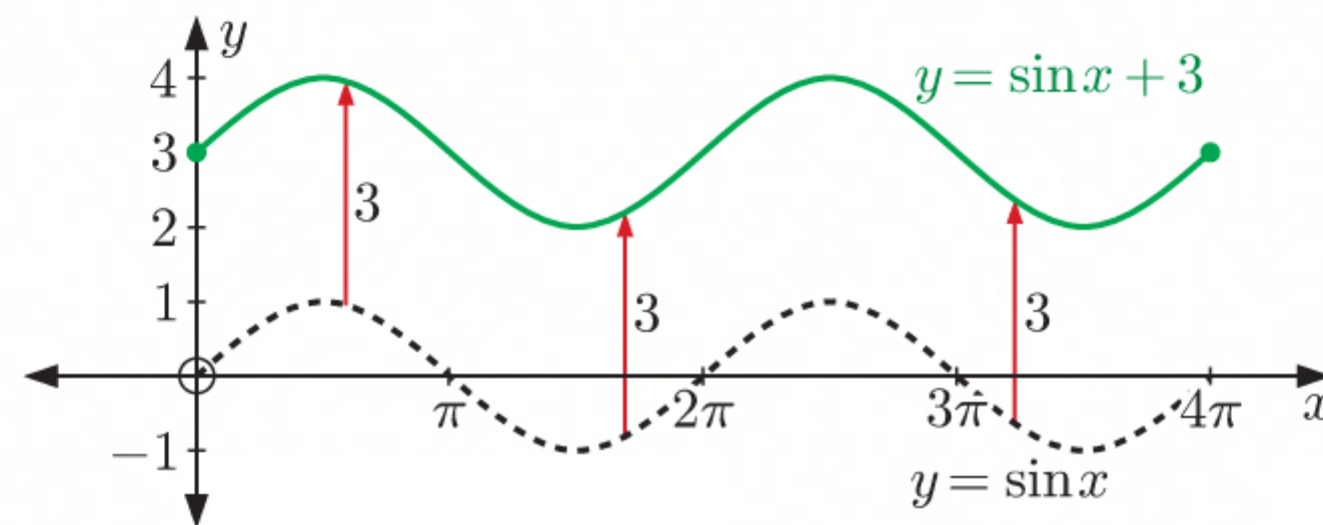
c controls the horizontal translation.

d controls the vertical translation.

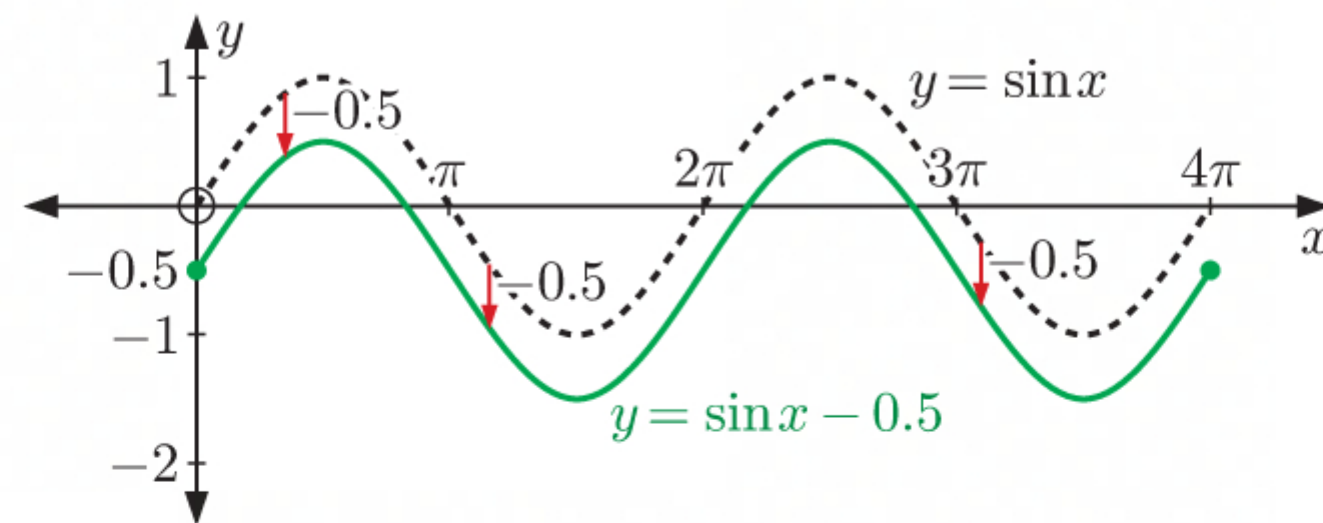
8 a $y = \sin x - 2$ is a vertical translation of $y = \sin x$ downwards by 2 units.



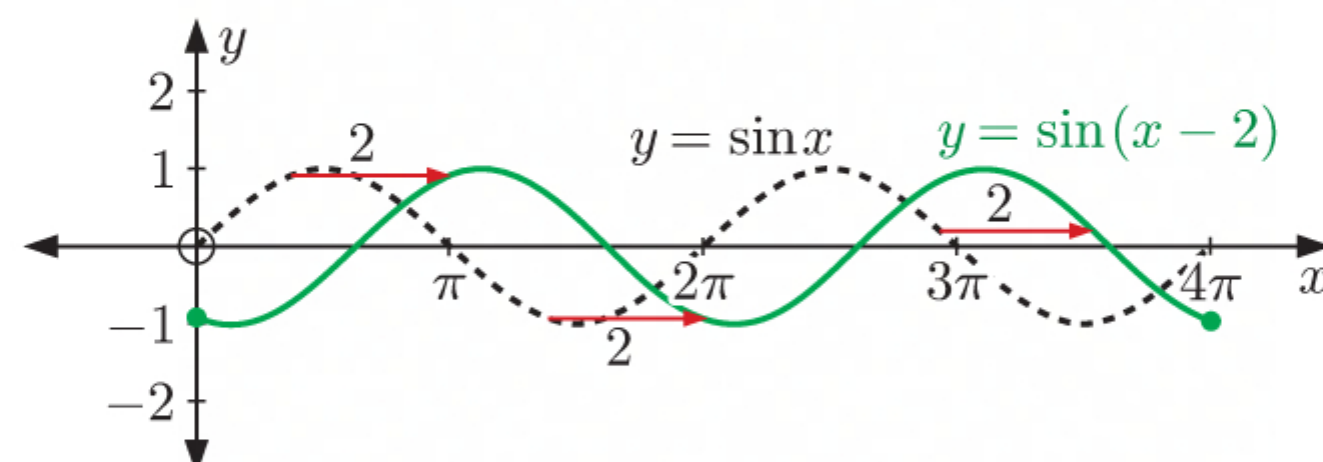
b $y = \sin x + 3$ is a vertical translation of $y = \sin x$ upwards by 3 units.



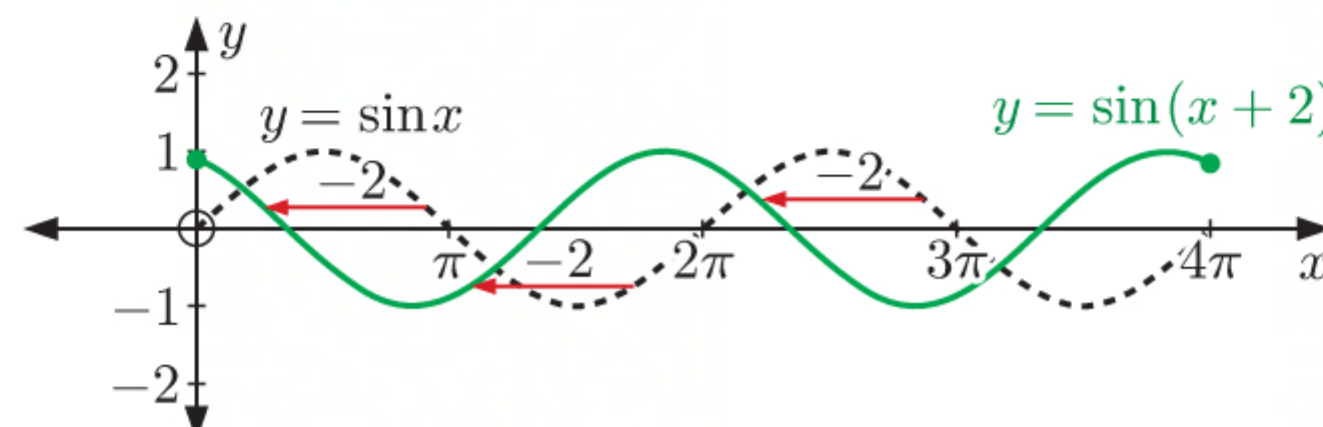
c $y = \sin x - 0.5$ is a vertical translation of $y = \sin x$ downwards by 0.5 units.



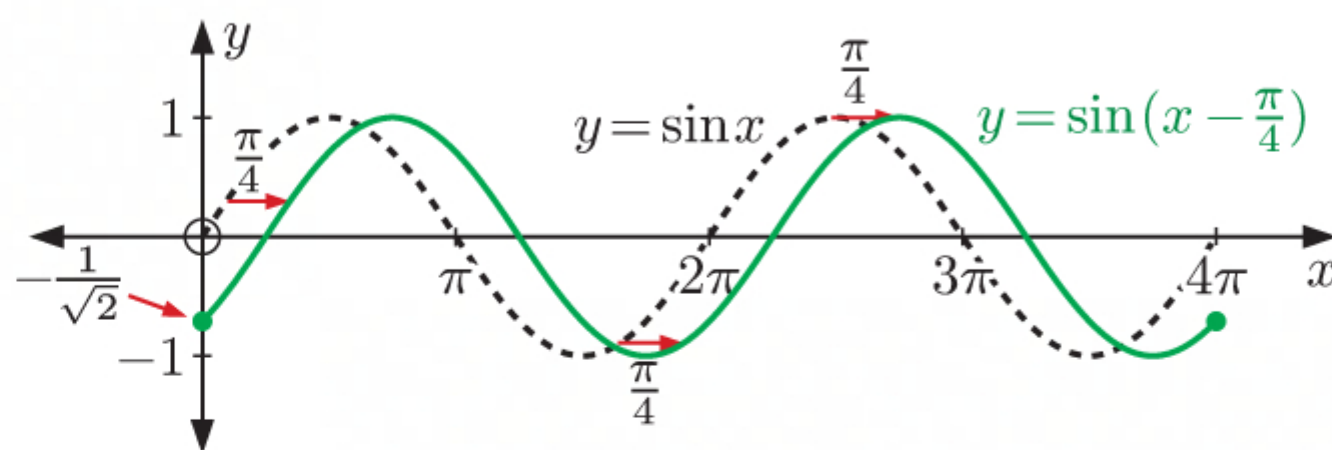
d $y = \sin(x - 2)$ is a horizontal translation of $y = \sin x$ to the right by 2 units.



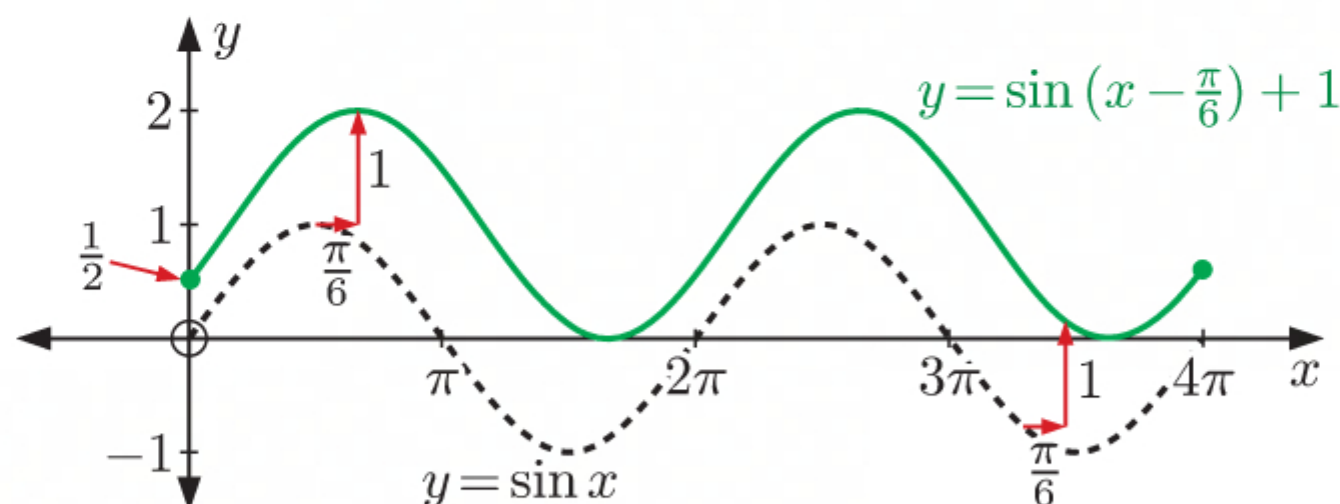
e $y = \sin(x + 2)$ is a horizontal translation of $y = \sin x$ to the left by 2 units.



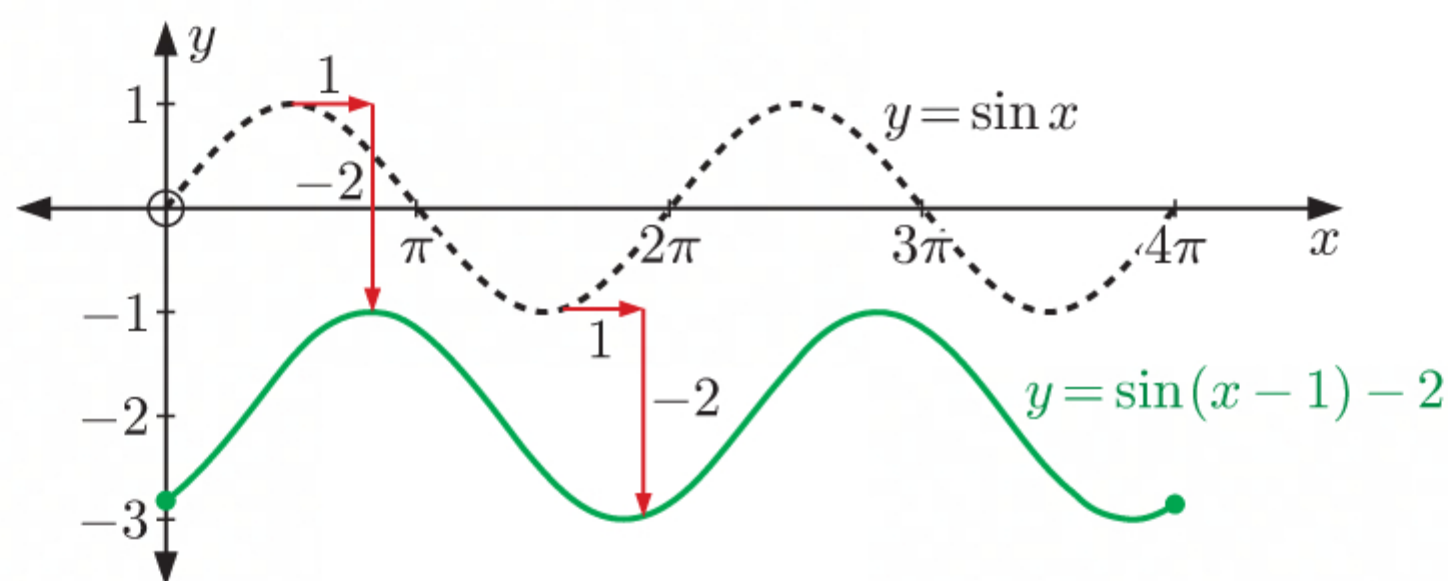
- f** $y = \sin\left(x - \frac{\pi}{4}\right)$ is a horizontal translation of $y = \sin x$ to the right by $\frac{\pi}{4}$ units.



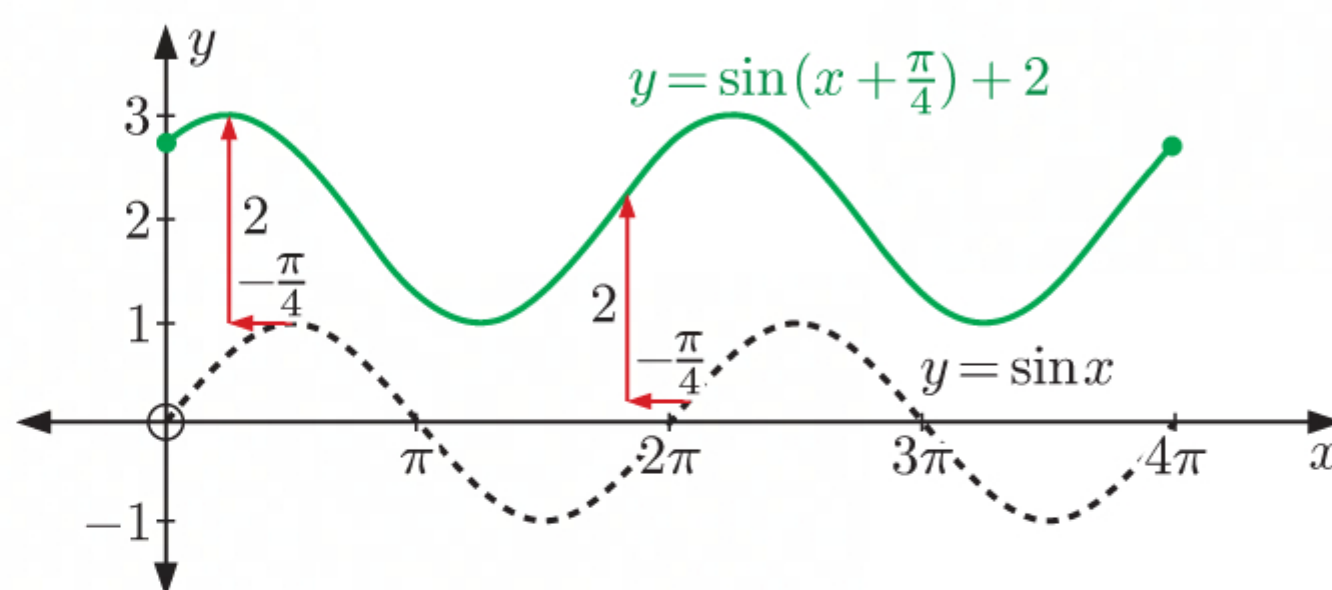
- g** $y = \sin\left(x - \frac{\pi}{6}\right) + 1$ is a translation of $y = \sin x$ to the right by $\frac{\pi}{6}$ units and upwards by 1 unit.



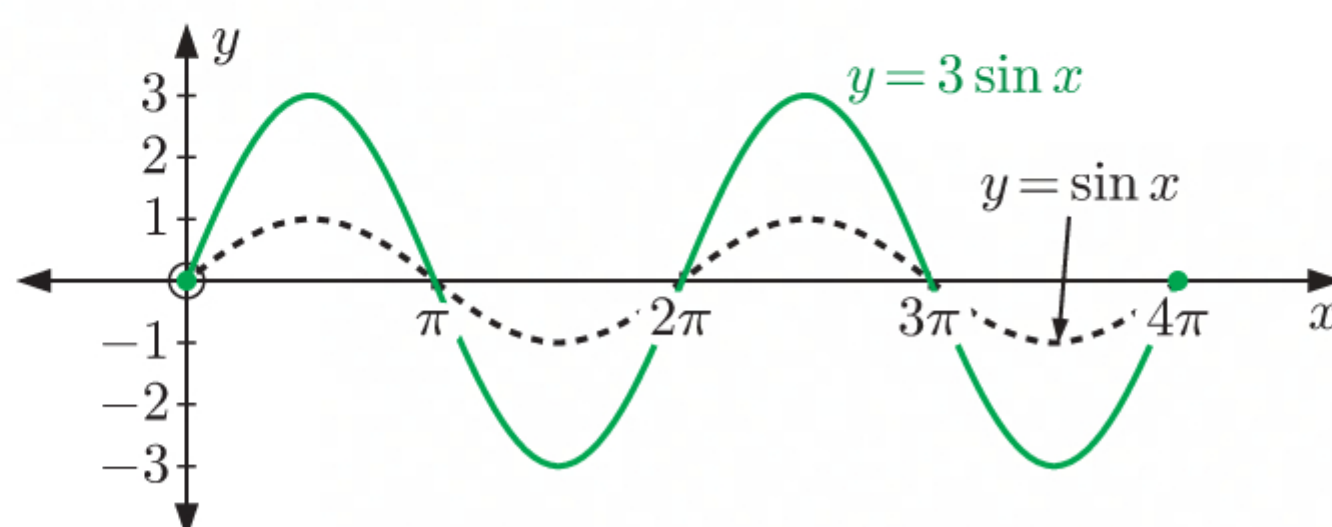
- h** $y = \sin(x - 1) - 2$ is a translation of $y = \sin x$ to the right by 1 unit and downwards by 2 units.



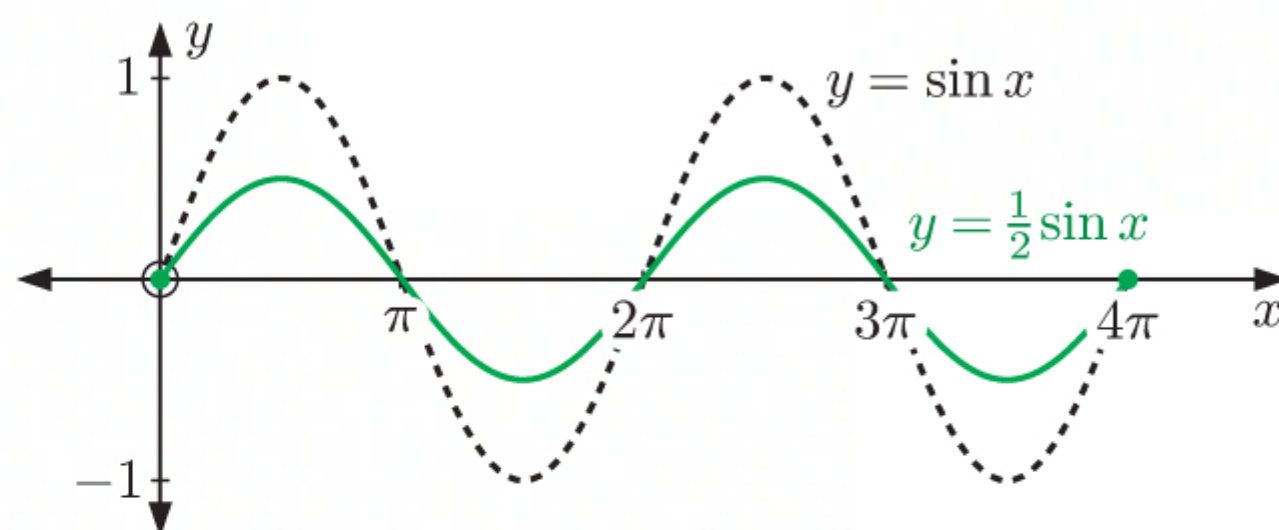
- i** $y = \sin\left(x + \frac{\pi}{4}\right) + 2$ is a translation of $y = \sin x$ to the left by $\frac{\pi}{4}$ units and upwards by 2 units.



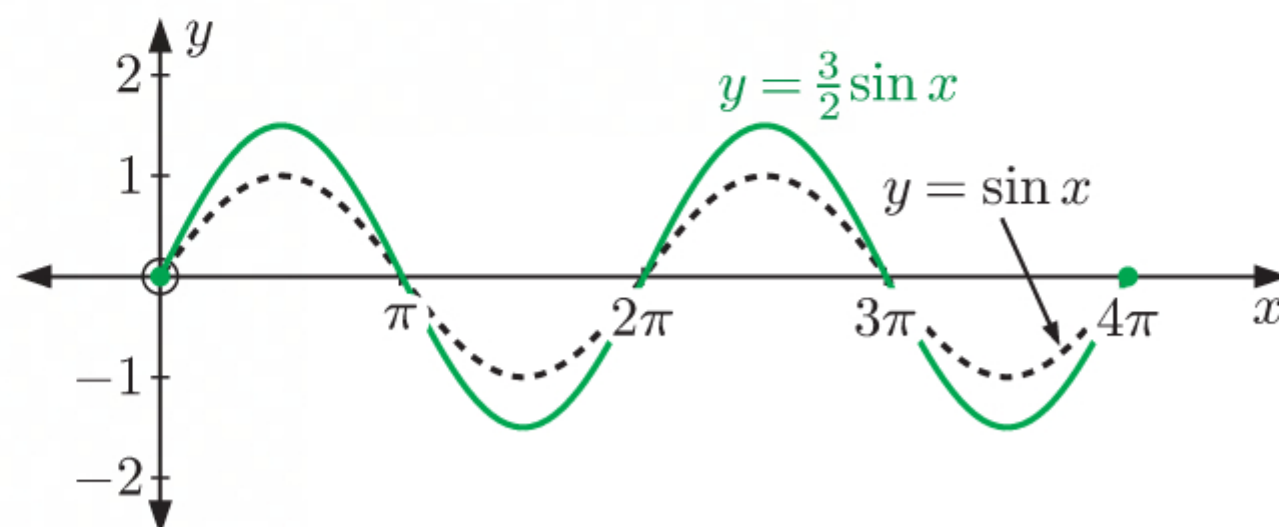
- j** $y = 3 \sin x$ is a vertical stretch of $y = \sin x$ with scale factor 3. The amplitude is 3 and the period is 2π .



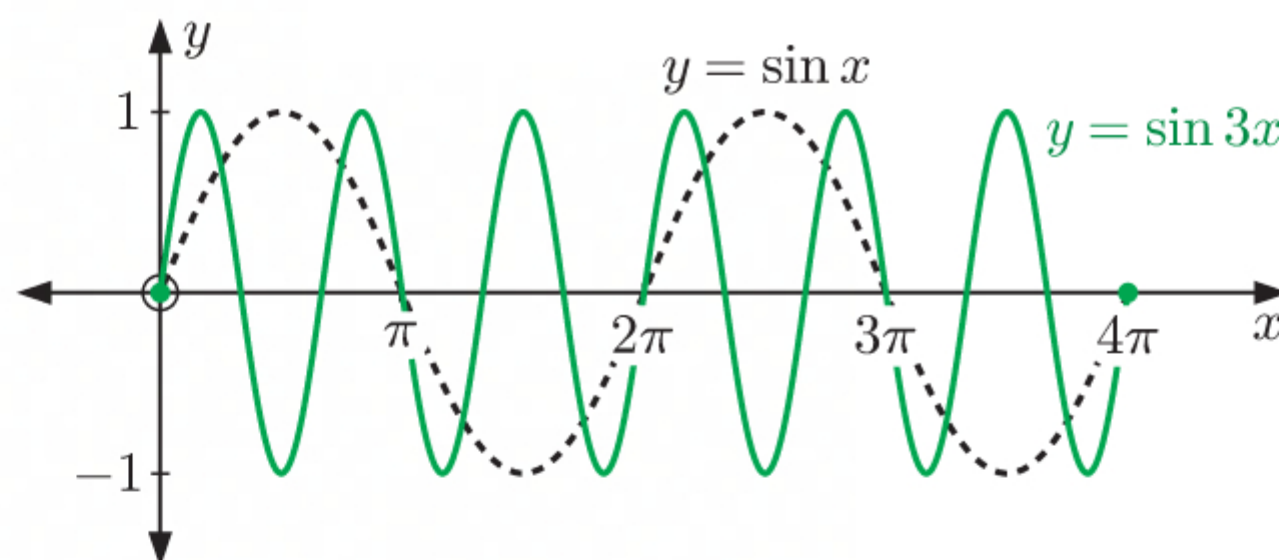
- k** $y = \frac{1}{2} \sin x$ is a vertical stretch of $y = \sin x$ with scale factor $\frac{1}{2}$.
The amplitude is $\frac{1}{2}$ and the period is 2π .



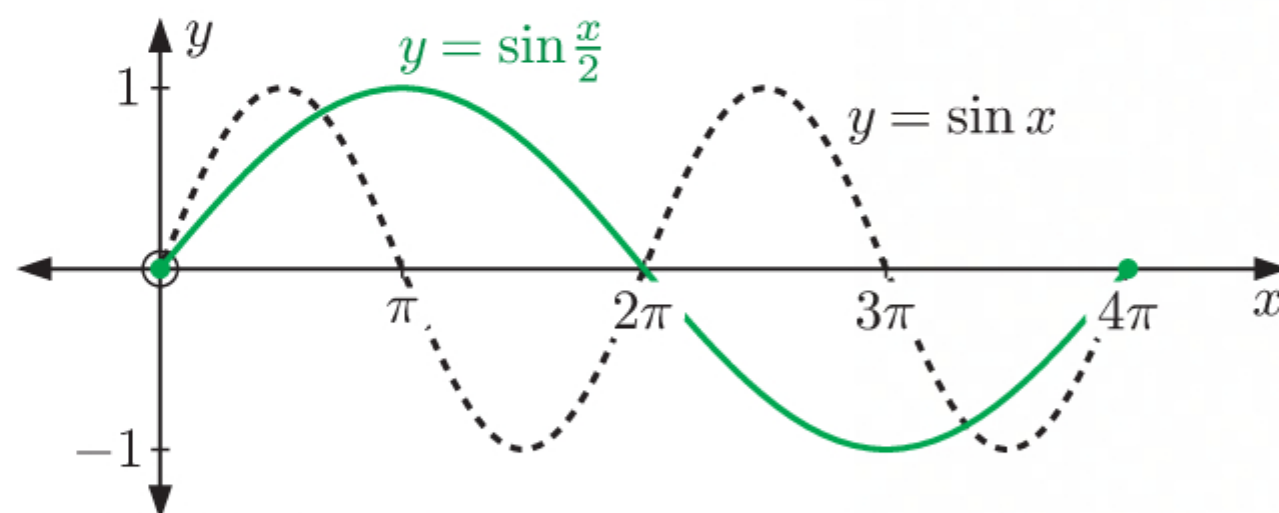
- l** $y = \frac{3}{2} \sin x$ is a vertical stretch of $y = \sin x$ with scale factor $\frac{3}{2}$.
The amplitude is $\frac{3}{2}$ and the period is 2π .



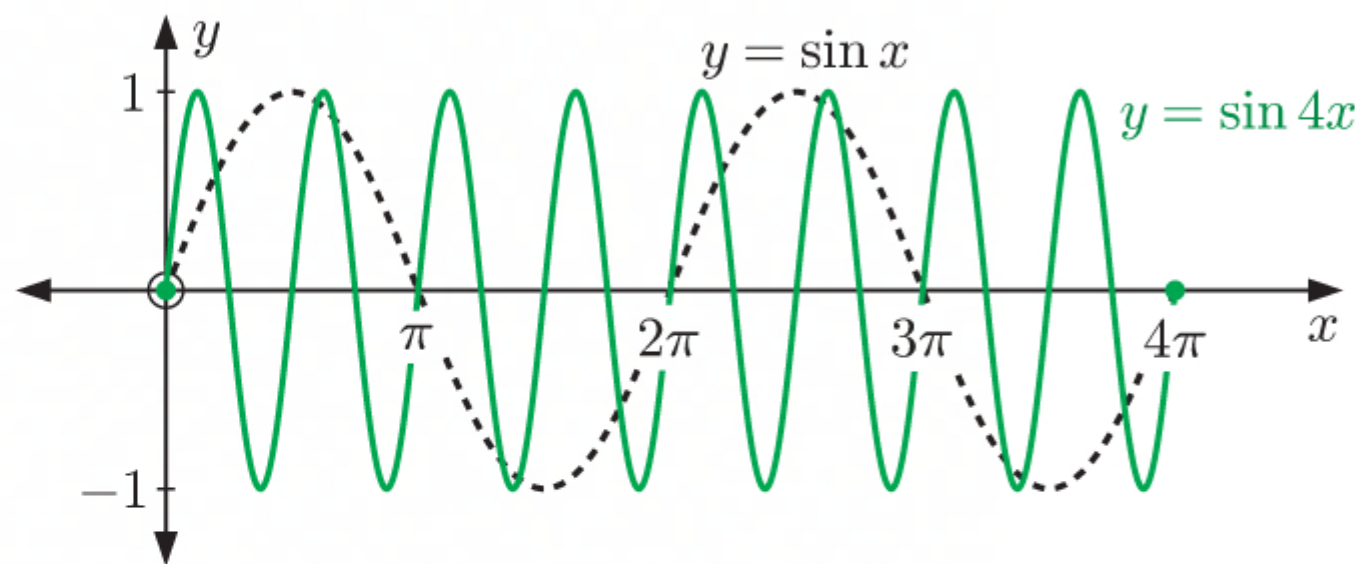
- m** $y = \sin 3x$ is a horizontal stretch of $y = \sin x$ with scale factor $\frac{1}{3}$.
The period is $\frac{2\pi}{3}$. \therefore the maximum values are $\frac{2\pi}{3}$ units apart.



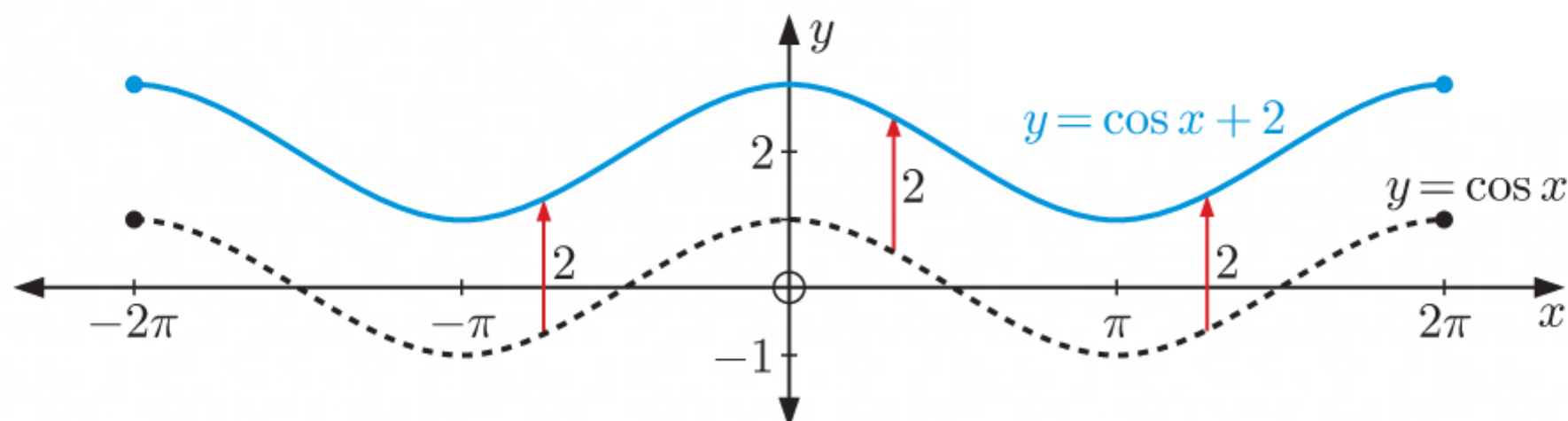
- n** $y = \sin \frac{x}{2}$ is a horizontal stretch of $y = \sin x$ with scale factor $\frac{1}{\frac{1}{2}} = 2$.
The period is $\frac{2\pi}{\frac{1}{2}} = 4\pi$. \therefore the maximum values are 4π units apart.



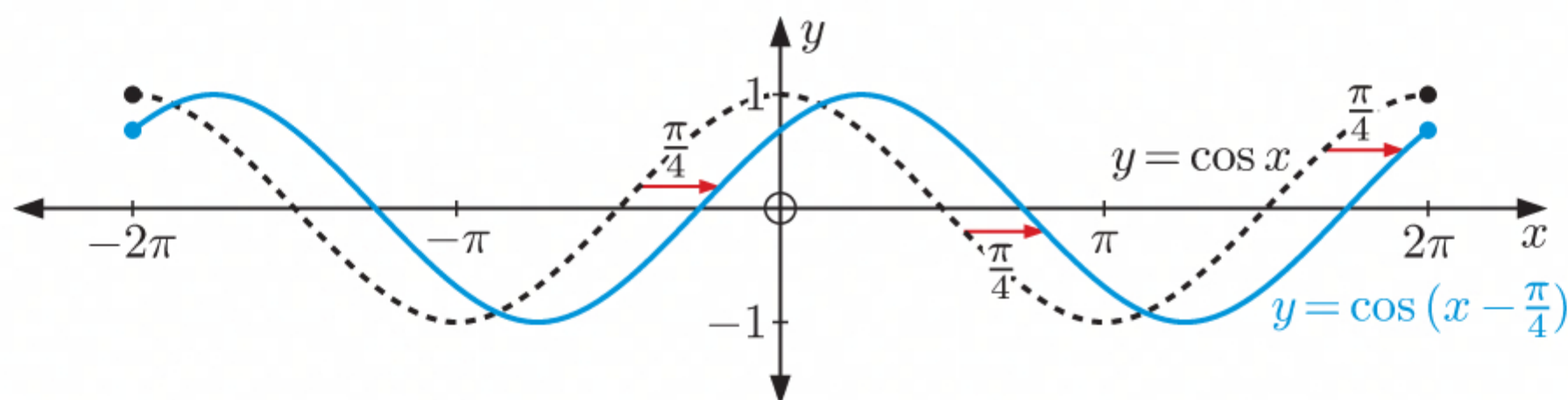
- $y = \sin 4x$ is a horizontal stretch of $y = \sin x$ with scale factor $\frac{1}{4}$.
The period is $\frac{2\pi}{4} = \frac{\pi}{2}$. \therefore the maximum values are $\frac{\pi}{2}$ units apart.



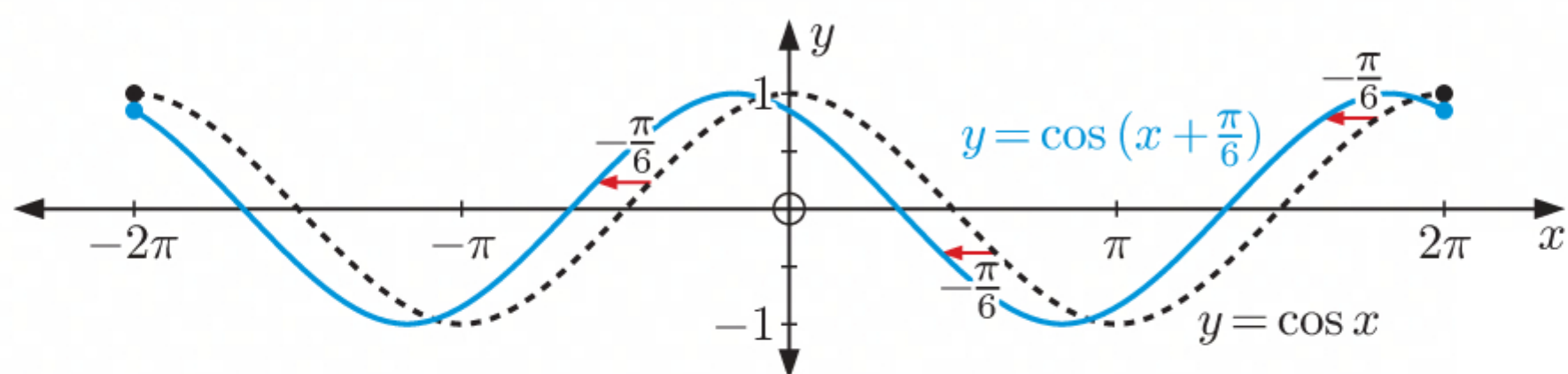
- 9 a $y = \cos x + 2$ is a vertical translation of $y = \cos x$ upwards by 2 units.



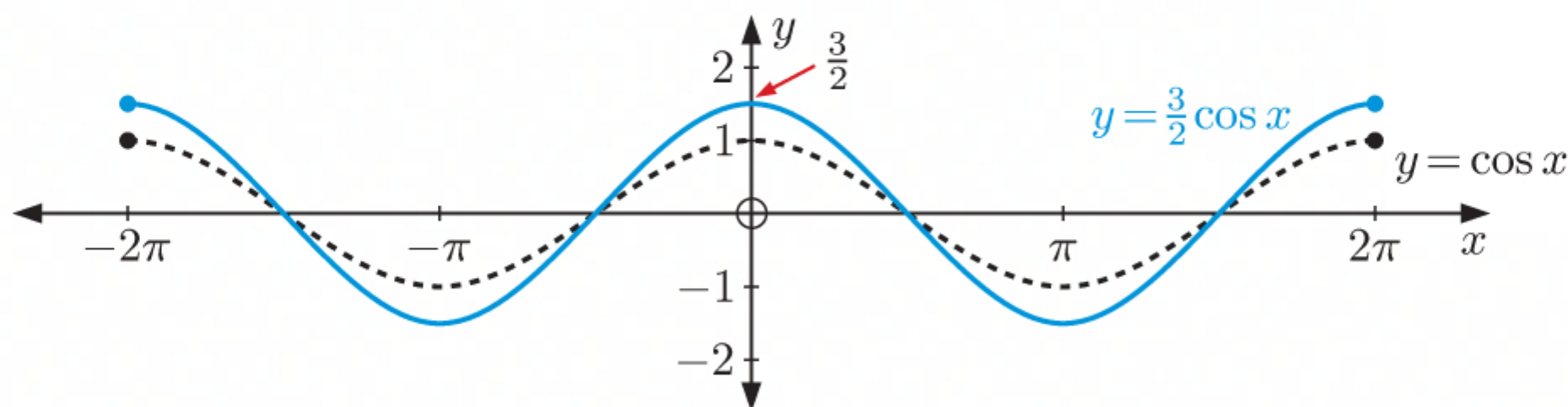
- b $y = \cos(x - \frac{\pi}{4})$ is a horizontal translation of $y = \cos x$ to the right by $\frac{\pi}{4}$ units.



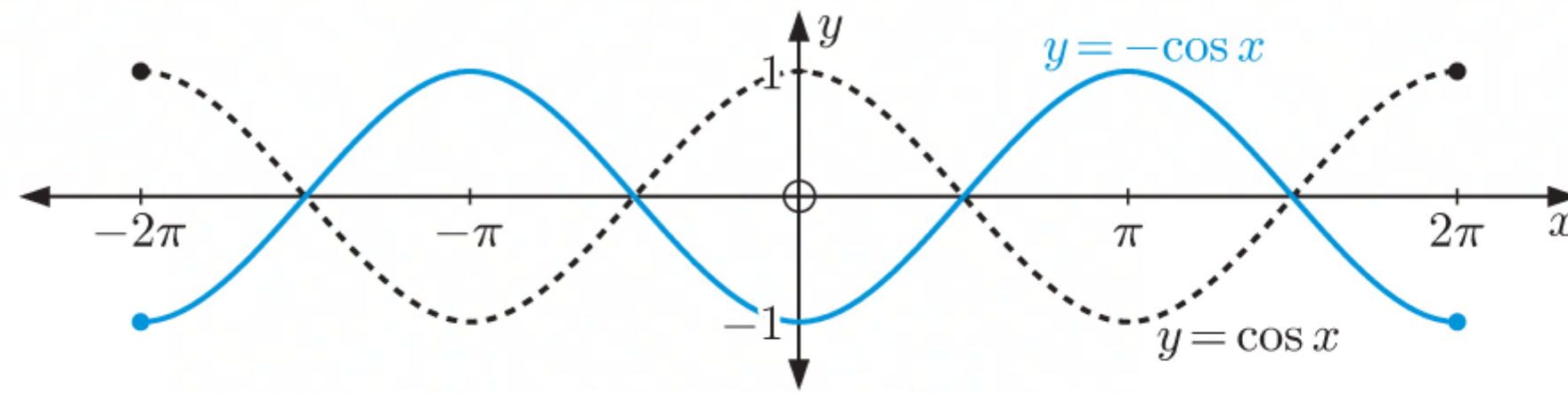
- c $y = \cos(x + \frac{\pi}{6})$ is a horizontal translation of $y = \cos x$ to the left by $\frac{\pi}{6}$ units.



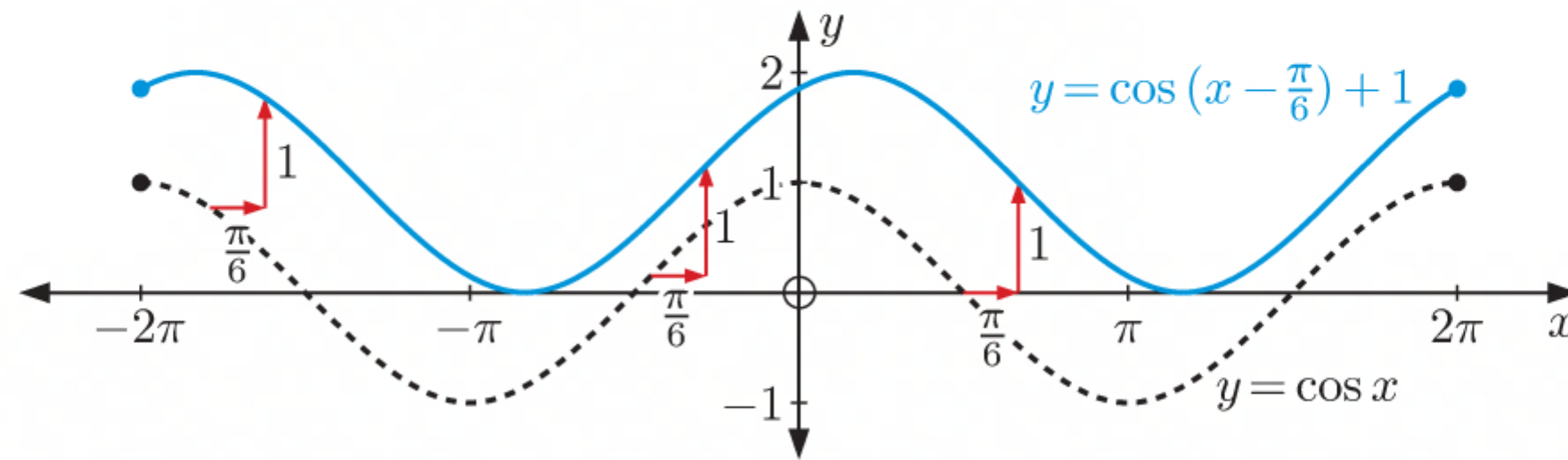
- d $y = \frac{3}{2} \cos x$ is a vertical stretch of $y = \cos x$ with scale factor $\frac{3}{2}$.
The amplitude is $\frac{3}{2}$ and the period is 2π .



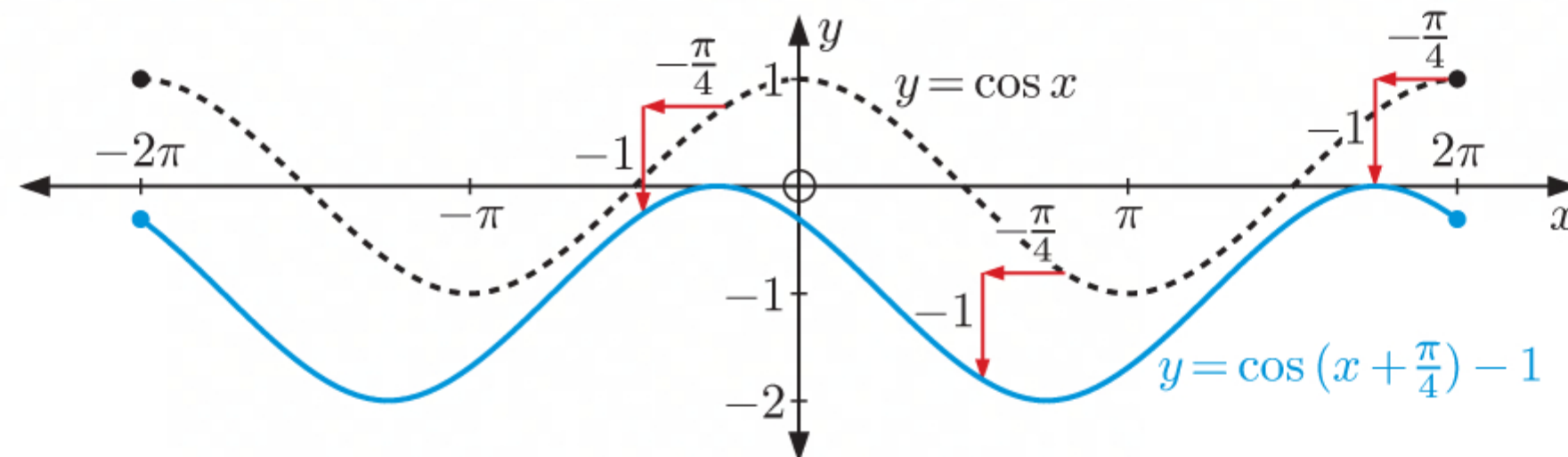
- e $y = -\cos x$ is the reflection of $y = \cos x$ in the x -axis.
The amplitude is 1 and the period is 2π .



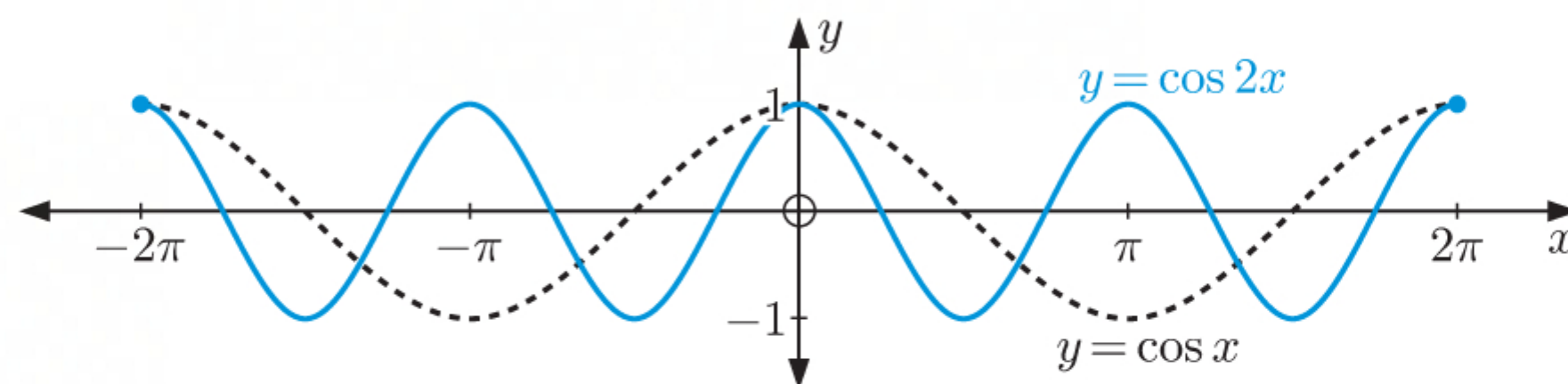
- f $y = \cos\left(x - \frac{\pi}{6}\right) + 1$ is a translation of $y = \cos x$ to the right by $\frac{\pi}{6}$ units and upwards by 1 unit.



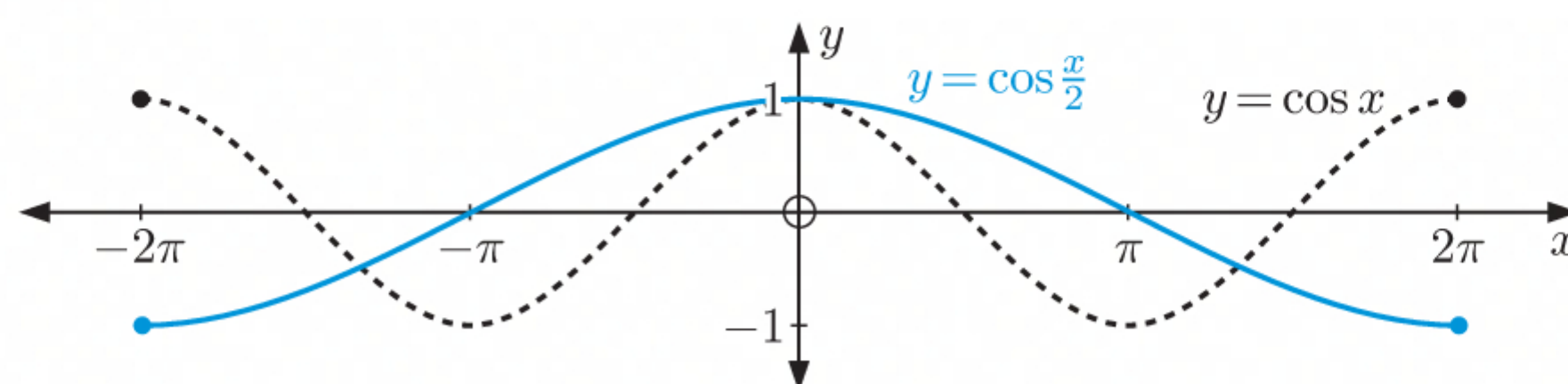
- g $y = \cos\left(x + \frac{\pi}{4}\right) - 1$ is a translation of $y = \cos x$ to the left by $\frac{\pi}{4}$ units and downwards by 1 unit.



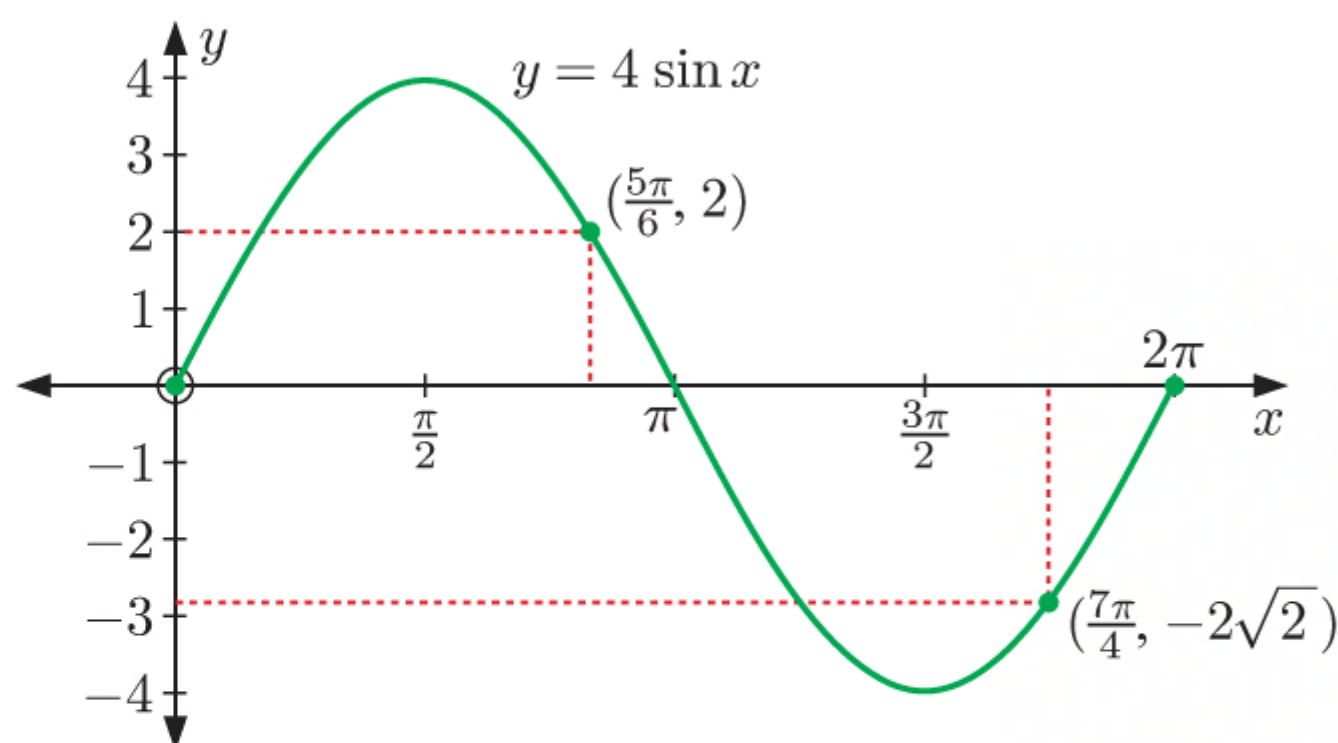
- h $y = \cos 2x$ is a horizontal stretch of $y = \cos x$ with scale factor $\frac{1}{2}$.
The period is $\frac{2\pi}{2} = \pi$. \therefore the maximum values are π units apart.



- i $y = \cos \frac{x}{2}$ is a horizontal stretch of $y = \cos x$ with scale factor $\frac{1}{\frac{1}{2}} = 2$.
The period is $\frac{2\pi}{\frac{1}{2}} = 4\pi$. \therefore the maximum values are 4π units apart.



- 10 a** $y = 4 \sin x$ is a vertical stretch of $y = \sin x$ with scale factor 4.
The amplitude is 4 and the period is 2π .



- b i** When $x = \frac{5\pi}{6}$, $y = 4 \sin \frac{5\pi}{6}$
 $= 4 \times \frac{1}{2}$
 $= 2$
- ii** When $x = \frac{7\pi}{4}$, $y = 4 \sin \frac{7\pi}{4}$
 $= 4 \times \left(-\frac{1}{\sqrt{2}}\right)$
 $= -2\sqrt{2}$
 ≈ -2.83
- 11** $y = 3 \cos x$ has minimum value -3 and maximum value 3 .
Now, $y = 3 \cos x + d$ is a vertical translation of $y = 3 \cos x$ by d units.
- a** $y = 3 \cos x + d$ will lie entirely above the x -axis when $y = 3 \cos x$ has been translated more than 3 units upwards.
 $\therefore d > 3$
- b** $y = 3 \cos x + d$ will lie entirely below the x -axis when $y = 3 \cos x$ has been translated more than 3 units downwards.
 $\therefore d < -3$
- c** $y = 3 \cos x + d$ will lie partially above and partially below the x -axis when $y = 3 \cos x$ has been translated between 0 and 3 units upwards or downwards.
 $\therefore -3 < d < 3$

12 a $\sin x \xrightarrow[\text{scale factor } \frac{1}{3}]{\text{horizontal stretch}} \sin 3x \xrightarrow[\text{scale factor 2}]{\text{vertical stretch}} 2 \sin 3x$

A horizontal stretch with scale factor $\frac{1}{3}$, then a vertical stretch with scale factor 2 maps $y = \sin x$ onto $y = 2 \sin 3x$.

b $\cos x \xrightarrow[\text{scale factor 2}]{\text{vertical stretch}} 2 \cos x \xrightarrow[\text{reflection in } x\text{-axis}]{\text{reflection}} -2 \cos x$

A vertical stretch with scale factor 2, then a reflection in the x -axis maps $y = \cos x$ onto $y = -2 \cos x$.

c $\sin x \xrightarrow[\text{scale factor 3}]{\text{vertical stretch}} 3 \sin x \xrightarrow[\text{translation } \begin{pmatrix} 0 \\ -5 \end{pmatrix}]{\text{translation}} 3 \sin x - 5$

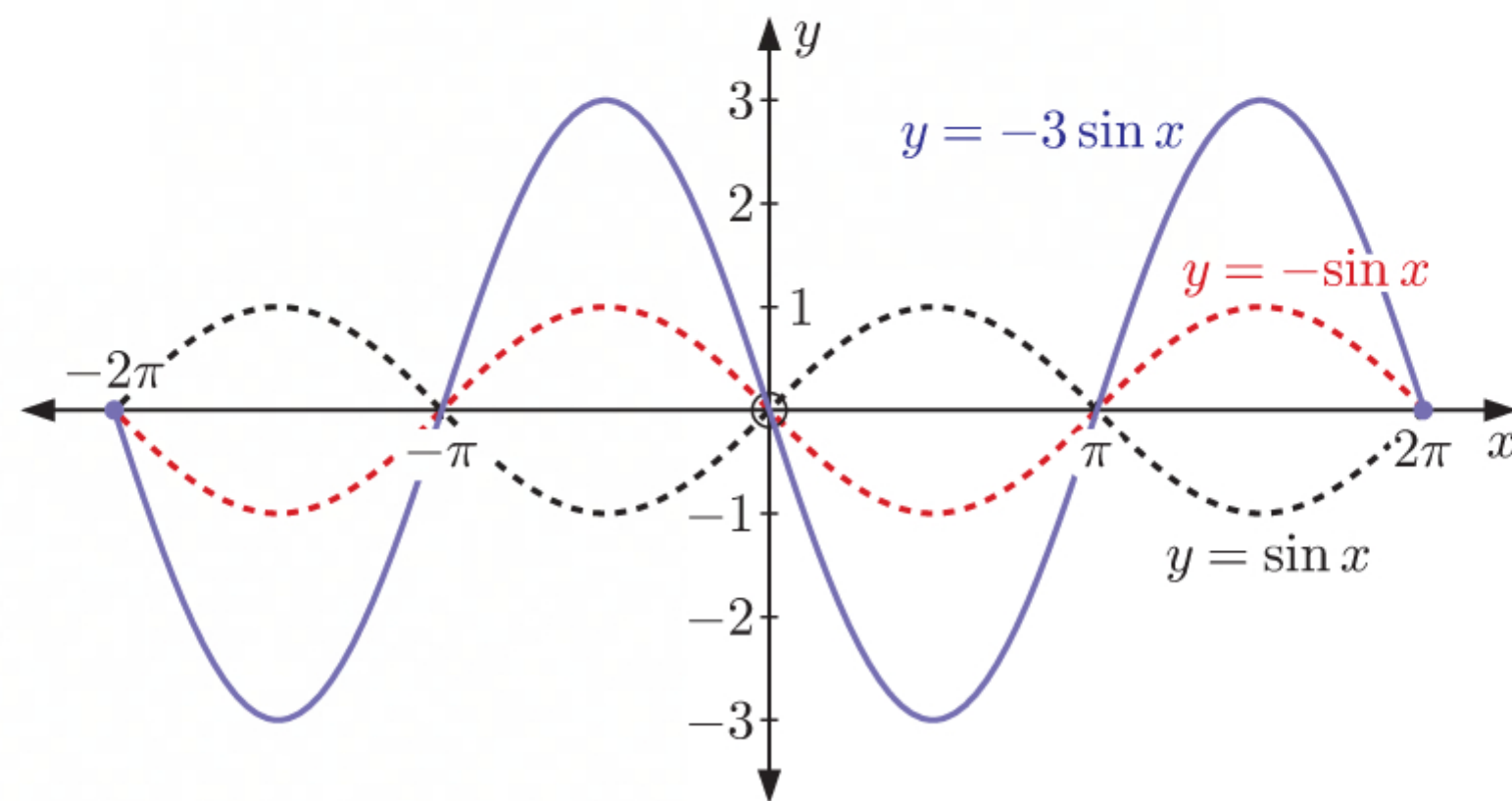
A vertical stretch with scale factor 3, then a translation 5 units downwards maps $y = \sin x$ onto $y = 3 \sin x - 5$.

d $\cos x \xrightarrow[\text{scale factor } \frac{1}{2}]{\text{horizontal stretch}} \cos 2x \xrightarrow[\text{translation } \begin{pmatrix} -\frac{\pi}{6} \\ 0 \end{pmatrix}]{\text{}} \cos\left(2\left(x + \frac{\pi}{6}\right)\right)$

A horizontal stretch with scale factor $\frac{1}{2}$, then a translation $\frac{\pi}{6}$ units left maps $y = \cos x$ onto $y = \cos\left(2\left(x + \frac{\pi}{6}\right)\right)$.

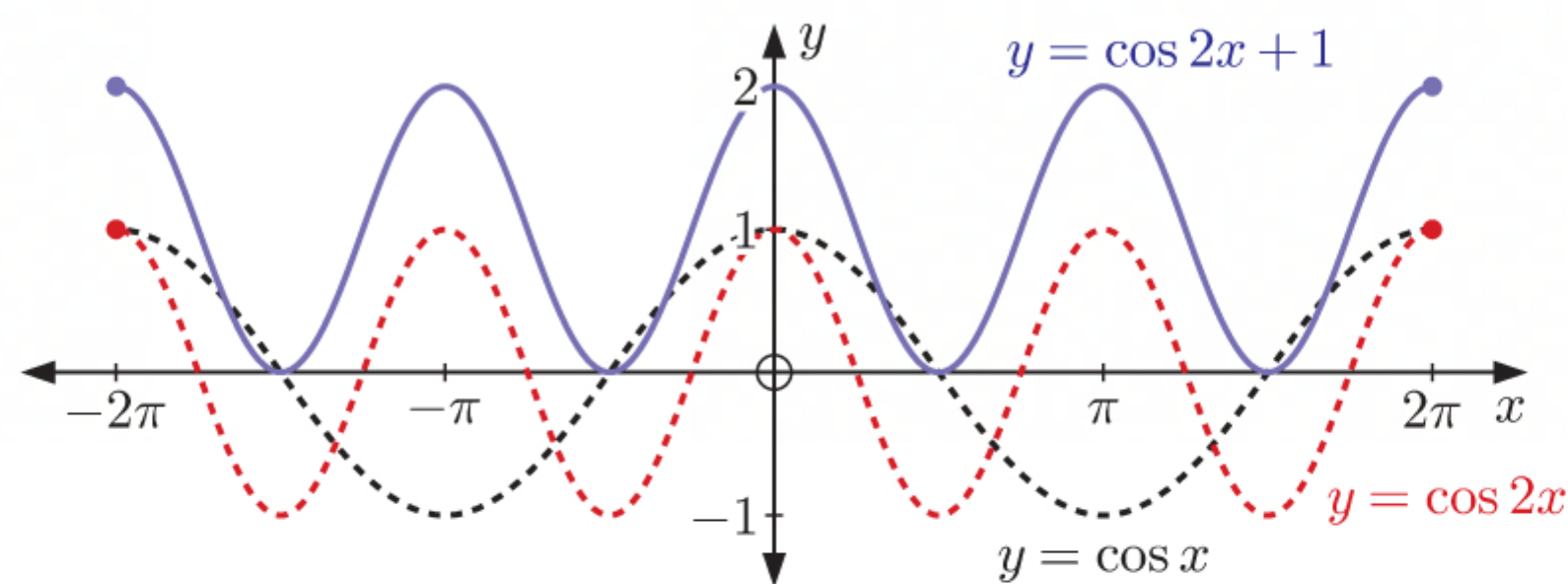
- 13 a $a = -3$, so the amplitude is $|-3| = 3$.

We reflect $y = \sin x$ in the x -axis to give $y = -\sin x$, then stretch $y = -\sin x$ vertically with scale factor 3.



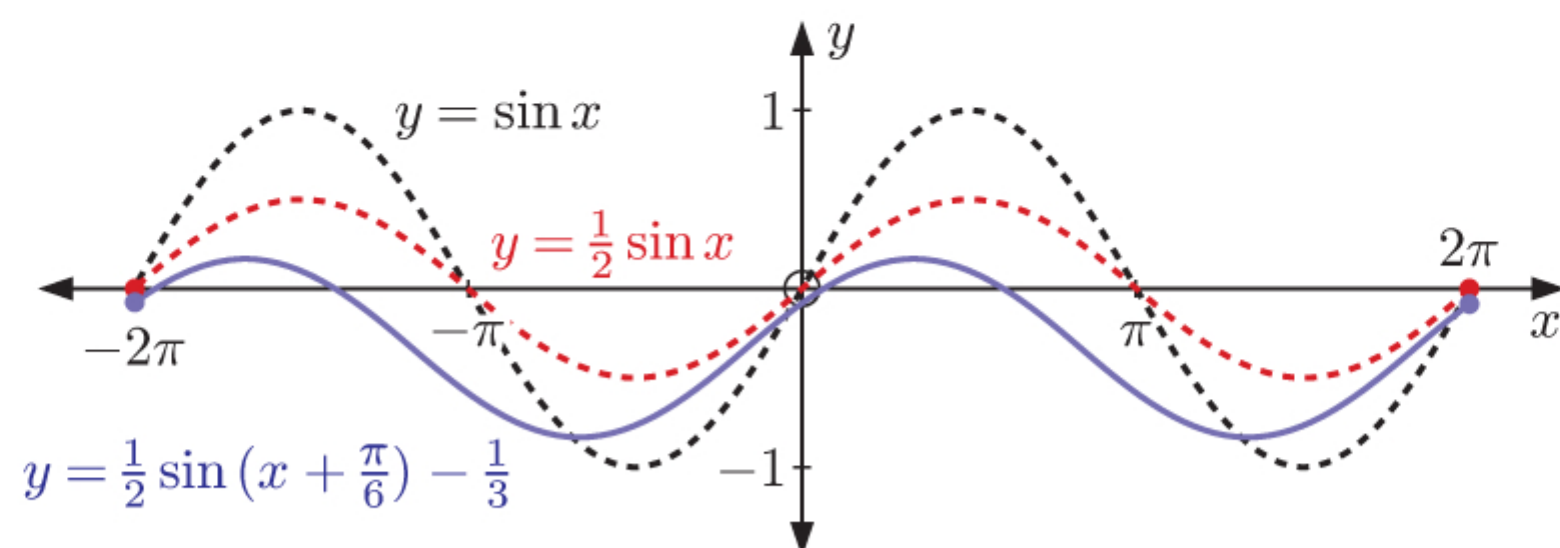
- b $b = 2$, so the period is $\frac{2\pi}{b} = \frac{2\pi}{2} = \pi$.

We stretch $y = \cos x$ horizontally with scale factor $\frac{1}{2}$ to give $y = \cos 2x$, then translate $y = \cos 2x$ 1 unit upwards to give $y = \cos 2x + 1$.



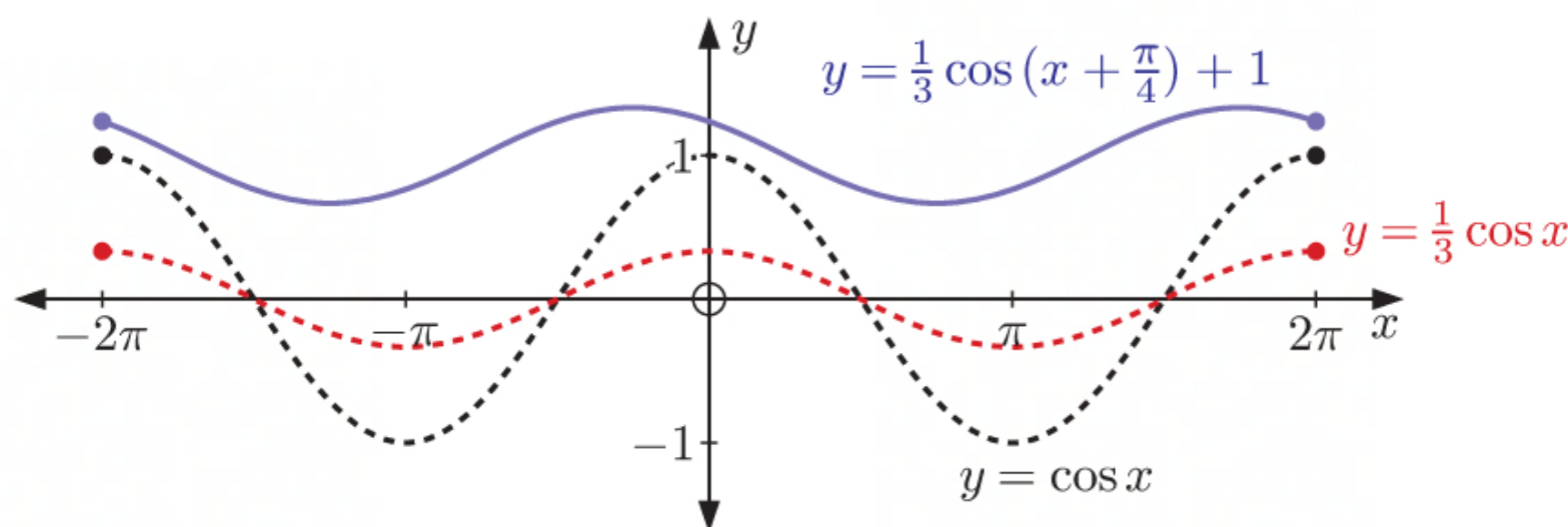
- c $a = \frac{1}{2}$, so the amplitude is $\left|\frac{1}{2}\right| = \frac{1}{2}$.

We stretch $y = \sin x$ vertically with scale factor $\frac{1}{2}$ to give $y = \frac{1}{2} \sin x$, then translate $y = \frac{1}{2} \sin x$ $\frac{\pi}{6}$ units to the left and $\frac{1}{3}$ units downwards to give $y = \frac{1}{2} \sin\left(x + \frac{\pi}{6}\right) - \frac{1}{3}$.



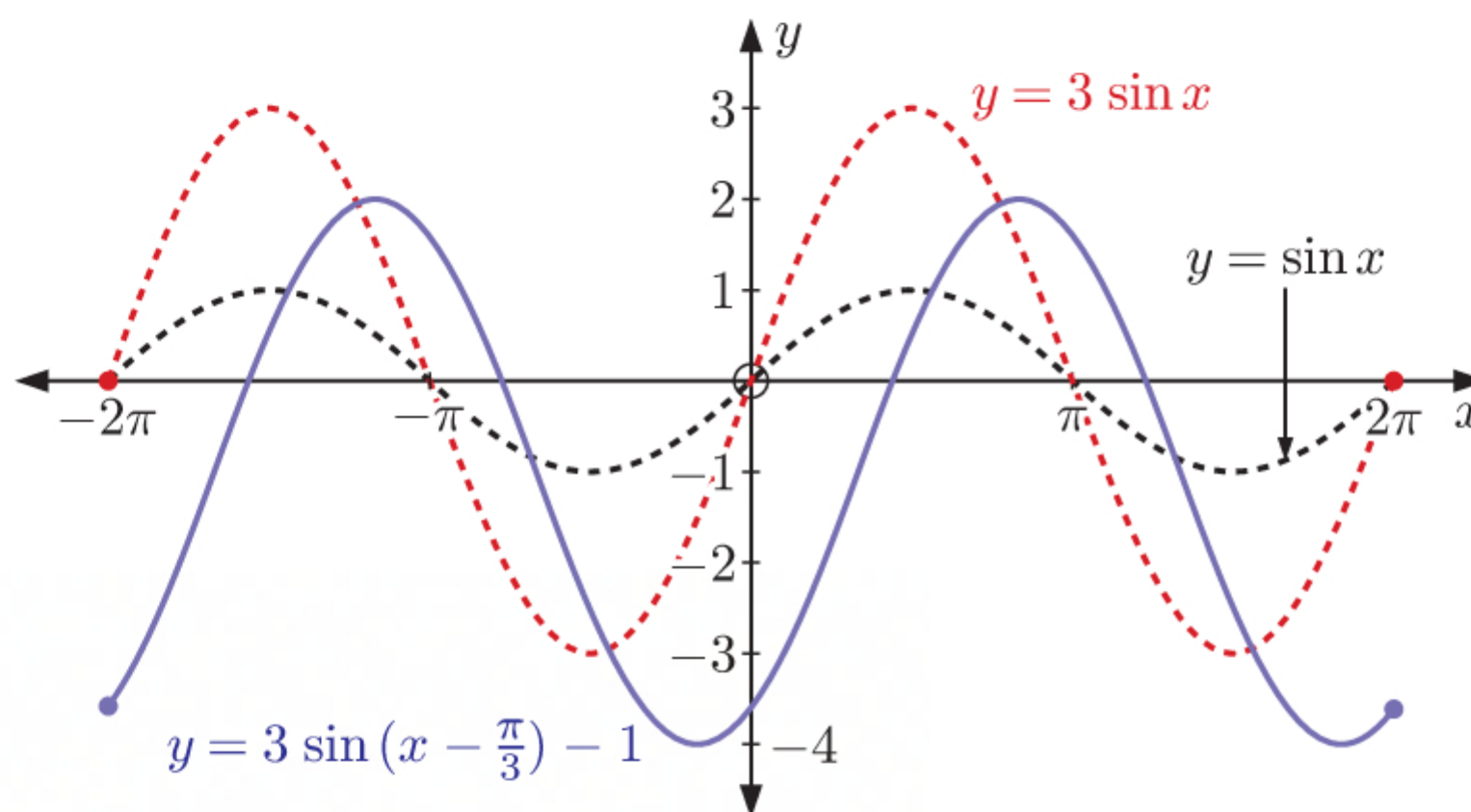
- d** $a = \frac{1}{3}$, so the amplitude is $|\frac{1}{3}| = \frac{1}{3}$.

We stretch $y = \cos x$ vertically with scale factor $\frac{1}{3}$ to give $y = \frac{1}{3} \cos x$, then translate $y = \frac{1}{3} \cos x$ $\frac{\pi}{4}$ units to the left and 1 unit upwards to give $y = \frac{1}{3} \cos(x + \frac{\pi}{4}) + 1$.



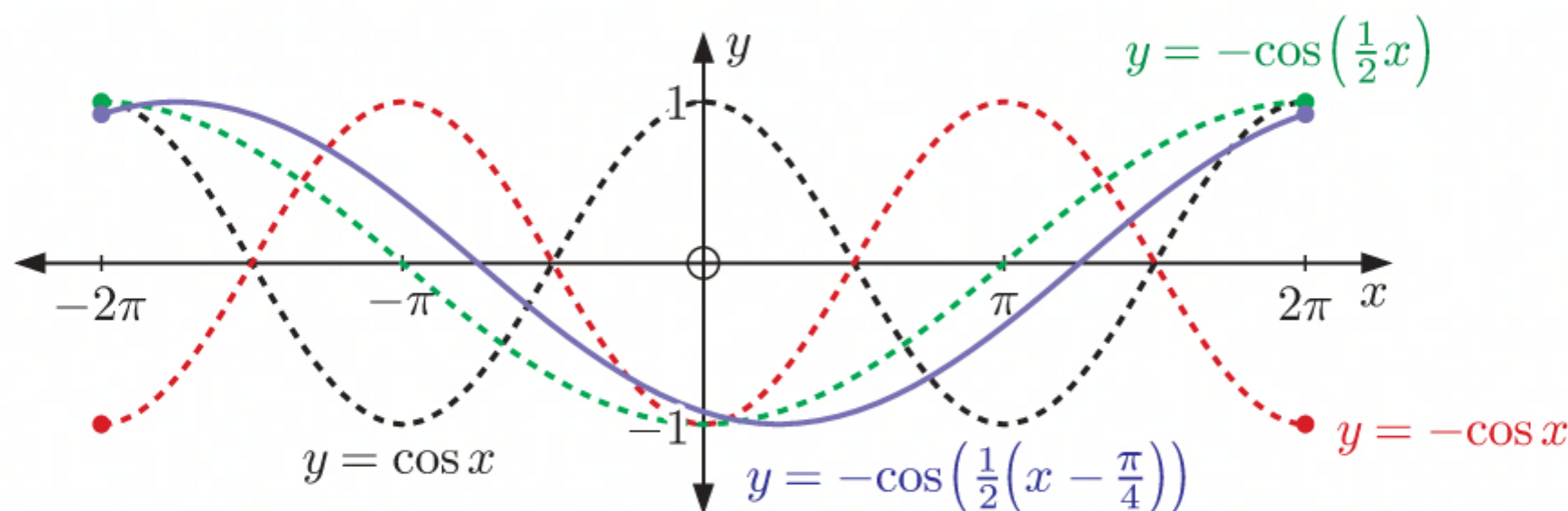
- e** $a = 3$, so the amplitude is $|3| = 3$.

We stretch $y = \sin x$ vertically with scale factor 3 to give $y = 3 \sin x$, then translate $y = 3 \sin x$ $\frac{\pi}{3}$ units to the right and 1 unit downwards to give $y = 3 \sin(x - \frac{\pi}{3}) - 1$.



- f** $a = -1$, so the amplitude is $|-1| = 1$. $b = \frac{1}{2}$, so the period is $\frac{2\pi}{b} = \frac{2\pi}{\frac{1}{2}} = 4\pi$.

We reflect $y = \cos x$ in the x -axis to give $y = -\cos x$, then stretch $y = -\cos x$ horizontally with scale factor 2 to give $y = -\cos(\frac{1}{2}x)$, then translate $y = -\cos(\frac{1}{2}x)$ $\frac{\pi}{4}$ units to the right to give $y = -\cos(\frac{1}{2}(x - \frac{\pi}{4}))$.



14 $y = a \sin(b(x - c)) + d$

- a** b affects the period of the function. So, a change in b will produce a change in the x -intercepts of the function.
- c affects horizontal translation. So, a change in c will produce a change in the x -intercepts of the function (provided the change in c is not a multiple of π).
- d affects vertical translation. So, a change in d will produce a change in the x -intercepts of the function.

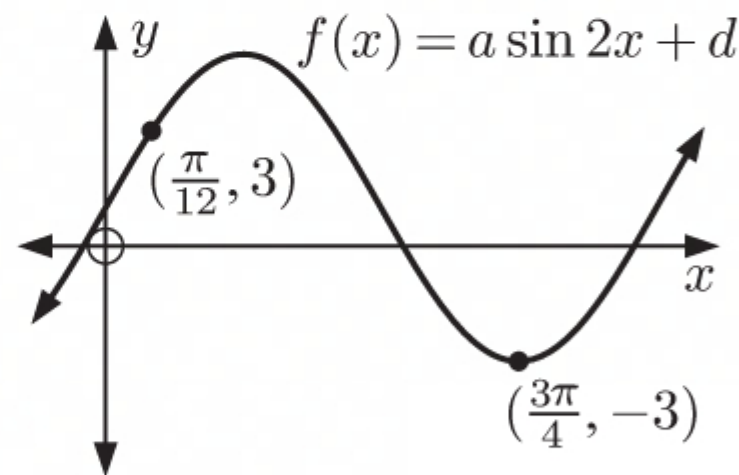
b c affects horizontal translation. So, a change in c will produce a change in the y -intercept of the function (provided the change in c is not a multiple of π).

d affects vertical translation. So, a change in d will produce a change in the y -intercept of the function.

c a affects the amplitude of the function. So, a change in a will produce a change in the range of the function (provided a is not changed to -1).

d affects vertical translation. So, a change in d will produce a change in the range of the function.

15 a



$$f(x) = a \sin 2x + d$$

$$f\left(\frac{\pi}{12}\right) = 3$$

$$\therefore a \sin\left(2 \times \frac{\pi}{12}\right) + d = 3$$

$$\therefore a \sin \frac{\pi}{6} + d = 3$$

$$\therefore \frac{1}{2}a + d = 3$$

$$\therefore d = 3 - \frac{1}{2}a \quad \dots (*)$$

and

$$f\left(\frac{3\pi}{4}\right) = -3$$

$$\therefore a \sin\left(2 \times \frac{3\pi}{4}\right) + d = -3$$

$$\therefore a \sin \frac{3\pi}{2} + 3 - \frac{1}{2}a = -3 \quad \{\text{using } (*)\}$$

$$\therefore -a + 3 - \frac{1}{2}a = -3$$

$$\therefore -\frac{3}{2}a = -6$$

$$\therefore a = 4$$

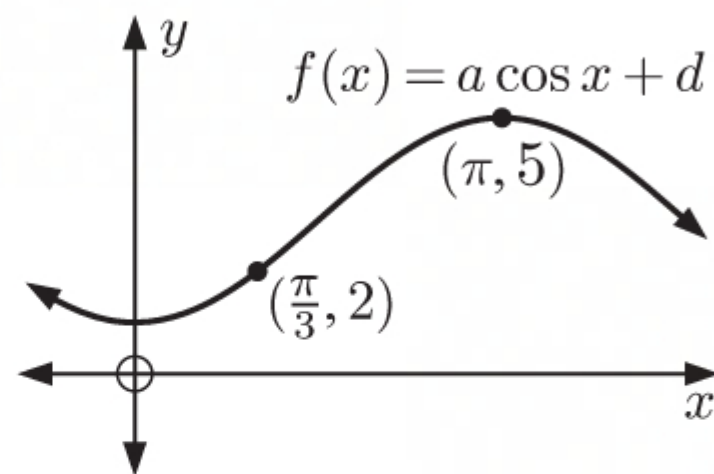
Substituting $a = 4$ into $(*)$ gives $d = 3 - \frac{1}{2}(4)$

$$= 3 - 2$$

$$= 1$$

$$\therefore a = 4, d = 1$$

b



$$f(x) = a \cos x + d$$

$$f\left(\frac{\pi}{3}\right) = 2$$

$$\therefore a \cos \frac{\pi}{3} + d = 2$$

$$\therefore \frac{1}{2}a + d = 2$$

$$\therefore d = 2 - \frac{1}{2}a \quad \dots (*)$$

and

$$f(\pi) = 5$$

$$\therefore a \cos \pi + d = 5$$

$$\therefore -a + 2 - \frac{1}{2}a = 5 \quad \{\text{using } (*)\}$$

$$\therefore -\frac{3}{2}a = 3$$

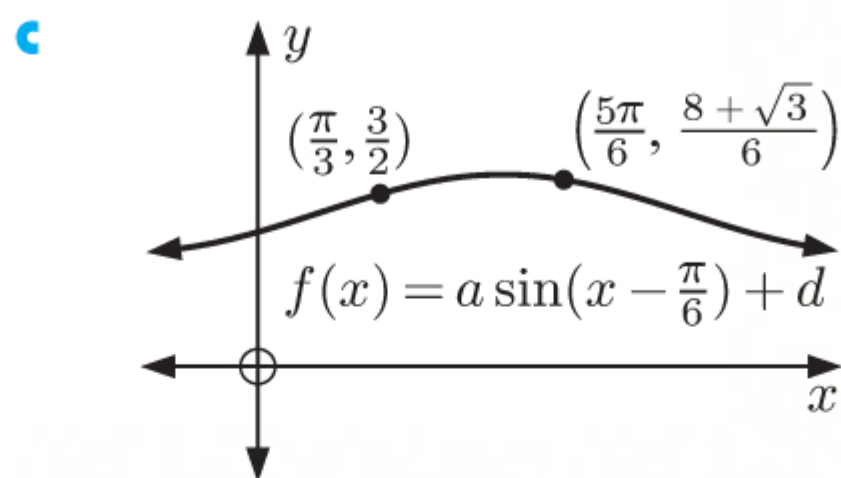
$$\therefore a = -2$$

Substituting $a = -2$ into $(*)$ gives $d = 2 - \frac{1}{2}(-2)$

$$= 2 + 1$$

$$= 3$$

$$\therefore a = -2, d = 3$$



$$f(x) = a \sin\left(x - \frac{\pi}{6}\right) + d$$

$$f\left(\frac{\pi}{3}\right) = \frac{3}{2}$$

$$\therefore a \sin\left(\frac{\pi}{3} - \frac{\pi}{6}\right) + d = \frac{3}{2}$$

$$\therefore a \sin \frac{\pi}{6} + d = \frac{3}{2}$$

$$\therefore \frac{1}{2}a + d = \frac{3}{2}$$

$$\therefore d = \frac{3}{2} - \frac{1}{2}a \quad \dots (*)$$

and

$$f\left(\frac{5\pi}{6}\right) = \frac{8 + \sqrt{3}}{6}$$

$$\therefore a \sin\left(\frac{5\pi}{6} - \frac{\pi}{6}\right) + d = \frac{8 + \sqrt{3}}{6}$$

$$\therefore a \sin \frac{2\pi}{3} + d = \frac{8 + \sqrt{3}}{6}$$

$$\therefore \frac{\sqrt{3}}{2}a + \frac{3}{2} - \frac{1}{2}a = \frac{8 + \sqrt{3}}{6}$$

$$\therefore \left(\frac{-1 + \sqrt{3}}{2}\right)a = \frac{-1 + \sqrt{3}}{6}$$

$$\therefore a = \frac{1}{3}$$

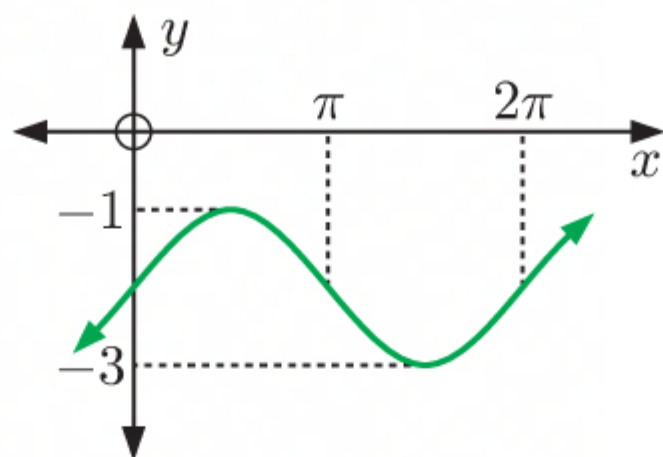
Substituting $a = \frac{1}{3}$ into (*) gives $d = \frac{3}{2} - \frac{1}{2}\left(\frac{1}{3}\right)$

$$= \frac{3}{2} - \frac{1}{6}$$

$$= \frac{4}{3}$$

$$\therefore a = \frac{1}{3}, \quad d = \frac{4}{3}$$

16 a



The amplitude is 1, so $a = 1$.

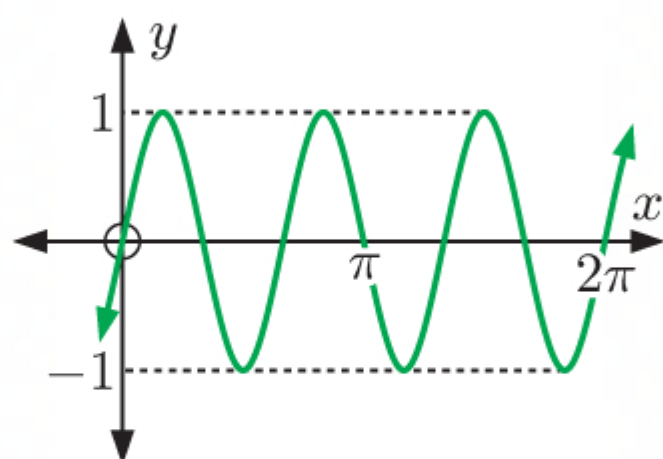
The period is 2π , so $\frac{2\pi}{b} = 2\pi$ and $\therefore b = 1$.

There is no horizontal translation, so $c = 0$.

The principal axis is $y = -2$, so $d = -2$.

\therefore the equation of the function is $y = \sin x - 2$.

b



The amplitude is 1, so $a = 1$.

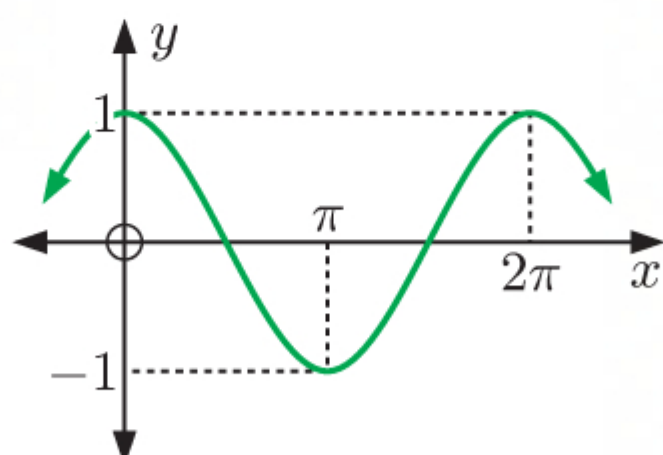
The period is $\frac{2\pi}{3}$, so $\frac{2\pi}{b} = \frac{2\pi}{3}$ and $\therefore b = 3$.

There is no horizontal translation, so $c = 0$.

The principal axis is $y = 0$, so $d = 0$.

\therefore the equation of the function is $y = \sin 3x$.

c



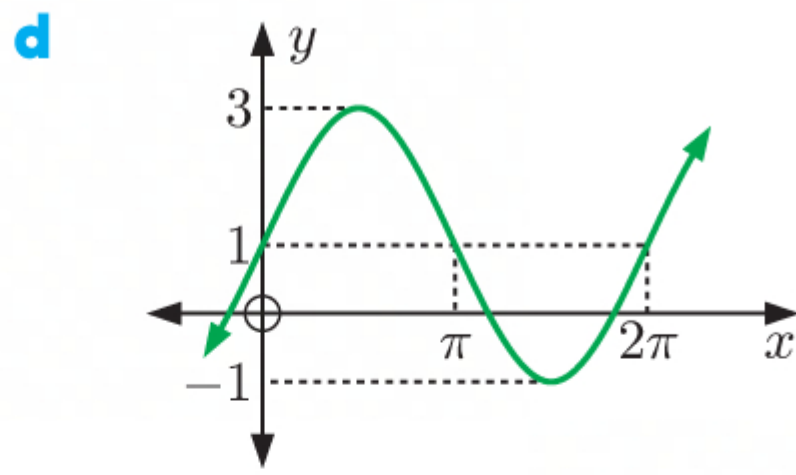
The amplitude is 1, so $a = 1$.

The period is 2π , so $\frac{2\pi}{b} = 2\pi$ and $\therefore b = 1$.

There is a horizontal translation of $\frac{\pi}{2}$ units to the left, so $c = -\frac{\pi}{2}$.

The principal axis is $y = 0$, so $d = 0$.

\therefore the equation of the function is $y = \sin\left(x + \frac{\pi}{2}\right)$.



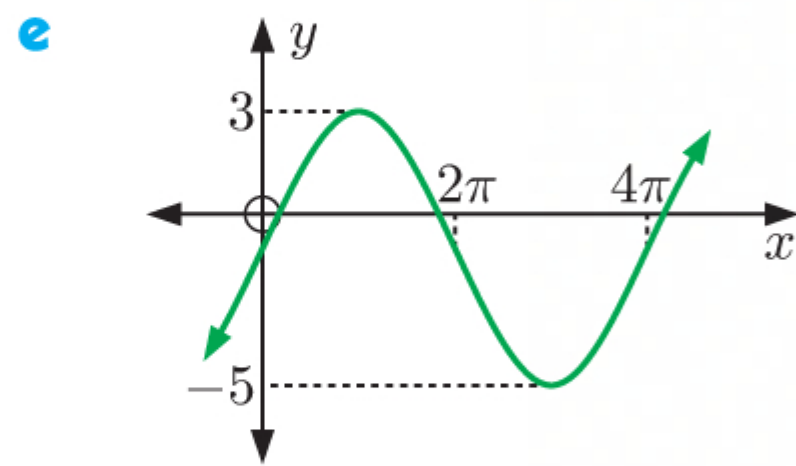
The amplitude is 2, so $a = 2$.

The period is 2π , so $\frac{2\pi}{b} = 2\pi$ and $\therefore b = 1$.

There is no horizontal translation, so $c = 0$.

The principal axis is $y = 1$, so $d = 1$.

\therefore the equation of the function is $y = 2 \sin x + 1$.



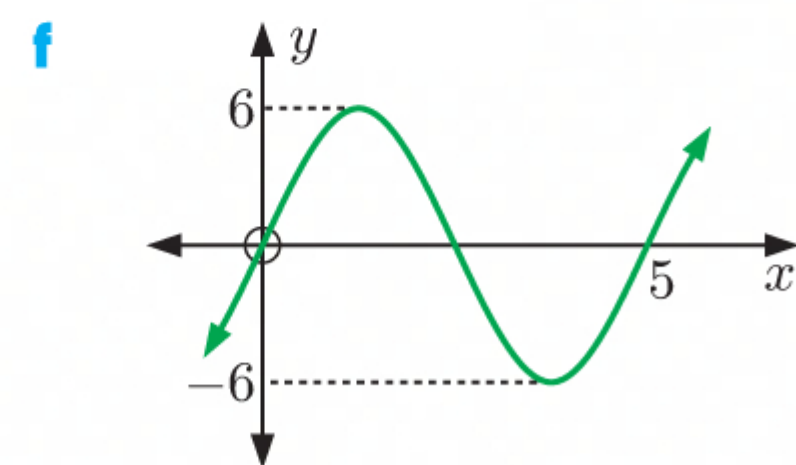
The amplitude is 4, so $a = 4$.

The period is 4π , so $\frac{2\pi}{b} = 4\pi$ and $\therefore b = \frac{1}{2}$.

There is no horizontal translation, so $c = 0$.

The principal axis is $y = -1$, so $d = -1$.

\therefore the equation of the function is $y = 4 \sin \frac{x}{2} - 1$.



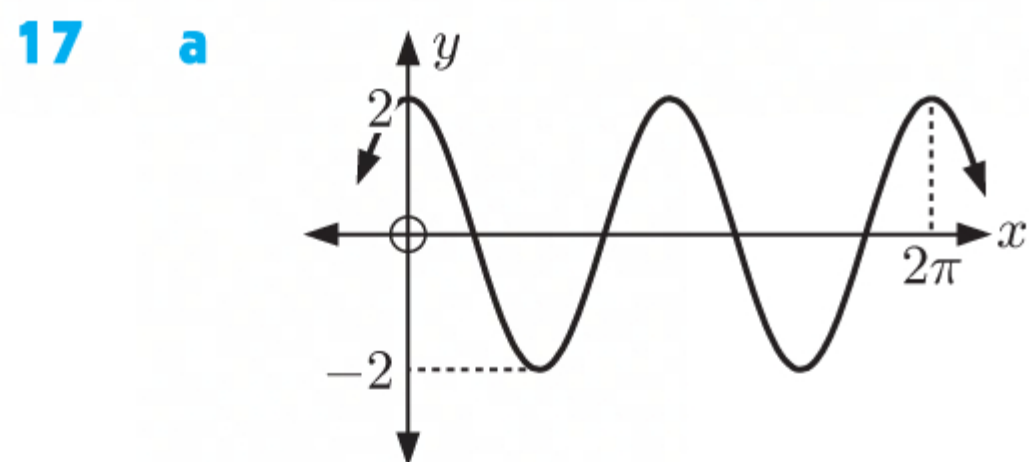
The amplitude is 6, so $a = 6$.

The period is 5, so $\frac{2\pi}{b} = 5$ and $\therefore b = \frac{2\pi}{5}$.

There is no horizontal translation, so $c = 0$.

The principal axis is $y = 0$, so $d = 0$.

\therefore the equation of the function is $y = 6 \sin \frac{2\pi x}{5}$.



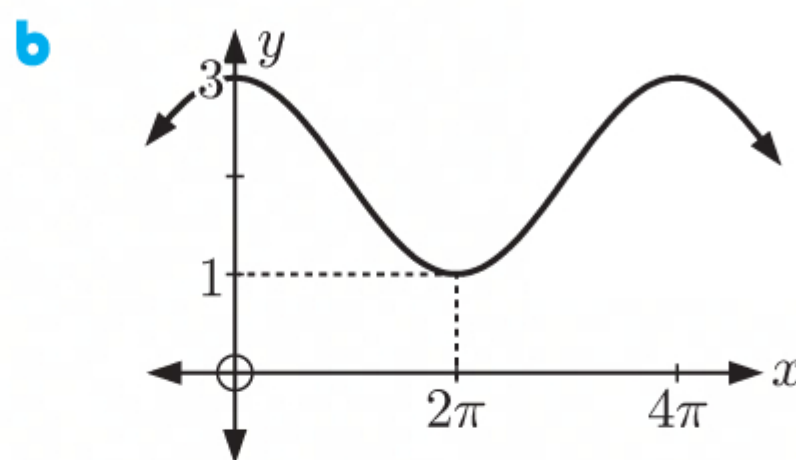
The amplitude is 2, so $a = 2$.

The period is π , so $\frac{2\pi}{b} = \pi$ and $\therefore b = 2$.

There is no horizontal translation, so $c = 0$.

The principal axis is $y = 0$, so $d = 0$.

\therefore the equation of the function is $y = 2 \cos 2x$.



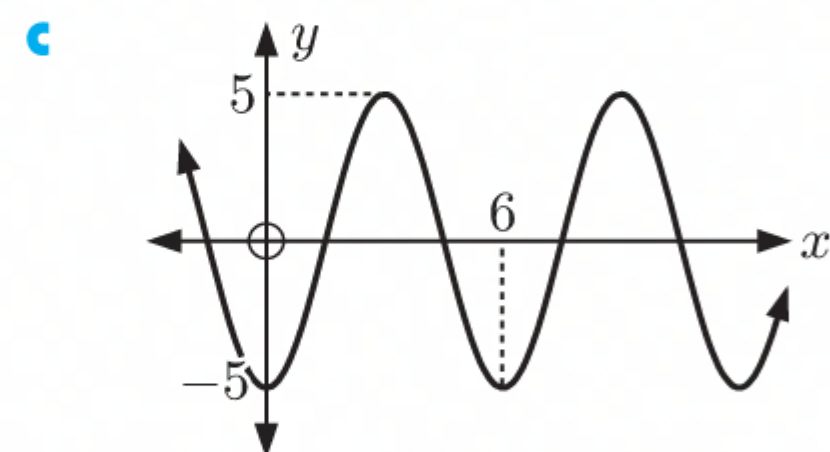
The amplitude is 1, so $a = 1$.

The period is 4π , so $\frac{2\pi}{b} = 4\pi$ and $\therefore b = \frac{1}{2}$.

There is no horizontal translation, so $c = 0$.

The principal axis is $y = 2$, so $d = 2$.

\therefore the equation of the function is $y = \cos \frac{x}{2} + 2$.



The amplitude is 5 and the function has been mirrored in the x -axis, so $a = -5$.

The period is 6, so $\frac{2\pi}{b} = 6$ and $\therefore b = \frac{\pi}{3}$.

There is no horizontal translation, so $c = 0$.

The principal axis is $y = 0$, so $d = 0$.

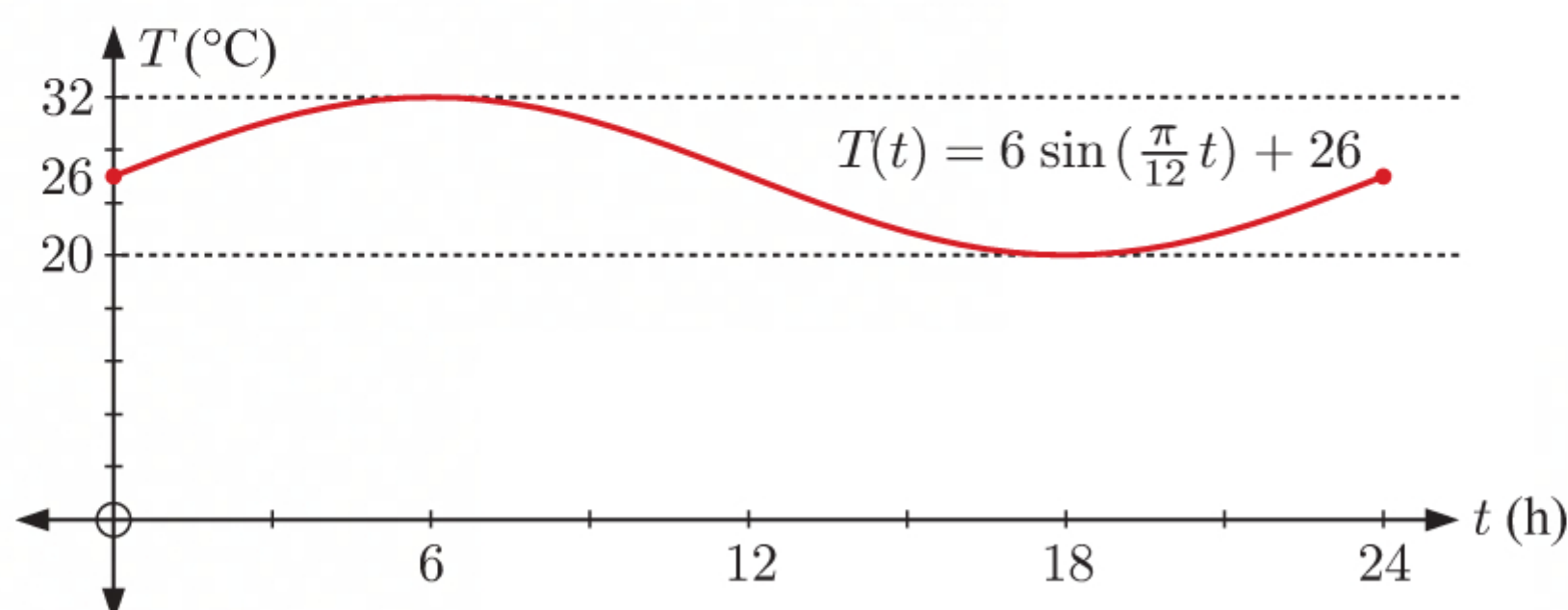
The graph has been reflected in the x -axis.

\therefore the equation of the function is $y = -5 \cos \frac{\pi x}{3}$.

EXERCISE 8D.1

1 a For $T(t) = 6 \sin\left(\frac{\pi}{12}t\right) + 26$:

- the amplitude is 6
- the period is $\frac{2\pi}{(\frac{\pi}{12})} = 24$ hours
- the principal axis is $T = 26$.



b i Midnight is 12 hours after midday.

When $t = 12$,

$$\begin{aligned} T &= 6 \sin\left(\frac{\pi}{12} \times 12\right) + 26 \\ &= 6 \sin \pi + 26 \\ &= 6 \times 0 + 26 \\ &= 26 \end{aligned}$$

\therefore at midnight the temperature inside Vanessa's house is 26°C .

ii 2 pm is 2 hours after midday.

When $t = 2$,

$$\begin{aligned} T &= 6 \sin\left(\frac{\pi}{12} \times 2\right) + 26 \\ &= 6 \sin \frac{\pi}{6} + 26 \\ &= 6 \times \frac{1}{2} + 26 \\ &= 29 \end{aligned}$$

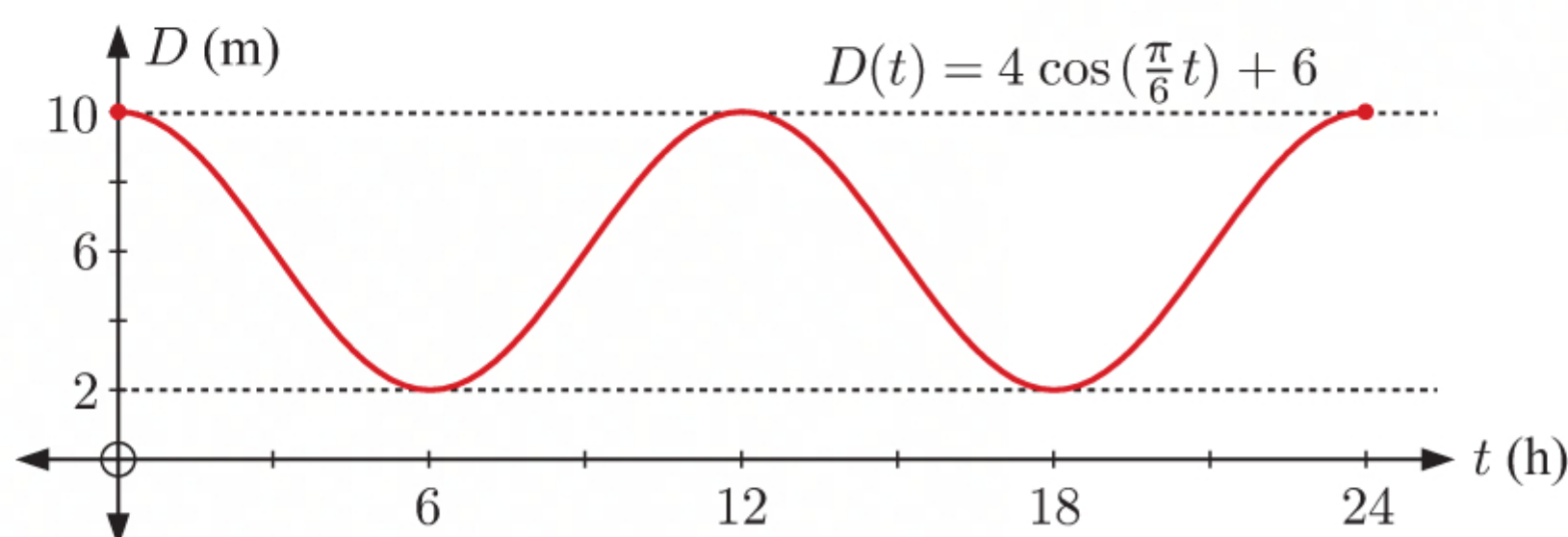
\therefore at 2 pm the temperature inside Vanessa's house is 29°C .

c The maximum temperature inside Vanessa's house is $26 + 6 = 32^\circ\text{C}$, which occurs when $t = 6$.

So, the maximum temperature inside Vanessa's house occurs at 6 pm.

2 a For $D(t) = 4 \cos\left(\frac{\pi}{6}t\right) + 6$:

- the amplitude is 4
- the period is $\frac{2\pi}{(\frac{\pi}{6})} = 12$ hours
- the principal axis is $D = 6$.



- b** The highest water depth is $6 + 4 = 10$ metres, which occurs when $t = 0, 12$, or 24 .
So, the highest water depth occurs at midnight, midday, and midnight the next day.
The lowest water depth is $6 - 4 = 2$ metres, which occurs when $t = 6$ or 18 .
So, the lowest water depth occurs at 6 am or 6 pm.

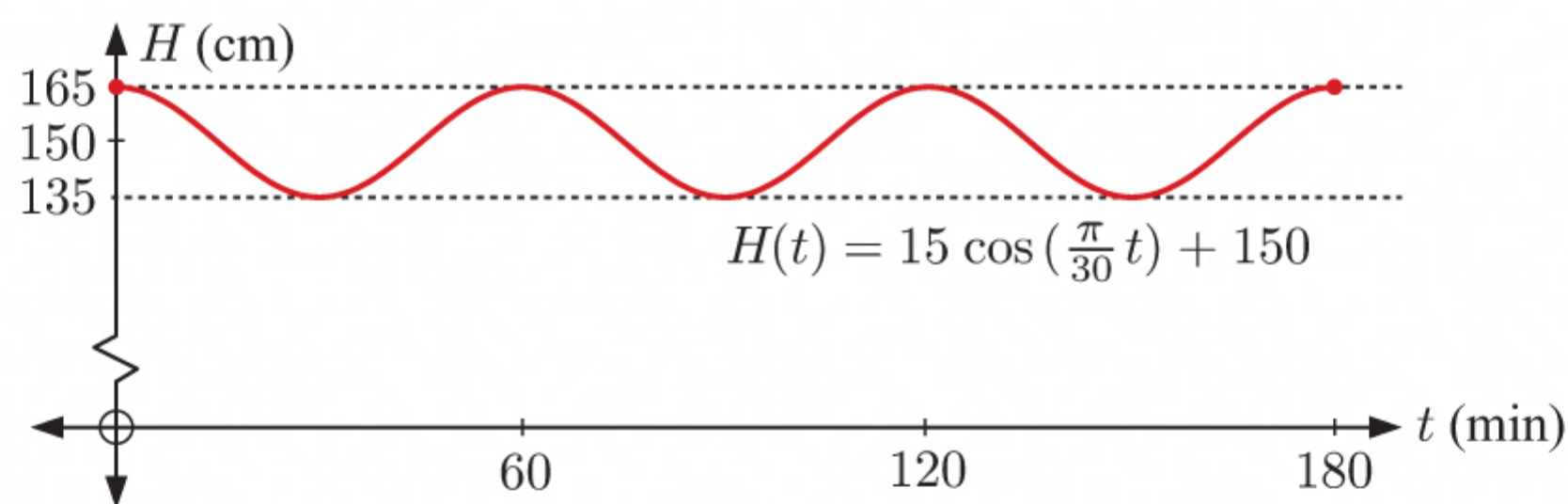
- c** 8 pm is 20 hours after midnight.

$$\begin{aligned}\text{When } t = 20, \quad D &= 4 \cos\left(\frac{\pi}{6} \times 20\right) + 6 \\ &= 4 \cos\left(\frac{10\pi}{3}\right) + 6 \\ &= 4 \times \left(-\frac{1}{2}\right) + 6 \\ &= 4\end{aligned}$$

\therefore at 8 pm the water depth is 4 metres, but the boat requires 5 metres, so it cannot enter the harbour at that time.

- 3 a** For $H(t) = 15 \cos\left(\frac{\pi}{30}t\right) + 150$:

- the amplitude is 15
- the period is $\frac{2\pi}{(\frac{\pi}{30})} = 60$ minutes
- the principal axis is $H = 150$.



- b** The minute hand's length is represented by the amplitude in the function.
 \therefore the minute hand's length is 15 cm.

- c i** 5:08 pm is 8 minutes after 5 pm.

When $t = 8$,

$$\begin{aligned}H &= 15 \cos\left(\frac{\pi}{30} \times 8\right) + 150 \\ &\approx 160.037\end{aligned}$$

\therefore at 5:08 pm, the minute hand's tip is approximately 160.0 cm above ground level.

- ii** 5:37 pm is 37 minutes after 5 pm.

When $t = 37$,

$$\begin{aligned}H &= 15 \cos\left(\frac{\pi}{30} \times 37\right) + 150 \\ &\approx 138.853\end{aligned}$$

\therefore at 5:37 pm, the minute hand's tip is approximately 138.9 cm above ground level.

- iii** 5:51 pm is 51 minutes after 5 pm.

When $t = 51$,

$$\begin{aligned}H &= 15 \cos\left(\frac{\pi}{30} \times 51\right) + 150 \\ &\approx 158.817\end{aligned}$$

\therefore at 5:51 pm, the minute hand's tip is approximately 158.8 cm above ground level.

- iv** 6:23 pm is 83 minutes after 5 pm.

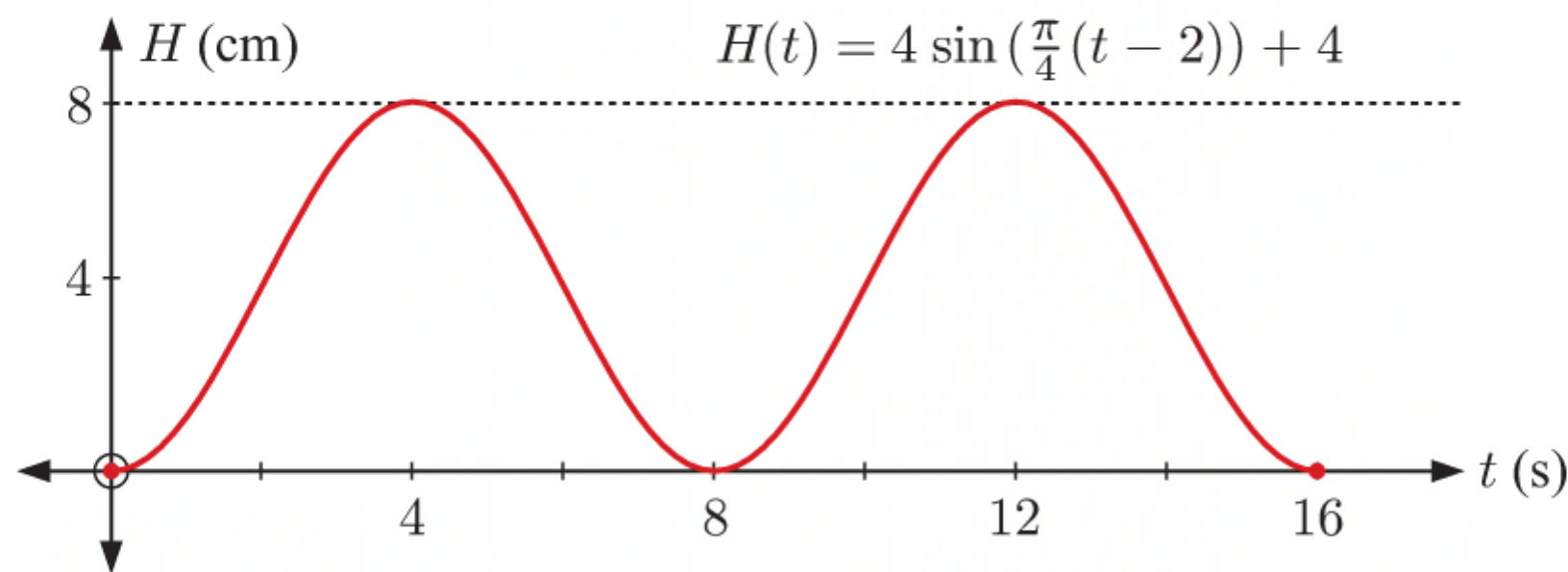
When $t = 83$,

$$\begin{aligned}H &= 15 \cos\left(\frac{\pi}{30} \times 83\right) + 150 \\ &\approx 138.853\end{aligned}$$

\therefore at 6:23 pm, the minute hand's tip is approximately 138.9 cm above ground level.

4 a For $H(t) = 4 \sin\left(\frac{\pi}{4}(t - 2)\right) + 4$:

- the amplitude is 4
- the period is $\frac{2\pi}{(\frac{\pi}{4})} = 8$ seconds
- the horizontal translation is 2 seconds to the right
- the principal axis is $H = 4$.



b When $t = 2$,

$$\begin{aligned} H &= 4 \sin\left(\frac{\pi}{4}(2 - 2)\right) + 4 \\ &= 4 \sin\left(\frac{\pi}{4} \times 0\right) + 4 \\ &= 4 \sin 0 + 4 \\ &= 4 \times 0 + 4 \\ &= 4 \end{aligned}$$

\therefore 2 seconds after the gate touches the ground it is 4 cm above ground level.

c When $t = 5.3 + 1 = 6.3$,

$$\begin{aligned} H &= 4 \sin\left(\frac{\pi}{4}(6.3 - 2)\right) + 4 \\ &= 4 \sin\left(\frac{\pi}{4} \times 4.3\right) + 4 \\ &\approx 3.066 \end{aligned}$$

\therefore the ball will not pass through the entrance as its diameter is $2 \times 2.14 = 4.28$ cm but the gate height is only approximately 3.07 cm above ground level.

5 The mean temperature $= \frac{15.8 + 5.4}{2} = 10.6^\circ\text{C}$, so $d = 10.6$.

$$\begin{aligned} \text{The amplitude} &= \frac{15.8 - 5.4}{2} = 5.2^\circ\text{C} \\ \therefore a &= 5.2 \end{aligned}$$

The period is 24 hours, so $b = \frac{2\pi}{24} = \frac{\pi}{12}$.

The maximum occurs at 2 pm, so we assume the temperature passed its mean value 6 hours earlier, at 8 am.

The day begins at midnight, so the function is shifted 8 hours to the right, thus $c = 8$.

If t is the number of hours after midnight, the temperature T is modelled by

$$T(t) = 5.2 \sin\left(\frac{\pi}{12}(t - 8)\right) + 10.6^\circ\text{C}.$$

6 The mean height $= \frac{1.36 + 0.16}{2} = 0.76$ m, so $d = 0.76$.

$$\begin{aligned} \text{The amplitude} &= \frac{1.36 - 0.16}{2} = 0.6 \text{ m} \\ \therefore a &= 0.6 \end{aligned}$$

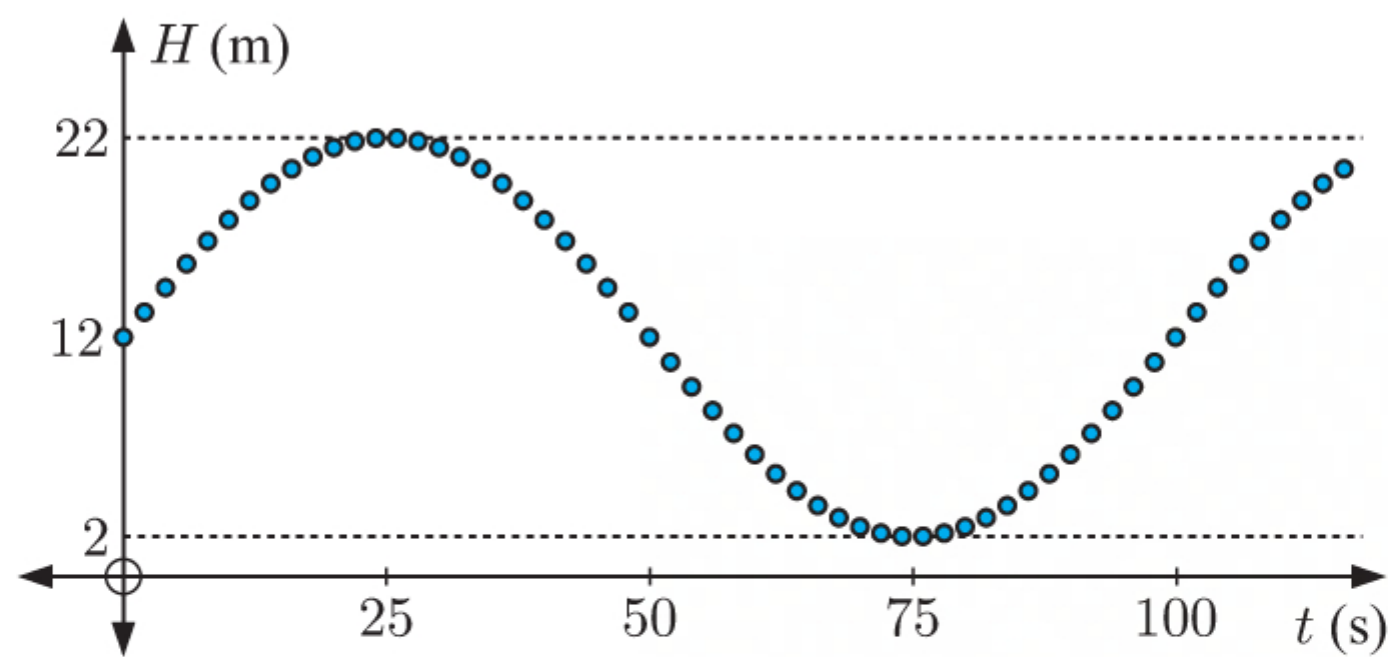
The period is 12.4 hours, so $b = \frac{2\pi}{12.4} = \frac{5\pi}{31}$.

The maximum occurs at 1:30 am, so the function is shifted $1\frac{1}{2}$ hours to the right, thus $c = 1.5$.

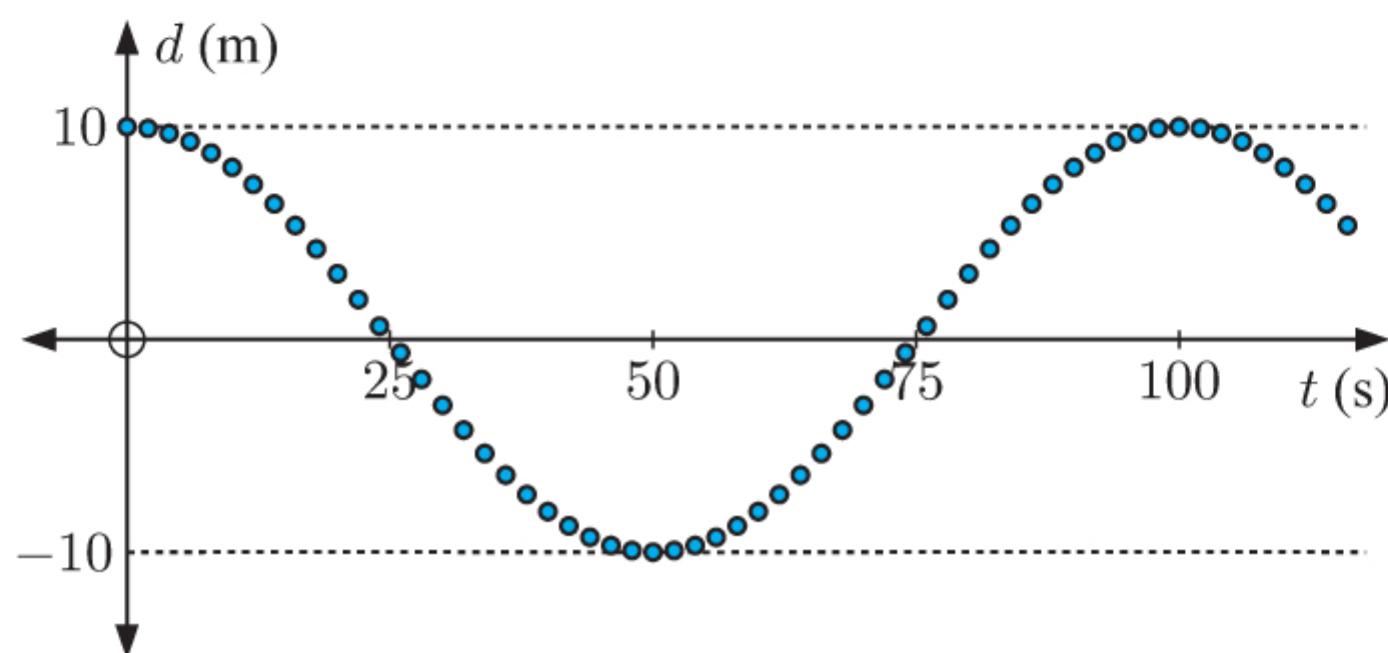
If t is the number of hours after midnight, the height H is modelled by

$$H(t) = 0.6 \cos\left(\frac{5\pi}{31}(t - 1.5)\right) + 0.76 \text{ m}.$$

- 7 a** The light is initially $10 + 2 = 12$ m above the ground (its mean height) and oscillates with amplitude 10 m and period 100 seconds.



- b** Assume that when the light is directly above or below the centre of the Ferris wheel, its horizontal position is 0 m. The horizontal position of the light oscillates with amplitude 10 m and period 100 seconds. The light initially has horizontal position 10 m.



- c** Both graphs are periodic with an amplitude of 10 m and a period of 100 s. The graphs differ by a horizontal translation of 25 s, and the principal axis is translated by 12 m.

- d i** For the height, $H(t)$:

The amplitude is 10, so $a = 10$.

The period is $\frac{2\pi}{b} = 100$,

$$\therefore 100b = 2\pi$$

$$\therefore b = \frac{\pi}{50}$$

The mean height occurs at midnight, so there is no horizontal translation, thus $c = 0$.

The principal axis is $H = 12$, so $d = 12$.

$$\therefore H(t) = 10 \sin\left(\frac{\pi}{50}t\right) + 12 \text{ m}$$

- ii** The graph of $d(t)$ has the same amplitude and period as the graph of $H(t)$.

The graph of $d(t)$ is obtained by translating the graph of $H(t)$ 25 units to the left and 12 units downwards.

$$\therefore d(t) = 10 \sin\left(\frac{\pi}{50}(t + 25)\right) \text{ m}$$

- 8 a** The mean height $= \frac{16.2 + (16.2 - 14)}{2}$
 $= \frac{16.2 + 2.2}{2}$
 $= 9.2 \text{ m}$
 $\therefore d = 9.2$

$$\begin{aligned} \text{The amplitude} &= \frac{14}{2} \\ &= 7 \text{ m} \\ \therefore a &= 7 \end{aligned}$$

The period is 12.4 hours, so $b = \frac{2\pi}{12.4} = \frac{5\pi}{31}$.

The maximum occurs at 9 am, so we assume the height passed its mean height

$$\frac{12.4}{4} = 3.1 \text{ hours earlier, or 5.9 hours after midnight.}$$

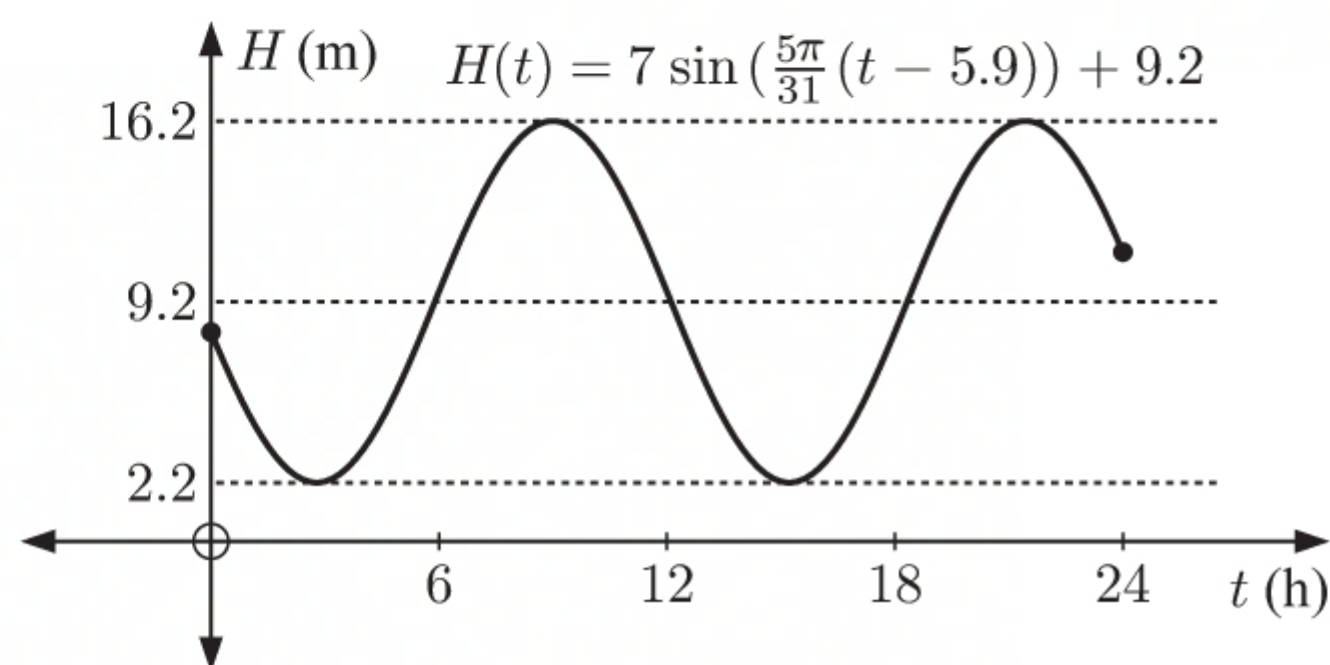
The day begins at midnight, so the function is shifted 5.9 hours to the right, thus $c = 5.9$.

If t is the number of hours after midnight, the height H is modelled by

$$H(t) = 7 \sin\left(\frac{5\pi}{31}(t - 5.9)\right) + 9.2 \text{ m.}$$

b For $H(t) = 7 \sin\left(\frac{5\pi}{31}(t - 5.9)\right) + 9.2$:

- the amplitude is 7
- the period is 12.4 hours
- the horizontal translation is 5.9 hours to the right
- the principal axis is $H = 9.2$.



9 a The mean height of the tip of the hour hand $= \frac{6 + (-6)}{2}$
 $= 0 \text{ cm}$
 $\therefore d = 0$

The amplitude of the height of the tip of the hour hand $= 6 \text{ cm}$

$$\therefore a = 6$$

The period is 12 hours, so $b = \frac{2\pi}{12} = \frac{\pi}{6}$.

The maximum occurs at midnight, so there is no translation.

If t is the number of hours after midnight, the height of the tip of the hour hand relative to the centre of the clock is modelled by $H(t) = 6 \cos\left(\frac{\pi}{6}t\right) \text{ cm}$.



b The mean horizontal displacement of the tip of the minute hand $= \frac{12 + (-12)}{2}$
 $= 0 \text{ cm}$
 $\therefore d = 0$

The amplitude of the horizontal displacement of the tip of the minute hand $= 12 \text{ cm}$

$$\therefore a = 12$$

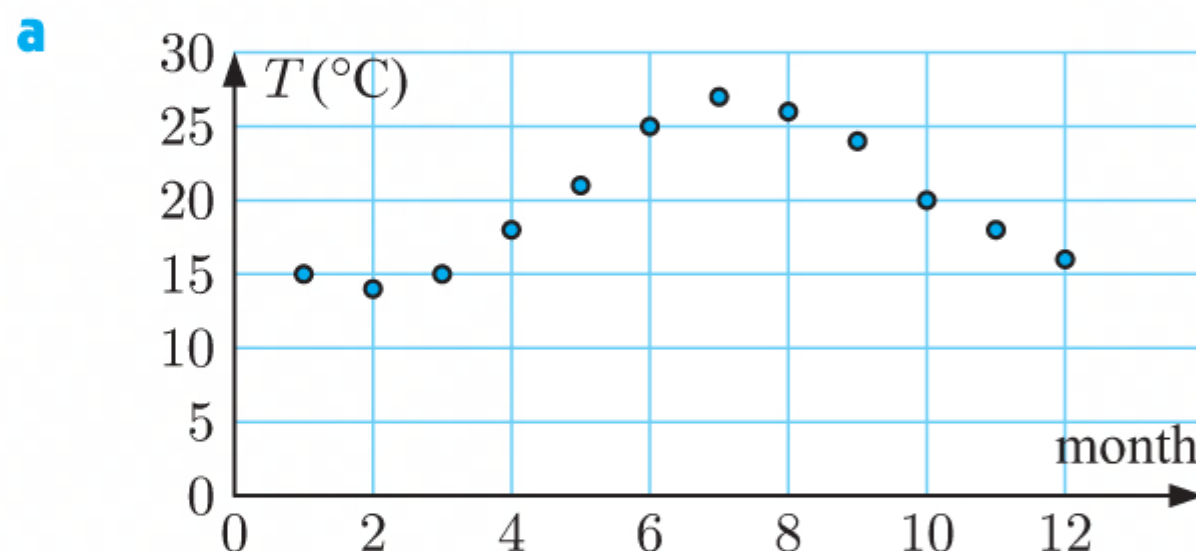
The period is 1 hour, so $b = \frac{2\pi}{1} = 2\pi$.

The mean horizontal displacement occurs at midnight, so there is no translation.

If t is the number of hours after midnight, the horizontal displacement of the tip of the minute hand relative to the centre of the clock is modelled by $d(t) = 12 \sin 2\pi t \text{ cm}$.

EXERCISE 8D.2

1	Month	Jan	Feb	Mar	Apr	May	Jun	July	Aug	Sept	Oct	Nov	Dec
	Temperature ($^{\circ}\text{C}$)	15	14	15	18	21	25	27	26	24	20	18	16



b The data appears to be periodic, so it is appropriate to fit a trigonometric model.

c i The period is 12 months, so $\frac{2\pi}{b} = 12$ and $\therefore b = \frac{\pi}{6}$.

ii The amplitude $= \frac{\max - \min}{2} \approx \frac{27 - 14}{2} \approx 6.5$, so $a \approx 6.5$.

iii The principal axis is midway between the maximum and minimum, so $d \approx \frac{27 + 14}{2} \approx 20.5$.

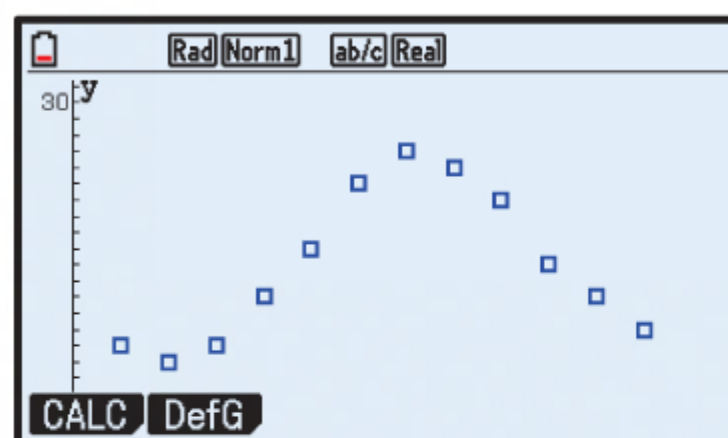
iv The model is $T \approx 6.5 \sin\left(\frac{\pi}{6}(t - c)\right) + 20.5$ for some constant c .

On the original graph, the sine function starts a new period between months 4 and 5.

We estimate that $c \approx 4.5$.

d From c, our model is $T \approx 6.5 \sin\left(\frac{\pi}{6}(t - 4.5)\right) + 20.5$
 $\approx 6.5 \sin(0.524t - 2.36) + 20.5$

	List 1	List 2	List 3	List 4
SUB				
1	1	15		
2	2	14		
3	3	15		
4	4	18		



	List 1	List 2	List 3	List 4
SUB				
1	1	15		
2	2	14		
3	3	15		
4	4	18		

Using technology, $T \approx 6.15 \sin(0.575t - 2.69) + 20.4$.

Our model was a reasonable fit.

2	Month	Jan	Feb	Mar	Apr	May	Jun	July	Aug	Sept	Oct	Nov	Dec
	Temperature ($^{\circ}\text{C}$)	15	16	$14\frac{1}{2}$	12	10	$7\frac{1}{2}$	7	$7\frac{1}{2}$	$8\frac{1}{2}$	$10\frac{1}{2}$	$12\frac{1}{2}$	14

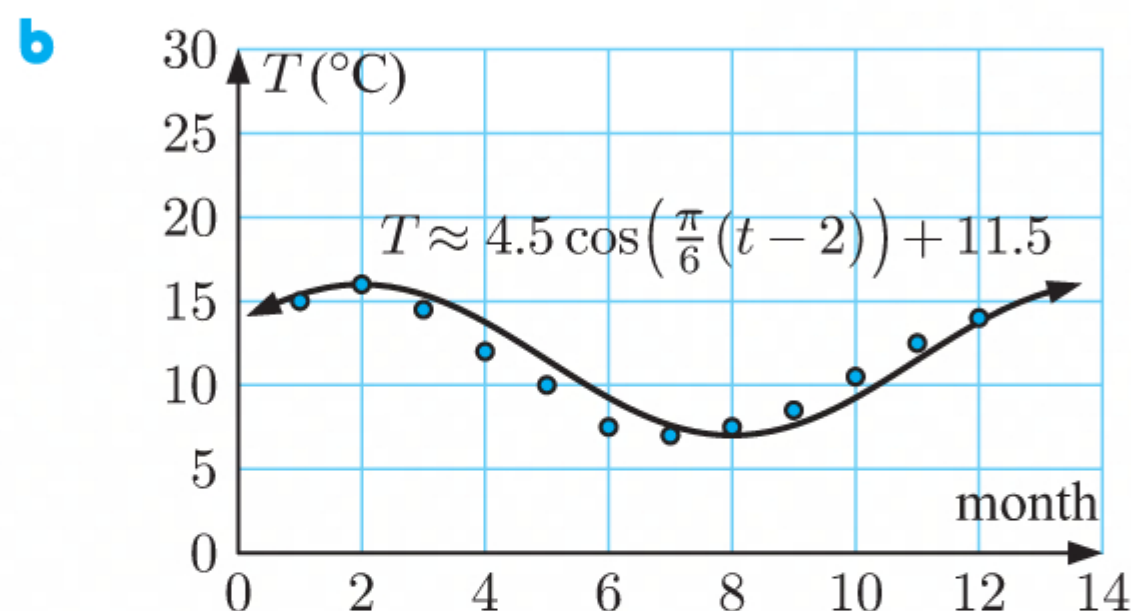
a The period is 12 months, so $\frac{2\pi}{b} = 12$ and $\therefore b = \frac{\pi}{6}$.

The amplitude $= \frac{\max - \min}{2} \approx \frac{16 - 7}{2} \approx 4.5$, so $a \approx 4.5$.

The principal axis is midway between the maximum and minimum, so $d \approx \frac{16 + 7}{2} \approx 11.5$.

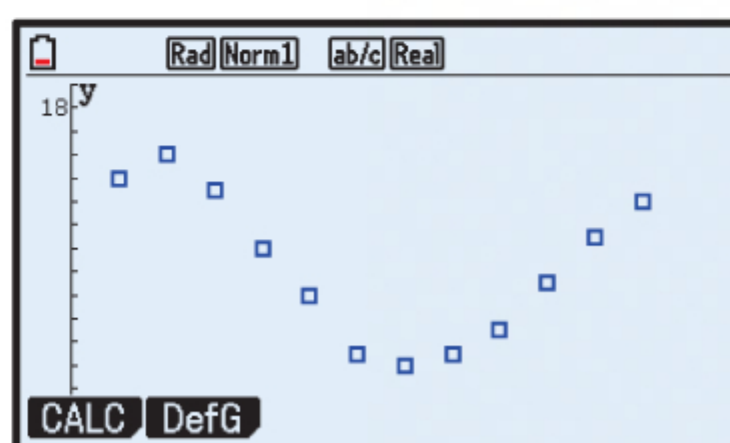
The maximum occurs in February, which is the 2nd month. We estimate that the translation $c \approx 2$.

$$\therefore T \approx 4.5 \cos\left(\frac{\pi}{6}(t-2)\right) + 11.5$$



c From **b**, our model is $T \approx 4.5 \cos\left(\frac{\pi}{6}(t-2)\right) + 11.5$
 $\approx 4.5 \cos(0.524t - 1.05) + 11.5$

	List 1	List 2	List 3	List 4
SUB				
1	1	15		
2	2	16		
3	3	14.5		
4	4	12		



	Rad	Norm1	ab/c	Real
SinReg				
a	=	4.28622831		
b	=	0.53287379		
c	=	0.76548952		
d	=	11.1831837		
MSe	=	0.14318243		
y=	a	·	sin(bx+c)+d	

Using technology, $T \approx 4.29 \sin(0.533t + 0.765) + 11.2$
 $\approx 4.29 \cos(0.533t + 0.76 - \frac{\pi}{2}) + 11.2$ $\{\cos x = \sin(x - \frac{\pi}{2})\}$
 $\approx 4.29 \cos(0.533t - 0.805) + 11.2$

3

Month	Jan	Feb	Mar	Apr	May	Jun	July	Aug	Sept	Oct	Nov	Dec
Temperature (°C)	0	0	-4	-9	-14	-17	-18	-19	-17	-13	-6	-2

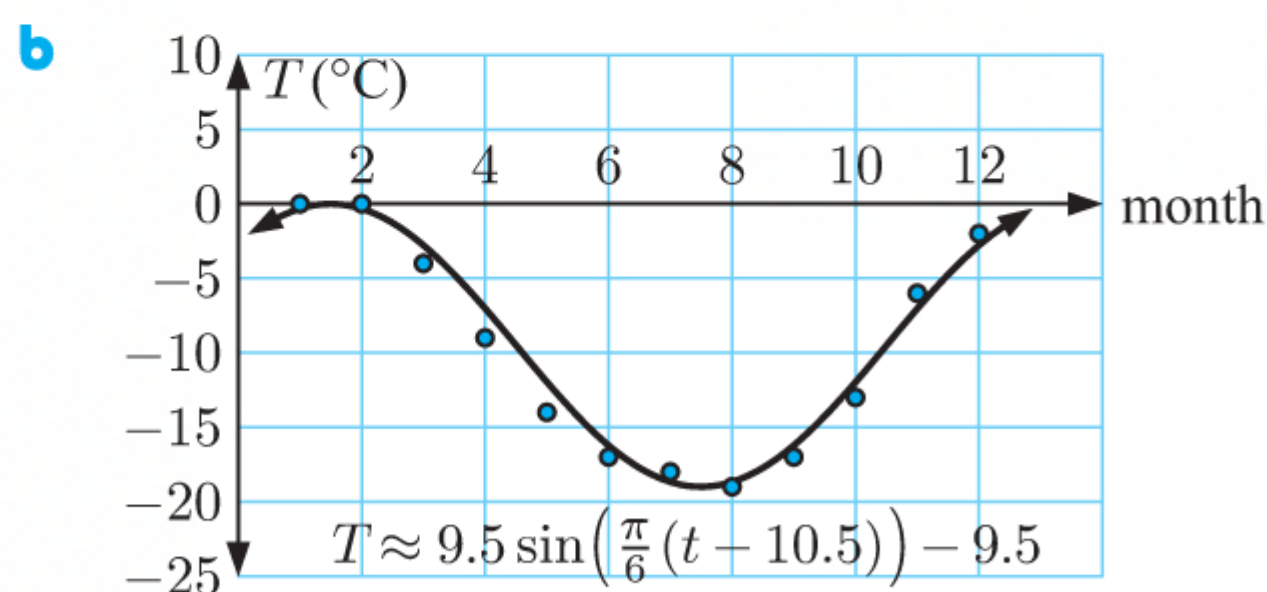
a The period is 12 months, so $\frac{2\pi}{b} = 12$ and $\therefore b = \frac{\pi}{6}$.

The amplitude $= \frac{\max - \min}{2} \approx \frac{0 - (-19)}{2} \approx 9.5$, so $a \approx 9.5$.

The principal axis is midway between the maximum and minimum, so $d \approx \frac{0 + (-19)}{2} \approx -9.5$.

The sine function starts a new period midway between August and the following January (months 8 and 13). We estimate that $c \approx \frac{8+13}{2} \approx 10.5$.

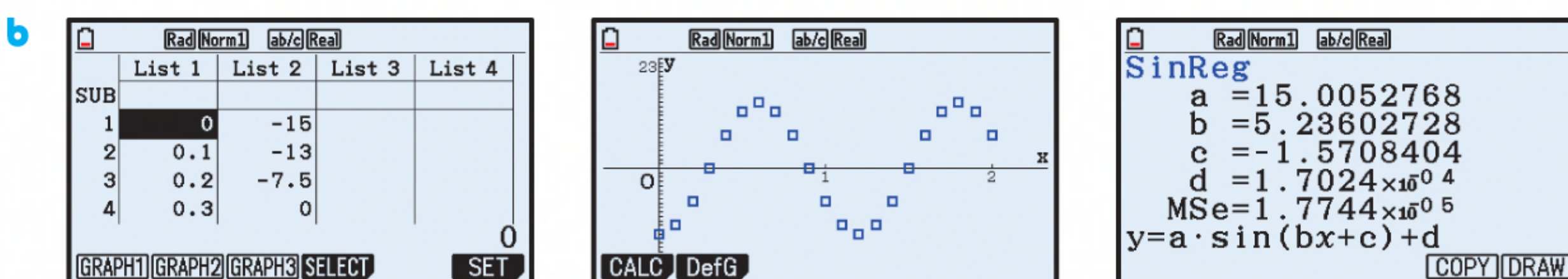
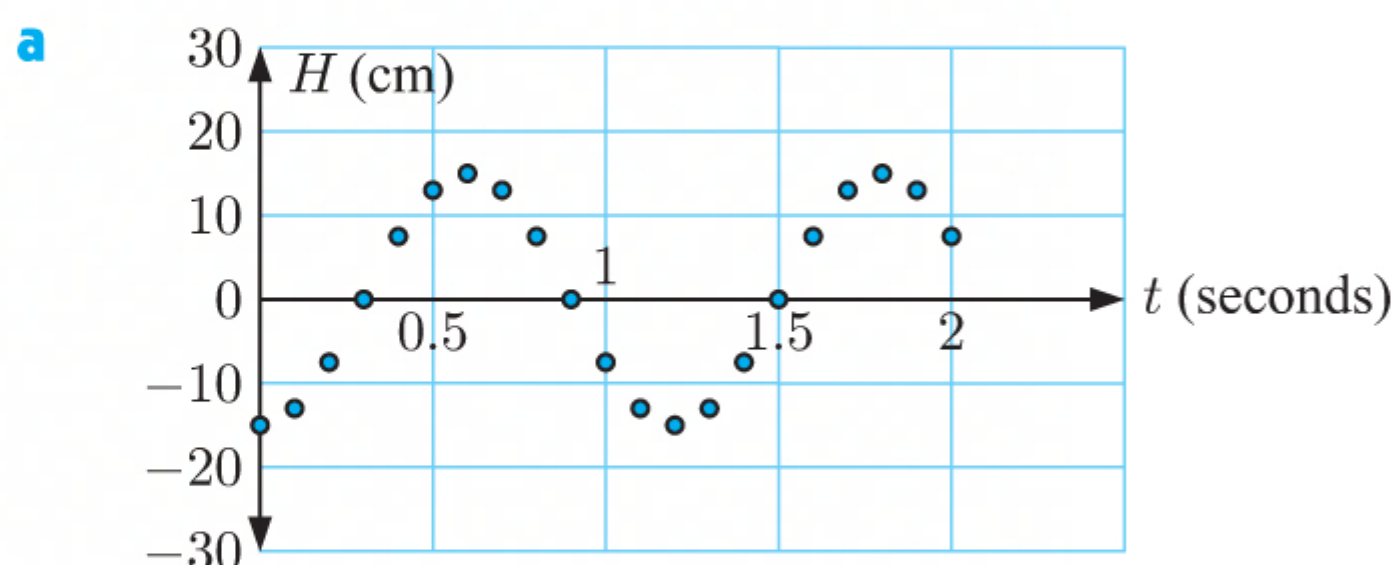
$$\therefore T \approx 9.5 \sin\left(\frac{\pi}{6}(t-10.5)\right) - 9.5$$



c The model is not perfect, but it is a reasonable fit.

4	Time (t seconds)	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
	Height (H cm)	-15	-13	-7.5	0	7.5	13	15	13	7.5	0

Time (t seconds)	1	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2.0
Height (H cm)	-7.5	-13	-15	-13	-7.5	0	7.5	13	15	13	7.5



Using technology, $H \approx 15.0 \sin(5.24t - 1.57) + 0.000170$

- c When $t = 4.25$, $H \approx 15.0 \sin(5.24(4.25) - 1.57) + 0.000170$
 ≈ 14.5

\therefore the height of the object after 4.25 seconds will be about 14.5 cm.

- d This model is unrealistic since the spring will not oscillate indefinitely. It will slow down and come to a stop.

EXERCISE 8E

- 1 a $y = \tan\left(x - \frac{\pi}{2}\right)$ is a horizontal translation of $y = \tan x$ to the right by $\frac{\pi}{2}$ units.
 So, a horizontal translation of $\frac{\pi}{2}$ units to the right will map $y = \tan x$ onto $y = \tan\left(x - \frac{\pi}{2}\right)$.
- b $y = 4 \tan x$ is a vertical stretch of $y = \tan x$ with scale factor 4.
 So, a vertical stretch with scale factor 4 will map $y = \tan x$ onto $y = 4 \tan x$.
- c $y = \tan\left(\frac{\pi}{2}x\right)$ is a horizontal stretch of $y = \tan x$ with scale factor $\frac{2}{\pi}$.
 So, a horizontal stretch with scale factor $\frac{2}{\pi}$ will map $y = \tan x$ onto $y = \tan\left(\frac{\pi}{2}x\right)$.

d $\tan x \xrightarrow[\text{horizontal stretch}]{\text{scale factor } \frac{1}{2}} \tan 2x \xrightarrow[\text{translation}]{\begin{pmatrix} 0 \\ -1 \end{pmatrix}} \tan 2x - 1$

So, a horizontal stretch with scale factor $\frac{1}{2}$, then a translation 1 unit downwards will map $y = \tan x$ onto $y = \tan 2x - 1$.

$$\text{e } \tan x \xrightarrow[\text{vertical stretch}]{\text{scale factor } \frac{1}{2}} \frac{1}{2} \tan x \xrightarrow[\text{reflection}]{\text{in } x\text{-axis}} -\frac{1}{2} \tan x$$

So, a vertical stretch with scale factor $\frac{1}{2}$, then a reflection in the x -axis will map $y = \tan x$ onto $y = -\frac{1}{2} \tan x$.

$$\text{f } \tan(x + \pi) = \tan x \quad \{\text{period is } \pi\}$$

$\therefore y = \tan(x + \pi) + 2 = \tan x + 2$ is a vertical translation of $y = \tan x$ 2 units upwards.

So, a vertical translation of 2 units upwards will map $y = \tan x$ onto $y = \tan(x + \pi) + 2$.

$$\text{2 a } y = \tan 3x \text{ has period } \frac{\pi}{b} = \frac{\pi}{3}$$

$$\text{b } y = \tan \frac{x}{4} \text{ has period } \frac{\pi}{b} = \frac{\pi}{(\frac{1}{4})} = 4\pi$$

$$\text{c } y = \tan \pi x \text{ has period } \frac{\pi}{b} = \frac{\pi}{\pi} = 1$$

$$\text{d } y = -\tan\left(\frac{\pi}{2}x\right) \text{ has period } \frac{\pi}{b} = \frac{\pi}{(\frac{\pi}{2})} = 2$$

$$\text{e } y = \tan\left(\frac{2x}{3} - \frac{\pi}{3}\right) \text{ has period } \frac{\pi}{b} = \frac{\pi}{(\frac{2}{3})} = \frac{3\pi}{2}$$

$$\text{f } y = \tan nx, \quad n \neq 0 \text{ has period } \frac{\pi}{b} = \frac{\pi}{n}$$

$$\text{3 a i } y = \tan 2x = 0 \text{ when}$$

$$2x = k\pi, \quad k \in \mathbb{Z}$$

$$\therefore x = \frac{k\pi}{2}, \quad k \in \mathbb{Z}$$

$$\text{ii } y = \tan 2x \text{ is undefined when}$$

$$2x = \frac{\pi}{2} + k\pi, \quad k \in \mathbb{Z}$$

$$\therefore x = \frac{\pi}{4} + \frac{k\pi}{2}, \quad k \in \mathbb{Z}$$

$$\text{b i } y = \tan\left(x + \frac{\pi}{3}\right) = 0 \text{ when}$$

$$x + \frac{\pi}{3} = k\pi, \quad k \in \mathbb{Z}$$

$$\therefore x = \frac{2\pi}{3} + k\pi, \quad k \in \mathbb{Z}$$

$$\text{ii } y = \tan\left(x + \frac{\pi}{3}\right) \text{ is undefined when}$$

$$x + \frac{\pi}{3} = \frac{\pi}{2} + k\pi, \quad k \in \mathbb{Z}$$

$$\therefore x = \frac{\pi}{6} + k\pi, \quad k \in \mathbb{Z}$$

$$\text{c i } y = \frac{1}{2} \tan\left(\frac{1}{2}\left(x - \frac{\pi}{6}\right)\right) = 0 \text{ when}$$

$$\frac{1}{2}\left(x - \frac{\pi}{6}\right) = k\pi, \quad k \in \mathbb{Z}$$

$$\therefore x - \frac{\pi}{6} = 2k\pi, \quad k \in \mathbb{Z}$$

$$\therefore x = \frac{\pi}{6} + 2k\pi, \quad k \in \mathbb{Z}$$

$$\text{ii } y = \frac{1}{2} \tan\left(\frac{1}{2}\left(x - \frac{\pi}{6}\right)\right) \text{ is undefined}$$

$$\text{when } \frac{1}{2}\left(x - \frac{\pi}{6}\right) = \frac{\pi}{2} + k\pi, \quad k \in \mathbb{Z}$$

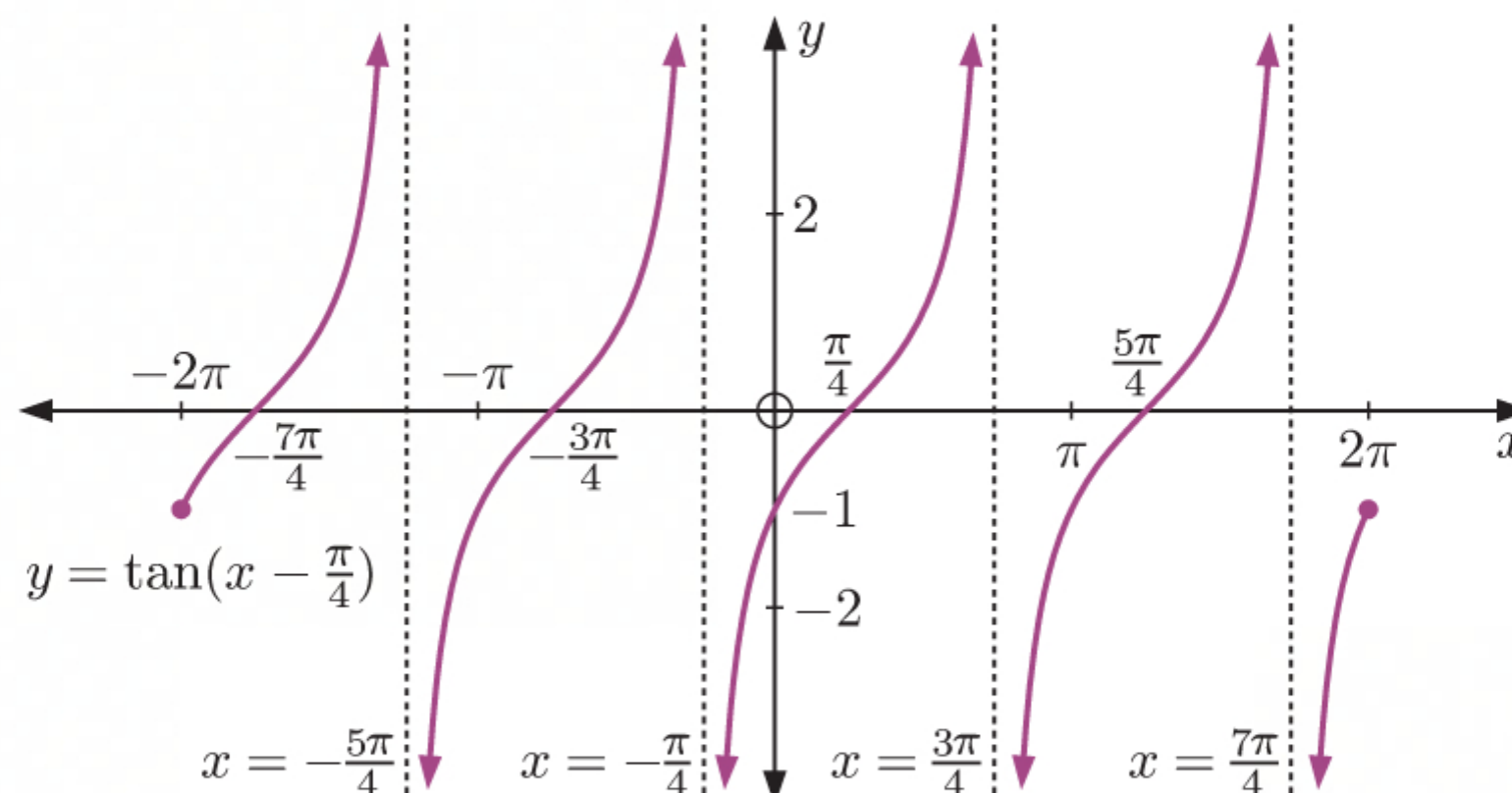
$$\therefore x - \frac{\pi}{6} = \pi + 2k\pi, \quad k \in \mathbb{Z}$$

$$\therefore x = \frac{7\pi}{6} + 2k\pi, \quad k \in \mathbb{Z}$$

- 4 a** $y = \tan\left(x - \frac{\pi}{4}\right)$ is a horizontal translation of $y = \tan x$ to the right by $\frac{\pi}{4}$ units.

$y = \tan x$ has vertical asymptotes $x = -\frac{3\pi}{2}$, $x = -\frac{\pi}{2}$, $x = \frac{\pi}{2}$, $x = \frac{3\pi}{2}$, and its x -intercepts are -2π , $-\pi$, 0 , π , and 2π .

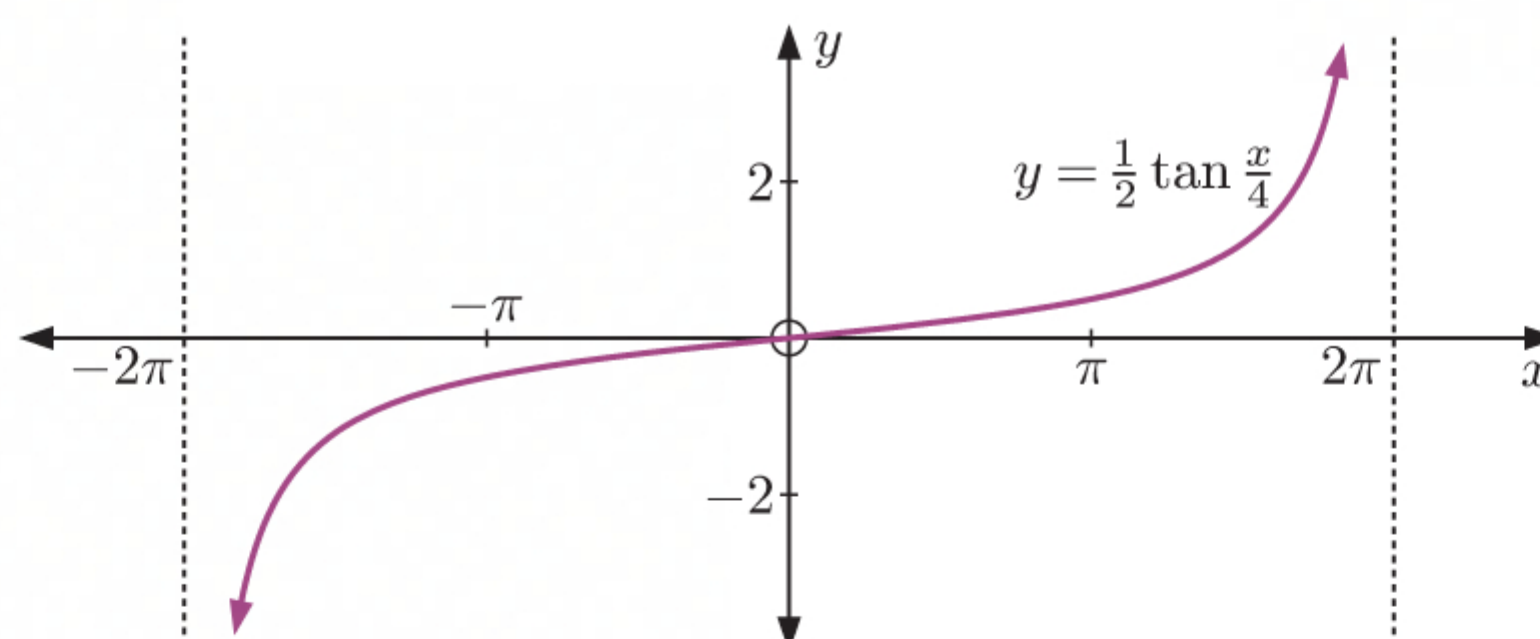
$\therefore y = \tan\left(x - \frac{\pi}{4}\right)$ has vertical asymptotes $x = -\frac{5\pi}{4}$, $x = -\frac{\pi}{4}$, $x = \frac{3\pi}{4}$, $x = \frac{7\pi}{4}$, and x -intercepts $-\frac{7\pi}{4}$, $-\frac{3\pi}{4}$, $\frac{\pi}{4}$, $\frac{5\pi}{4}$.



- b** $y = \frac{1}{2} \tan \frac{x}{4}$ is a horizontal stretch of $y = \tan x$ with scale factor 4, followed by a vertical stretch with scale factor $\frac{1}{2}$.

Since $b = \frac{1}{4}$, the period is 4π .

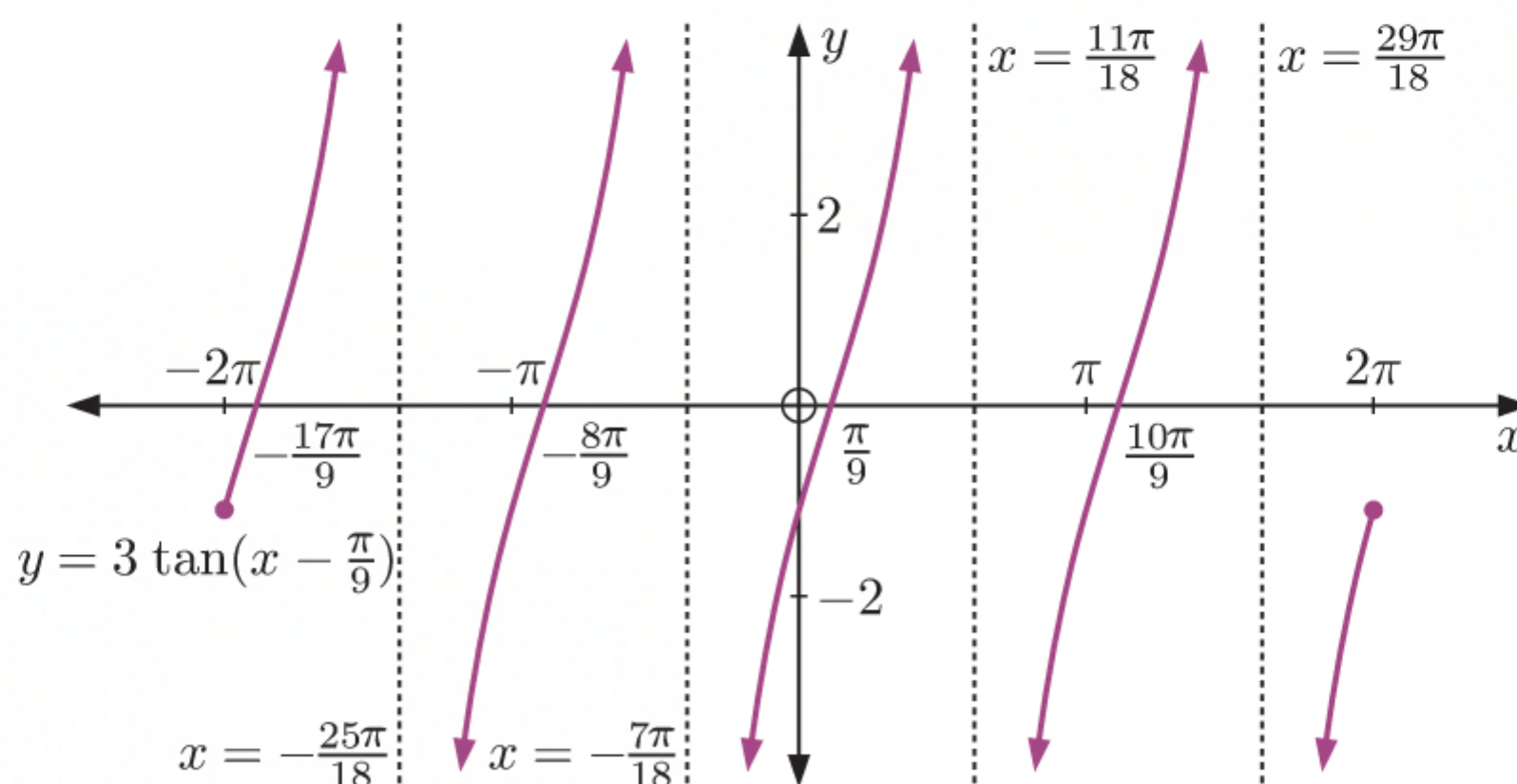
$y = \frac{1}{2} \tan \frac{x}{4}$ has vertical asymptotes $x = \pm 2\pi$, and x -intercept 0 .



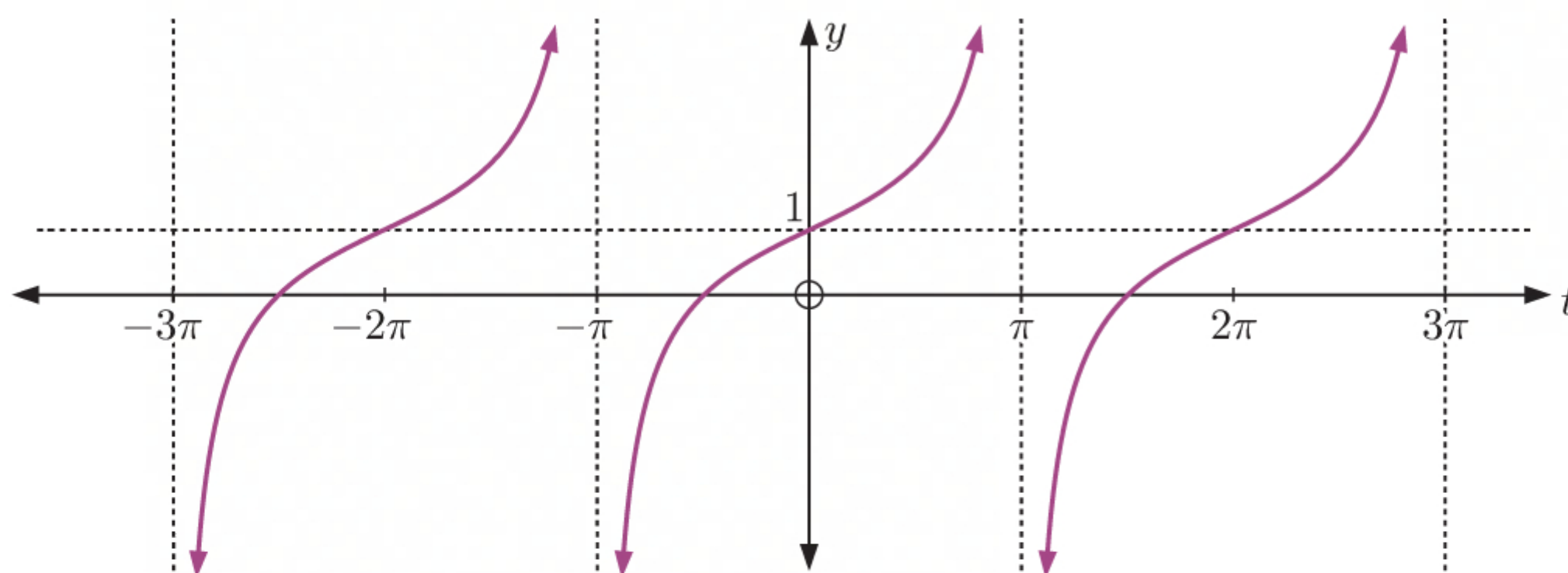
- c** $y = 3 \tan\left(x - \frac{\pi}{9}\right)$ is a horizontal translation of $y = \tan x$ to the right by $\frac{\pi}{9}$ units, followed by a vertical stretch with scale factor 3.

$y = \tan x$ has vertical asymptotes $x = -\frac{3\pi}{2}$, $x = -\frac{\pi}{2}$, $x = \frac{\pi}{2}$, $x = \frac{3\pi}{2}$, and x -intercepts -2π , $-\pi$, 0 , π , and 2π .

$\therefore y = 3 \tan\left(x - \frac{\pi}{9}\right)$ has vertical asymptotes $x = -\frac{25\pi}{18}$, $x = -\frac{7\pi}{18}$, $x = \frac{11\pi}{18}$, $x = \frac{29\pi}{18}$, and x -intercepts $-\frac{17\pi}{9}$, $-\frac{8\pi}{9}$, $\frac{\pi}{9}$, and $\frac{10\pi}{9}$.



5



The y -intercept is 1, so $y = \tan pt$ has been translated upwards by 1 unit.

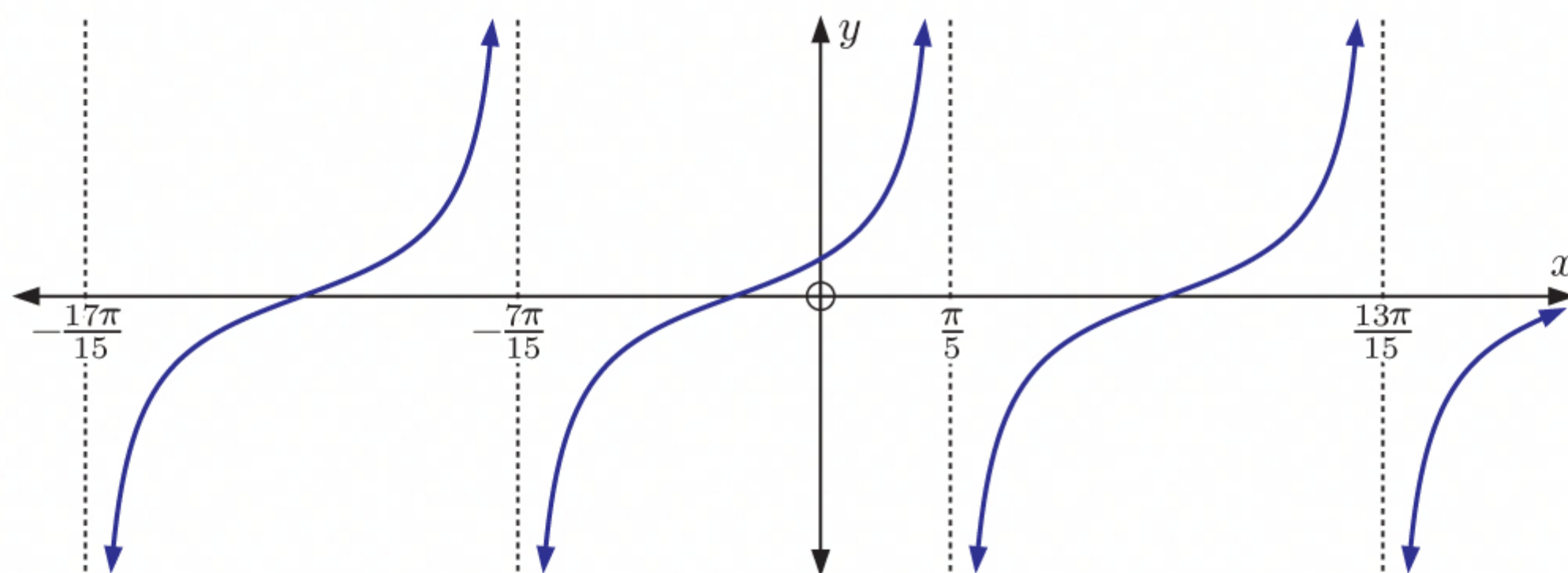
$$\therefore q = 1$$

$$y = \tan pt + 1 \text{ has period } \frac{\pi}{p} = 2\pi$$

$$\therefore p = \frac{1}{2}$$

So, $p = \frac{1}{2}$, $q = 1$.

6



$$y = \tan a(x - b) \text{ has period } \frac{\pi}{a} = \frac{\pi}{5} - \left(-\frac{7\pi}{15}\right) = \frac{2\pi}{3}$$

$$\therefore \frac{a}{\pi} = \frac{3}{2\pi}$$

$$\therefore a = \frac{3}{2}$$

$y = \tan x$ has vertical asymptotes $x = \frac{\pi}{2} + k\pi$, $k \in \mathbb{Z}$

$\therefore y = \tan \frac{3}{2}x$ has vertical asymptotes $x = \frac{\pi}{3} + \frac{2k\pi}{3}$, $k \in \mathbb{Z}$

$\therefore y = \tan \frac{3}{2}(x - b)$ has vertical asymptotes $x = \frac{\pi}{3} + b + \frac{2k\pi}{3}$, $k \in \mathbb{Z}$

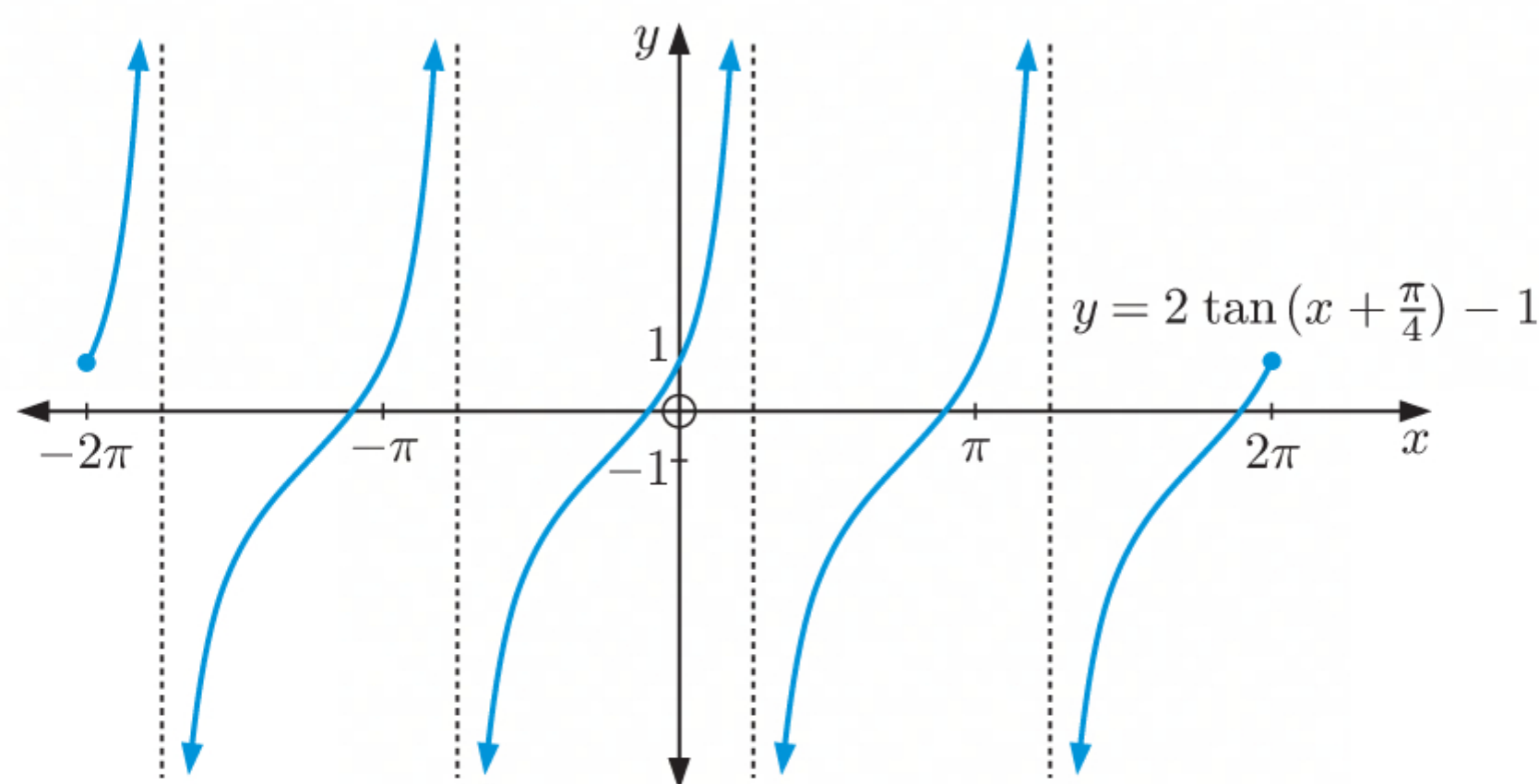
From the graph, $\frac{\pi}{3} + b + \frac{2k_1\pi}{3} = \frac{\pi}{5} + \frac{2k_2\pi}{3}$, $k_1, k_2 \in \mathbb{Z}$

$$\therefore b = -\frac{2\pi}{15} + \frac{2k\pi}{3}, \quad k \in \mathbb{Z} \quad \{k = k_2 - k_1\}$$

7

$$\text{a } \tan x \xrightarrow{\substack{\text{vertical stretch} \\ \text{scale factor 2}}} 2 \tan x \xrightarrow{\substack{\text{translation } \begin{pmatrix} -\frac{\pi}{4} \\ -1 \end{pmatrix}}} 2 \tan\left(x + \frac{\pi}{4}\right) - 1$$

So, a vertical stretch with scale factor 2, then a translation $\frac{\pi}{4}$ units left and 1 unit downwards will map $y = \tan x$ onto $y = 2 \tan\left(x + \frac{\pi}{4}\right) - 1$.

b

8 $f(x) = \tan x$, $g(x) = 2x - \frac{\pi}{2}$

a **i** $(f \circ g)(x) = f(g(x))$

$$= f\left(2x - \frac{\pi}{2}\right)$$

$$= \tan\left(2x - \frac{\pi}{2}\right)$$

ii $(g \circ f)(x) = g(f(x))$

$$= g(\tan x)$$

$$= 2 \tan x - \frac{\pi}{2}$$

b **i** $(f \circ g)\left(\frac{\pi}{3}\right) = \tan\left(2 \times \frac{\pi}{3} - \frac{\pi}{2}\right)$

$$= \tan\left(\frac{2\pi}{3} - \frac{\pi}{2}\right)$$

$$= \tan \frac{\pi}{6}$$

$$= \frac{1}{\sqrt{3}}$$

ii $(g \circ f)(\pi) = 2 \tan \pi - \frac{\pi}{2}$

$$= -\frac{\pi}{2}$$

c **i** $(f \circ g)(x) = \tan\left(2x - \frac{\pi}{2}\right)$ has period $\frac{\pi}{2}$.

$$\tan\left(2x - \frac{\pi}{2}\right) \text{ is undefined when } 2x - \frac{\pi}{2} = \frac{\pi}{2} + k\pi, \quad k \in \mathbb{Z}$$

$$\therefore 2x = \pi + k\pi, \quad k \in \mathbb{Z}$$

$$\therefore 2x = k\pi, \quad k \in \mathbb{Z}$$

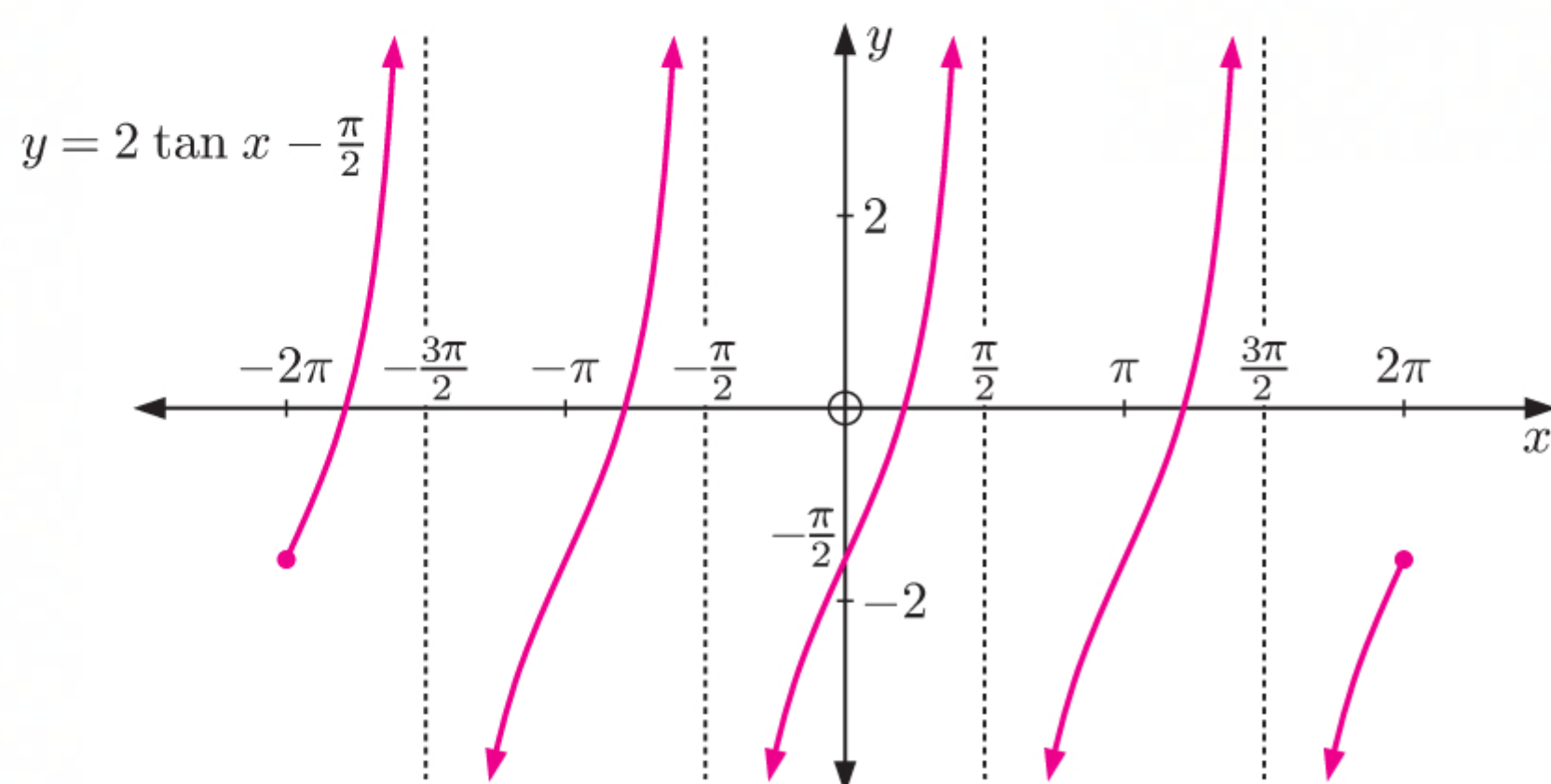
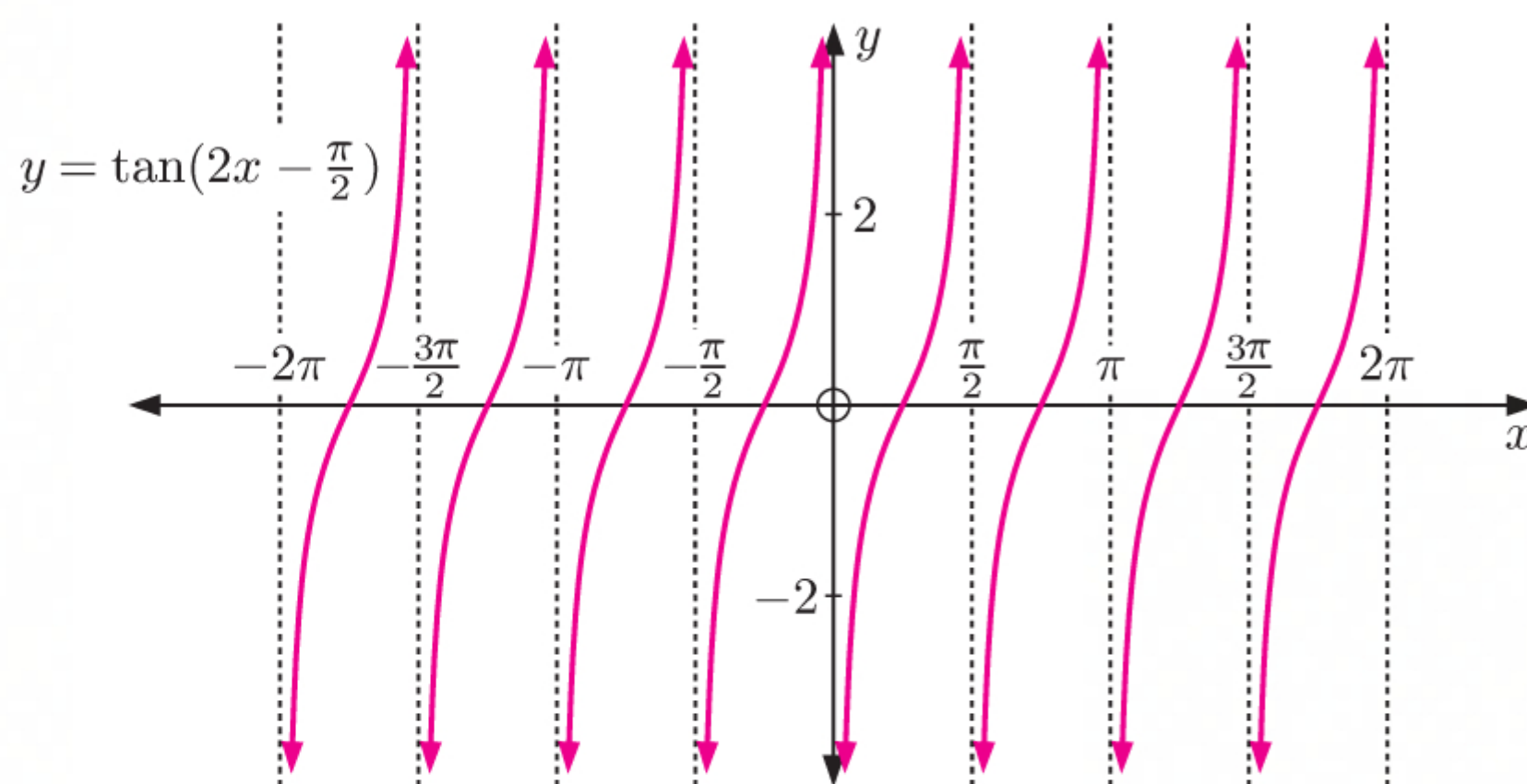
$$\therefore x = \frac{k\pi}{2}, \quad k \in \mathbb{Z}$$

$$\therefore \text{the vertical asymptotes of } (f \circ g)(x) \text{ are } x = \frac{k\pi}{2}, \quad k \in \mathbb{Z}.$$

ii $(g \circ f)(x) = 2 \tan x - \frac{\pi}{2}$ has period π .

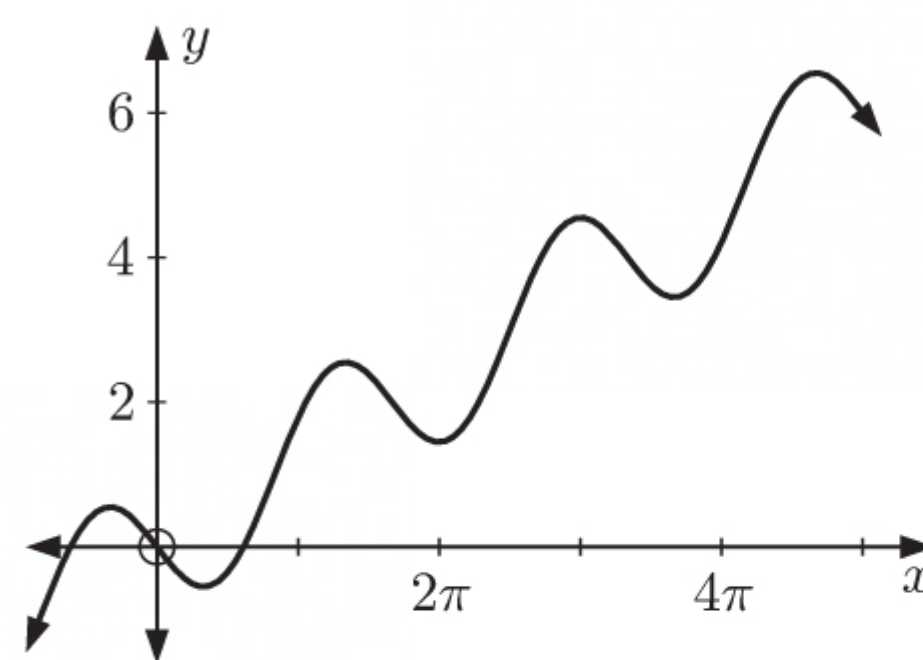
$$\tan x \text{ is undefined when } x = \frac{\pi}{2} + k\pi, \quad k \in \mathbb{Z}$$

$$\therefore \text{the vertical asymptotes of } (g \circ f)(x) \text{ are } x = \frac{\pi}{2} + k\pi, \quad k \in \mathbb{Z}.$$

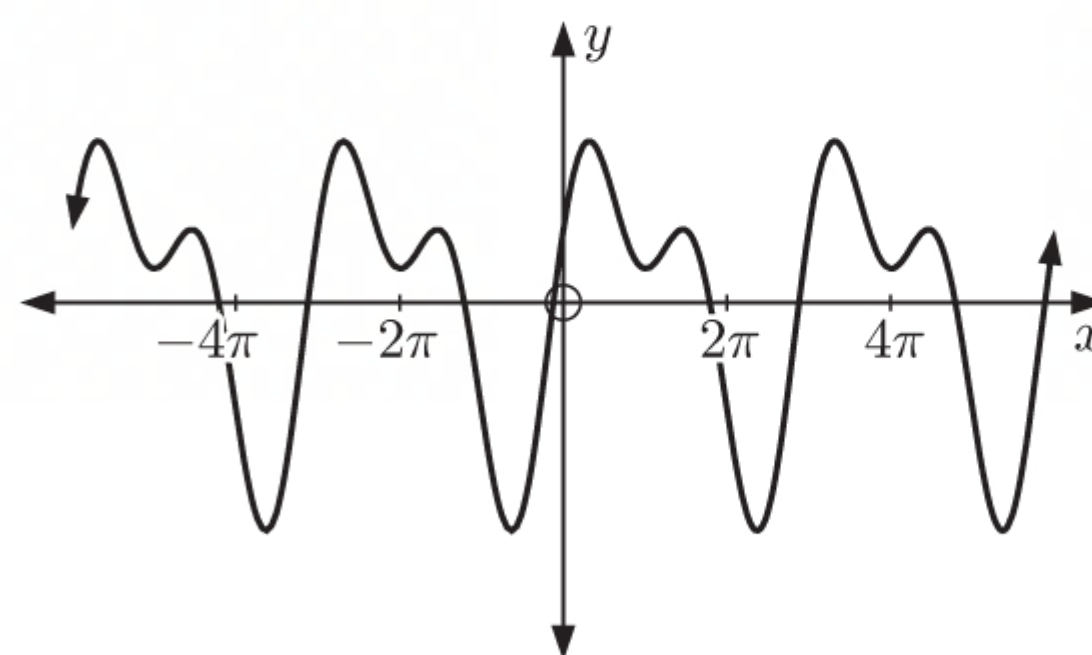
d

REVIEW SET 8A

- 1 a** This graph repeats itself over and over in intervals of the same length, but not in a horizontal direction.
 \therefore this graph does not show periodic behaviour.



- b** This graph repeats itself over and over in a horizontal direction, in intervals of the same length.
 \therefore this graph shows periodic behaviour.



- 2 a** $1 + \sin x$ has minimum value $1 + (-1) = 0$ {when $\sin x = -1$ }
 and maximum value $1 + 1 = 2$ {when $\sin x = 1$ }

- b** $-2 \cos 3x$ has minimum value $-2(1) = -2$ {when $\cos 3x = 1$ }
and maximum value $-2(-1) = 2$ {when $\cos 3x = -1$ }

3 a $y = 4 \sin \frac{x}{5}$ has period $\frac{2\pi}{b} = \frac{2\pi}{(\frac{1}{5})}$
 $= 10\pi$

b $y = -2 \cos 4x$ has period $\frac{2\pi}{b} = \frac{2\pi}{4}$
 $= \frac{\pi}{2}$

c $y = 4 \cos \frac{x}{2} + 4$ has period $\frac{2\pi}{b} = \frac{2\pi}{(\frac{1}{2})}$
 $= 4\pi$

d $y = \frac{1}{2} \tan 3x$ has period $\frac{\pi}{b} = \frac{\pi}{3}$

4 $y = -3 \sin \frac{x}{4} + 1$ has period $= \frac{2\pi}{b} = \frac{2\pi}{(\frac{1}{4})} = 8\pi$

amplitude $= |-3| = 3$

maximum value $= -3(-1) + 1 = 4$ {when $\sin \frac{x}{4} = -1$ }

and minimum value $= -3(1) + 1 = -2$ {when $\sin \frac{x}{4} = 1$ }

$y = 3 \cos \pi x$ has period $= \frac{2\pi}{b} = \frac{2\pi}{\pi} = 2$

amplitude $= 3$

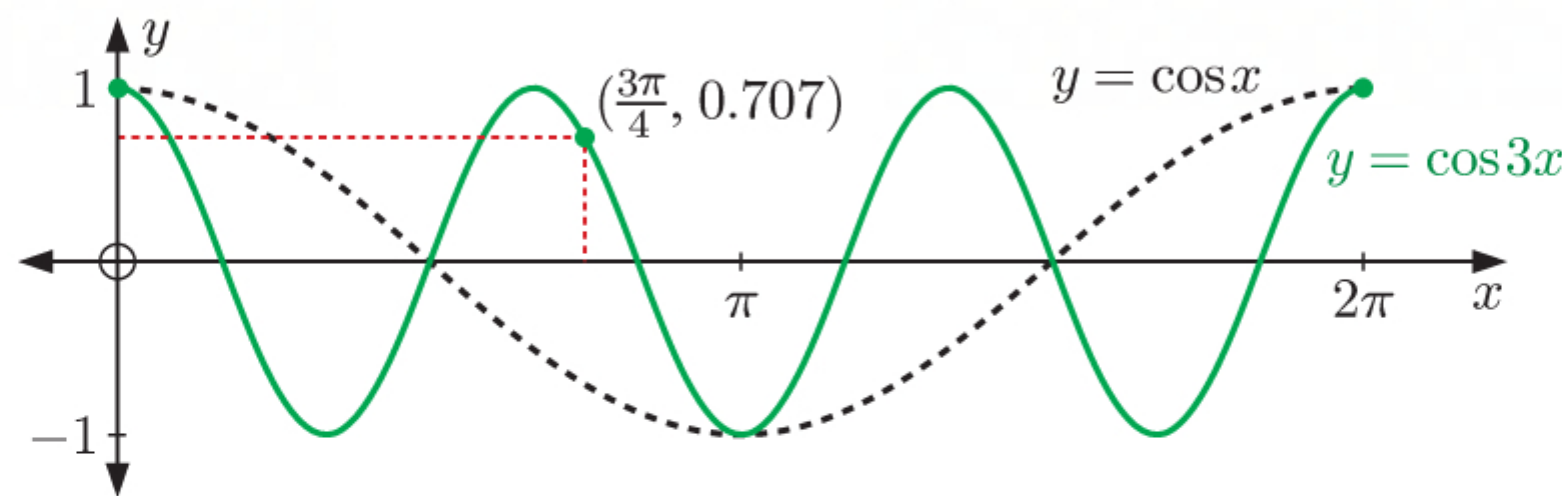
maximum value $= 3(1) = 3$ {when $\cos \pi x = 1$ }

and minimum value $= 3(-1) = -3$ {when $\cos \pi x = -1$ }

So,

Function	Period	Amplitude	Range
$y = -3 \sin \frac{x}{4} + 1$	8π	3	$-2 \leq y \leq 4$
$y = 3 \cos \pi x$	2	3	$-3 \leq y \leq 3$

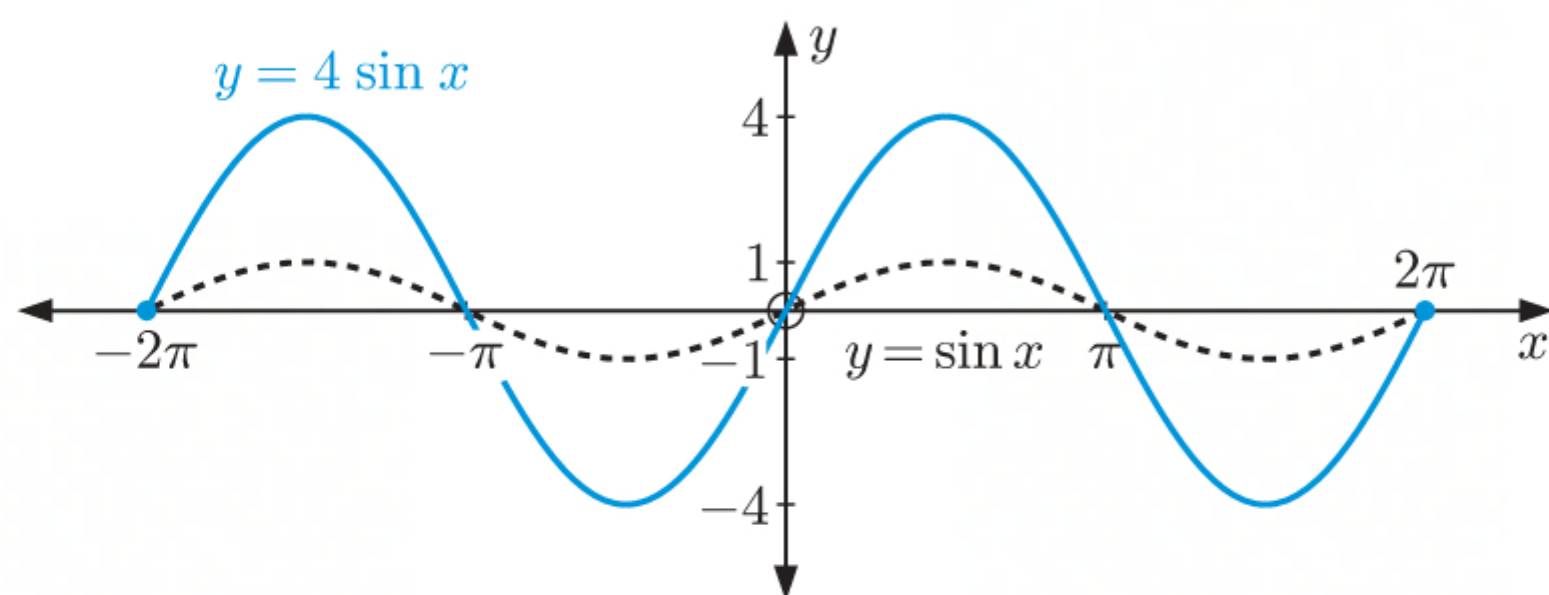
- 5 a** $y = \cos 3x$ is a horizontal stretch of $y = \cos x$ with scale factor $\frac{1}{3}$.



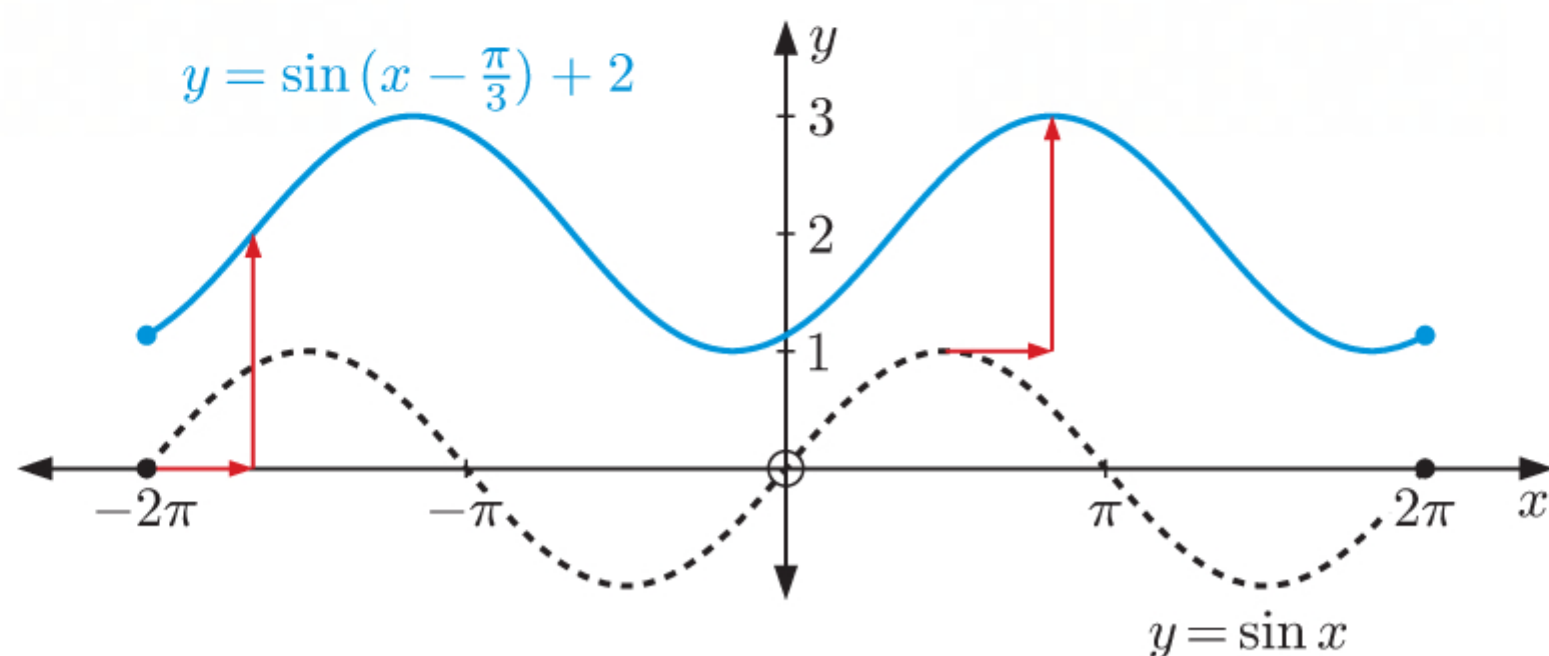
b When $x = \frac{3\pi}{4}$, $y = \cos 3\left(\frac{3\pi}{4}\right)$
 $= \cos \frac{9\pi}{4}$
 $= \cos\left(\frac{\pi}{4} + 2\pi\right)$
 $= \cos \frac{\pi}{4}$ { $\cos(\theta + 2\pi) = \cos \theta$ }
 $= \frac{1}{\sqrt{2}} \approx 0.707$

- 6 a** $a = 4$, so the amplitude is $|4| = 4$.

We stretch $y = \sin x$ vertically with scale factor 4 to give $y = 4 \sin x$.

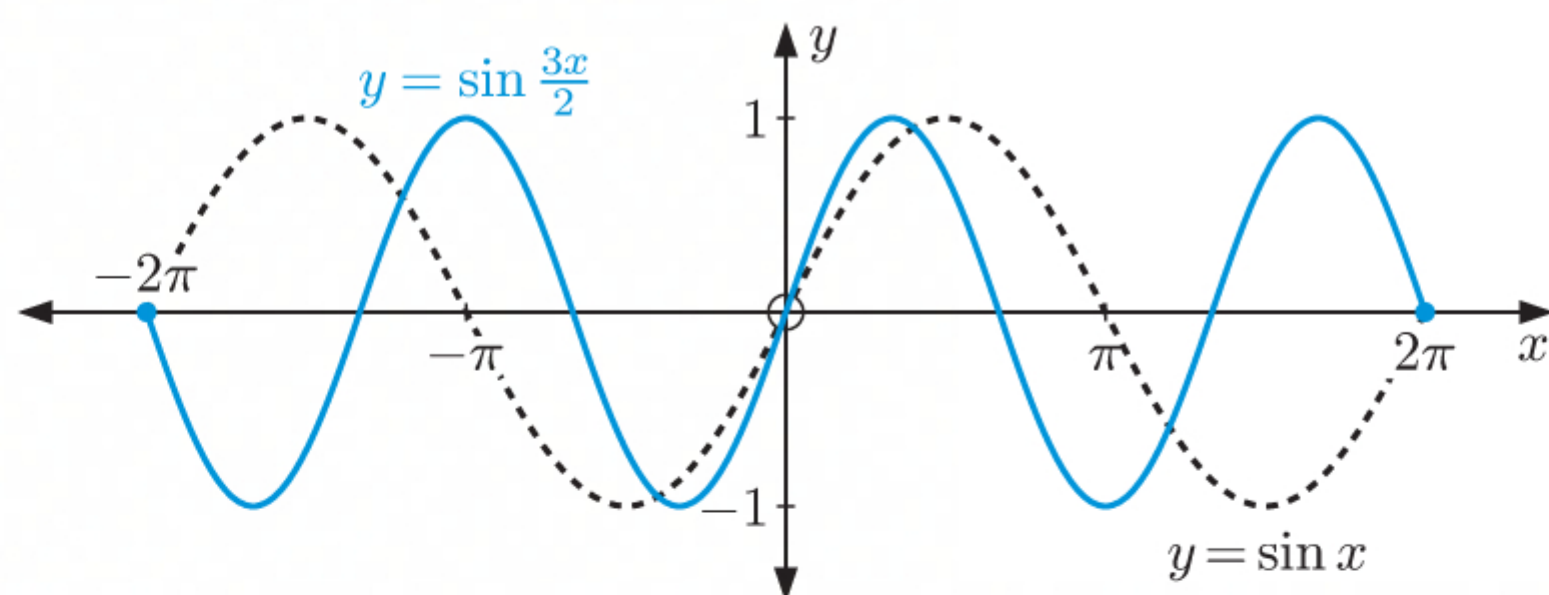


- b** We translate $y = \sin x$ $\frac{\pi}{3}$ units to the right and 2 units upwards to give $y = \sin(x - \frac{\pi}{3}) + 2$.

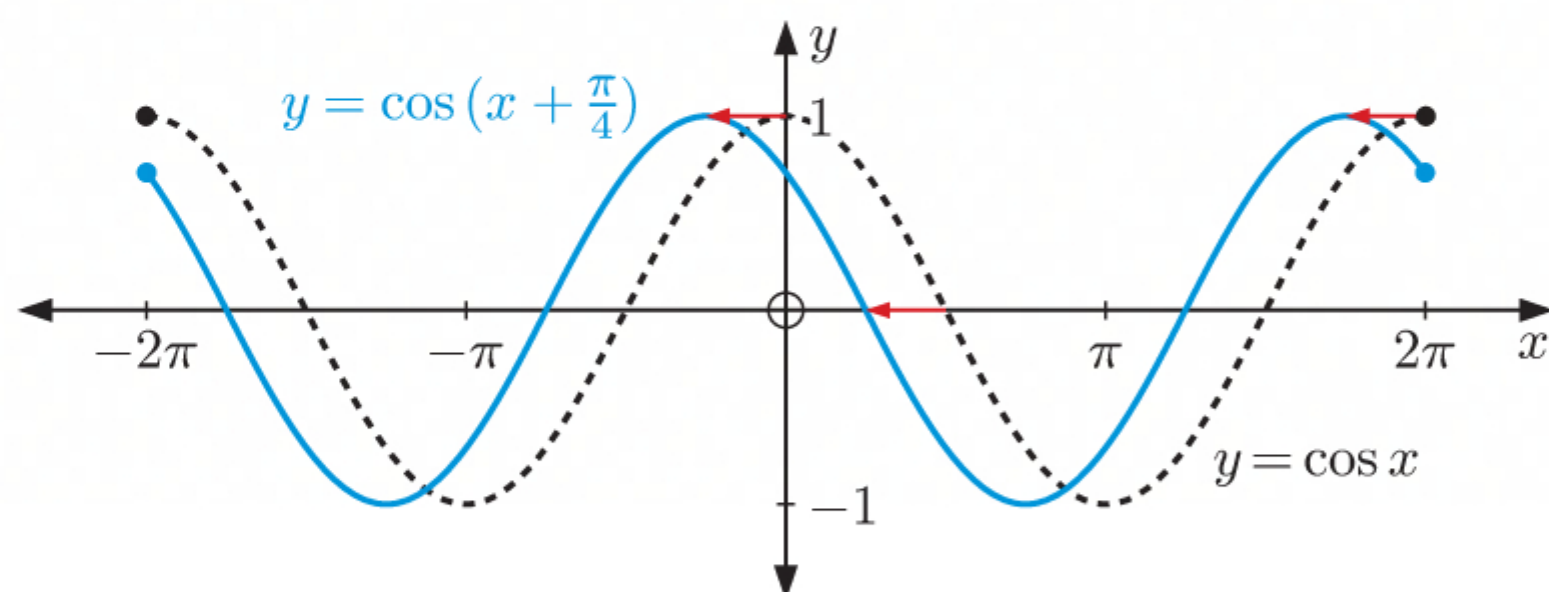


- c** $b = \frac{3}{2}$, so the period is $\frac{2\pi}{b} = \frac{2\pi}{(\frac{3}{2})} = \frac{4\pi}{3}$.

We stretch $y = \sin x$ horizontally with scale factor $\frac{2}{3}$ to give $y = \sin \frac{3x}{2}$.

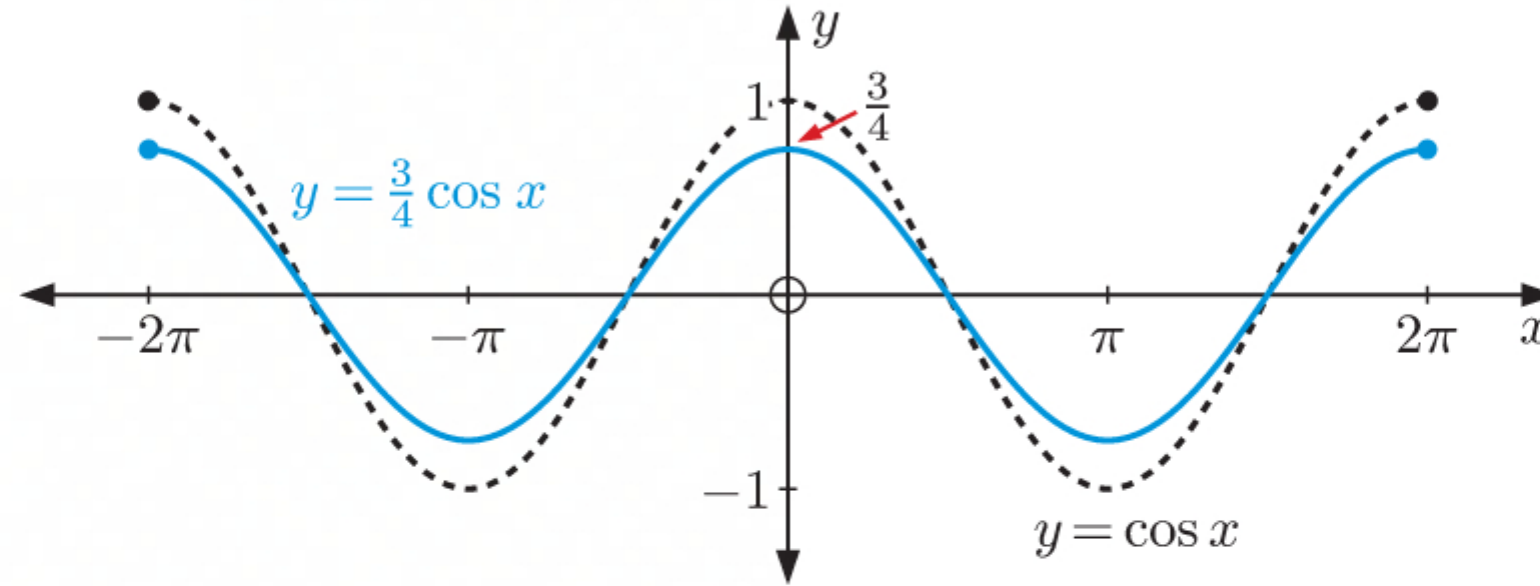


- d** We translate $y = \cos x$ $\frac{\pi}{4}$ units to the left to give $y = \cos(x + \frac{\pi}{4})$.



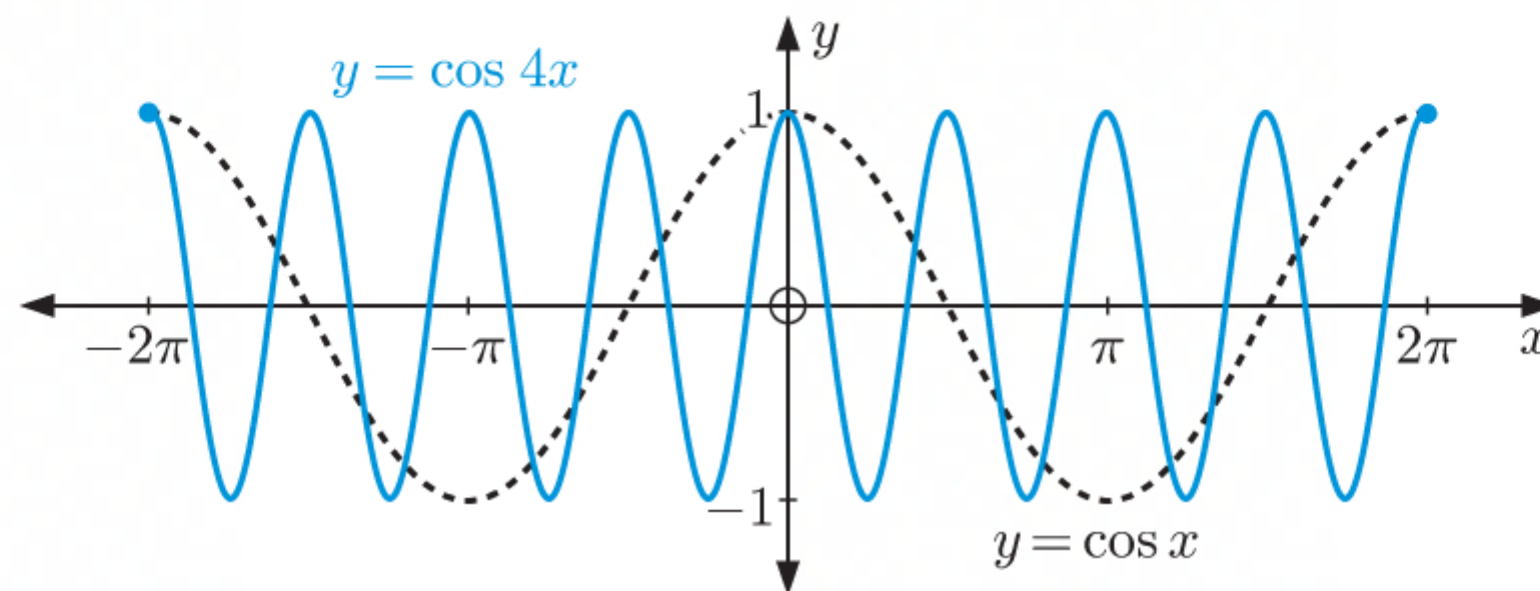
- e** $a = \frac{3}{4}$, so the amplitude is $\left| \frac{3}{4} \right| = \frac{3}{4}$.

We stretch $y = \cos x$ vertically with scale factor $\frac{3}{4}$ to give $y = \frac{3}{4} \cos x$.



- f** $b = 4$, so the period is $\frac{2\pi}{b} = \frac{2\pi}{4} = \frac{\pi}{2}$.

We stretch $y = \cos x$ horizontally with scale factor $\frac{1}{4}$ to give $y = \cos 4x$.

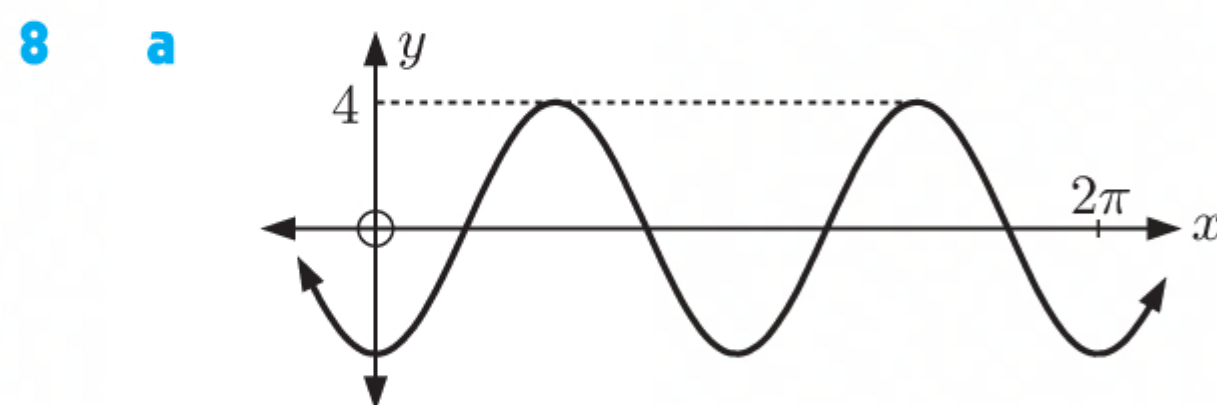


- 7 a** $\sin x \xrightarrow[\text{vertical stretch}]{\text{scale factor } 3} 3 \sin x \xrightarrow[\text{horizontal stretch}]{\text{scale factor } \frac{1}{2}} 3 \sin 2x$

A vertical stretch with scale factor 3, then a horizontal stretch with scale factor $\frac{1}{2}$ maps $y = \sin x$ onto $y = 3 \sin 2x$.

- b** $\cos x \xrightarrow[\text{translation } \left(\begin{smallmatrix} \frac{\pi}{3} \\ -1 \end{smallmatrix} \right)]{} \cos\left(x - \frac{\pi}{3}\right) - 1$

A translation $\frac{\pi}{3}$ units right and 1 unit downwards maps $y = \cos x$ onto $y = \cos\left(x - \frac{\pi}{3}\right) - 1$.



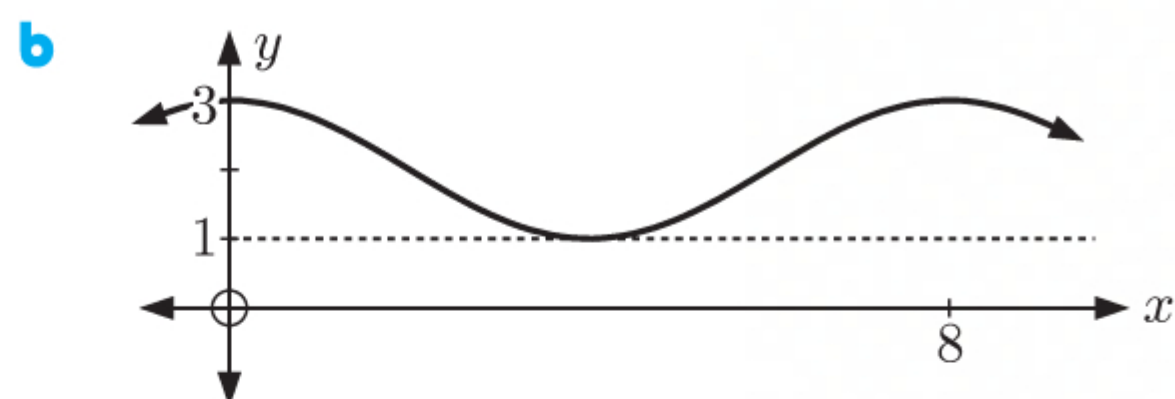
The amplitude is 4, and $y < 0$ when $x = 0$, so $a = -4$.

The period is π , so $\frac{2\pi}{b} = \pi$ and $\therefore b = 2$.

There is no horizontal translation, so $c = 0$.

The principal axis is $y = 0$, so $d = 0$.

\therefore the equation of the function is $y = -4 \cos 2x$.



The amplitude is 1, so $a = 1$.

The period is 8, so $\frac{2\pi}{b} = 8$ and $\therefore b = \frac{\pi}{4}$.

There is no horizontal translation, so $c = 0$.

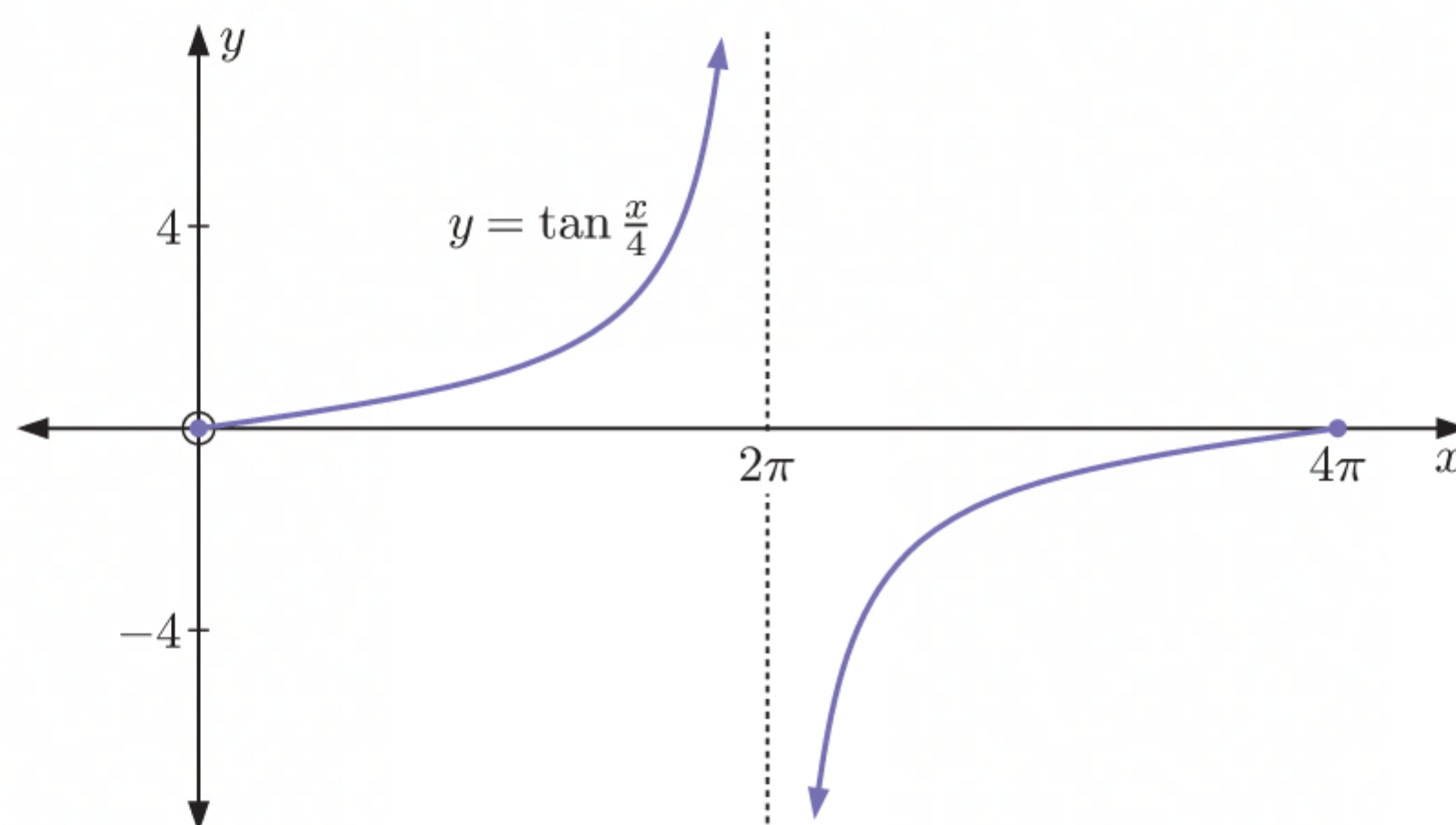
The principal axis is $y = 2$, so $d = 2$.

\therefore the equation of the function is $y = \cos \frac{\pi x}{4} + 2$.

9 a $y = \tan \frac{x}{4}$ is a horizontal stretch of $y = \tan x$ with scale factor 4.

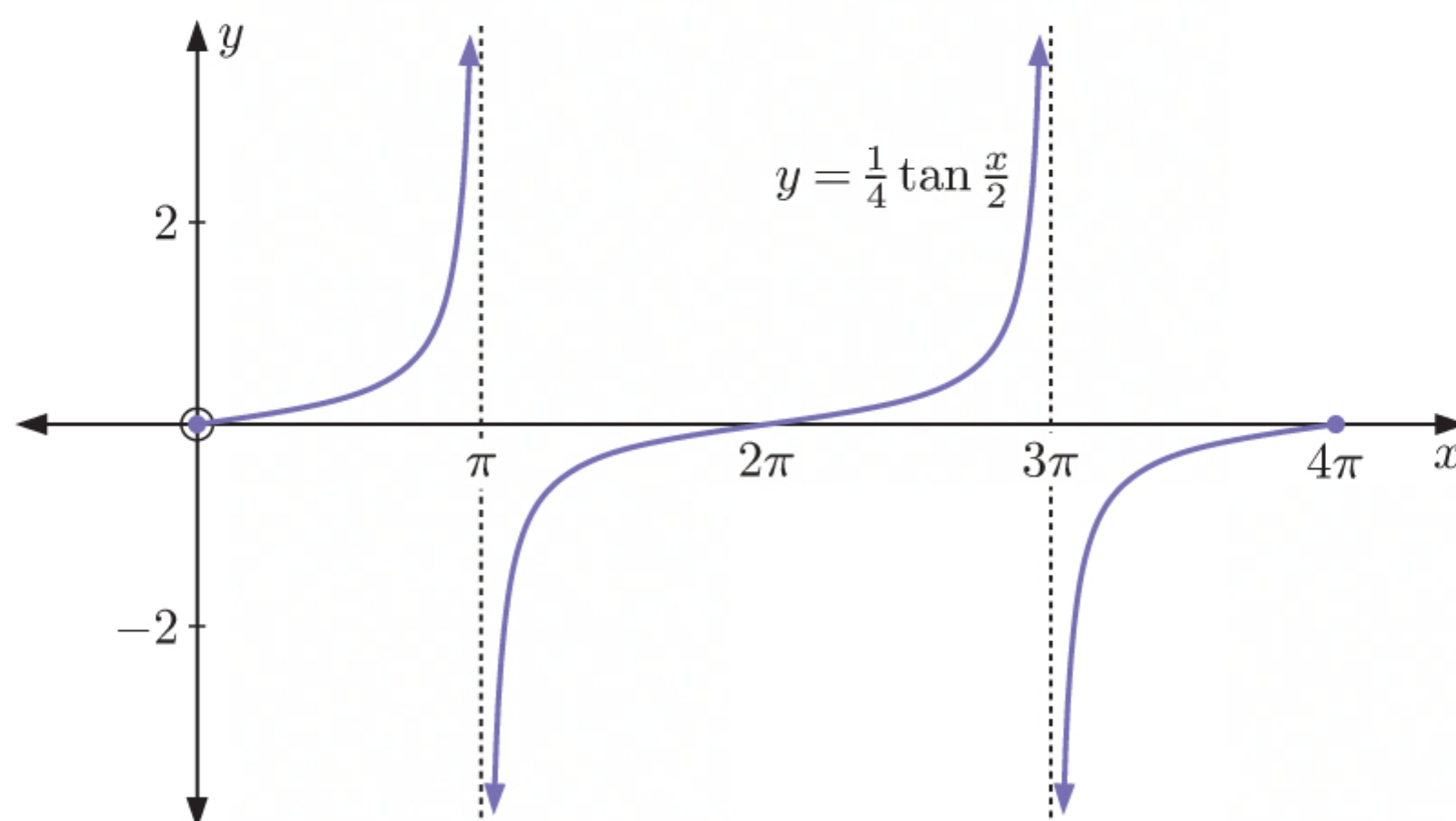
$y = \tan x$ has vertical asymptotes $x = \frac{\pi}{2}$, $x = \frac{3\pi}{2}$, $x = \frac{5\pi}{2}$, $x = \frac{7\pi}{2}$, and x -intercepts $0, \pi, 2\pi, 3\pi$, and 4π .

$\therefore y = \tan \frac{x}{4}$ has vertical asymptote $x = 2\pi$, and x -intercepts 0 and 4π .



b $y = \frac{1}{4} \tan \frac{x}{2}$ is a horizontal stretch of $y = \tan x$ with scale factor 2, followed by a vertical stretch with scale factor $\frac{1}{4}$.

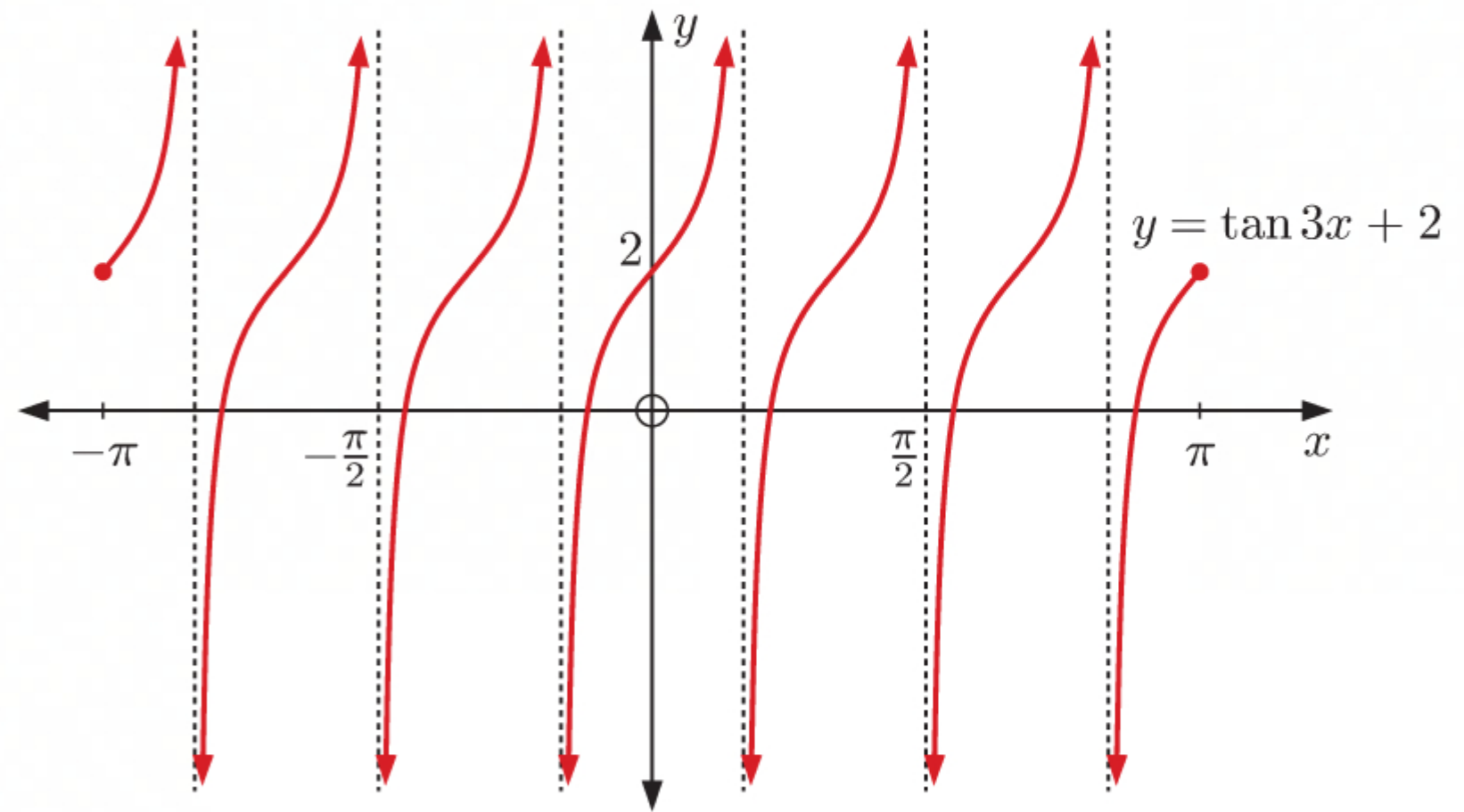
$y = \frac{1}{4} \tan \frac{x}{2}$ has vertical asymptotes $x = \pi$, $x = 3\pi$, and x -intercepts $0, 2\pi$, and 4π .



10 a $\tan x \xrightarrow[\text{horizontal stretch}]{\text{scale factor } \frac{1}{3}} \tan 3x \xrightarrow[\text{translation } \begin{pmatrix} 0 \\ 2 \end{pmatrix}]{} \tan 3x + 2$

A horizontal stretch with scale factor $\frac{1}{3}$, then a vertical translation 2 units upwards maps $y = \tan x$ onto $y = \tan 3x + 2$.

b $y = \tan 3x + 2$ has **c**
period $\frac{\pi}{b} = \frac{\pi}{3}$



11 a The amplitude $= \frac{\max - \min}{2} = \frac{17 - 3}{2} = 7$, so $a = 7$.

The period is $\frac{2\pi}{b} = 13 - (-3)$

$$\therefore \frac{2\pi}{b} = 16$$

$$\therefore \frac{b}{2\pi} = \frac{1}{16}$$

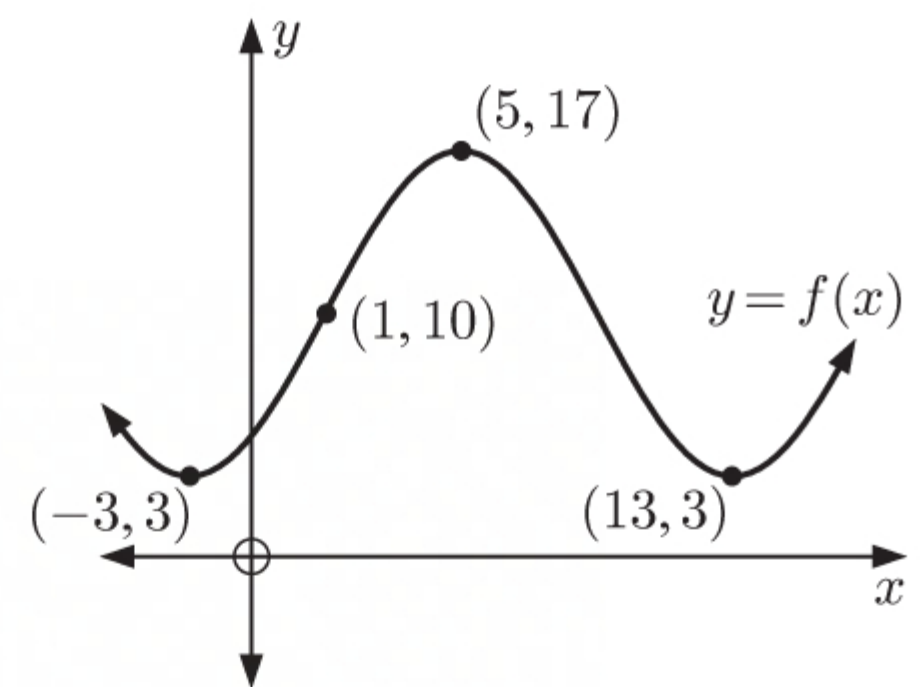
$$\therefore b = \frac{\pi}{8}$$

The principal axis is $y = \frac{\max + \min}{2} = \frac{17 + 3}{2} = 10$,
so $d = 10$.

The point midway between $(-3, 3)$ and $(5, 17)$ is $(1, 10)$, so there is a horizontal translation of 1 unit to the right, and thus $c = 1$.

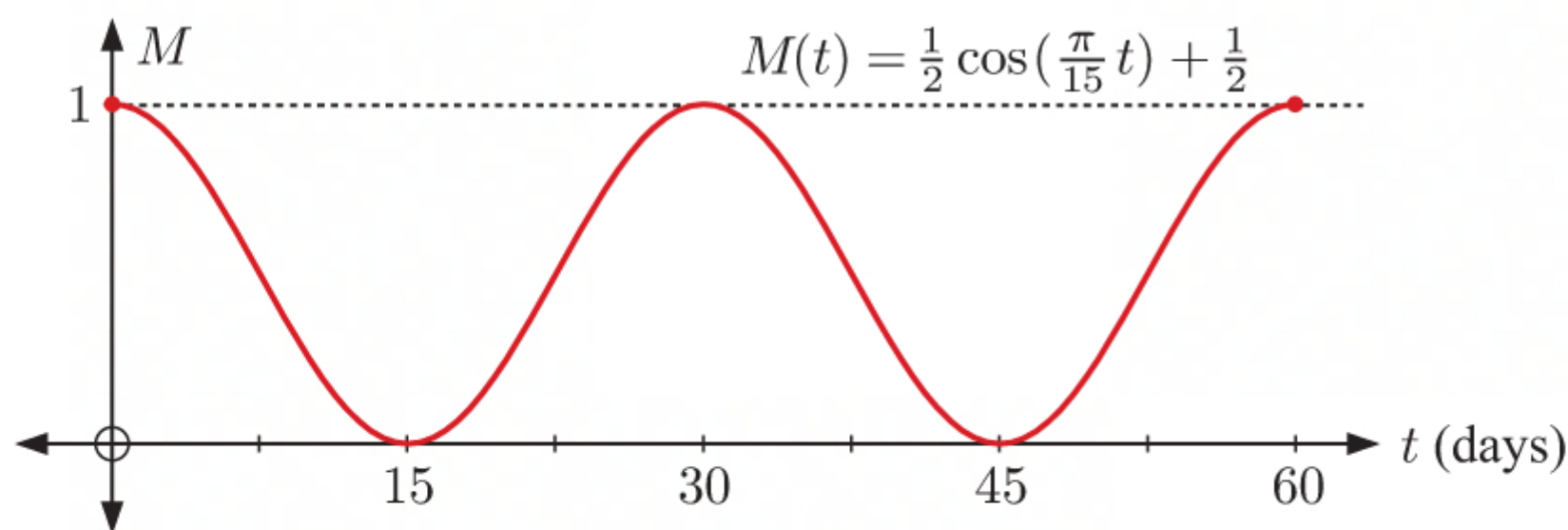
So, $a = 7$, $b = \frac{\pi}{8}$, $c = 1$, and $d = 10$.

b $f(x) = 7 \sin\left(\frac{\pi}{8}(x - 1)\right) + 10$



$f(x) \xrightarrow[\text{translation } \begin{pmatrix} 2 \\ -3 \end{pmatrix}]{} f(x - 2) - 3 \xrightarrow[\text{vertical stretch}]{\text{scale factor } 2} 2[f(x - 2) - 3]$

$$\begin{aligned} \text{So, } g(x) &= 2[f(x - 2) - 3] \\ &= 2[7 \sin\left(\frac{\pi}{8}((x - 2) - 1)\right) + 10 - 3] \\ &= 2[7 \sin\left(\frac{\pi}{8}(x - 3)\right) + 7] \\ &= 14 \sin\left(\frac{\pi}{8}(x - 3)\right) + 14 \end{aligned}$$

12 a**b i** January 6th is 5 days after January 1st, so $t = 5$.

$$\begin{aligned}
 M(5) &= \frac{1}{2} \cos\left(\frac{\pi}{15} \times 5\right) + \frac{1}{2} \\
 &= \frac{1}{2} \cos \frac{\pi}{3} + \frac{1}{2} \\
 &= \frac{1}{2} \times \frac{1}{2} + \frac{1}{2} \\
 &= \frac{1}{4} + \frac{1}{2} \\
 &= \frac{3}{4} = 0.75
 \end{aligned}$$

 \therefore the proportion of the moon illuminated on the night of January 6th is 0.75.**ii** January 21st is 20 days after January 1st, so $t = 20$.

$$\begin{aligned}
 M(20) &= \frac{1}{2} \cos\left(\frac{\pi}{15} \times 20\right) + \frac{1}{2} \\
 &= \frac{1}{2} \cos \frac{4\pi}{3} + \frac{1}{2} \\
 &= \frac{1}{2} \times \left(-\frac{1}{2}\right) + \frac{1}{2} \\
 &= -\frac{1}{4} + \frac{1}{2} \\
 &= \frac{1}{4} = 0.25
 \end{aligned}$$

 \therefore the proportion of the moon illuminated on the night of January 21st is 0.25.**iii** January 27th is 26 days after January 1st, so $t = 26$.

$$\begin{aligned}
 M(26) &= \frac{1}{2} \cos\left(\frac{\pi}{15} \times 26\right) + \frac{1}{2} \\
 &\approx 0.835
 \end{aligned}$$

 \therefore the proportion of the moon illuminated on the night of January 27th is 0.835.**iv** February 19th is 49 days after January 1st, so $t = 49$.

$$\begin{aligned}
 M(49) &= \frac{1}{2} \cos\left(\frac{\pi}{15} \times 49\right) + \frac{1}{2} \\
 &\approx 0.165
 \end{aligned}$$

 \therefore the proportion of the moon illuminated on the night of February 19th is 0.165.**c** The period is $\frac{2\pi}{b} = \frac{2\pi}{(\frac{\pi}{15})} = 30$ days. \therefore a full moon occurs once every 30 days.

$$\begin{aligned}
 \text{d } M(t) = 0 \text{ when } \frac{1}{2} \cos\left(\frac{\pi}{15}t\right) + \frac{1}{2} &= 0 \\
 \therefore \frac{1}{2} \cos\left(\frac{\pi}{15}t\right) &= -\frac{1}{2} \\
 \therefore \cos\left(\frac{\pi}{15}t\right) &= -1 \\
 \therefore \frac{\pi}{15}t &= \pi, 3\pi, 5\pi, \dots
 \end{aligned}$$

which is true when $t = 15$ or 45 $\{0 \leq t \leq 60\}$ $t = 15$ corresponds to January 16, and $t = 45$ corresponds to February 15. \therefore the moon is not illuminated at all on January 16th and February 15th.

- 13 a** The mean temperature $= \frac{14.1 + 6.7}{2} = 10.4^\circ\text{C}$, so $d = 10.4$.

$$\text{The amplitude} = \frac{14.1 - 6.7}{2} = 3.7^\circ\text{C}$$

$$\therefore a = 3.7$$

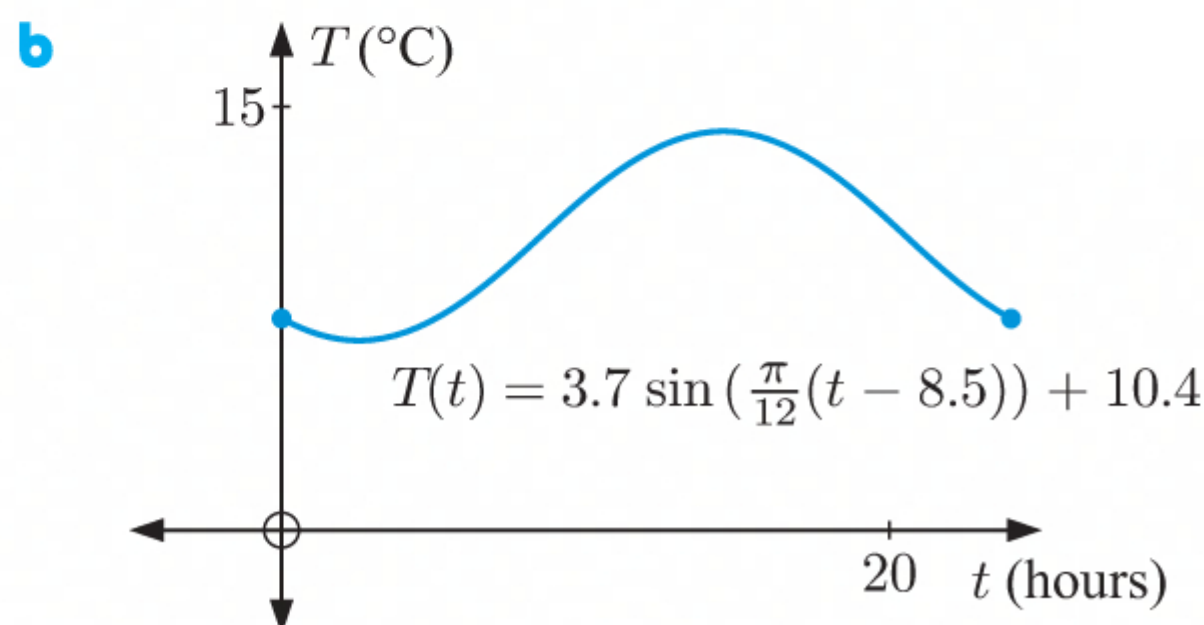
The period is 24 hours, so $b = \frac{2\pi}{24} = \frac{\pi}{12}$.

The maximum occurs at 2:30 pm, so we assume the temperature passed its mean value 6 hours earlier, at 8:30 am.

The day begins at midnight, so the function is shifted $8\frac{1}{2}$ hours to the right, thus $c = 8.5$.

If t is the number of hours after midnight, the temperature T is modelled by

$$T(t) = 3.7 \sin\left(\frac{\pi}{12}(t - 8.5)\right) + 10.4^\circ\text{C}.$$



14

Number of Mars days (n)	0	100	200	300	400	500	600	700	800	900	1000	1100	1200	1300
Temp. ($^\circ\text{C}$)	-43	-15	-5	-21	-59	-79	-68	-50	-27	-8	-15	-70	-78	-68

- a** The maximum temperature recorded was -5°C , the minimum temperature recorded was -79°C .
- b** The time between maximum values is $900 - 200 = 700$ so we estimate the length of a Mars year to be about 700 Mars days.
- c** We estimate that the period is 700 Mars days, so $b \approx \frac{2\pi}{700}$.

$$\text{The amplitude} = \frac{\max - \min}{2} \approx \frac{-5 - (-79)}{2} \approx 37, \text{ so } a \approx 37.$$

The principal axis is midway between the maximum and minimum, so

$$d \approx \frac{-5 + (-79)}{2} \approx -42.$$

The first minimum occurs at $n \approx 500$ and the following maximum occurs at $n \approx 900$.

The sine function starts a new period midway between these two points.

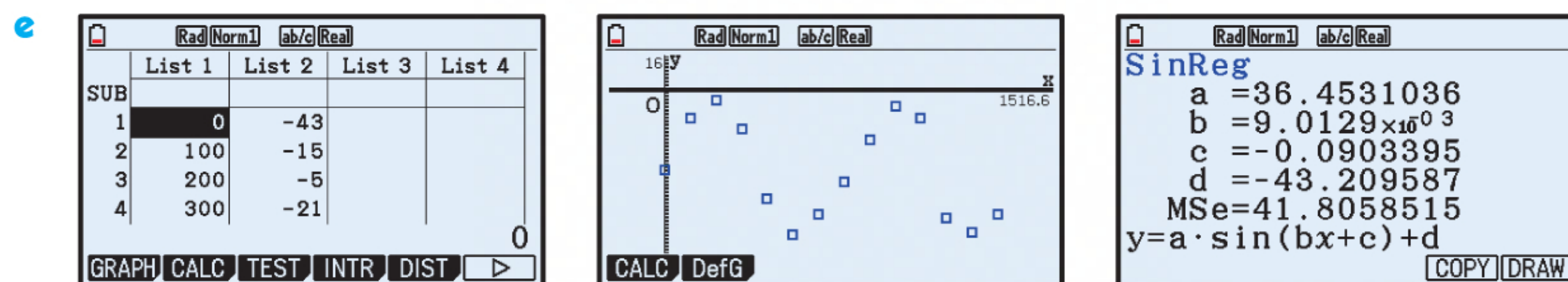
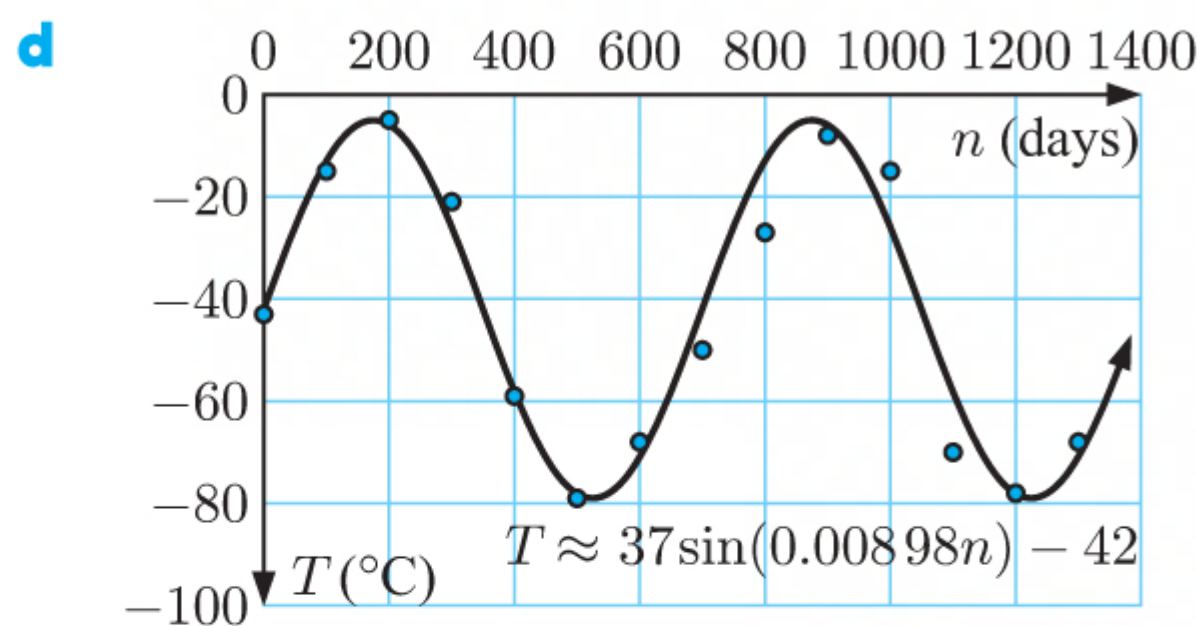
$$\therefore c \approx \frac{500 + 900}{2} \approx 700$$

$$\therefore T \approx 37 \sin\left(\frac{2\pi}{700}(n - 700)\right) - 42$$

$$\approx 37 \sin\left(\frac{2\pi n}{700} - 2\pi\right) - 42$$

$$\approx 37 \sin \frac{2\pi n}{700} - 42$$

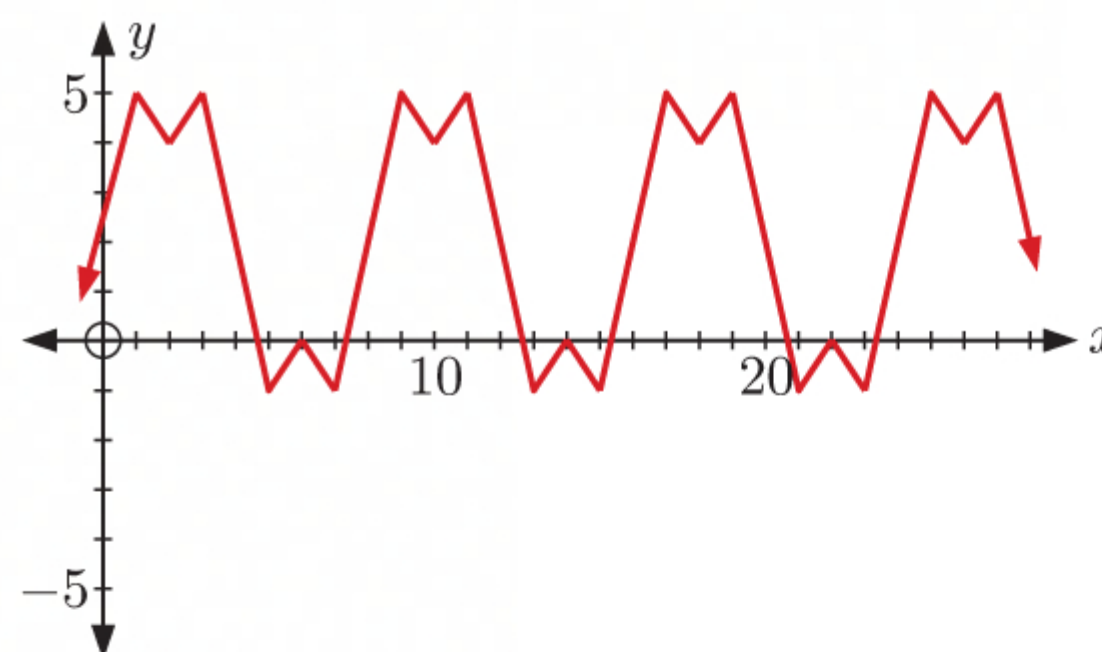
$$\approx 37 \sin(0.00898n) - 42$$



Using technology, $T \approx 36.5 \sin(0.00901n - 0.0903) - 43.2$.
Our model fits the data reasonably well.

REVIEW SET 8B

- 1 a** The graph is periodic because it repeats itself over and over in a horizontal direction in intervals of the same length.



- b i** period = 8 **ii** maximum value = 5 **iii** minimum value = -1
- 2 a** $y = \cos\left(x + \frac{\pi}{4}\right) + 1$ is a horizontal translation to the left by $\frac{\pi}{4}$ units followed by a vertical translation upwards by 1 unit.
So, a translation of $\begin{pmatrix} -\frac{\pi}{4} \\ 1 \end{pmatrix}$ will map $y = \cos x$ onto $y = \cos\left(x + \frac{\pi}{4}\right) + 1$.
- b** $y = \sin 3x$ is a horizontal stretch of $y = \sin x$ with scale factor $\frac{1}{3}$.
So, a horizontal stretch with scale factor $\frac{1}{3}$ will map $y = \sin x$ onto $y = \sin 3x$.
- 3 a** $y = 4 \sin \frac{x}{3}$ has period $\frac{2\pi}{\frac{1}{3}} = \frac{2\pi}{1/3} = 6\pi$ **b** $y = \tan 4x$ has period $\frac{\pi}{4}$
- 4** $y = \sin bx, b > 0$
- a** period = $\frac{2\pi}{b} = 6\pi$ **b** period = $\frac{2\pi}{b} = \frac{\pi}{12}$ **c** period = $\frac{2\pi}{b} = 9$
 $\therefore b = \frac{1}{3}$ $\therefore b = 24$ $\therefore b = \frac{2\pi}{9}$

5 a $y = 5 \sin x - 3$ has minimum value $5(-1) - 3 = -8$ {when $\sin x = -1$ }
and maximum value $5(1) - 3 = 2$ {when $\sin x = 1$ }

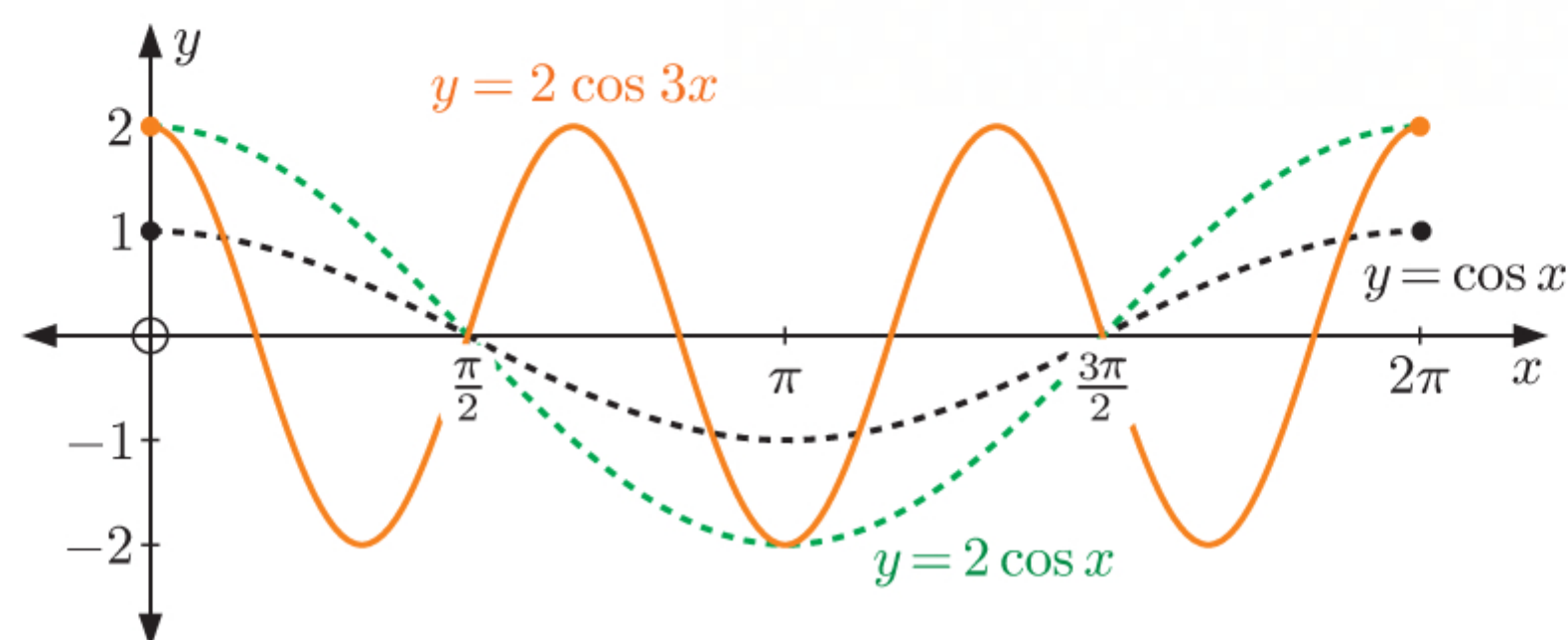
b $y = \frac{1}{3} \cos x + 1$ has minimum value $\frac{1}{3}(-1) + 1 = \frac{2}{3}$ {when $\cos x = -1$ }
and maximum value $\frac{1}{3}(1) + 1 = 1\frac{1}{3}$ {when $\cos x = 1$ }

6 a $y = -\frac{1}{3} \sin(x - \frac{\pi}{4}) + 5$ has principal axis $y = 5$.

b $y = 2 \cos \frac{x}{3} - 4$ has principal axis $y = -4$.

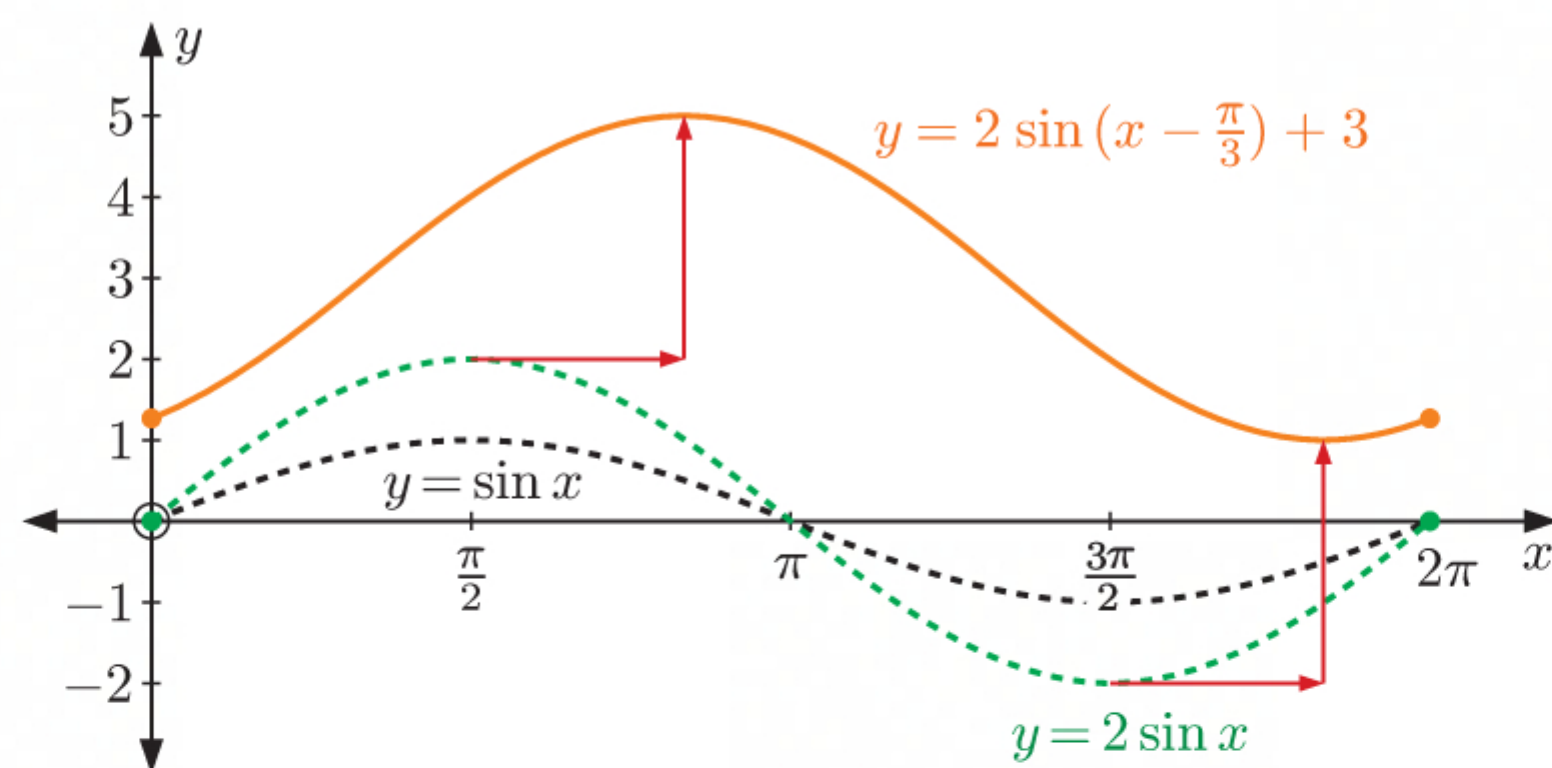
7 a $a = 2$, so the amplitude is $|2| = 2$. $b = 3$, so the period is $\frac{2\pi}{b} = \frac{2\pi}{3}$.

We stretch $y = \cos x$ vertically with scale factor 2 to give $y = 2 \cos x$, then stretch $y = 2 \cos x$ horizontally with scale factor $\frac{1}{3}$ to give $y = 2 \cos 3x$.



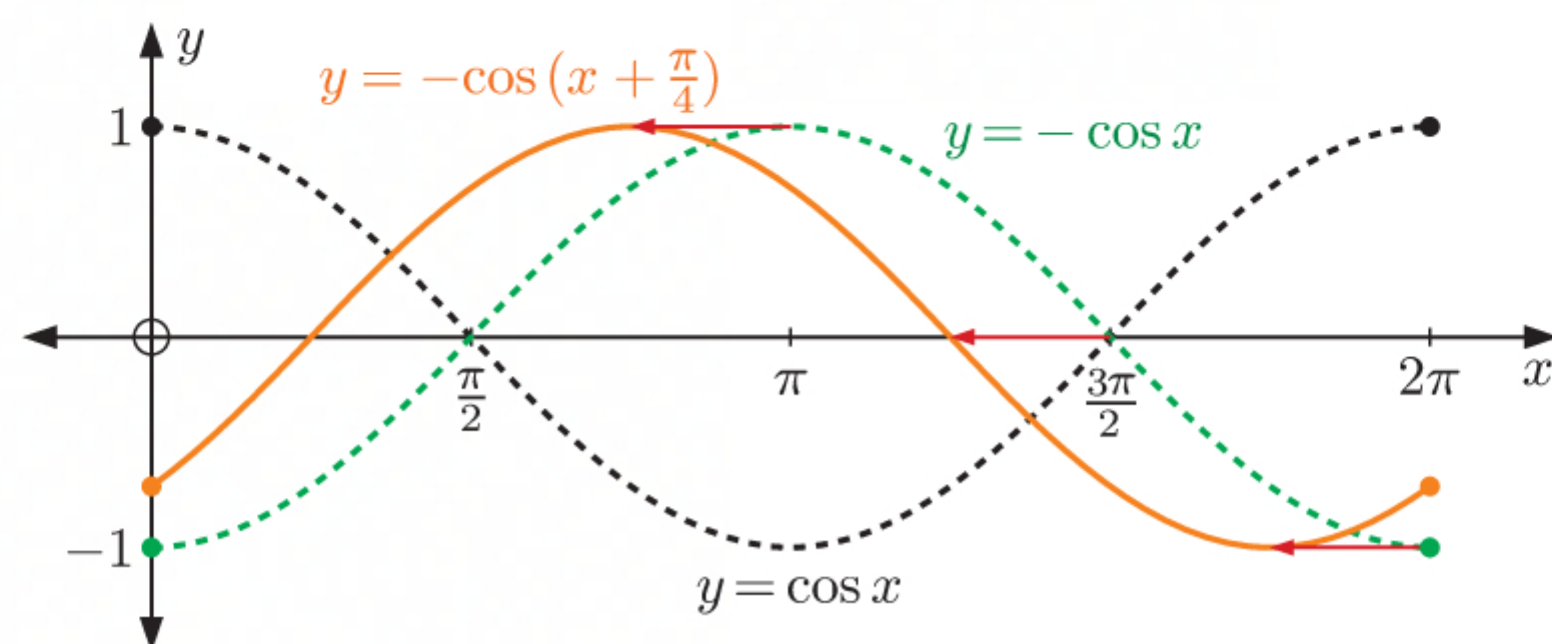
b $a = 2$, so the amplitude is $|2| = 2$.

We stretch $y = \sin x$ vertically with scale factor 2 to give $y = 2 \sin x$, then translate $y = 2 \sin x$ $\frac{\pi}{3}$ units to the right and 3 units upwards to give $y = 2 \sin(x - \frac{\pi}{3}) + 3$.



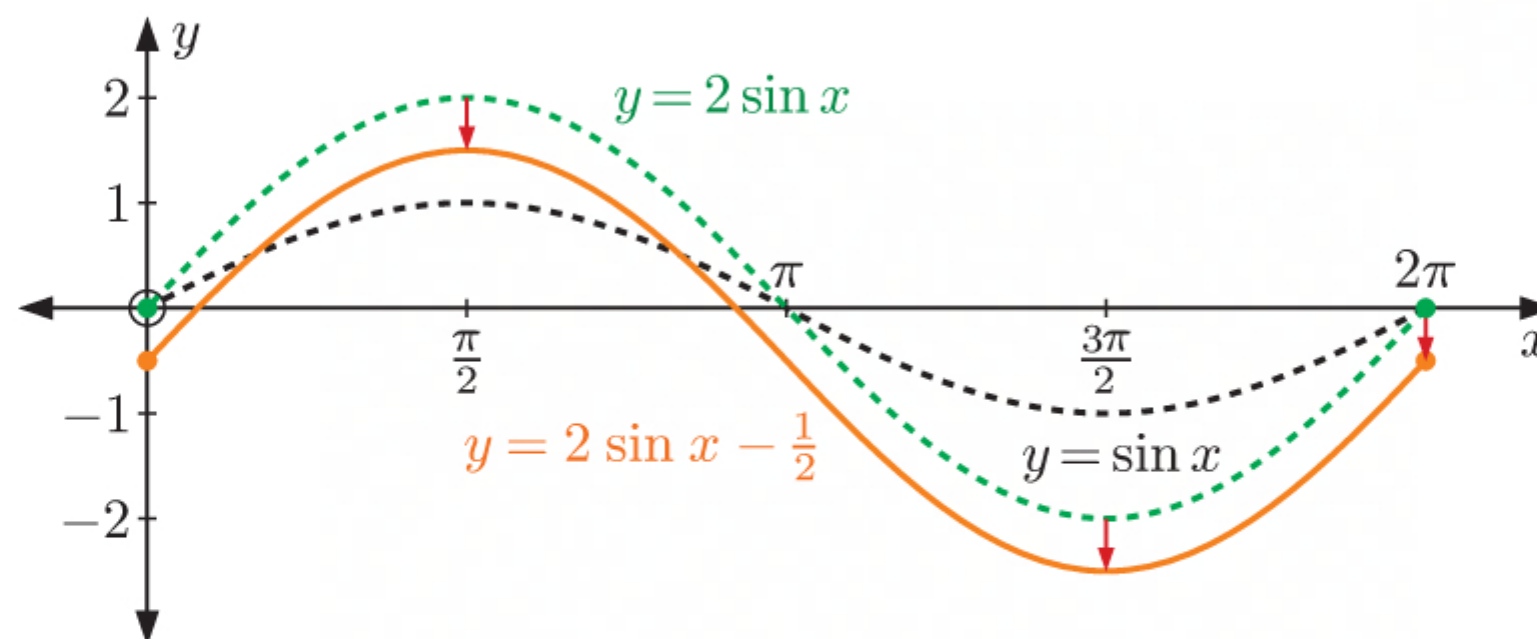
c $a = -1$, so the amplitude is $|-1| = 1$.

We reflect $y = \cos x$ in the x -axis to give $y = -\cos x$, then translate $y = -\cos x$ $\frac{\pi}{4}$ units to the left to give $y = -\cos(x + \frac{\pi}{4})$.

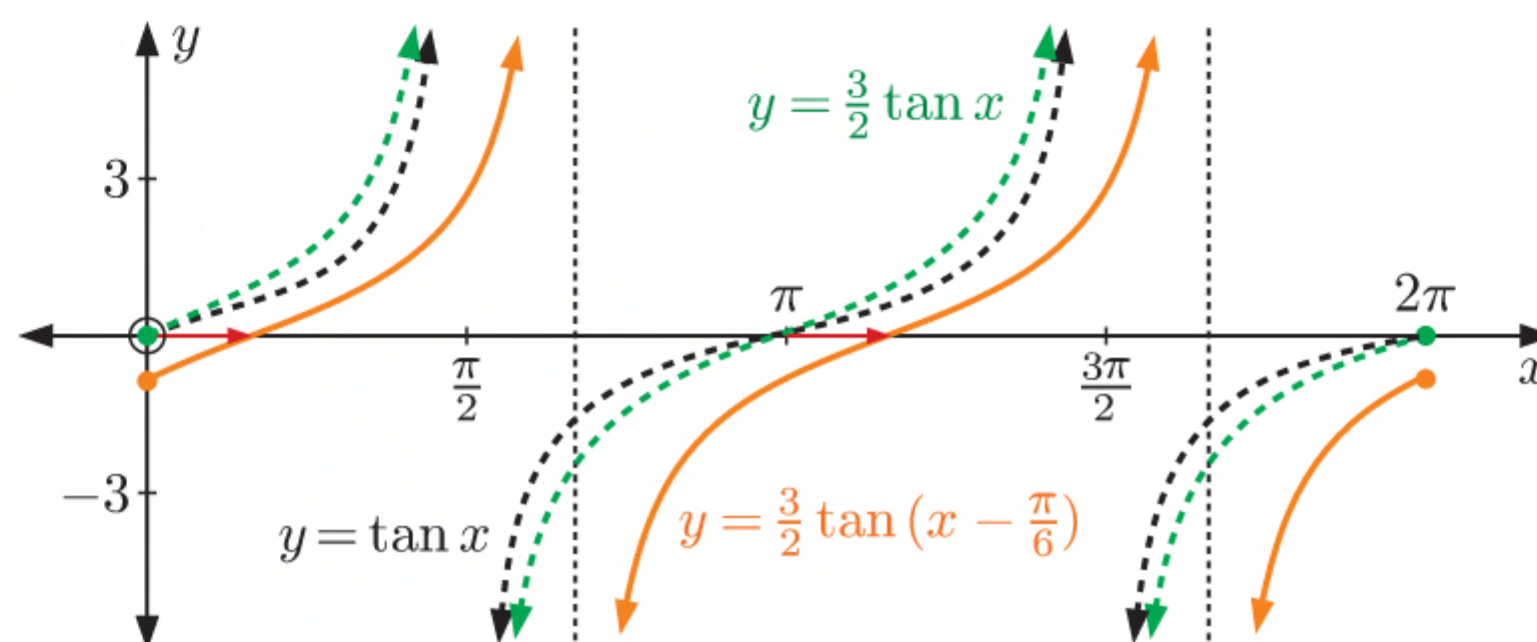


- d** $a = 2$, so the amplitude is $|2| = 2$.

We stretch $y = \sin x$ vertically with scale factor 2 to give $y = 2 \sin x$, then translate $y = 2 \sin x$ $\frac{1}{2}$ unit downwards to give $y = 2 \sin x - \frac{1}{2}$.

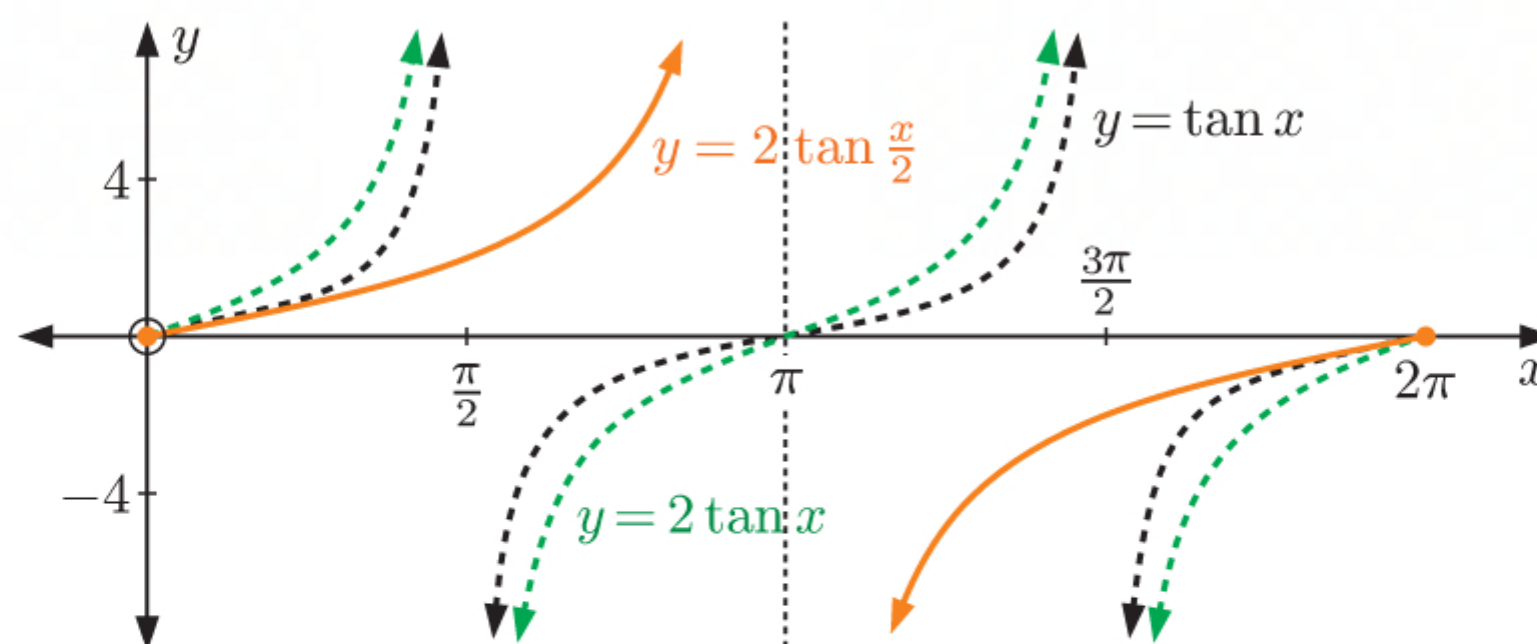


- e** We stretch $y = \tan x$ vertically with scale factor $\frac{3}{2}$ to give $y = \frac{3}{2} \tan x$, then translate $y = \frac{3}{2} \tan x$ $\frac{\pi}{6}$ units to the right to give $y = \frac{3}{2} \tan(x - \frac{\pi}{6})$.



- f** $b = \frac{1}{2}$, so the period is $\frac{\pi}{b} = \frac{\pi}{(\frac{1}{2})} = 2\pi$.

We stretch $y = \tan x$ vertically with scale factor 2 to give $y = 2 \tan x$, then stretch $y = 2 \tan x$ horizontally with scale factor 2 to give $y = 2 \tan \frac{x}{2}$.



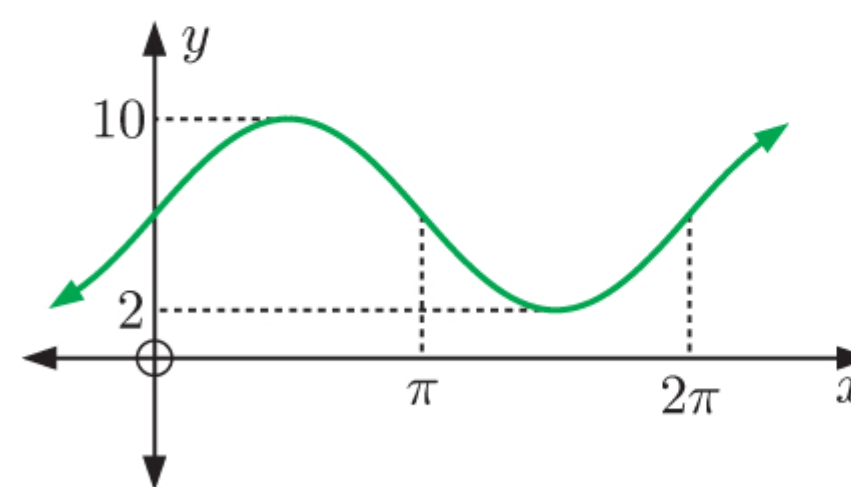
- 8 a** The amplitude is 4, so $a = 4$.

The period is 2π , so $\frac{2\pi}{b} = 2\pi$ and $\therefore b = 1$.

There is no horizontal translation, so $c = 0$.

The principal axis is $y = 6$, so $d = 6$.

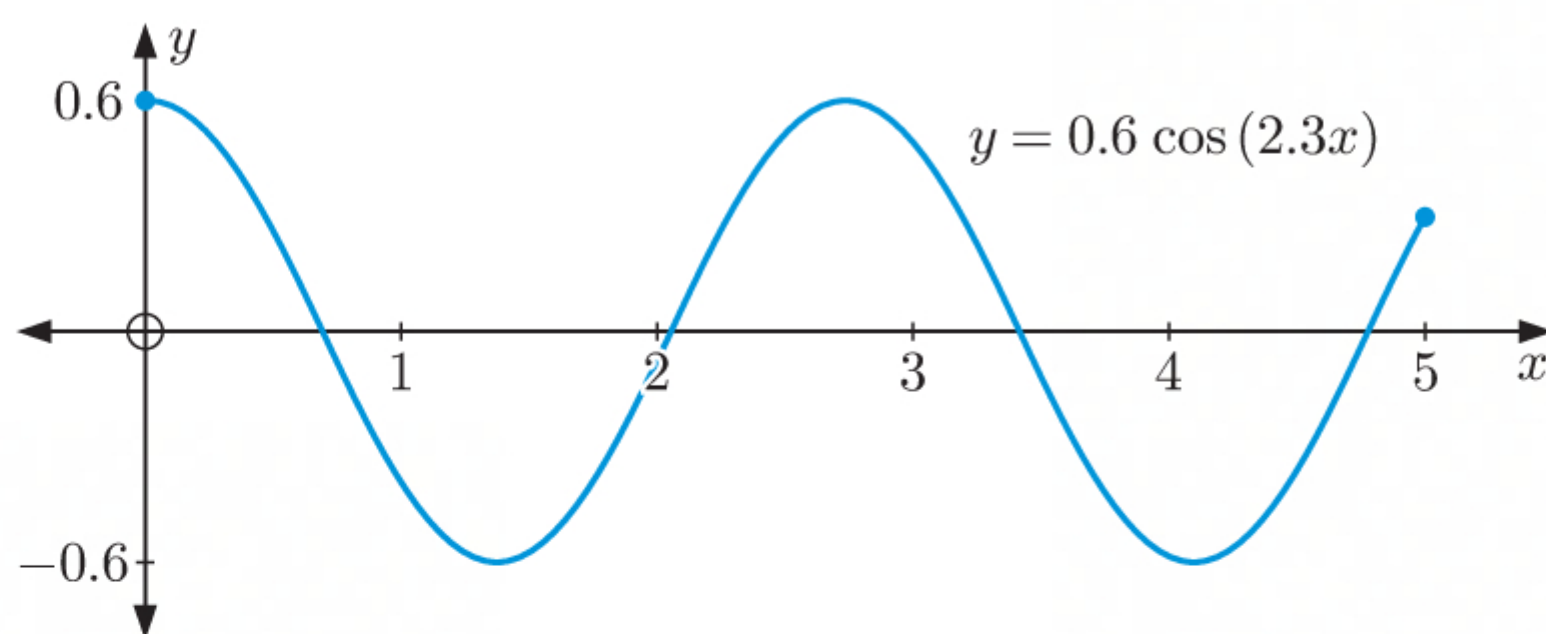
The equation of the function is $y = 4 \sin x + 6$.



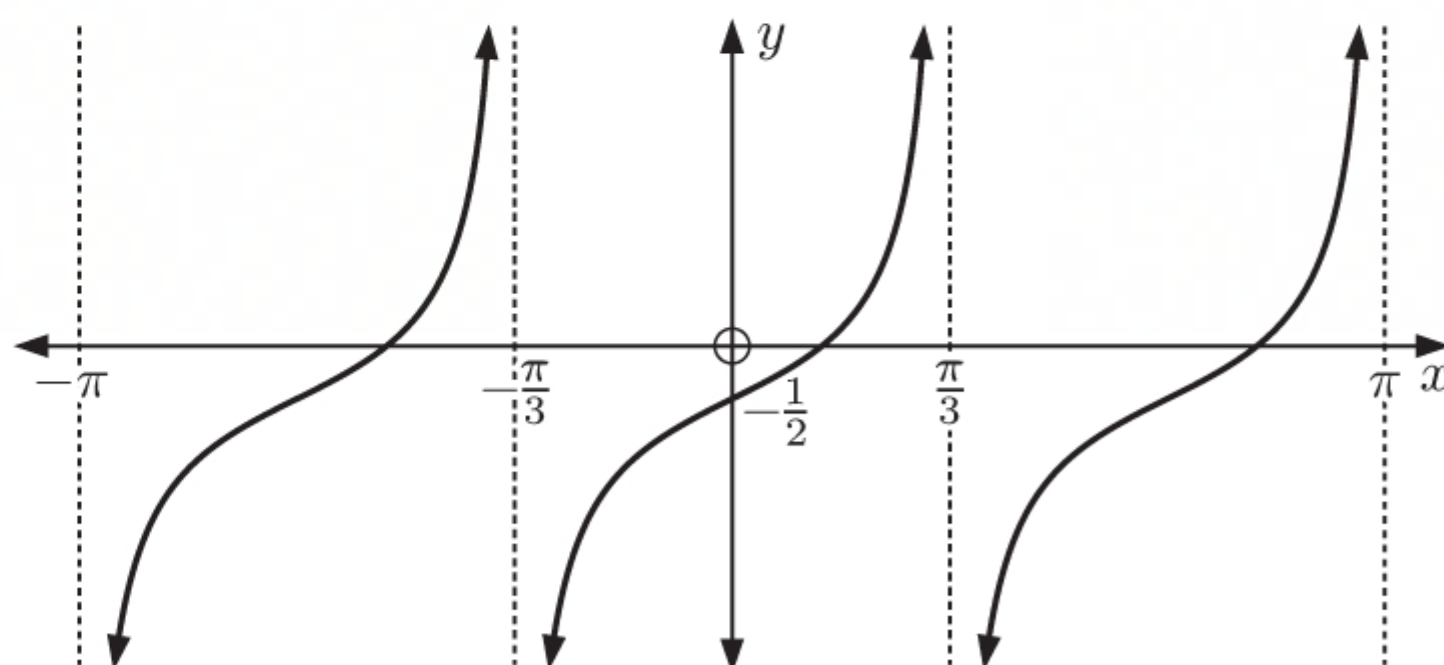
- b** The amplitude, period, and principal axis are the same for an equivalent cosine function, however there is a horizontal translation of $\frac{\pi}{2}$ units to the right, so $c = \frac{\pi}{2}$.

The equation of the function is $y = 4 \cos(x - \frac{\pi}{2}) + 6$.

9



10



$$y = \tan ax + b \text{ has period } \frac{\pi}{a} = \frac{2\pi}{3}$$

$$\therefore \frac{a}{\pi} = \frac{3}{2\pi}$$

$$\therefore a = \frac{3}{2}$$

$$\text{When } x = 0, y = -\frac{1}{2}$$

$$\therefore -\frac{1}{2} = \tan 0 + b$$

$$\therefore b = -\frac{1}{2}$$

$$\text{So, } a = \frac{3}{2}, b = -\frac{1}{2}$$

11

$$\text{a } \tan x \xrightarrow[\text{reflection in } x\text{-axis}]{\text{horizontal stretch scale factor } \frac{1}{2}} -\tan x \xrightarrow{\text{horizontal stretch scale factor } \frac{1}{2}} -\tan 2x$$

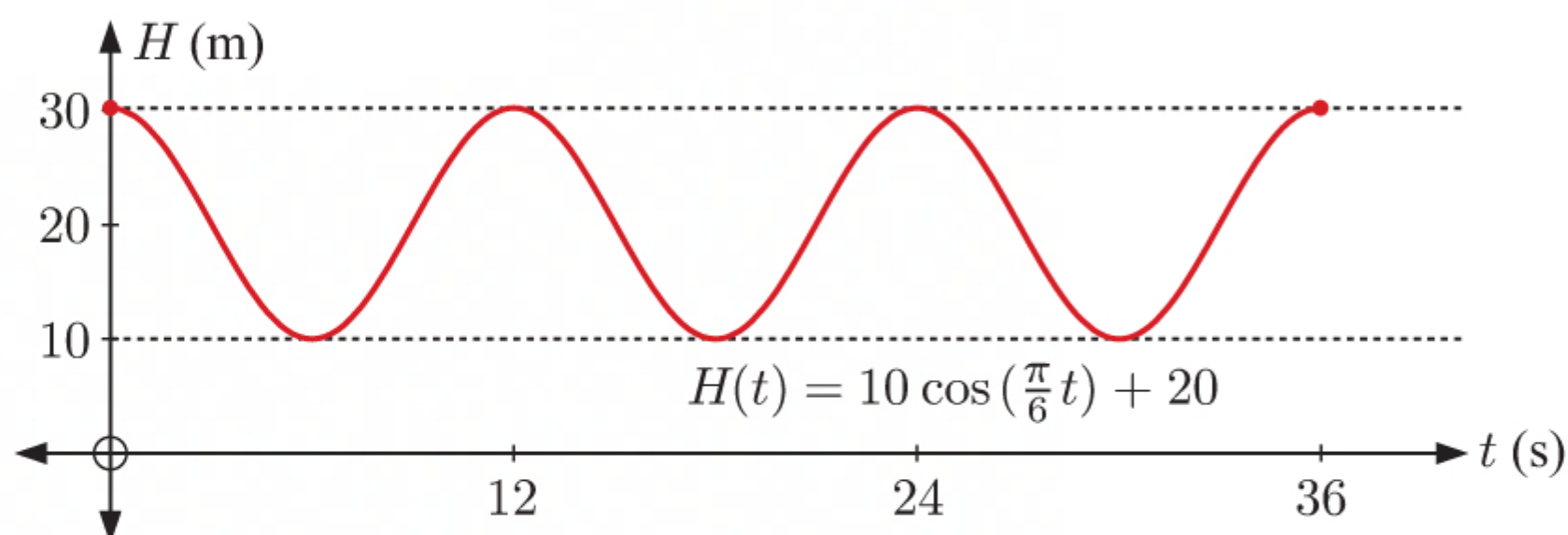
A reflection in the x -axis, then a horizontal stretch with scale factor $\frac{1}{2}$ will map $y = \tan x$ onto $y = -\tan 2x$.

$$\text{b } \sin x \xrightarrow[\text{vertical stretch scale factor 2}]{\text{horizontal stretch scale factor 2}} 2 \sin x \xrightarrow[\text{translation } \begin{pmatrix} \frac{\pi}{2} \\ \frac{1}{2} \end{pmatrix}]{\text{horizontal stretch scale factor 2}} 2 \sin \frac{1}{2}x \xrightarrow{\text{translation } \begin{pmatrix} \frac{\pi}{2} \\ \frac{1}{2} \end{pmatrix}} 2 \sin\left(\frac{1}{2}\left(x - \frac{\pi}{2}\right)\right) + \frac{1}{2}$$

A vertical stretch with scale factor 2, then a horizontal stretch with scale factor 2, then a translation $\frac{\pi}{2}$ units right and $\frac{1}{2}$ unit upwards will map $y = \sin x$ onto $y = 2 \sin\left(\frac{1}{2}\left(x - \frac{\pi}{2}\right)\right) + \frac{1}{2}$.

12 a $H(0) = 10 \cos 0 + 20$
 $= 30$

$H(t) = 10 \cos\left(\frac{\pi}{6}t\right) + 20$ has period $\frac{2\pi}{b} = \frac{2\pi}{\frac{\pi}{6}} = 12$ seconds



b After 9 seconds, $t = 9$ and $H(9) = 10 \cos\left(\frac{\pi}{6} \times 9\right) + 20$
 $= 10 \cos \frac{3\pi}{2} + 20$
 $= 0 + 20$
 $= 20$

So, the height of the blade's tip after 9 seconds is 20 m.

c $H(t) = 10 \cos\left(\frac{\pi}{6}t\right) + 20$ has minimum value $10(-1) + 20 = 10$ {when $\cos\left(\frac{\pi}{6}t\right) = -1$ }
 So, the minimum height of the blade's tip is 10 m.

d From part **a**, the period is 12 seconds, so it takes 12 seconds for the blade to complete a full revolution.

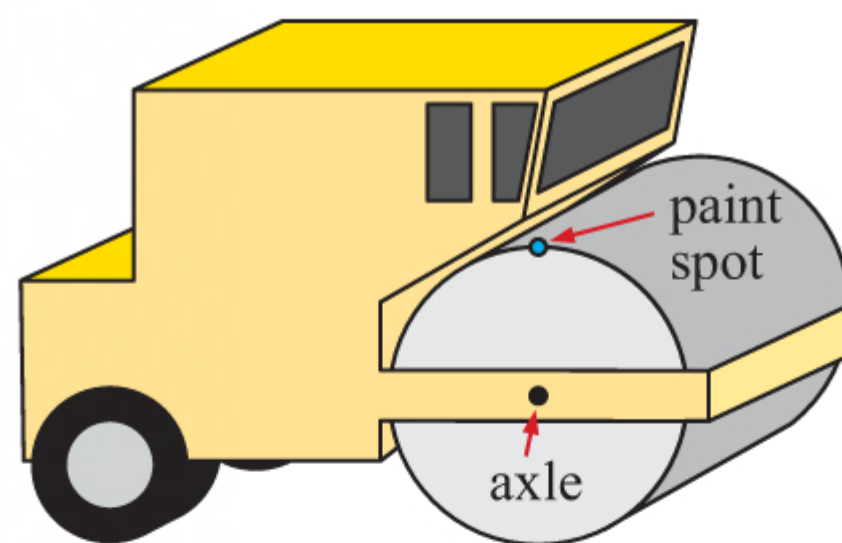
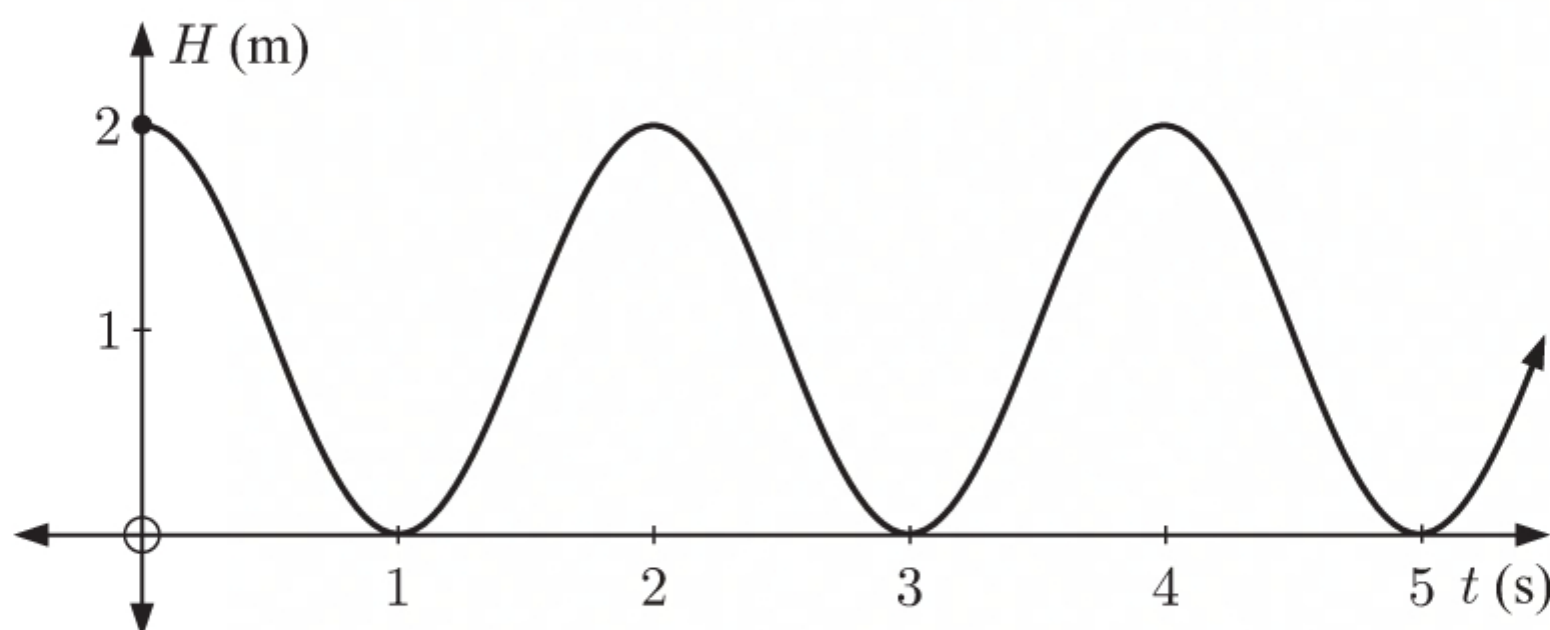
13 a The amplitude $a = 1$.

The period is $\frac{2\pi}{b} = 2$ s $\therefore b = \pi$

The principal axis is $y = \frac{\max + \min}{2} = \frac{2 + 0}{2} = 1$
 $\therefore d = 1$

Assume that the paint spot is initially at its highest point.

The graph is:



b The graph in **a** is a horizontal translation of $H(t) = \sin \pi t + 1$ 1.5 units to the right.
 So, the height of the paint spot at time t is $H(t) = \sin(\pi(t - 1.5)) + 1$ m.
 Alternatively, $H(t) = \cos \pi t + 1$ m.

14

Month (t)	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
Temperature ($^{\circ}\text{C}$)	31.5	31.8	29.5	25.4	21.5	18.8	17.7	18.3	20.1	22.4	25.5	28.8

- a** The period is 12 months, so $\frac{2\pi}{b} = 12$ and $\therefore b = \frac{\pi}{6}$.

The amplitude $= \frac{\max - \min}{2} \approx \frac{31.8 - 17.7}{2} \approx 7.05$, so $a \approx 7.05$.

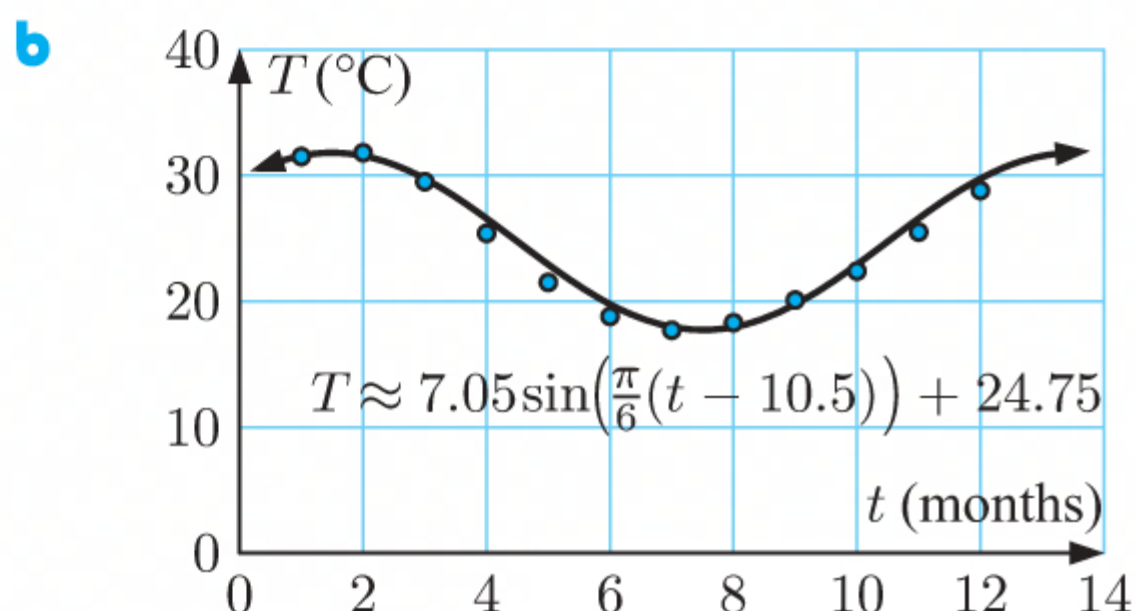
The principal axis is midway between the maximum and minimum, so

$$d \approx \frac{31.8 + 17.7}{2} \approx 24.75.$$

The sine function starts a new period between July and the following February (months 7 and 14). We estimate that $c \approx \frac{7 + 14}{2} \approx 10.5$.

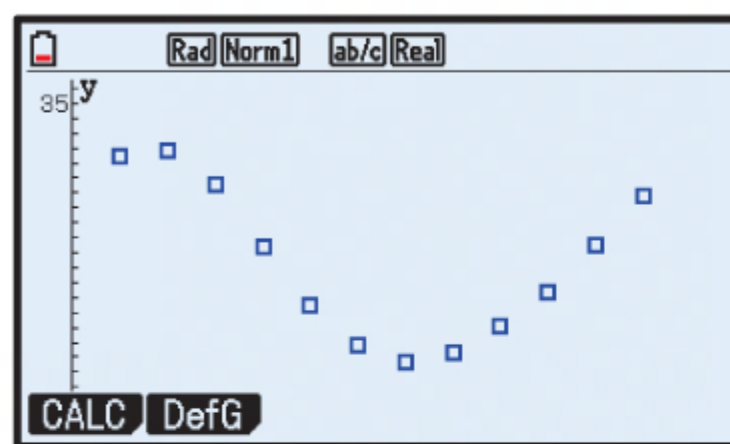
So, $a \approx 7.05$, $b \approx \frac{\pi}{6}$, $c \approx 10.5$, and $d \approx 24.75$.

$$\therefore T \approx 7.05 \sin\left(\frac{\pi}{6}(t - 10.5)\right) + 24.75$$



- c** From **a**, our model is $T \approx 7.05 \sin\left(\frac{\pi}{6}(t - 10.5)\right) + 24.75$
 $\approx 7.05 \sin(0.524t - 5.50) + 24.75$

	List 1	List 2	List 3	List 4
SUB				
1	1	31.5		
2	2	31.8		
3	3	29.5		
4	4	25.4		



	Rad	Norm1	ab/c	Real
SinReg				
a	7.19713743			
b	0.48790404			
c	1.07783236			
d	24.7471359			
MSe	0.25907655			
y=a·sin(bx+c)+d				

Using technology, $T \approx 7.20 \sin(0.488t + 1.08) + 24.7$
 $\approx 7.20 \sin(0.488t + 1.08 - 2\pi) + 24.7 \quad \{\sin(\theta - 2\pi) = \sin \theta\}$
 $\approx 7.20 \sin(0.488t - 5.20) + 24.7$

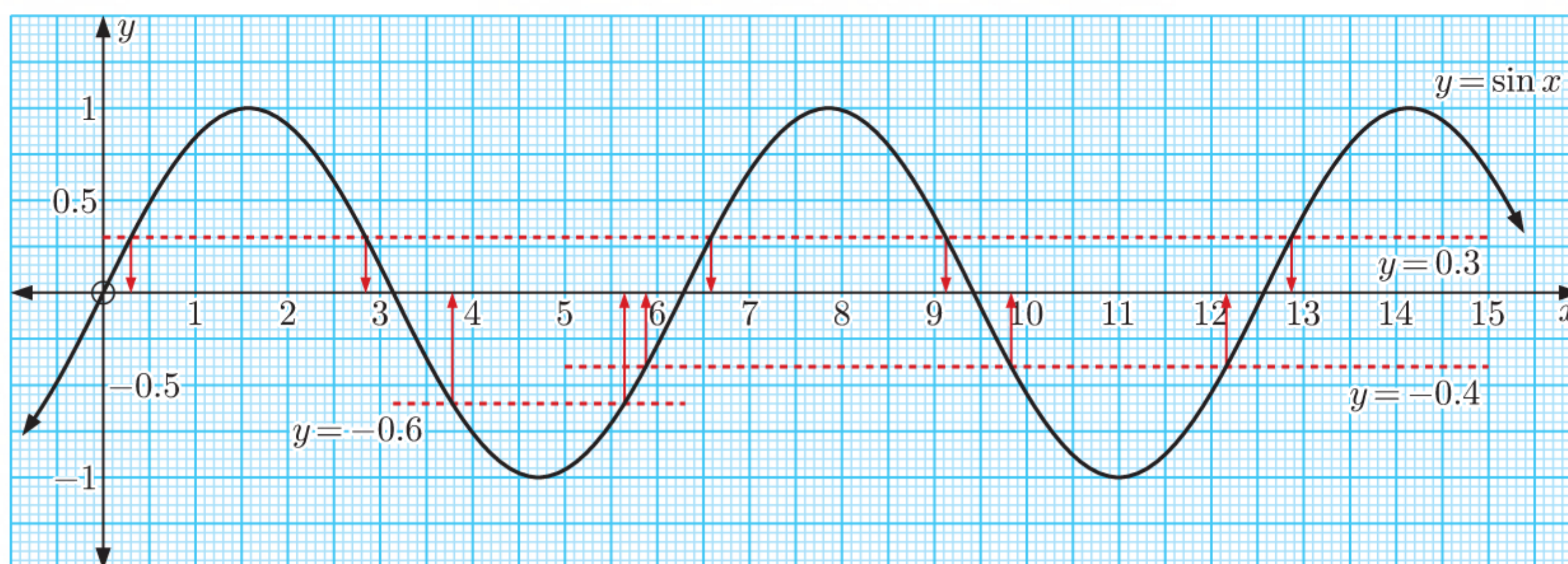
The model fits reasonably well but not perfectly.

Chapter 9

TRIGONOMETRIC EQUATIONS AND IDENTITIES

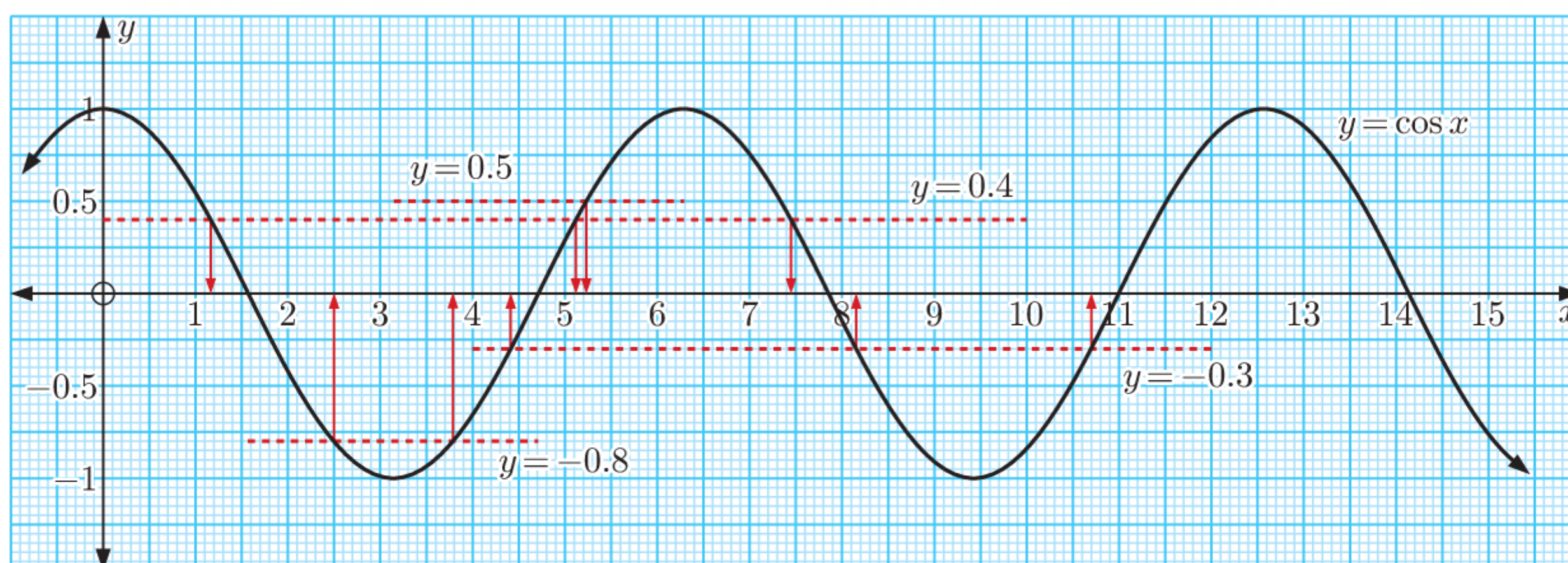
EXERCISE 9A.1

1



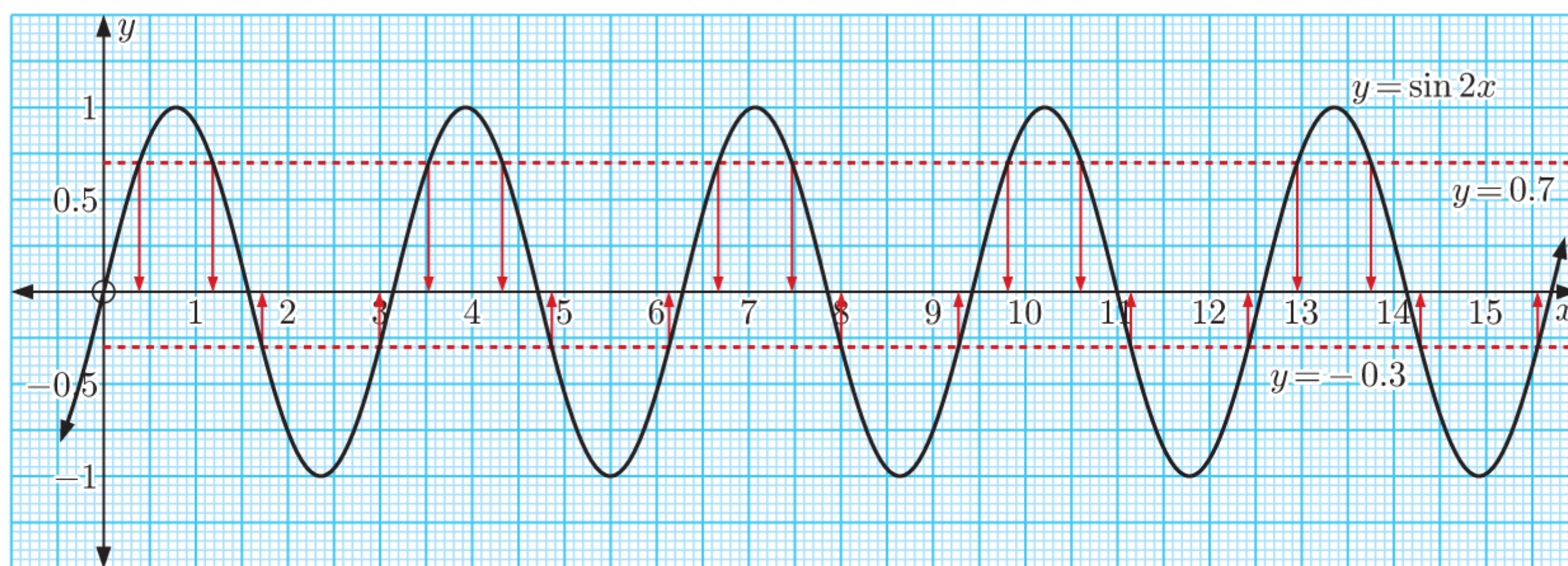
- a** When $\sin x = 0.3$, $0 \leq x \leq 15$, $x \approx 0.3, 2.8, 6.6, 9.1, 12.9$
- b** When $\sin x = -0.4$, $5 \leq x \leq 15$, $x \approx 5.9, 9.8, 12.2$
- c** When $\sin x = 0.3$, $0 \leq x \leq 2\pi$, $x \approx 0.3, 2.8$
- d** When $\sin x = -0.6$, $\pi \leq x \leq 2\pi$, $x \approx 3.8, 5.6$

2



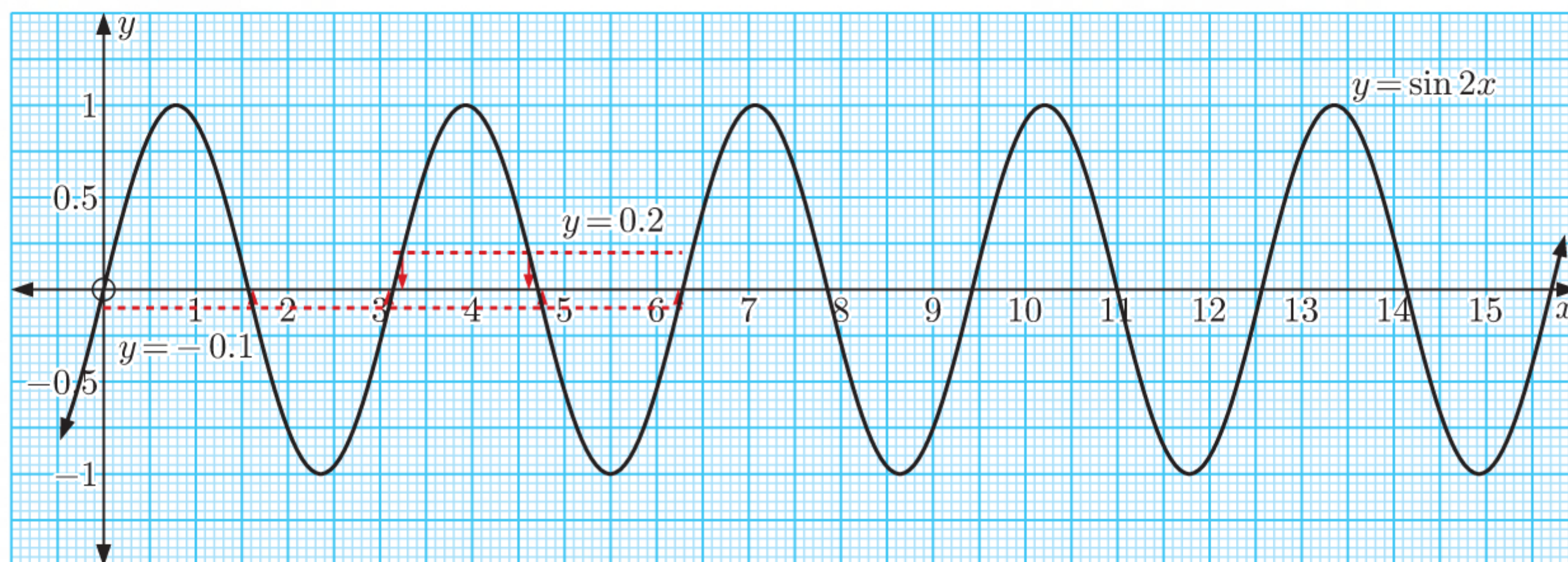
- a** When $\cos x = 0.4$, $0 \leq x \leq 10$, $x \approx 1.2, 5.1, 7.4$
- b** When $\cos x = -0.3$, $4 \leq x \leq 12$, $x \approx 4.4, 8.2, 10.7$
- c** When $\cos x = 0.5$, $\pi \leq x \leq 2\pi$, $x \approx 5.2$
- d** When $\cos x = -0.8$, $\frac{\pi}{2} \leq x \leq \frac{3\pi}{2}$, $x \approx 2.5, 3.8$

3



a When $\sin 2x = 0.7$, $0 \leq x \leq 16$, $x \approx 0.4, 1.2, 3.5, 4.3, 6.7, 7.5, 9.8, 10.6, 13.0, 13.7$

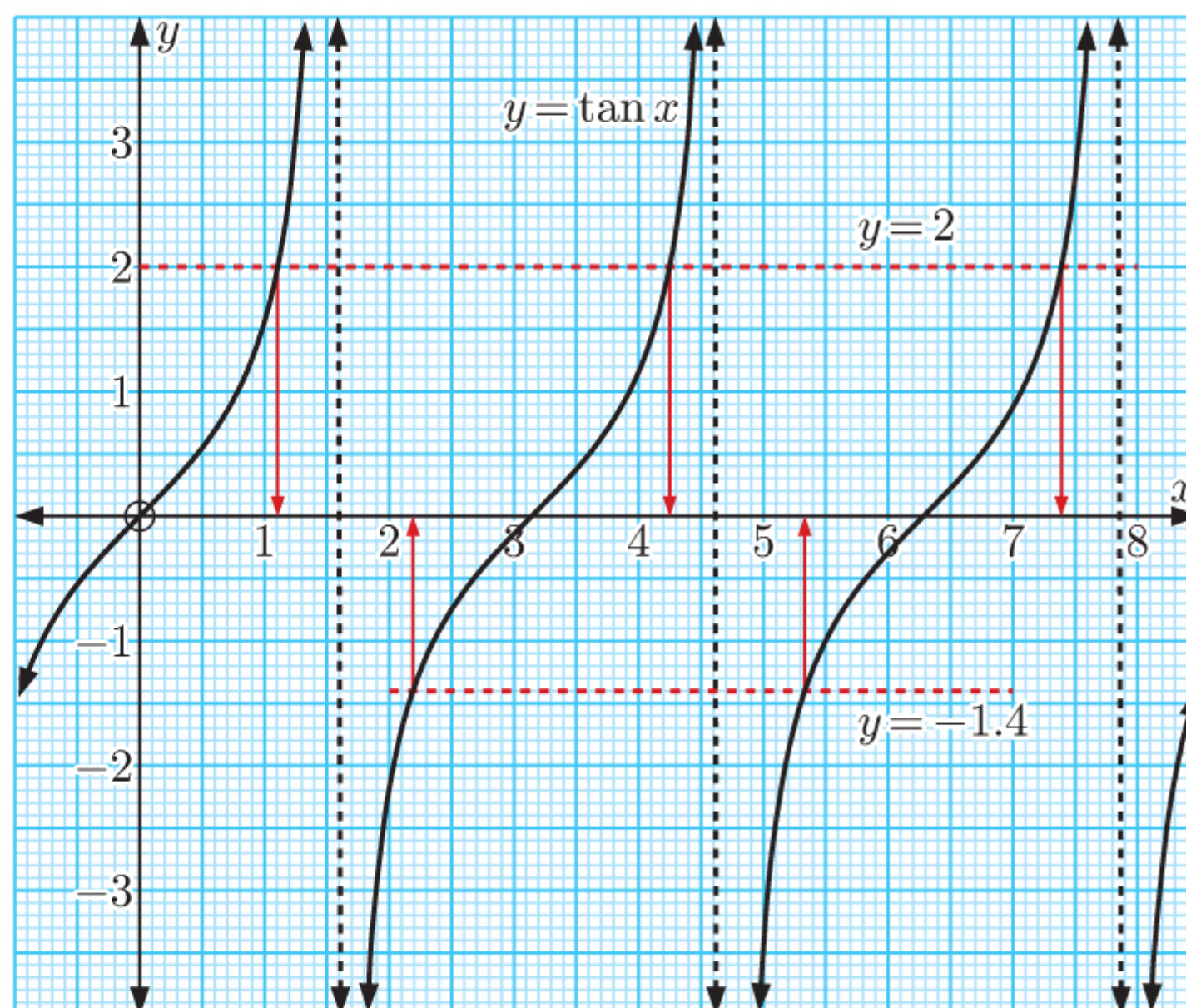
b When $\sin 2x = -0.3$, $0 \leq x \leq 16$, $x \approx 1.7, 3.0, 4.9, 6.1, 8.0, 9.3, 11.1, 12.4, 14.3, 15.6$



c When $\sin 2x = 0.2$, $\pi \leq x \leq 2\pi$, $x \approx 3.2, 4.6$

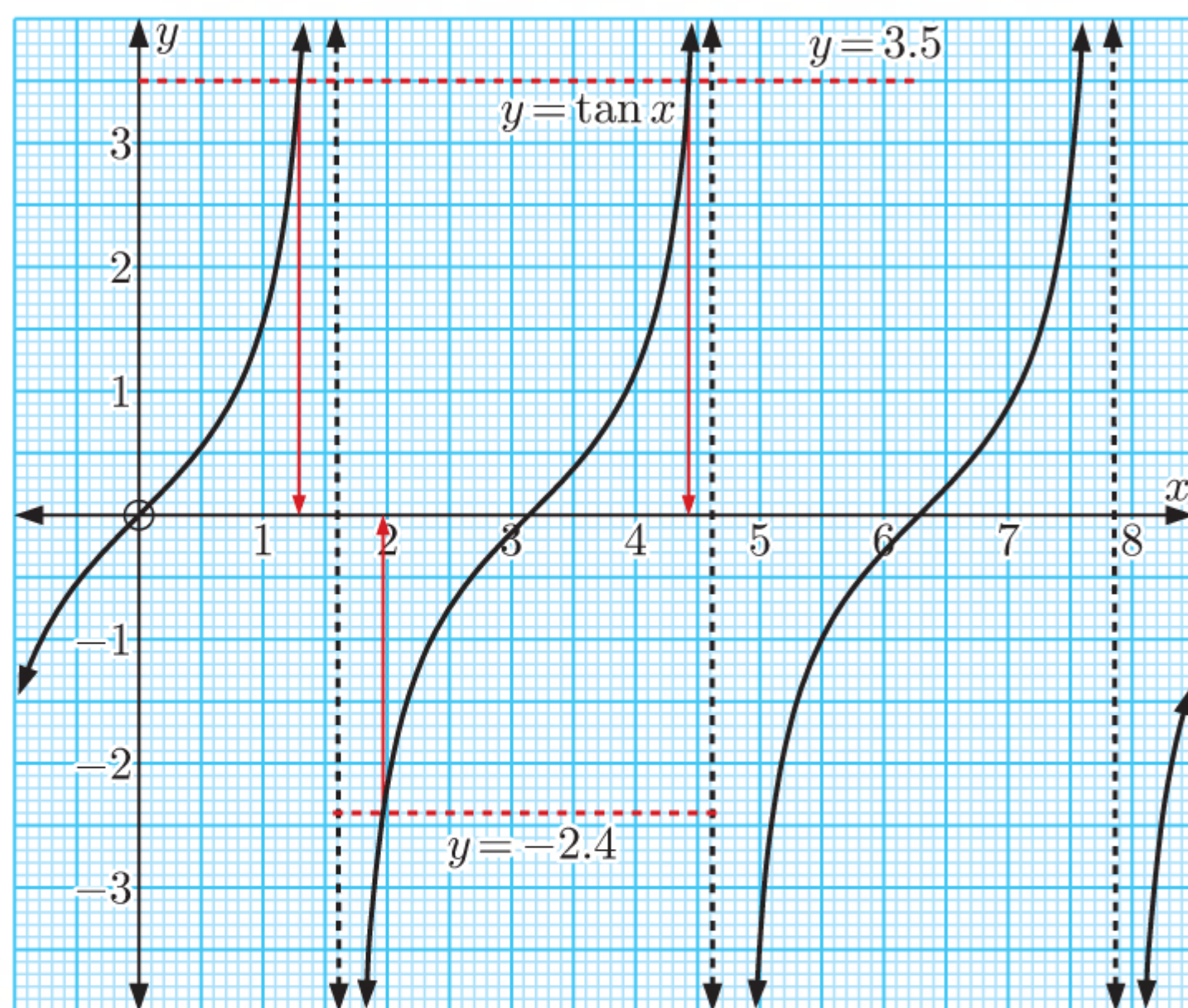
d When $\sin 2x = -0.1$, $0 \leq x \leq 2\pi$, $x \approx 1.6, 3.1, 4.8, 6.2$

4



a When $\tan x = 2$, $0 \leq x \leq 8$, $x \approx 1.1, 4.2, 7.4$

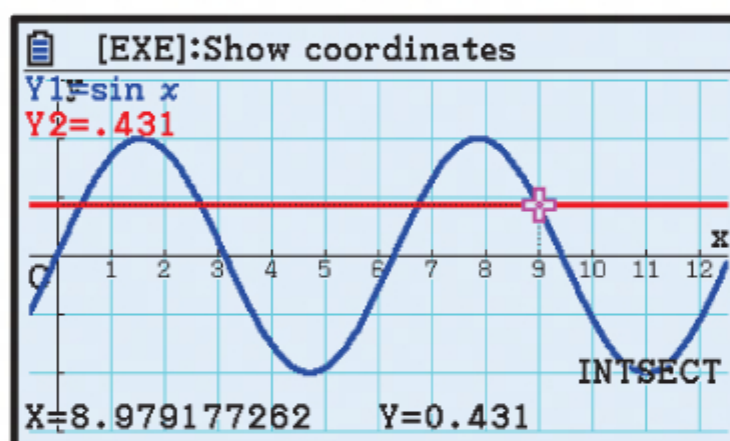
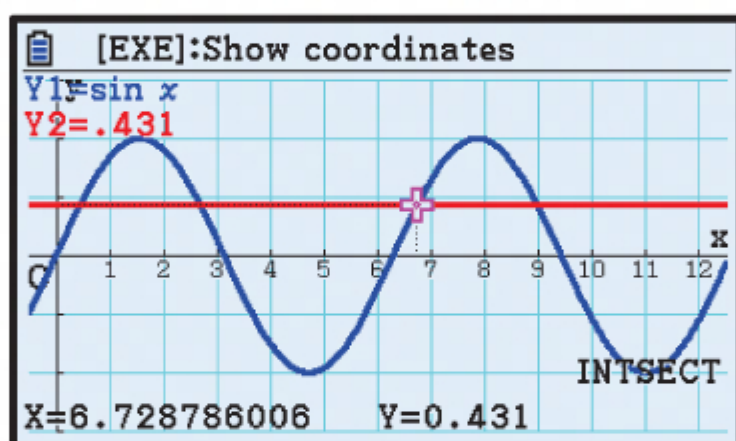
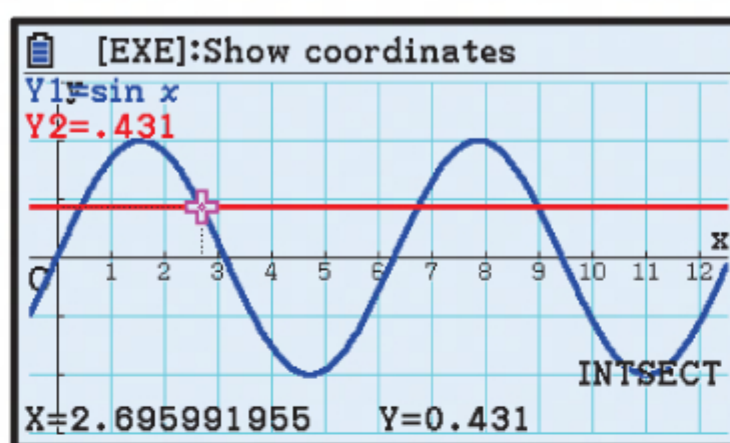
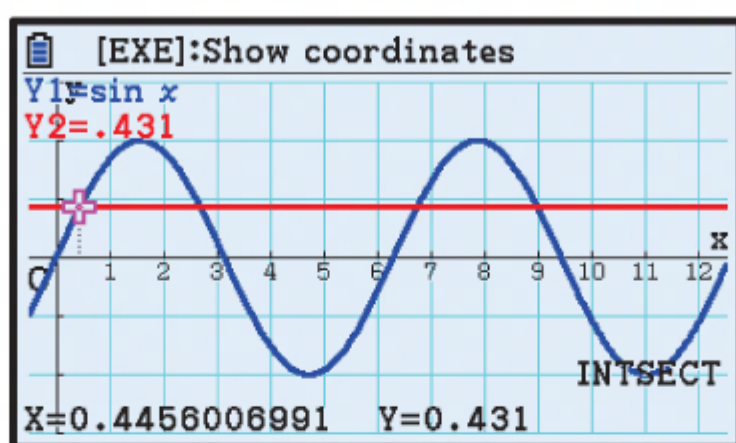
b When $\tan x = -1.4$, $2 \leq x \leq 7$, $x \approx 2.2, 5.3$



- c When $\tan x = 3.5$, $0 \leq x \leq 2\pi$, $x \approx 1.3, 4.4$
- d When $\tan x = -2.4$, $\frac{\pi}{2} \leq x \leq \frac{3\pi}{2}$, $x \approx 2.0$

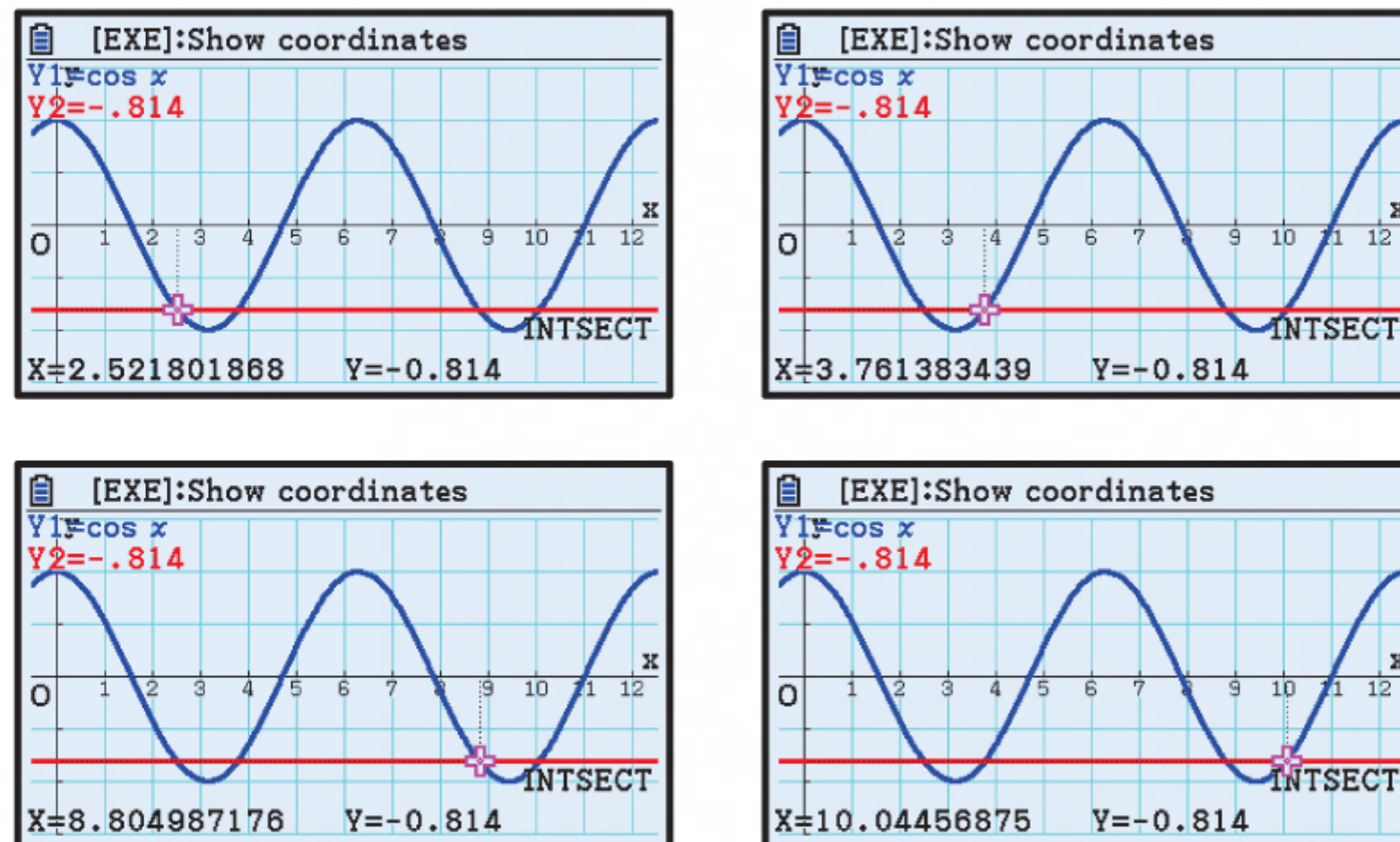
EXERCISE 9A.2

- 1 a We graph the functions $Y_1 = \sin X$ and $Y_2 = 0.431$ on the same set of axes. We need to use **window** settings just larger than the domain. In this case, $X_{\min} = -0.5$, $X_{\max} = 12.5$, $X_{\text{scale}} = 1$.



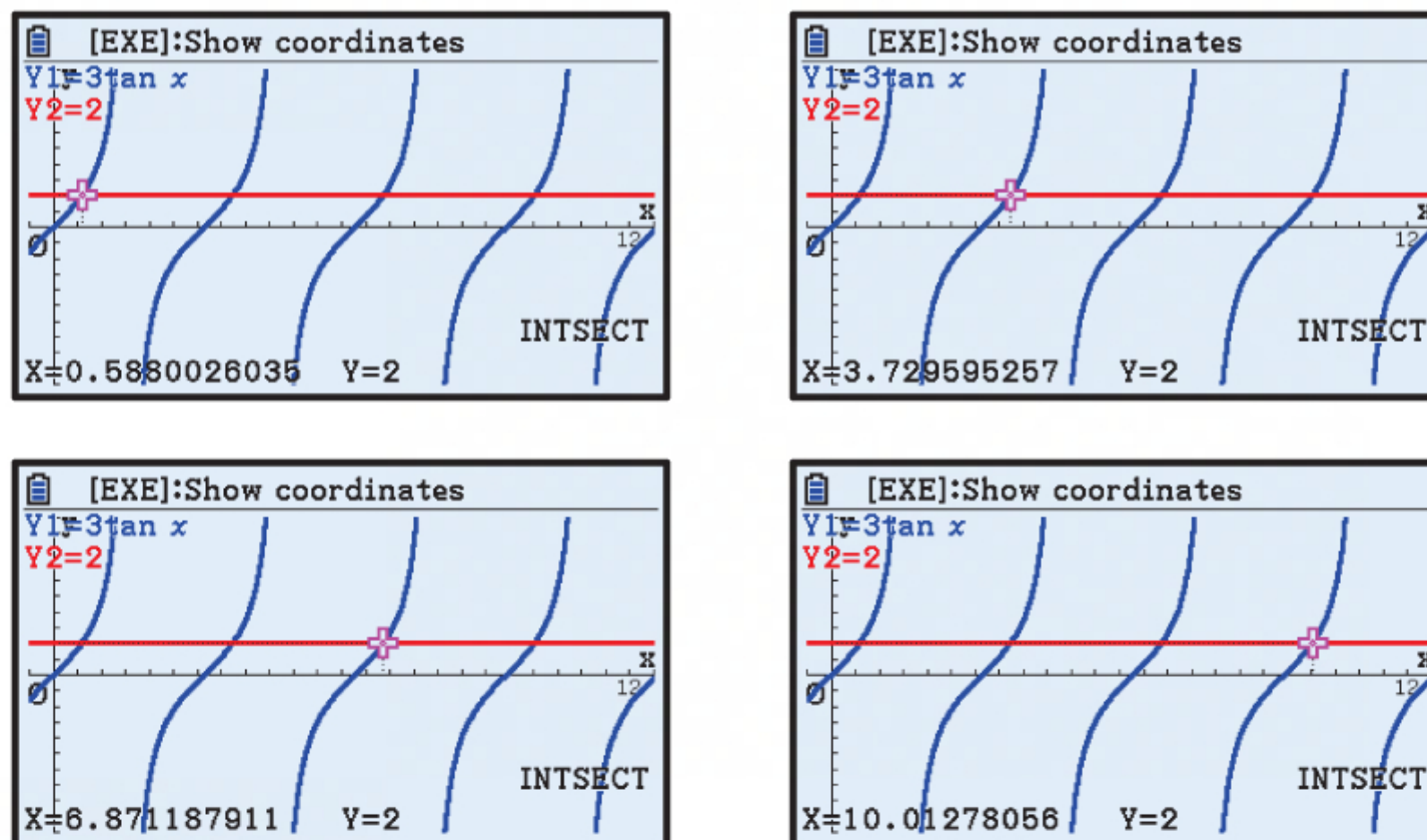
The solutions are $x \approx 0.446, 2.70, 6.73, 8.98$.

- b** We graph the functions $Y_1 = \cos X$ and $Y_2 = -0.814$ on the same set of axes. We need to use **window** settings just larger than the domain. In this case, $X_{\min} = -0.5$, $X_{\max} = 12.5$, $X_{\text{scale}} = 1$.



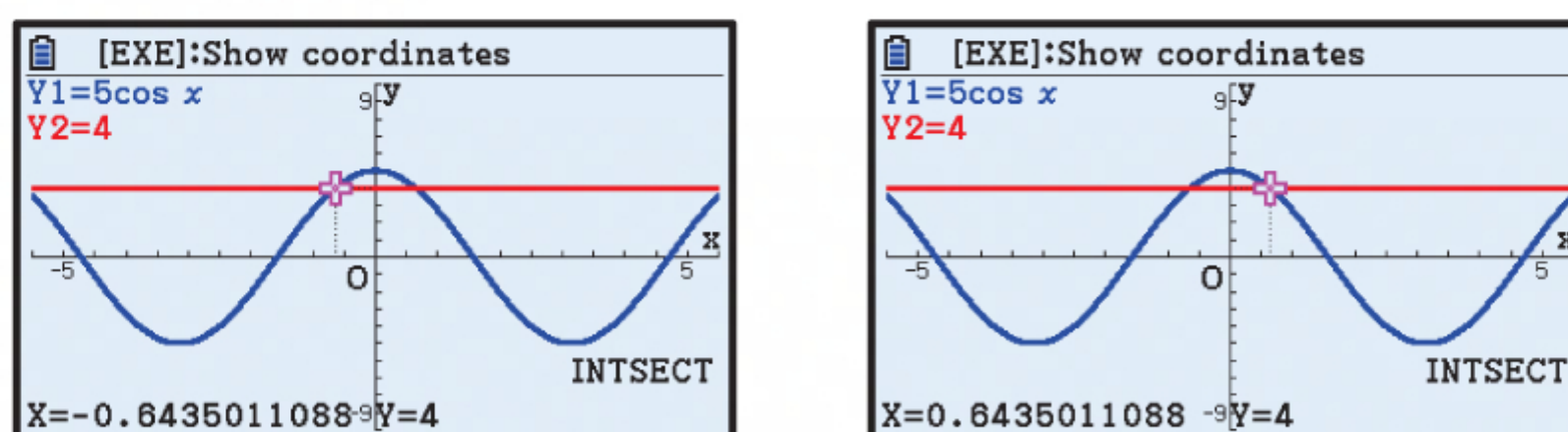
The solutions are $x \approx 2.52, 3.76, 8.80, 10.0$.

- c** We graph the functions $Y_1 = 3 \tan X$ and $Y_2 = 2$ on the same set of axes. We need to use **window** settings just larger than the domain. In this case, $X_{\min} = -0.5$, $X_{\max} = 12.5$, $X_{\text{scale}} = 0.5$.



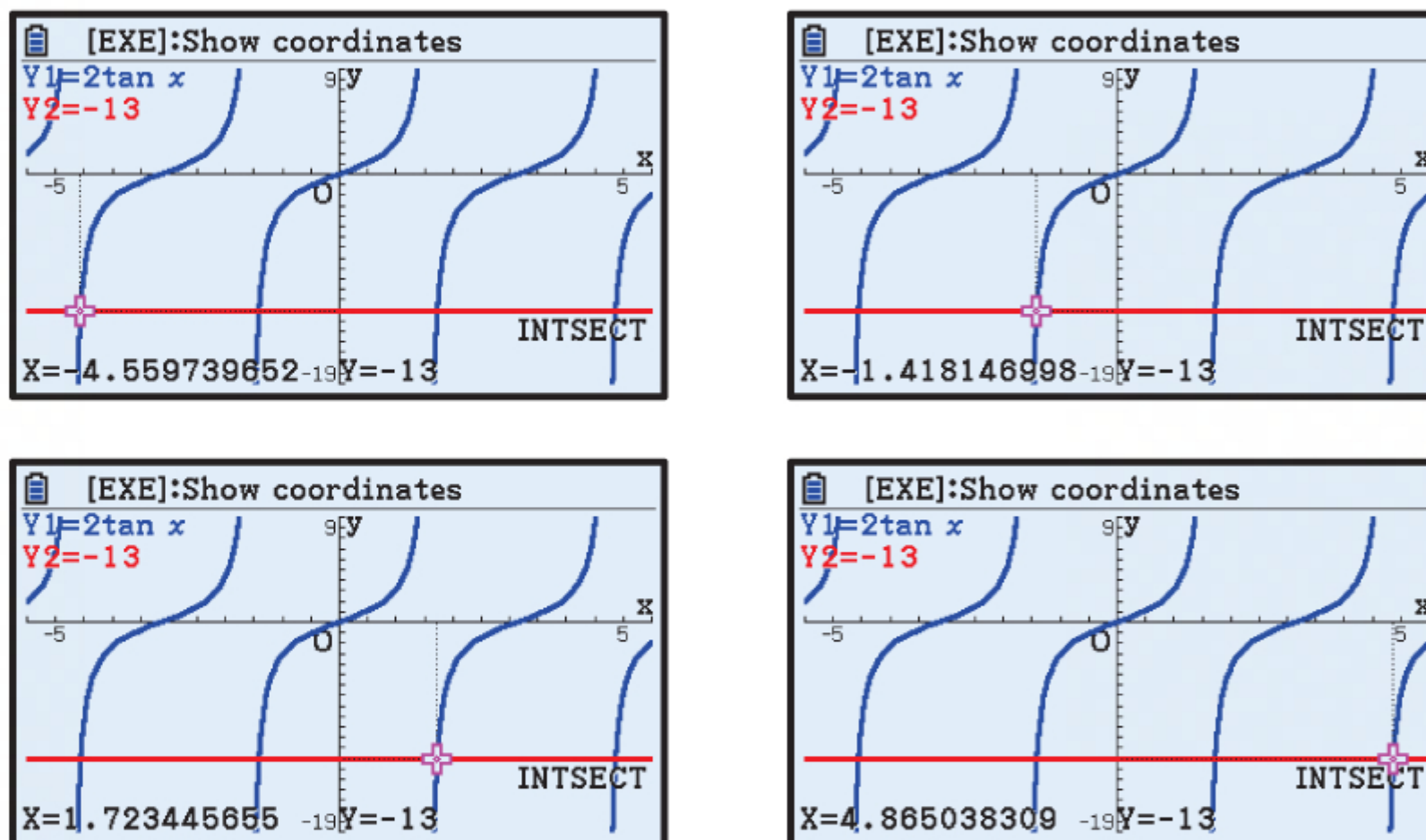
The solutions are $x \approx 0.588, 3.73, 6.87, 10.0$.

- 2 a** We graph the functions $Y_1 = 5 \cos X$ and $Y_2 = 4$ on the same set of axes. We need to use **window** settings just larger than the domain. In this case, $X_{\min} = -5.5$, $X_{\max} = 5.5$, $X_{\text{scale}} = 0.5$.



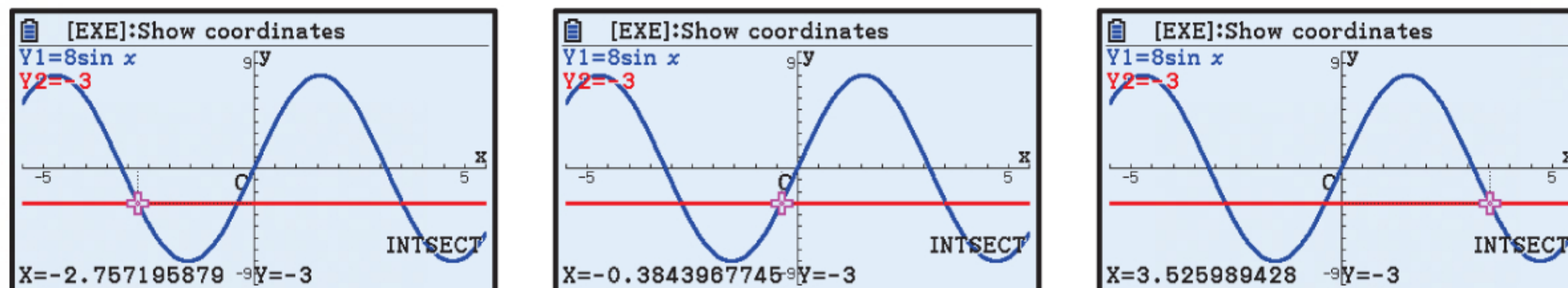
The solutions are $x \approx -0.644, 0.644$.

- b** We graph the functions $Y_1 = 2 \tan X$ and $Y_2 = -13$ on the same set of axes. We need to use **window** settings just larger than the domain. In this case, $X_{\min} = -5.5$, $X_{\max} = 5.5$, $X_{\text{scale}} = 0.5$.



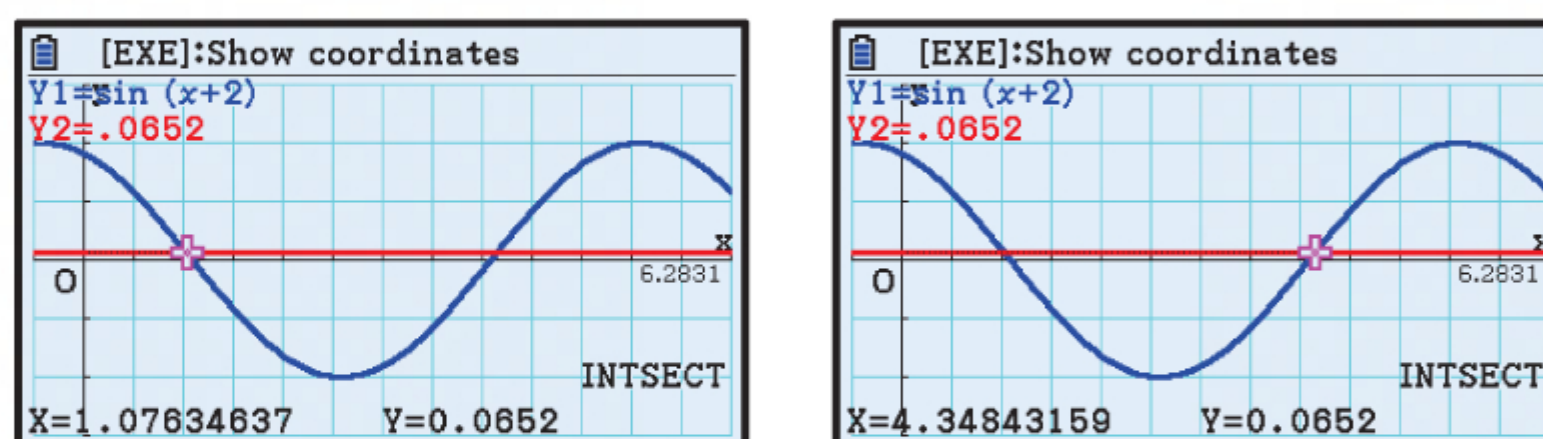
The solutions are $x \approx -4.56$, -1.42 , 1.72 , 4.87 .

- c** We graph the functions $Y_1 = 8 \sin X$ and $Y_2 = -3$ on the same set of axes. We need to use **window** settings just larger than the domain. In this case, $X_{\min} = -5.5$, $X_{\max} = 5.5$, $X_{\text{scale}} = 0.5$.



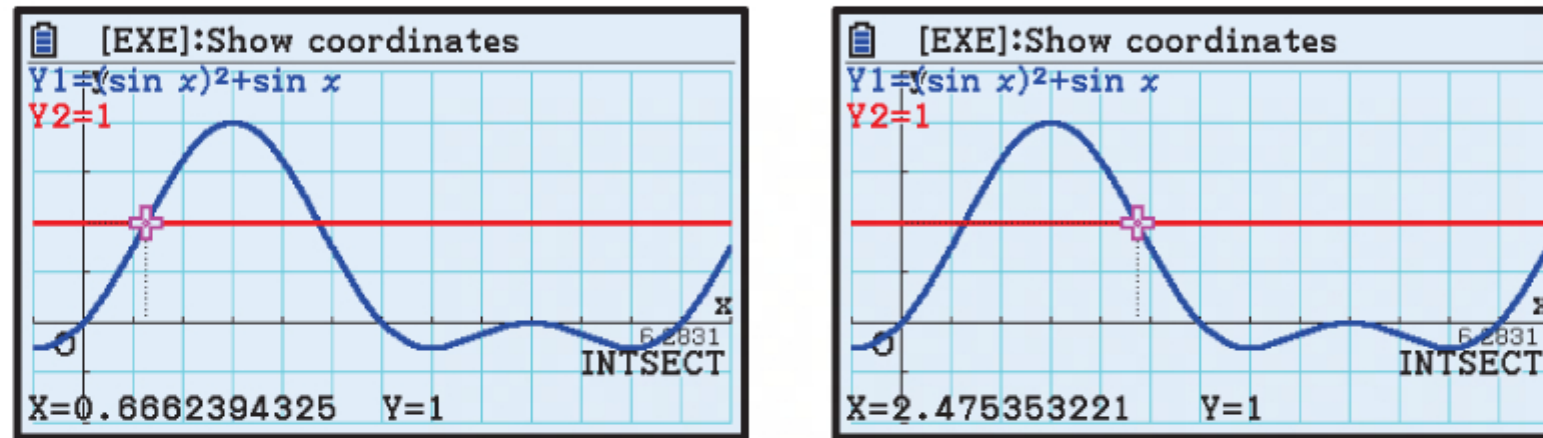
The solutions are $x \approx -2.76$, -0.384 , 3.53 .

- 3 a** We graph the functions $Y_1 = \sin(X + 2)$ and $Y_2 = 0.0652$ on the same set of axes. We need to use **window** settings just larger than the domain. In this case, $X_{\min} = -\frac{\pi}{6}$, $X_{\max} = \frac{13\pi}{6}$, $X_{\text{scale}} = \frac{\pi}{6}$.



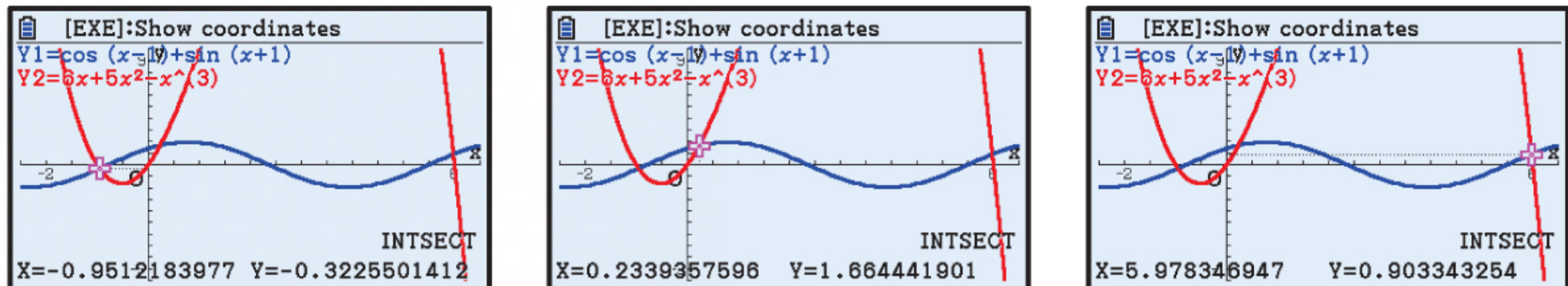
The solutions are $x \approx 1.08$, 4.35 .

- b** We graph the functions $Y_1 = (\sin X)^2 + \sin X$ and $Y_2 = 1$ on the same set of axes. We need to use **window** settings just larger than the domain. In this case, $X_{\min} = -\frac{\pi}{6}$, $X_{\max} = \frac{13\pi}{6}$, $X_{\text{scale}} = \frac{\pi}{6}$.



The solutions are $x \approx 0.666, 2.48$.

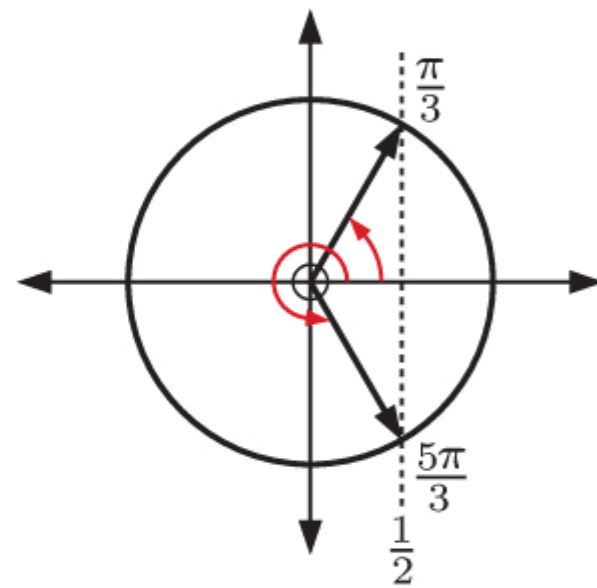
- 4** We graph the functions $Y_1 = \cos(X - 1) + \sin(X + 1)$ and $Y_2 = 6X + 5X^2 - X^3$ on the same set of axes. We need to use **window** settings just larger than the domain. In this case, $X_{\min} = -2.5$, $X_{\max} = 6.5$, $X_{\text{scale}} = 0.5$.



The solutions are $x \approx -0.951, 0.234, 5.98$.

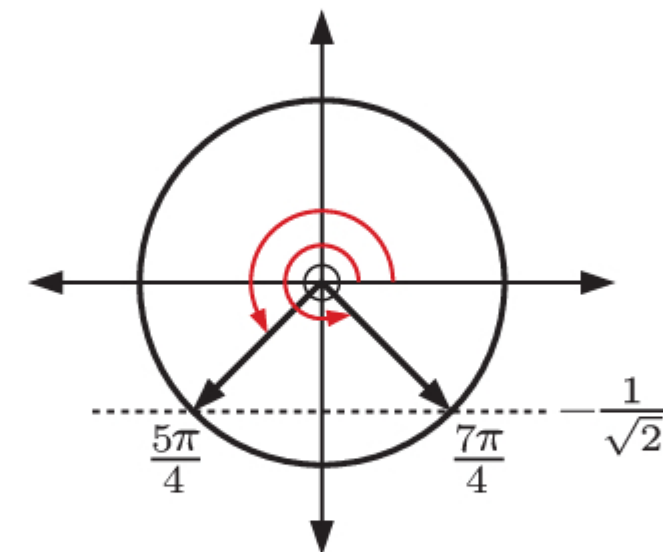
EXERCISE 9A.3

- 1 a** On $0 \leq x \leq 2\pi$, the angles with cosine $\frac{1}{2}$ are $\frac{\pi}{3}$ and $\frac{5\pi}{3}$.



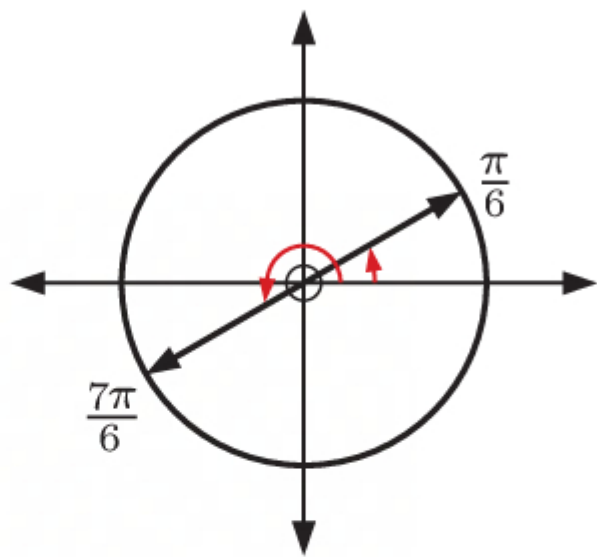
\therefore the solutions are $x = \frac{\pi}{3}$ or $\frac{5\pi}{3}$.

- b** On $0 \leq x \leq 2\pi$, the angles with sine $-\frac{1}{\sqrt{2}}$ are $\frac{5\pi}{4}$ and $\frac{7\pi}{4}$.



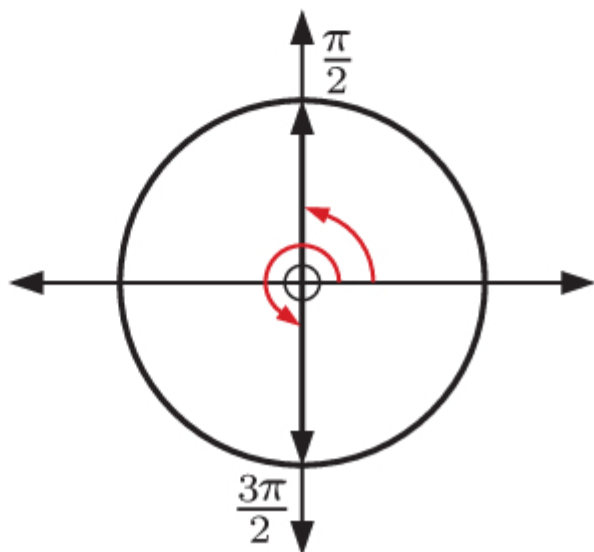
\therefore the solutions are $x = \frac{5\pi}{4}$ or $\frac{7\pi}{4}$.

- c** On $0 \leq x \leq 2\pi$, the angles with tangent $\frac{1}{\sqrt{3}}$ are $\frac{\pi}{6}$ and $\frac{7\pi}{6}$.



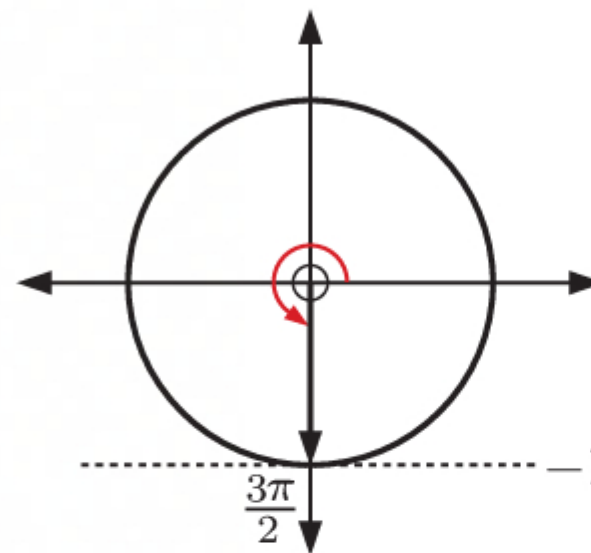
\therefore the solutions are $x = \frac{\pi}{6}$ or $\frac{7\pi}{6}$.

- e** On $0 \leq x \leq 2\pi$, the angles with cosine 0 are $\frac{\pi}{2}$ and $\frac{3\pi}{2}$.



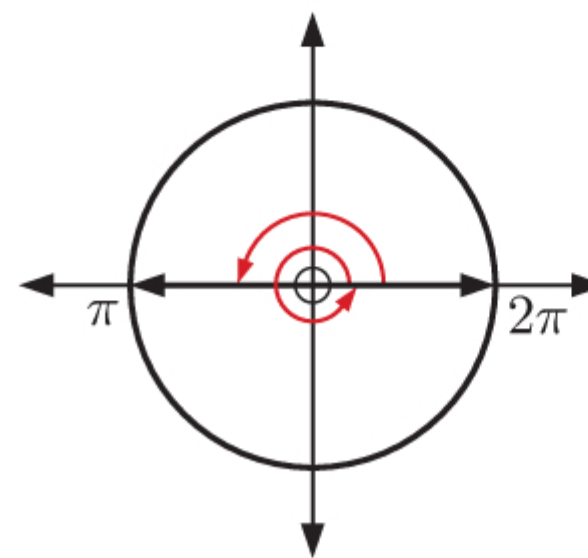
\therefore the solutions are $x = \frac{\pi}{2}$ or $\frac{3\pi}{2}$.

- d** On $0 \leq x \leq 2\pi$, the angle with sine -1 is $\frac{3\pi}{2}$.



\therefore the solution is $x = \frac{3\pi}{2}$.

- f** On $0 \leq x \leq 2\pi$, the angles with tangent 0 are 0, π , and 2π .



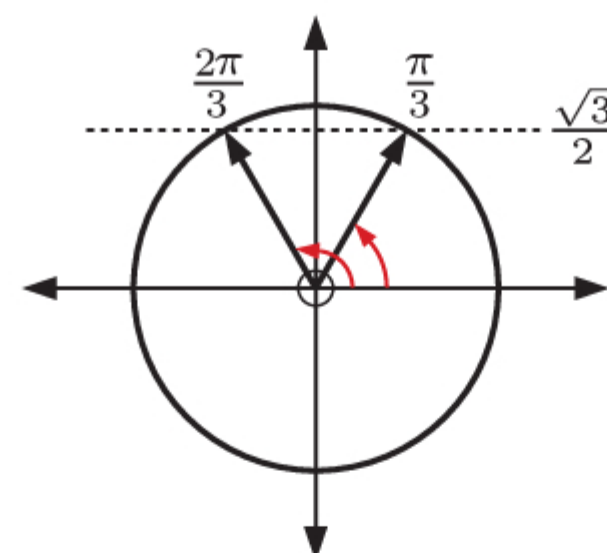
\therefore the solutions are $x = 0$, π , or 2π .

2 a $2 \sin x = \sqrt{3}$

$\therefore \sin x = \frac{\sqrt{3}}{2}$

On $0 \leq x \leq 2\pi$, the angles with sine $\frac{\sqrt{3}}{2}$ are $\frac{\pi}{3}$ and $\frac{2\pi}{3}$.

\therefore the solutions are $x = \frac{\pi}{3}$ or $\frac{2\pi}{3}$.

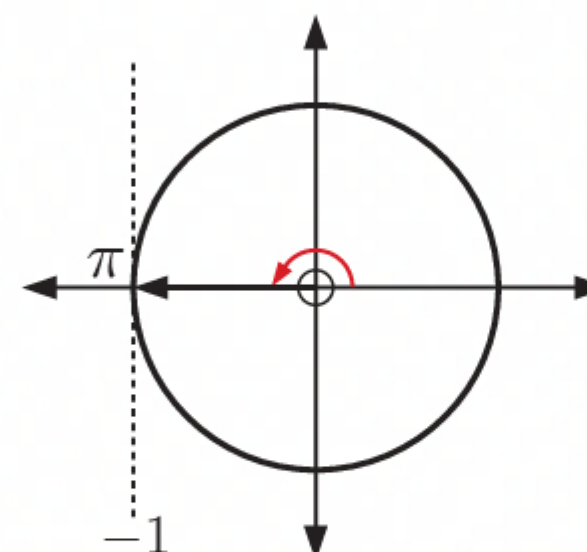


b $3 \cos x + 3 = 0$

$\therefore \cos x = -\frac{3}{3} = -1$

On $0 \leq x \leq 2\pi$, the angle with cosine -1 is π .

\therefore the solution is $x = \pi$.

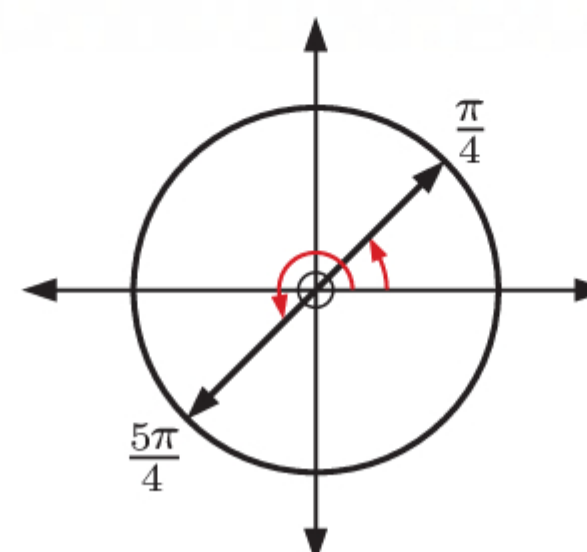


c $2 \tan x - 2 = 0$

$\therefore \tan x = \frac{2}{2} = 1$

On $0 \leq x \leq 2\pi$, the angles with tangent 1 are $\frac{\pi}{4}$ and $\frac{5\pi}{4}$.

\therefore the solutions are $x = \frac{\pi}{4}$ or $\frac{5\pi}{4}$.



3 a $2 \cos x + 1 = 0$

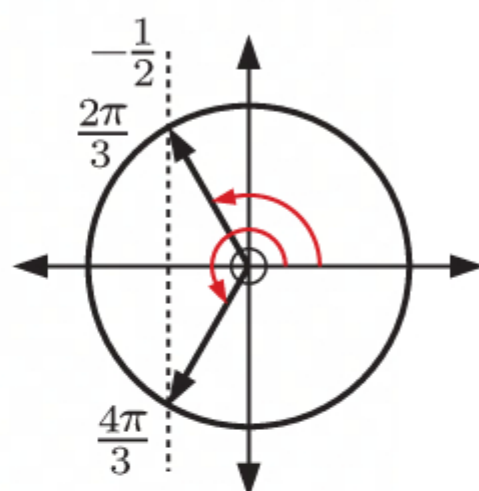
$$\therefore \cos x = -\frac{1}{2}$$

There are two points on the unit circle with cosine $-\frac{1}{2}$.

They correspond to angles $\frac{2\pi}{3}$ and $\frac{4\pi}{3}$.

For the domain $0 \leq x \leq 4\pi$ we have

4 solutions: $x = \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{8\pi}{3}, \text{ or } \frac{10\pi}{3}$.



b $\sqrt{2} \sin x = 1$

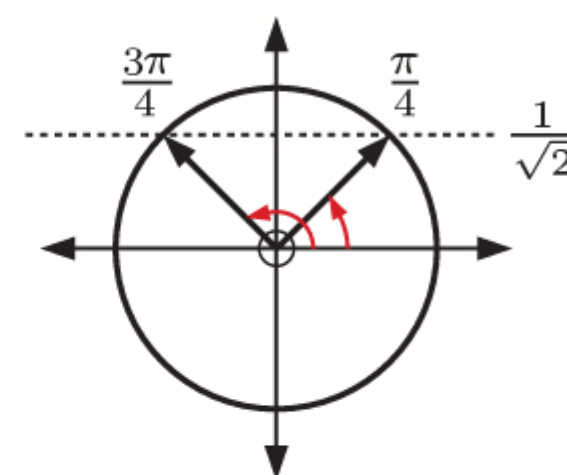
$$\therefore \sin x = \frac{1}{\sqrt{2}}$$

There are two points on the unit circle with sine $\frac{1}{\sqrt{2}}$.

They correspond to angles $\frac{\pi}{4}$ and $\frac{3\pi}{4}$.

For the domain $0 \leq x \leq 4\pi$ we have

4 solutions: $x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{9\pi}{4}, \text{ or } \frac{11\pi}{4}$.



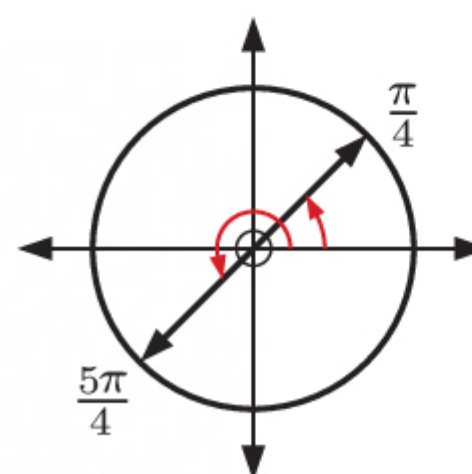
c $\tan x = 1$

There are two points on the unit circle with tangent 1.

They correspond to angles $\frac{\pi}{4}$ and $\frac{5\pi}{4}$.

For the domain $0 \leq x \leq 4\pi$ we have

4 solutions: $x = \frac{\pi}{4}, \frac{5\pi}{4}, \frac{9\pi}{4}, \text{ or } \frac{13\pi}{4}$.



4 a $2 \sin x + \sqrt{3} = 0$

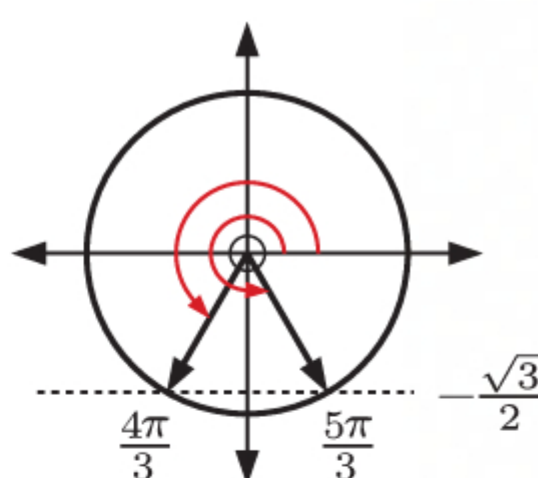
$$\therefore \sin x = -\frac{\sqrt{3}}{2}$$

There are two points on the unit circle with sine $-\frac{\sqrt{3}}{2}$.

They correspond to angles $\frac{4\pi}{3}$ and $\frac{5\pi}{3}$.

For the domain $-2\pi \leq x \leq 2\pi$ we have 4 solutions:

$x = -\frac{2\pi}{3}, -\frac{\pi}{3}, \frac{4\pi}{3}, \text{ or } \frac{5\pi}{3}$.



b $\sqrt{2} \cos x + 1 = 0$

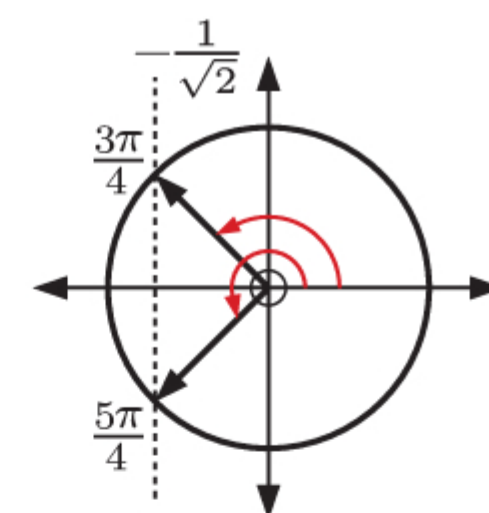
$$\therefore \cos x = -\frac{1}{\sqrt{2}}$$

There are two points on the unit circle with cosine $-\frac{1}{\sqrt{2}}$.

They correspond to angles $\frac{3\pi}{4}$ and $\frac{5\pi}{4}$.

For the domain $-2\pi \leq x \leq 2\pi$ we have 4 solutions:

$x = -\frac{5\pi}{4}, -\frac{3\pi}{4}, \frac{3\pi}{4}, \text{ or } \frac{5\pi}{4}$.



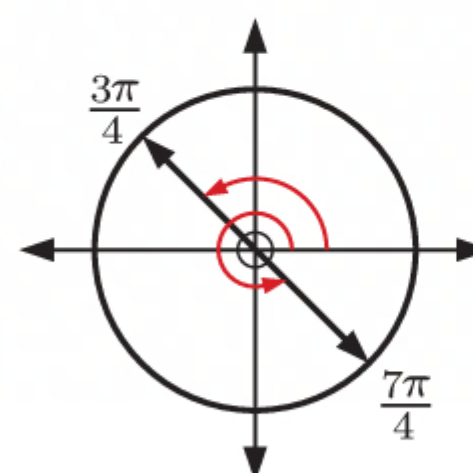
c $\tan x = -1$

There are two points on the unit circle with tangent -1 .

They correspond to angles $\frac{3\pi}{4}$ and $\frac{7\pi}{4}$.

For the domain $-2\pi \leq x \leq 2\pi$ we have 4 solutions:

$x = -\frac{5\pi}{4}, -\frac{\pi}{4}, \frac{3\pi}{4}, \text{ or } \frac{7\pi}{4}$.

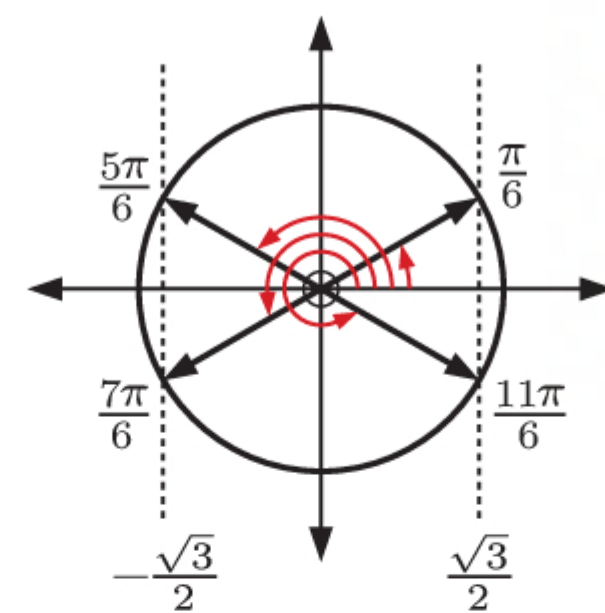


5 a $\cos^2 x = \frac{3}{4}$

$$\therefore \cos x = \pm \frac{\sqrt{3}}{2}$$

On $0 \leq x \leq 2\pi$, the angles with cosine $\frac{\sqrt{3}}{2}$ are $\frac{\pi}{6}$ and $\frac{11\pi}{6}$,
and the angles with cosine $-\frac{\sqrt{3}}{2}$ are $\frac{5\pi}{6}$ and $\frac{7\pi}{6}$.

\therefore the solutions are $x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6},$ or $\frac{11\pi}{6}$.

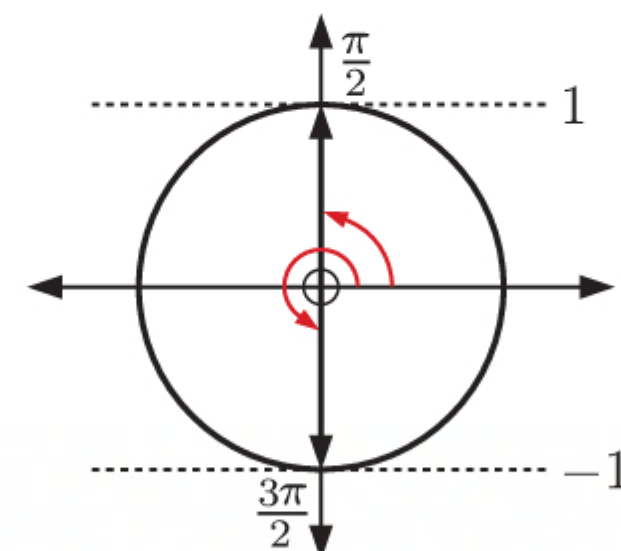


b $\sin^2 x = 1$

$$\therefore \sin x = \pm 1$$

On $0 \leq x \leq 2\pi$, the angle with sine 1 is $\frac{\pi}{2}$ and the angle
with sine -1 is $\frac{3\pi}{2}$.

\therefore the solutions are $x = \frac{\pi}{2}$ or $\frac{3\pi}{2}$.

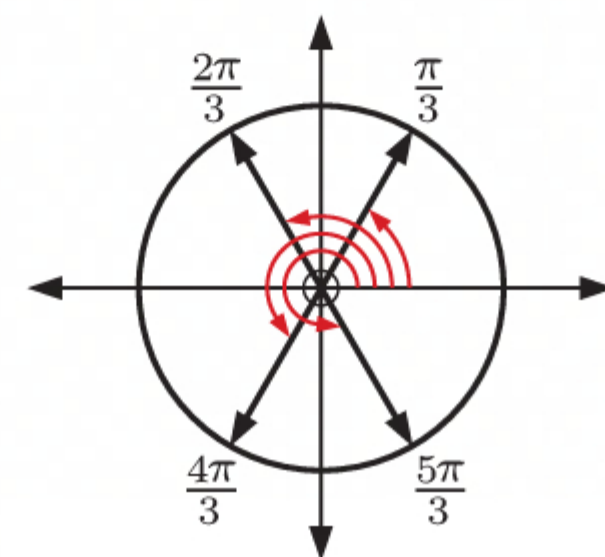


c $\tan^2 x = 3$

$$\therefore \tan x = \pm \sqrt{3}$$

On $0 \leq x \leq 2\pi$, the angles with tangent $\sqrt{3}$ are $\frac{\pi}{3}$ and $\frac{4\pi}{3}$,
and the angles with tangent $-\sqrt{3}$ are $\frac{2\pi}{3}$ and $\frac{5\pi}{3}$.

\therefore the solutions are $x = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3},$ or $\frac{5\pi}{3}$.



6 a If $0 \leq x \leq 2\pi$

then $0 \leq 2x \leq 4\pi$

\therefore the domain is $0 \leq 2x \leq 4\pi$.

b If $0 \leq x \leq 2\pi$

then $0 \leq \frac{x}{4} \leq \frac{2\pi}{4}$

$\therefore 0 \leq \frac{x}{4} \leq \frac{\pi}{2}$

\therefore the domain is $0 \leq \frac{x}{4} \leq \frac{\pi}{2}$.

c If $0 \leq x \leq 2\pi$

then $\frac{\pi}{2} \leq x + \frac{\pi}{2} \leq \frac{5\pi}{2}$

\therefore the domain is $\frac{\pi}{2} \leq x + \frac{\pi}{2} \leq \frac{5\pi}{2}$.

d If $0 \leq x \leq 2\pi$

then $-\frac{\pi}{6} \leq x - \frac{\pi}{6} \leq \frac{11\pi}{6}$

\therefore the domain is $-\frac{\pi}{6} \leq x - \frac{\pi}{6} \leq \frac{11\pi}{6}$.

e If $0 \leq x \leq 2\pi$

then $-\frac{\pi}{4} \leq x - \frac{\pi}{4} \leq \frac{7\pi}{4}$

and so $-\frac{\pi}{2} \leq 2(x - \frac{\pi}{4}) \leq \frac{7\pi}{2}$

\therefore the domain is $-\frac{\pi}{2} \leq 2(x - \frac{\pi}{4}) \leq \frac{7\pi}{2}$.

f If $0 \leq x \leq 2\pi$

then $-2\pi \leq -x \leq 0$

\therefore the domain is $-2\pi \leq -x \leq 0$.

7 a If $-\pi \leq x \leq \pi$

then $-3\pi \leq 3x \leq 3\pi$

\therefore the domain is $-3\pi \leq 3x \leq 3\pi$.

b If $-\pi \leq x \leq \pi$

then $-\frac{\pi}{4} \leq \frac{x}{4} \leq \frac{\pi}{4}$

\therefore the domain is $-\frac{\pi}{4} \leq \frac{x}{4} \leq \frac{\pi}{4}$.

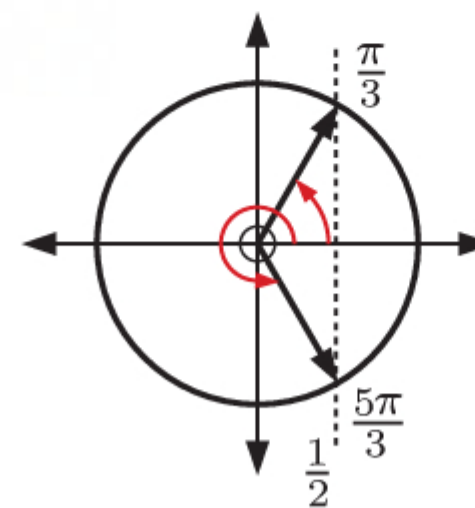
c If $-\pi \leq x \leq \pi$
 then $-\frac{3\pi}{2} \leq x - \frac{\pi}{2} \leq \frac{\pi}{2}$
 \therefore the domain is $-\frac{3\pi}{2} \leq x - \frac{\pi}{2} \leq \frac{\pi}{2}$.

e If $-\pi \leq x \leq \pi$
 then $2\pi \geq -2x \geq -2\pi$
 and so $-2\pi \leq -2x \leq 2\pi$
 \therefore the domain is $-2\pi \leq -2x \leq 2\pi$.

d If $-\pi \leq x \leq \pi$
 then $-2\pi \leq 2x \leq 2\pi$
 and so $-\frac{3\pi}{2} \leq 2x + \frac{\pi}{2} \leq \frac{5\pi}{2}$
 \therefore the domain is $-\frac{3\pi}{2} \leq 2x + \frac{\pi}{2} \leq \frac{5\pi}{2}$.

f If $-\pi \leq x \leq \pi$
 then $\pi \geq -x \geq -\pi$
 and so $\pi - \pi \leq \pi - x \leq \pi + \pi$
 $\therefore 0 \leq \pi - x \leq 2\pi$
 \therefore the domain is $0 \leq \pi - x \leq 2\pi$.

- 8** The three equations all have the form $\cos \theta = \frac{1}{2}$.
 There are two points on the unit circle with cosine $\frac{1}{2}$.
 They correspond to angles $\frac{\pi}{3}$ and $\frac{5\pi}{3}$.

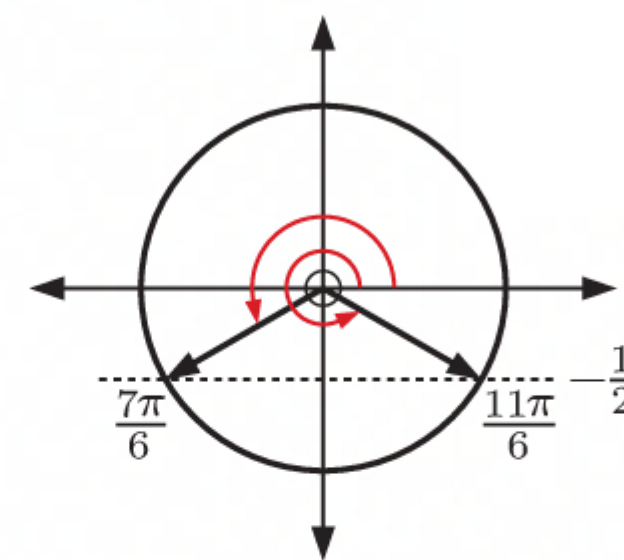


- a** In this case θ is simply x , so we have the domain $0 \leq x \leq 3\pi$.
 The solutions for this domain are $x = \frac{\pi}{3}, \frac{5\pi}{3}, \text{ or } \frac{7\pi}{3}$.

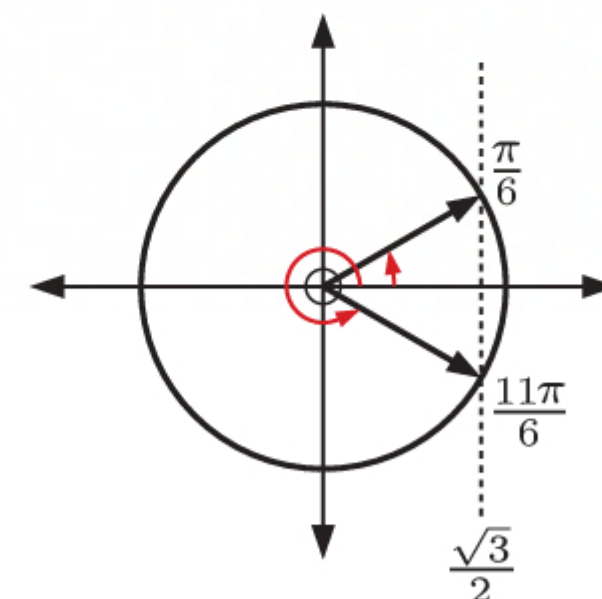
- b** In this case θ is $2x$.
 If $0 \leq x \leq 3\pi$ then $0 \leq 2x \leq 6\pi$.
 $\therefore 2x = \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}, \frac{11\pi}{3}, \frac{13\pi}{3}, \text{ or } \frac{17\pi}{3}$
 $\therefore x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}, \frac{13\pi}{6}, \text{ or } \frac{17\pi}{6}$

- c** In this case θ is $x + \frac{\pi}{3}$.
 If $0 \leq x \leq 3\pi$ then $\frac{\pi}{3} \leq x + \frac{\pi}{3} \leq \frac{10\pi}{3}$.
 $\therefore x + \frac{\pi}{3} = \frac{\pi}{3}, \frac{5\pi}{3}, \text{ or } \frac{7\pi}{3}$
 $\therefore x = 0, \frac{4\pi}{3}, \text{ or } 2\pi$

- 9 a** If $0 \leq x \leq 2\pi$, then $0 \leq 2x \leq 4\pi$.
 \therefore the angles between 0 and 4π with sine $-\frac{1}{2}$ are
 $\frac{7\pi}{6}, \frac{11\pi}{6}, \frac{19\pi}{6}, \text{ and } \frac{23\pi}{6}$
 $\therefore 2x = \frac{7\pi}{6}, \frac{11\pi}{6}, \frac{19\pi}{6}, \text{ or } \frac{23\pi}{6}$
 \therefore the solutions are $x = \frac{7\pi}{12}, \frac{11\pi}{12}, \frac{19\pi}{12}, \text{ or } \frac{23\pi}{12}$.



- b** If $0 \leq x \leq 2\pi$, then $0 \leq 3x \leq 6\pi$.
 \therefore the angles between 0 and 6π with cosine $\frac{\sqrt{3}}{2}$ are
 $\frac{\pi}{6}, \frac{11\pi}{6}, \frac{13\pi}{6}, \frac{23\pi}{6}, \frac{25\pi}{6}, \text{ and } \frac{35\pi}{6}$
 $\therefore 3x = \frac{\pi}{6}, \frac{11\pi}{6}, \frac{13\pi}{6}, \frac{23\pi}{6}, \frac{25\pi}{6}, \text{ or } \frac{35\pi}{6}$
 \therefore the solutions are $x = \frac{\pi}{18}, \frac{11\pi}{18}, \frac{13\pi}{18}, \frac{23\pi}{18}, \frac{25\pi}{18}, \text{ or } \frac{35\pi}{18}$.



c $\tan 2x - \sqrt{3} = 0$

$$\therefore \tan 2x = \sqrt{3}$$

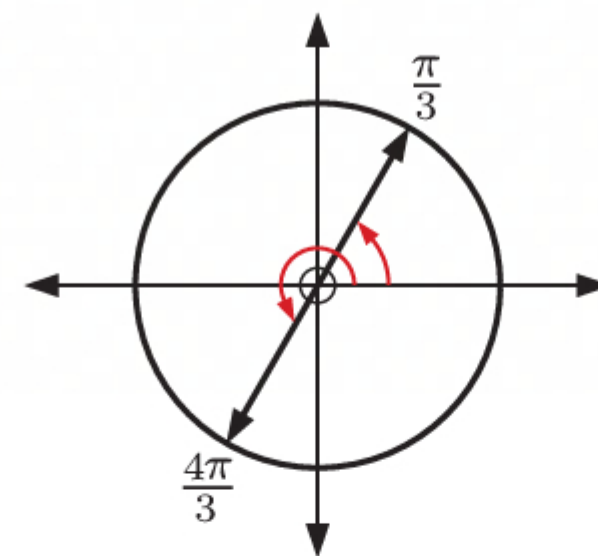
If $0 \leq x \leq 2\pi$, then $0 \leq 2x \leq 4\pi$.

The angles between 0 and 4π with tangent $\sqrt{3}$ are

$$\frac{\pi}{3}, \frac{4\pi}{3}, \frac{7\pi}{3}, \text{ and } \frac{10\pi}{3}$$

$$\therefore 2x = \frac{\pi}{3}, \frac{4\pi}{3}, \frac{7\pi}{3}, \text{ or } \frac{10\pi}{3}$$

$$\therefore \text{the solutions are } x = \frac{\pi}{6}, \frac{2\pi}{3}, \frac{7\pi}{6}, \text{ or } \frac{5\pi}{3}.$$

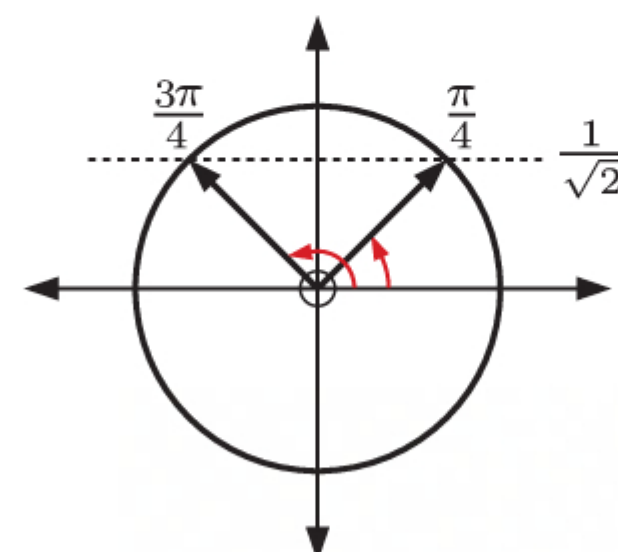


d If $0 \leq x \leq 2\pi$, then $0 \leq \frac{x}{2} \leq \pi$.

The angles between 0 and π with sine $\frac{1}{\sqrt{2}}$ are $\frac{\pi}{4}$ and $\frac{3\pi}{4}$.

$$\therefore \frac{x}{2} = \frac{\pi}{4} \text{ or } \frac{3\pi}{4}$$

$$\therefore \text{the solutions are } x = \frac{\pi}{2} \text{ or } \frac{3\pi}{2}.$$



e $2 \cos \frac{x}{2} + 1 = 0$

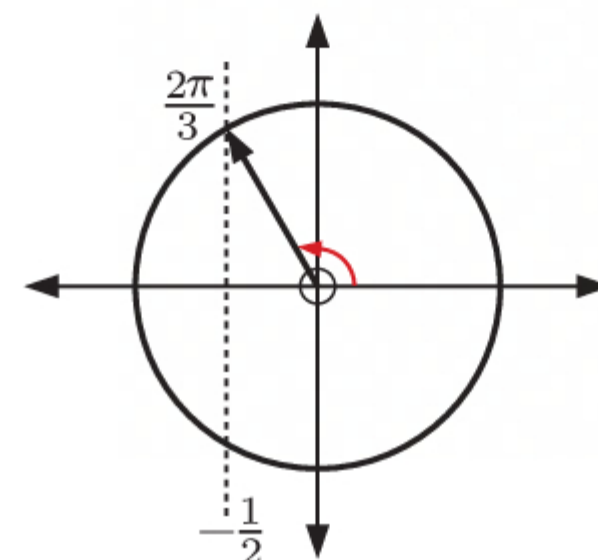
$$\therefore \cos \frac{x}{2} = -\frac{1}{2}$$

If $0 \leq x \leq 2\pi$, then $0 \leq \frac{x}{2} \leq \pi$.

The angle between 0 and π with cosine $-\frac{1}{2}$ is $\frac{2\pi}{3}$.

$$\therefore \frac{x}{2} = \frac{2\pi}{3}$$

$$\therefore \text{the solution is } x = \frac{4\pi}{3}.$$



f $3 \tan \frac{x}{3} - 3 = 0$

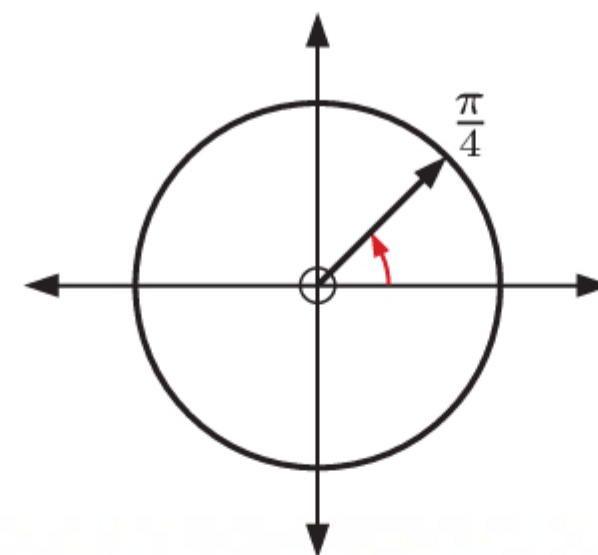
$$\therefore \tan \frac{x}{3} = \frac{3}{3} = 1$$

If $0 \leq x \leq 2\pi$, then $0 \leq \frac{x}{3} \leq \frac{2\pi}{3}$.

The angle between 0 and $\frac{2\pi}{3}$ with tangent 1 is $\frac{\pi}{4}$.

$$\therefore \frac{x}{3} = \frac{\pi}{4}$$

$$\therefore \text{the solution is } x = \frac{3\pi}{4}.$$



10 a $\cos^2 3x = \frac{1}{4}$

$$\therefore \cos 3x = \pm \frac{1}{2}$$

If $0 \leq x \leq 2\pi$, then $0 \leq 3x \leq 6\pi$.

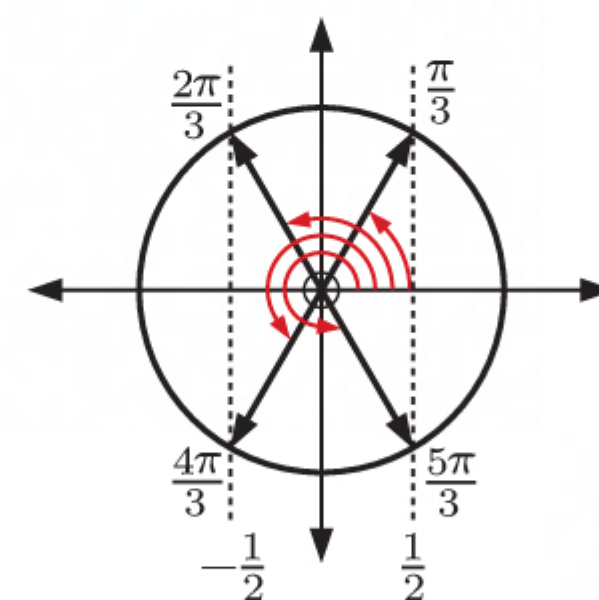
On $0 \leq 3x \leq 6\pi$, the angles with cosine $\frac{1}{2}$ are $\frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3},$

$\frac{11\pi}{3}, \frac{13\pi}{3},$ and $\frac{17\pi}{3}$, and the angles with cosine $-\frac{1}{2}$ are $\frac{2\pi}{3},$

$\frac{4\pi}{3}, \frac{8\pi}{3}, \frac{10\pi}{3}, \frac{14\pi}{3},$ and $\frac{16\pi}{3}$.

$$\therefore 3x = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}, \frac{8\pi}{3}, \frac{10\pi}{3}, \frac{11\pi}{3}, \frac{13\pi}{3}, \frac{14\pi}{3}, \frac{16\pi}{3}, \text{ or } \frac{17\pi}{3}$$

$$\therefore \text{the solutions are } x = \frac{\pi}{9}, \frac{2\pi}{9}, \frac{4\pi}{9}, \frac{5\pi}{9}, \frac{7\pi}{9}, \frac{8\pi}{9}, \frac{10\pi}{9}, \frac{11\pi}{9}, \frac{13\pi}{9}, \frac{14\pi}{9}, \frac{16\pi}{9}, \text{ or } \frac{17\pi}{9}.$$



b $\sin^2 2x = 1$

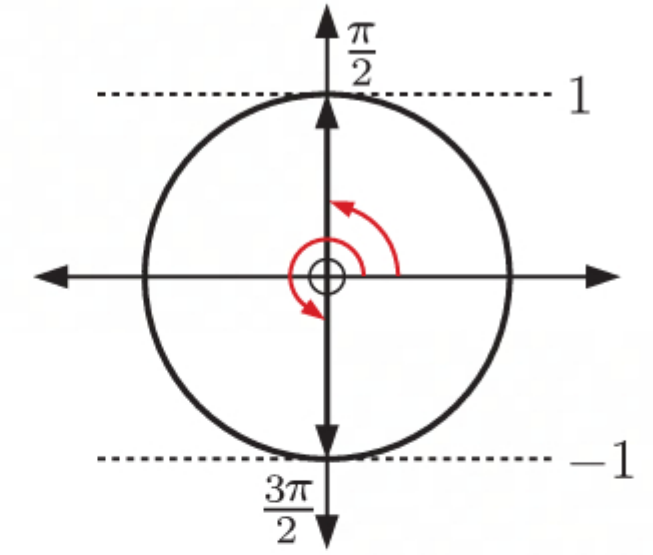
$\therefore \sin 2x = \pm 1$

If $0 \leq x \leq 2\pi$, then $0 \leq 2x \leq 4\pi$.

On $0 \leq 2x \leq 4\pi$, the angles with sine 1 are $\frac{\pi}{2}$ and $\frac{5\pi}{2}$,
and the angles with sine -1 are $\frac{3\pi}{2}$ and $\frac{7\pi}{2}$.

$\therefore 2x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \text{ or } \frac{7\pi}{2}$

\therefore the solutions are $x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \text{ or } \frac{7\pi}{4}$.



c $\tan^2\left(\frac{x}{2}\right) = \frac{1}{3}$

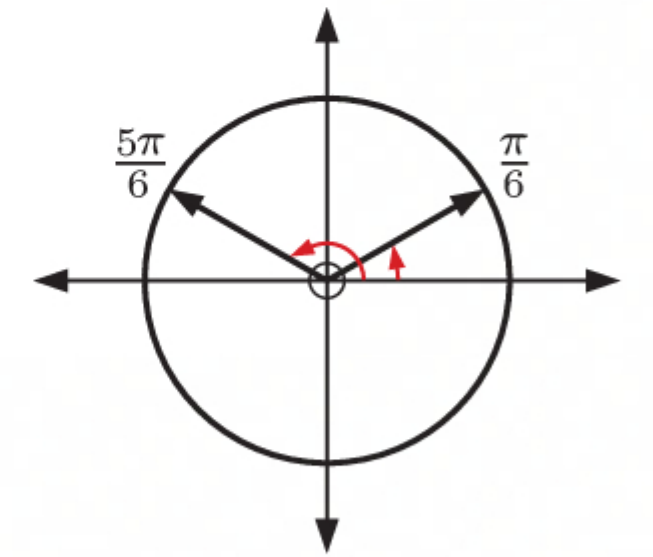
$\therefore \tan \frac{x}{2} = \pm \frac{1}{\sqrt{3}}$

If $0 \leq x \leq 2\pi$, then $0 \leq \frac{x}{2} \leq \pi$.

On $0 \leq \frac{x}{2} \leq \pi$, the angle with tangent $\frac{1}{\sqrt{3}}$ is $\frac{\pi}{6}$, and the angle
with tangent $-\frac{1}{\sqrt{3}}$ is $\frac{5\pi}{6}$.

$\therefore \frac{x}{2} = \frac{\pi}{6} \text{ or } \frac{5\pi}{6}$

\therefore the solutions are $x = \frac{\pi}{3} \text{ or } \frac{5\pi}{3}$.



11 a $\sin x = -\cos x$

$\therefore \frac{\sin x}{\cos x} = -1$

$\therefore \tan x = -1$

$\therefore x = \frac{3\pi}{4} \text{ or } \frac{7\pi}{4}$

b $0 \leq x \leq 2\pi \therefore 0 \leq 3x \leq 6\pi$

$\sin 3x = \cos 3x$

$\therefore \frac{\sin 3x}{\cos 3x} = 1$

$\therefore \tan 3x = 1$

$\therefore 3x = \frac{\pi}{4}, \frac{5\pi}{4}, \frac{9\pi}{4}, \frac{13\pi}{4}, \frac{17\pi}{4}, \text{ or } \frac{21\pi}{4}$

$\therefore x = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{3\pi}{4}, \frac{13\pi}{12}, \frac{17\pi}{12}, \text{ or } \frac{7\pi}{4}$

c $0 \leq x \leq 2\pi \therefore 0 \leq 2x \leq 4\pi$

$\sin 2x = \sqrt{3} \cos 2x$

$\therefore \frac{\sin 2x}{\cos 2x} = \sqrt{3}$

$\therefore \tan 2x = \sqrt{3}$

$\therefore 2x = \frac{\pi}{3}, \frac{4\pi}{3}, \frac{7\pi}{3}, \text{ or } \frac{10\pi}{3}$

$\therefore x = \frac{\pi}{6}, \frac{2\pi}{3}, \frac{7\pi}{6}, \text{ or } \frac{5\pi}{3}$

12 a $\cos\left(x - \frac{2\pi}{3}\right) = \frac{1}{2}, \quad -2\pi \leq x \leq 2\pi$

There are two points on the unit circle with cosine $\frac{1}{2}$.

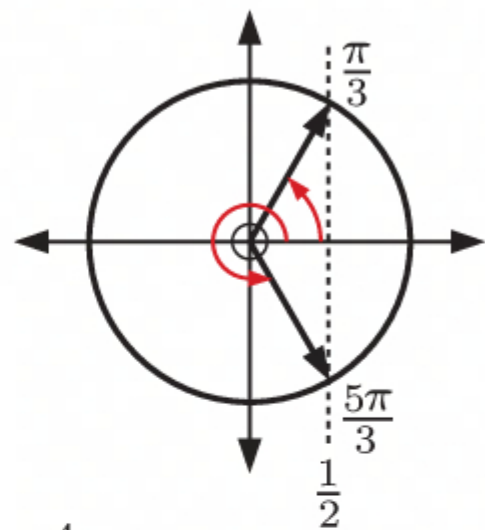
They correspond to angles $\frac{\pi}{3}$ and $\frac{5\pi}{3}$.

Since $-2\pi \leq x \leq 2\pi$

$$-\frac{8\pi}{3} \leq x - \frac{2\pi}{3} \leq \frac{4\pi}{3}$$

So, $x - \frac{2\pi}{3} = -\frac{7\pi}{3}, -\frac{5\pi}{3}, -\frac{\pi}{3}, \text{ or } \frac{\pi}{3}$

$$\therefore x = -\frac{5\pi}{3}, -\pi, \frac{\pi}{3}, \text{ or } \pi$$



b $\sqrt{2}\sin\left(x - \frac{\pi}{4}\right) + 1 = 0, \quad 0 \leq x \leq 3\pi$

$$\therefore \sin\left(x - \frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}}$$

There are two points on the unit circle with sine $-\frac{1}{\sqrt{2}}$.

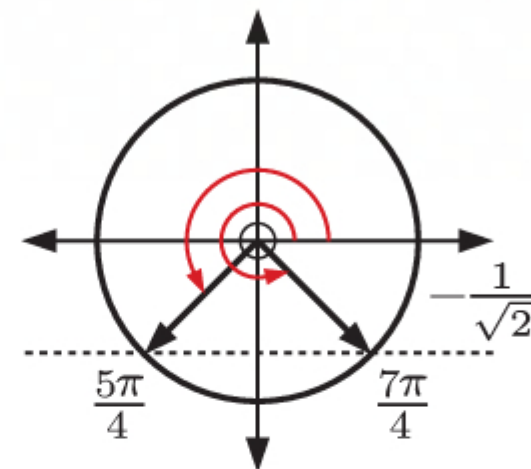
They correspond to angles $\frac{5\pi}{4}$ and $\frac{7\pi}{4}$.

Since $0 \leq x \leq 3\pi$

$$-\frac{\pi}{4} \leq x - \frac{\pi}{4} \leq \frac{11\pi}{4}$$

So, $x - \frac{\pi}{4} = -\frac{\pi}{4}, \frac{5\pi}{4}, \text{ or } \frac{7\pi}{4}$

$$\therefore x = 0, \frac{3\pi}{2}, \text{ or } 2\pi$$



c $\sin\left(4\left(x - \frac{\pi}{4}\right)\right) = 0, \quad 0 \leq x \leq \pi$

There are two points on the unit circle with sine 0.

They correspond to angles 0 and π .

Since $0 \leq x \leq \pi$

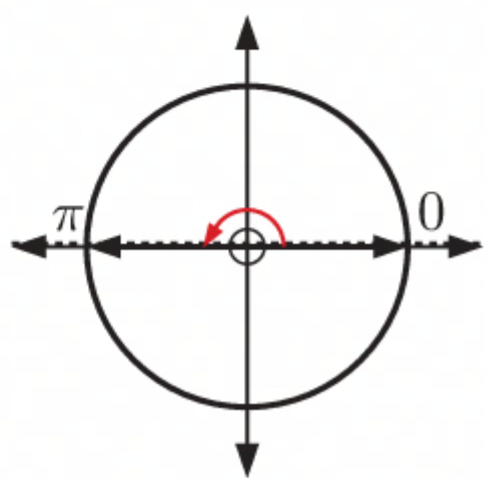
$$-\frac{\pi}{4} \leq x - \frac{\pi}{4} \leq \frac{3\pi}{4}$$

$$\therefore -\pi \leq 4\left(x - \frac{\pi}{4}\right) \leq 3\pi$$

So, $4\left(x - \frac{\pi}{4}\right) = -\pi, 0, \pi, 2\pi, \text{ or } 3\pi$

$$\therefore x - \frac{\pi}{4} = -\frac{\pi}{4}, 0, \frac{\pi}{4}, \frac{\pi}{2}, \text{ or } \frac{3\pi}{4}$$

$$\therefore x = 0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \text{ or } \pi$$



d $2\sin\left(2\left(x - \frac{\pi}{3}\right)\right) = -\sqrt{3}, \quad 0 \leq x \leq 2\pi$

$$\therefore \sin\left(2\left(x - \frac{\pi}{3}\right)\right) = -\frac{\sqrt{3}}{2}$$

There are two points on the unit circle with sine $-\frac{\sqrt{3}}{2}$.

They correspond to angles $\frac{4\pi}{3}$ and $\frac{5\pi}{3}$.

Since $0 \leq x \leq 2\pi$

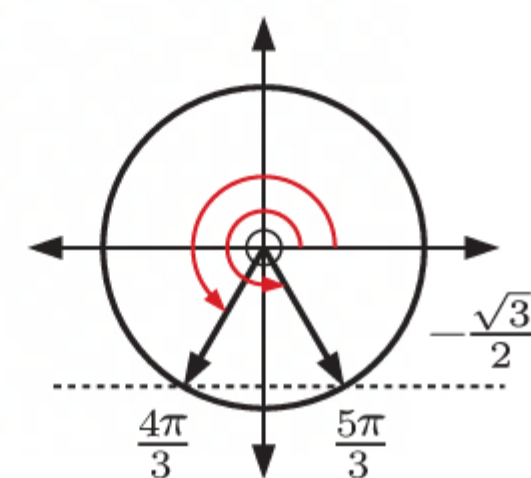
$$-\frac{\pi}{3} \leq x - \frac{\pi}{3} \leq \frac{5\pi}{3}$$

$$\therefore -\frac{2\pi}{3} \leq 2\left(x - \frac{\pi}{3}\right) \leq \frac{10\pi}{3}$$

So, $2\left(x - \frac{\pi}{3}\right) = -\frac{2\pi}{3}, -\frac{\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}, \text{ or } \frac{10\pi}{3}$

$$\therefore x - \frac{\pi}{3} = -\frac{\pi}{3}, -\frac{\pi}{6}, \frac{2\pi}{3}, \frac{5\pi}{6}, \text{ or } \frac{5\pi}{3}$$

$$\therefore x = 0, \frac{\pi}{6}, \pi, \frac{7\pi}{6}, \text{ or } 2\pi$$



13 a $y = 3\sin bx + d$

When $x = 0, y = 2$

$$\therefore 2 = 3\sin 0 + d$$

$$\therefore d = 2$$

When $x = \frac{\pi}{2}, y = \frac{7}{2}$

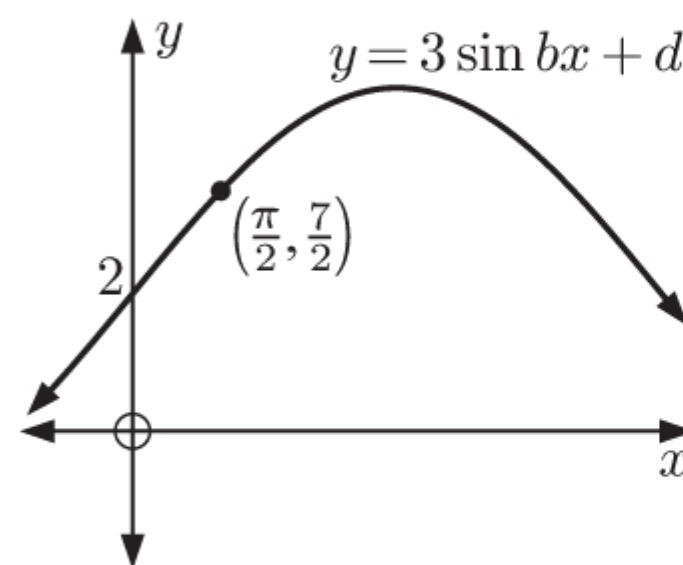
$$\therefore \frac{7}{2} = 3\sin\left(b \times \frac{\pi}{2}\right) + 2$$

$$\therefore \frac{3}{2} = 3\sin \frac{b\pi}{2}$$

$$\therefore \sin \frac{b\pi}{2} = \frac{1}{2}$$

$$\therefore \frac{b\pi}{2} = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}, \dots$$

$$\therefore b = \frac{1}{3}, \frac{5}{3}, \frac{7}{3}, \frac{11}{3}, \dots$$



From the graph, we require the value of b which gives the largest period, because $x = \frac{\pi}{2}$ is the *first* value for which $y = \frac{7}{2}$. This is the smallest value of b .

So, $b = \frac{1}{3}, d = 2$.

b $y = a \cos bx$

When $x = 0$, $y = -2$

$$\therefore -2 = a \cos 0$$

$$\therefore a = -2$$

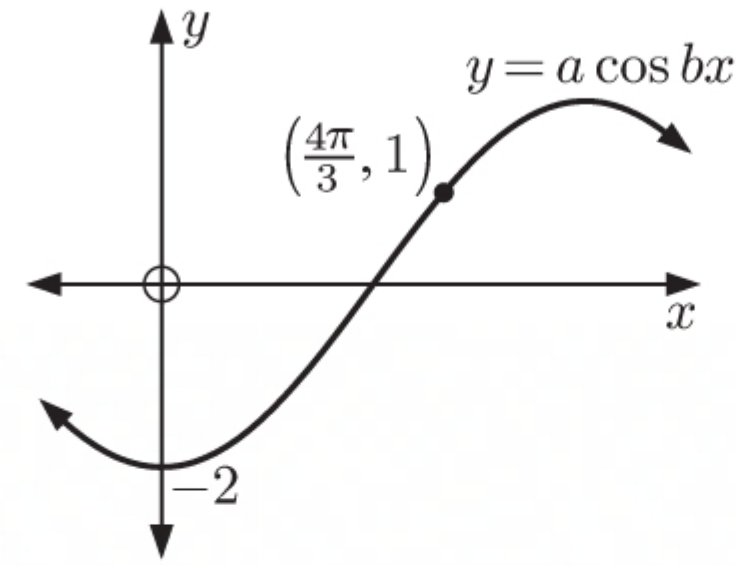
When $x = \frac{4\pi}{3}$, $y = 1$

$$\therefore 1 = -2 \cos\left(b \times \frac{4\pi}{3}\right)$$

$$\therefore \cos \frac{4b\pi}{3} = -\frac{1}{2}$$

$$\therefore \frac{4b\pi}{3} = \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{8\pi}{3}, \frac{10\pi}{3}, \dots$$

$$\therefore b = \frac{1}{2}, 1, 2, \frac{5}{2}, \dots$$



From the graph, we require the value of b which gives the largest period, because $x = \frac{4\pi}{3}$ is the *first* value for which $y = 1$. This is the smallest value of b .

So, $a = -2$, $b = \frac{1}{2}$.

c $y = \frac{1}{2} \cos bx + d$

When $x = 0$, $y = -0.5$

$$\therefore -\frac{1}{2} = \frac{1}{2} \cos 0 + d$$

$$\therefore d = -1$$

When $x = \frac{2\pi}{3}$, $y = -\frac{5}{4}$

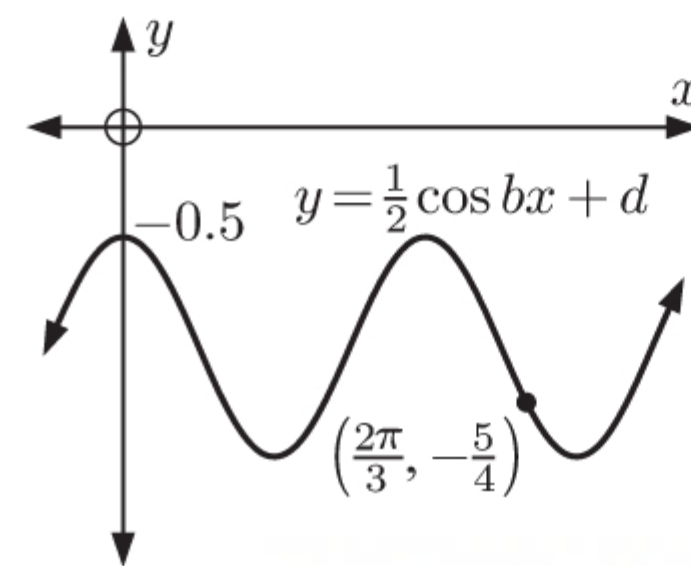
$$\therefore -\frac{5}{4} = \frac{1}{2} \cos\left(b \times \frac{2\pi}{3}\right) - 1$$

$$\therefore -\frac{1}{4} = \frac{1}{2} \cos \frac{2b\pi}{3}$$

$$\therefore \cos \frac{2b\pi}{3} = -\frac{1}{2}$$

$$\therefore \frac{2b\pi}{3} = \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{8\pi}{3}, \frac{10\pi}{3}, \dots$$

$$\therefore b = 1, 2, 4, 5, \dots$$



From the graph, we require the value of b which gives the 3rd largest period, because $x = \frac{2\pi}{3}$ is the *third* value for which $y = -\frac{5}{4}$. This is the 3rd smallest value of b .

So, $b = 4$, $d = -1$.

d $y = 5 \cos\left(b\left(x - \frac{\pi}{4}\right)\right) + d$

When $x = \frac{\pi}{4}$, $y = 1$

$$\therefore 1 = 5 \cos(b \times 0) + d$$

$$\therefore d = -4$$

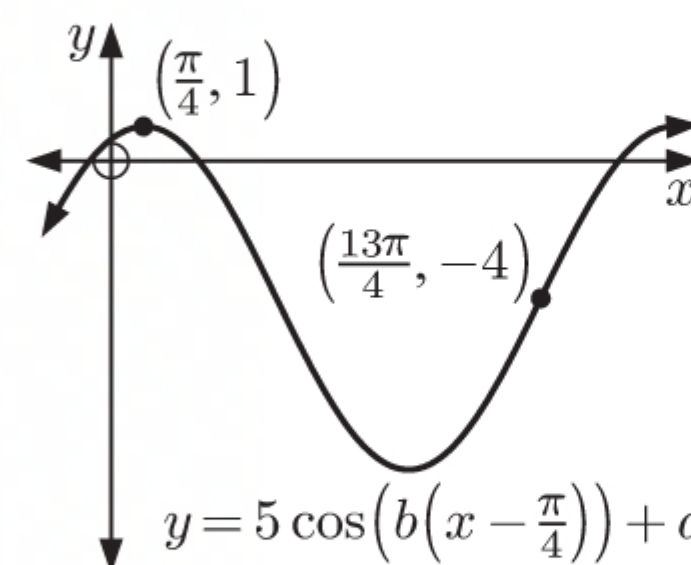
When $x = \frac{13\pi}{4}$, $y = -4$

$$\therefore -4 = 5 \cos(b \times 3\pi) - 4$$

$$\therefore \cos 3b\pi = 0$$

$$\therefore 3b\pi = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}, \dots$$

$$\therefore b = \frac{1}{6}, \frac{1}{2}, \frac{5}{6}, \frac{7}{6}, \dots$$



From the graph, we require the value of b which gives the 2nd largest period, because $x = \frac{13\pi}{4}$ is the *second* value for which $y = -4$. This is the 2nd smallest value of b .

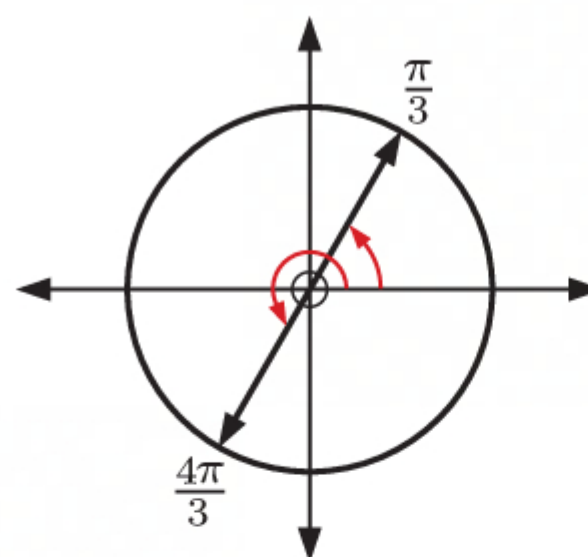
So, $b = \frac{1}{2}$, $d = -4$.

14 $\tan x = \sqrt{3}, \quad 0 \leq x \leq 2\pi$

There are two points on the unit circle with tangent $\sqrt{3}$.

They correspond to angles $\frac{\pi}{3}$ and $\frac{4\pi}{3}$.

$\therefore x = \frac{\pi}{3} \text{ or } \frac{4\pi}{3}$



a Since $0 \leq x \leq 2\pi$

$$-\frac{\pi}{6} \leq x - \frac{\pi}{6} \leq \frac{11\pi}{6}$$

So, $x - \frac{\pi}{6} = \frac{\pi}{3} \text{ or } \frac{4\pi}{3}$

$\therefore x = \frac{\pi}{2} \text{ or } \frac{3\pi}{2}$

b Since $0 \leq x \leq 2\pi$

$$0 \leq 4x \leq 8\pi$$

So, $4x = \frac{\pi}{3}, \frac{4\pi}{3}, \frac{7\pi}{3}, \frac{10\pi}{3}, \frac{13\pi}{3}, \frac{16\pi}{3}, \frac{19\pi}{3}, \text{ or } \frac{22\pi}{3}$

$\therefore x = \frac{\pi}{12}, \frac{\pi}{3}, \frac{7\pi}{12}, \frac{5\pi}{6}, \frac{13\pi}{12}, \frac{4\pi}{3}, \frac{19\pi}{12}, \text{ or } \frac{11\pi}{6}$

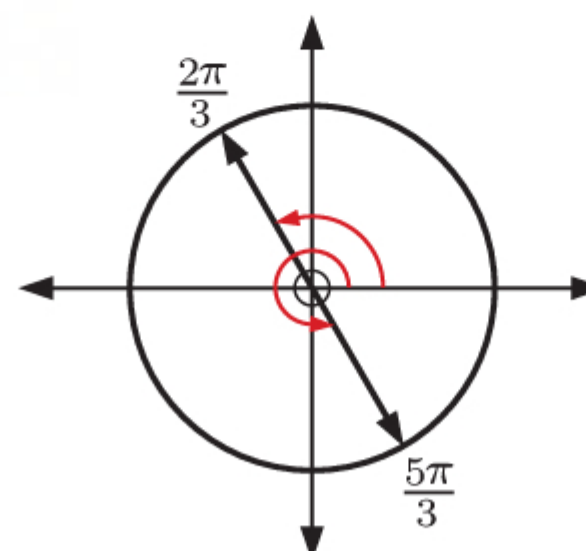
c $\tan^2 x = 3, \quad 0 \leq x \leq 2\pi$

$\therefore \tan x = \pm\sqrt{3}$

There are two points on the unit circle with tangent $-\sqrt{3}$.

They correspond to angles $\frac{2\pi}{3}$ and $\frac{5\pi}{3}$.

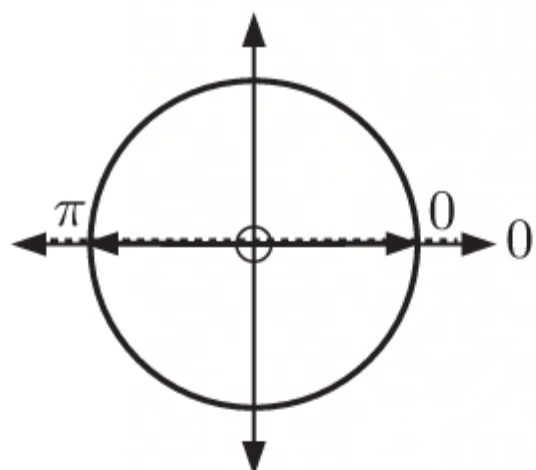
So, $x = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \text{ or } \frac{5\pi}{3}$



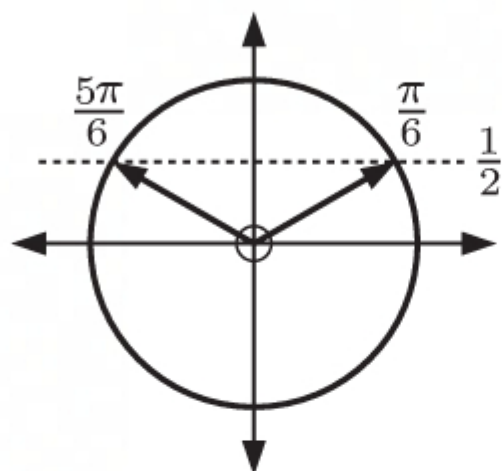
15 a $2\sin^2 x - \sin x = 0$

$\therefore \sin x(2\sin x - 1) = 0$

$\therefore \sin x = 0 \text{ or } \frac{1}{2}$



$\sin x = 0$ when
 $x = 0, \pi, \text{ or } 2\pi$
 $\{0 \leq x \leq 2\pi\}$



$\sin x = \frac{1}{2}$ when
 $x = \frac{\pi}{6} \text{ or } \frac{5\pi}{6}$
 $\{0 \leq x \leq 2\pi\}$

The solutions are:

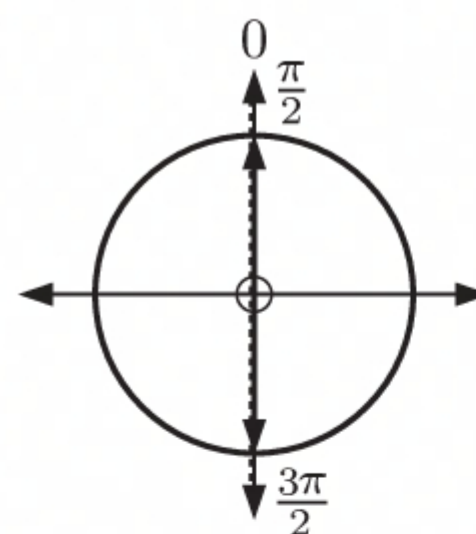
$x = 0, \frac{\pi}{6}, \frac{5\pi}{6}, \pi, \text{ or } 2\pi.$

b $2\cos^2 x = \cos x$

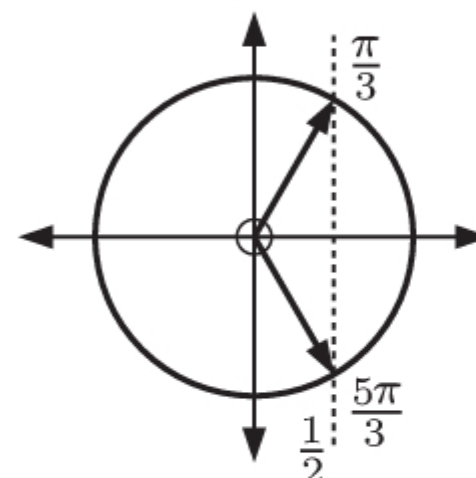
$\therefore 2\cos^2 x - \cos x = 0$

$\therefore \cos x(2\cos x - 1) = 0$

$\therefore \cos x = 0 \text{ or } \frac{1}{2}$



$\cos x = 0$ when
 $x = \frac{\pi}{2} \text{ or } \frac{3\pi}{2}$
 $\{0 \leq x \leq 2\pi\}$

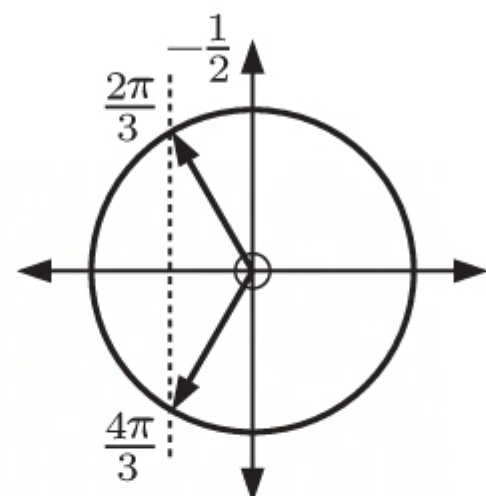


$\cos x = \frac{1}{2}$ when
 $x = \frac{\pi}{3} \text{ or } \frac{5\pi}{3}$
 $\{0 \leq x \leq 2\pi\}$

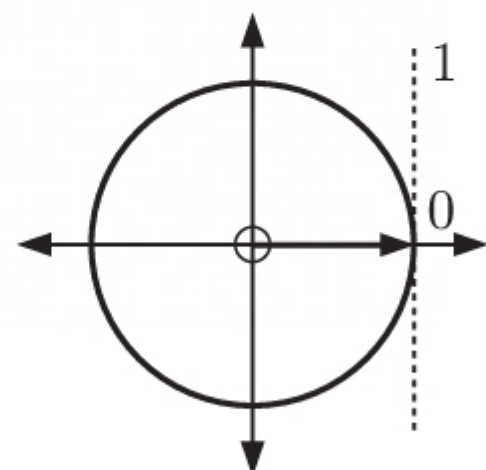
The solutions are:

$x = \frac{\pi}{3}, \frac{\pi}{2}, \frac{3\pi}{2}, \text{ or } \frac{5\pi}{3}.$

$$\begin{aligned} \text{c} \quad & 2\cos^2 x - \cos x - 1 = 0 \\ \therefore & (2\cos x + 1)(\cos x - 1) = 0 \\ \therefore & \cos x = -\frac{1}{2} \text{ or } 1 \end{aligned}$$



$$\begin{aligned} \cos x = -\frac{1}{2} \text{ when} \\ x = \frac{2\pi}{3} \text{ or } \frac{4\pi}{3} \\ \{0 \leq x \leq 2\pi\} \end{aligned}$$

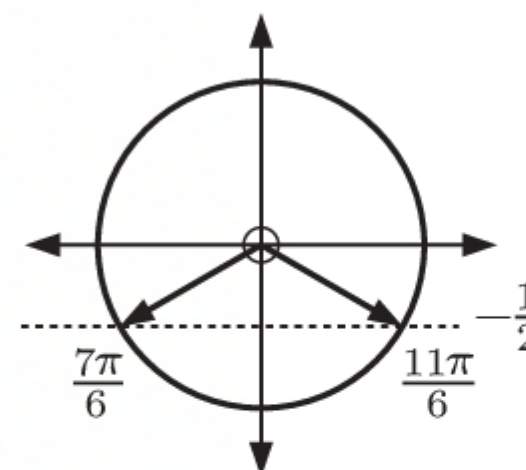


$$\begin{aligned} \cos x = 1 \text{ when} \\ x = 0 \text{ or } 2\pi \\ \{0 \leq x \leq 2\pi\} \end{aligned}$$

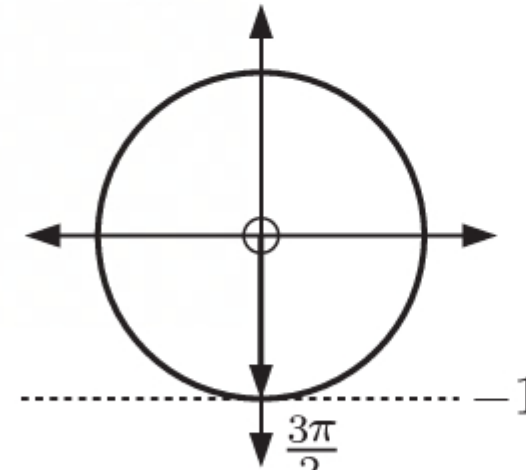
The solutions are:

$$x = 0, \frac{2\pi}{3}, \frac{4\pi}{3}, \text{ or } 2\pi.$$

$$\begin{aligned} \text{d} \quad & 2\sin^2 x + 3\sin x + 1 = 0 \\ \therefore & (2\sin x + 1)(\sin x + 1) = 0 \\ \therefore & \sin x = -\frac{1}{2} \text{ or } -1 \end{aligned}$$



$$\begin{aligned} \sin x = -\frac{1}{2} \text{ when} \\ x = \frac{7\pi}{6} \text{ or } \frac{11\pi}{6} \\ \{0 \leq x \leq 2\pi\} \end{aligned}$$

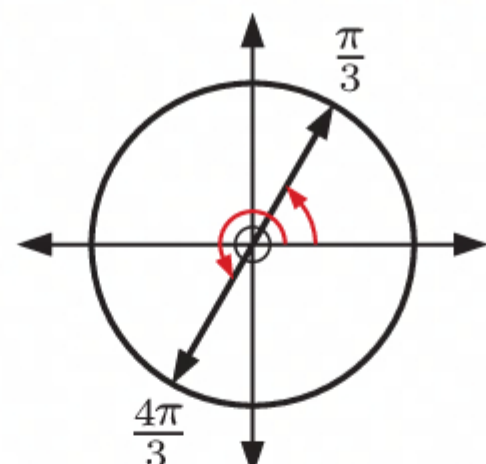


$$\begin{aligned} \sin x = -1 \text{ when} \\ x = \frac{3\pi}{2} \\ \{0 \leq x \leq 2\pi\} \end{aligned}$$

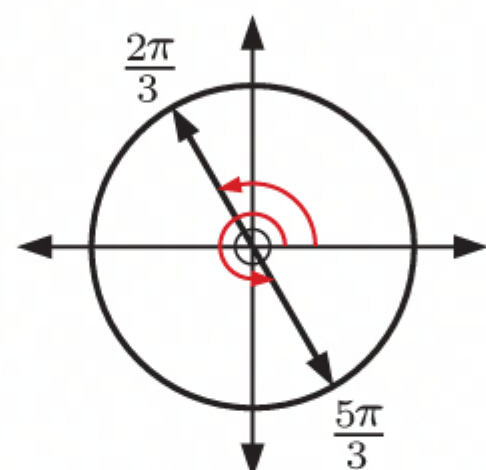
The solutions are:

$$x = \frac{7\pi}{6}, \frac{3\pi}{2}, \text{ or } \frac{11\pi}{6}.$$

$$\begin{aligned} \text{e} \quad & \tan^4 x - 2\tan^2 x - 3 = 0 \\ \therefore & (\tan^2 x + 1)(\tan^2 x - 3) = 0 \\ \therefore & \tan^2 x = -1 \text{ or } \tan^2 x = 3 \\ \therefore & \tan^2 x = 3 \quad \{\text{since } \tan^2 x \geq 0\} \\ \therefore & \tan x = \pm\sqrt{3} \end{aligned}$$



$$\begin{aligned} \tan x = \sqrt{3} \text{ when} \\ x = \frac{\pi}{3} \text{ or } \frac{4\pi}{3} \\ \{0 \leq x \leq 2\pi\} \end{aligned}$$



$$\begin{aligned} \tan x = -\sqrt{3} \text{ when} \\ x = \frac{2\pi}{3} \text{ or } \frac{5\pi}{3} \\ \{0 \leq x \leq 2\pi\} \end{aligned}$$

The solutions are:

$$x = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \text{ or } \frac{5\pi}{3}.$$

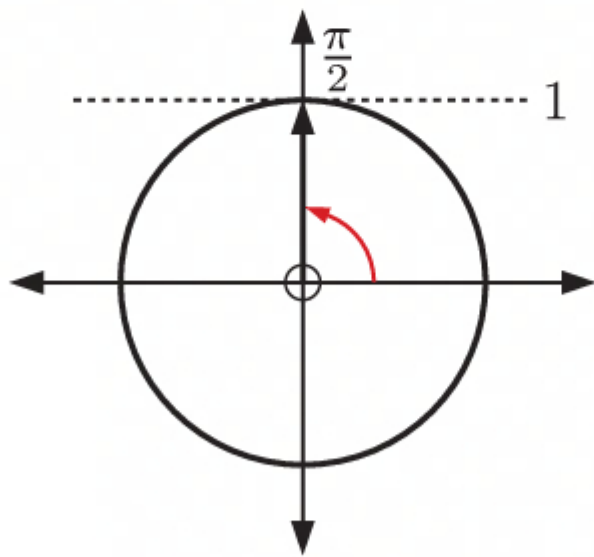
EXERCISE 9B

1 $P(t) = 7500 + 3000 \sin \frac{\pi t}{8}, \quad 0 \leq t \leq 12$

a i $P(0) = 7500 + 3000 \sin 0$
 $= 7500 + 0$
 $= 7500$ grasshoppers

ii $P(5) = 7500 + 3000 \sin \frac{5\pi}{8}$
 $\approx 10\,271.6$
 $\approx 10\,300$ grasshoppers

- b The greatest value of $P(t)$ occurs when $\sin \frac{\pi t}{8} = 1$, so the greatest population is $7500 + 3000 = 10\,500$ grasshoppers.



The point on the unit circle with sine 1 corresponds to angle $\frac{\pi}{2}$.

$$0 \leq t \leq 12 \quad \therefore \quad 0 \leq \frac{\pi t}{8} \leq \frac{3\pi}{2}$$

$$\text{So } \frac{\pi t}{8} = \frac{\pi}{2}$$

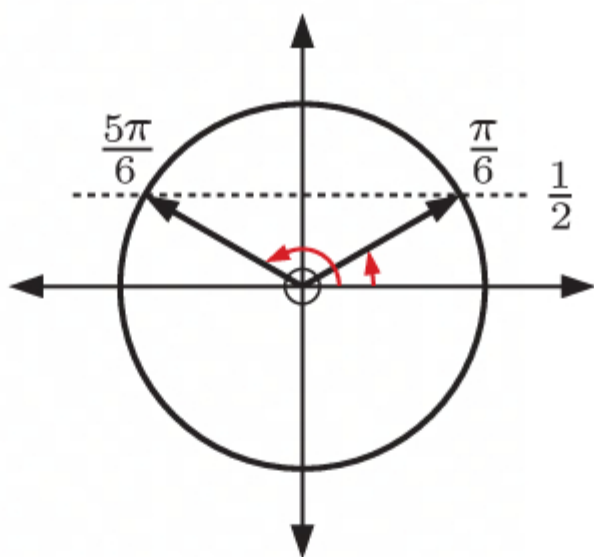
$$\therefore t = 4$$

So the greatest population occurs after 4 weeks.

c i When $P(t) = 9000$, $7500 + 3000 \sin \frac{\pi t}{8} = 9000$

$$\therefore 3000 \sin \frac{\pi t}{8} = 1500$$

$$\therefore \sin \frac{\pi t}{8} = \frac{1}{2}$$



The points on the unit circle with sine $\frac{1}{2}$ correspond to angles $\frac{\pi}{6}$ and $\frac{5\pi}{6}$.

$$0 \leq t \leq 12 \quad \therefore \quad 0 \leq \frac{\pi t}{8} \leq \frac{3\pi}{2}$$

$$\text{So } \frac{\pi t}{8} = \frac{\pi}{6}, \frac{5\pi}{6}$$

$$\therefore t = \frac{8}{6}, \frac{40}{6}$$

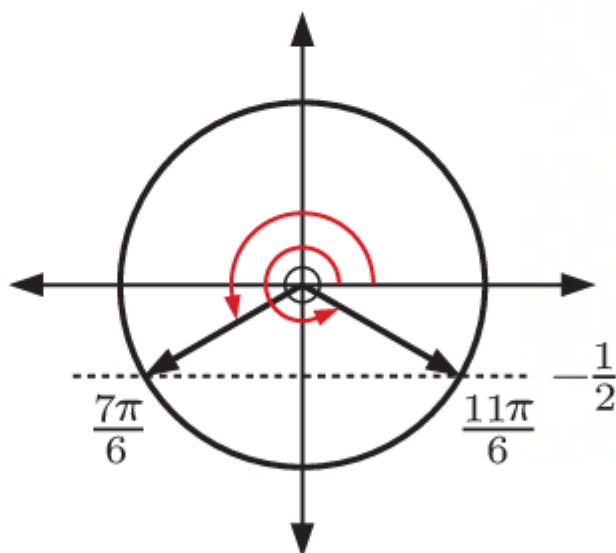
$$\therefore t = 1\frac{1}{3} \text{ or } 6\frac{2}{3}$$

So, the population is 9000 at $1\frac{1}{3}$ weeks and $6\frac{2}{3}$ weeks.

ii When $P(t) = 6000$, $7500 + 3000 \sin \frac{\pi t}{8} = 6000$

$$\therefore 3000 \sin \frac{\pi t}{8} = -1500$$

$$\therefore \sin \frac{\pi t}{8} = -\frac{1}{2}$$



The points on the unit circle with sine $-\frac{1}{2}$ correspond to angles $\frac{7\pi}{6}$ and $\frac{11\pi}{6}$.

$$0 \leq t \leq 12 \quad \therefore \quad 0 \leq \frac{\pi t}{8} \leq \frac{3\pi}{2}$$

$$\text{So } \frac{\pi t}{8} = \frac{7\pi}{6}, \frac{11\pi}{6}$$

$$\therefore t = \frac{56}{6}, \frac{44}{6}$$

$$\therefore t = 9\frac{1}{3}$$

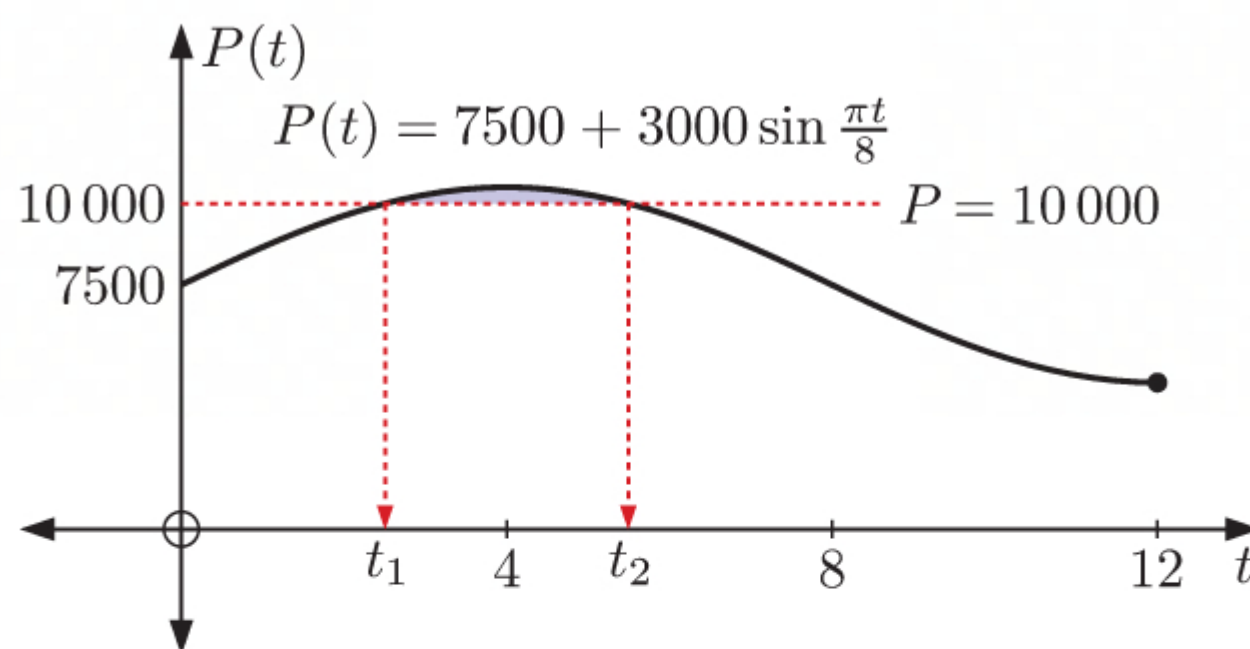
So, the population is 6000 at $9\frac{1}{3}$ weeks.

d We need to solve $P(t) = 10\,000$

$$\therefore 7500 + 3000 \sin \frac{\pi t}{8} = 10\,000$$

Using technology, we obtain $t_1 \approx 2.51$,
 $t_2 \approx 5.49$.

So, the population exceeds 10 000 for
 $2.51 \leq t \leq 5.49$ weeks.



2 $H(t) = 10 \sin\left(\frac{\pi}{50}t\right) + 12$ metres

a i $H(0) = 10 \sin\left(\frac{\pi}{50}(0)\right) + 12$
 $= 10 \times 0 + 12$
 $= 12$

The height of the green light is initially 12 metres.

ii $H(75) = 10 \sin\left(\frac{\pi}{50}(75)\right) + 12$
 $= 10 \times \sin\left(\frac{3\pi}{2}\right) + 12$
 $= 2$

The height of the green light after 75 seconds is 2 metres.

b A full circle is completed in one period.

$$\text{period} = \frac{2\pi}{\frac{\pi}{50}} = 2\pi \times \frac{50}{\pi} = 100 \text{ seconds}$$

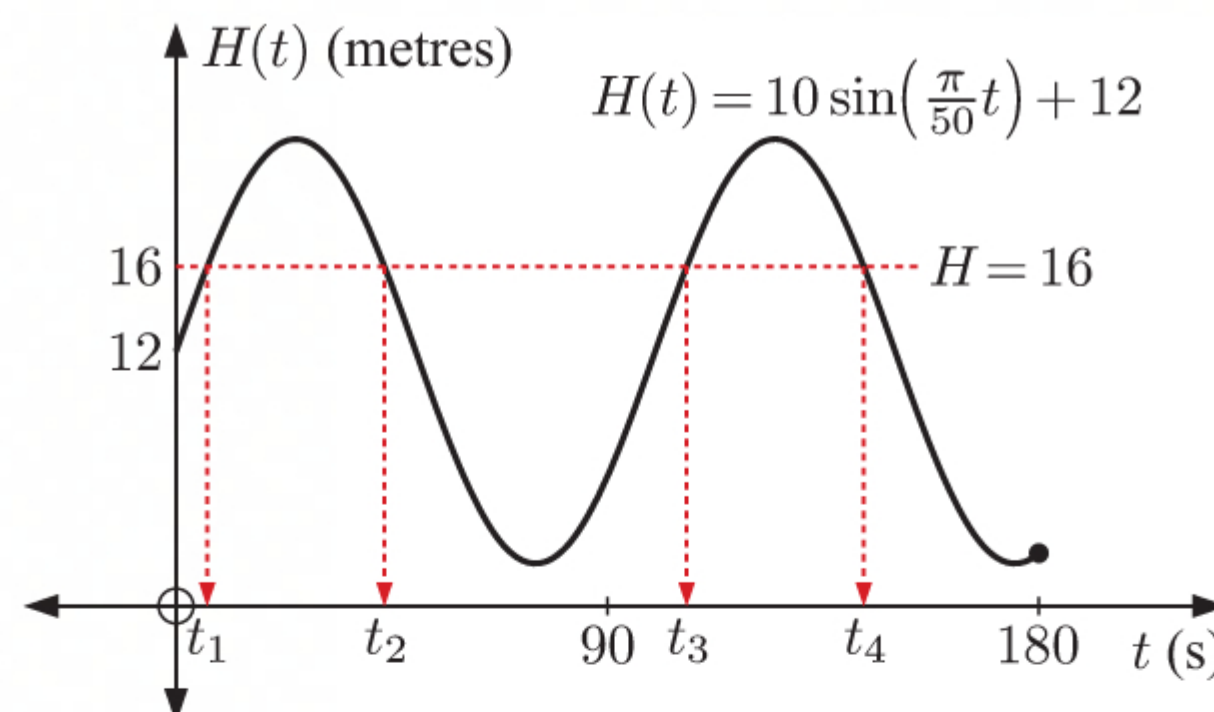
So, it takes 100 seconds for the wheel to complete a full circle.

c We need to solve $H(t) = 16$, $0 \leq t \leq 180$

$$\therefore 10 \sin\left(\frac{\pi}{50}t\right) + 12 = 16$$

Using technology, we obtain $t_1 \approx 6.55$,
 $t_2 \approx 43.5$, $t_3 \approx 107$, and $t_4 \approx 143$.

So, in the first three minutes, the green light is 16 metres above the ground at approximately 6.55, 43.5, 107, and 143 seconds.

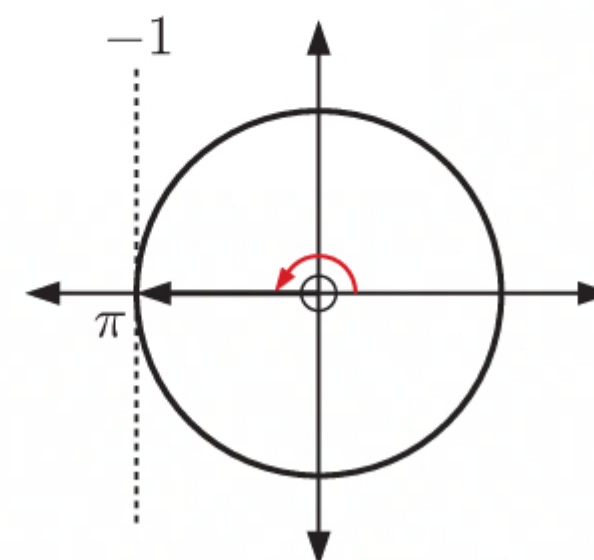


3 $H(t) = 20 - 19 \cos \frac{2\pi t}{3}$

a $H(0) = 20 - 19 \cos 0$
 $= 20 - 19$
 $= 1 \text{ m}$

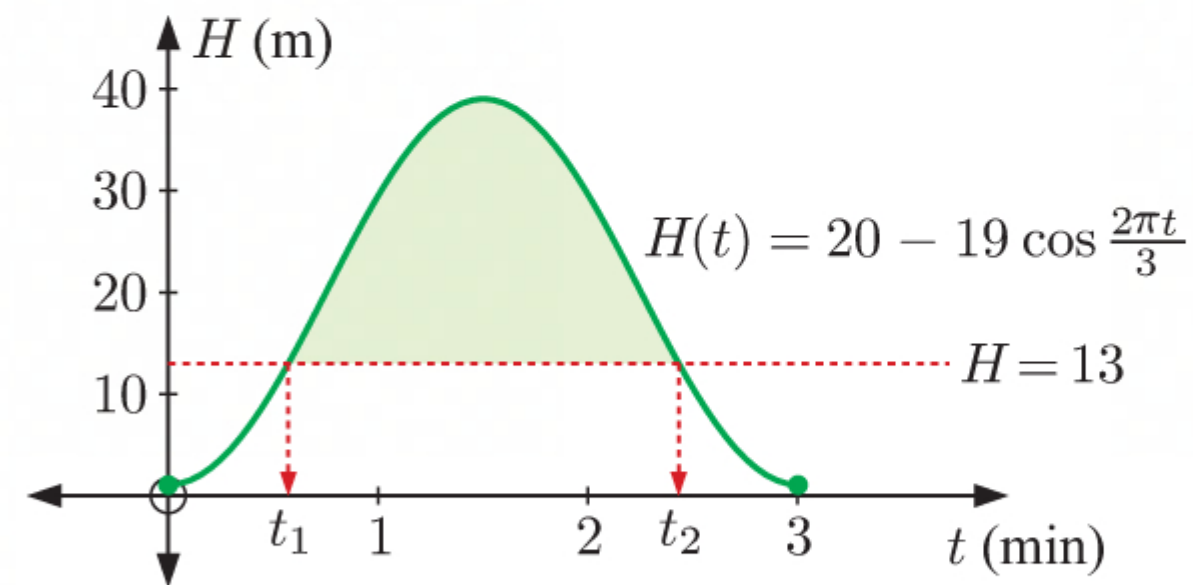
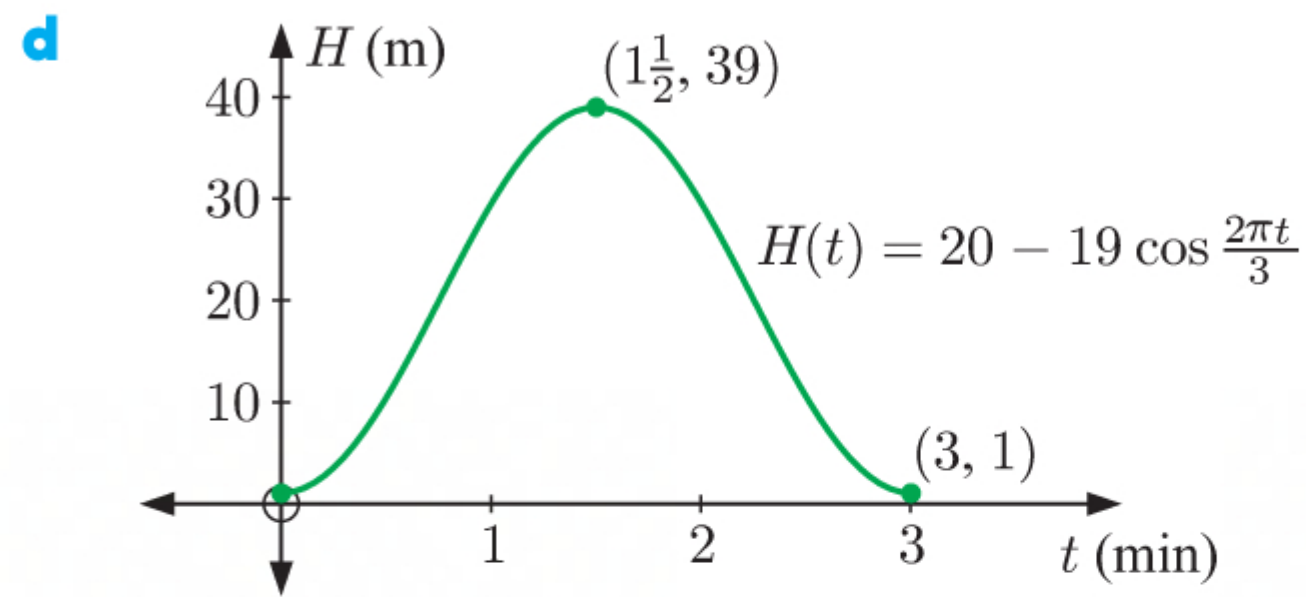
So, at time $t = 0$, the passenger is 1 m above the ground.

b H is a maximum when $\cos \frac{2\pi t}{3} = -1$
 $\therefore \frac{2\pi t}{3} = \pi + k2\pi$
 $\therefore \frac{2t}{3} = 1 + k(2)$
 $\therefore t = \frac{3}{2} + 3k$
 $\therefore t = 1\frac{1}{2} \text{ min} \quad \{\text{as } k = 0 \text{ for the first revolution}\}$



- c period = $\frac{2\pi}{\frac{2\pi}{3}} = 3$ min
 \therefore one revolution takes 3 minutes.

- e We need to solve $H(t) = 13$
 $\therefore 20 - 19 \cos \frac{2\pi t}{3} = 13$
 Using technology, we obtain $t_1 \approx 0.570$,
 $t_2 \approx 2.43$.
 So, the passenger can see his friend for
 $0.570 \leq t \leq 2.43$ minutes.



4 $P(t) = 400 + 250 \sin \frac{\pi t}{2}$

a $P(0) = 400 + 250 \sin 0$
 $= 400 + 250(0)$
 $= 400$ water buffalo

b i 6 months = $\frac{1}{2}$ year
 $P(\frac{1}{2}) = 400 + 250 \sin \left(\frac{\pi(\frac{1}{2})}{2} \right)$
 $= 400 + 250 \sin \frac{\pi}{4}$
 $= 400 + 250 \times \frac{1}{\sqrt{2}}$
 ≈ 577 water buffalo

c $P(1) = 400 + 250 \sin \frac{\pi}{2}$
 $= 400 + 250 \times 1$
 $= 650$ water buffalo

This is the maximum herd size.

- d $P(t)$ is smallest when $\sin \frac{\pi t}{2} = -1$
 and is $400 - 250 = 150$ water buffalo.

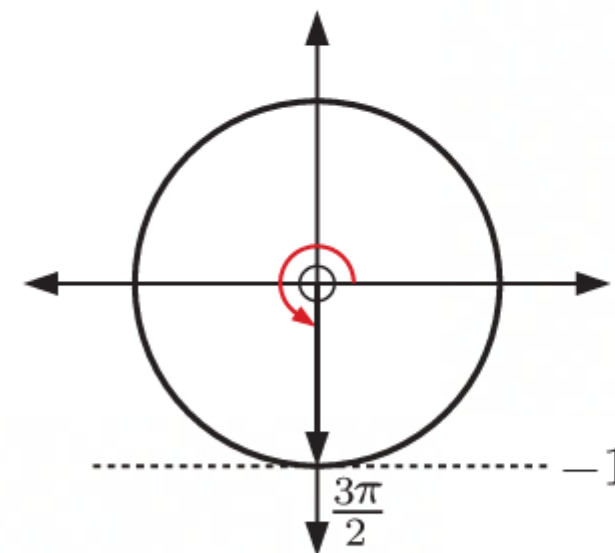
It occurs when $\frac{\pi t}{2} = \frac{3\pi}{2} + k2\pi$

$$\therefore \frac{t}{2} = \frac{3}{2} + k(2)$$

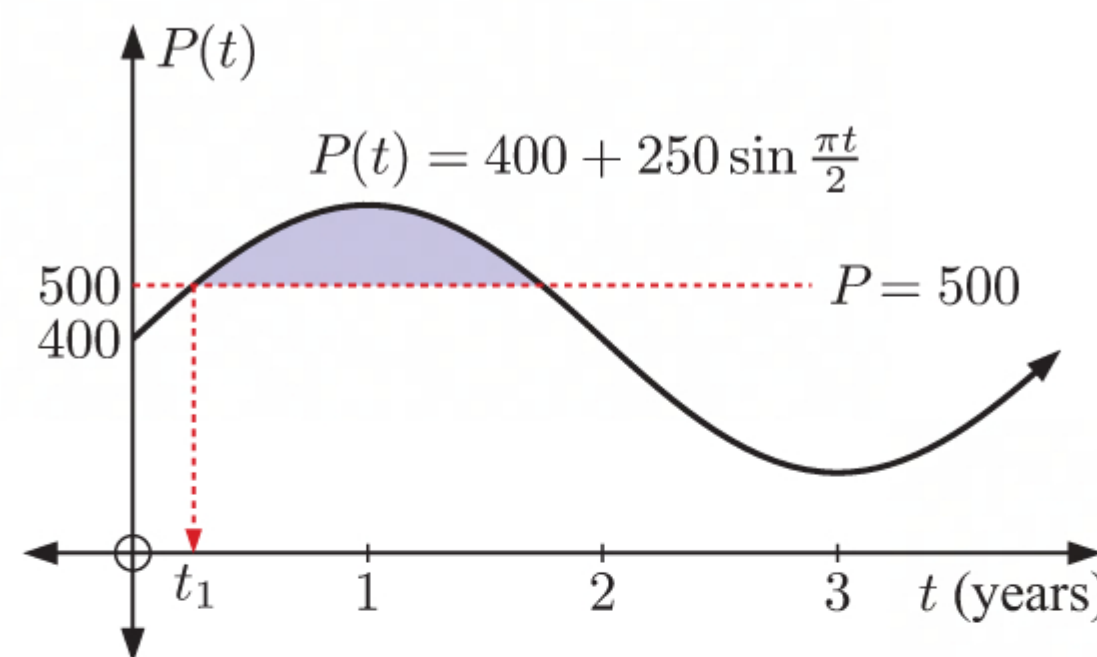
$$\therefore t = 3 + 4k$$

So, the first time is after 3 years.

ii $P(2) = 400 + 250 \sin \pi$
 $= 400 + 250(0)$
 $= 400$ water buffalo



- e** We need to solve $P(t) = 500$
 $\therefore 400 + 250 \sin \frac{\pi t}{2} = 500$
 Using technology, we obtain $t_1 \approx 0.262$.
 \therefore the herd first exceeded 500 when
 $t \approx 0.262$ years.



5 $C(t) = 9.2 \sin\left(\frac{\pi}{7}(t-4)\right) + 107.8$ cents L^{-1}

- a i** 107.8 is the median value.

Values are between $107.8 - 9.2 = 98.6$ cents L^{-1} {when $\sin\left(\frac{\pi}{7}(t-4)\right) = -1$ }

↑
min.

and $107.8 + 9.2 = 117.0$ cents L^{-1} {when $\sin\left(\frac{\pi}{7}(t-4)\right) = 1$ }

↑
max.

\therefore the statement is true.

ii period $= \frac{2\pi}{\frac{\pi}{7}} = 14$ days \therefore true

b $C(7) = 9.2 \sin\left(\frac{\pi}{7}(3)\right) + 107.8$
 ≈ 116.8 cents L^{-1}

c When $C(t) = \$1.10 L^{-1}$

then $9.2 \sin\left(\frac{\pi}{7}(t-4)\right) + 107.8 = 110$

$$\therefore \sin\left(\frac{\pi}{7}(t-4)\right) = \frac{2.2}{9.2}$$

$$\therefore t \approx 4.538, 10.462, 18.538, 24.462 \text{ using technology}$$

So, the price was \$1.10 per litre on the 5th, 11th, 19th, and 25th days.

d The minimum cost per litre is $-9.2 + 107.8 = 98.6$ cents L^{-1}

when $\sin\left(\frac{\pi}{7}(t-4)\right) = -1$

$$\therefore \frac{\pi}{7}(t-4) = \frac{3\pi}{2}$$

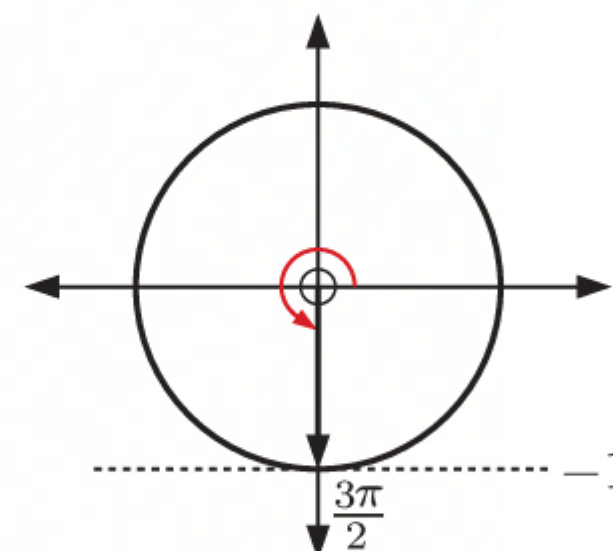
$$\therefore \frac{t-4}{7} = \frac{3}{2}$$

$$\therefore 2t - 8 = 21$$

$$\therefore 2t = 29$$

$$\therefore t = 14.5 \pm 14k \text{ \{period is 14 days\}}$$

So, the minimum occurred on the 1st day and the 15th day.



- 6 a For $H(t) = a \cos(b(t + c)) + d$:

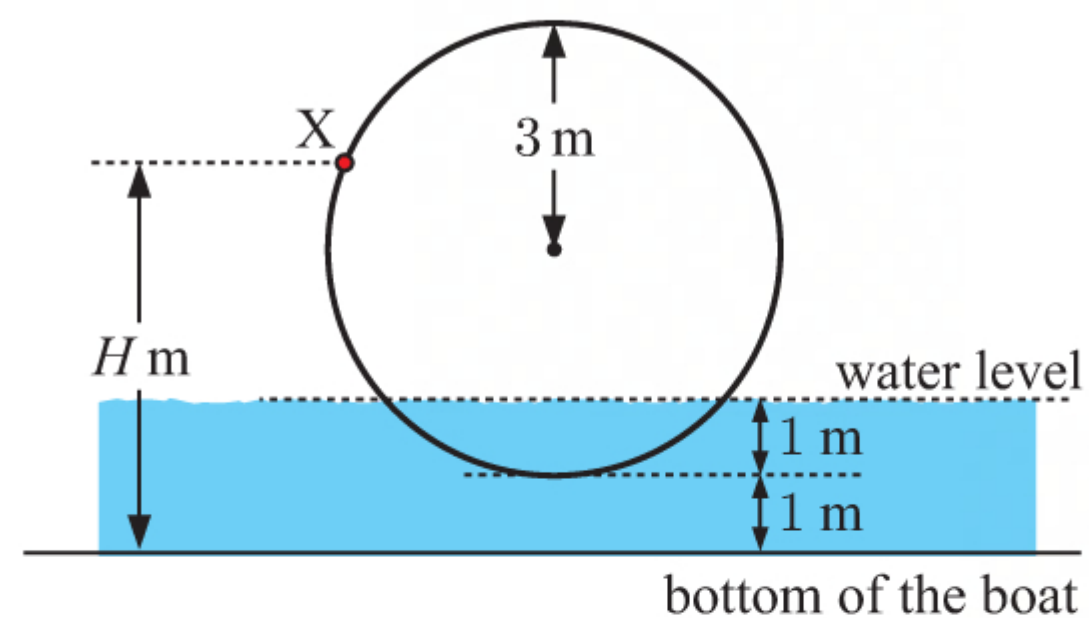
The period is $\frac{2\pi}{b} = 4 \text{ s} \therefore b = \frac{\pi}{2}$

The amplitude $a = 3$

There is a vertical translation of $+4 \therefore d = 4$

There is no horizontal translation $\therefore c = 0$

$$\therefore H(t) = 3 \cos\left(\frac{\pi}{2}t\right) + 4$$

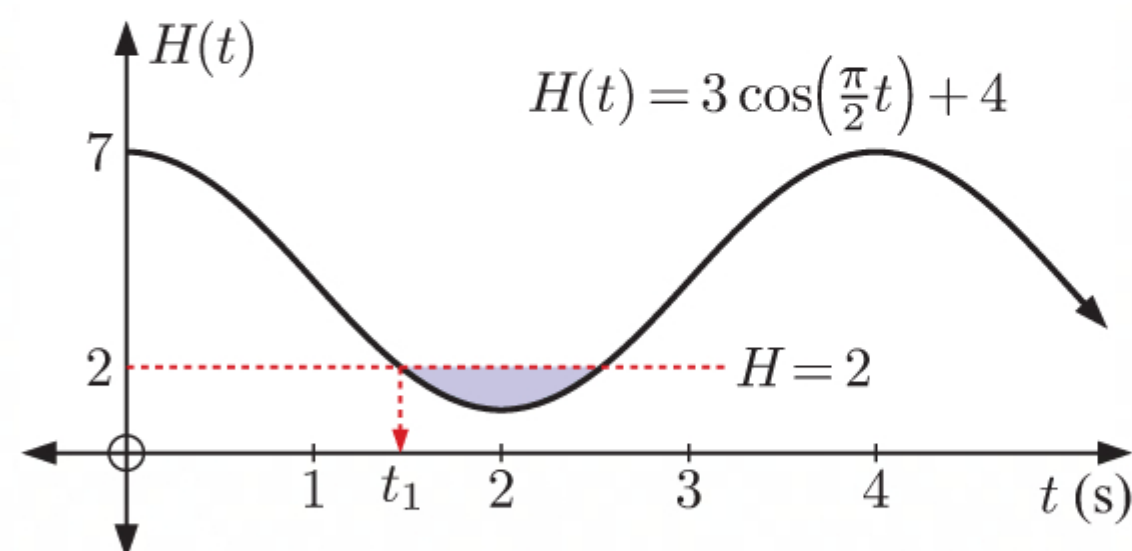


- b X first enters the water when $H(t) = 2$

$$\therefore 3 \cos\left(\frac{\pi}{2}t\right) + 4 = 2$$

Using technology, we obtain $t_1 \approx 1.46$.

X first enters the water after about 1.46 seconds.



EXERCISE 9C.1

1 a $\sin \theta + \sin \theta$
 $= 2 \sin \theta$

d $3 \sin \theta - 2 \sin \theta$
 $= \sin \theta$

2 a $3 \tan x - \frac{\sin x}{\cos x}$
 $= 3 \tan x - \tan x$
 $= 2 \tan x$

d $\frac{\sin x}{\tan x}$
 $= \sin x \div \frac{\sin x}{\cos x}$
 $= \sin x \times \frac{\cos x}{\sin x}$
 $= \cos x$

3 a $3 \cos \theta - \cos(-\theta)$
 $= 3 \cos \theta - \cos \theta$
 $= 2 \cos \theta$

b $2 \cos \theta + \cos \theta$
 $= 3 \cos \theta$

e $\tan \theta - 3 \tan \theta$
 $= -2 \tan \theta$

b $\frac{\sin^2 x}{\cos^2 x}$
 $= \left(\frac{\sin x}{\cos x}\right)^2$
 $= \tan^2 x$

e $3 \sin x + 2 \cos x \tan x$
 $= 3 \sin x + 2 \cos x \times \frac{\sin x}{\cos x}$
 $= 3 \sin x + 2 \sin x$
 $= 5 \sin x$

b $\tan(-\theta)$
 $= \frac{\sin(-\theta)}{\cos(-\theta)}$
 $= \frac{-\sin \theta}{\cos \theta}$
 $= -\tan \theta$

c $3 \sin \theta - \sin \theta$
 $= 2 \sin \theta$

f $2 \cos^2 \theta - 5 \cos^2 \theta$
 $= -3 \cos^2 \theta$

c $\tan x \cos x$
 $= \frac{\sin x}{\cos x} \times \cos x$
 $= \sin x$

f $\frac{2 \tan x}{\sin x}$
 $= 2 \left(\frac{\sin x}{\cos x}\right) \times \frac{1}{\sin x}$
 $= \frac{2}{\cos x}$

c $\sin(-\theta) + \cos\left(\frac{\pi}{2} - \theta\right)$
 $= -\sin \theta + \sin \theta$
 $= 0$

$$\begin{aligned}
 \text{d} \quad & \tan(\pi - \theta) \\
 &= \frac{\sin(\pi - \theta)}{\cos(\pi - \theta)} \\
 &= \frac{\sin \theta}{-\cos \theta} \\
 &= -\tan \theta
 \end{aligned}$$

$$\begin{aligned}
 \text{g} \quad & \frac{\sin(-\theta)}{\cos(\pi - \theta)} \\
 &= \frac{-\sin \theta}{-\cos \theta} \\
 &= \tan \theta
 \end{aligned}$$

$$\begin{aligned}
 \text{e} \quad & \tan\left(\frac{\pi}{2} - \theta\right) \\
 &= \frac{\sin\left(\frac{\pi}{2} - \theta\right)}{\cos\left(\frac{\pi}{2} - \theta\right)} \\
 &= \frac{\cos \theta}{\sin \theta} \\
 &= \frac{1}{\tan \theta}
 \end{aligned}$$

$$\begin{aligned}
 \text{h} \quad & \frac{\cos\left(\frac{\pi}{2} - \theta\right)}{\cos(-\theta)} \\
 &= \frac{\sin \theta}{\cos \theta} \\
 &= \tan \theta
 \end{aligned}$$

$$\begin{aligned}
 \text{f} \quad & \sin\left(\frac{\pi}{2} - \theta\right) - \cos(\pi - \theta) \\
 &= \cos \theta - (-\cos \theta) \\
 &= \cos \theta + \cos \theta \\
 &= 2 \cos \theta
 \end{aligned}$$

$$\begin{aligned}
 \text{i} \quad & \frac{\sin(\pi - \theta) - \sin(-\theta)}{\cos(-\theta)} \\
 &= \frac{\sin \theta - (-\sin \theta)}{\cos \theta} \\
 &= \frac{2 \sin \theta}{\cos \theta} \\
 &= 2 \tan \theta
 \end{aligned}$$

EXERCISE 9C.2

$$\begin{aligned}
 \text{1 a} \quad & 3 \sin^2 \theta + 3 \cos^2 \theta \\
 &= 3(\sin^2 \theta + \cos^2 \theta) \\
 &= 3(1) \\
 &= 3
 \end{aligned}$$

$$\begin{aligned}
 \text{d} \quad & 3 - 3 \sin^2 \theta \\
 &= 3(1 - \sin^2 \theta) \\
 &= 3 \cos^2 \theta
 \end{aligned}$$

$$\begin{aligned}
 \text{g} \quad & \cos^2 \theta - 1 \\
 &= 1 - \sin^2 \theta - 1 \\
 &= -\sin^2 \theta
 \end{aligned}$$

$$\begin{aligned}
 \text{j} \quad & \frac{1 - \sin^2 \theta}{\cos^2 \theta} \\
 &= \frac{\cos^2 \theta}{\cos^2 \theta} \\
 &= 1
 \end{aligned}$$

$$\begin{aligned}
 \text{b} \quad & -2 \sin^2 \theta - 2 \cos^2 \theta \\
 &= -2(\sin^2 \theta + \cos^2 \theta) \\
 &= -2(1) \\
 &= -2
 \end{aligned}$$

$$\begin{aligned}
 \text{e} \quad & 4 - 4 \cos^2 \theta \\
 &= 4(1 - \cos^2 \theta) \\
 &= 4 \sin^2 \theta
 \end{aligned}$$

$$\begin{aligned}
 \text{h} \quad & \sin^2 \theta - 1 \\
 &= 1 - \cos^2 \theta - 1 \\
 &= -\cos^2 \theta
 \end{aligned}$$

$$\begin{aligned}
 \text{k} \quad & \frac{1 - \cos^2 \theta}{\sin \theta} \\
 &= \frac{\sin^2 \theta}{\sin \theta} \\
 &= \sin \theta
 \end{aligned}$$

$$\begin{aligned}
 \text{c} \quad & -\cos^2 \theta - \sin^2 \theta \\
 &= -(\cos^2 \theta + \sin^2 \theta) \\
 &= -(1) \\
 &= -1
 \end{aligned}$$

$$\begin{aligned}
 \text{f} \quad & \cos^3 \theta + \cos \theta \sin^2 \theta \\
 &= \cos \theta (\cos^2 \theta + \sin^2 \theta) \\
 &= \cos \theta (1) \\
 &= \cos \theta
 \end{aligned}$$

$$\begin{aligned}
 \text{i} \quad & 2 \cos^2 \theta - 2 \\
 &= -2(1 - \cos^2 \theta) \\
 &= -2 \sin^2 \theta
 \end{aligned}$$

$$\begin{aligned}
 \text{l} \quad & \frac{\cos^2 \theta - 1}{-\sin \theta} \\
 &= \frac{1 - \sin^2 \theta - 1}{-\sin \theta} \\
 &= \frac{-\sin^2 \theta}{-\sin \theta} \\
 &= \sin \theta
 \end{aligned}$$

$$\begin{aligned}
 \text{2 a} \quad & (1 + \sin \theta)^2 \\
 &= 1 + 2 \sin \theta + \sin^2 \theta
 \end{aligned}$$

$$\begin{aligned}
 \text{b} \quad & (\sin \alpha - 2)^2 \\
 &= \sin^2 \alpha - 4 \sin \alpha + 4
 \end{aligned}$$

$$\begin{aligned}
 \text{c} \quad & (\tan \alpha - 1)^2 \\
 &= \tan^2 \alpha - 2 \tan \alpha + 1 \\
 &= \frac{\sin^2 \alpha}{\cos^2 \alpha} - 2 \tan \alpha + \frac{\cos^2 \alpha}{\cos^2 \alpha} \\
 &= \frac{1}{\cos^2 \alpha} - 2 \tan \alpha
 \end{aligned}$$

$$\begin{aligned}
 \text{e} \quad & (\sin \beta - \cos \beta)^2 \\
 &= \sin^2 \beta - 2 \sin \beta \cos \beta + \cos^2 \beta \\
 &= \sin^2 \beta + \cos^2 \beta - 2 \sin \beta \cos \beta \\
 &= 1 - 2 \sin \beta \cos \beta
 \end{aligned}$$

$$\begin{aligned}
 \text{d} \quad & (\sin \alpha + \cos \alpha)^2 \\
 &= \sin^2 \alpha + 2 \sin \alpha \cos \alpha + \cos^2 \alpha \\
 &= \sin^2 \alpha + \cos^2 \alpha + 2 \sin \alpha \cos \alpha \\
 &= 1 + 2 \sin \alpha \cos \alpha
 \end{aligned}$$

$$\begin{aligned}
 \text{f} \quad & -(2 - \cos \alpha)^2 \\
 &= -(4 - 4 \cos \alpha + \cos^2 \alpha) \\
 &= -4 + 4 \cos \alpha - \cos^2 \alpha
 \end{aligned}$$

$$\begin{aligned}
 3 \quad \text{a} \quad & (\sin x + \tan x)(\sin x - \tan x) = \sin^2 x - \tan^2 x \\
 &= \sin^2 x - \frac{\sin^2 x}{\cos^2 x} \\
 &= \sin^2 x \left(1 - \frac{1}{\cos^2 x} \right) \\
 &= \sin^2 x \left(\frac{\cos^2 x - 1}{\cos^2 x} \right) \\
 &= \sin^2 x \left(\frac{-\sin^2 x}{\cos^2 x} \right) \\
 &= -\sin^2 x \tan^2 x
 \end{aligned}$$

$$\begin{aligned}
 \text{b} \quad & (2 \sin \theta + 3 \cos \theta)^2 + (3 \sin \theta - 2 \cos \theta)^2 \\
 &= 4 \sin^2 \theta + 12 \sin \theta \cos \theta + 9 \cos^2 \theta + 9 \sin^2 \theta - 12 \sin \theta \cos \theta + 4 \cos^2 \theta \\
 &= 13 \sin^2 \theta + 13 \cos^2 \theta \\
 &= 13(\sin^2 \theta + \cos^2 \theta) \\
 &= 13
 \end{aligned}$$

EXERCISE 9C.3

$$\begin{aligned}
 1 \quad \text{a} \quad & 1 - \sin^2 \theta \\
 &= (1 + \sin \theta)(1 - \sin \theta)
 \end{aligned}$$

$$\begin{aligned}
 \text{c} \quad & \tan^2 \alpha - 1 \\
 &= (\tan \alpha + 1)(\tan \alpha - 1)
 \end{aligned}$$

$$\begin{aligned}
 \text{e} \quad & 2 \cos \phi + 3 \cos^2 \phi \\
 &= \cos \phi(2 + 3 \cos \phi)
 \end{aligned}$$

$$\begin{aligned}
 \text{g} \quad & \tan^2 \theta + 5 \tan \theta + 6 \\
 &= (\tan \theta + 2)(\tan \theta + 3)
 \end{aligned}$$

$$\begin{aligned}
 \text{i} \quad & 6 \cos^2 \alpha - \cos \alpha - 1 \\
 &= (3 \cos \alpha + 1)(2 \cos \alpha - 1)
 \end{aligned}$$

$$\begin{aligned}
 \text{k} \quad & 2 \sin^2 x + 7 \sin x \cos x + 3 \cos^2 x \\
 &= (2 \sin x + \cos x)(\sin x + 3 \cos x)
 \end{aligned}$$

$$\begin{aligned}
 \text{b} \quad & \sin^2 \alpha - \cos^2 \alpha \\
 &= (\sin \alpha + \cos \alpha)(\sin \alpha - \cos \alpha)
 \end{aligned}$$

$$\begin{aligned}
 \text{d} \quad & 2 \sin^2 \beta - \sin \beta \\
 &= \sin \beta(2 \sin \beta - 1)
 \end{aligned}$$

$$\begin{aligned}
 \text{f} \quad & 3 \sin^2 \theta - 6 \sin \theta \\
 &= 3 \sin \theta(\sin \theta - 2)
 \end{aligned}$$

$$\begin{aligned}
 \text{h} \quad & 2 \cos^2 \theta + 7 \cos \theta + 3 \\
 &= (2 \cos \theta + 1)(\cos \theta + 3)
 \end{aligned}$$

$$\begin{aligned}
 \text{j} \quad & 3 \tan^2 \alpha - 2 \tan \alpha \\
 &= \tan \alpha(3 \tan \alpha - 2)
 \end{aligned}$$

2 a

$$2 \cos^2 x = \sin x + 1$$

$$\therefore 2(1 - \sin^2 x) = \sin x + 1$$

$$\therefore 2 - 2 \sin^2 x = \sin x + 1$$

$$\therefore 2 \sin^2 x + \sin x - 1 = 0$$

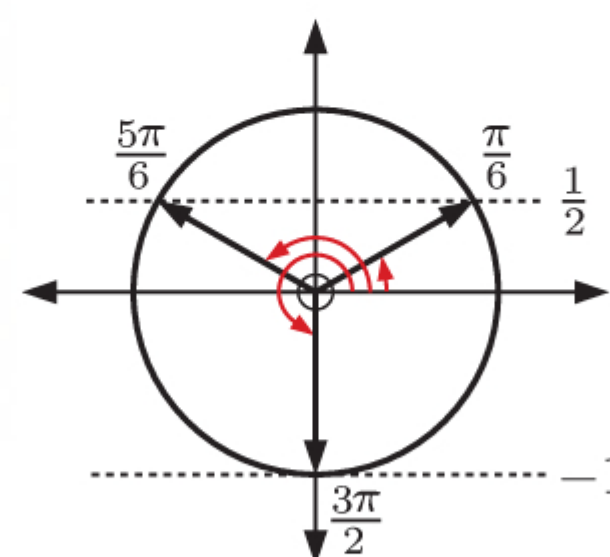
$$\therefore (2 \sin x - 1)(\sin x + 1) = 0$$

$$\therefore 2 \sin x - 1 = 0 \quad \text{or} \quad \sin x + 1 = 0$$

$$\therefore \sin x = \frac{1}{2} \quad \text{or} \quad \sin x = -1$$

On $0 \leq x \leq 2\pi$, the angles with sine $\frac{1}{2}$ are $\frac{\pi}{6}$ and $\frac{5\pi}{6}$,
and the angle with sine -1 is $\frac{3\pi}{2}$.

\therefore the solutions are $x = \frac{\pi}{6}, \frac{5\pi}{6}, \text{ or } \frac{3\pi}{2}$.

**b**

$$\sin^2 x = 2 - \cos x$$

$$\therefore 1 - \cos^2 x = 2 - \cos x$$

$$\therefore \cos^2 x - \cos x = -1$$

$$\therefore \cos^2 x - \cos x + \left(-\frac{1}{2}\right)^2 = -1 + \left(-\frac{1}{2}\right)^2 \quad \{\text{completing the square}\}$$

$$\therefore \left(\cos x - \frac{1}{2}\right)^2 = -\frac{3}{4}$$

which has no real solutions as $\left(\cos x - \frac{1}{2}\right)^2$ cannot be negative.

c

$$2 \cos^2 x = 3 \sin x$$

$$\therefore 2(1 - \sin^2 x) = 3 \sin x$$

$$\therefore 2 - 2 \sin^2 x = 3 \sin x$$

$$\therefore 2 \sin^2 x + 3 \sin x - 2 = 0$$

$$\therefore (2 \sin x - 1)(\sin x + 2) = 0$$

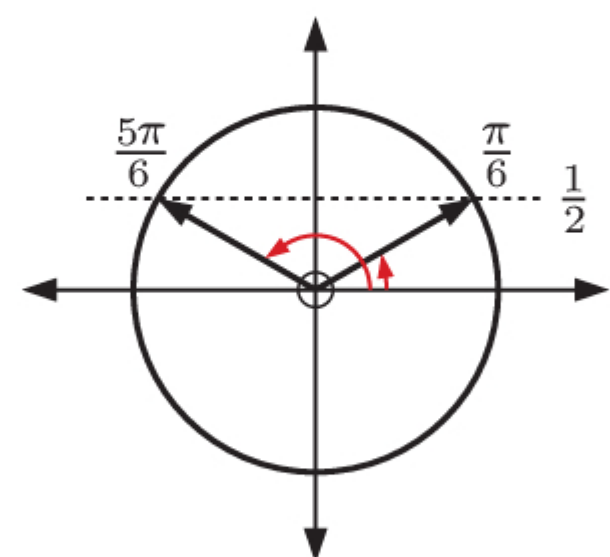
$$\therefore 2 \sin x - 1 = 0 \quad \text{or} \quad \sin x + 2 = 0$$

$$\therefore \sin x = \frac{1}{2} \quad \text{or} \quad \sin x = -2$$

Now $-1 \leq \sin x \leq 1$ for all real values of x , so $\sin x = -2$ has no solutions.

On $0 \leq x \leq 2\pi$, the angles with sine $\frac{1}{2}$ are $\frac{\pi}{6}$ and $\frac{5\pi}{6}$.

\therefore the solutions are $x = \frac{\pi}{6}$ or $\frac{5\pi}{6}$.

**3 a**

$$\begin{aligned} & \frac{1 - \sin^2 \alpha}{1 - \sin \alpha} \\ &= \frac{(1 + \sin \alpha)(1 - \sin \alpha)}{1 - \sin \alpha} \\ &= 1 + \sin \alpha \end{aligned}$$

b

$$\begin{aligned} & \frac{\tan^2 \beta - 1}{\tan \beta + 1} \\ &= \frac{(\tan \beta + 1)(\tan \beta - 1)}{\tan \beta + 1} \\ &= \tan \beta - 1 \end{aligned}$$

$$\begin{aligned}
 \text{c} \quad & \frac{\cos^2 \phi - \sin^2 \phi}{\cos \phi + \sin \phi} \\
 &= \frac{(\cos \phi + \sin \phi)(\cos \phi - \sin \phi)}{\cos \phi + \sin \phi} \\
 &= \cos \phi - \sin \phi
 \end{aligned}$$

$$\begin{aligned}
 \text{e} \quad & \frac{\sin \alpha + \cos \alpha}{\sin^2 \alpha - \cos^2 \alpha} \\
 &= \frac{\sin \alpha + \cos \alpha}{(\sin \alpha + \cos \alpha)(\sin \alpha - \cos \alpha)} \\
 &= \frac{1}{\sin \alpha - \cos \alpha}
 \end{aligned}$$

$$\begin{aligned}
 \text{d} \quad & \frac{\cos^2 \phi - \sin^2 \phi}{\cos \phi - \sin \phi} \\
 &= \frac{(\cos \phi + \sin \phi)(\cos \phi - \sin \phi)}{\cos \phi - \sin \phi} \\
 &= \cos \phi + \sin \phi
 \end{aligned}$$

$$\begin{aligned}
 \text{f} \quad & \frac{3 - 3 \sin^2 \theta}{6 \cos \theta} = \frac{3(1 - \sin^2 \theta)}{6 \cos \theta} \\
 &= \frac{3 \cos^2 \theta}{6 \cos \theta} \\
 &= \frac{\cos \theta}{2}
 \end{aligned}$$

$$\begin{aligned}
 4 \quad \text{a} \quad & (\cos \theta + \sin \theta)^2 + (\cos \theta - \sin \theta)^2 \\
 &= \cos^2 \theta + 2 \cos \theta \sin \theta + \sin^2 \theta + \cos^2 \theta - 2 \cos \theta \sin \theta + \sin^2 \theta \\
 &= 2 \cos^2 \theta + 2 \sin^2 \theta \\
 &= 2(\cos^2 \theta + \sin^2 \theta) \\
 &= 2
 \end{aligned}$$

$$\begin{aligned}
 \text{b} \quad & (\sin \theta + 4 \cos \theta)^2 + (4 \sin \theta - \cos \theta)^2 \\
 &= \sin^2 \theta + 8 \sin \theta \cos \theta + 16 \cos^2 \theta + 16 \sin^2 \theta - 8 \sin \theta \cos \theta + \cos^2 \theta \\
 &= 17 \sin^2 \theta + 17 \cos^2 \theta \\
 &= 17(\sin^2 \theta + \cos^2 \theta) \\
 &= 17
 \end{aligned}$$

$$\begin{aligned}
 \text{c} \quad & (1 - \cos \theta) \left(1 + \frac{1}{\cos \theta} \right) \\
 &= 1 + \frac{1}{\cos \theta} - \cos \theta - 1 \\
 &= \frac{1}{\cos \theta} - \cos \theta \\
 &= \frac{1 - \cos^2 \theta}{\cos \theta} \\
 &= \frac{\sin^2 \theta}{\cos \theta} \\
 &= \tan \theta \sin \theta
 \end{aligned}$$

$$\begin{aligned}
 \text{d} \quad & \left(1 + \frac{1}{\sin \theta} \right) (\sin \theta - \sin^2 \theta) \\
 &= \sin \theta - \sin^2 \theta + 1 - \sin \theta \\
 &= 1 - \sin^2 \theta \\
 &= \cos^2 \theta
 \end{aligned}$$

$$\begin{aligned}
 \text{e} \quad & \frac{\cos \alpha}{1 - \tan \alpha} + \frac{\sin^2 \alpha}{\sin \alpha - \cos \alpha} \\
 &= \frac{\cos \alpha}{1 - \frac{\sin \alpha}{\cos \alpha}} + \frac{\sin^2 \alpha}{\sin \alpha - \cos \alpha} \\
 &= \frac{\cos \alpha}{\frac{\cos \alpha - \sin \alpha}{\cos \alpha}} + \frac{\sin^2 \alpha}{\sin \alpha - \cos \alpha} \\
 &= \frac{\cos^2 \alpha}{\cos \alpha - \sin \alpha} - \frac{\sin^2 \alpha}{\cos \alpha - \sin \alpha} \\
 &= \frac{\cos^2 \alpha - \sin^2 \alpha}{\cos \alpha - \sin \alpha} \\
 &= \frac{(\cos \alpha + \sin \alpha)(\cos \alpha - \sin \alpha)}{\cos \alpha - \sin \alpha} \\
 &= \sin \alpha + \cos \alpha
 \end{aligned}$$

$$\begin{aligned}
 \text{g} \quad & \frac{\sin \theta}{1 - \cos \theta} - \frac{\sin \theta}{1 + \cos \theta} \\
 &= \frac{\sin \theta(1 + \cos \theta) - \sin \theta(1 - \cos \theta)}{(1 - \cos \theta)(1 + \cos \theta)} \\
 &= \frac{\sin \theta + \sin \theta \cos \theta - \sin \theta + \sin \theta \cos \theta}{1 - \cos^2 \theta} \\
 &= \frac{2 \sin \theta \cos \theta}{\sin^2 \theta} \\
 &= \frac{2 \cos \theta}{\sin \theta} \\
 &= \frac{2}{\tan \theta}
 \end{aligned}$$

$$\begin{aligned}
 \text{f} \quad & \frac{\sin \theta}{1 + \cos \theta} + \frac{1 + \cos \theta}{\sin \theta} \\
 &= \frac{\sin^2 \theta + (1 + \cos \theta)^2}{\sin \theta(1 + \cos \theta)} \\
 &= \frac{\sin^2 \theta + 1 + 2 \cos \theta + \cos^2 \theta}{\sin \theta(1 + \cos \theta)} \\
 &= \frac{1 + 1 + 2 \cos \theta}{\sin \theta(1 + \cos \theta)} \\
 &= \frac{2(1 + \cos \theta)}{\sin \theta(1 + \cos \theta)} \\
 &= \frac{2}{\sin \theta}
 \end{aligned}$$

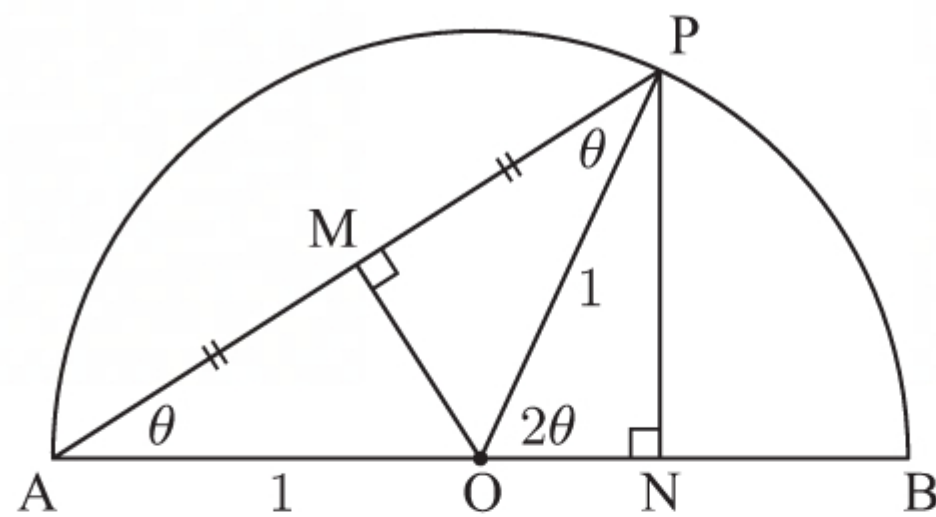
$$\begin{aligned}
 \text{h} \quad & \frac{1}{1 - \sin \theta} + \frac{1}{1 + \sin \theta} \\
 &= \frac{1 + \sin \theta + 1 - \sin \theta}{(1 - \sin \theta)(1 + \sin \theta)} \\
 &= \frac{2}{1 - \sin^2 \theta} \\
 &= \frac{2}{\cos^2 \theta}
 \end{aligned}$$

INVESTIGATION 1**DOUBLE ANGLE IDENTITIES**

θ	$\sin 2\theta$	$2 \sin \theta$	$2 \sin \theta \cos \theta$	$\cos 2\theta$	$2 \cos \theta$	$\cos^2 \theta - \sin^2 \theta$
0.631	0.953	1.180	0.953	0.304	1.615	0.304
57.81°	0.902	1.693	0.902	-0.432	1.065	-0.432
-3.697	-0.896	1.055	-0.896	0.444	-1.699	0.444
1.234	0.624	1.888	0.624	-0.782	0.661	-0.782
2.236	-0.971	1.574	-0.971	-0.238	-1.234	-0.238

2 We can see that $\sin 2\theta = 2 \sin \theta \cos \theta$ and $\cos 2\theta = \cos^2 \theta - \sin^2 \theta$.

$$\begin{array}{ll}
 \text{3 a i} & \sin \theta = \frac{OM}{1} \\
 & = OM \\
 & \therefore OM = \sin \theta \\
 \text{ii} & \cos \theta = \frac{AM}{1} \\
 & = AM \\
 & \therefore AM = \cos \theta \\
 \text{iii} & \cos 2\theta = \frac{ON}{1} \\
 & = ON \\
 & \therefore ON = \cos 2\theta \\
 \text{iv} & \sin 2\theta = \frac{PN}{1} \\
 & = PN \\
 & \therefore PN = \sin 2\theta
 \end{array}$$



$$\begin{array}{l}
 \text{b i In } \triangle ANP, \sin \theta = \frac{PN}{AP} \\
 = \frac{PN}{AM + MP} \\
 = \frac{\sin 2\theta}{\cos \theta + \cos \theta} \\
 \therefore \sin \theta = \frac{\sin 2\theta}{2 \cos \theta} \\
 \therefore \cos \theta = \frac{\sin 2\theta}{2 \sin \theta}
 \end{array}$$

$$\begin{array}{l}
 \text{ii In } \triangle ANP, \cos \theta = \frac{AN}{AP} \\
 = \frac{AO + ON}{AM + MP} \\
 = \frac{1 + \cos 2\theta}{\cos \theta + \cos \theta} \\
 \therefore \cos \theta = \frac{1 + \cos 2\theta}{2 \cos \theta}
 \end{array}$$

$$\begin{array}{l}
 \text{c i From b i, } \cos \theta = \frac{\sin 2\theta}{2 \sin \theta} \\
 \therefore \sin 2\theta = 2 \sin \theta \cos \theta
 \end{array}$$

$$\begin{array}{l}
 \text{ii From b ii, } \cos \theta = \frac{1 + \cos 2\theta}{2 \cos \theta} \\
 \therefore 2 \cos^2 \theta = 1 + \cos 2\theta \\
 \therefore \cos 2\theta = 2 \cos^2 \theta - 1
 \end{array}$$

d Since θ is the size of the base angles of an isosceles triangle, we have proven the identities for $0 \leq \theta \leq \frac{\pi}{2}$.

EXERCISE 9D

$$\begin{array}{l}
 \text{1 a If } \theta = 30^\circ, \sin 2\theta = \sin 60^\circ = \frac{\sqrt{3}}{2} \\
 \text{and } 2 \sin \theta \cos \theta = 2 \times \sin 30^\circ \times \cos 30^\circ \\
 = 2 \times \frac{1}{2} \times \frac{\sqrt{3}}{2} \\
 = \frac{\sqrt{3}}{2}
 \end{array}$$

$$\therefore \sin 2\theta = 2 \sin \theta \cos \theta \text{ for } \theta = 30^\circ.$$

$$\begin{array}{l}
 \text{b If } \theta = 30^\circ, \cos 2\theta = \cos 60^\circ = \frac{1}{2} \\
 \text{and } \cos^2 \theta - \sin^2 \theta = \cos^2 30^\circ - \sin^2 30^\circ \\
 = \left(\frac{\sqrt{3}}{2}\right)^2 - \left(\frac{1}{2}\right)^2 \\
 = \frac{3}{4} - \frac{1}{4} \\
 = \frac{1}{2}
 \end{array}$$

$$\therefore \cos 2\theta = \cos^2 \theta - \sin^2 \theta \text{ for } \theta = 30^\circ.$$

2 $\sin \theta = \frac{4}{5}$ and $\cos \theta = \frac{3}{5}$

a $\sin 2\theta = 2 \sin \theta \cos \theta$
 $= 2\left(\frac{4}{5}\right)\left(\frac{3}{5}\right)$
 $= \frac{24}{25}$

b $\cos 2\theta = \cos^2 \theta - \sin^2 \theta$
 $= \left(\frac{3}{5}\right)^2 - \left(\frac{4}{5}\right)^2$
 $= \frac{9}{25} - \frac{16}{25}$
 $= -\frac{7}{25}$

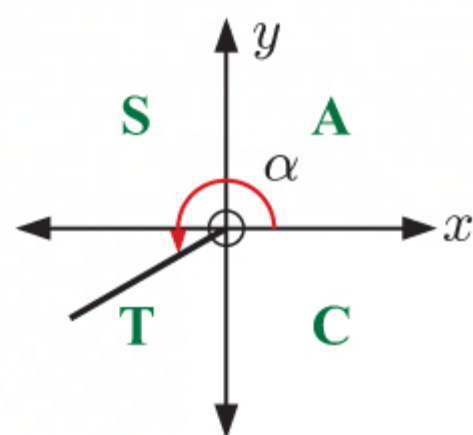
c $\tan 2\theta = \frac{\sin 2\theta}{\cos 2\theta}$
 $= \frac{\frac{24}{25}}{-\frac{7}{25}}$
 {using **a** and **b**}
 $= -\frac{24}{7}$

3 a If $\cos A = \frac{1}{3}$, $\cos 2A = 2 \cos^2 A - 1$
 $= 2\left(\frac{1}{3}\right)^2 - 1$
 $= 2 \times \frac{1}{9} - 1$
 $= \frac{2}{9} - 1$
 $= -\frac{7}{9}$

b If $\sin \phi = -\frac{2}{3}$, $\cos 2\phi = 1 - 2 \sin^2 \phi$
 $= 1 - 2\left(-\frac{2}{3}\right)^2$
 $= 1 - 2\left(\frac{4}{9}\right)$
 $= 1 - \frac{8}{9}$
 $= \frac{1}{9}$

4 $\sin \alpha = -\frac{2}{3}$ and $\pi < \alpha < \frac{3\pi}{2}$

a α is in quadrant 3, so $\cos \alpha$ is negative.

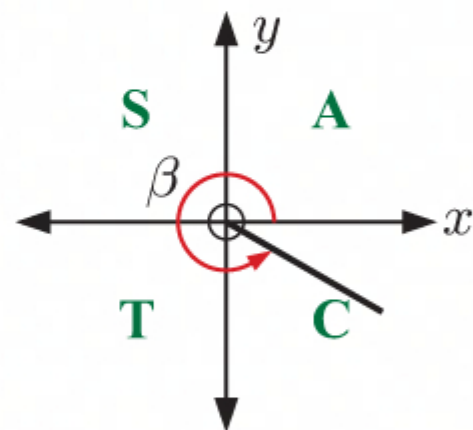


Now $\cos^2 \alpha + \sin^2 \alpha = 1$
 $\therefore \cos^2 \alpha + \frac{4}{9} = 1$
 $\therefore \cos^2 \alpha = \frac{5}{9}$
 $\therefore \cos \alpha = -\frac{\sqrt{5}}{3}$ { $\cos \alpha < 0$ }

b Using the double angle identity, $\sin 2\alpha = 2 \sin \alpha \cos \alpha$
 $= 2\left(-\frac{2}{3}\right)\left(-\frac{\sqrt{5}}{3}\right)$ {using **a**}
 $= \frac{4\sqrt{5}}{9}$

5 $\cos \beta = \frac{2}{5}$ and $270^\circ < \beta < 360^\circ$

a β is in quadrant 4, so $\sin \beta$ is negative.



Now $\cos^2 \beta + \sin^2 \beta = 1$
 $\therefore \frac{4}{25} + \sin^2 \beta = 1$
 $\therefore \sin^2 \beta = \frac{21}{25}$
 $\therefore \sin \beta = -\frac{\sqrt{21}}{5}$ { $\sin \beta < 0$ }

b Using the double angle identity, $\sin 2\beta = 2 \sin \beta \cos \beta$
 $= 2\left(-\frac{\sqrt{21}}{5}\right)\left(\frac{2}{5}\right)$ {using **a**}
 $= -\frac{4\sqrt{21}}{25}$

6 α is acute and $\cos 2\alpha = -\frac{7}{9}$ \therefore $\cos \alpha$ and $\sin \alpha$ are positive

a $\cos 2\alpha = 2\cos^2 \alpha - 1$

$$\therefore -\frac{7}{9} = 2\cos^2 \alpha - 1$$

$$\therefore 2\cos^2 \alpha = \frac{2}{9}$$

$$\therefore \cos^2 \alpha = \frac{1}{9}$$

$$\therefore \cos \alpha = \frac{1}{3} \quad \{\text{since } \cos \alpha > 0\}$$

b $\sin^2 \alpha = 1 - \cos^2 \alpha$

$$\therefore \sin \alpha = \sqrt{1 - \cos^2 \alpha} \quad \{\text{since } \sin \alpha > 0\}$$

$$= \sqrt{1 - \frac{1}{9}} \quad \{\text{using a}\}$$

$$= \sqrt{\frac{8}{9}}$$

$$= \frac{2\sqrt{2}}{3}$$

7 θ is obtuse and $\cos 2\theta = -\frac{1}{3}$ \therefore $\cos \theta$ is negative and $\sin \theta$ is positive

a $\cos 2\theta = 2\cos^2 \theta - 1$

$$\therefore -\frac{1}{3} = 2\cos^2 \theta - 1$$

$$\therefore 2\cos^2 \theta = \frac{2}{3}$$

$$\therefore \cos^2 \theta = \frac{1}{3}$$

$$\therefore \cos \theta = -\frac{1}{\sqrt{3}} \quad \{\text{since } \cos \theta < 0\}$$

b $\sin^2 \theta = 1 - \cos^2 \theta$

$$\therefore \sin \theta = \sqrt{1 - \cos^2 \theta} \quad \{\text{since } \sin \theta > 0\}$$

$$= \sqrt{1 - \frac{1}{3}} \quad \{\text{using a}\}$$

$$= \sqrt{\frac{2}{3}} = \frac{\sqrt{2}}{\sqrt{3}}$$

8 $\tan 2\theta = \frac{\sin 2\theta}{\cos 2\theta}$

$$= \frac{2\sin \theta \cos \theta}{\cos^2 \theta - \sin^2 \theta}$$

$$= \frac{\frac{2\sin \theta \cos \theta}{\cos^2 \theta}}{\frac{\cos^2 \theta - \sin^2 \theta}{\cos^2 \theta}}$$

$$= \frac{\frac{2\sin \theta}{\cos \theta}}{1 - \frac{\sin^2 \theta}{\cos^2 \theta}}$$

$$= \frac{2\tan \theta}{1 - \tan^2 \theta}$$

9 a $2\sin \alpha \cos \alpha$
 $= \sin 2\alpha$

b $4\cos \alpha \sin \alpha$
 $= 2(2\sin \alpha \cos \alpha)$
 $= 2\sin 2\alpha$

c $\sin \alpha \cos \alpha$
 $= \frac{1}{2}(2\sin \alpha \cos \alpha)$
 $= \frac{1}{2}\sin 2\alpha$

d $2\cos^2 \beta - 1$
 $= \cos 2\beta$

e $1 - 2\cos^2 \phi$
 $= -(2\cos^2 \phi - 1)$
 $= -\cos 2\phi$

f $1 - 2\sin^2 N$
 $= \cos 2N$

g $2\sin^2 M - 1$
 $= -(1 - 2\sin^2 M)$
 $= -\cos 2M$

h $\cos^2 \alpha - \sin^2 \alpha$
 $= \cos 2\alpha$

i $\sin^2 \alpha - \cos^2 \alpha$
 $= -(\cos^2 \alpha - \sin^2 \alpha)$
 $= -\cos 2\alpha$

j $2\sin 2A \cos 2A$
 $= \sin[2(2A)]$
 $= \sin 4A$

k $2\cos 3\alpha \sin 3\alpha$
 $= \sin[2(3\alpha)]$
 $= \sin 6\alpha$

l $2\cos^2 4\theta - 1$
 $= \cos[2(4\theta)]$
 $= \cos 8\theta$

$$\begin{aligned}
 \text{m} \quad & 1 - 2\cos^2 3\beta \\
 &= -(2\cos^2 3\beta - 1) \\
 &= -\cos[2(3\beta)] \\
 &= -\cos 6\beta
 \end{aligned}$$

$$\begin{aligned}
 \text{n} \quad & 1 - 2\sin^2 5\alpha \\
 &= \cos[2(5\alpha)] \\
 &= \cos 10\alpha
 \end{aligned}$$

$$\begin{aligned}
 \text{o} \quad & 2\sin^2 3D - 1 \\
 &= -(1 - 2\sin^2 3D) \\
 &= -\cos[2(3D)] \\
 &= -\cos 6D
 \end{aligned}$$

$$\begin{aligned}
 \text{p} \quad & \cos^2 2A - \sin^2 2A \\
 &= \cos[2(2A)] \\
 &= \cos 4A
 \end{aligned}$$

$$\begin{aligned}
 \text{q} \quad & \cos^2\left(\frac{\alpha}{2}\right) - \sin^2\left(\frac{\alpha}{2}\right) \\
 &= \cos\left[2\left(\frac{\alpha}{2}\right)\right] \\
 &= \cos \alpha
 \end{aligned}$$

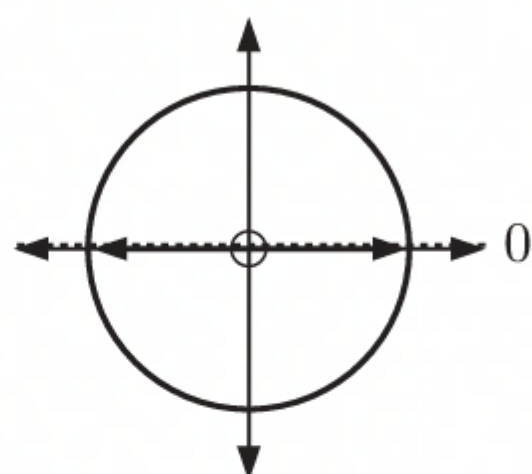
$$\begin{aligned}
 \text{r} \quad & 2\sin^2 3P - 2\cos^2 3P \\
 &= -2(\cos^2 3P - \sin^2 3P) \\
 &= -2\cos[2(3P)] \\
 &= -2\cos 6P
 \end{aligned}$$

$$\begin{aligned}
 10 \quad & \left[\cos \frac{\pi}{12} + \sin \frac{\pi}{12}\right]^2 \\
 &= \cos^2\left(\frac{\pi}{12}\right) + 2\cos \frac{\pi}{12} \sin \frac{\pi}{12} + \sin^2\left(\frac{\pi}{12}\right) \\
 &= 1 + 2\cos \frac{\pi}{12} \sin \frac{\pi}{12} \\
 &= 1 + \sin \frac{\pi}{6} \quad \{2\cos A \sin A = \sin 2A\} \\
 &= 1 + \frac{1}{2} \\
 &= \frac{3}{2}
 \end{aligned}$$

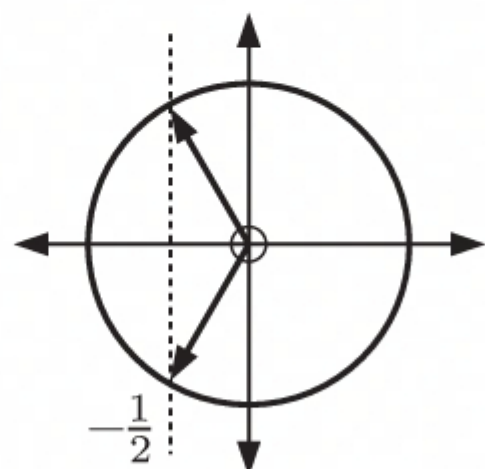
$$\begin{aligned}
 11 \quad \text{a} \quad & (\sin \theta + \cos \theta)^2 \\
 &= \sin^2 \theta + 2\sin \theta \cos \theta + \cos^2 \theta \\
 &= \sin^2 \theta + \cos^2 \theta + 2\sin \theta \cos \theta \\
 &= 1 + \sin 2\theta
 \end{aligned}$$

$$\begin{aligned}
 \text{b} \quad & \cos^4 \theta - \sin^4 \theta \\
 &= (\cos^2 \theta + \sin^2 \theta)(\cos^2 \theta - \sin^2 \theta) \\
 &= 1 \times \cos 2\theta \\
 &= \cos 2\theta
 \end{aligned}$$

$$\begin{aligned}
 12 \quad \text{a} \quad & \sin 2x + \sin x = 0 \\
 \therefore & 2\sin x \cos x + \sin x = 0 \\
 \therefore & \sin x(2\cos x + 1) = 0 \\
 \therefore & \sin x = 0 \quad \text{or} \quad \cos x = -\frac{1}{2}
 \end{aligned}$$



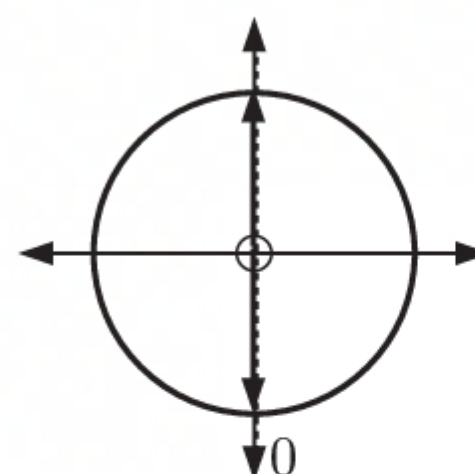
$$\begin{aligned}
 \sin x = 0 \quad \text{when} \\
 x = 0, \pi, \text{ or } 2\pi \\
 \{0 \leq x \leq 2\pi\}
 \end{aligned}$$



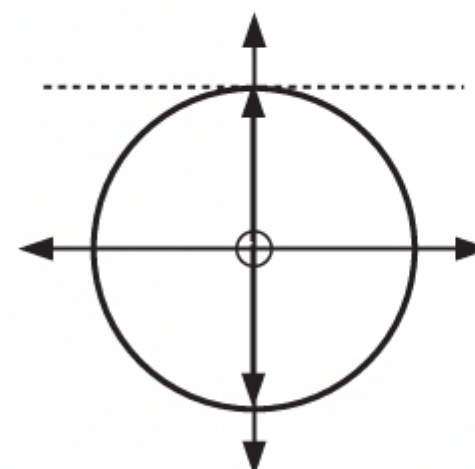
$$\begin{aligned}
 \cos x = -\frac{1}{2} \quad \text{when} \\
 x = \frac{2\pi}{3} \text{ or } \frac{4\pi}{3} \\
 \{0 \leq x \leq 2\pi\}
 \end{aligned}$$

$$\therefore x = 0, \frac{2\pi}{3}, \pi, \frac{4\pi}{3}, 2\pi$$

$$\begin{aligned}
 \text{b} \quad & \sin 2x - 2\cos x = 0 \\
 \therefore & 2\sin x \cos x - 2\cos x = 0 \\
 \therefore & 2\cos x(\sin x - 1) = 0 \\
 \therefore & \cos x = 0 \quad \text{or} \quad \sin x = 1
 \end{aligned}$$



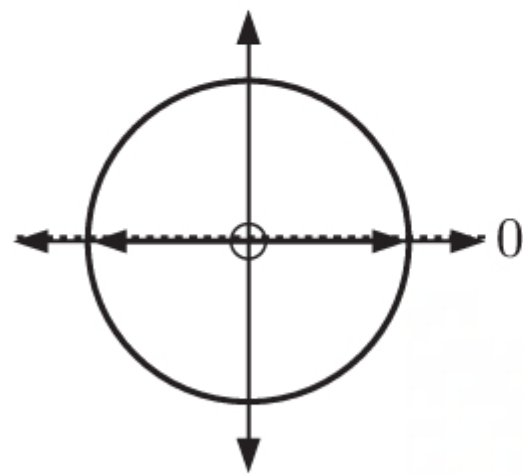
$$\begin{aligned}
 \cos x = 0 \quad \text{when} \\
 x = \frac{\pi}{2} \text{ or } \frac{3\pi}{2} \\
 \{0 \leq x \leq 2\pi\}
 \end{aligned}$$



$$\begin{aligned}
 \sin x = 1 \quad \text{when} \\
 x = \frac{\pi}{2} \\
 \{0 \leq x \leq 2\pi\}
 \end{aligned}$$

$$\therefore x = \frac{\pi}{2}, \frac{3\pi}{2}$$

$$\begin{aligned}
 \sin 2x + 3 \sin x &= 0 \\
 \therefore 2 \sin x \cos x + 3 \sin x &= 0 \\
 \therefore \sin x(2 \cos x + 3) &= 0 \\
 \therefore \sin x = 0 \quad \{-1 \leq \cos x \leq 1\}
 \end{aligned}$$



$$\therefore x = 0, \pi, 2\pi$$

$$\begin{aligned}
 \text{13 a} \quad \frac{1}{2} - \frac{1}{2} \cos 2\theta & \\
 &= \frac{1}{2} - \frac{1}{2}(1 - 2 \sin^2 \theta) \\
 &= \frac{1}{2} - \frac{1}{2} + \sin^2 \theta \\
 &= \sin^2 \theta
 \end{aligned}$$

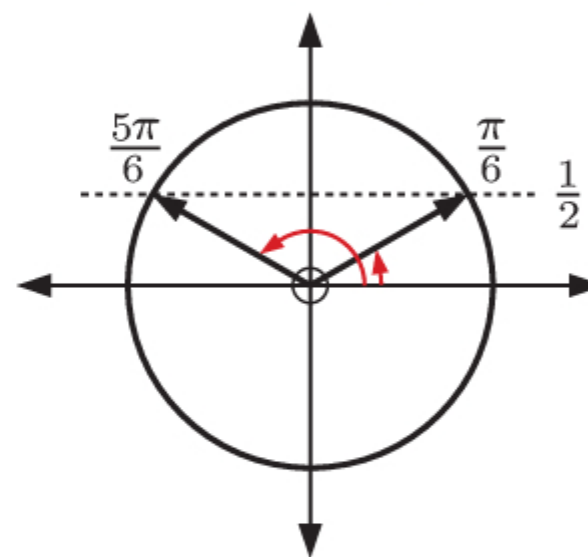
$$\begin{aligned}
 \text{b} \quad \frac{1}{2} + \frac{1}{2} \cos 2\theta & \\
 &= \frac{1}{2} + \frac{1}{2}(2 \cos^2 \theta - 1) \\
 &= \frac{1}{2} + \cos^2 \theta - \frac{1}{2} \\
 &= \cos^2 \theta
 \end{aligned}$$

$$\begin{aligned}
 \text{14} \quad \sin \theta \cos \theta &= \frac{1}{4}, \quad -\pi \leq \theta \leq \pi \\
 \therefore \frac{1}{2}(2 \sin \theta \cos \theta) &= \frac{1}{4} \\
 \therefore \frac{1}{2} \sin 2\theta &= \frac{1}{4} \\
 \therefore \sin 2\theta &= \frac{1}{2}
 \end{aligned}$$

There are two points on the unit circle with sine $\frac{1}{2}$.
They correspond to angles $\frac{\pi}{6}$ and $\frac{5\pi}{6}$.

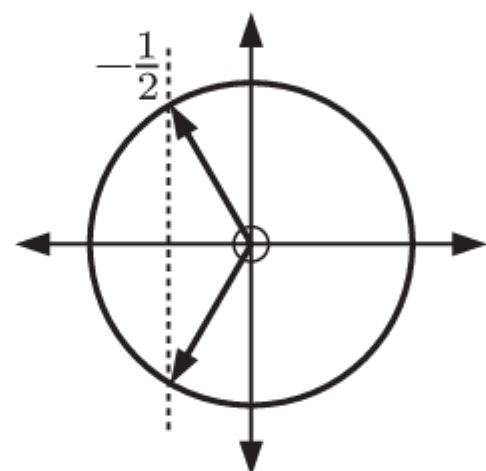
$$\begin{aligned}
 \text{Since } -\pi \leq \theta \leq \pi \\
 \therefore -2\pi \leq 2\theta \leq 2\pi
 \end{aligned}$$

$$\begin{aligned}
 \text{So, } 2\theta &= -\frac{11\pi}{6}, -\frac{7\pi}{6}, \frac{\pi}{6}, \text{ or } \frac{5\pi}{6} \\
 \therefore \theta &= -\frac{11\pi}{12}, -\frac{7\pi}{12}, \frac{\pi}{12}, \text{ or } \frac{5\pi}{12}
 \end{aligned}$$

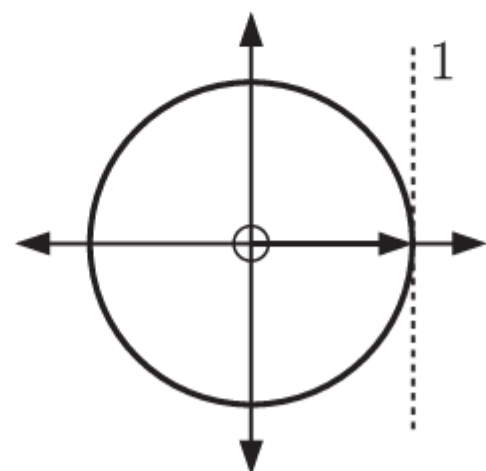


15 a

$$\begin{aligned}
 \cos 2x - \cos x &= 0 \\
 \therefore (2\cos^2 x - 1) - \cos x &= 0 \\
 \therefore 2\cos^2 x - \cos x - 1 &= 0 \\
 \therefore (2\cos x + 1)(\cos x - 1) &= 0 \\
 \therefore \cos x &= -\frac{1}{2} \text{ or } 1
 \end{aligned}$$



$$\begin{aligned}
 \cos x &= -\frac{1}{2} \text{ when} \\
 x &= \frac{2\pi}{3} \text{ or } \frac{4\pi}{3} \\
 \{0 \leq x \leq 2\pi\}
 \end{aligned}$$

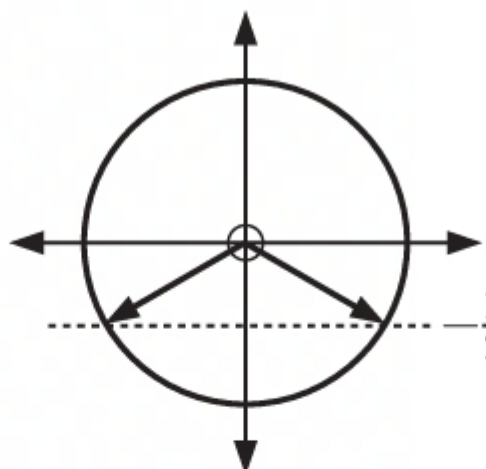


$$\begin{aligned}
 \cos x &= 1 \text{ when} \\
 x &= 0 \text{ or } 2\pi \\
 \{0 \leq x \leq 2\pi\}
 \end{aligned}$$

$$\therefore x = 0, \frac{2\pi}{3}, \frac{4\pi}{3}, \text{ or } 2\pi$$

c

$$\begin{aligned}
 \cos 2x + \sin x &= 0 \\
 \therefore (1 - 2\sin^2 x) + \sin x &= 0 \\
 \therefore -2\sin^2 x + \sin x + 1 &= 0 \\
 \therefore 2\sin^2 x - \sin x - 1 &= 0 \\
 \therefore (2\sin x + 1)(\sin x - 1) &= 0 \\
 \therefore \sin x &= -\frac{1}{2} \text{ or } 1
 \end{aligned}$$

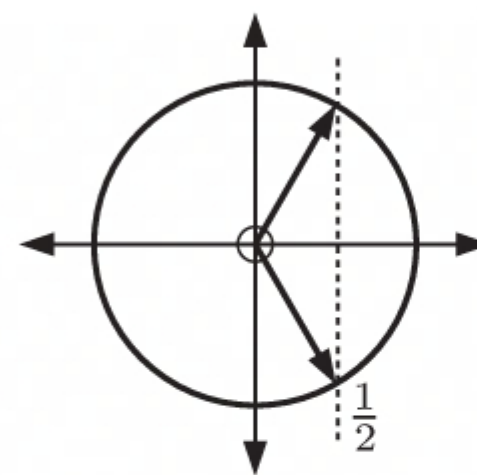


$$\begin{aligned}
 \sin x &= -\frac{1}{2} \text{ when} \\
 x &= \frac{7\pi}{6} \text{ or } \frac{11\pi}{6} \\
 \{0 \leq x \leq 2\pi\}
 \end{aligned}$$

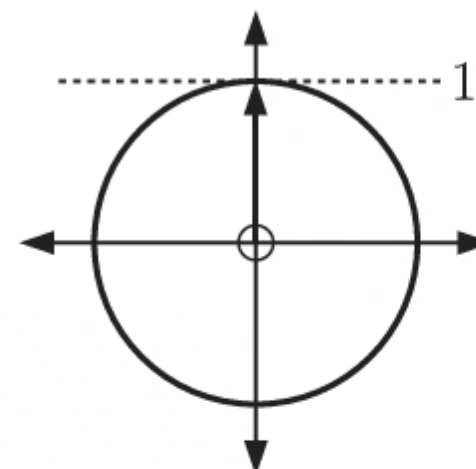
$$\therefore x = \frac{\pi}{2}, \frac{7\pi}{6}, \text{ or } \frac{11\pi}{6}$$

b

$$\begin{aligned}
 \cos 2x + 3\cos x &= 1 \\
 \therefore (2\cos^2 x - 1) + 3\cos x &= 1 \\
 \therefore 2\cos^2 x + 3\cos x - 2 &= 0 \\
 \therefore (2\cos x - 1)(\cos x + 2) &= 0 \\
 \therefore \cos x &= \frac{1}{2} \\
 \{-1 \leq \cos x \leq 1\}
 \end{aligned}$$

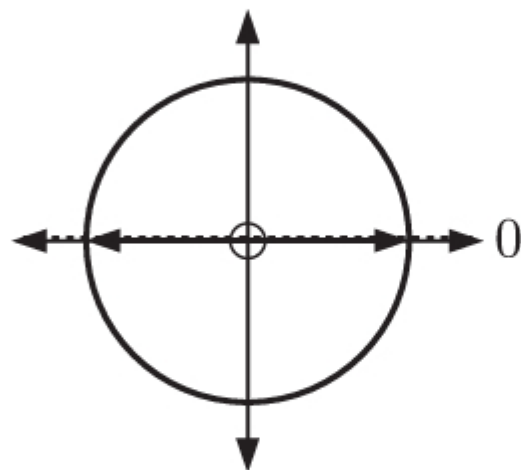


$$\therefore x = \frac{\pi}{3} \text{ or } \frac{5\pi}{3}$$

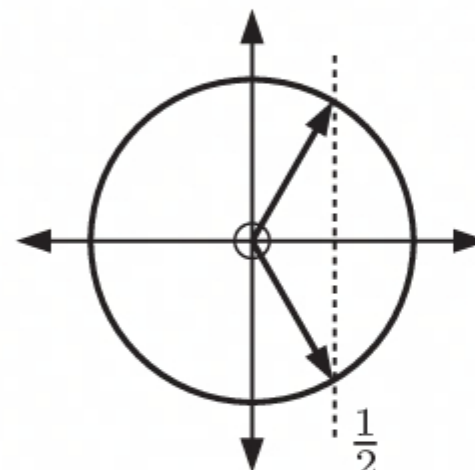


$$\begin{aligned}
 \sin x &= 1 \text{ when} \\
 x &= \frac{\pi}{2} \\
 \{0 \leq x \leq 2\pi\}
 \end{aligned}$$

d $\sin 4x = \sin 2x$
 $\therefore 2 \sin 2x \cos 2x = \sin 2x$
 $\therefore 2 \sin 2x \cos 2x - \sin 2x = 0$
 $\therefore \sin 2x(2 \cos 2x - 1) = 0$
 $\therefore \sin 2x = 0 \quad \text{or} \quad \cos 2x = \frac{1}{2}$



$\sin 2x = 0$ when
 $2x = 0, \pi, 2\pi, 3\pi,$
 or 4π
 $\{0 \leq 2x \leq 4\pi\}$



$\cos 2x = \frac{1}{2}$ when
 $2x = \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3},$
 or $\frac{11\pi}{3}$
 $\{0 \leq 2x \leq 4\pi\}$

$\therefore 2x = 0, \frac{\pi}{3}, \pi, \frac{5\pi}{3}, 2\pi, \frac{7\pi}{3}, 3\pi, \frac{11\pi}{3}, \text{ or } 4\pi$
 $\therefore x = 0, \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}, \pi, \frac{7\pi}{6}, \frac{3\pi}{2}, \frac{11\pi}{6}, \text{ or } 2\pi$

e $\sin x + \cos x = \sqrt{2}$
 Squaring both sides gives:

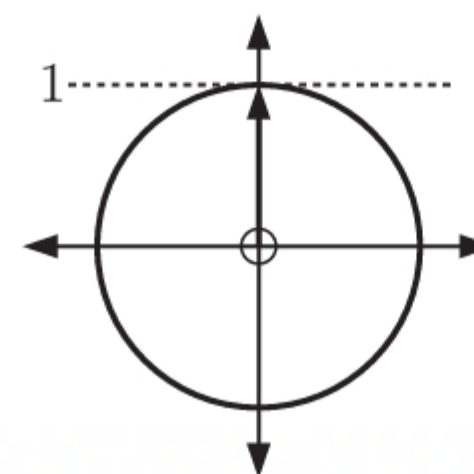
$$\sin^2 x + 2 \sin x \cos x + \cos^2 x = 2$$

$$\therefore \sin 2x + 1 = 2$$

$$\therefore \sin 2x = 1$$

$$\therefore 2x = \frac{\pi}{2} \text{ or } \frac{5\pi}{2} \quad \{0 \leq 2x \leq 4\pi\}$$

$$\therefore x = \frac{\pi}{4} \text{ or } \frac{5\pi}{4}$$



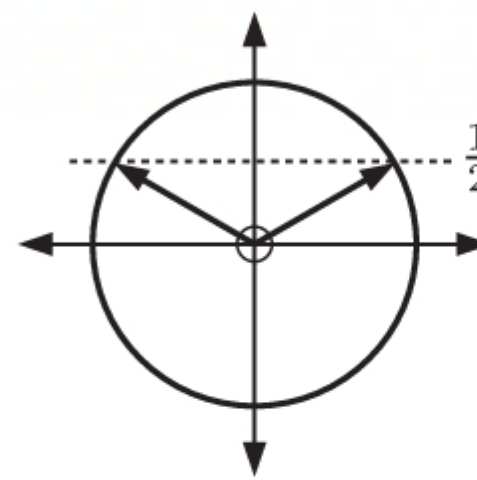
Since we squared the original equation, we must check our answers.

$$\sin \frac{\pi}{4} + \cos \frac{\pi}{4} = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \frac{2}{\sqrt{2}} = \sqrt{2} \quad \checkmark$$

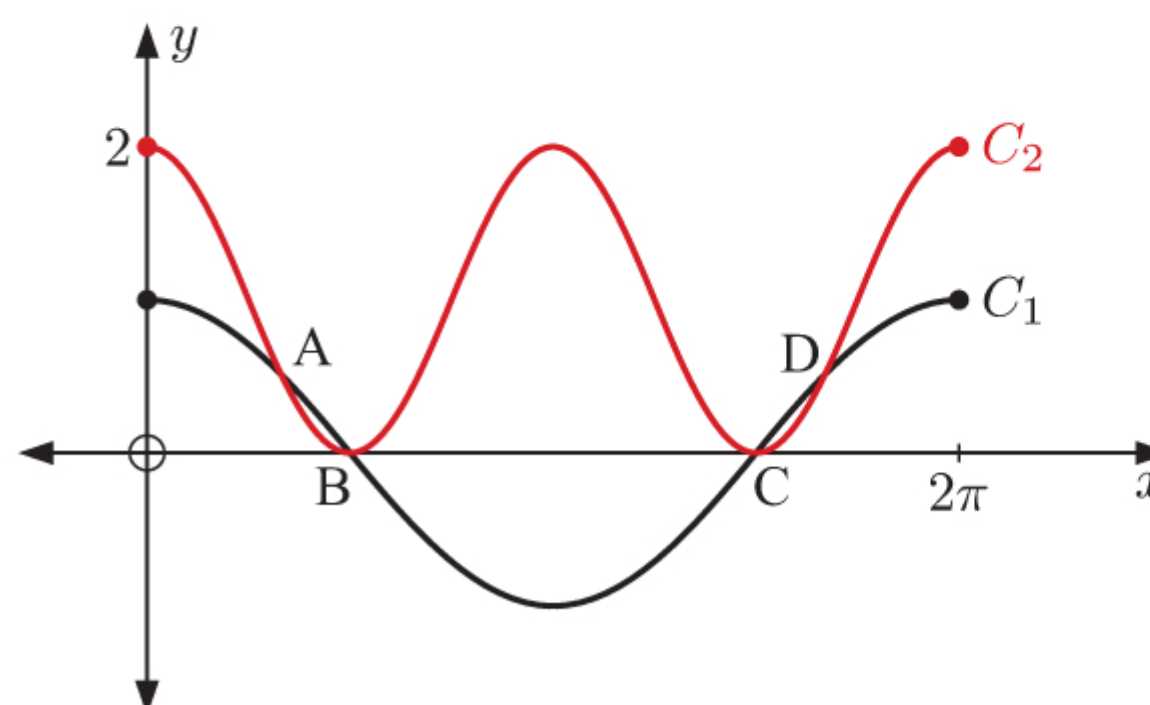
$$\sin \frac{5\pi}{4} + \cos \frac{5\pi}{4} = -\frac{1}{\sqrt{2}} + \left(-\frac{1}{\sqrt{2}}\right) = -\frac{2}{\sqrt{2}} = -\sqrt{2} \quad \times$$

$$\therefore x = \frac{\pi}{4} \text{ is the only solution}$$

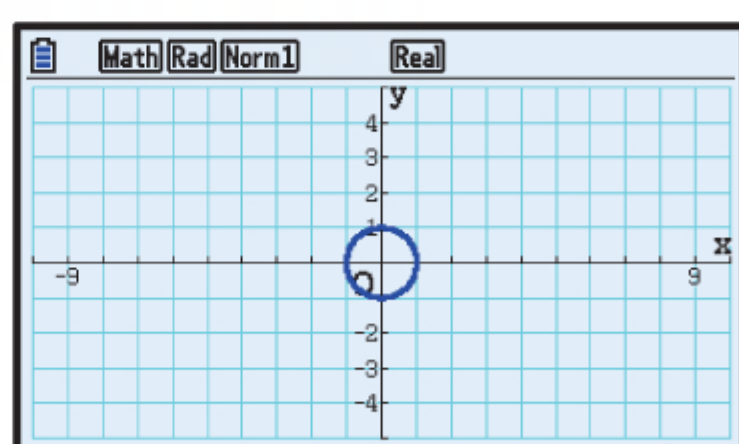
f $2 \cos^2 x = 3 \sin x$
 $\therefore 2(1 - \sin^2 x) = 3 \sin x$
 $\therefore 2 \sin^2 x + 3 \sin x - 2 = 0$
 $\therefore (2 \sin x - 1)(\sin x + 2) = 0$
 $\therefore \sin x = \frac{1}{2} \quad \{-1 \leq \sin x \leq 1\}$
 $\therefore x = \frac{\pi}{6} \text{ or } \frac{5\pi}{6} \quad \{0 \leq x \leq 2\pi\}$



16 a $-1 \leq \cos 2x \leq 1$
 $\therefore 0 \leq \cos 2x + 1 \leq 2$
 So, the graph of $y = \cos 2x + 1$ lies entirely on or above the x -axis.
 $\therefore C_2$ is $y = \cos 2x + 1$
 and C_1 is $y = \cos x$



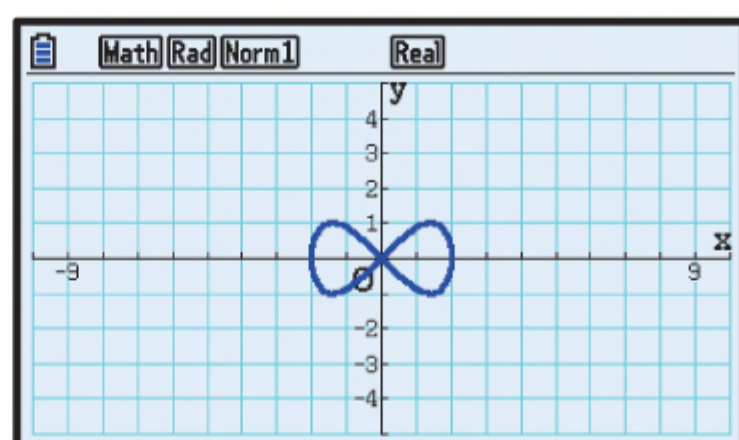
- b** We solve $\cos x = \cos 2x + 1$, for $0 \leq x \leq 2\pi$
- $$\therefore \cos x = 2\cos^2 x - 1 + 1$$
- $$\therefore 2\cos^2 x - \cos x = 0$$
- $$\therefore \cos x(2\cos x - 1) = 0$$
- $$\therefore \cos x = 0 \quad \text{or} \quad \cos x = \frac{1}{2}$$
- $$\therefore x = \frac{\pi}{2}, \frac{3\pi}{2} \quad \text{or} \quad x = \frac{\pi}{3}, \frac{5\pi}{3}$$
- \therefore A is $(\frac{\pi}{3}, \frac{1}{2})$, B is $(\frac{\pi}{2}, 0)$, C is $(\frac{3\pi}{2}, 0)$, and D is $(\frac{5\pi}{3}, \frac{1}{2})$.

INVESTIGATION 2**PARAMETRIC EQUATIONS****1 a**

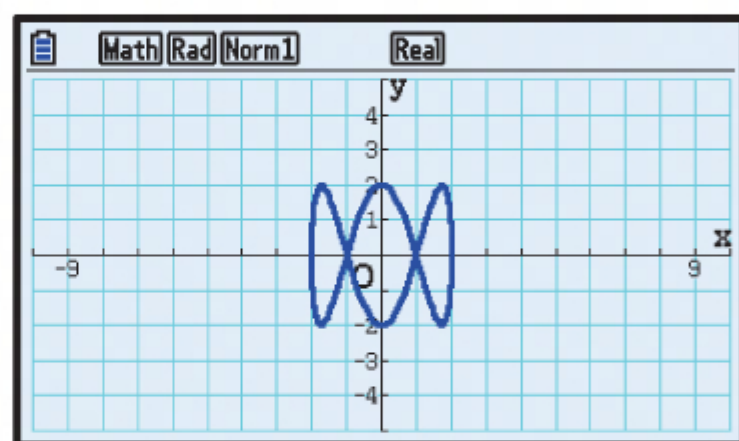
$$\{(x, y) \mid x = \cos t, y = \sin t, 0 \leq t \leq 2\pi\}$$

- b** The graph is not a function since the graph does not pass the vertical line test.

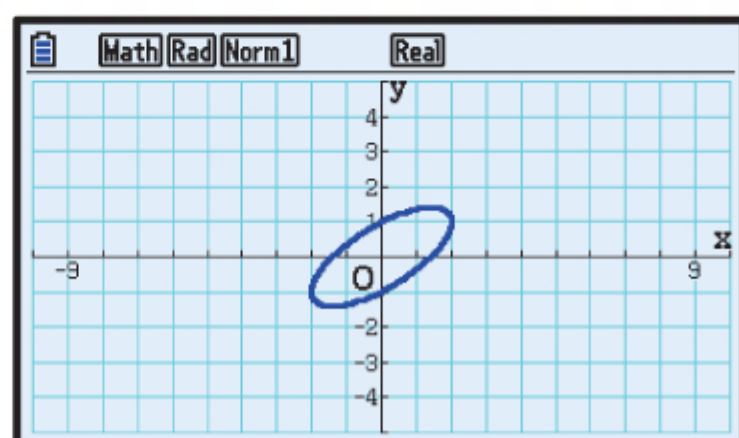
c $x^2 + y^2 = \cos^2 t + \sin^2 t$
 $\therefore x^2 + y^2 = 1$

2 a

$$\{(x, y) \mid x = 2 \cos t, y = \sin 2t, 0 \leq t \leq 2\pi\}$$

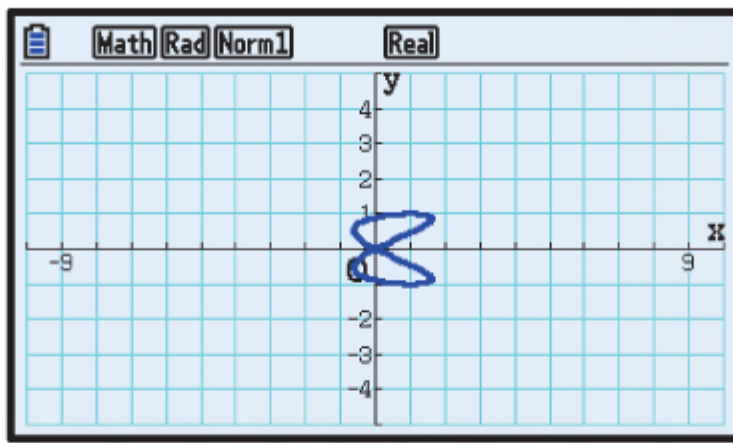
b

$$\{(x, y) \mid x = 2 \cos t, y = 2 \sin 3t, 0 \leq t \leq 2\pi\}$$

c

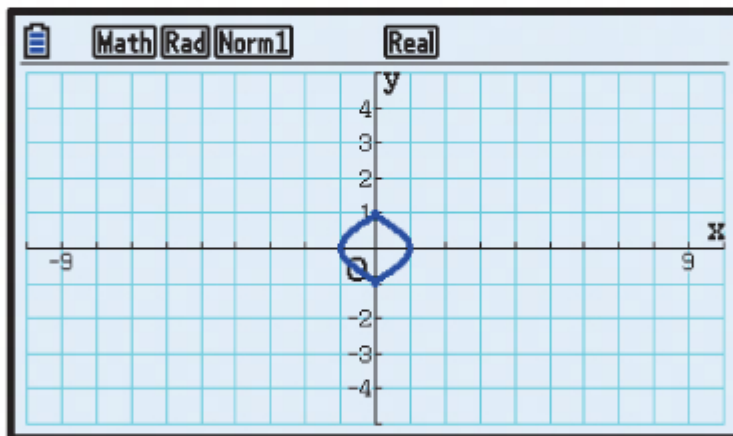
$$\{(x, y) \mid x = 2 \cos t, y = \cos t - \sin t, 0 \leq t \leq 2\pi\}$$

d



$$\{(x, y) \mid x = \cos^2 t + \sin 2t, y = \cos t, 0 \leq t \leq 2\pi\}$$

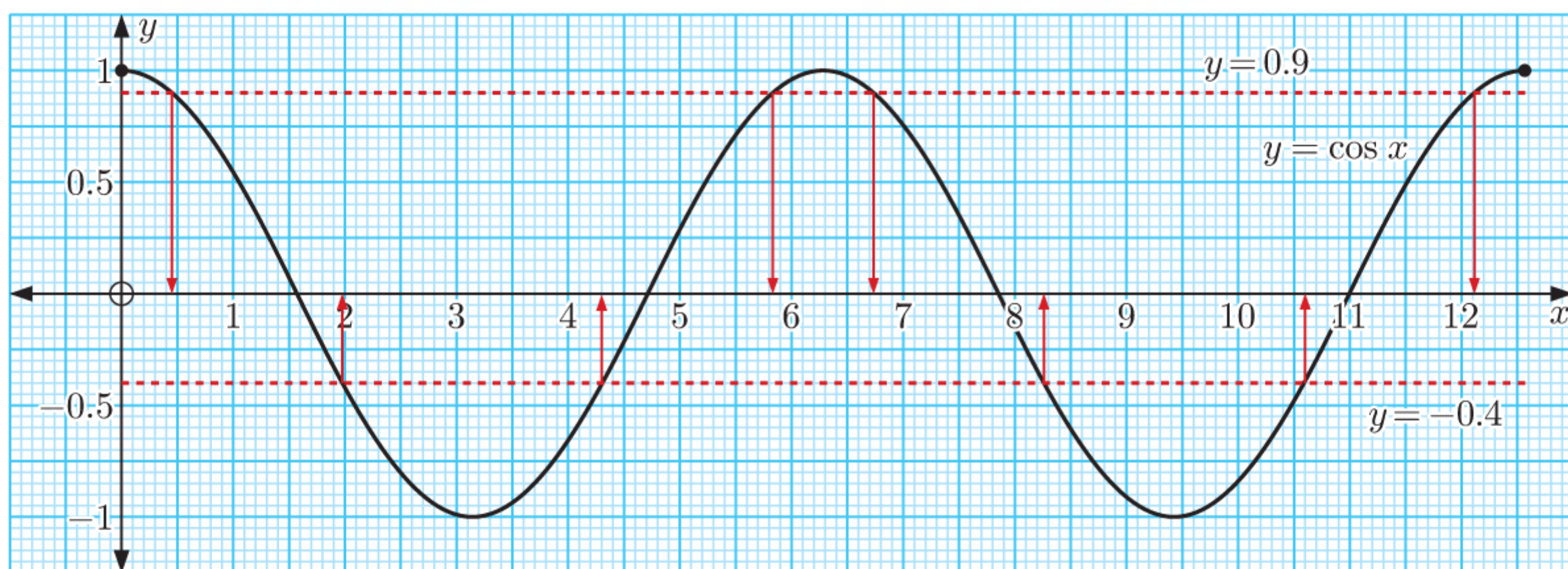
e



$$\{(x, y) \mid x = \cos^3 t, y = \sin t, 0 \leq t \leq 2\pi\}$$

REVIEW SET 9A

1



a When $\cos x = -0.4$, $0 \leq x \leq 4\pi$, $x \approx 2.0, 4.3, 8.3, 10.6$

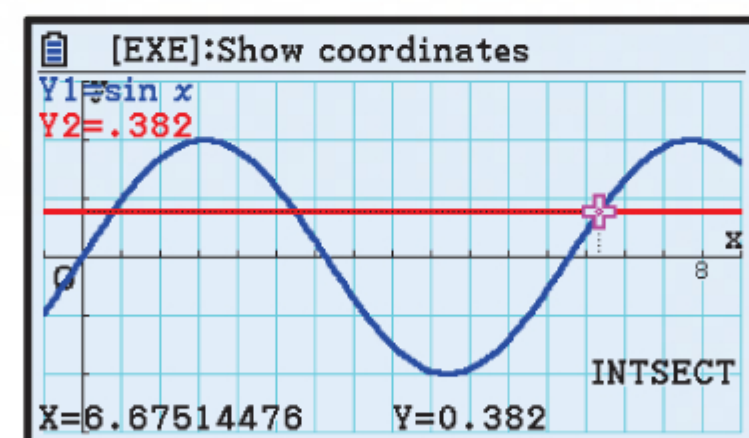
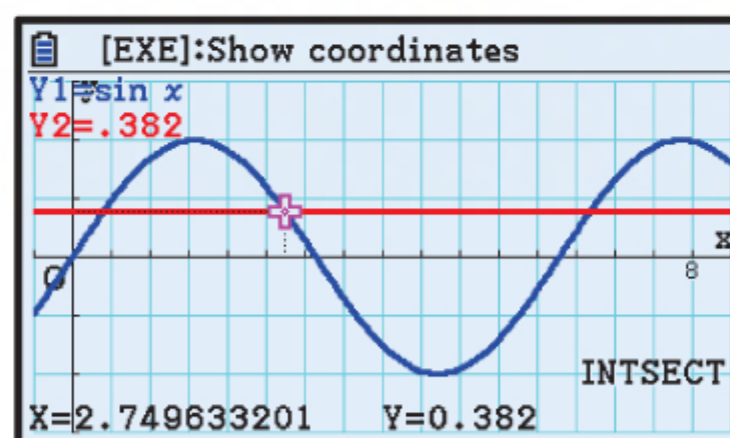
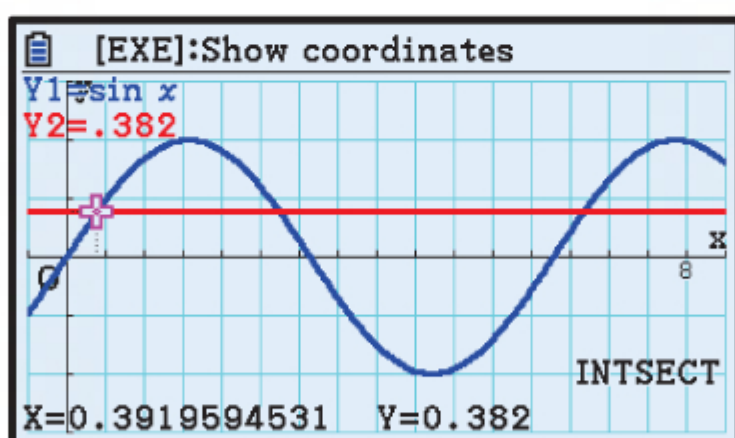
b When $\cos x = 0.9$, $0 \leq x \leq 4\pi$, $x \approx 0.5, 5.8, 6.7, 12.1$

2

a We graph the functions $Y_1 = \sin X$ and $Y_2 = 0.382$ on the same set of axes.

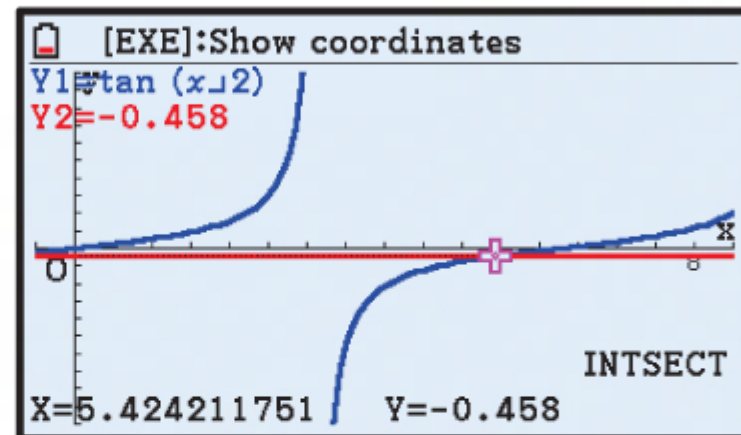
We use **window** settings just larger than the domain.

In this case, $X_{\min} = -0.5$, $X_{\max} = 8.5$, $X_{\text{scale}} = 0.5$.



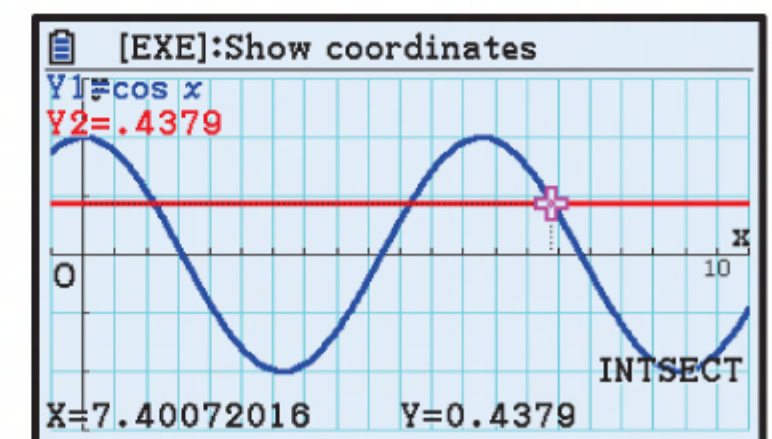
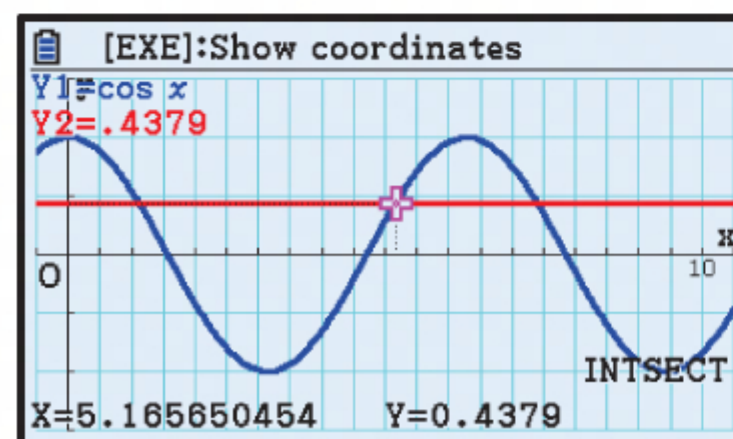
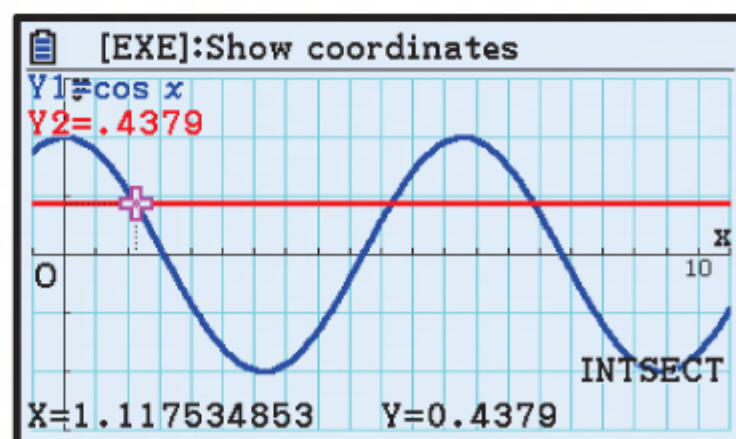
The solutions are $x \approx 0.392, 2.75, 6.68$.

- b** We graph the functions $Y_1 = \tan \frac{x}{2}$ and $Y_2 = -0.458$ on the same set of axes. We use **window** settings just larger than the domain. In this case, $X_{\min} = -0.5$, $X_{\max} = 8.5$, $X_{\text{scale}} = 0.5$.



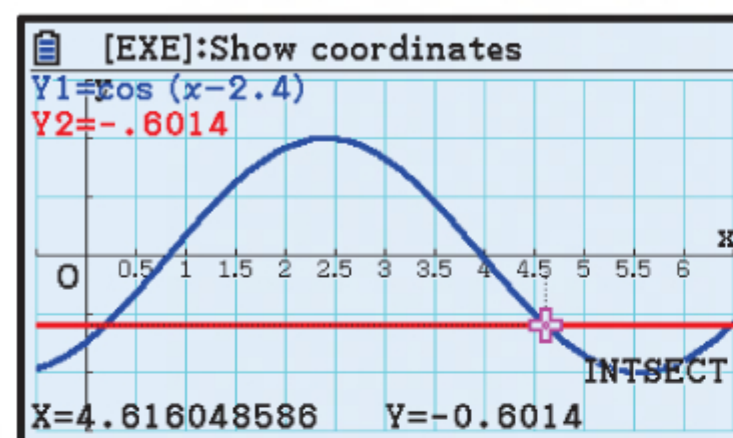
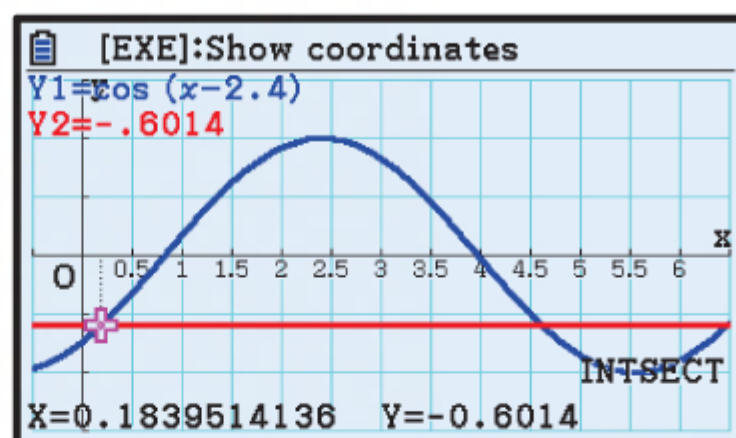
The solution is $x \approx 5.42$.

- 3 a** We graph the functions $Y_1 = \cos X$ and $Y_2 = 0.4379$ on the same set of axes. We need to use **window** settings just larger than the domain. In this case, $X_{\min} = -0.5$, $X_{\max} = 10.5$, $X_{\text{scale}} = 0.5$.



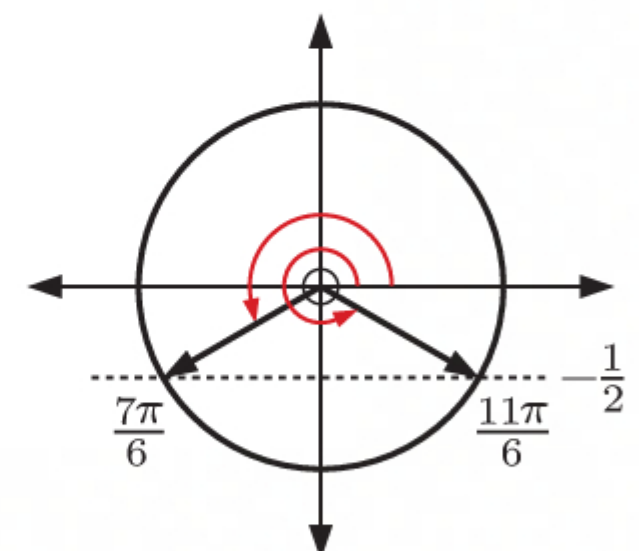
The solutions are $x \approx 1.12, 5.17, 7.40$.

- b** We graph the functions $Y_1 = \cos(X - 2.4)$ and $Y_2 = -0.6014$ on the same set of axes. We need to use **window** settings just larger than the domain. In this case, $X_{\min} = -0.5$, $X_{\max} = 6.5$, $X_{\text{scale}} = 0.5$.



The solutions are $x \approx 0.184, 4.62$.

- 4 a** $2 \sin x = -1$
 $\therefore \sin x = -\frac{1}{2}$
 On $0 \leq x \leq 2\pi$, the angles with sine $-\frac{1}{2}$ are $\frac{7\pi}{6}$ and $\frac{11\pi}{6}$.
 \therefore the solutions are $x = \frac{7\pi}{6}$ or $\frac{11\pi}{6}$.

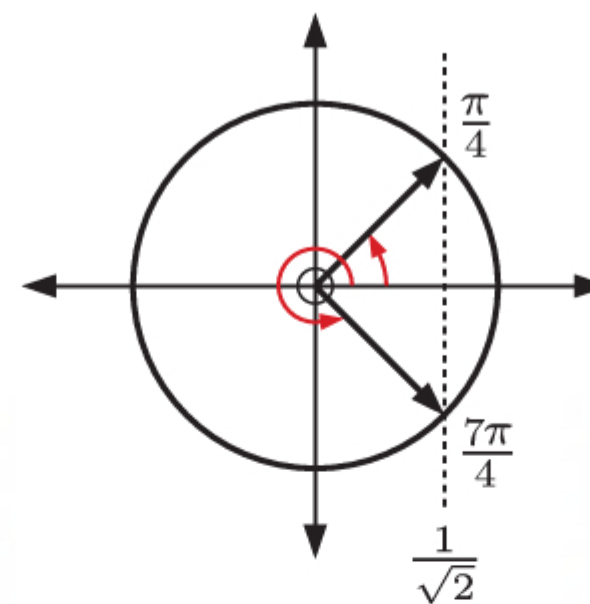


b $\sqrt{2} \cos x - 1 = 0$

$$\therefore \cos x = \frac{1}{\sqrt{2}}$$

On $0 \leq x \leq 2\pi$, the angles with cosine $\frac{1}{\sqrt{2}}$ are $\frac{\pi}{4}$ and $\frac{7\pi}{4}$.

\therefore the solutions are $x = \frac{\pi}{4}$ or $\frac{7\pi}{4}$.



c $2 \cos 2x + 1 = 0$

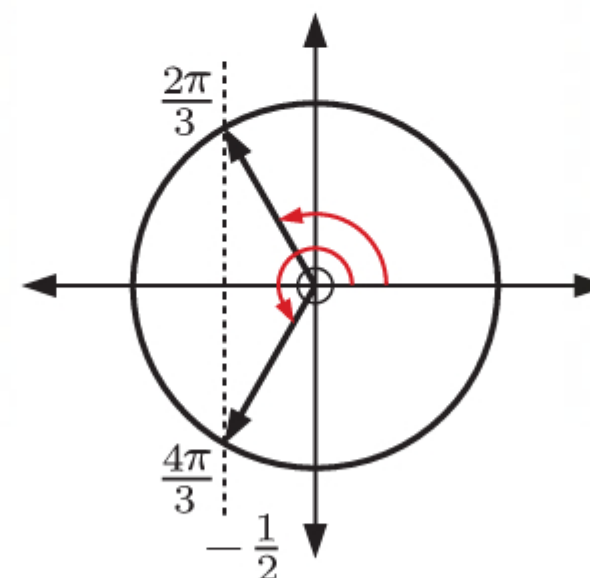
$$\therefore \cos 2x = -\frac{1}{2}$$

If $0 \leq x \leq 2\pi$, then $0 \leq 2x \leq 4\pi$.

On $0 \leq 2x \leq 4\pi$, the angles with cosine $-\frac{1}{2}$ are $\frac{2\pi}{3}$, $\frac{4\pi}{3}$, $\frac{8\pi}{3}$, and $\frac{10\pi}{3}$.

\therefore the solutions are $2x = \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{8\pi}{3}, \text{ or } \frac{10\pi}{3}$

$$\therefore x = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \text{ or } \frac{5\pi}{3}$$



5 a $\tan^2 2x = 1$

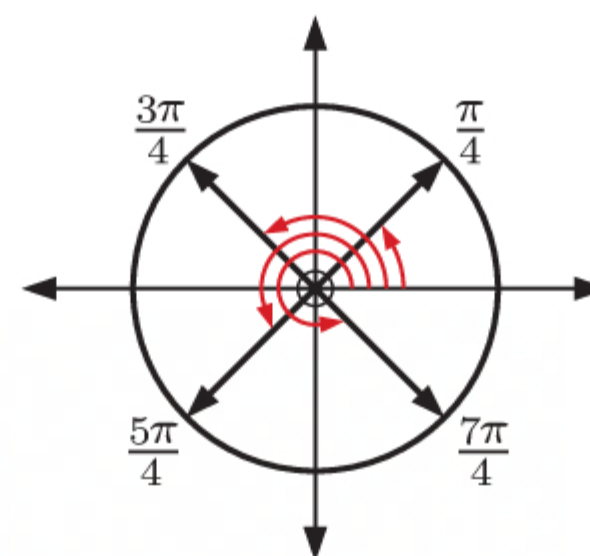
$$\therefore \tan 2x = \pm 1$$

If $0 \leq x \leq 2\pi$, then $0 \leq 2x \leq 4\pi$.

On $0 \leq 2x \leq 4\pi$, the angles with tangent 1 are $\frac{\pi}{4}$, $\frac{5\pi}{4}$, $\frac{9\pi}{4}$, and $\frac{13\pi}{4}$, and the angles with tangent -1 are $\frac{3\pi}{4}$, $\frac{7\pi}{4}$, $\frac{11\pi}{4}$, and $\frac{15\pi}{4}$.

$$\therefore 2x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}, \frac{9\pi}{4}, \frac{11\pi}{4}, \frac{13\pi}{4}, \text{ or } \frac{15\pi}{4}$$

$$\therefore x = \frac{\pi}{8}, \frac{3\pi}{8}, \frac{5\pi}{8}, \frac{7\pi}{8}, \frac{9\pi}{8}, \frac{11\pi}{8}, \frac{13\pi}{8}, \text{ or } \frac{15\pi}{8}$$



b $\sin^2 x - \sin x - 2 = 0, \quad 0 \leq x \leq 2\pi$

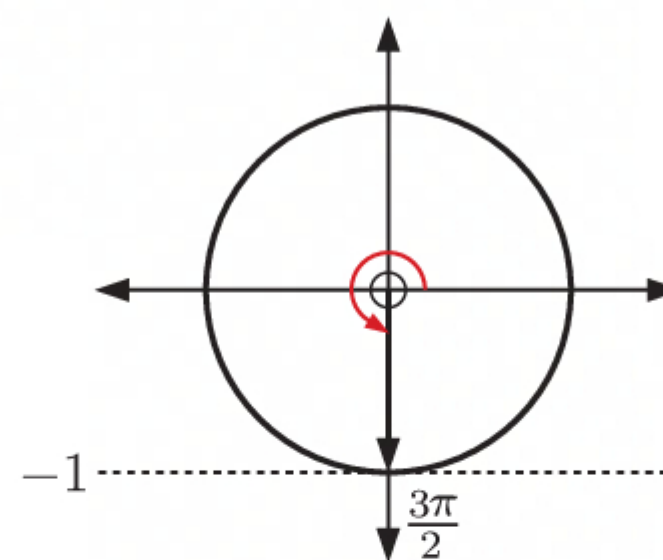
$$\therefore (\sin x - 2)(\sin x + 1) = 0$$

$$\therefore \sin x = 2 \text{ or } -1$$

But $\sin x$ values lie between -1 and 1 inclusive.

$$\therefore \sin x = -1$$

$$\therefore x = \frac{3\pi}{2}$$

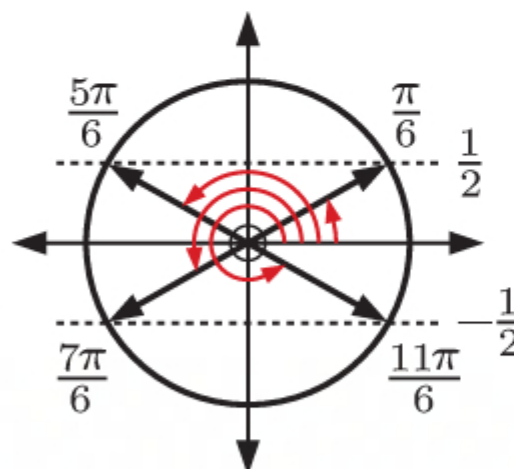


c $4 \sin^2 x = 1, \quad 0 \leq x \leq 2\pi$

$$\therefore \sin^2 x = \frac{1}{4}$$

$$\therefore \sin x = \pm \frac{1}{2}$$

$$\therefore x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \text{ or } \frac{11\pi}{6}$$



6 a $\sqrt{2} \cos\left(x + \frac{\pi}{4}\right) - 1 = 0, \quad 0 \leq x \leq 4\pi$

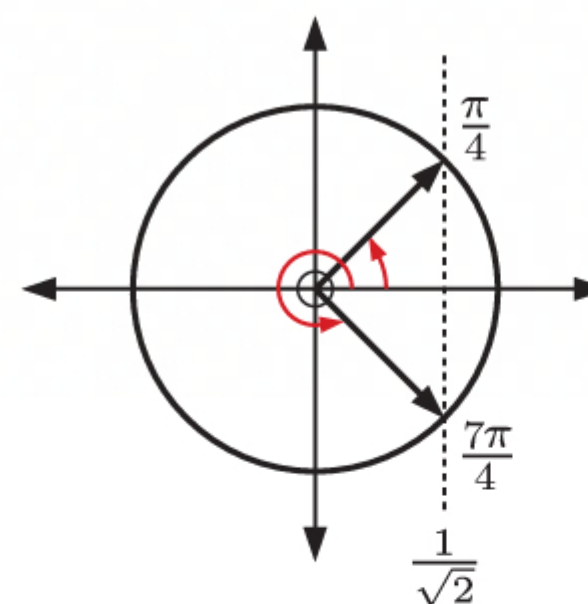
$$\therefore \cos\left(x + \frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$$

Since $0 \leq x \leq 4\pi$,

$$\frac{\pi}{4} \leq x + \frac{\pi}{4} \leq \frac{17\pi}{4}$$

So, $x + \frac{\pi}{4} = \frac{\pi}{4}, \frac{7\pi}{4}, \frac{9\pi}{4}, \frac{15\pi}{4}, \text{ or } \frac{17\pi}{4}$

$$\therefore x = 0, \frac{3\pi}{2}, 2\pi, \frac{7\pi}{2}, \text{ or } 4\pi$$



b $\tan 2x - \sqrt{3} = 0, \quad 0 \leq x \leq 2\pi$

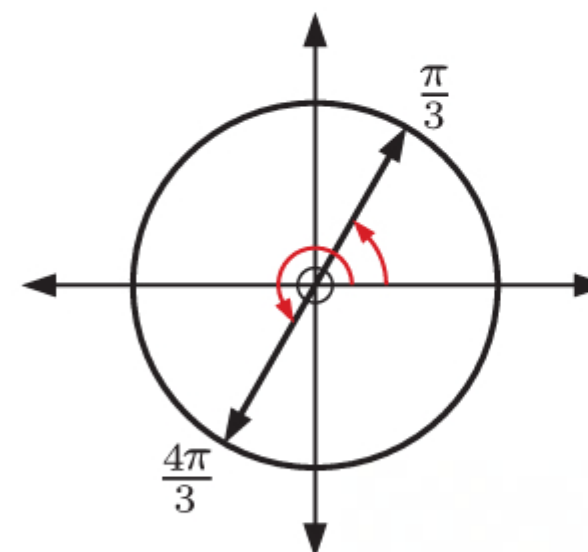
$$\therefore \tan 2x = \sqrt{3}$$

Since $0 \leq x \leq 2\pi$,

$$0 \leq 2x \leq 4\pi$$

So, $2x = \frac{\pi}{3}, \frac{4\pi}{3}, \frac{7\pi}{3}, \text{ or } \frac{10\pi}{3}$

$$\therefore x = \frac{\pi}{6}, \frac{2\pi}{3}, \frac{7\pi}{6}, \text{ or } \frac{5\pi}{3}$$



7 $P(t) = 5 + 2 \sin \frac{\pi t}{3}, \quad 0 \leq t \leq 8$, where $P(t)$ is in thousands of water beetles.

a $P(0) = 5 + 2 \sin 0$
 $= 5$

The initial population was 5000 water beetles.

b Smallest $P = 5 + 2(-1) = 3$ {when $\sin \frac{\pi t}{3} = -1$ }

Largest $P = 5 + 2(1) = 7$ {when $\sin \frac{\pi t}{3} = 1$ }

\therefore the smallest population was 3000 water beetles and the largest population was 7000 water beetles.

c If the population is > 6000 , then $P(t) > 6$

$$\therefore 5 + 2 \sin \frac{\pi t}{3} > 6$$

$$\therefore 2 \sin \frac{\pi t}{3} > 1$$

$$\therefore \sin \frac{\pi t}{3} > \frac{1}{2}$$

The points on the unit circle with sine $\frac{1}{2}$ correspond to angles $\frac{\pi}{6}$ and $\frac{5\pi}{6}$.

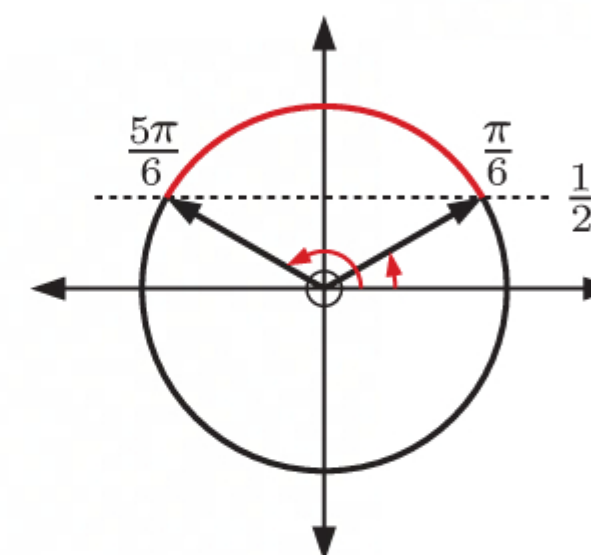
$$0 \leq t \leq 8 \quad \therefore 0 \leq \frac{\pi t}{3} \leq \frac{8\pi}{3}$$

$$\text{So, } \frac{\pi}{6} < \frac{\pi t}{3} < \frac{5\pi}{6}, \quad \frac{13\pi}{6} < \frac{\pi t}{3} \leq \frac{8\pi}{3}$$

$$\therefore \frac{1}{2} < t < \frac{5}{2}, \quad \frac{13}{2} < t \leq 8$$

$$\therefore 0.5 < t < 2.5, \quad 6.5 < t \leq 8$$

So, the population of water beetles was greater than 6000 when $0.5 < t < 2.5$ weeks, and $6.5 < t \leq 8$ weeks.



8 a $3 \cos(-\theta) - 2 \cos \theta$
 $= 3 \cos \theta - 2 \cos \theta$
 $= \cos \theta$

b $\cos\left(\frac{3\pi}{2} - \theta\right)$
 $= \cos \frac{3\pi}{2} \cos \theta + \sin \frac{3\pi}{2} \sin \theta$
 $= (0) \cos \theta + (-1) \sin \theta$
 $= -\sin \theta$

$$\begin{aligned}
 \text{c} \quad & \sin\left(\theta + \frac{\pi}{2}\right) \\
 &= \sin \theta \cos \frac{\pi}{2} + \cos \theta \sin \frac{\pi}{2} \\
 &= \sin \theta(0) + \cos \theta(1) \\
 &= \cos \theta
 \end{aligned}$$

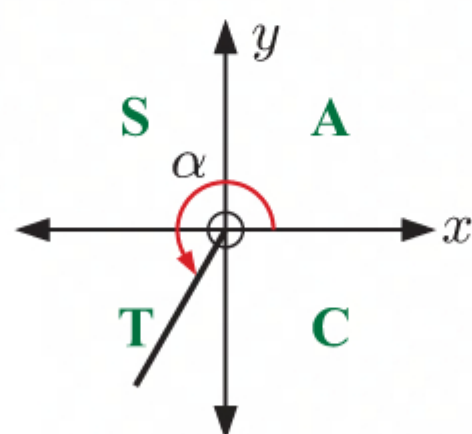
$$\begin{aligned}
 \text{e} \quad & \frac{\sin \alpha - \cos \alpha}{\sin^2 \alpha - \cos^2 \alpha} \\
 &= \frac{\sin \alpha - \cos \alpha}{(\sin \alpha + \cos \alpha)(\sin \alpha - \cos \alpha)} \\
 &= \frac{1}{\sin \alpha + \cos \alpha}
 \end{aligned}$$

$$\begin{aligned}
 \text{d} \quad & \frac{1 - \cos^2 \theta}{1 + \cos \theta} \\
 &= \frac{(1 + \cos \theta)(1 - \cos \theta)}{1 + \cos \theta} \\
 &= 1 - \cos \theta
 \end{aligned}$$

$$\begin{aligned}
 \text{f} \quad & \frac{4 \sin^2 \alpha - 4}{8 \cos \alpha} \\
 &= \frac{-4(1 - \sin^2 \alpha)}{8 \cos \alpha} \\
 &= \frac{-4 \cos^2 \alpha}{8 \cos \alpha} \\
 &= \frac{-\cos \alpha}{2}
 \end{aligned}$$

9 $\sin \alpha = -\frac{3}{4}$ and $\pi \leq \alpha \leq \frac{3\pi}{2}$

a α is in quadrant 3, so $\cos \alpha$ is negative.



Now $\cos^2 \alpha + \sin^2 \alpha = 1$

$$\therefore \cos^2 \alpha + \frac{9}{16} = 1$$

$$\therefore \cos^2 \alpha = \frac{7}{16}$$

$$\therefore \cos \alpha = -\frac{\sqrt{7}}{4} \quad \{\cos \alpha < 0\}$$

b Using the double angle identity, $\sin 2\alpha = 2 \sin \alpha \cos \alpha$

$$\begin{aligned}
 &= 2\left(-\frac{3}{4}\right)\left(-\frac{\sqrt{7}}{4}\right) \quad \{\text{using a}\} \\
 &= \frac{3\sqrt{7}}{8}
 \end{aligned}$$

c Using the double angle identity, $\cos 2\alpha = \cos^2 \alpha - \sin^2 \alpha$

$$\begin{aligned}
 &= \left(-\frac{\sqrt{7}}{4}\right)^2 - \left(-\frac{3}{4}\right)^2 \quad \{\text{using a}\} \\
 &= \frac{7}{16} - \frac{9}{16} \\
 &= -\frac{2}{16} \\
 &= -\frac{1}{8}
 \end{aligned}$$

10 $\frac{\sin 2\alpha - \sin \alpha}{\cos 2\alpha - \cos \alpha + 1} = \frac{2 \sin \alpha \cos \alpha - \sin \alpha}{2 \cos^2 \alpha - 1 - \cos \alpha + 1}$

$$\begin{aligned}
 &= \frac{\sin \alpha(2 \cos \alpha - 1)}{\cos \alpha(2 \cos \alpha - 1)} \\
 &= \frac{\sin \alpha}{\cos \alpha} \\
 &= \tan \alpha
 \end{aligned}$$

11 $f(x) = \cos x$, $g(x) = 2x$, $0 \leq x \leq 2\pi$

a $(f \circ g)(x) = 1$

$$\therefore f(g(x)) = 1$$

$$\therefore f(2x) = 1$$

$$\therefore \cos 2x = 1$$

If $0 \leq x \leq 2\pi$, then $0 \leq 2x \leq 4\pi$.

On $0 \leq 2x \leq 4\pi$, the angles with cosine 1 are 0, 2π , and 4π .

So, $2x = 0, 2\pi$, or 4π

$$\therefore x = 0, \pi, \text{ or } 2\pi$$

b $(g \circ f)(x) = 1$

$$\therefore g(f(x)) = 1$$

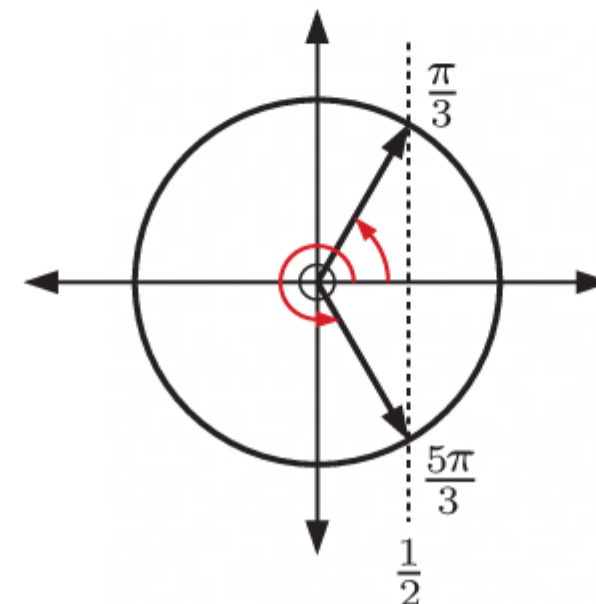
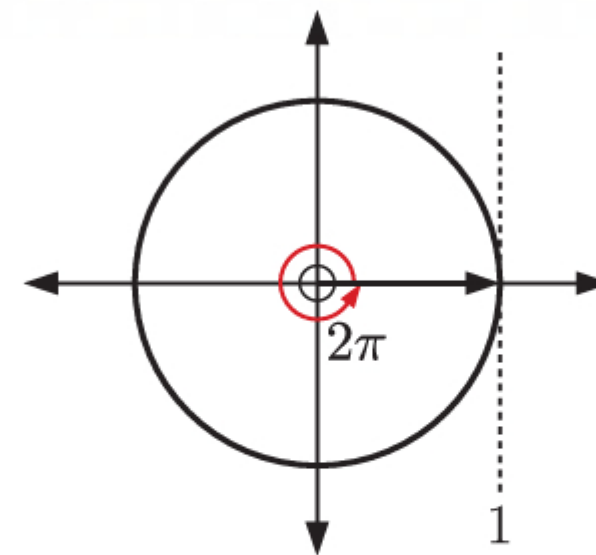
$$\therefore g(\cos x) = 1$$

$$\therefore 2 \cos x = 1$$

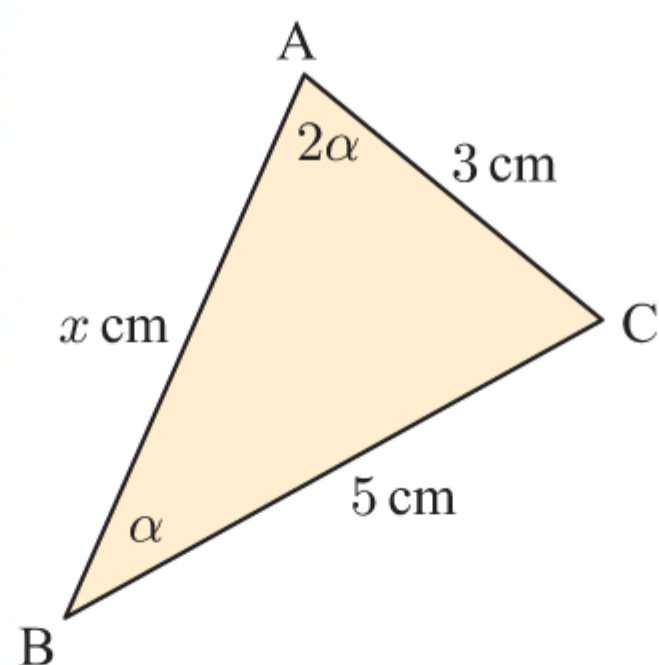
$$\therefore \cos x = \frac{1}{2}$$

On $0 \leq x \leq 2\pi$, the angles with cosine $\frac{1}{2}$ are $\frac{\pi}{3}$ and $\frac{5\pi}{3}$.

$$\therefore x = \frac{\pi}{3} \text{ or } \frac{5\pi}{3}$$



12



a By the sine rule, $\frac{\sin 2\alpha}{5} = \frac{\sin \alpha}{3}$

$$\therefore \frac{2 \sin \alpha \cos \alpha}{\sin \alpha} = \frac{5}{3}$$

$$\therefore 2 \cos \alpha = \frac{5}{3}$$

$$\{\sin \alpha \neq 0 \text{ as } 0 < \alpha < \pi\}$$

$$\therefore \cos \alpha = \frac{5}{6}$$

b Using the cosine rule,

$$3^2 = x^2 + 5^2 - 2 \times x \times 5 \times \cos \alpha$$

$$\therefore 9 = x^2 + 25 - 10x \left(\frac{5}{6}\right) \quad \{\text{using a}\}$$

$$\therefore x^2 - \frac{25}{3}x + 16 = 0$$

$$\therefore 3x^2 - 25x + 48 = 0$$

c $(3x - 16)(x - 3) = 0$

$$\therefore x = \frac{16}{3} \text{ or } 3$$

If $x = 3$, triangle ABC is isosceles.

$$\therefore \widehat{ACB} = \widehat{BAC} = \alpha \quad \{\text{base angles}\}$$

$$\therefore 2\alpha + \alpha + \alpha = \pi \quad \{\text{angles in a triangle}\}$$

$$\therefore 4\alpha = \pi$$

$$\therefore 2\alpha = \frac{\pi}{2}$$

So, triangle ABC is right angled at B.

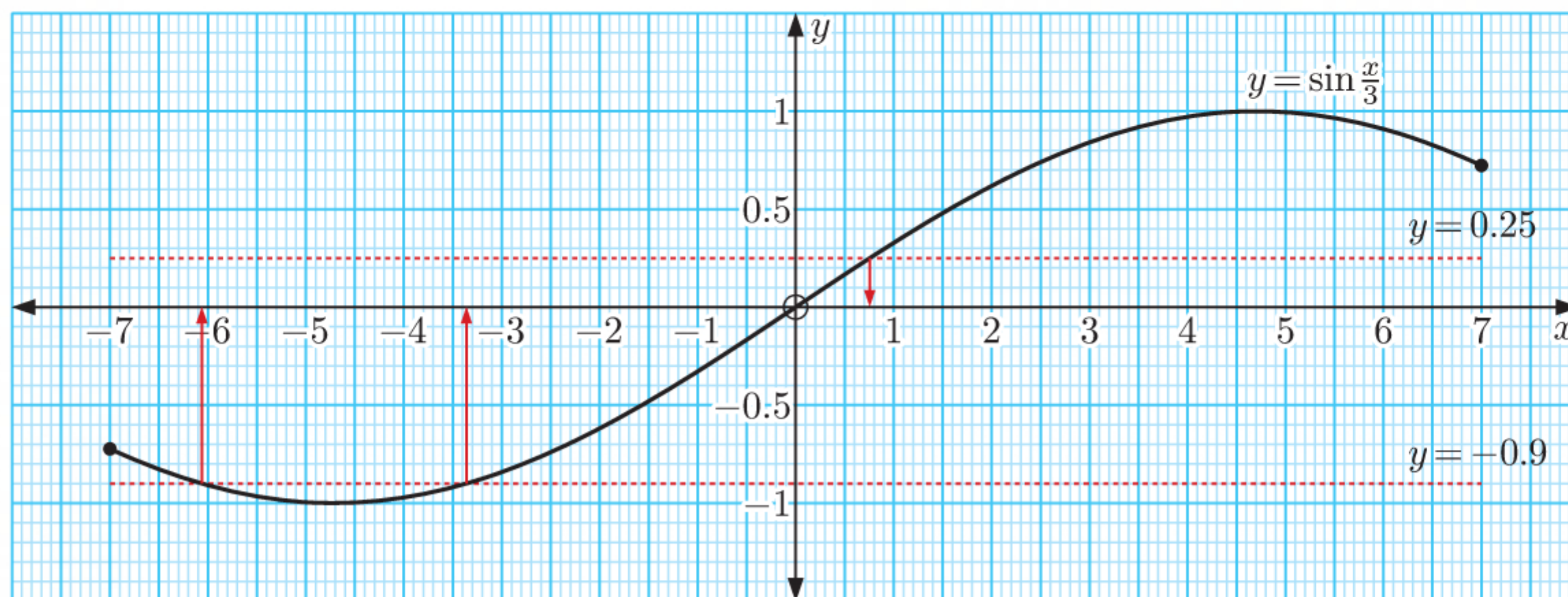
However, $3^2 + 3^2 = 9 + 9 = 18 \neq 25 = 5^2$.

$\therefore x = 3$ is not a valid solution

$\therefore x = \frac{16}{3}$ is the only solution

REVIEW SET 9B

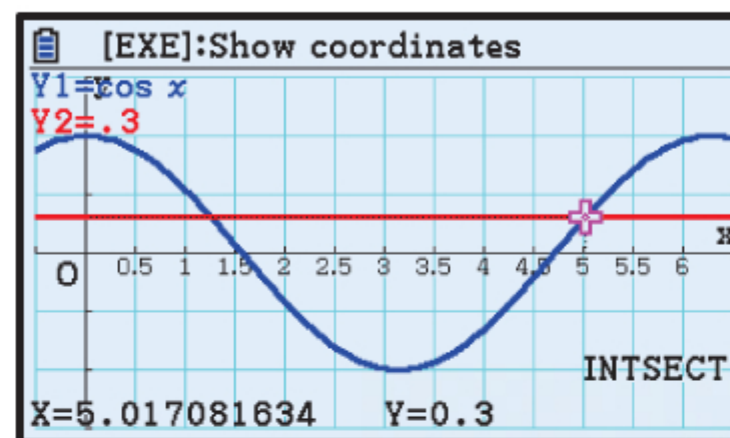
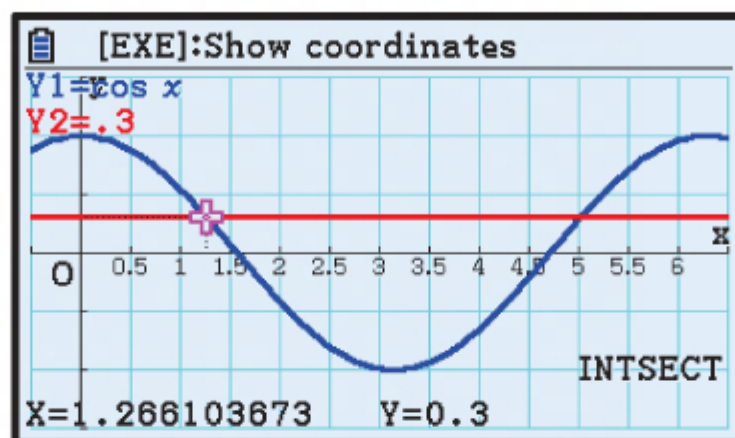
1



a When $\sin \frac{x}{3} = -0.9$, $-7 \leq x \leq 7$,
 $x \approx -6.1, -3.4$

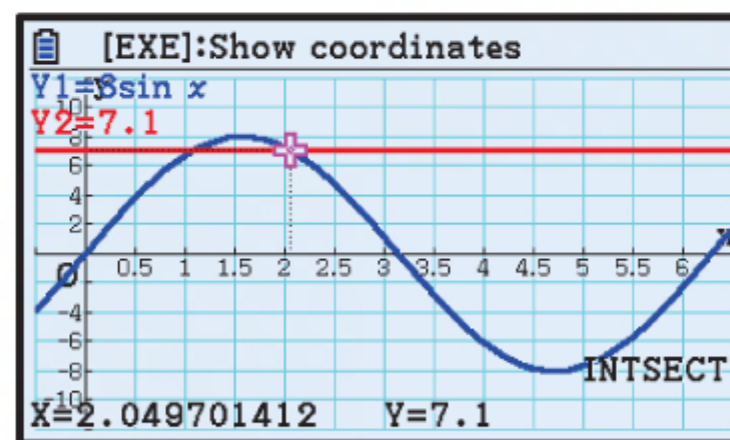
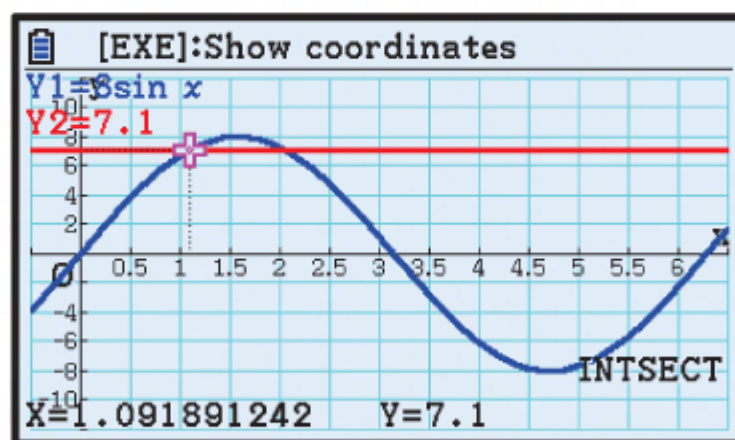
b When $\sin \frac{x}{3} = \frac{1}{4}$, $-7 \leq x \leq 7$,
 $x \approx 0.8$

- 2 a** We graph the functions $Y_1 = \cos X$ and $Y_2 = 0.3$ on the same set of axes. We need to use **window** settings just larger than the domain. In this case, $X_{\min} = -\frac{\pi}{6}$, $X_{\max} = \frac{13\pi}{6}$, $X_{\text{scale}} = \frac{\pi}{6}$.



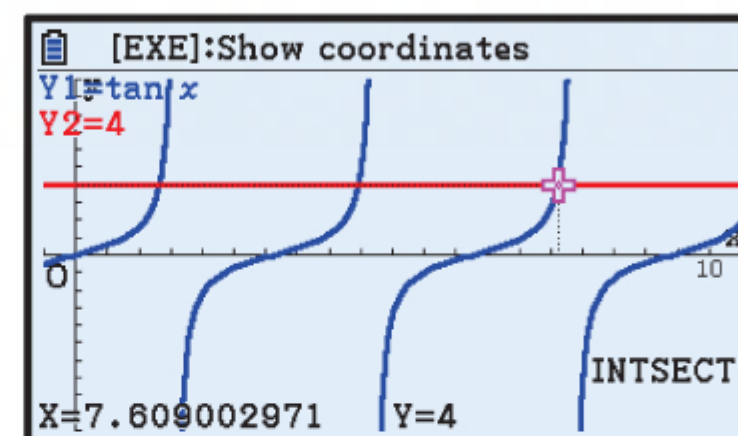
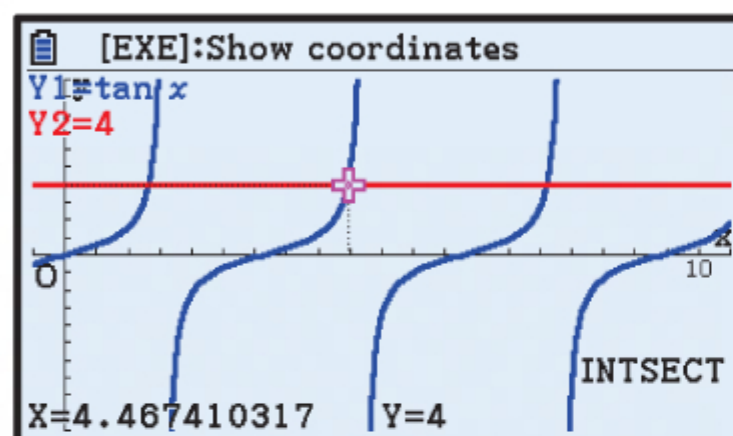
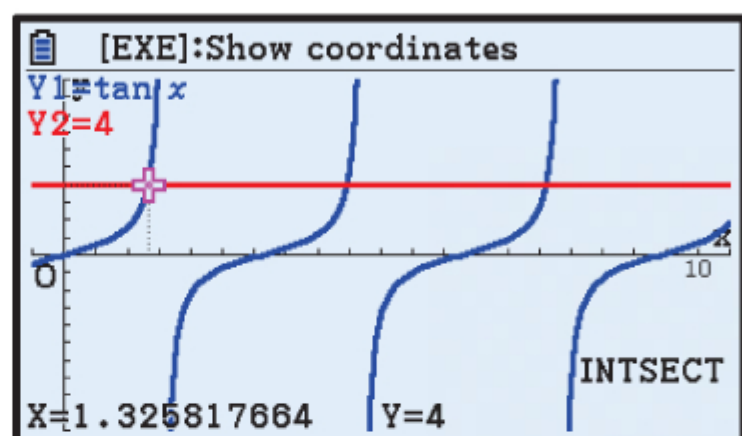
The solutions are $x \approx 1.27, 5.02$.

- b** We graph the functions $Y_1 = 8 \sin X$ and $Y_2 = 7.1$ on the same set of axes. We need to use the **window** settings just larger than the domain. In this case, $X_{\min} = -\frac{\pi}{6}$, $X_{\max} = \frac{13\pi}{6}$, $X_{\text{scale}} = \frac{\pi}{6}$.



The solutions are $x \approx 1.09, 2.05$.

- 3 a** We graph the functions $Y_1 = \tan X$ and $Y_2 = 4$ on the same set of axes. We need to use the **window** settings just larger than the domain. In this case, $X_{\min} = -0.5$, $X_{\max} = 10.5$, $X_{\text{scale}} = 0.5$.



The solutions are $x \approx 1.33, 4.47, 7.61$.

b $\tan \frac{x}{4} = 4$

Since $0 \leq x \leq 10$

$$\therefore 0 \leq \frac{x}{4} \leq \frac{5}{2}$$

So, $\frac{x}{4} \approx 1.33$ {using **a**}

$$\therefore x \approx 5.30$$

c $\tan(x - 1.5) = 4$

Since $0 \leq x \leq 10$

$$\therefore -1.5 \leq x - 1.5 \leq 8.5$$

So, $x - 1.5 \approx 1.33, 4.47, \text{ or } 7.61$ {using **a**}

$$\therefore x \approx 2.83, 5.97, \text{ or } 9.11$$

4 a $2 \sin 3x = -\sqrt{3}, \quad 0 \leq x \leq 2\pi$

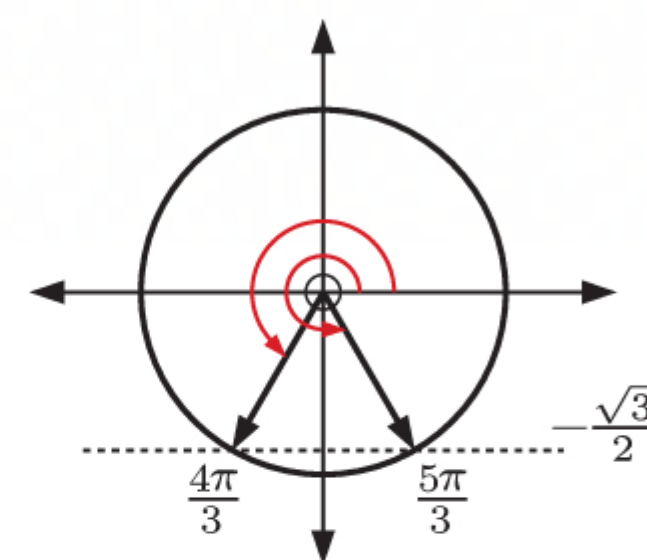
$$\therefore \sin 3x = -\frac{\sqrt{3}}{2}$$

If $0 \leq x \leq 2\pi$, then $0 \leq 3x \leq 6\pi$.

On $0 \leq 3x \leq 6\pi$, the angles with sine $-\frac{\sqrt{3}}{2}$ are $\frac{4\pi}{3}, \frac{5\pi}{3}, \frac{10\pi}{3}, \frac{11\pi}{3}, \frac{16\pi}{3}$, and $\frac{17\pi}{3}$.

So, $3x = \frac{4\pi}{3}, \frac{5\pi}{3}, \frac{10\pi}{3}, \frac{11\pi}{3}, \frac{16\pi}{3}, \text{ or } \frac{17\pi}{3}$

$$\therefore x = \frac{4\pi}{9}, \frac{5\pi}{9}, \frac{10\pi}{9}, \frac{11\pi}{9}, \frac{16\pi}{9}, \text{ or } \frac{17\pi}{9}$$



b $\sqrt{3} \tan \frac{x}{2} = -1, \quad 0 \leq x \leq 2\pi$

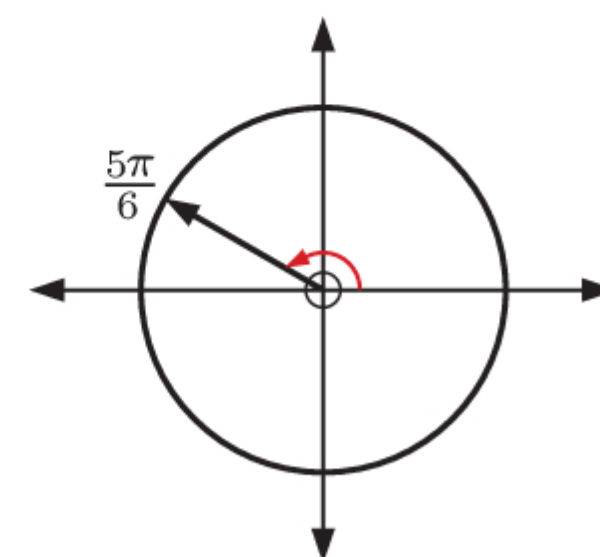
$$\therefore \tan \frac{x}{2} = -\frac{1}{\sqrt{3}}$$

If $0 \leq x \leq 2\pi$, then $0 \leq \frac{x}{2} \leq \pi$.

On $0 \leq \frac{x}{2} \leq \pi$, the angle with tangent $-\frac{1}{\sqrt{3}}$ is $\frac{5\pi}{6}$.

So, $\frac{x}{2} = \frac{5\pi}{6}$

$$\therefore x = \frac{5\pi}{3}$$



$$\text{c} \quad \cos 2x = \sqrt{3} \sin 2x, \quad 0 \leq x \leq 2\pi$$

$$\therefore \frac{1}{\sqrt{3}} = \frac{\sin 2x}{\cos 2x}$$

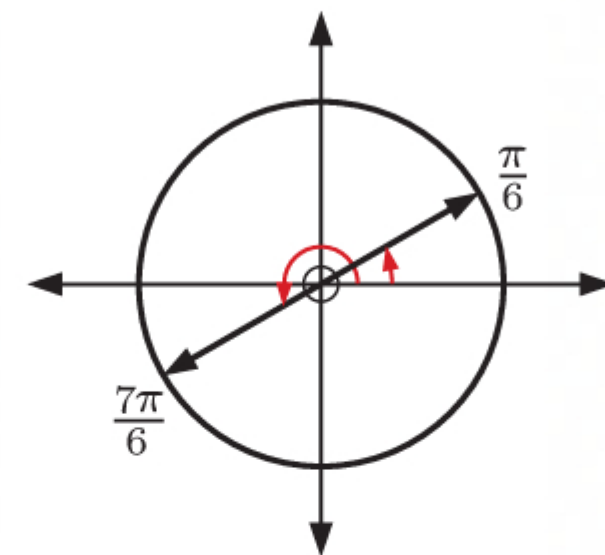
$$\therefore \tan 2x = \frac{1}{\sqrt{3}}$$

If $0 \leq x \leq 2\pi$, then $0 \leq 2x \leq 4\pi$.

On $0 \leq 2x \leq 4\pi$, the angles with tangent $\frac{1}{\sqrt{3}}$ are $\frac{\pi}{6}$, $\frac{7\pi}{6}$, $\frac{13\pi}{6}$, and $\frac{19\pi}{6}$.

$$\text{So, } 2x = \frac{\pi}{6}, \frac{7\pi}{6}, \frac{13\pi}{6}, \text{ or } \frac{19\pi}{6}$$

$$\therefore x = \frac{\pi}{12}, \frac{7\pi}{12}, \frac{13\pi}{12}, \text{ or } \frac{19\pi}{12}$$



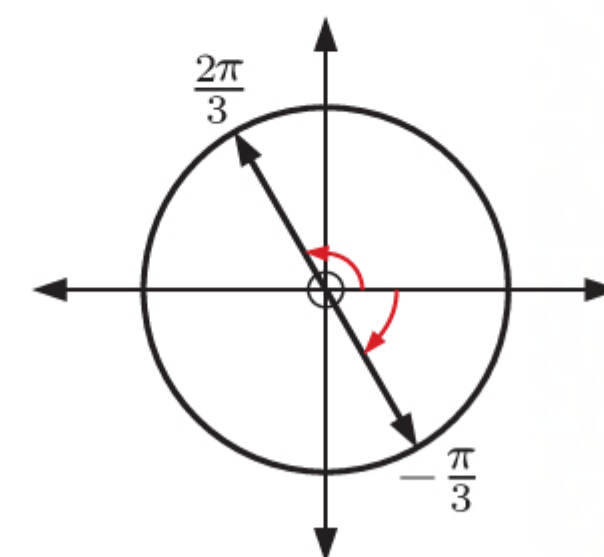
$$\text{5 a} \quad \tan\left(x + \frac{\pi}{6}\right) = -\sqrt{3}, \quad -\pi \leq x \leq \pi$$

If $-\pi \leq x \leq \pi$, then $-\frac{5\pi}{6} \leq x + \frac{\pi}{6} \leq \frac{7\pi}{6}$.

On $-\frac{5\pi}{6} \leq x + \frac{\pi}{6} \leq \frac{7\pi}{6}$, the angles with tangent $-\sqrt{3}$ are $-\frac{\pi}{3}$ and $\frac{2\pi}{3}$.

$$\text{So, } x + \frac{\pi}{6} = -\frac{\pi}{3} \text{ or } \frac{2\pi}{3}$$

$$\therefore x = -\frac{\pi}{2} \text{ or } \frac{\pi}{2}$$



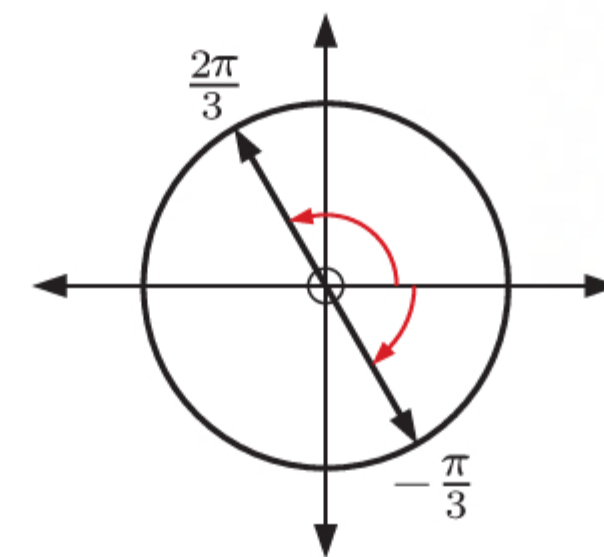
$$\text{b} \quad \tan 2x = -\sqrt{3}, \quad -\pi \leq x \leq \pi$$

If $-\pi \leq x \leq \pi$, then $-2\pi \leq 2x \leq 2\pi$.

On $-2\pi \leq 2x \leq 2\pi$, the angles with tangent $-\sqrt{3}$ are $-\frac{4\pi}{3}$, $-\frac{\pi}{3}$, $\frac{2\pi}{3}$, and $\frac{5\pi}{3}$.

$$\text{So, } 2x = -\frac{4\pi}{3}, -\frac{\pi}{3}, \frac{2\pi}{3}, \text{ or } \frac{5\pi}{3}$$

$$\therefore x = -\frac{2\pi}{3}, -\frac{\pi}{6}, \frac{\pi}{3}, \text{ or } \frac{5\pi}{6}$$



$$\text{c} \quad \tan^2 x - 3 = 0, \quad -\pi \leq x \leq \pi$$

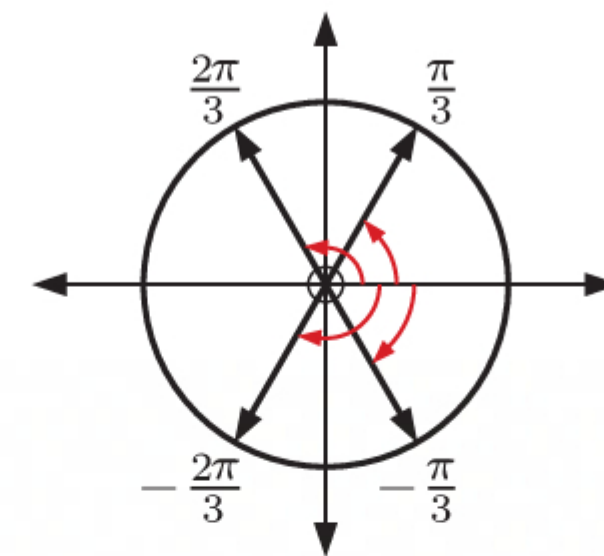
$$\therefore \tan^2 x = 3$$

$$\therefore \tan x = \pm\sqrt{3}$$

On $-\pi \leq x \leq \pi$, the angles with tangent $\sqrt{3}$ are $-\frac{2\pi}{3}$ and $\frac{\pi}{3}$.

The angles with tangent $-\sqrt{3}$ are $-\frac{\pi}{3}$ and $\frac{2\pi}{3}$.

$$\therefore x = -\frac{2\pi}{3}, -\frac{\pi}{3}, \frac{\pi}{3}, \text{ or } \frac{2\pi}{3}$$



$$\text{6 a} \quad y = 2 \sin 3x + \sqrt{3}, \quad 0 \leq x \leq 2\pi$$

The x -intercepts are the values of x such that

$$2 \sin 3x + \sqrt{3} = 0$$

$$\therefore \sin 3x = -\frac{\sqrt{3}}{2}$$

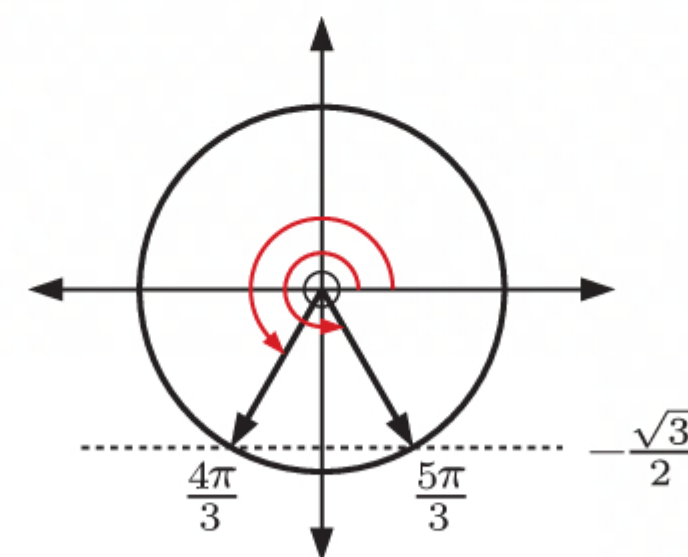
If $0 \leq x \leq 2\pi$, then $0 \leq 3x \leq 6\pi$.

On $0 \leq 3x \leq 6\pi$, the angles with sine $-\frac{\sqrt{3}}{2}$ are $\frac{4\pi}{3}$, $\frac{5\pi}{3}$, $\frac{10\pi}{3}$, $\frac{11\pi}{3}$, $\frac{16\pi}{3}$, and $\frac{17\pi}{3}$.

$$\text{So, } 3x = \frac{4\pi}{3}, \frac{5\pi}{3}, \frac{10\pi}{3}, \frac{11\pi}{3}, \frac{16\pi}{3}, \text{ or } \frac{17\pi}{3}$$

$$\therefore x = \frac{4\pi}{9}, \frac{5\pi}{9}, \frac{10\pi}{9}, \frac{11\pi}{9}, \frac{16\pi}{9}, \text{ or } \frac{17\pi}{9}$$

\therefore the x -intercepts of $y = 2 \sin 3x + \sqrt{3}$, $0 \leq x \leq 2\pi$ are $\frac{4\pi}{9}$, $\frac{5\pi}{9}$, $\frac{10\pi}{9}$, $\frac{11\pi}{9}$, $\frac{16\pi}{9}$, and $\frac{17\pi}{9}$.



b $y = \sqrt{2} \sin\left(x + \frac{\pi}{4}\right), \quad 0 \leq x \leq 3\pi$

The x -intercepts are the values of x such that

$$\sqrt{2} \sin\left(x + \frac{\pi}{4}\right) = 0$$

$$\therefore \sin\left(x + \frac{\pi}{4}\right) = 0$$

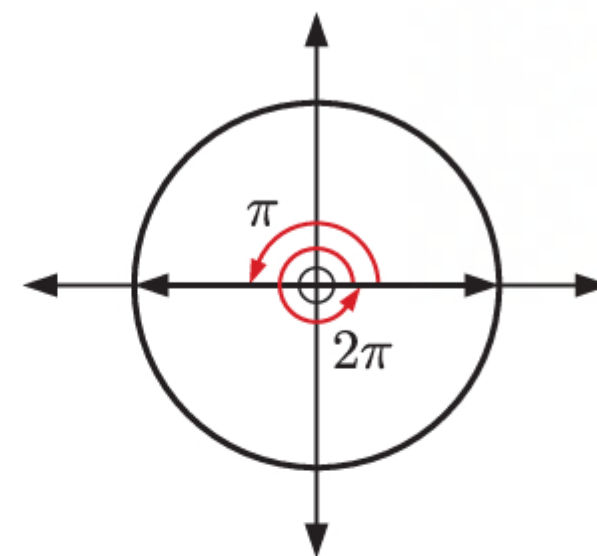
If $0 \leq x \leq 3\pi$, then $\frac{\pi}{4} \leq x + \frac{\pi}{4} \leq \frac{13\pi}{4}$.

On $\frac{\pi}{4} \leq x + \frac{\pi}{4} \leq \frac{13\pi}{4}$, the angles with sine 0 are π , 2π , and 3π .

So, $x + \frac{\pi}{4} = \pi, 2\pi$, or 3π

$$\therefore x = \frac{3\pi}{4}, \frac{7\pi}{4}, \text{ or } \frac{11\pi}{4}$$

\therefore the x -intercepts of $y = \sqrt{2} \sin\left(x + \frac{\pi}{4}\right), \quad 0 \leq x \leq 3\pi$ are $\frac{3\pi}{4}, \frac{7\pi}{4}$, and $\frac{11\pi}{4}$.



7 $P(t) = 40 + 12 \sin\left(\frac{2\pi}{7}\left(t - \frac{37}{12}\right)\right)$ mg

a $P(t)$ has a minimum of $40 + 12(-1) = 28$ mg per m^3 {when $\sin\left(\frac{2\pi}{7}\left(t - \frac{37}{12}\right)\right) = -1$ }

b The minimum level of pollution occurs when

$$\sin\left(\frac{2\pi}{7}\left(t - \frac{37}{12}\right)\right) = -1$$

$$\therefore \frac{2\pi}{7}\left(t - \frac{37}{12}\right) = \frac{3\pi}{2} + k2\pi$$

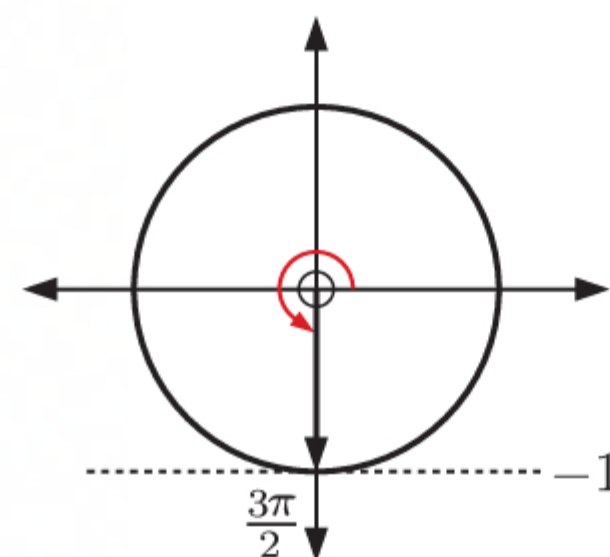
$$\therefore \frac{2}{7}\left(t - \frac{37}{12}\right) = \frac{3}{2} + k(2)$$

$$\text{So, } t - \frac{37}{12} = \frac{21}{4} + k(7)$$

$$\therefore t = 8\frac{1}{3} + k(7)$$

$$\therefore t = 1\frac{1}{3}, 8\frac{1}{3}, 15\frac{1}{3}, \text{ and so on}$$

\therefore it occurs on Mondays at 8:00 am. $\{1\frac{1}{3}$ days after midnight on Saturday}



8 a $\cos^3 \theta + \sin^2 \theta \cos \theta$
 $= \cos \theta (\cos^2 \theta + \sin^2 \theta)$
 $= \cos \theta$

c $5 - 5 \sin^2 \theta$
 $= 5(1 - \sin^2 \theta)$
 $= 5 \cos^2 \theta$

b $\frac{\cos^2 \theta - 1}{\sin \theta}$
 $= \frac{-(1 - \cos^2 \theta)}{\sin \theta}$
 $= -\frac{\sin^2 \theta}{\sin \theta}$
 $= -\sin \theta$

d $\frac{\sin^2 \theta - 1}{\cos \theta}$
 $= -\frac{(1 - \sin^2 \theta)}{\cos \theta}$
 $= -\frac{\cos^2 \theta}{\cos \theta}$
 $= -\cos \theta$

$$9 \quad \sin A = \frac{5}{13} \quad \text{and} \quad \cos A = \frac{12}{13}$$

$$\begin{aligned} \mathbf{a} \quad \sin 2A &= 2 \sin A \cos A \\ &= 2\left(\frac{5}{13}\right)\left(\frac{12}{13}\right) \\ &= \frac{120}{169} \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad \cos 2A &= \cos^2 A - \sin^2 A \\ &= \left(\frac{12}{13}\right)^2 - \left(\frac{5}{13}\right)^2 \\ &= \frac{144-25}{169} \\ &= \frac{119}{169} \end{aligned}$$

$$\begin{aligned} 10 \quad \mathbf{a} \quad \frac{\cos \theta}{1 + \sin \theta} + \frac{1 + \sin \theta}{\cos \theta} &= \frac{\cos^2 \theta + (1 + \sin \theta)^2}{(1 + \sin \theta) \cos \theta} \\ &= \frac{\cos^2 \theta + 1 + 2 \sin \theta + \sin^2 \theta}{(1 + \sin \theta) \cos \theta} \\ &= \frac{2 + 2 \sin \theta}{(1 + \sin \theta) \cos \theta} \quad \{\cos^2 \theta + \sin^2 \theta = 1\} \\ &= \frac{2(1 + \sin \theta)}{(1 + \sin \theta) \cos \theta} \\ &= \frac{2}{\cos \theta} \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad \left(1 + \frac{1}{\cos \theta}\right)(\cos \theta - \cos^2 \theta) &= \cos \theta - \cos^2 \theta + 1 - \cos \theta \\ &= 1 - \cos^2 \theta \\ &= \sin^2 \theta \end{aligned}$$

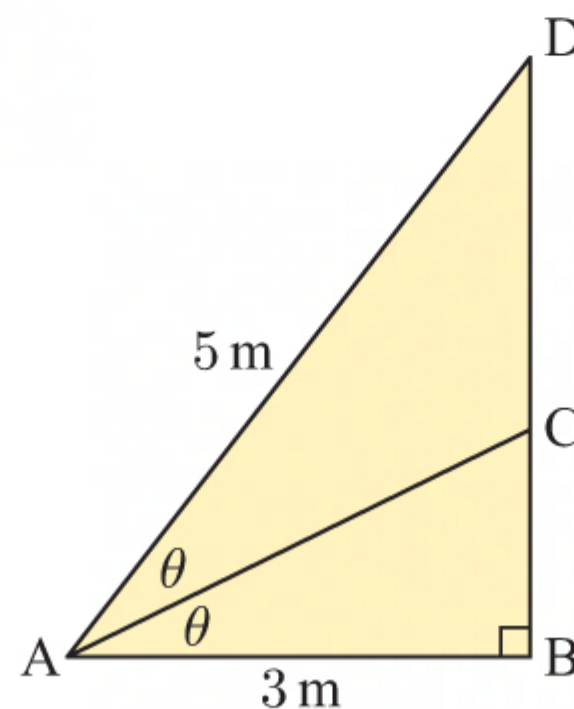
$$\begin{aligned} 11 \quad \frac{1 - \cos 2\theta}{\sin 2\theta} &= \sqrt{3}, \quad 0 < \theta < \frac{\pi}{2} \\ \therefore \frac{1 - (1 - 2 \sin^2 \theta)}{2 \sin \theta \cos \theta} &= \sqrt{3} \\ \therefore \frac{2 \sin^2 \theta}{2 \sin \theta \cos \theta} &= \sqrt{3} \\ \therefore \frac{\sin \theta}{\cos \theta} &= \sqrt{3} \quad \{\sin \theta \neq 0 \text{ on } 0 < \theta < \frac{\pi}{2}\} \\ \therefore \tan \theta &= \sqrt{3} \\ \therefore \theta &= \frac{\pi}{3} \end{aligned}$$

12 θ is acute $\therefore \cos \theta$ and $\sin \theta$ are positive.

$$\begin{aligned} \text{In } \triangle ABD, \quad \cos 2\theta &= \frac{3}{5} \\ \text{So, } 1 - 2 \sin^2 \theta &= \frac{3}{5} \\ \therefore 2 \sin^2 \theta &= \frac{2}{5} \\ \therefore \sin^2 \theta &= \frac{1}{5} \\ \therefore \sin \theta &= \frac{1}{\sqrt{5}} \quad \{\sin \theta > 0\} \end{aligned}$$

$$\begin{aligned} \text{or } 2 \cos^2 \theta - 1 &= \frac{3}{5} \\ \therefore 2 \cos^2 \theta &= \frac{8}{5} \\ \therefore \cos^2 \theta &= \frac{4}{5} \\ \therefore \cos \theta &= \frac{2}{\sqrt{5}} \quad \{\cos \theta > 0\} \end{aligned}$$

$$\begin{aligned} \text{In } \triangle ABC, \quad \tan \theta &= \frac{BC}{3} \\ \therefore BC &= 3 \tan \theta \\ &= 3 \times \frac{\sin \theta}{\cos \theta} \\ &= 3 \times \frac{\left(\frac{1}{\sqrt{5}}\right)}{\left(\frac{2}{\sqrt{5}}\right)} \\ &= \frac{3}{2} = 1.5 \end{aligned} \quad \therefore [BC] \text{ is } 1.5 \text{ m long.}$$



Chapter 10

REASONING AND PROOF

EXERCISE 10A

- 1**
 - a** The negation of “The cat is black” is “The cat is not black”.
 - b** The negation of “ x is prime” is “ x is not prime”, since x might be 1, or it might be a non-integer.
 - c** The negation of “The tree is deciduous” is “The tree is not deciduous”.
- 2**
 - a** The statement “If $x^2 = 9$ then $x = 3$ ” is false, since x may be -3 .
 - b** The statement “If $x = 3$ then $x^2 = 9$ ” is true, since $3^2 = 9$.
 - c** The statement “ $x = 3$ if and only if $x^2 = 9$ ” is false, since $x^2 = 9$ does not imply that $x = 3$ (from **a**).
- 3**
 - a** The statement “If x is positive then $\sqrt{x} \in \mathbb{R}$ ” is true, as the square root of any positive number is real.
 - b** The statement “If $\sqrt{x} \in \mathbb{R}$ then x is positive” is false, since $\sqrt{0} = 0 \in \mathbb{R}$, but 0 is not positive.
 - c** The statement “ x is positive if and only if $\sqrt{x} \in \mathbb{R}$ ” is false, since $\sqrt{x} \in \mathbb{R}$ does not imply that x is positive (from **b**).
- 4**
 - a** The converse of the statement “If Socrates is a cat then Socrates is an animal” is “If Socrates is an animal, then Socrates is a cat”.
 - b** The converse is false. If Socrates is an animal he is not necessarily a cat.
- 5**
 - a** $A: xyz = 0, \quad B: (x = 0) \vee (y = 0) \vee (z = 0)$
If $xyz = 0$, then one of x, y , or z must be zero. $\therefore A \Rightarrow B$
If one of x, y , or z is 0, then $xyz = 0$. $\therefore B \Rightarrow A$
 $A \Rightarrow B$ and $B \Rightarrow A$, $\therefore A$ and B are equivalent.
 - b** $A: x$ is even, $B: x^2$ is even
If x is even, then x^2 is even. $\therefore A \Rightarrow B$
However, if x^2 is even but not a perfect square, then x is irrational and hence not even.
For example, if $x^2 = 2$, then $x = \pm\sqrt{2}$ neither of which are even.
 $\therefore B \not\Rightarrow A$
 $\therefore A$ and B are not equivalent.

6 The statement which we are trying to establish the truth about is:

“Every card which has a D on one side has a 7 on the other”.

We are **only** concerned with cards that have (or might have) a D on one side.



We need to turn this card to make sure that the other side is a 7. Otherwise the statement would be false.



We need to turn this card to make sure that the other side is **not** a D. Otherwise the statement would be false.



We do not need to turn this card since the statement says nothing about cards with a letter other than D.



If we turn this card and the other side is a D, then the statement remains valid.

If it is **not** a D, it is of no concern to us.

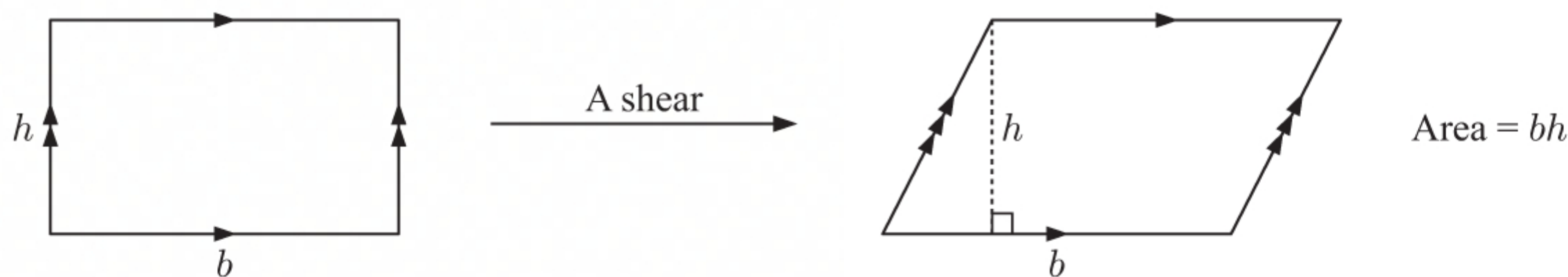
In either case, the statement remains valid, so we do not need to turn this card.

\therefore we need to turn cards D and 3. We do not need to turn cards K and 7.

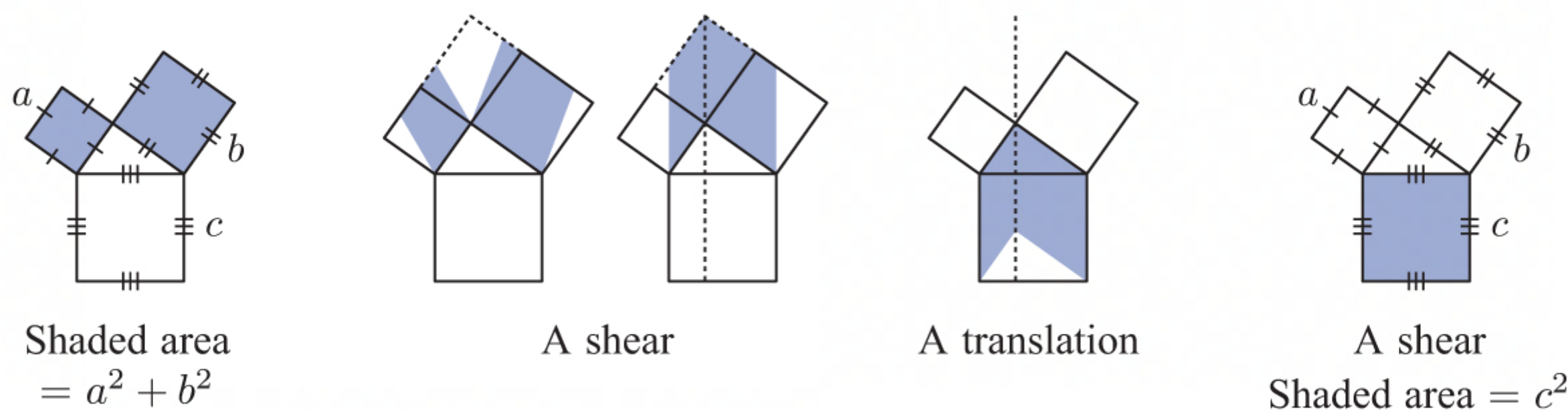
INVESTIGATION

PYTHAGORAS' THEOREM

1 When a shear is applied to a plane figure, the area does not change. For example, when a shear is applied to a parallelogram, the base length and height of the parallelogram do not change. Hence a shear does not change the area of a parallelogram.



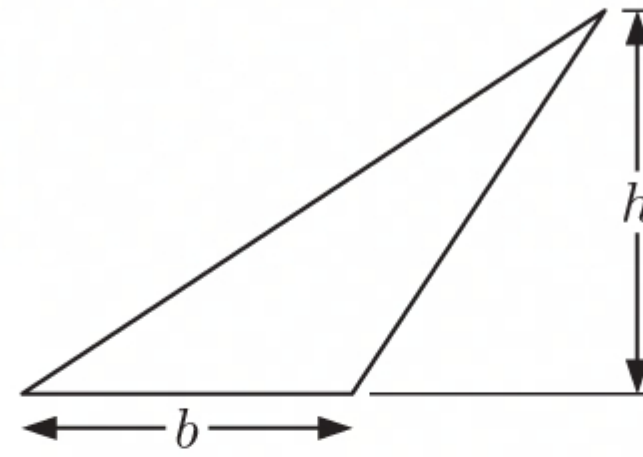
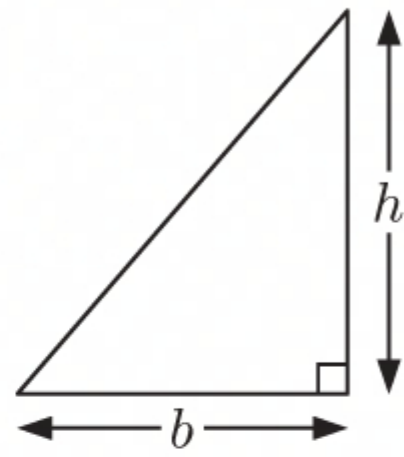
When a translation is applied to a plane figure, the area does not change, as the object and image are congruent.



The shaded area does not change in the sequence of shears and translations above.

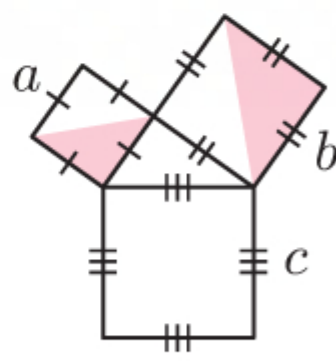
$$\therefore a^2 + b^2 = c^2$$

- 2 When a shear is applied to a triangle, the base length and height of the triangle do not change. Hence a shear does not change the area of a triangle.

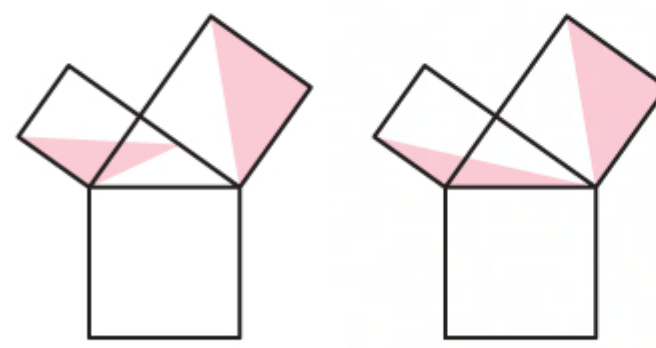


When a rotation is applied to a plane figure, the area does not change, as the object and image are congruent.

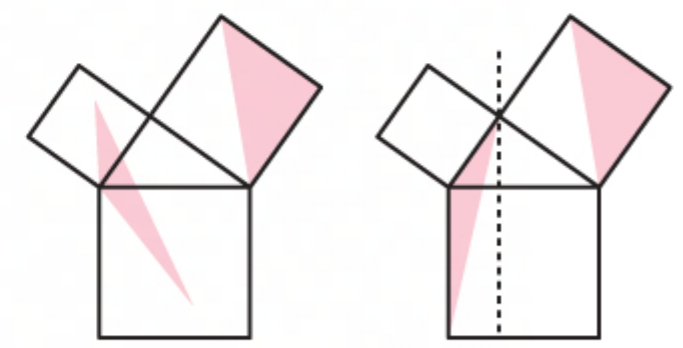
Using shears and rotations, we make the side lengths of the triangles match the side lengths of the area we are trying to fit them into.



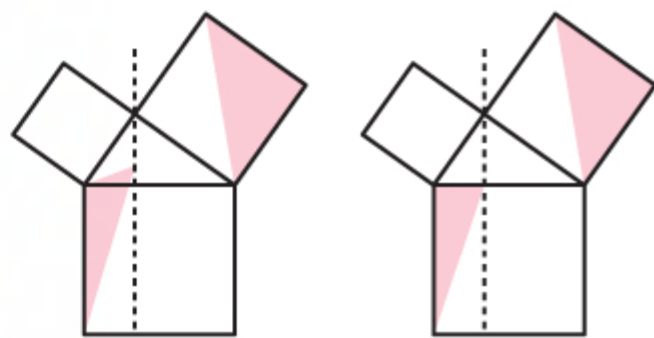
$$\text{Shaded area} = \frac{1}{2}a^2 + \frac{1}{2}b^2$$



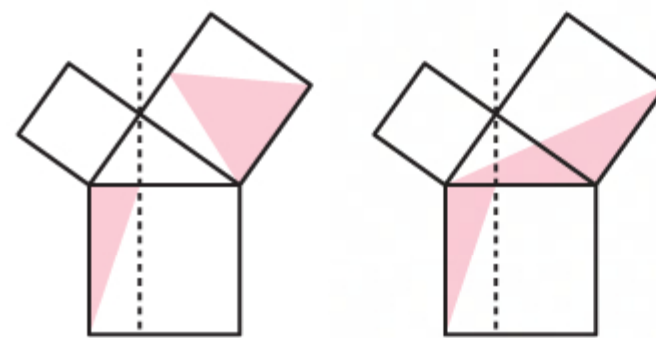
A shear



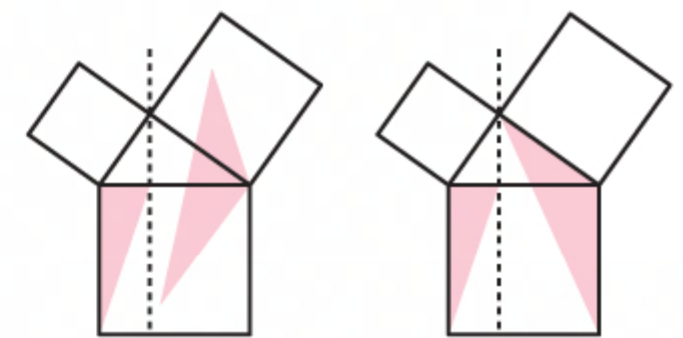
A rotation



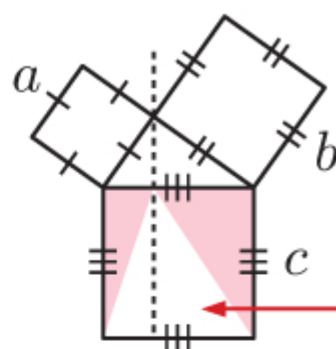
A shear



A shear



A rotation



A shear

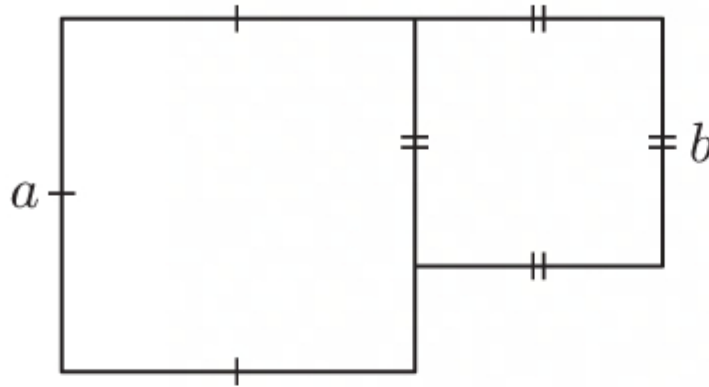
$$\begin{aligned} \text{Shaded area} &= c^2 - \frac{1}{2}c^2 \\ &= \frac{1}{2}c^2 \end{aligned}$$

The shaded area does not change in the sequence of shears and rotations above.

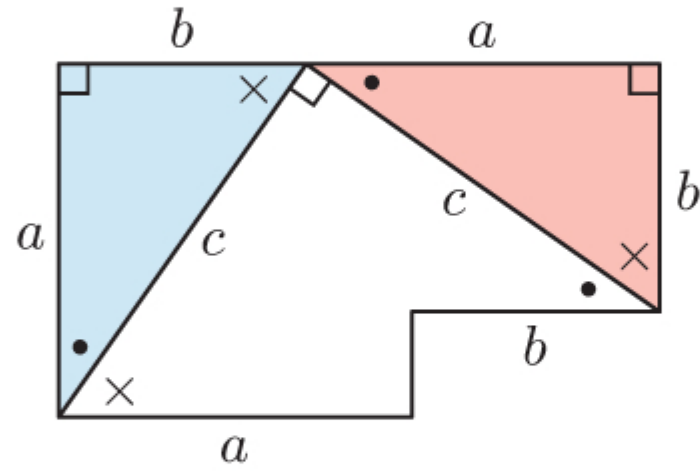
$$\therefore \frac{1}{2}a^2 + \frac{1}{2}b^2 = \frac{1}{2}c^2$$

$$\therefore a^2 + b^2 = c^2$$

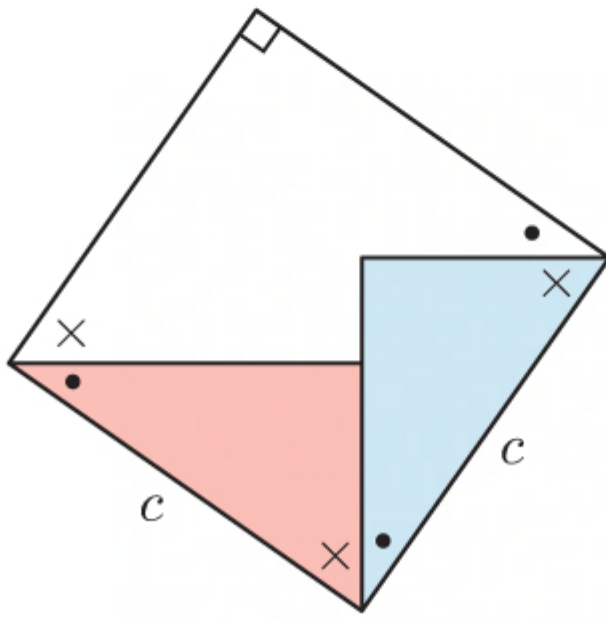
3



We have started with two squares with areas a^2 and b^2 . The total area is $a^2 + b^2$.



We have constructed the figure alongside (which has the same area as the figure above) with the side lengths shown. The hypotenuse of the right angled triangles is c , and $\bullet + \times = 90^\circ$.



We translate the red and blue triangles to the positions shown, which does not change the area of the figure.

We now have a square with side length c , and area c^2 .

$$\therefore a^2 + b^2 = c^2$$

EXERCISE 10B

1 a If $x = -2$ then $x^2 - x - 6 = (-2)^2 - (-2) - 6$
 $= 4 + 2 - 6$
 $= 0$

b The converse is “If $x^2 - x - 6 = 0$ then $x = -2$.”

However if $x^2 - x - 6 = 0$ then $(x + 2)(x - 3) = 0$
 $\therefore x = -2$ or 3

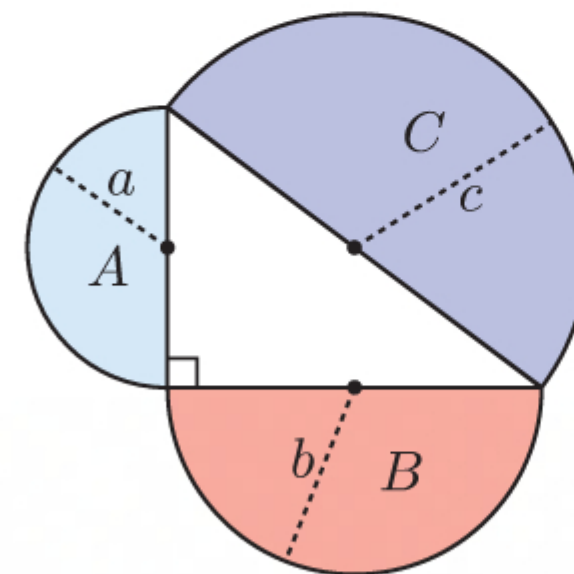
The converse is false, since x could be 3 .

2 The lengths of the sides of the triangle are $2a$, $2b$, and $2c$.

$$\begin{aligned} \therefore (2a)^2 + (2b)^2 &= (2c)^2 && \{\text{Pythagoras}\} \\ \therefore 4a^2 + 4b^2 &= 4c^2 \\ \therefore a^2 + b^2 &= c^2 \end{aligned}$$

Now $A = \frac{\pi a^2}{2}$, $B = \frac{\pi b^2}{2}$, $C = \frac{\pi c^2}{2}$

$$\begin{aligned} \therefore A + B &= \frac{\pi a^2}{2} + \frac{\pi b^2}{2} \\ &= \frac{\pi}{2}(a^2 + b^2) \\ &= \frac{\pi}{2} \times c^2 \\ &= C \end{aligned}$$



3 $x = a^2 - b^2, \quad y = 2ab, \quad z = a^2 + b^2, \quad a, b \in \mathbb{N}$

$$\begin{aligned} x^2 + y^2 &= (a^2 - b^2)^2 + (2ab)^2 \\ &= a^4 - 2a^2b^2 + b^4 + 4a^2b^2 \\ &= a^4 + 2a^2b^2 + b^4 \\ &= (a^2 + b^2)^2 \\ &= z^2 \end{aligned}$$

4 Let the middle number be x .

\therefore the sum of the three consecutive integers is $(x-1) + x + (x+1) = 3x$ which is divisible by 3.

5 Let the middle number be x .

\therefore the product of the three consecutive integers, increased by the middle integer is

$$\begin{aligned} (x-1)x(x+1) + x &= (x^2 - x)(x+1) + x \\ &= x^3 + x^2 - x^2 - x + x \\ &= x^3 \quad \text{which is a perfect cube.} \end{aligned}$$

6 $(a-b)^2 \geq 0$ for all $a, b \in \mathbb{R}$

$$\begin{aligned} \therefore a^2 - 2ab + b^2 &\geq 0 \\ \therefore a^2 + b^2 &\geq 2ab \\ \therefore \frac{a^2 + b^2}{2} &\geq ab \end{aligned}$$

7 $\sin 2\theta \tan \theta = 2 \sin \theta \cos \theta \times \frac{\sin \theta}{\cos \theta} \quad \{\text{double angle formula, definition of } \tan \theta\}$

$$= 2 \sin^2 \theta$$

8 Let the 3-digit number be “ abc ” which has value $100a + 10b + c$.

This number written backwards is “ cba ” which has value $100c + 10b + a$.

If “ abc ” $>$ “ cba ”, then $S = \text{“}abc\text{”} - \text{“}cba\text{”}$

$$\begin{aligned} &= 100a + 10b + c - (100c + 10b + a) \\ &= 100a + 10b + c - 100c - 10b - a \\ &= 99a - 99c \\ &= 99(a - c) \end{aligned}$$

Similarly, if “ cba ” $>$ “ abc ”, then $S = 99(c - a)$.

$$\therefore S = 99|a - c|$$

Now $0 < |a - c| \leq 9$ since $a \neq c$.

Let S' be S written backwards.

Consider the following table with each possible value of $|a - c|$:

$ a - c $	S	S'	$S + S'$
1	$99 \times 1 = 099$	990	1089
2	$99 \times 2 = 198$	891	1089
3	$99 \times 3 = 297$	792	1089
4	$99 \times 4 = 396$	693	1089
5	$99 \times 5 = 495$	594	1089
6	$99 \times 6 = 594$	495	1089
7	$99 \times 7 = 693$	396	1089
8	$99 \times 8 = 792$	297	1089
9	$99 \times 9 = 891$	198	1089

For each value of $|a - c|$, $S + S' = 1089$.

\therefore when S is written backwards and added to S , the result is always 1089.

9 a $4x^2 = 3x$

$\therefore 4x = 3$  Incorrect step

$\therefore x = \frac{3}{4}$

Correct solution:

$$4x^2 = 3x$$

$$\therefore 4x^2 - 3x = 0$$


$$\therefore x(4x - 3) = 0$$

$$\therefore x = 0 \quad \text{or} \quad 4x - 3 = 0$$

$$\therefore x = 0 \quad \text{or} \quad 4x = 3$$

$$\therefore x = 0 \quad \text{or} \quad x = \frac{3}{4}$$

b $(x + 3)(2 - x) = 4$

$\therefore x + 3 = 4 \quad \text{or} \quad 2 - x = 4$  Incorrect step

$\therefore x = 1 \quad \text{or} \quad x = -2$

Correct solution:

$$(x + 3)(2 - x) = 4$$

$$\therefore 2x - x^2 + 6 - 3x = 4$$

$$\therefore x^2 + x - 2 = 0$$

$$\therefore (x + 2)(x - 1) = 0$$

$$\therefore x = -2 \quad \text{or} \quad 1$$

EXERCISE 10C

1 a $(a + b)^2 - (a - b)^2 = [a + b + (a - b)][a + b - (a - b)] \quad \{\text{difference of two squares}\}$
 $= 2a \times 2b$
 $= 4ab$

b $(a + b)^2 - 4(a - b)^2 = (a + b)^2 - [2(a - b)]^2$
 $= [a + b + 2(a - b)][a + b - 2(a - b)] \quad \{\text{difference of two squares}\}$
 $= (a + b + 2a - 2b)(a + b - 2a + 2b)$
 $= (3a - b)(3b - a)$
 $= (3b - a)(3a - b)$

$$\begin{aligned}
 & \mathbf{2} \quad x^2 + (a-3)x + 2a = 0 \\
 \Leftrightarrow & \left(x + \frac{a-3}{2}\right)^2 - \left(\frac{a-3}{2}\right)^2 + 2a = 0 \quad \{\text{completing the square}\} \\
 \Leftrightarrow & \left(x + \frac{a-3}{2}\right)^2 = \frac{(a-3)^2}{4} - 2a
 \end{aligned}$$

The equation has real solutions if and only if

$$\begin{aligned}
 & \frac{(a-3)^2}{4} - 2a \geq 0 \\
 \Leftrightarrow & (a-3)^2 - 8a \geq 0 \\
 \Leftrightarrow & a^2 - 6a + 9 - 8a \geq 0 \\
 \Leftrightarrow & a^2 - 14a + 9 \geq 0 \\
 \Leftrightarrow & (a-7)^2 - 49 + 9 \geq 0 \\
 \Leftrightarrow & (a-7)^2 \geq 40 \\
 \Leftrightarrow & a-7 \geq \sqrt{40} \quad \text{or} \quad a-7 \leq -\sqrt{40} \\
 \Leftrightarrow & a \geq 7 + \sqrt{40} \approx 13.3 \quad \text{or} \quad a \leq 7 - \sqrt{40} \approx 0.675
 \end{aligned}$$

\therefore the smallest positive integer a for which the equation has real solutions is $a = 14$.

$$\begin{aligned}
 & \mathbf{3} \quad (x-y)^5 + (x-y)^3 = 0 \\
 \Leftrightarrow & (x-y)^3[(x-y)^2 + 1] = 0 \\
 \Leftrightarrow & (x-y)^3 = 0 \quad \text{or} \quad (x-y)^2 + 1 = 0 \\
 \Leftrightarrow & x = y \quad \text{or} \quad (x-y)^2 = -1 \\
 & \quad \quad \quad \text{which is not possible}
 \end{aligned}$$

$$\therefore (x-y)^5 + (x-y)^3 = 0 \Leftrightarrow x = y$$

$$\begin{aligned}
 & \mathbf{4} \quad \mathbf{a} \quad (n^2 - 2n + 2)(n^2 + 2n + 2) = n^4 + \cancel{2n^3} + \cancel{2n^2} - \cancel{2n^3} - \cancel{4n^2} - \cancel{4n} + \cancel{2n^2} + \cancel{4n} + 4 \\
 & \quad \quad \quad = n^4 + 4
 \end{aligned}$$

$$\begin{aligned}
 & \mathbf{b} \quad \text{Notice that } n^2 - 2n + 2 = n^2 - 2n + 1 + 1 = (n-1)^2 + 1 \\
 & \quad \quad \text{and } n^2 + 2n + 2 = n^2 + 2n + 1 + 1 = (n+1)^2 + 1
 \end{aligned}$$

$$\begin{aligned}
 & \text{For } n \neq 1, \quad (n-1)^2 > 0 \quad \text{and for } n \neq -1, \quad (n+1)^2 > 0 \\
 & \quad \quad \therefore (n-1)^2 + 1 > 1 \quad \quad \quad \therefore (n+1)^2 + 1 > 1 \\
 & \quad \quad \therefore n^2 - 2n + 2 > 1 \quad \quad \quad \therefore n^2 + 2n + 2 > 1
 \end{aligned}$$

So, for $n \neq \pm 1$, $n^4 + 4 = (n^2 - 2n + 2)(n^2 + 2n + 2)$ is composite, since $n^2 - 2n + 2 > 1$ and $n^2 + 2n + 2 > 1$.

Now, for $n = 1$, $n^4 + 4 = 1^4 + 4 = 5$ which is prime
and for $n = -1$, $n^4 + 4 = (-1)^4 + 4 = 5$ which is also prime.

$$\therefore n^4 + 4 \text{ is prime} \Leftrightarrow n = \pm 1$$

$$\begin{aligned}
 & \mathbf{5} \quad \mathbf{a} \quad (k+1)^2 - (k-1)^2 = [k+1 + (k-1)][k+1 - (k-1)] \quad \{\text{difference of two squares}\} \\
 & \quad \quad \quad = 2k \times 2 \\
 & \quad \quad \quad = 4k
 \end{aligned}$$

$$\begin{aligned}
 & \mathbf{b} \quad \mathbf{i} \quad \text{Substituting } 4k = 40 \text{ or } k = 10 \text{ gives } (10+1)^2 - (10-1)^2 = 40 \\
 & \quad \quad \quad \therefore 11^2 - 9^2 = 40 \\
 & \quad \quad \quad \therefore 121 - 81 = 40
 \end{aligned}$$

So, 121 and 81 are two square numbers with difference 40.

- ii Substituting $4k = 100$ or $k = 25$ gives $(25 + 1)^2 - (25 - 1)^2 = 100$
 $\therefore 26^2 - 24^2 = 100$
 $\therefore 676 - 576 = 100$

So, 676 and 576 are two square numbers with difference 100.

6 a

$$\begin{aligned}
 & a = b \\
 \Leftrightarrow & a^2 = ab \\
 \Leftrightarrow & a^2 - b^2 = ab - b^2 \\
 \Leftrightarrow & (a - b)(a + b) = b(a - b) \\
 \Leftrightarrow & a + b = b \quad \leftarrow \text{incorrect step, if } a = b \text{ then } a - b = 0, \text{ and we} \\
 \Leftrightarrow & 2a = a \quad \text{cannot divide both sides by } a - b. \\
 \Leftrightarrow & 2 = 1 \quad \leftarrow \text{incorrect step, if } a = 0 \text{ then we cannot divide} \\
 & \quad \text{both sides by } a.
 \end{aligned}$$

b

$$\begin{aligned}
 & \frac{x + 10}{x - 6} - 5 = \frac{4x - 40}{13 - x} \\
 \Leftrightarrow & \frac{x + 10 - 5(x - 6)}{x - 6} = \frac{4x - 40}{13 - x} \\
 \Leftrightarrow & \frac{4x - 40}{6 - x} = \frac{4x - 40}{13 - x} \\
 \Leftrightarrow & 6 - x = 13 - x \quad \leftarrow \text{incorrect step, if } 4x - 40 = 0 \text{ then we cannot} \\
 \Leftrightarrow & 6 = 13 \quad \text{divide both sides by } 4x - 40 \text{ and take the} \\
 & \quad \text{reciprocal.}
 \end{aligned}$$

Correct solution:

$$\begin{aligned}
 & \frac{4x - 40}{6 - x} = \frac{4x - 40}{13 - x} \\
 \Leftrightarrow & (4x - 40)(13 - x) = (6 - x)(4x - 40) \\
 \Leftrightarrow & 52x - \cancel{4x^2} - 520 + \cancel{40x} = 24x - 240 - \cancel{4x^2} + \cancel{40x} \\
 \Leftrightarrow & 28x = 280 \\
 \Leftrightarrow & x = 10
 \end{aligned}$$

7 a

$$\begin{aligned}
 & 6x - 12 = 3(x - 2) \\
 \Leftrightarrow & 6x - 12 + 3(x - 2) = 0 \quad \leftarrow \text{incorrect step, we should have subtracted} \\
 \Leftrightarrow & 12x - 24 = 0 \quad 3(x - 2) \text{ from both sides.} \\
 \Leftrightarrow & x = 2
 \end{aligned}$$

Correct solution:

$$\begin{aligned}
 & 6x - 12 = 3(x - 2) \\
 \Leftrightarrow & 6x - 12 - 3(x - 2) = 0 \\
 \Leftrightarrow & 6x - 12 - 3x + 6 = 0 \\
 \Leftrightarrow & 3x = 6 \\
 \Leftrightarrow & x = 2
 \end{aligned}$$

$$\begin{aligned}
 \text{b} \quad & x^2 - 6x + 9 = 0 \\
 \Leftrightarrow & x^2 - 6x = -9 \\
 \Leftrightarrow & x(x - 6) = 3(-3) \\
 \Leftrightarrow & x = 3 \text{ or } x - 6 = -3 \quad \leftarrow \text{incorrect step, in general this is not true.} \\
 \Leftrightarrow & x = 3
 \end{aligned}$$

Correct solution:

$$\begin{aligned}
 & x^2 - 6x + 9 = 0 \\
 \Leftrightarrow & (x - 3)^2 = 0 \\
 \Leftrightarrow & x = 3
 \end{aligned}$$

$$\begin{aligned}
 8 \quad \text{a} \quad \text{i} \quad & \frac{1}{m} + \frac{1}{3m} = \frac{3}{3m} + \frac{1}{3m} \\
 & = \frac{4}{3m}
 \end{aligned}$$

$$\begin{aligned}
 \text{ii} \quad \text{For } n \neq 0, -1, \quad & \frac{1}{n+1} + \frac{1}{n^2+n} = \frac{n}{n(n+1)} + \frac{1}{n(n+1)} \\
 & = \frac{n+1}{n(n+1)} \\
 & = \frac{1}{n}
 \end{aligned}$$

b No, $\frac{1}{n+1} + \frac{1}{n^2+n}$ is undefined for $n = 0, -1$ while $\frac{1}{n}$ is only undefined for $n = 0$. It would be incorrect to say that $\frac{1}{n+1} + \frac{1}{n^2+n}$ and $\frac{1}{n}$ are equivalent even though $\frac{1}{n+1} + \frac{1}{n^2+n} = \frac{1}{n}$ for $n \neq 0, -1$.

EXERCISE 10D

1 Let $x = 0.\overline{9} = 0.999\dots$

$$\begin{aligned}
 \Rightarrow & 10x = 9.999\dots \\
 \Rightarrow & 10x = 9 + x \\
 \Rightarrow & 9x = 9 \\
 \Rightarrow & x = 1 \\
 \Rightarrow & x \in \mathbb{Z} \\
 \Rightarrow & 0.\overline{9} \in \mathbb{Z}
 \end{aligned}$$

2 a Let $x = 0.\overline{4} = 0.444\dots$

$$\begin{aligned}
 \Rightarrow & 10x = 4.444\dots \\
 \Rightarrow & 10x = 4 + x \\
 \Rightarrow & 9x = 4 \\
 \Rightarrow & x = \frac{4}{9} \\
 \Rightarrow & x \in \mathbb{Q} \\
 \Rightarrow & 0.\overline{4} \in \mathbb{Q}
 \end{aligned}$$

b Let $x = 0.\overline{23} = 0.2323\dots$

$$\begin{aligned}
 \Rightarrow & 100x = 23.2323\dots \\
 \Rightarrow & 100x = 23 + x \\
 \Rightarrow & 99x = 23 \\
 \Rightarrow & x = \frac{23}{99} \\
 \Rightarrow & x \in \mathbb{Q} \\
 \Rightarrow & 0.\overline{23} \in \mathbb{Q}
 \end{aligned}$$

$$\begin{aligned}
 \bullet \text{ Let } x &= 0.0\overline{79} = 0.079\,79\dots \\
 \Rightarrow 1000x &= 79.7979\dots \quad \text{and} \quad 10x = 0.7979\dots \\
 \Rightarrow 1000x &= 79 + 10x \\
 \Rightarrow 990x &= 79 \\
 \Rightarrow x &= \frac{79}{990} \\
 \Rightarrow x &\in \mathbb{Q} \\
 \Rightarrow 0.0\overline{79} &\in \mathbb{Q}
 \end{aligned}$$

$$\mathbf{3} \quad (4 - \sqrt{2}) + \frac{4 - \sqrt{2}}{3 - \sqrt{2}} + \frac{4 - \sqrt{2}}{(3 - \sqrt{2})^2} + \frac{4 - \sqrt{2}}{(3 - \sqrt{2})^3} + \dots$$

is a geometric series with $u_1 = 4 - \sqrt{2}$ and $r = \frac{1}{3 - \sqrt{2}}$.

$$\begin{aligned}
 \therefore (4 - \sqrt{2}) + \frac{4 - \sqrt{2}}{3 - \sqrt{2}} + \frac{4 - \sqrt{2}}{(3 - \sqrt{2})^2} + \frac{4 - \sqrt{2}}{(3 - \sqrt{2})^3} + \dots &= \frac{4 - \sqrt{2}}{1 - \left(\frac{1}{3 - \sqrt{2}}\right)} \quad \left\{ \frac{u_1}{1 - r} \right\} \\
 &= \frac{4 - \sqrt{2}}{\frac{3 - \sqrt{2}}{3 - \sqrt{2}} - \left(\frac{1}{3 - \sqrt{2}}\right)} \\
 &= \frac{4 - \sqrt{2}}{\left(\frac{2 - \sqrt{2}}{3 - \sqrt{2}}\right)} \\
 &= (4 - \sqrt{2}) \times \frac{3 - \sqrt{2}}{2 - \sqrt{2}} \\
 &= \frac{12 - 4\sqrt{2} - 3\sqrt{2} + 2}{2 - \sqrt{2}} \\
 &= \frac{14 - 7\sqrt{2}}{2 - \sqrt{2}} \\
 &= \frac{7(2 - \sqrt{2})}{2 - \sqrt{2}}
 \end{aligned}$$

$$\therefore (4 - \sqrt{2}) + \frac{4 - \sqrt{2}}{3 - \sqrt{2}} + \frac{4 - \sqrt{2}}{(3 - \sqrt{2})^2} + \frac{4 - \sqrt{2}}{(3 - \sqrt{2})^3} + \dots = 7 \in \mathbb{Z}$$

$$\therefore (4 - \sqrt{2}) + \frac{4 - \sqrt{2}}{3 - \sqrt{2}} + \frac{4 - \sqrt{2}}{(3 - \sqrt{2})^2} + \frac{4 - \sqrt{2}}{(3 - \sqrt{2})^3} + \dots \in \mathbb{Z}$$

$\mathbf{4}$ Let x and y be rational numbers.

By definition, there exists $p, q \in \mathbb{Z}$, $q \neq 0$ so that $x = \frac{p}{q}$.

By definition, there exists $r, s \in \mathbb{Z}$, $s \neq 0$ so that $y = \frac{r}{s}$.

$$\text{So, } x - y = \frac{p}{q} - \frac{r}{s} = \frac{ps - qr}{qs}$$

Since p, q, r, s are all integers, $ps - qr$ is an integer which we call P .

Since q, s are non-zero integers, qs is a non-zero integer which we call Q .

$$\therefore x - y = \frac{ps - qr}{qs} = \frac{P}{Q} \quad \text{where } P, Q \in \mathbb{Z}, Q \neq 0$$

\therefore by definition, $x - y$ is a rational number.

\therefore the difference between any two rational numbers is also a rational number.

- 5** Let x and y be rational numbers.

By definition, there exists $p, q \in \mathbb{Z}$, $q \neq 0$ so that $x = \frac{p}{q}$.

By definition, there exists $r, s \in \mathbb{Z}$, $s \neq 0$ so that $y = \frac{r}{s}$.

$$\text{So, } xy = \frac{pr}{qs}$$

Since p and r are integers, pr is an integer which we call P .

Since q and s are non-zero integers, qs is a non-zero integer which we call Q .

$$\therefore xy = \frac{pr}{qs} = \frac{P}{Q} \text{ where } P, Q \in \mathbb{Z}, Q \neq 0$$

\therefore by definition, xy is a rational number.

\therefore the product of any two rational numbers is also a rational number.

- 6** Let the two odd integers be $2a + 1$ and $2b + 1$, $a, b \in \mathbb{Z}$.

$$\begin{aligned} \text{Now } (2a + 1)(2b + 1) &= 4ab + 2a + 2b + 1 \\ &= 2(2ab + a + b) + 1 \end{aligned}$$

Since a and b are integers, $2ab + a + b$ is an integer which we call c .

$$\therefore (2a + 1)(2b + 1) = 2c + 1 \text{ which is odd.}$$

\therefore the product of two odd integers is odd.

- 7** Let the two odd integers be $p = 2a + 1$ and $q = 2b + 1$, $a, b \in \mathbb{Z}$.

$$\begin{aligned} \text{Now } p^2 - q^2 &= (p + q)(p - q) \\ &= (2a + 1 + 2b + 1)(2a + 1 - 2b - 1) \\ &= (2a + 2b + 2)(2a - 2b) \\ &= 4(a + b + 1)(a - b) \\ &= 4(a^2 - ab + ba - b^2 + a - b) \\ &= 4(a^2 + a - b^2 - b) \\ &= 4[a(a + 1) - b(b + 1)] \end{aligned}$$

As a and $a + 1$ are consecutive integers, a or $a + 1$ is even, so $a(a + 1)$ is even.

Similarly, $b(b + 1)$ is even.

$$\text{Let } a(a + 1) = 2c, \quad c \in \mathbb{Z}$$

$$\text{and } b(b + 1) = 2d, \quad d \in \mathbb{Z}$$

$$\begin{aligned} \therefore p^2 - q^2 &= 4(2c - 2d) \\ &= 8(c - d) \text{ where } c - d \text{ is an integer.} \end{aligned}$$

$$\therefore p^2 - q^2 \text{ is divisible by 8.}$$

- 8** Let the two consecutive odd integers be $p = 2a + 3$ and $q = 2a + 1$, $a \in \mathbb{Z}$.

$$\begin{aligned} \text{Now } p^3 - q^3 - 2 &= (2a + 3)^3 - (2a + 1)^3 - 2 \\ &= (2a)^3 + 3(2a)^2(3) + 3(2a)(3)^2 + (3)^3 - [(2a)^3 + 3(2a)^2(1) + 3(2a)(1)^2 + (1)^3] - 2 \\ &= 8a^3 + 36a^2 + 54a + 27 - (8a^3 + 12a^2 + 6a + 1) - 2 \\ &= \cancel{8a^3} + 36a^2 + 54a + 27 - \cancel{8a^3} - 12a^2 - 6a - 1 - 2 \\ &= 24a^2 + 48a + 24 \\ &= 24(a^2 + 2a + 1) \\ &= 24(a + 1)^2 \text{ where } (a + 1)^2 \text{ is an integer.} \end{aligned}$$

$$\therefore p^3 - q^3 - 2 \text{ is divisible by 24.}$$

9 $ax^2 + bx + c = 0$ has rational root $\frac{r}{s}$, $r, s \in \mathbb{Z}$, $s \neq 0$.

\therefore the other root (which may be repeated) must also be rational, say $\frac{t}{u}$, $t, u \in \mathbb{Z}$, $u \neq 0$.

$$\begin{aligned}\therefore ax^2 + bx + c &= (sx - r)(ux - t), \quad r, s, t, u \in \mathbb{Z} \\ &= (su)x^2 - (st + ru)x + rt\end{aligned}$$

Equating coefficients of x^2 , $a = su$, $u \in \mathbb{Z}$

$\therefore s$ is a factor of a .

Equating constant terms, $c = rt$, $t \in \mathbb{Z}$

$\therefore r$ is a factor of c .

REVIEW SET 10A

1 $f(x) = x^2 + px + q$ has roots a and b

$$\therefore x^2 + px + q = (x - a)(x - b)$$

$$\therefore x^2 + px + q = x^2 - bx - ax + ab$$

$$\therefore x^2 + px + q = x^2 - (a + b)x + ab$$

Equating coefficients of x , $p = -(a + b)$

Equating constant terms, $q = ab$

2 a Let $x = 2.\overline{9} = 2.999\dots$

$$\Rightarrow 10x = 29.999\dots$$

$$\Rightarrow 10x = 27 + x$$

$$\Rightarrow 9x = 27$$

$$\Rightarrow x = 3$$

$$\Rightarrow x \in \mathbb{Z}$$

$$\Rightarrow 2.\overline{9} \in \mathbb{Z}$$

b Let $x = 0.\overline{38} = 0.3838\dots$

$$\Rightarrow 100x = 38.3838\dots$$

$$\Rightarrow 100x = 38 + x$$

$$\Rightarrow 99x = 38$$

$$\Rightarrow x = \frac{38}{99}$$

$$\Rightarrow x \in \mathbb{Q}$$

$$\Rightarrow 0.\overline{38} \in \mathbb{Q}$$

3 a The negation of “The boy has blue eyes” is “The boy does not have blue eyes”.

b The negation of “ x is larger than 4” is “ x is not larger than 4”.

4 a If a function f is periodic with period p , then $f(x + p) = f(x)$ for all x , by definition.

\therefore the statement is true.

b If $f(x + p) = f(x)$ for all x , the period of f could be p , $\frac{p}{2}$, or $\frac{p}{3}$, and so on.

\therefore the statement is false.

c The statement “ f is periodic with period p if and only if $f(x + p) = f(x)$ for all x ” is false, since $f(x + p) = f(x)$ for all $x \not\Rightarrow f$ is periodic with period p (from b).

5 $(3a - b)^2 \geq 0$, $a, b \in \mathbb{R}$

$$\Leftrightarrow (3a)^2 - 2(3a)(b) + (-b)^2 \geq 0$$

$$\Leftrightarrow 9a^2 - 6ab + b^2 \geq 0$$

$$\Leftrightarrow 9a^2 + b^2 \geq 6ab$$

$$\begin{aligned}
6 \quad a \quad & x = \tan^2 \theta - \sin^2 \theta \\
& \Leftrightarrow x = \frac{\sin^2 \theta}{\cos^2 \theta} - \sin^2 \theta \\
& \Leftrightarrow x = \frac{\sin^2 \theta - \sin^2 \theta \cos^2 \theta}{\cos^2 \theta} \\
& \Leftrightarrow x = \frac{\sin^2 \theta (1 - \cos^2 \theta)}{\cos^2 \theta} \\
& \Leftrightarrow x = \frac{\sin^2 \theta}{\cos^2 \theta} \times \sin^2 \theta \\
& \Leftrightarrow x = \tan^2 \theta \sin^2 \theta \\
& \therefore \tan^2 \theta - \sin^2 \theta = \tan^2 \theta \sin^2 \theta
\end{aligned}$$

$$\begin{aligned}
b \quad i \quad & \frac{1 - \sin^2 \theta}{\cos \theta} = \frac{\cos^2 \theta}{\cos \theta} \\
& = \cos \theta \\
ii \quad & \frac{1 - \sin^2 \theta}{\cos \theta} \text{ is undefined for } \theta = \frac{(2n+1)\pi}{2}, \quad n \in \mathbb{Z}, \text{ while } \cos \theta \text{ is defined for all } \\
& \theta \in \mathbb{R}. \text{ It would be incorrect to say that } \frac{1 - \sin^2 \theta}{\cos \theta} \text{ and } \cos \theta \text{ are equivalent, even} \\
& \text{though } \frac{1 - \sin^2 \theta}{\cos \theta} = \cos \theta \text{ for all } \theta \neq \frac{(2n+1)\pi}{2}, \quad n \in \mathbb{Z}.
\end{aligned}$$

7 Let x and y be non-zero rational numbers.

By definition, there exists $p, q \in \mathbb{Z}$, $p, q \neq 0$ so that $x = \frac{p}{q}$.

By definition, there exists $r, s \in \mathbb{Z}$, $r, s \neq 0$ so that $y = \frac{r}{s}$.

$$\begin{aligned}
\text{So, } \frac{x}{y} &= \frac{\frac{p}{q}}{\frac{r}{s}} \\
&= \frac{p}{q} \times \frac{s}{r} \\
&= \frac{ps}{qr}
\end{aligned}$$

Since p and s are non-zero integers, ps is a non-zero integer which we call P .

Since q and r are non-zero integers, qr is a non-zero integer which we call Q .

$$\therefore \frac{x}{y} = \frac{ps}{qr} = \frac{P}{Q} \text{ where } P, Q \in \mathbb{Z}, P, Q \neq 0$$

\therefore by definition, $\frac{x}{y}$ is a rational number.

\therefore the quotient of two rational numbers is also a rational number.

$$\begin{aligned}
8 \quad a \quad i \quad & 1^3 + 2^3 = 1 + 8 \\
& = 9
\end{aligned}$$

which is composite since $9 = 3 \times 3$.

$$\begin{aligned}
ii \quad & 2^3 + 3^3 = 8 + 27 \\
& = 35
\end{aligned}$$

which is composite since $35 = 5 \times 7$.

$$\text{iii } 3^3 + 4^3 = 27 + 64 \\ = 91$$

which is composite since $91 = 7 \times 13$.

$$\text{iv } 4^3 + 5^3 = 64 + 125 \\ = 189$$

which is composite since $189 = 9 \times 21$.

$$\begin{aligned} \text{b } k^3 + (k+1)^3 &= k^3 + k^3 + 3k^2 + 3k + 1 \\ &= 2k^3 + 3k^2 + 3k + 1 \\ &= 2k^3 + 2k^2 + 2k + k^2 + k + 1 \\ &= 2k(k^2 + k + 1) + k^2 + k + 1 \\ &= (2k+1)(k^2 + k + 1) \end{aligned}$$

- c For all $k \in \mathbb{Z}^+$, $k^3 + (k+1)^3$ will always have factors $(2k+1)$ and $(k^2 + k + 1)$.
 \therefore the sum of two consecutive positive cubes is always composite.

REVIEW SET 10B

$$\begin{aligned} \text{1 } p(x) &= x^2 + 2bx + c, \quad q(x) = p(x-b) \\ &= (x-b)^2 + 2b(x-b) + c \\ &= x^2 - 2bx + b^2 + 2bx - 2b^2 + c \\ &= x^2 - b^2 + c \\ q(-x) &= p(-x-b) \\ &= (-x-b)^2 + 2b(-x-b) + c \\ &= (-x)^2 + 2(-x)(-b) + (-b)^2 - 2bx - 2b^2 + c \\ &= x^2 + 2bx + b^2 - 2bx - 2b^2 + c \\ &= x^2 - b^2 + c \end{aligned}$$

$\therefore q(x) = q(-x)$ for all x .

$$\begin{aligned} \text{2 } (\sqrt{a} - \sqrt{b})^2 &\geq 0 \quad \text{for all } a, b \in \mathbb{R}^+ \\ \Leftrightarrow (\sqrt{a})^2 - 2\sqrt{a}\sqrt{b} + (\sqrt{b})^2 &\geq 0 \\ \Leftrightarrow a - 2\sqrt{ab} + b &\geq 0 \\ \Leftrightarrow a + b &\geq 2\sqrt{ab} \\ \Leftrightarrow \frac{a+b}{2} &\geq \sqrt{ab} \end{aligned}$$

- 3 a The statement “If x is acute then $\sin x$ is positive” is true as $\sin x > 0$ whenever $0 < x < \frac{\pi}{2}$.
 b The converse of the statement “If x is acute then $\sin x$ is positive” is “If $\sin x$ is positive then x is acute”.
 c The converse is false. If $\frac{\pi}{2} < x < \pi$, then $\sin x > 0$. So, x could be obtuse and $\sin x$ would still be positive.

- 4 a A: x is not prime, B: x is composite
 A and B are not equivalent since $x = 1$ is neither prime nor composite.

b A : x and y are both odd integers, B : xy is odd

If x and y are odd integers then xy is odd. {from **Exercise 10D 6**}

$\therefore A \Rightarrow B$

If, for example, $xy = 3$, then $x = y = \sqrt{3}$ is a possible solution, but $\sqrt{3}$ is not odd.

$\therefore B \not\Rightarrow A$

So A and B are not equivalent.

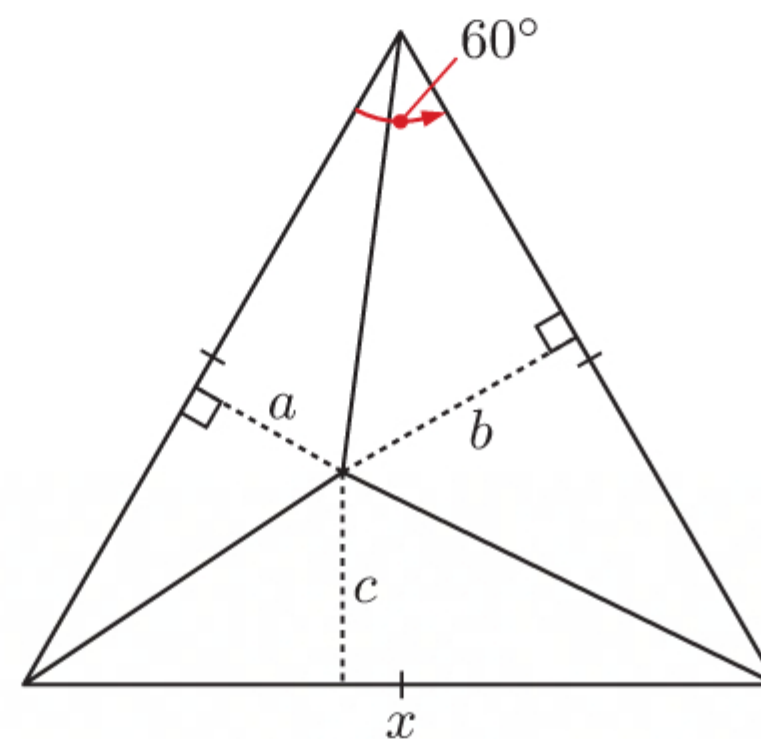
5 Let $k = 2a + 1$, $a \in \mathbb{Z}$

$$\begin{aligned} \therefore k^3 + k^2 - k - 1 &= (k^2 - 1)(k + 1) \\ &= [(2a + 1)^2 - 1][2a + 1 + 1] \\ &= (4a^2 + 4a + 1 - 1)(2a + 2) \\ &= (4a^2 + 4a)(2a + 2) \\ &= 8(a^2 + a)(a + 1) \quad \text{where } (a^2 + a) \text{ and } (a + 1) \text{ are integers.} \end{aligned}$$

$\therefore k^3 + k^2 - k - 1$ is divisible by 8.

6 Given a point in an equilateral triangle, let a , b , and c be the distances from the point to each side of the triangle, as shown alongside. We draw lines from the point to each corner which divides the triangle into three smaller triangles with heights a , b , and c , and with total area

$$\begin{aligned} & \left(\frac{1}{2} \times x \times a\right) + \left(\frac{1}{2} \times x \times b\right) + \left(\frac{1}{2} \times x \times c\right) \\ &= \frac{1}{2}x(a + b + c) \end{aligned}$$



$$\begin{aligned} \text{But the whole triangle has area } & \frac{1}{2} \times x \times x \times \sin 60^\circ \quad \{\text{sine rule}\} \\ &= \frac{1}{2}x^2 \times \frac{\sqrt{3}}{2} \\ &= \frac{\sqrt{3}}{4}x^2 \end{aligned}$$

$$\therefore \frac{1}{2}x(a + b + c) = \frac{\sqrt{3}}{4}x^2$$

$$\therefore a + b + c = \frac{\sqrt{3}}{2}x \quad \text{which is constant for a fixed value of } x.$$

\therefore in an equilateral triangle the sum of the distances from any point in the triangle to the three sides is a constant.

7 The two digit number “ ab ” has value $10a + b$ and the two digit number “ ba ” has value $10b + a$.

$$\begin{aligned} \therefore \text{“}ab\text{”} - \text{“}ba\text{”} &= 10a + b - (10b + a) \\ &= 10a + b - 10b - a \\ &= 9a - 9b \\ &= 9(a - b) \quad \text{where } (a - b) \text{ is an integer.} \end{aligned}$$

\therefore the difference between the two digit numbers “ ab ” and “ ba ” is always divisible by 9.

8

$$-6 = -6$$

$$\Leftrightarrow 9 - 15 = 4 - 10$$

$$\Leftrightarrow 3^2 - 3 \times 5 = 2^2 - 2 \times 5$$

$$\Leftrightarrow 3^2 - 2 \times 3 \times \frac{5}{2} + \left(\frac{5}{2}\right)^2 = 2^2 - 2 \times 2 \times \frac{5}{2} + \left(\frac{5}{2}\right)^2$$

$$\Leftrightarrow \left(3 - \frac{5}{2}\right)^2 = \left(2 - \frac{5}{2}\right)^2$$

$$\Leftrightarrow 3 - \frac{5}{2} = 2 - \frac{5}{2} \quad \leftarrow \text{incorrect step, } a^2 = b^2 \not\Rightarrow a = b$$

$$\Leftrightarrow 3 = 2 \quad \begin{array}{l} \text{If } a^2 = b^2, \text{ then } a^2 - b^2 = 0, \\ (a + b)(a - b) = 0. \end{array}$$

Chapter 11

INTRODUCTION TO DIFFERENTIAL CALCULUS

EXERCISE 11A.1

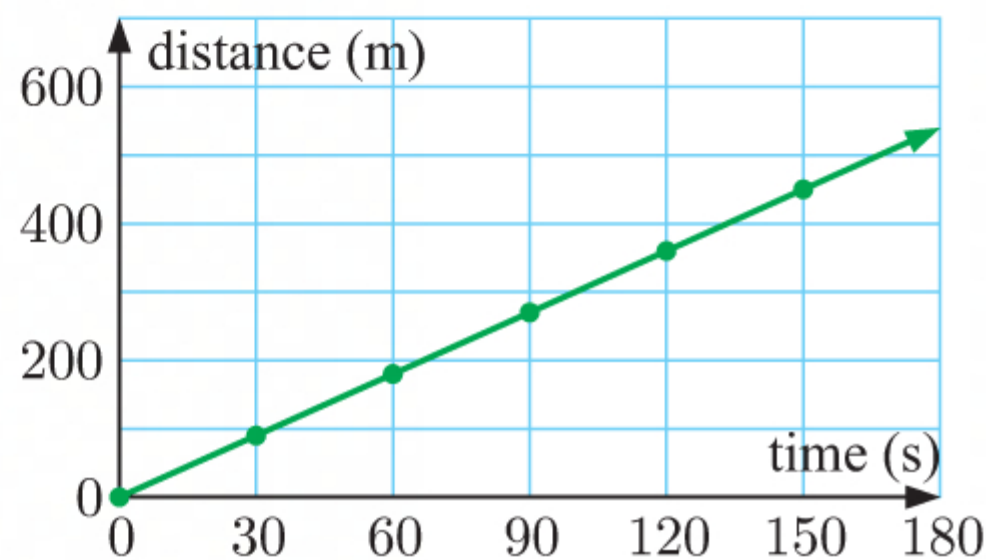
1 a

Time (seconds)	0	30	60	90	120	150
Distance (metres)	0	90	180	270	360	450

$\xrightarrow{+90}$ $\xrightarrow{+90}$ $\xrightarrow{+90}$ $\xrightarrow{+90}$ $\xrightarrow{+90}$

The distance travelled increases by the same amount each time interval.
 \therefore the jogger is travelling at a constant speed.

b

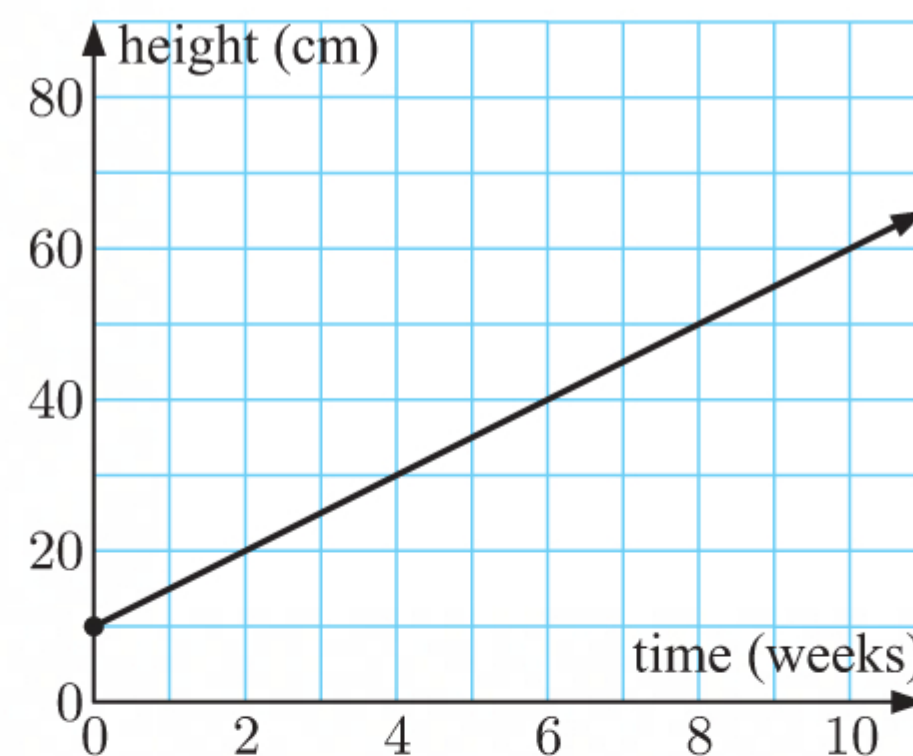


$$\begin{aligned} \text{c speed} &= \frac{(90 - 0) \text{ m}}{(30 - 0) \text{ s}} \\ &= 3 \text{ m per s} \end{aligned}$$

2

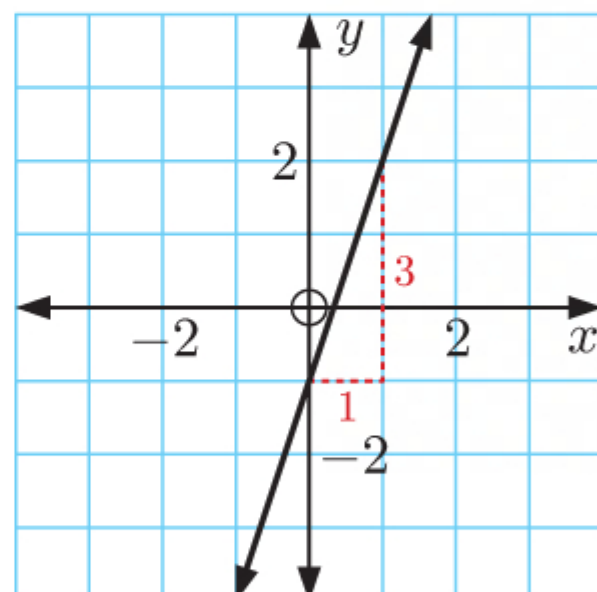
a The graph of height against time is a straight line.
 \therefore the rate of change in height is constant.

$$\begin{aligned} \text{b rate of change} &= \frac{(60 - 10) \text{ cm}}{(10 - 0) \text{ weeks}} \\ &= \frac{50 \text{ cm}}{10 \text{ weeks}} \\ &= 5 \text{ cm per week} \end{aligned}$$



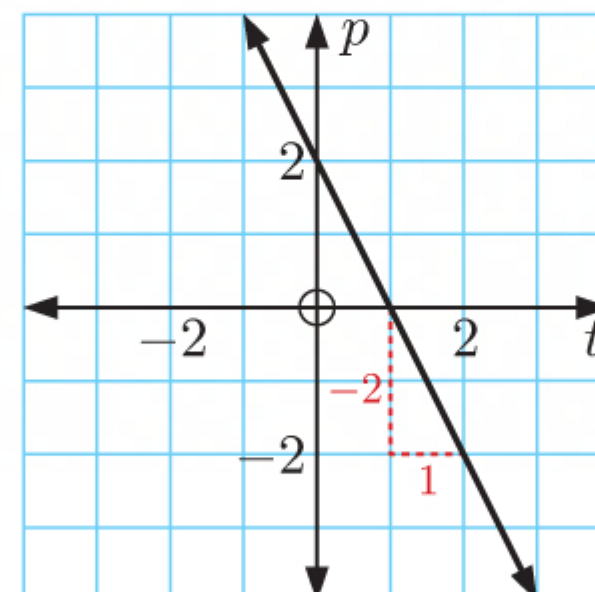
3

a

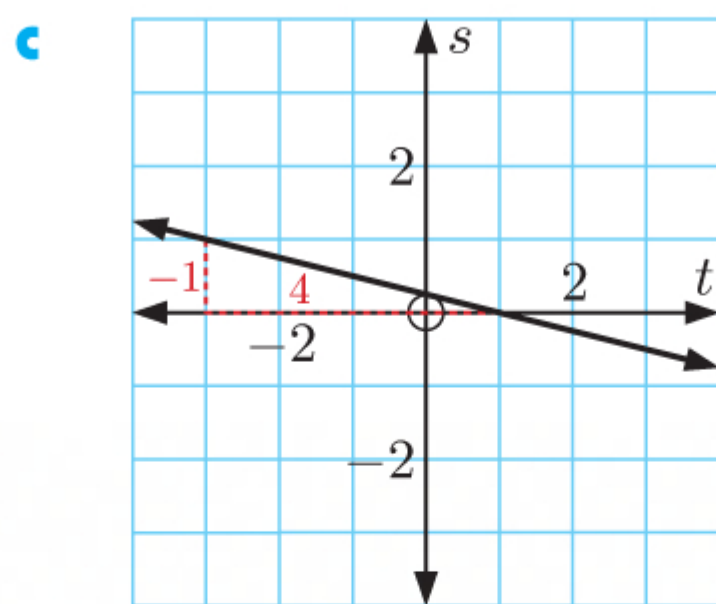


$$\begin{aligned} \text{rate of change} &= \text{gradient of line} \\ &= \frac{3}{1} \\ &= 3 \end{aligned}$$

b



$$\begin{aligned} \text{rate of change} &= \text{gradient of line} \\ &= \frac{-2}{1} \\ &= -2 \end{aligned}$$



rate of change = gradient of line

$$= \frac{-1}{4}$$

$$= -\frac{1}{4}$$

- 4** For the function $f(x) = \frac{5}{2}x - 3$, the gradient is $\frac{5}{2}$, so the rate of change is $\frac{5}{2}$.

EXERCISE 11A.2

- 1 a** The graph of distance against time is not a straight line.

\therefore Aileen did not travel at a constant speed.

- b i** average speed from $t = 0$ to $t = 5$ h

$$= \frac{(300 - 0) \text{ km}}{(5 - 0) \text{ h}}$$

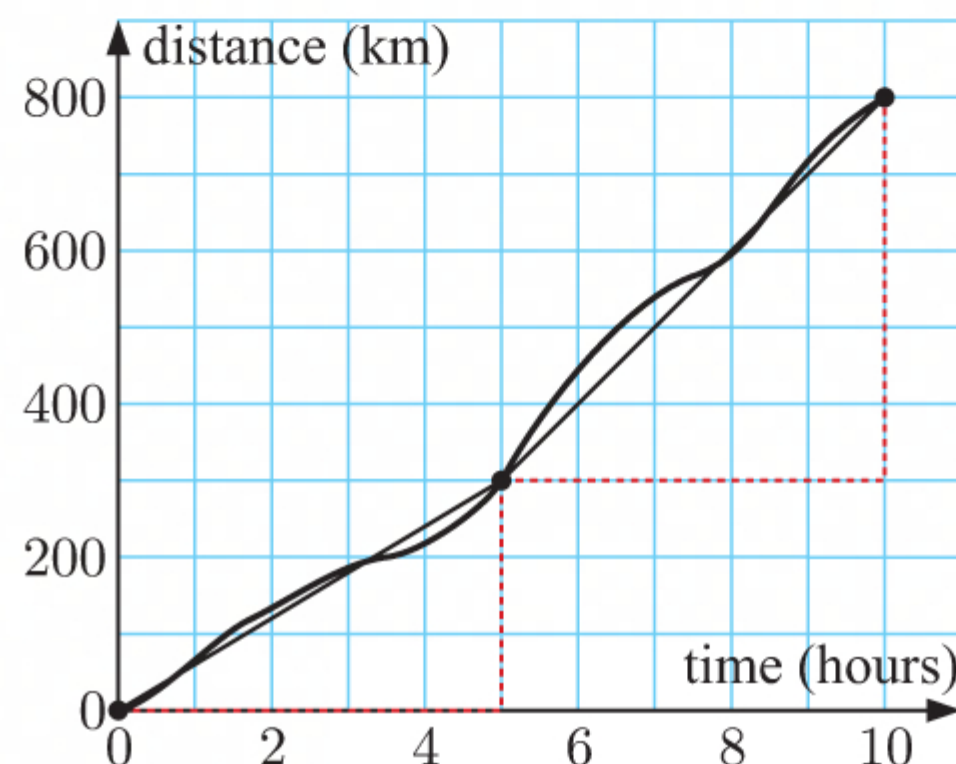
$$= 60 \text{ km per hour}$$

- ii** average speed from $t = 5$ h to $t = 10$ h

$$= \frac{(800 - 300) \text{ km}}{(10 - 5) \text{ h}}$$

$$= \frac{500}{5} \text{ km per hour}$$

$$= 100 \text{ km per hour}$$



- 2 a** average rate of change from $t = 1$ h to $t = 2.5$ h

$$= \frac{(250 - 100) \text{ m}}{(2.5 - 1) \text{ h}}$$

$$= \frac{150}{1.5} \text{ m per hour}$$

$$= 100 \text{ m per hour}$$

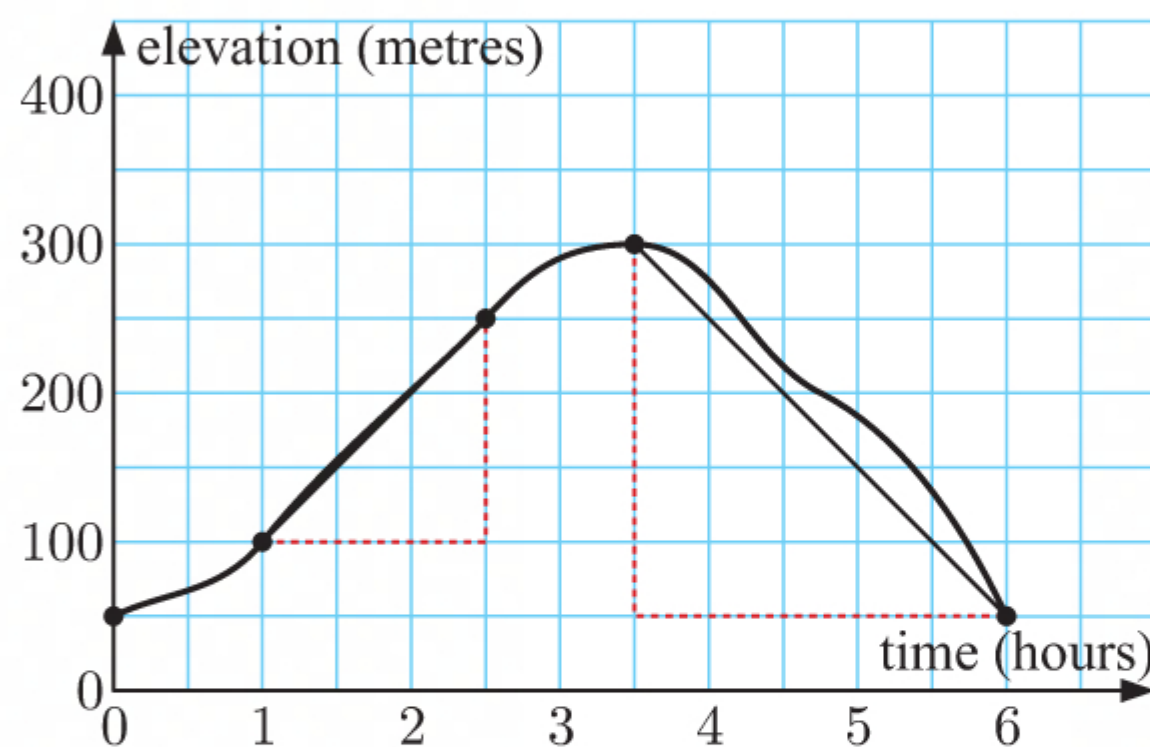
- b** average rate of change from $t = 3.5$ h to $t = 6$ h

$$= \frac{(50 - 300) \text{ m}}{(6 - 3.5) \text{ h}}$$

$$= \frac{-250}{2.5} \text{ m per hour}$$

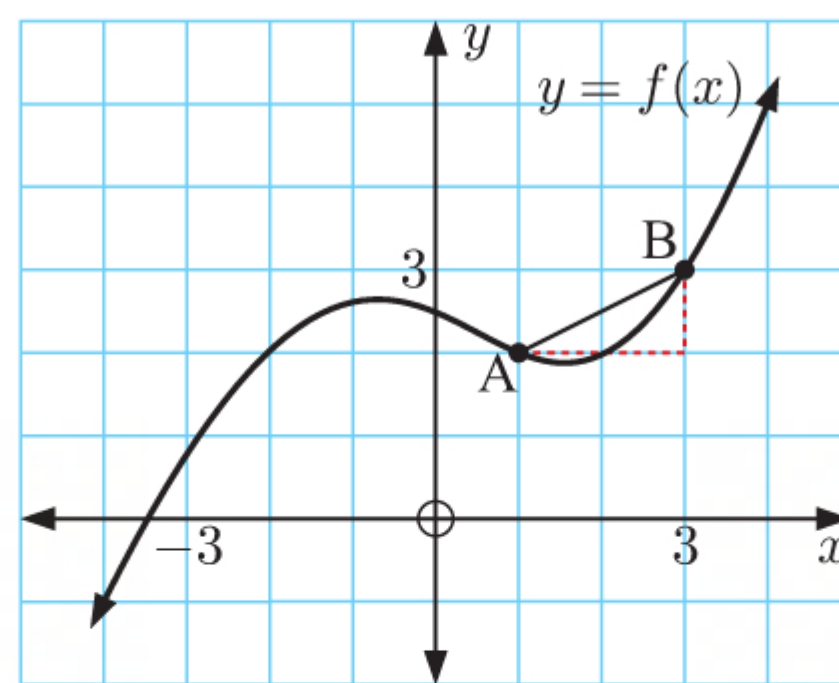
$$= -100 \text{ m per hour}$$

$$= 100 \text{ m per hour (downwards)}$$



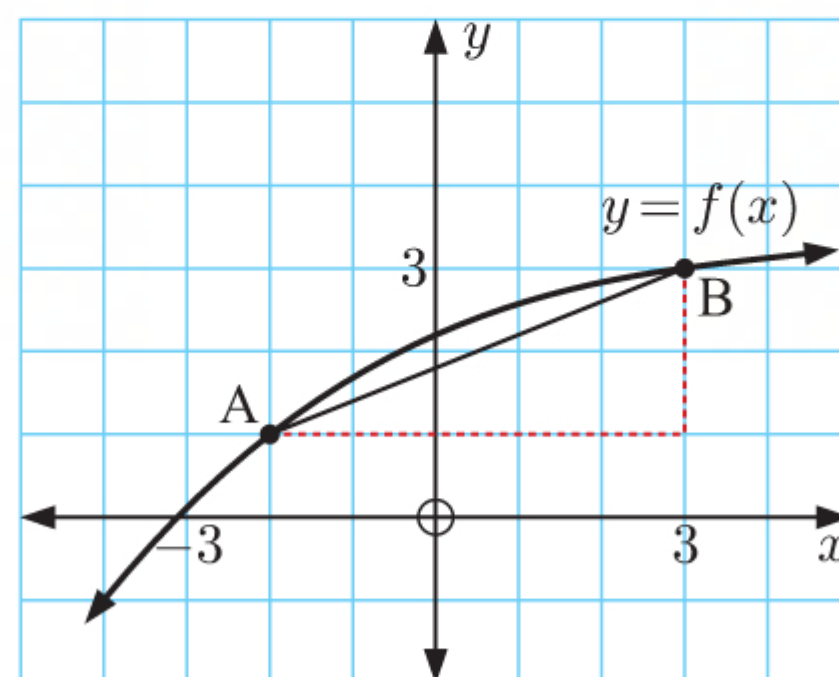
- 3 a** average rate of change in $f(x)$ from A to B

$$\begin{aligned}
 &= \frac{f(b) - f(a)}{b - a} \\
 &= \frac{3 - 2}{3 - 1} \\
 &= \frac{1}{2}
 \end{aligned}$$



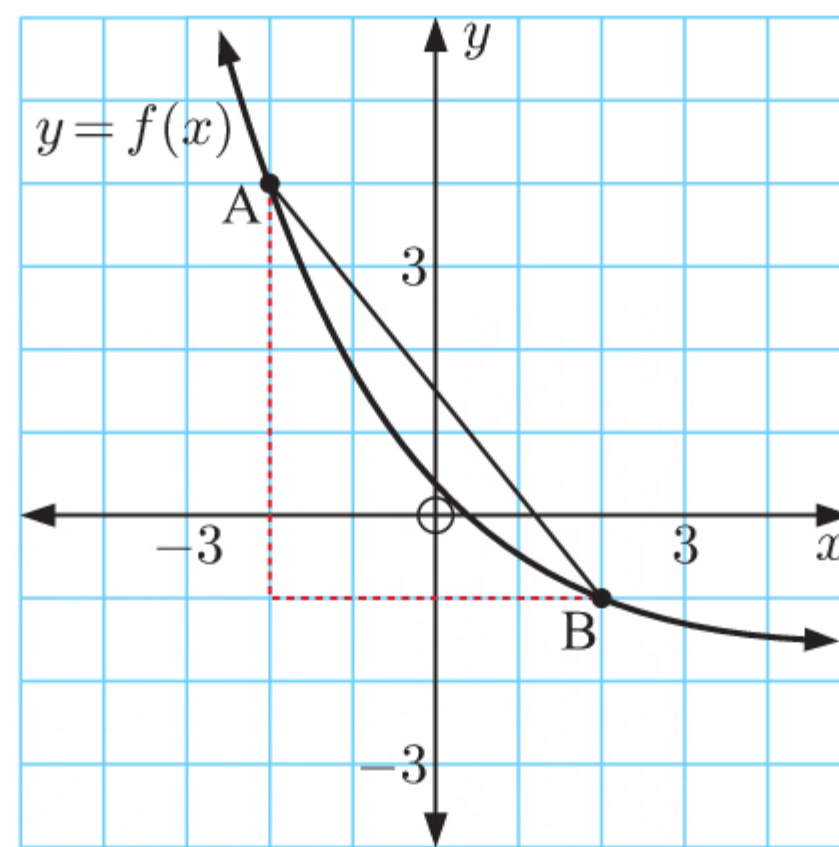
- b** average rate of change in $f(x)$ from A to B

$$\begin{aligned}
 &= \frac{f(b) - f(a)}{b - a} \\
 &= \frac{3 - 1}{3 - (-2)} \\
 &= \frac{2}{5}
 \end{aligned}$$



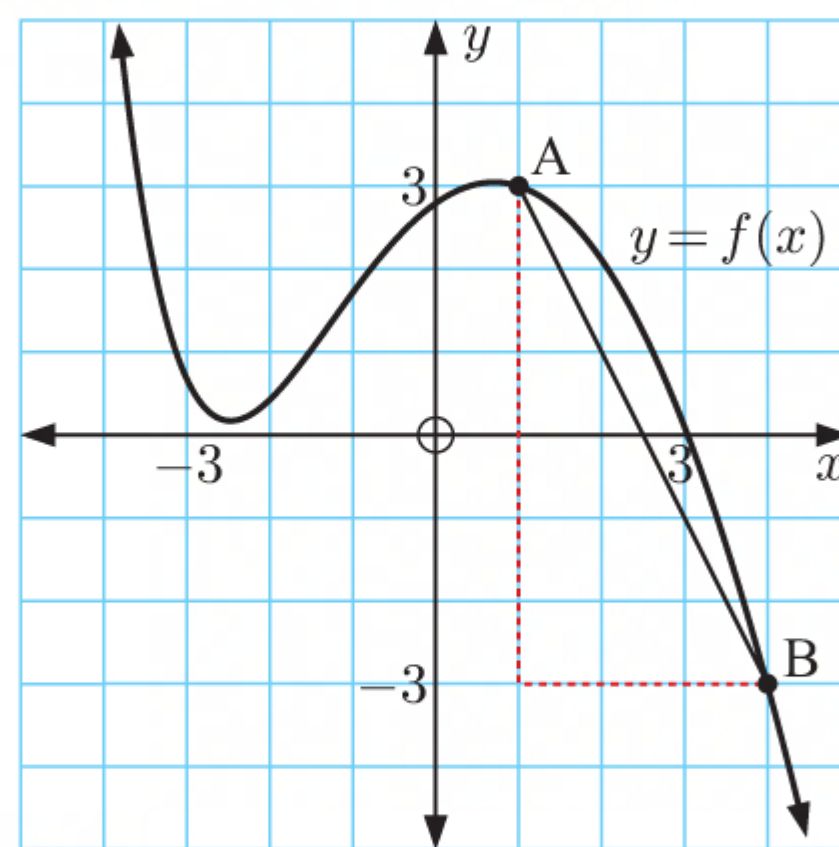
- c** average rate of change in $f(x)$ from A to B

$$\begin{aligned}
 &= \frac{f(b) - f(a)}{b - a} \\
 &= \frac{-1 - 4}{2 - (-2)} \\
 &= -\frac{5}{4}
 \end{aligned}$$



- d** average rate of change in $f(x)$ from A to B

$$\begin{aligned}
 &= \frac{f(b) - f(a)}{b - a} \\
 &= \frac{-3 - 3}{4 - 1} \\
 &= \frac{-6}{3} \\
 &= -2
 \end{aligned}$$



- 4 a i** average rate of change in $f(x)$ from $x = 1$ to $x = 2$

$$\begin{aligned} &= \frac{f(2) - f(1)}{2 - 1} \\ &= \frac{4 - 1}{1} \\ &= 3 \end{aligned}$$

- ii** average rate of change in $f(x)$ from $x = 1$ to $x = 1.5$

$$\begin{aligned} &= \frac{f(1.5) - f(1)}{1.5 - 1} \\ &= \frac{2.25 - 1}{0.5} \\ &= 2.5 \end{aligned}$$

- iii** average rate of change in $f(x)$ from $x = 1$ to $x = 1.1$

$$\begin{aligned} &= \frac{f(1.1) - f(1)}{1.1 - 1} \\ &= \frac{1.21 - 1}{0.1} \\ &= 2.1 \end{aligned}$$

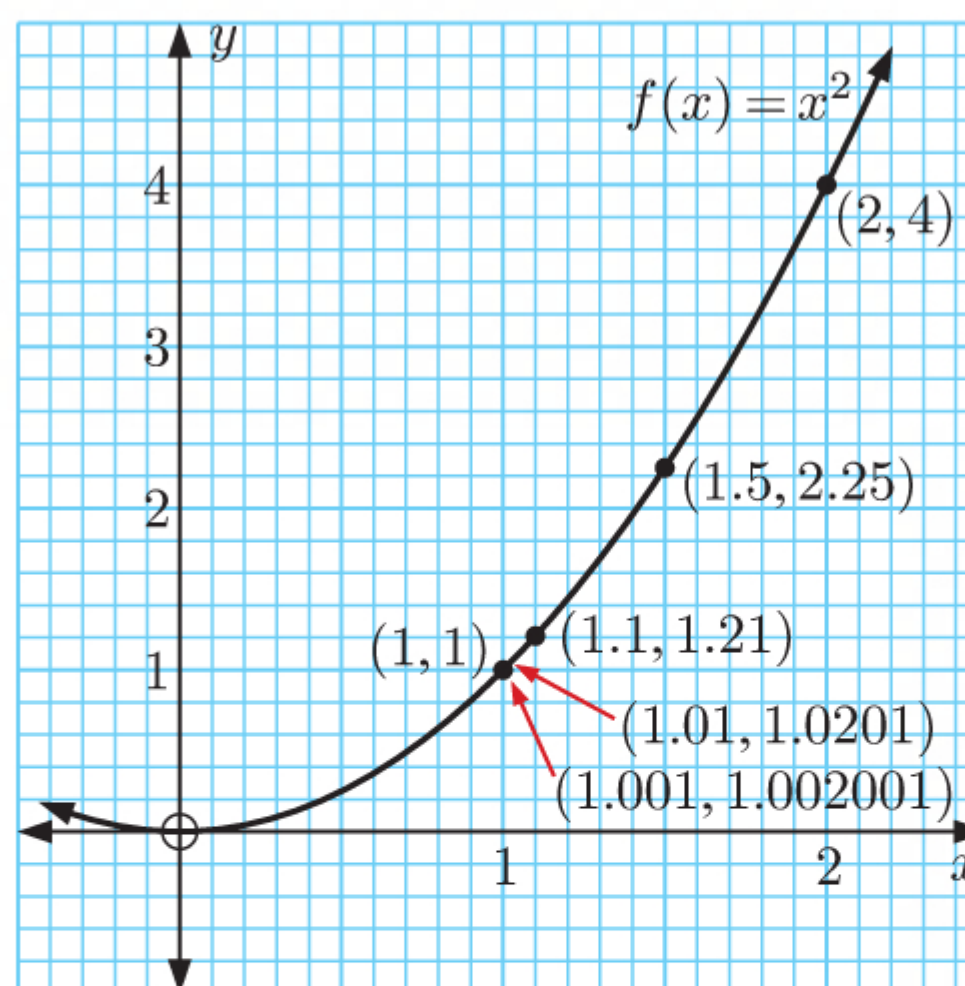
- iv** average rate of change in $f(x)$ from $x = 1$ to $x = 1.01$

$$\begin{aligned} &= \frac{f(1.01) - f(1)}{1.01 - 1} \\ &= \frac{1.0201 - 1}{0.01} \\ &= 2.01 \end{aligned}$$

- v** average rate of change in $f(x)$ from $x = 1$ to $x = 1.001$

$$\begin{aligned} &= \frac{f(1.001) - f(1)}{1.001 - 1} \\ &= \frac{1.002001 - 1}{0.001} \\ &= 2.001 \end{aligned}$$

- b** The average rate of change gets closer and closer to 2.



INVESTIGATION 1

INSTANTANEOUS SPEED

t	gradient of [FM]
4	30
3	25
2.5	22.5
2.1	20.5
2.01	20.05

- 3** As M approaches F , the gradient of [FM] approaches 20. However, when M reaches F , the gradient is undefined since we cannot divide by zero.

- 4** As t approaches 2 from the right, the gradient of [FM] approaches 20.

We suspect that the instantaneous speed of the ball bearing when $t = 2$ seconds is 20 m s^{-1} .

5

t	gradient of [FM]
0	10
1.5	17.5
1.9	19.5
1.99	19.95

- 6** As t approaches 2 from the left, the gradient of [FM] approaches 20.

The instantaneous speed of the ball bearing when $t = 2$ seconds appears to be 20 m s^{-1} , which agrees with our result in **4**.

EXERCISE 11B

- 1 a** The tangent at A has gradient

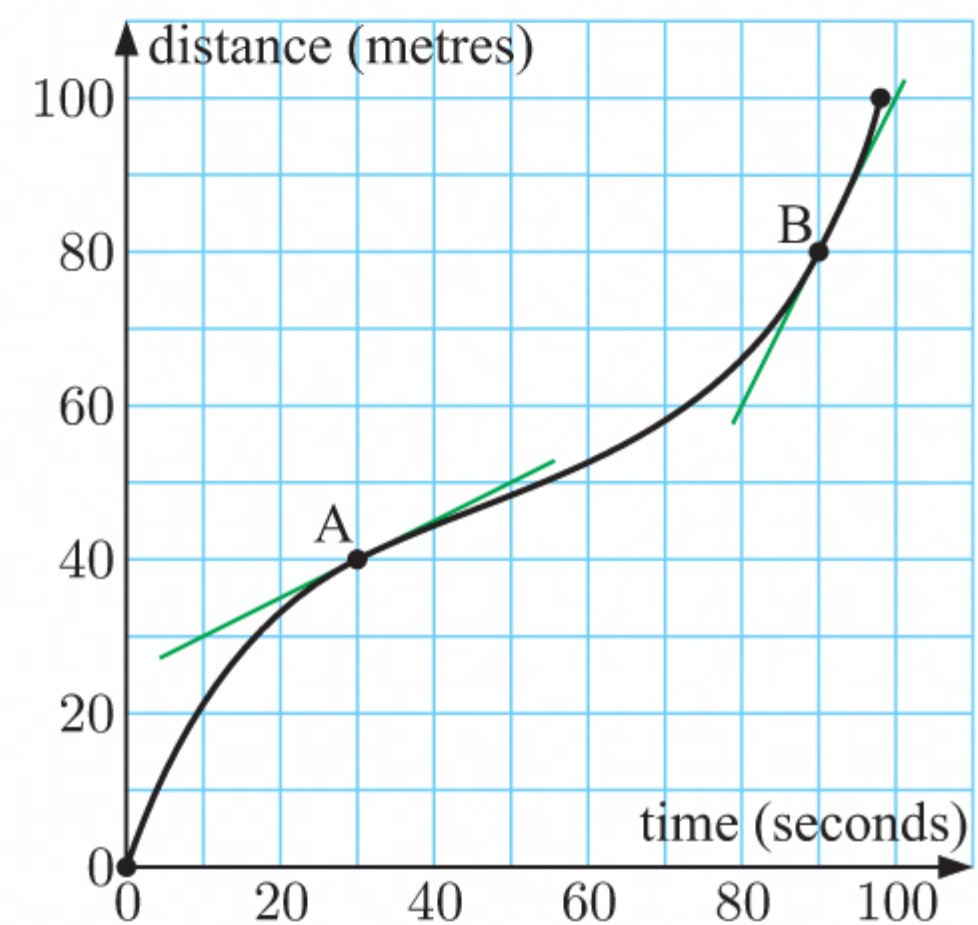
$$\frac{50 - 40}{50 - 30} = \frac{10}{20} = \frac{1}{2}.$$

\therefore the swimmer's instantaneous speed after 30 seconds is 0.5 m s^{-1} .

- b** The tangent at B has gradient

$$\frac{80 - 60}{90 - 80} = \frac{20}{10} = 2.$$

\therefore the swimmer's instantaneous speed after 90 seconds is 2 m s^{-1} .



- 2 a** The tangent at $x = -1$ has gradient

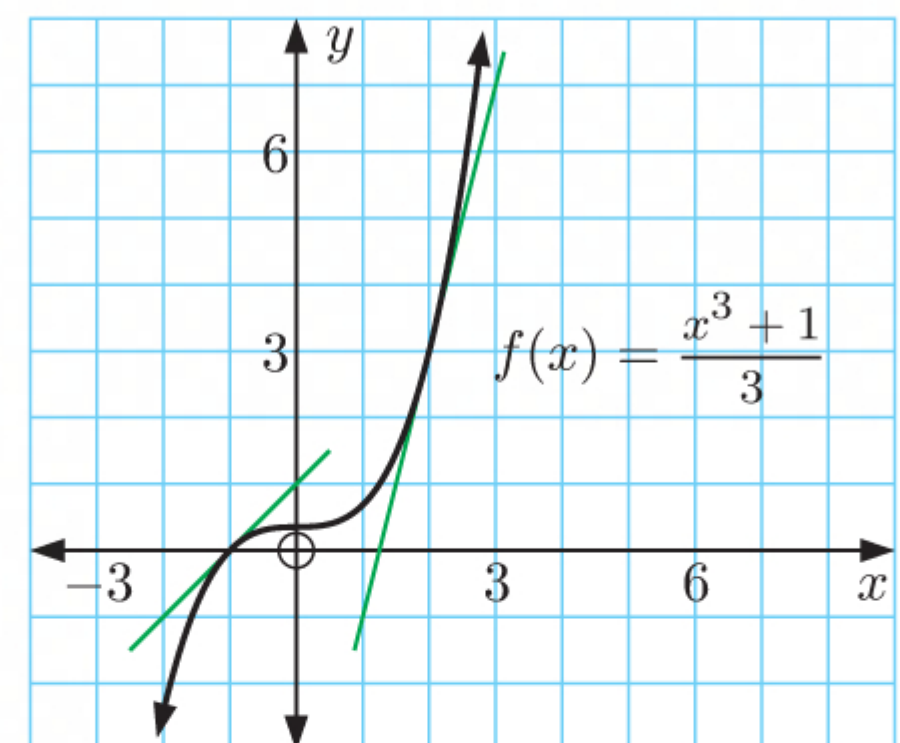
$$\frac{1 - (-1)}{0 - (-2)} = \frac{2}{2} = 1.$$

\therefore the instantaneous rate of change in $f(x)$ at $x = -1$ is 1.

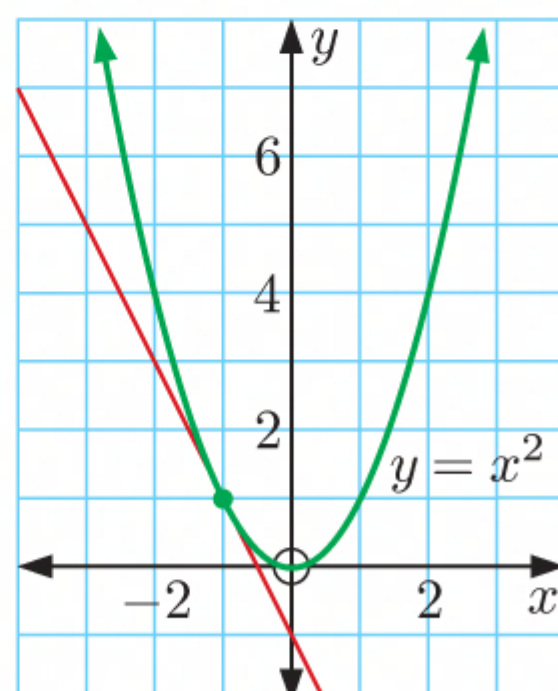
- b** The tangent at $x = 2$ has gradient

$$\frac{3 - (-1)}{2 - 1} = \frac{4}{1} = 4.$$

\therefore the instantaneous rate of change in $f(x)$ at $x = 2$ is 4.



- 3 a, b**



c The tangent at $x = -1$ has gradient $\frac{1 - (-1)}{-1 - 0} = -2$.

\therefore the instantaneous rate of change in $y = x^2$ when $x = -1$ is -2 .

EXERCISE 11C.1

1 a As $x \rightarrow 3$, $x + 4 \rightarrow 7$
 $\therefore \lim_{x \rightarrow 3} (x + 4) = 7$

c As $x \rightarrow 4$, $3x - 1 \rightarrow 11$
 $\therefore \lim_{x \rightarrow 4} (3x - 1) = 11$

d As $x \rightarrow 2$, $5x^2 - 3x + 2 \rightarrow 5(4) - 3(2) + 2 = 16$
 $\therefore \lim_{x \rightarrow 2} (5x^2 - 3x + 2) = 16$

e As $h \rightarrow 0$, $h^2 \rightarrow 0$ and $1 - h \rightarrow 1$
 $\therefore \lim_{h \rightarrow 0} h^2(1 - h) = 0 \times 1 = 0$

b As $x \rightarrow -1$, $5 - 2x \rightarrow 7$
 $\therefore \lim_{x \rightarrow -1} (5 - 2x) = 7$

f As $x \rightarrow 0$, $x^2 + 5 \rightarrow 5$
 $\therefore \lim_{x \rightarrow 0} (x^2 + 5) = 5$

2 a $\lim_{x \rightarrow 0} 5 = 5$

b $\lim_{h \rightarrow 2} 7 = 7$

c $\lim_{x \rightarrow 0} c = c$ (when c is a constant)

3 a $\frac{x^2 - 3x}{x}$ can be made as close as we like to -2 by making x sufficiently close to 1.

$$\therefore \lim_{x \rightarrow 1} \frac{x^2 - 3x}{x} = -2$$

b $\frac{h^2 + 5h}{h}$ can be made as close as we like to 7 by making h sufficiently close to 2.

$$\therefore \lim_{h \rightarrow 2} \frac{h^2 + 5h}{h} = 7$$

c $\frac{x - 1}{x + 1}$ can be made as close as we like to -1 by making x sufficiently close to 0.

$$\therefore \lim_{x \rightarrow 0} \frac{x - 1}{x + 1} = -1$$

4 $\lim_{x \rightarrow 0} \frac{x}{x} = \lim_{x \rightarrow 0} 1$ {since $x \neq 0$ }
 $= 1$

5 a

x	$\frac{x^2 - 4}{x - 2}$
1.9	3.9
1.99	3.99
1.999	3.999
1.9999	3.9999
1.99999	3.99999

x	$\frac{x^2 - 4}{x - 2}$
2.1	4.1
2.01	4.01
2.001	4.001
2.0001	4.0001
2.00001	4.00001

b $\lim_{x \rightarrow 2} \frac{x^2 - 4}{x - 2} = 4$

c $\lim_{x \rightarrow 2} \frac{x^2 - 4}{x - 2}$
 $= \lim_{x \rightarrow 2} \frac{(x + 2)(x - 2)}{x - 2}$
 $= \lim_{x \rightarrow 2} (x + 2)$ {since $x \neq 2$ }
 $= 4$

$$\begin{aligned}
 \text{6 a} \quad & \lim_{x \rightarrow 0} \frac{x^2 - 3x}{x} \\
 &= \lim_{x \rightarrow 0} \frac{\cancel{x}(x - 3)}{\cancel{x}} \\
 &= \lim_{x \rightarrow 0} (x - 3) \quad \{\text{since } x \neq 0\} \\
 &= -3
 \end{aligned}$$

$$\begin{aligned}
 \text{c} \quad & \lim_{x \rightarrow 0} \frac{2x^2 - x}{x} \\
 &= \lim_{x \rightarrow 0} \frac{\cancel{x}(2x - 1)}{\cancel{x}} \\
 &= \lim_{x \rightarrow 0} (2x - 1) \quad \{\text{since } x \neq 0\} \\
 &= -1
 \end{aligned}$$

$$\begin{aligned}
 \text{e} \quad & \lim_{h \rightarrow 0} \frac{3h^2 - 4h}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\cancel{h}(3h - 4)}{\cancel{h}} \\
 &= \lim_{h \rightarrow 0} (3h - 4) \quad \{\text{since } h \neq 0\} \\
 &= -4
 \end{aligned}$$

$$\begin{aligned}
 \text{g} \quad & \lim_{x \rightarrow 1} \frac{x^2 - x}{x - 1} \\
 &= \lim_{x \rightarrow 1} \frac{x(\cancel{x - 1})}{\cancel{x - 1}} \\
 &= \lim_{x \rightarrow 1} x \quad \{\text{since } x \neq 1\} \\
 &= 1
 \end{aligned}$$

$$\begin{aligned}
 \text{i} \quad & \lim_{x \rightarrow 3} \frac{x^2 - x - 6}{x - 3} \\
 &= \lim_{x \rightarrow 3} \frac{(x + 2)(\cancel{x - 3})}{\cancel{x - 3}} \\
 &= \lim_{x \rightarrow 3} (x + 2) \quad \{\text{since } x \neq 3\} \\
 &= 5
 \end{aligned}$$

$$\begin{aligned}
 \text{b} \quad & \lim_{x \rightarrow 0} \frac{x^2 + 5x}{x} \\
 &= \lim_{x \rightarrow 0} \frac{\cancel{x}(x + 5)}{\cancel{x}} \\
 &= \lim_{x \rightarrow 0} (x + 5) \quad \{\text{since } x \neq 0\} \\
 &= 5
 \end{aligned}$$

$$\begin{aligned}
 \text{d} \quad & \lim_{h \rightarrow 0} \frac{2h^2 + 6h}{h} \\
 &= \lim_{h \rightarrow 0} \frac{2\cancel{h}(h + 3)}{\cancel{h}} \\
 &= \lim_{h \rightarrow 0} 2(h + 3) \quad \{\text{since } h \neq 0\} \\
 &= 6
 \end{aligned}$$

$$\begin{aligned}
 \text{f} \quad & \lim_{h \rightarrow 0} \frac{h^3 - 8h}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\cancel{h}(h^2 - 8)}{\cancel{h}} \\
 &= \lim_{h \rightarrow 0} (h^2 - 8) \quad \{\text{since } h \neq 0\} \\
 &= -8
 \end{aligned}$$

$$\begin{aligned}
 \text{h} \quad & \lim_{x \rightarrow 2} \frac{x^2 - 2x}{x - 2} \\
 &= \lim_{x \rightarrow 2} \frac{x(\cancel{x - 2})}{\cancel{x - 2}} \\
 &= \lim_{x \rightarrow 2} x \quad \{\text{since } x \neq 2\} \\
 &= 2
 \end{aligned}$$

EXERCISE 11C.2

1 a $f(x) = \frac{1}{x}$

i As $x \rightarrow 0^-$, $f(x) \rightarrow -\infty$

As $x \rightarrow 0^+$, $f(x) \rightarrow \infty$

As $x \rightarrow -\infty$, $f(x) \rightarrow 0^-$

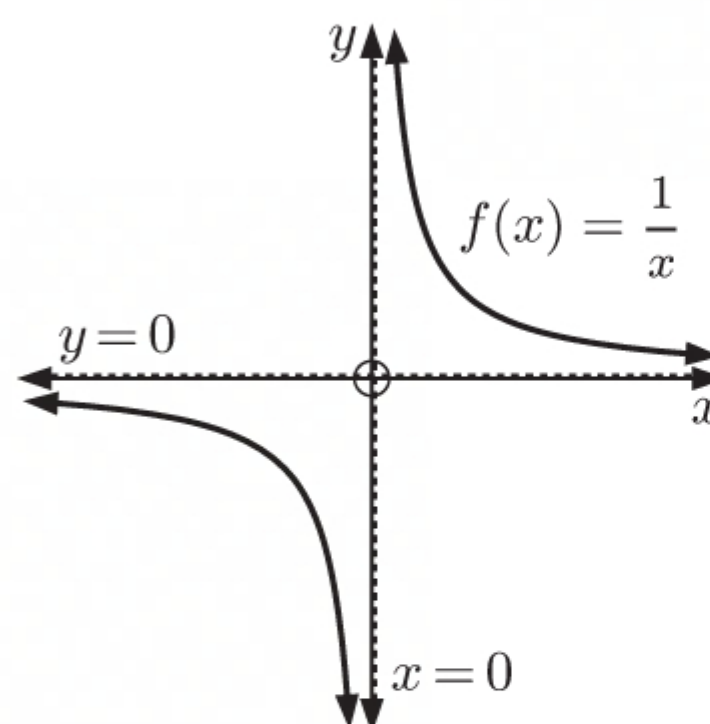
As $x \rightarrow \infty$, $f(x) \rightarrow 0^+$

The vertical asymptote is $x = 0$.

The horizontal asymptote is $y = 0$.

ii $\lim_{x \rightarrow -\infty} f(x) = 0$,

$\lim_{x \rightarrow \infty} f(x) = 0$



b $f(x) = \frac{3x-2}{x+3}$

i As $x \rightarrow -3^-$, $f(x) \rightarrow \infty$

As $x \rightarrow -3^+$, $f(x) \rightarrow -\infty$

As $x \rightarrow -\infty$, $f(x) \rightarrow 3^+$

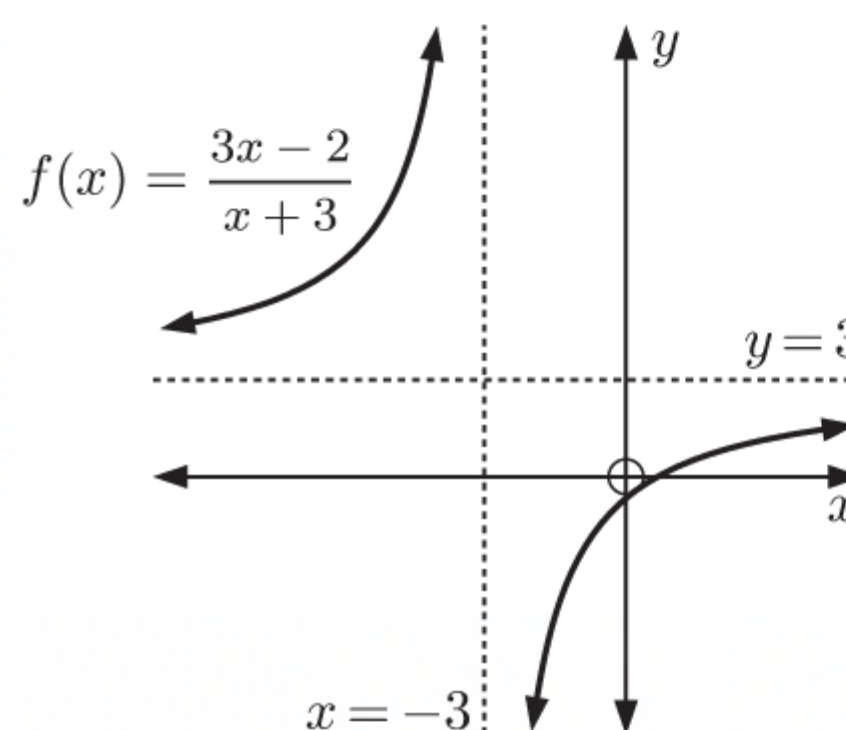
As $x \rightarrow \infty$, $f(x) \rightarrow 3^-$

The vertical asymptote is $x = -3$.

The horizontal asymptote is $y = 3$.

ii $\lim_{x \rightarrow -\infty} f(x) = 3$,

$\lim_{x \rightarrow \infty} f(x) = 3$



c $f(x) = \frac{1-2x}{3x+2}$

i As $x \rightarrow -\frac{2}{3}^-$, $f(x) \rightarrow -\infty$

As $x \rightarrow -\frac{2}{3}^+$, $f(x) \rightarrow \infty$

As $x \rightarrow -\infty$, $f(x) \rightarrow -\frac{2}{3}^-$

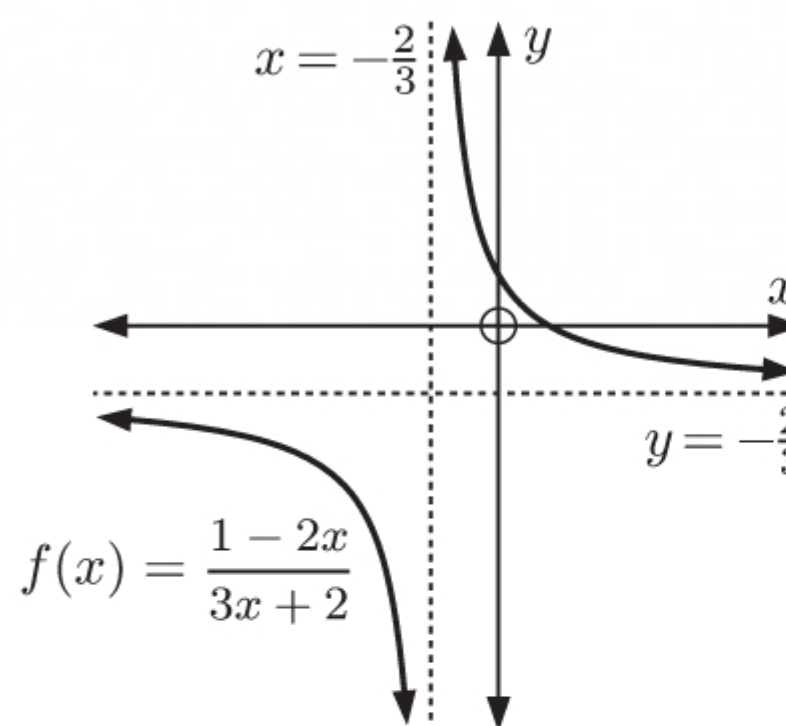
As $x \rightarrow \infty$, $f(x) \rightarrow -\frac{2}{3}^+$

The vertical asymptote is $x = -\frac{2}{3}$.

The horizontal asymptote is $y = -\frac{2}{3}$.

ii $\lim_{x \rightarrow -\infty} f(x) = -\frac{2}{3}$,

$\lim_{x \rightarrow \infty} f(x) = -\frac{2}{3}$



d $f(x) = \frac{x}{1-x}$

i As $x \rightarrow 1^-$, $f(x) \rightarrow \infty$

As $x \rightarrow 1^+$, $f(x) \rightarrow -\infty$

As $x \rightarrow -\infty$, $f(x) \rightarrow -1^+$

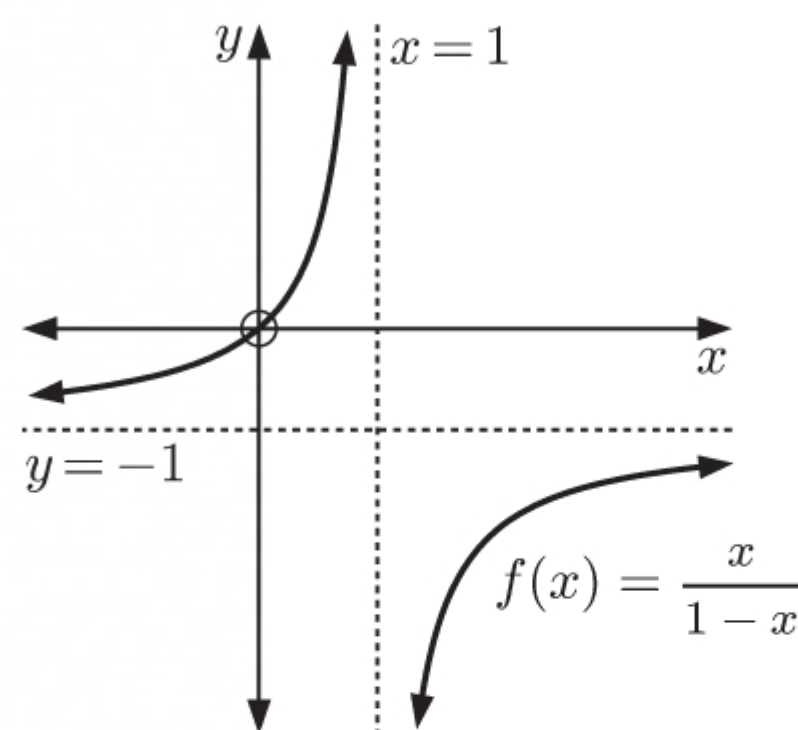
As $x \rightarrow \infty$, $f(x) \rightarrow -1^-$

The vertical asymptote is $x = 1$.

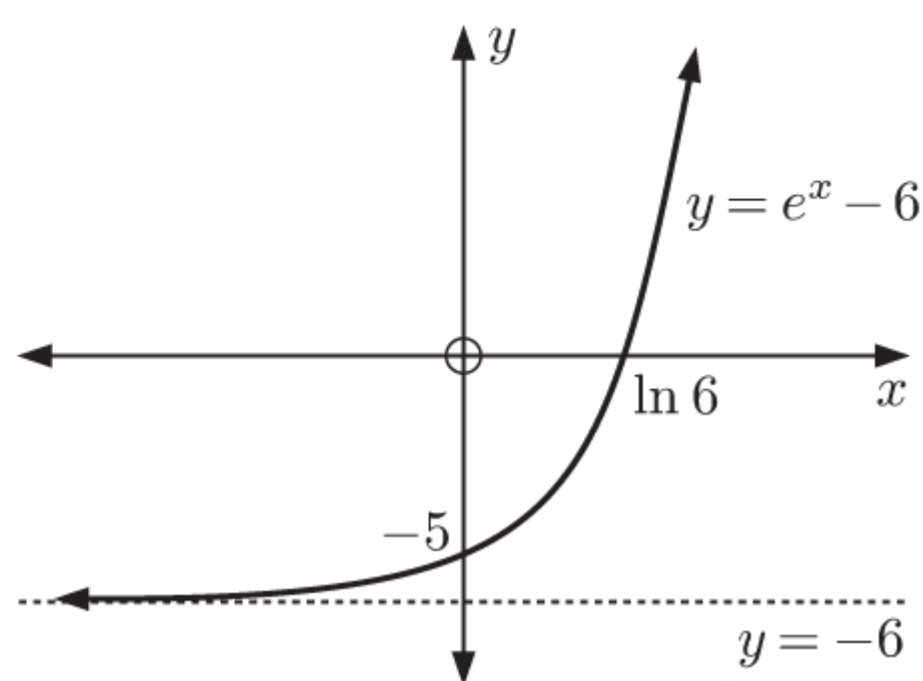
The horizontal asymptote is $y = -1$.

ii $\lim_{x \rightarrow -\infty} f(x) = -1$,

$\lim_{x \rightarrow \infty} f(x) = -1$



2 a



b i As $x \rightarrow -\infty$, $e^x - 6 \rightarrow -6^+$

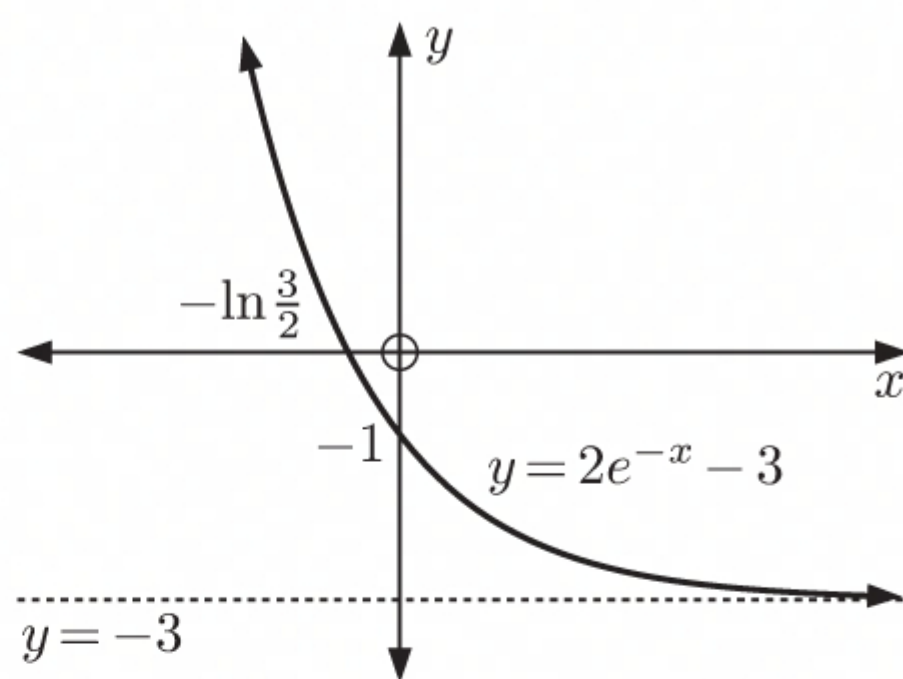
$\therefore \lim_{x \rightarrow -\infty} (e^x - 6) = -6$

\therefore the function has horizontal asymptote $y = -6$.

ii As $x \rightarrow \infty$, $e^x - 6 \rightarrow \infty$

$\therefore \lim_{x \rightarrow \infty} (e^x - 6)$ does not exist.

3 We sketch the graph of $y = 2e^{-x} - 3$:



a As $x \rightarrow -\infty$, $2e^{-x} - 3 \rightarrow \infty$

$\therefore \lim_{x \rightarrow -\infty} (2e^{-x} - 3)$ does not exist.

b As $x \rightarrow \infty$, $2e^{-x} - 3 \rightarrow -3^+$

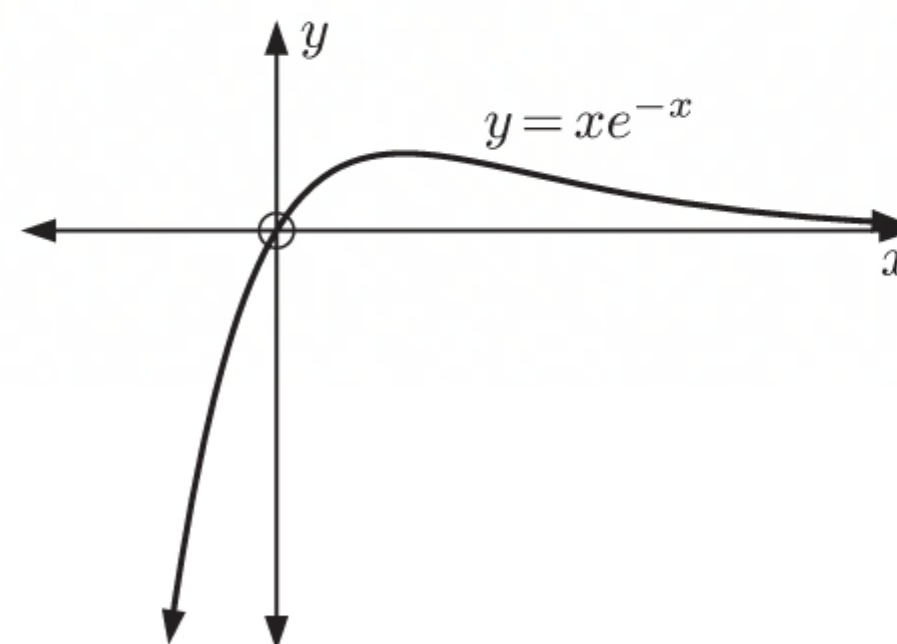
$\therefore \lim_{x \rightarrow \infty} (2e^{-x} - 3) = -3$.

4 a

x	xe^{-x}
10	$\approx 0.000\,454$
50	$\approx 9.64 \times 10^{-21}$
100	$\approx 3.72 \times 10^{-42}$
200	$\approx 2.77 \times 10^{-85}$

b We predict that $\lim_{x \rightarrow \infty} xe^{-x} = 0$.

c



As $x \rightarrow \infty$, $y \rightarrow 0^+$

The graph supports our prediction in **b**.

INVESTIGATION 2**LIMITS IN NUMBER SEQUENCES**

$x_n = 0.333 \dots 3$ where there are n 3s after the decimal point, $n \in \mathbb{Z}^+$.

n	x_n	$3x_n$	$1 - 3x_n$
1	0.3	0.9	0.1
2	0.33	0.99	0.01
3	0.333	0.999	0.001
4	0.3333	0.9999	0.0001
5	0.33333	0.99999	0.00001
10	0.3333333333	0.9999999999	0.0000000001

2 $(1 - 3x_{100})$ has 99 0s between the decimal point and the 1.

3 $(1 - 3x_n)$ has $(n - 1)$ 0s between the decimal point and the 1.

4 $\lim_{n \rightarrow \infty} (1 - 3x_n) = 0$ since the number of 0s between the decimal point and the 1 approaches infinity, and the number approaches zero.

5 $\lim_{n \rightarrow \infty} (1 - 3x_n) = 0$

$$\therefore \lim_{n \rightarrow \infty} 1 - \lim_{n \rightarrow \infty} 3x_n = 0$$

$$\therefore 1 - 3 \lim_{n \rightarrow \infty} x_n = 0$$

$$\therefore 3 \lim_{n \rightarrow \infty} x_n = 1$$

$$\therefore \lim_{n \rightarrow \infty} x_n = \frac{1}{3}$$

EXERCISE 11D

- 1 a** M has x -coordinate $3 + h$ and lies on the graph of $f(x) = x^2$.
 \therefore its y -coordinate is $(3 + h)^2$.

b The gradient of [FM] = $\frac{y_M - y_F}{x_M - x_F}$

$$= \frac{(3 + h)^2 - 9}{(3 + h) - 3}$$

$$= \frac{\cancel{9} + 6h + h^2 - \cancel{9}}{h}$$

$$= \frac{6h + h^2}{h}$$

$$= \frac{\cancel{h}(6 + h)}{\cancel{h}}$$

$$= 6 + h \quad \text{provided } h \neq 0$$

- c i** M has x -coordinate $3 + h$.

$$\therefore \text{ at the point } (4, 16), \quad 3 + h = 4$$

$$\therefore h = 1$$

The gradient of [FM] is $6 + h$.
{from **b**}

$$\therefore \text{ the gradient of [FM] at } (4, 16) \text{ is}$$

$$6 + 1 = 7.$$

- iii** M has x -coordinate $3 + h$.

$$\therefore \text{ at the point } (3.1, 9.61),$$

$$3 + h = 3.1$$

$$\therefore h = 0.1$$

The gradient of [FM] is $6 + h$.
{from **b**}

$$\therefore \text{ the gradient of [FM] at } (3.1, 9.61)$$

$$\text{is } 6 + 0.1 = 6.1.$$

- ii** M has x -coordinate $3 + h$.

$$\therefore \text{ at the point } (3.5, 12.25),$$

$$3 + h = 3.5$$

$$\therefore h = 0.5$$

The gradient of [FM] is $6 + h$.
{from **b**}

$$\therefore \text{ the gradient of [FM] at}$$

$$(3.5, 12.25) \text{ is } 6 + 0.5 = 6.5.$$

- iv** M has x -coordinate $3 + h$.

$$\therefore \text{ at the point } (3.01, 9.0601),$$

$$3 + h = 3.01$$

$$\therefore h = 0.01$$

The gradient of [FM] is $6 + h$.
{from **b**}

$$\therefore \text{ the gradient of [FM] at}$$

$$(3.01, 9.0601) \text{ is}$$

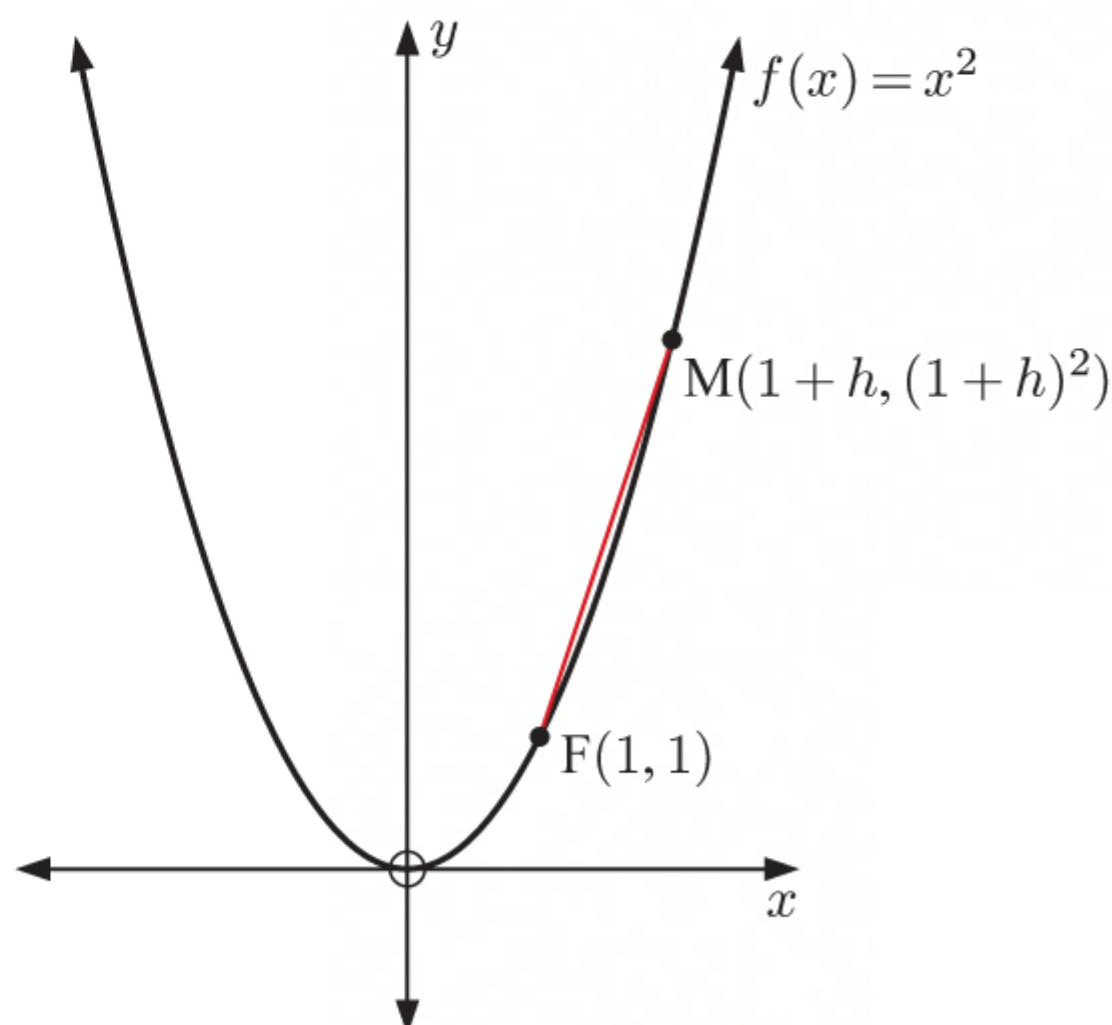
$$6 + 0.01 = 6.01.$$

- d** Using limit theory, the gradient of the tangent to $f(x) = x^2$ at the point $(3, 9)$ is

$$\begin{aligned} \lim_{h \rightarrow 0} \frac{f(3+h) - f(3)}{h} &= \lim_{h \rightarrow 0} \frac{(3+h)^2 - 3^2}{h} \\ &= \lim_{h \rightarrow 0} \frac{\cancel{9} + 6h + h^2 - \cancel{9}}{h} \\ &= \lim_{h \rightarrow 0} \frac{6h + h^2}{h} \\ &= \lim_{h \rightarrow 0} \frac{\cancel{h}(6+h)}{\cancel{h}} \\ &= \lim_{h \rightarrow 0} (6+h) \quad \{\text{as } h \neq 0\} \\ &= 6 \end{aligned}$$

- 2 a i** At $x = 1$, $f(1) = 1^2 = 1$.

Let F be the point $(1, 1)$ and M have x -coordinate $1 + h$, so M is $(1 + h, (1 + h)^2)$.

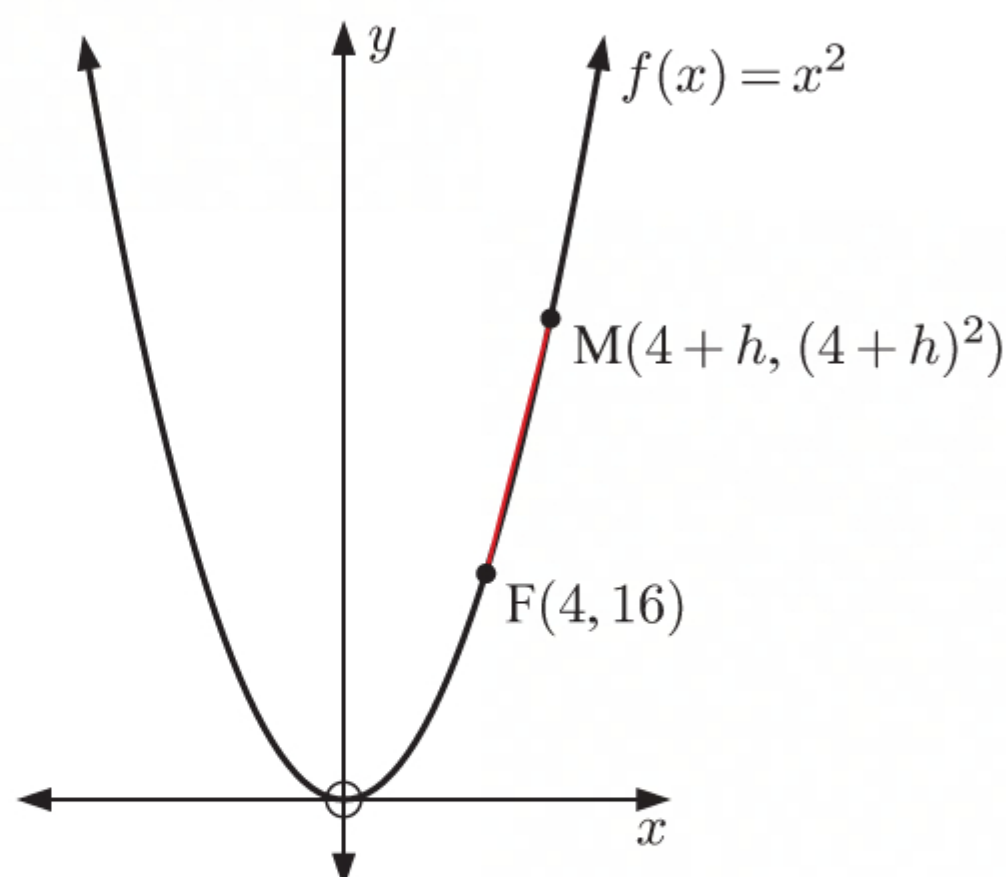


The gradient of the tangent at F

$$\begin{aligned} &= \lim_{h \rightarrow 0} \frac{f(1+h) - f(1)}{h} \\ &= \lim_{h \rightarrow 0} \frac{(1+h)^2 - 1^2}{h} \\ &= \lim_{h \rightarrow 0} \frac{\cancel{1} + 2h + h^2 - \cancel{1}}{h} \\ &= \lim_{h \rightarrow 0} \frac{2h + h^2}{h} \\ &= \lim_{h \rightarrow 0} \frac{\cancel{h}(2+h)}{\cancel{h}} \\ &= \lim_{h \rightarrow 0} (2+h) \quad \{\text{as } h \neq 0\} \\ &= 2 \end{aligned}$$

ii At $x = 4$, $f(4) = 4^2 = 16$.

Let F be the point $(4, 16)$ and M have x -coordinate $4 + h$, so M is $(4 + h, (4 + h)^2)$.



The gradient of the tangent at F

$$\begin{aligned}
 &= \lim_{h \rightarrow 0} \frac{f(4 + h) - f(4)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{(4 + h)^2 - 4^2}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\cancel{16} + 8h + h^2 - \cancel{16}}{h} \\
 &= \lim_{h \rightarrow 0} \frac{8h + h^2}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\cancel{h}(8 + h)}{\cancel{h}} \\
 &= \lim_{h \rightarrow 0} (8 + h) \quad \{\text{as } h \neq 0\} \\
 &= 8
 \end{aligned}$$

b The gradient of the tangent to $f(x) = x^2$ at the point where $x = 2$

$$\begin{aligned}
 &= \lim_{h \rightarrow 0} \frac{f(2 + h) - f(2)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{(2 + h)^2 - 2^2}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\cancel{4} + 4h + h^2 - \cancel{4}}{h} \\
 &= \lim_{h \rightarrow 0} \frac{4h + h^2}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\cancel{h}(4 + h)}{\cancel{h}} \\
 &= \lim_{h \rightarrow 0} (4 + h) \quad \{\text{as } h \neq 0\} \\
 &= 4
 \end{aligned}$$

The gradient of the tangent to $f(x) = x^2$ at the point where $x = 3$

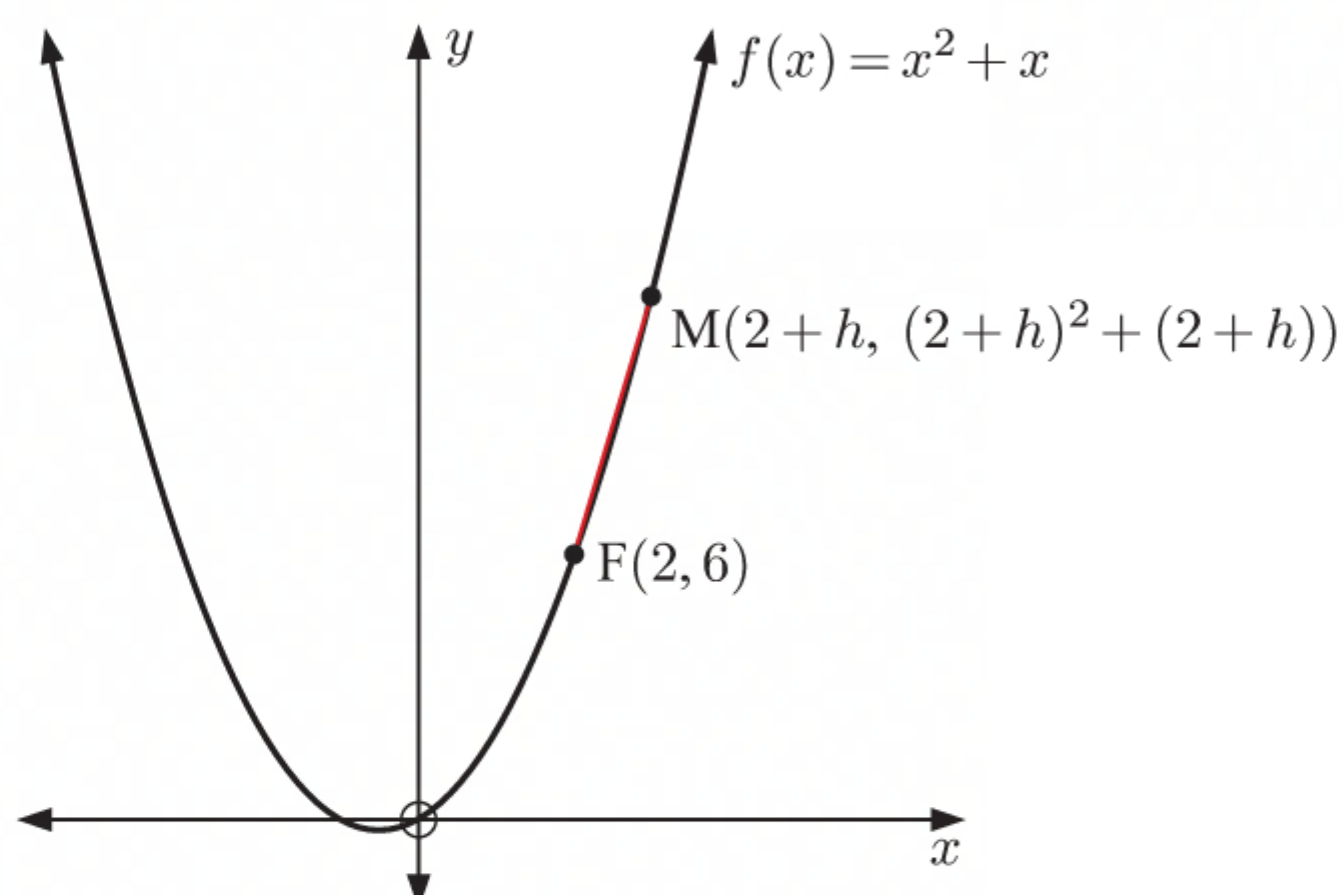
$$\begin{aligned}
 &= \lim_{h \rightarrow 0} \frac{f(3 + h) - f(3)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{(3 + h)^2 - 3^2}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\cancel{9} + 6h + h^2 - \cancel{9}}{h} \\
 &= \lim_{h \rightarrow 0} \frac{6h + h^2}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\cancel{h}(6 + h)}{\cancel{h}} \\
 &= \lim_{h \rightarrow 0} (6 + h) \quad \{\text{as } h \neq 0\} \\
 &= 6
 \end{aligned}$$

Using the results from a, the table is:

x -coordinate	Gradient of tangent to $f(x) = x^2$
1	2
2	4
3	6
4	8

c The gradient of the tangent is equal to twice the x -coordinate in each case in b. So, we predict the gradient of the tangent to $f(x) = x^2$ at the point where $x = a$ will be $2a$.

- 3 a** Let F be the point (2, 6) and M have x -coordinate $2 + h$,
so M is $(2 + h, (2 + h)^2 + (2 + h))$.

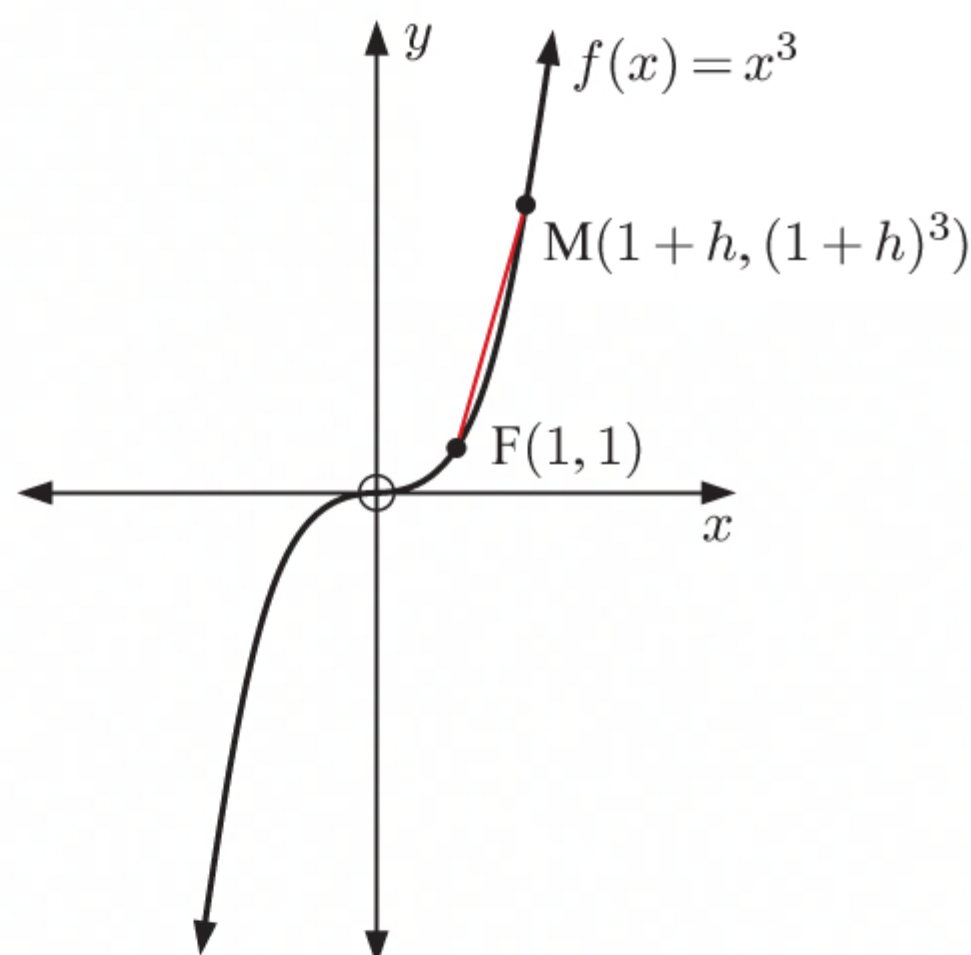


The gradient of the tangent at F

$$\begin{aligned}
 &= \lim_{h \rightarrow 0} \frac{f(2 + h) - f(2)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{(2 + h)^2 + (2 + h) - (2^2 + 2)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{4 + 4h + h^2 + 2 + h - 6}{h} \\
 &= \lim_{h \rightarrow 0} \frac{5h + h^2}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\cancel{h}(5 + h)}{\cancel{h}} \\
 &= \lim_{h \rightarrow 0} (5 + h) \quad \{\text{as } h \neq 0\} \\
 &= 5
 \end{aligned}$$

- b** At $x = 1$, $f(1) = 1^3 = 1$.

Let F be the point (1, 1) and M have x -coordinate $1 + h$, so M is $(1 + h, (1 + h)^3)$.

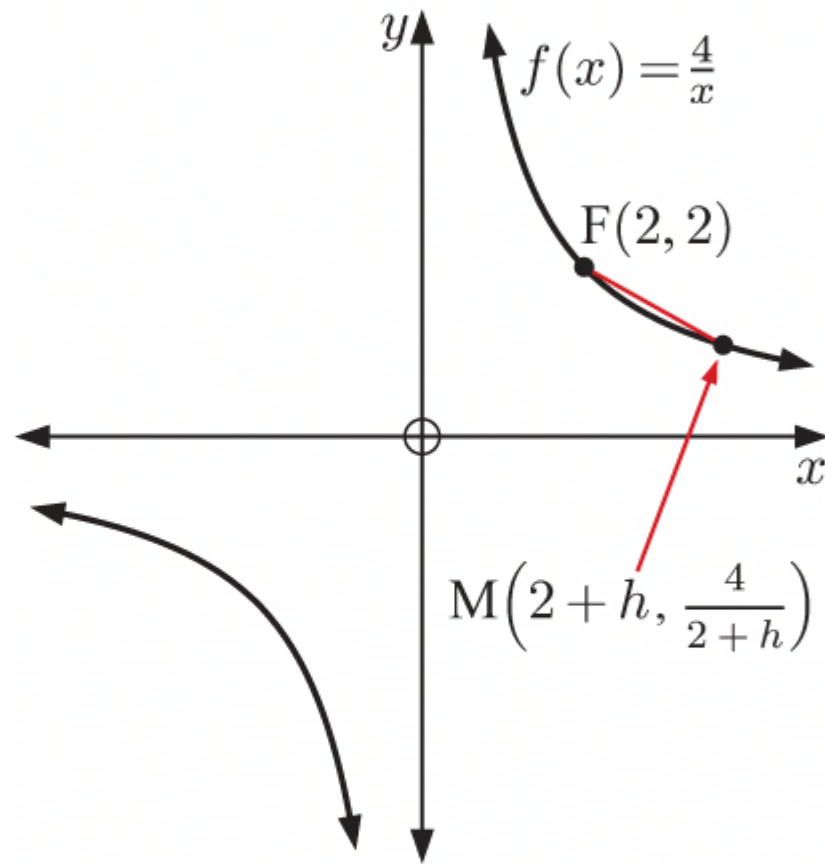


The gradient of the tangent at F

$$\begin{aligned}
 &= \lim_{h \rightarrow 0} \frac{f(1 + h) - f(1)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{(1 + h)^3 - 1^3}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\cancel{1} + 3h + 3h^2 + h^3 - \cancel{1}}{h} \\
 &\quad \{\text{binomial expansion}\} \\
 &= \lim_{h \rightarrow 0} \frac{3h + 3h^2 + h^3}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\cancel{h}(3 + 3h + h^2)}{\cancel{h}} \\
 &= \lim_{h \rightarrow 0} (3 + 3h + h^2) \quad \{\text{as } h \neq 0\} \\
 &= 3
 \end{aligned}$$

- At $x = 2$, $f(2) = \frac{4}{2} = 2$.

Let F be the point $(2, 2)$ and M have x -coordinate $2 + h$, so M is $\left(2 + h, \frac{4}{2 + h}\right)$.

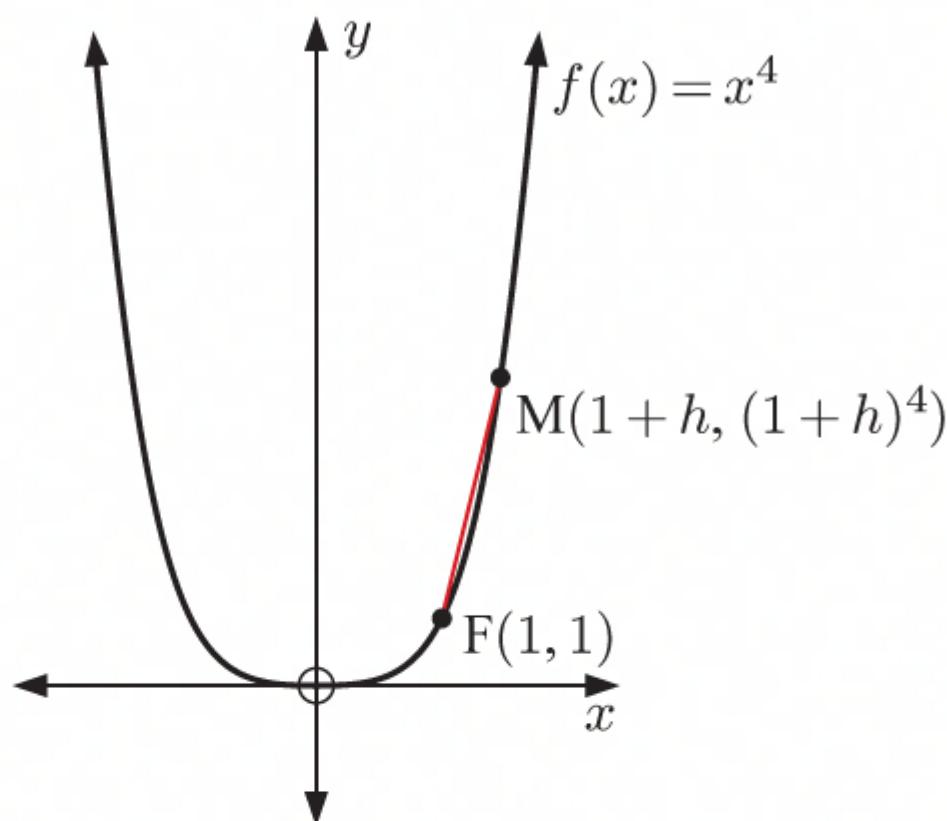


The gradient of the tangent at F

$$\begin{aligned}
 &= \lim_{h \rightarrow 0} \frac{f(2+h) - f(2)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\frac{4}{2+h} - \frac{4}{2}}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\frac{4}{2+h} - 2\left(\frac{2+h}{2+h}\right)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\frac{4 - 2(2+h)}{2+h}}{h} \\
 &= \lim_{h \rightarrow 0} \frac{4 - 4 - 2h}{h(2+h)} \\
 &= \lim_{h \rightarrow 0} \frac{-2\cancel{h}}{\cancel{h}(2+h)} \\
 &= \lim_{h \rightarrow 0} \frac{-2}{2+h} \quad \{\text{as } h \neq 0\} \\
 &= \frac{-2}{2} \\
 &= -1
 \end{aligned}$$

- At $x = 1$, $f(1) = 1^4 = 1$.

Let F be the point $(1, 1)$ and M have x -coordinate $1 + h$, so M is $(1 + h, (1 + h)^4)$.



The gradient of the tangent at F

$$\begin{aligned}
 &= \lim_{h \rightarrow 0} \frac{f(1+h) - f(1)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{(1+h)^4 - 1^4}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\cancel{1} + 4h + 6h^2 + 4h^3 + h^4 - \cancel{1}}{h} \\
 &\quad \{\text{binomial expansion}\} \\
 &= \lim_{h \rightarrow 0} \frac{4h + 6h^2 + 4h^3 + h^4}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\cancel{h}(4 + 6h + 4h^2 + h^3)}{\cancel{h}} \\
 &= \lim_{h \rightarrow 0} (4 + 6h + 4h^2 + h^3) \quad \{\text{as } h \neq 0\} \\
 &= 4
 \end{aligned}$$

EXERCISE 11E

1 a $f(0) = 4$

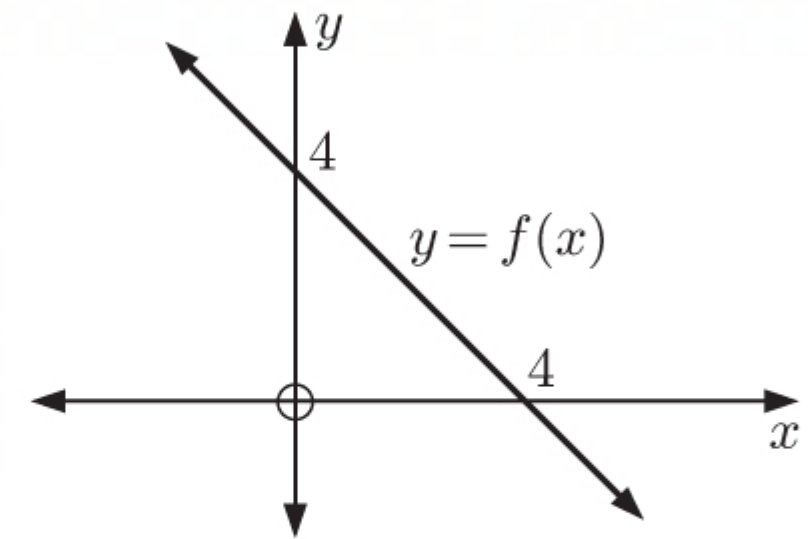
b $f'(0)$ is the gradient of the tangent to $f(x)$ at the point where $x = 0$.

Since $f(x)$ is a straight line, this is the same as the gradient of $f(x)$ itself.

$f(x)$ goes through $(0, 4)$ and $(4, 0)$, so it has

$$\text{gradient} = \frac{0 - 4}{4 - 0} = -1$$

$$\therefore f'(0) = -1$$

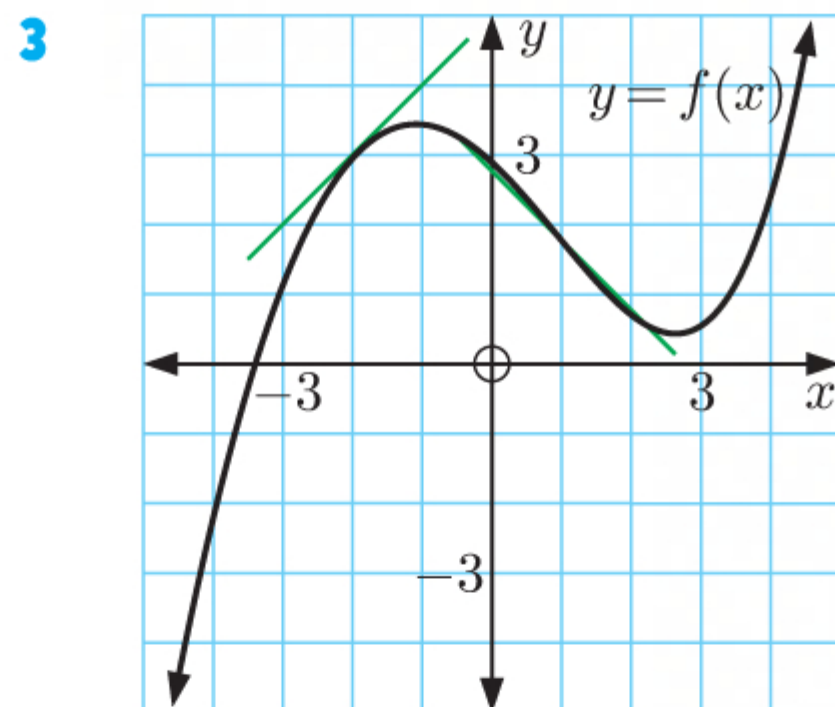
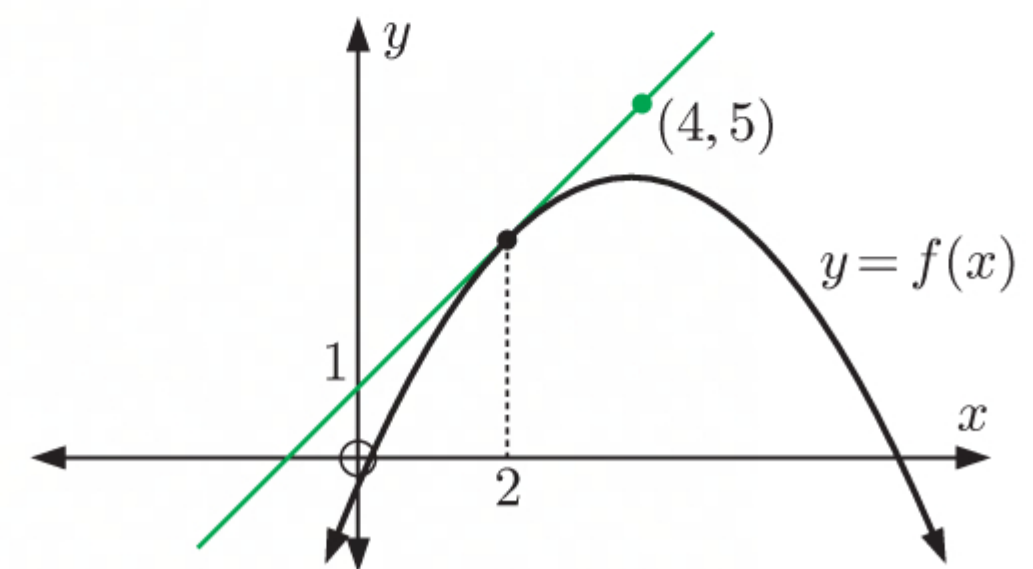


2 The graph shows the tangent to the curve $y = f(x)$ at the point where $x = 2$.

The tangent passes through $(0, 1)$ and $(4, 5)$.

$\therefore f'(2) = \text{gradient of the tangent}$

$$\begin{aligned} &= \frac{5 - 1}{4 - 0} \\ &= 1 \end{aligned}$$



a $f(3)$ is above the x -axis, so $f(3)$ is positive.

b $f'(1)$ is the gradient of the tangent to $f(x)$ at the point where $x = 1$. Since the curve is decreasing at $x = 1$, then $f'(1)$ is negative.

c $f(-4)$ is below the x -axis, so $f(-4)$ is negative.

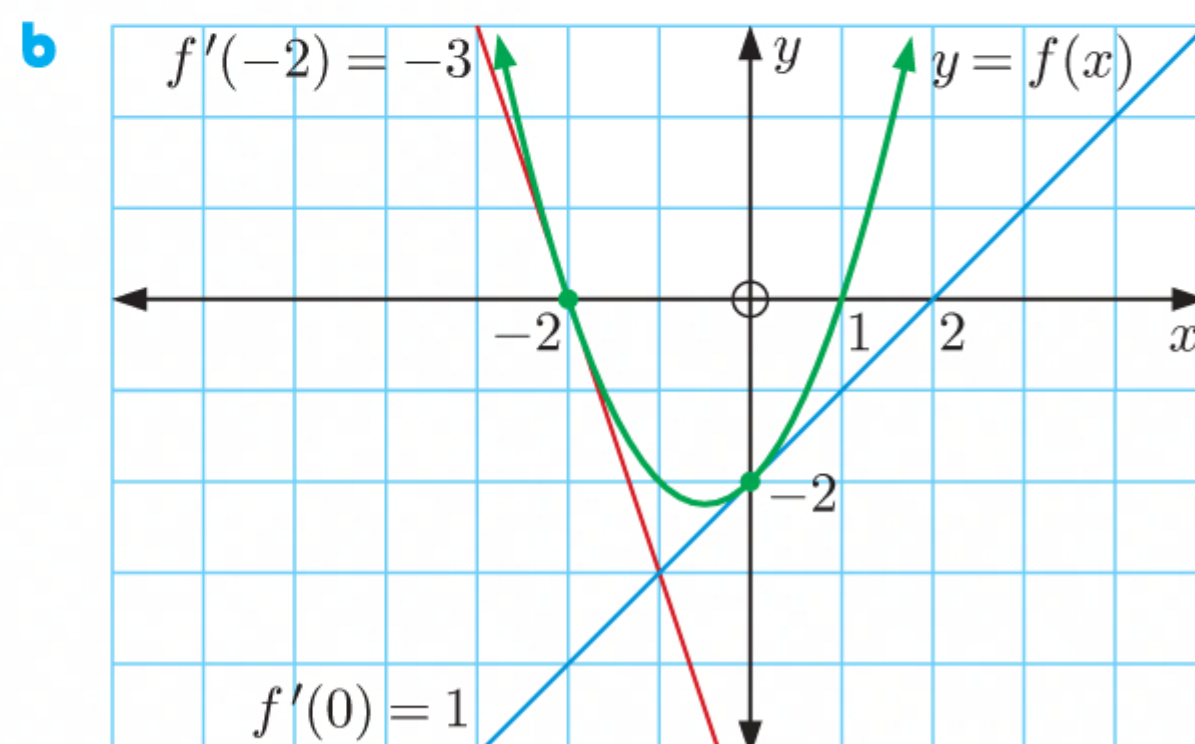
d $f'(-2)$ is the gradient of the tangent to $f(x)$ at the point where $x = -2$. Since the curve is increasing at $x = -2$, then $f'(-2)$ is positive.

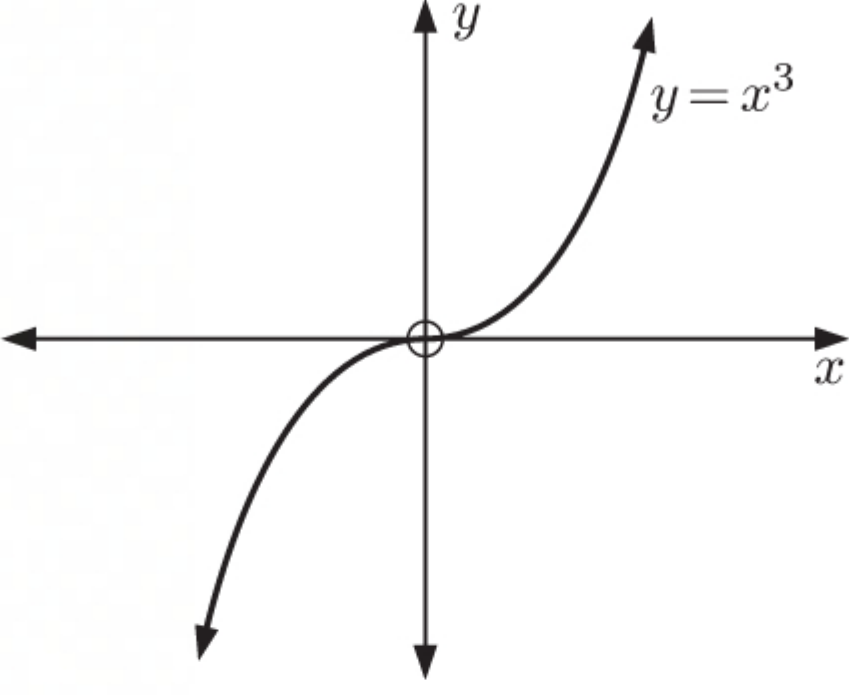
4 a i $f'(x) = 2x + 1$
 $f'(-2) = 2(-2) + 1$
 $= -3$

The gradient of the tangent to $y = f(x)$ at the point where $x = -2$ is -3 .

ii $f'(0) = 2(0) + 1$
 $= 1$

The gradient of the tangent to $y = f(x)$ at the point where $x = 0$ is 1 .

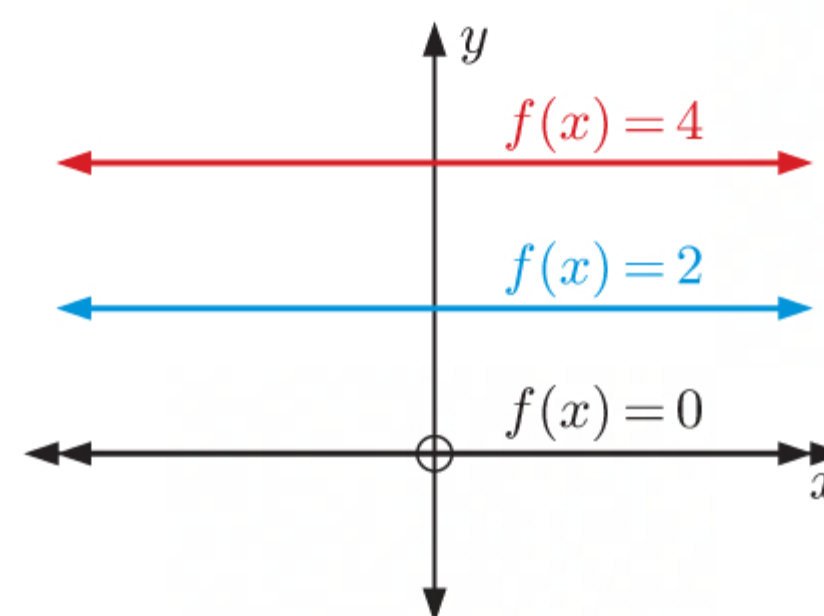


- 5**
- 
- $y = x^3$ is increasing for all $x < 0$ and $x > 0$.
 \therefore for all $x < 0$ and $x > 0$, $\frac{dy}{dx}$ is positive.
 $\therefore \frac{dy}{dx} \neq -x^2$ and $\frac{dy}{dx} \neq 4x$ as both of these derivative functions include negative values.
 When $x = 0$, $\frac{dy}{dx} = 0$ as the curve is horizontal at this point. $\therefore \frac{dy}{dx} \neq 3$
 Also, $\frac{dy}{dx} \neq 3(x-2)^2$ since $3(0-2)^2 = 12 \neq 0$.
 $\therefore \frac{dy}{dx} = 3x^2$ since $3(0)^2 = 0$.
 So, the answer is **C**.

INVESTIGATION 3

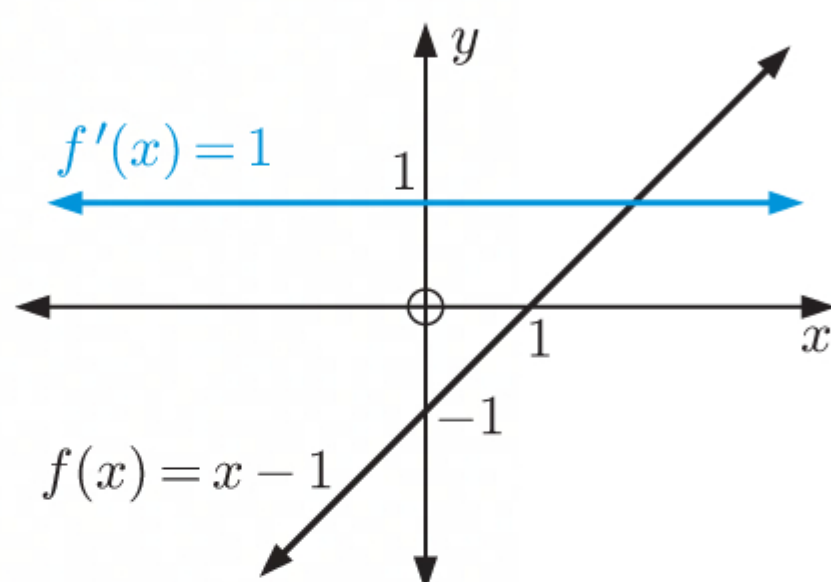
GRADIENT FUNCTIONS

- 1**
- a** $f(x) = 0$, $f(x) = 2$, and $f(x) = 4$ are all horizontal lines and hence all have gradient 0.
 - b** Yes, the gradient is constant for all values of x .

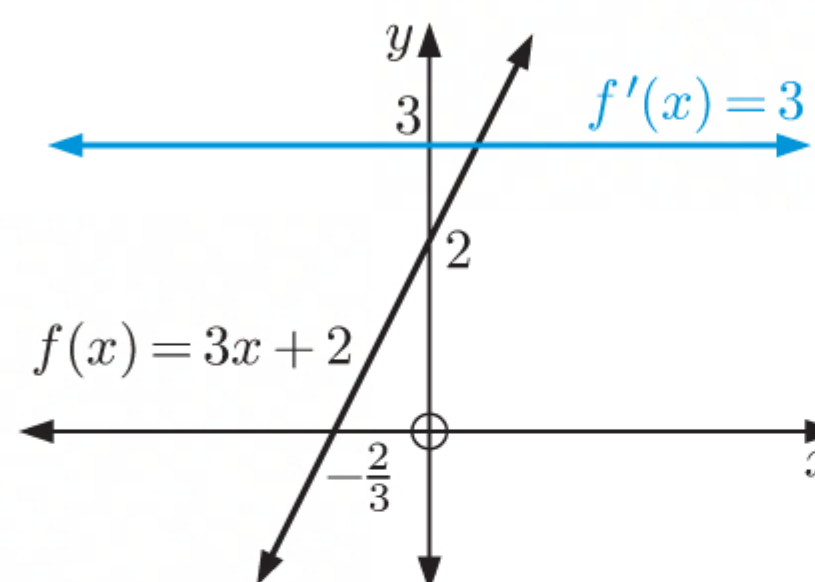


- 2**
- a** The gradient of $f(x) = mx + c$ is m .
 - b** The gradient m is constant for all values of x .

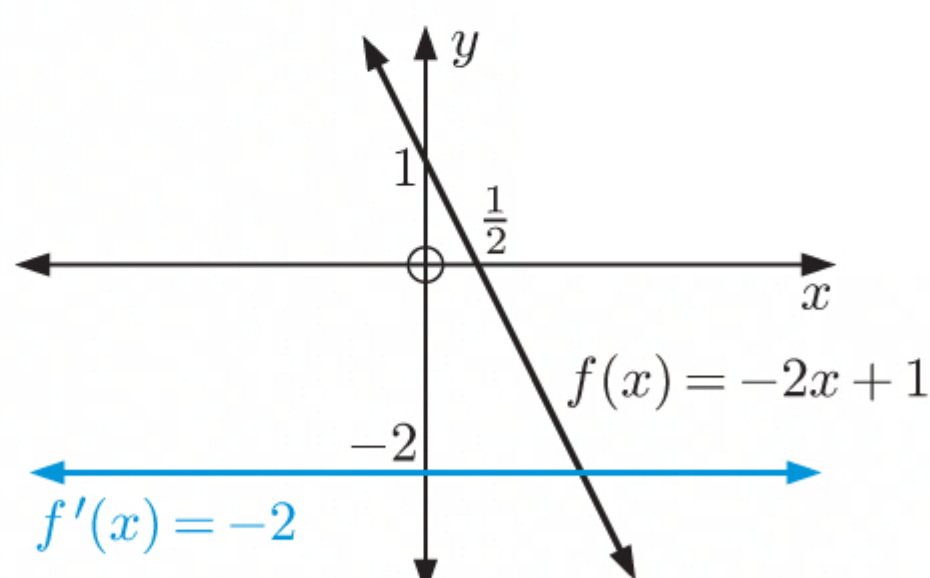
c i



ii

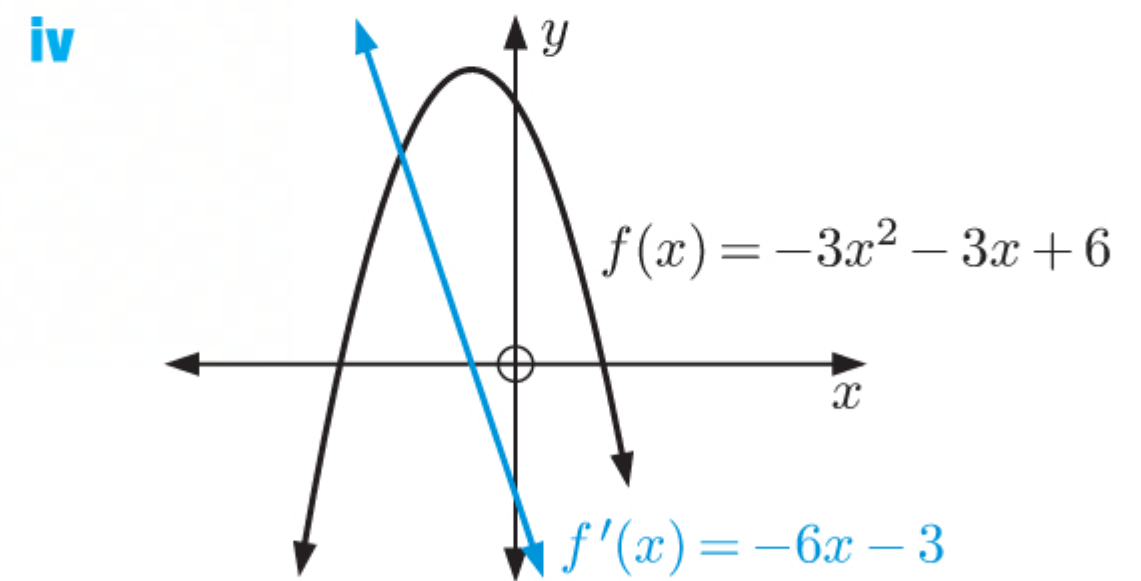
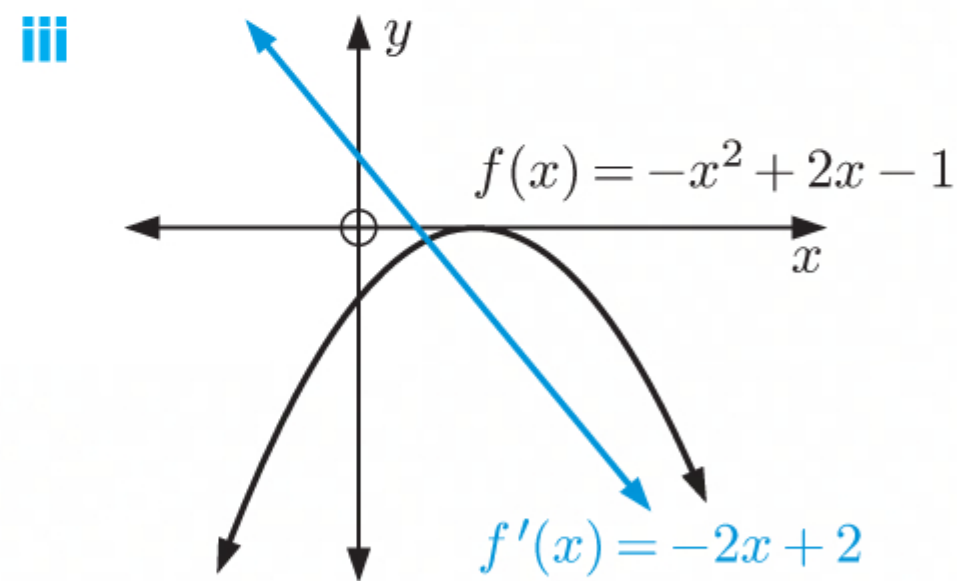
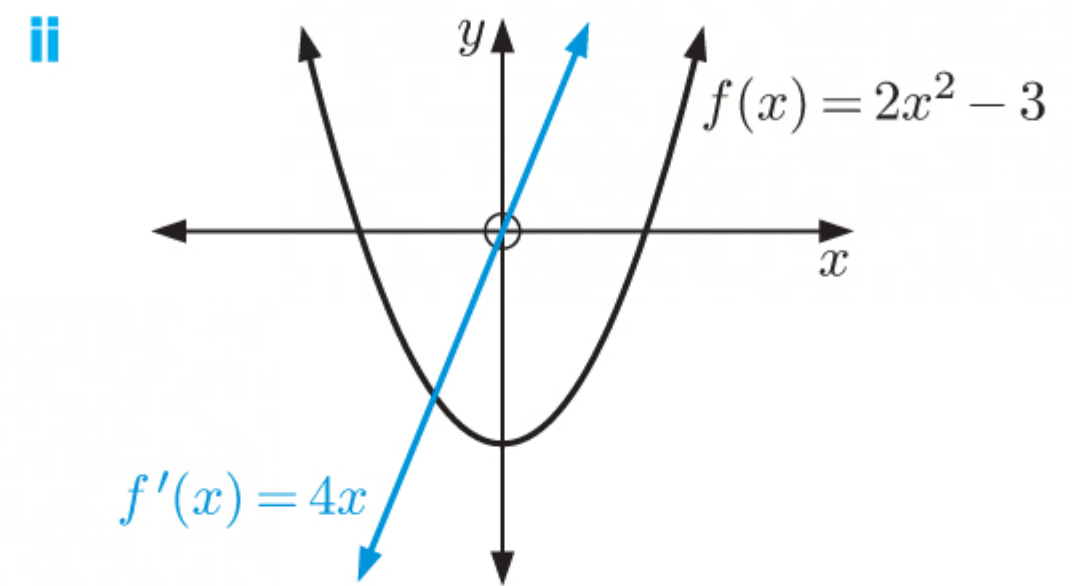
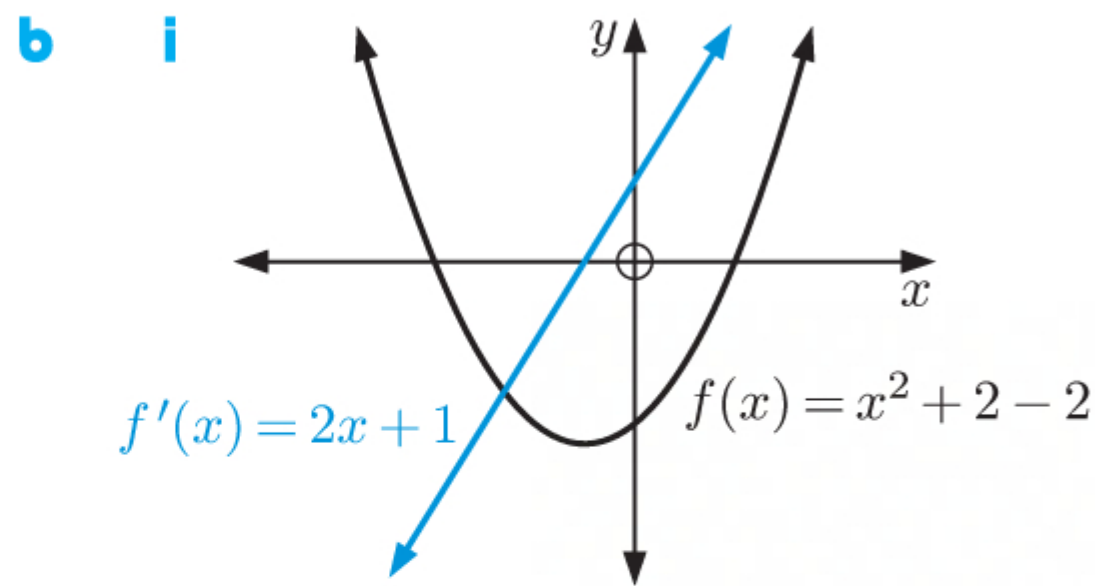


iii

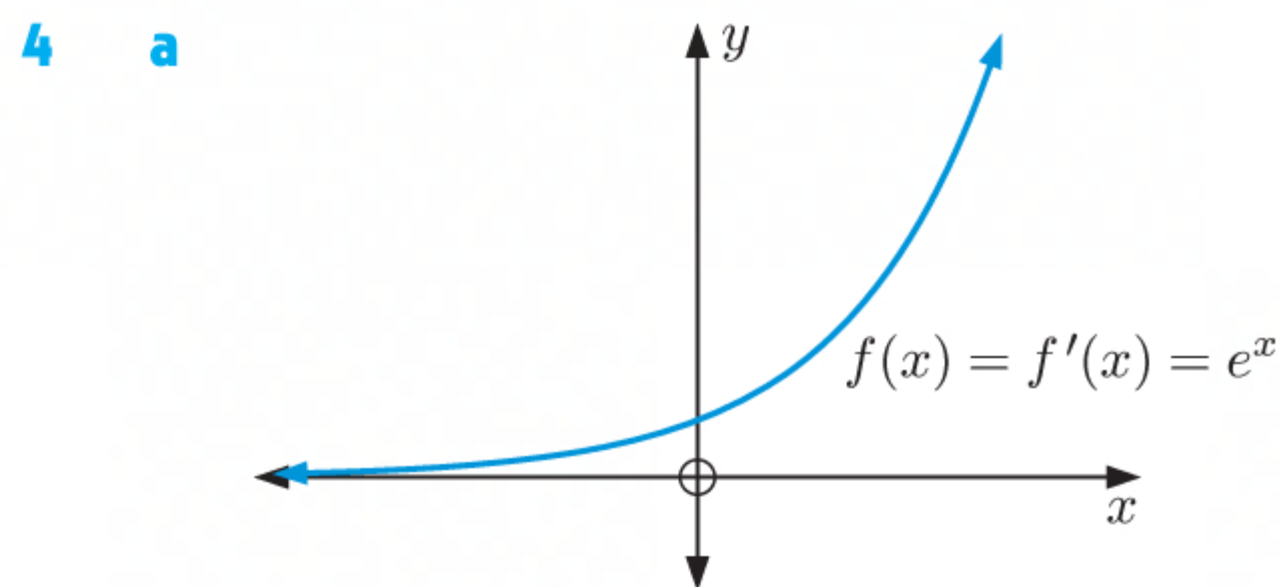


$f'(x)$ is constant for all x .

3 a $f'(x)$ is a linear function.



c The gradient functions $f'(x)$ in **b** are all linear functions.



b The gradient function is $f'(x) = f(x) = e^x$.

EXERCISE 11F

1 a $f(x) = x$

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{(x+h) - x}{h} \\ &= \lim_{h \rightarrow 0} \frac{h}{h} \\ &= \lim_{h \rightarrow 0} 1 \quad \{\text{as } h \neq 0\} \\ &= 1 \end{aligned}$$

b $f(x) = 1$

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{1 - 1}{h} \\ &= \lim_{h \rightarrow 0} \frac{0}{h} \\ &= \lim_{h \rightarrow 0} 0 \quad \{\text{as } h \neq 0\} \\ &= 0 \end{aligned}$$

c $f(x) = x^3$

$$\begin{aligned}
 f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{(x+h)^3 - x^3}{h} \\
 &= \lim_{h \rightarrow 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x^3}{h} \\
 &= \lim_{h \rightarrow 0} \frac{3x^2h + 3xh^2 + h^3}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\cancel{h}(3x^2 + 3xh + h^2)}{\cancel{h}} \\
 &= \lim_{h \rightarrow 0} (3x^2 + 3xh + h^2) \quad \{\text{as } h \neq 0\} \\
 &= 3x^2
 \end{aligned}$$

2 a $f(x) = 2x + 5$

$$\begin{aligned}
 f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{(2(x+h) + 5) - (2x + 5)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\cancel{2x} + 2h + \cancel{5} - \cancel{2x} - \cancel{5}}{h} \\
 &= \lim_{h \rightarrow 0} \frac{2\cancel{h}}{\cancel{h}} \\
 &= \lim_{h \rightarrow 0} 2 \quad \{\text{as } h \neq 0\} \\
 &= 2
 \end{aligned}$$

b $f(x) = x^2 - 3x$

$$\begin{aligned}
 f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{[(x+h)^2 - 3(x+h)] - [x^2 - 3x]}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\cancel{x^2} + 2xh + h^2 - \cancel{3x} - 3h - \cancel{x^2} + \cancel{3x}}{h} \\
 &= \lim_{h \rightarrow 0} \frac{2xh + h^2 - 3h}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\cancel{h}(2x + h - 3)}{\cancel{h}} \\
 &= \lim_{h \rightarrow 0} (2x + h - 3) \quad \{\text{as } h \neq 0\} \\
 &= 2x - 3
 \end{aligned}$$

$$\text{c} \quad f(x) = -x^2 + 5x - 3$$

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{[-(x+h)^2 + 5(x+h) - 3] - [-x^2 + 5x - 3]}{h} \\ &= \lim_{h \rightarrow 0} \frac{-\cancel{x^2} - 2xh - h^2 + \cancel{5x} + 5h - \cancel{3} + \cancel{x^2} - \cancel{5x} + \cancel{3}}{h} \\ &= \lim_{h \rightarrow 0} \frac{-2xh - h^2 + 5h}{h} \\ &= \lim_{h \rightarrow 0} \frac{\cancel{h}(-2x - h + 5)}{\cancel{h}} \\ &= \lim_{h \rightarrow 0} (-2x + 5 - h) \quad \{\text{as } h \neq 0\} \\ &= -2x + 5 \end{aligned}$$

$$\text{3} \quad \text{a} \quad y = f(x) = 4 - x$$

$$\begin{aligned} \frac{dy}{dx} &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{[4 - (x+h)] - [4 - x]}{h} \\ &= \lim_{h \rightarrow 0} \frac{\cancel{4} - \cancel{x} - h - \cancel{4} + \cancel{x}}{h} \\ &= \lim_{h \rightarrow 0} \frac{-h}{h} \\ &= \lim_{h \rightarrow 0} -1 \quad \{\text{as } h \neq 0\} \\ &= -1 \end{aligned}$$

$$\text{b} \quad y = f(x) = 2x^2 + x - 1$$

$$\begin{aligned} \frac{dy}{dx} &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{[2(x+h)^2 + (x+h) - 1] - [2x^2 + x - 1]}{h} \\ &= \lim_{h \rightarrow 0} \frac{\cancel{2x^2} + 4xh + 2h^2 + \cancel{x} + h - \cancel{1} - \cancel{2x^2} - \cancel{x} + \cancel{1}}{h} \\ &= \lim_{h \rightarrow 0} \frac{4xh + 2h^2 + h}{h} \\ &= \lim_{h \rightarrow 0} \frac{\cancel{h}(4x + 2h + 1)}{\cancel{h}} \\ &= \lim_{h \rightarrow 0} (4x + 1 + 2h) \quad \{\text{as } h \neq 0\} \\ &= 4x + 1 \end{aligned}$$

c $y = f(x) = x^3 - 2x^2 + 3$

$$\begin{aligned}
 \frac{dy}{dx} &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{[(x+h)^3 - 2(x+h)^2 + 3] - [x^3 - 2x^2 + 3]}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\cancel{x^3} + 3x^2h + 3xh^2 + h^3 - \cancel{2x^2} - 4xh - 2h^2 + \cancel{3} - \cancel{x^3} + \cancel{2x^2} - \cancel{3}}{h} \\
 &= \lim_{h \rightarrow 0} \frac{3x^2h + 3xh^2 + h^3 - 4xh - 2h^2}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\cancel{h}(3x^2 + 3xh + h^2 - 4x - 2h)}{\cancel{h}} \\
 &= \lim_{h \rightarrow 0} (3x^2 + 3xh + h^2 - 4x - 2h) \quad \{\text{as } h \neq 0\} \\
 &= 3x^2 - 4x
 \end{aligned}$$

4 a $f(x) = x^3$

$$\begin{aligned}
 f'(a) &= \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h} \\
 \therefore f'(2) &= \lim_{h \rightarrow 0} \frac{f(2+h) - f(2)}{h} \quad \text{where } f(2) = 2^3 = 8 \\
 &= \lim_{h \rightarrow 0} \frac{(2+h)^3 - 8}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\cancel{8} + 12h + 6h^2 + h^3 - \cancel{8}}{h} \\
 &= \lim_{h \rightarrow 0} \frac{12h + 6h^2 + h^3}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\cancel{h}(12 + 6h + h^2)}{\cancel{h}} \\
 &= \lim_{h \rightarrow 0} (12 + 6h + h^2) \quad \{\text{as } h \neq 0\} \\
 &= 12
 \end{aligned}$$

b $f(x) = x^4$

$$\begin{aligned}
 f'(a) &= \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h} \\
 \therefore f'(3) &= \lim_{h \rightarrow 0} \frac{f(3+h) - f(3)}{h} \quad \text{where } f(3) = 3^4 = 81 \\
 &= \lim_{h \rightarrow 0} \frac{(3+h)^4 - 81}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\cancel{81} + 108h + 54h^2 + 12h^3 + h^4 - \cancel{81}}{h} \\
 &= \lim_{h \rightarrow 0} \frac{108h + 54h^2 + 12h^3 + h^4}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\cancel{h}(108 + 54h + 12h^2 + h^3)}{\cancel{h}} \\
 &= \lim_{h \rightarrow 0} (108 + 54h + 12h^2 + h^3) \quad \{\text{as } h \neq 0\} \\
 &= 108
 \end{aligned}$$

5 a $f(x) = \frac{1}{x}$

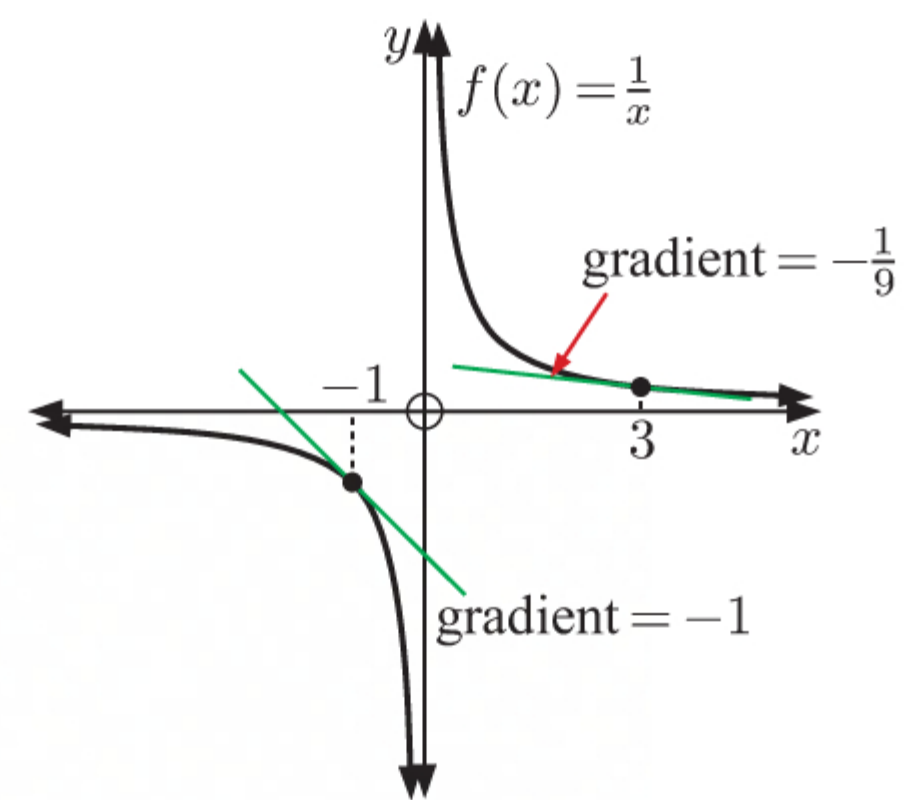
$$\begin{aligned}
 f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\left(\frac{x}{x}\right) \frac{1}{x+h} - \frac{1}{x} \left(\frac{x+h}{x+h}\right)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\frac{x - (x+h)}{x(x+h)}}{h} \\
 &= \lim_{h \rightarrow 0} \frac{-\cancel{h}}{\cancel{h}x(x+h)} \\
 &= \lim_{h \rightarrow 0} \frac{-1}{x(x+h)} \quad \{\text{as } h \neq 0\} \\
 &= -\frac{1}{x^2}
 \end{aligned}$$

b $f'(-1) = -\frac{1}{(-1)^2}$
 $= -1$

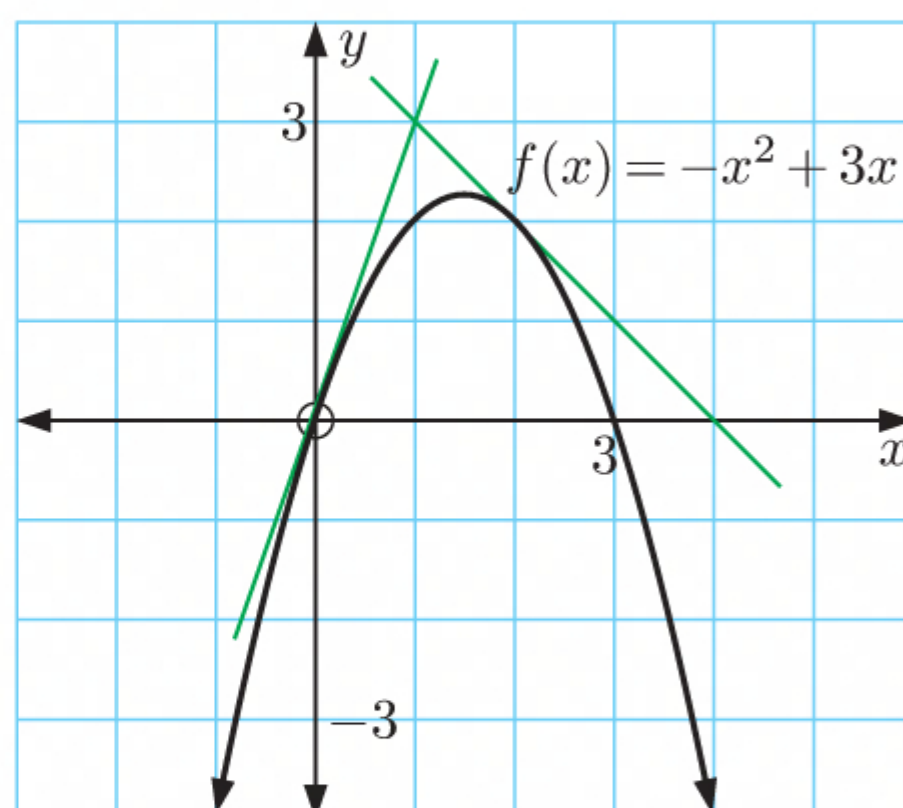
The tangent to $f(x) = \frac{1}{x}$ at the point where $x = -1$ has gradient -1 .

$$\begin{aligned}
 f'(3) &= -\frac{1}{3^2} \\
 &= -\frac{1}{9}
 \end{aligned}$$

The tangent to $f(x) = \frac{1}{x}$ at the point where $x = 3$ has gradient $-\frac{1}{9}$.



6 a



- i** The tangent to $f(x) = -x^2 + 3x$ at the point where $x = 0$ has gradient $\approx \frac{3-0}{1-0} \approx 3$.
- ii** The tangent to $f(x) = -x^2 + 3x$ at the point where $x = 2$ has gradient $\approx \frac{0-3}{4-1} \approx -1$.

$$\begin{aligned}
\text{b } f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h \rightarrow 0} \frac{-(x+h)^2 + 3(x+h) - (-x^2 + 3x)}{h} \\
&= \lim_{h \rightarrow 0} \frac{-\cancel{x^2} - 2xh - h^2 + \cancel{3x} + 3h + \cancel{x^2} - \cancel{3x}}{h} \\
&= \lim_{h \rightarrow 0} \frac{-2xh - h^2 + 3h}{h} \\
&= \lim_{h \rightarrow 0} \frac{\cancel{h}(-2x - h + 3)}{\cancel{h}} \\
&= \lim_{h \rightarrow 0} (-2x - h + 3) \quad \{\text{as } h \neq 0\} \\
&= -2x + 3
\end{aligned}$$

$$\begin{aligned}
\text{c } f'(0) &= -2(0) + 3 & f'(2) &= -2(2) + 3 \\
&= 3 & &= -1
\end{aligned}$$

Both values are the same as the estimates in **a**.

$$7 \quad \text{a } y = f(x) = x^3 - 3x$$

$$\begin{aligned}
\frac{dy}{dx} &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h \rightarrow 0} \frac{[(x+h)^3 - 3(x+h)] - [x^3 - 3x]}{h} \\
&= \lim_{h \rightarrow 0} \frac{\cancel{x^3} + 3x^2h + 3xh^2 + h^3 - \cancel{3x} - 3h - \cancel{x^3} + \cancel{3x}}{h} \\
&= \lim_{h \rightarrow 0} \frac{3x^2h + 3xh^2 + h^3 - 3h}{h} \\
&= \lim_{h \rightarrow 0} \frac{\cancel{h}(3x^2 + 3xh + h^2 - 3)}{\cancel{h}} \\
&= \lim_{h \rightarrow 0} (3x^2 + 3xh + h^2 - 3) \quad \{\text{as } h \neq 0\} \\
&= 3x^2 - 3
\end{aligned}$$

$$\begin{aligned}
\text{b } \text{The tangent has zero gradient when } \frac{dy}{dx} &= 0 \\
&\therefore 3x^2 - 3 = 0 \\
&\therefore 3x^2 = 3 \\
&\therefore x^2 = 1 \\
&\therefore x = \pm 1
\end{aligned}$$

$$\text{When } x = -1, \quad y = (-1)^3 - 3(-1) = 2$$

$$\text{When } x = 1, \quad y = (1)^3 - 3(1) = -2$$

So, the points on the graph at which the tangent has zero gradient are $(-1, 2)$ and $(1, -2)$.

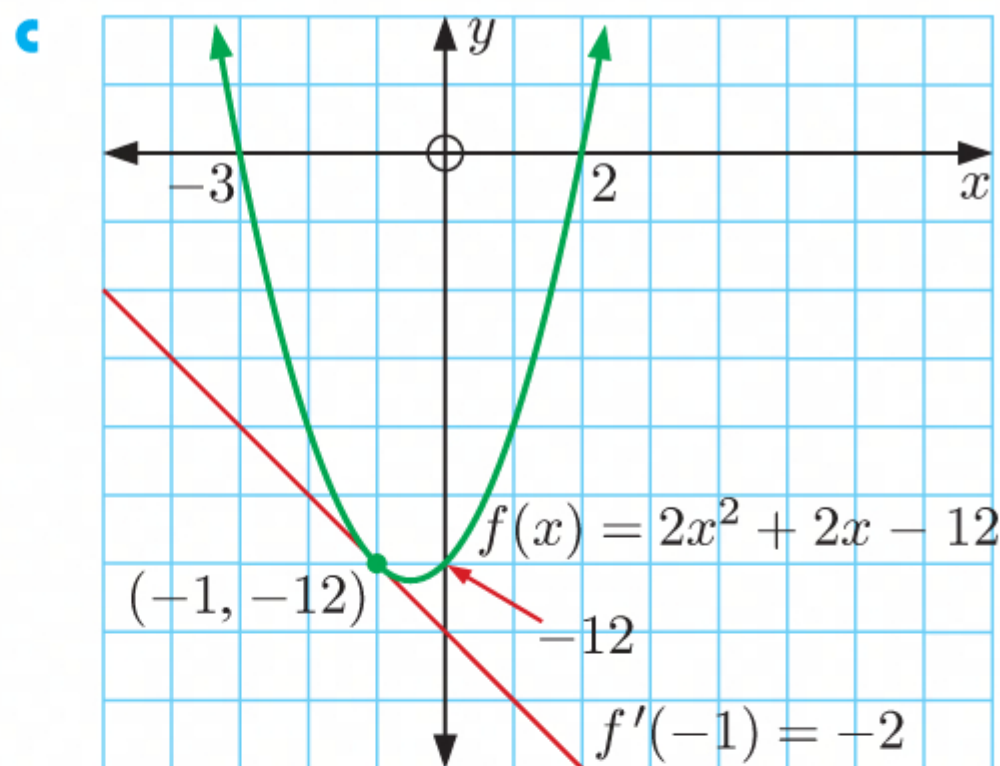
8 a $f(x) = 2x^2 + 2x - 12$

$$\begin{aligned}
 f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{[2(x+h)^2 + 2(x+h) - 12] - [2x^2 + 2x - 12]}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\cancel{2x^2} + 4xh + 2h^2 + \cancel{2x} + 2h - \cancel{12} - \cancel{2x^2} - \cancel{2x} + \cancel{12}}{h} \\
 &= \lim_{h \rightarrow 0} \frac{4xh + 2h^2 + 2h}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\cancel{h}(4x + 2h + 2)}{\cancel{h}} \\
 &= \lim_{h \rightarrow 0} (4x + 2h + 2) \quad \{\text{as } h \neq 0\} \\
 &= 4x + 2
 \end{aligned}$$

b The tangent has gradient -2 when $f'(x) = -2$
 $\therefore 4x + 2 = -2$
 $\therefore 4x = -4$
 $\therefore x = -1$

Now, $f(-1) = 2(-1)^2 + 2(-1) - 12$
 $= 2 - 2 - 12$
 $= -12$

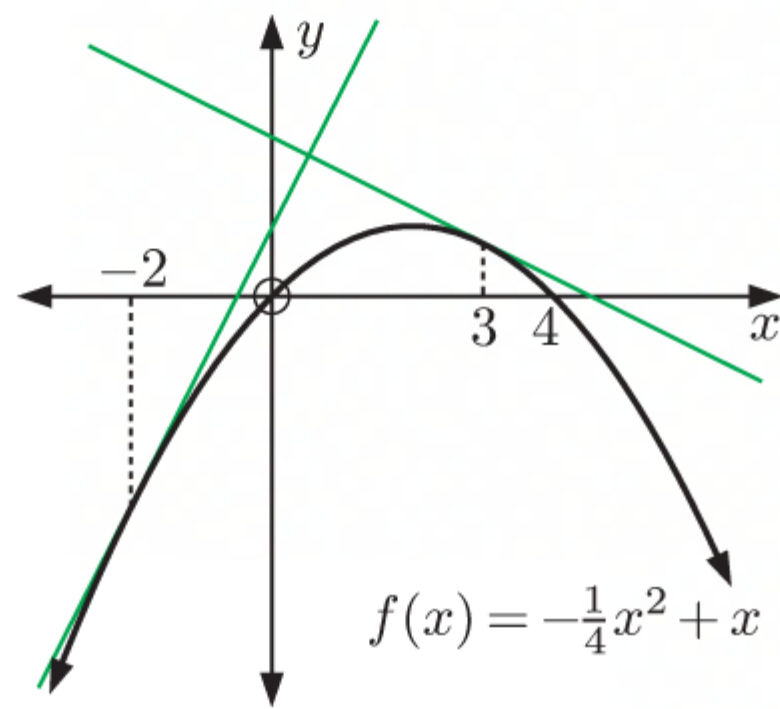
So, the tangent has gradient -2 at $(-1, -12)$.



9 a $f(x) = -\frac{1}{4}x^2 + x$

$$\begin{aligned}
 f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\left[-\frac{1}{4}(x+h)^2 + (x+h)\right] - \left[-\frac{1}{4}x^2 + x\right]}{h} \\
 &= \lim_{h \rightarrow 0} \frac{-\cancel{\frac{1}{4}x^2} - \frac{1}{2}xh - \frac{1}{4}h^2 + \cancel{x} + h + \cancel{\frac{1}{4}x^2} - \cancel{x}}{h} \\
 &= \lim_{h \rightarrow 0} \frac{-\frac{1}{2}xh - \frac{1}{4}h^2 + h}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\cancel{h}\left(-\frac{1}{2}x - \frac{1}{4}h + 1\right)}{\cancel{h}} \\
 &= \lim_{h \rightarrow 0} \left(-\frac{1}{2}x - \frac{1}{4}h + 1\right) \quad \{\text{as } h \neq 0\} \\
 &= -\frac{1}{2}x + 1
 \end{aligned}$$

b



The illustrated tangents are perpendicular if the product of their gradients is -1 .

One tangent passes through the point where $x = -2$ and the other tangent passes through the point where $x = 3$.

The tangent at $x = -2$ has gradient

$$\begin{aligned}
 f'(-2) &= -\frac{1}{2}(-2) + 1 \\
 &= 2
 \end{aligned}$$

and the tangent at $x = 3$ has gradient

$$\begin{aligned}
 f'(3) &= -\frac{1}{2}(3) + 1 \\
 &= -\frac{1}{2}
 \end{aligned}$$

Since $2 \times \left(-\frac{1}{2}\right) = -1$, the two tangents are perpendicular.

10 a

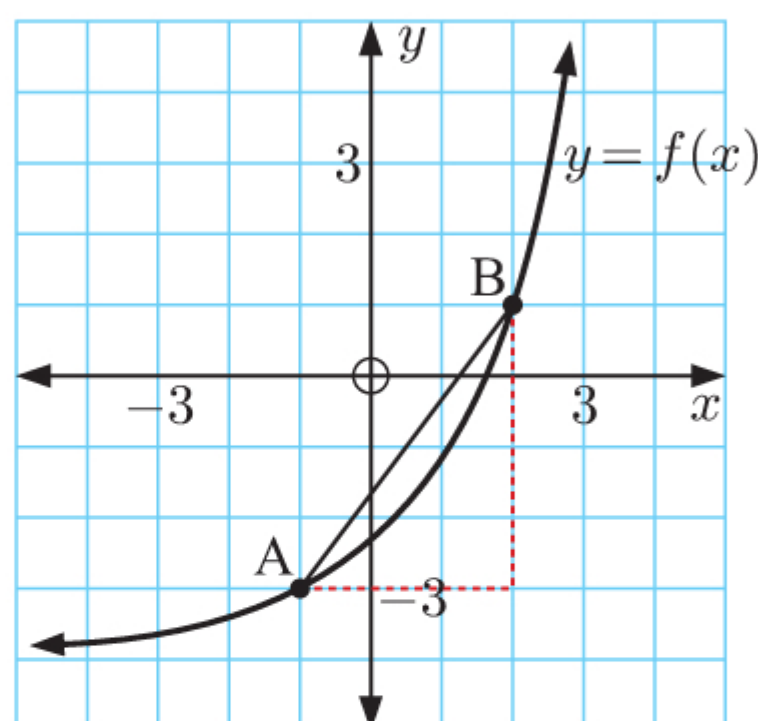
$f(x)$	$f'(x)$
x^1	1
x^2	$2x$
x^3	$3x^2$
x^4	$4x^3$
x^{-1}	$-x^{-2}$
x^0	0

b

If $f(x) = x^n$,
then $f'(x) = nx^{n-1}$.

REVIEW SET 11A

1

average rate of change in $f(x)$ from A to B

$$\begin{aligned}
 &= \frac{f(b) - f(a)}{b - a} \\
 &= \frac{1 - (-3)}{2 - (-1)} \\
 &= \frac{4}{3}
 \end{aligned}$$

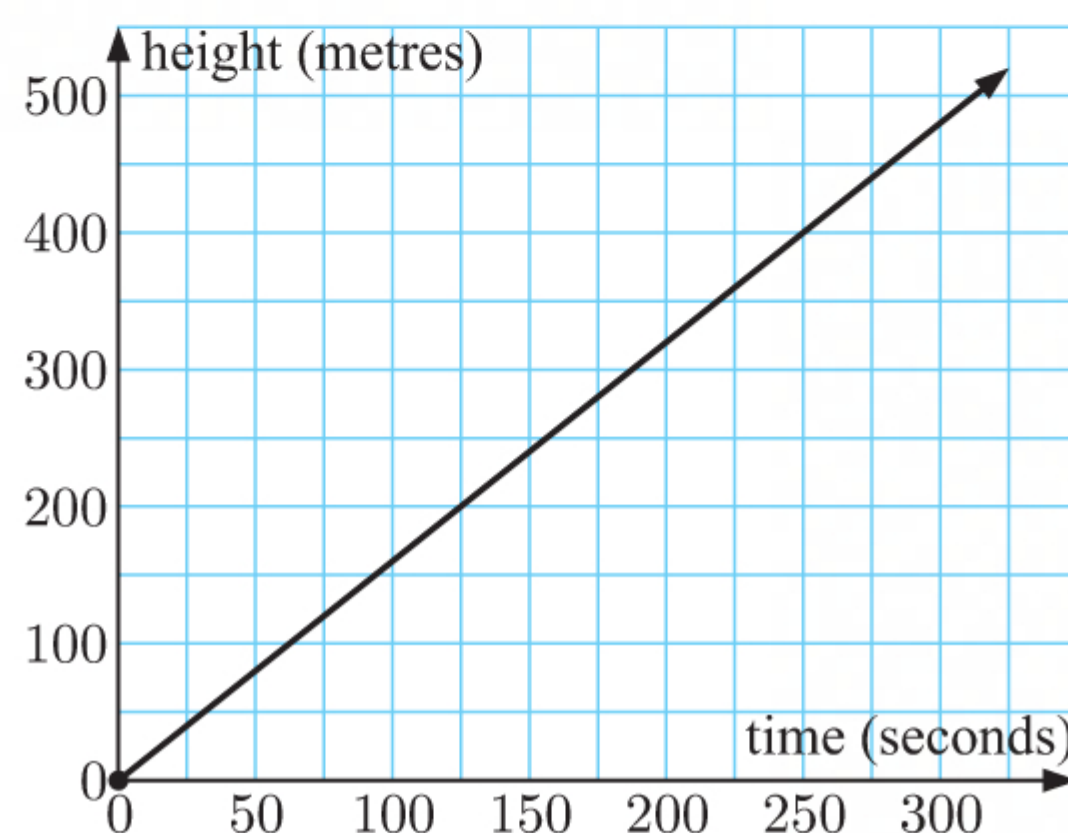
2

a The graph of height against time is a straight line.

\therefore the height increases by the same amount each time interval.

\therefore the ski-lift is increasing in height at a constant rate.

b rate of change = $\frac{(400 - 0) \text{ m}}{(250 - 0) \text{ s}}$
 $= 1.6 \text{ m s}^{-1}$



3

a We can make $6x - 7$ as close as we like to -1 by making x sufficiently close to 1.

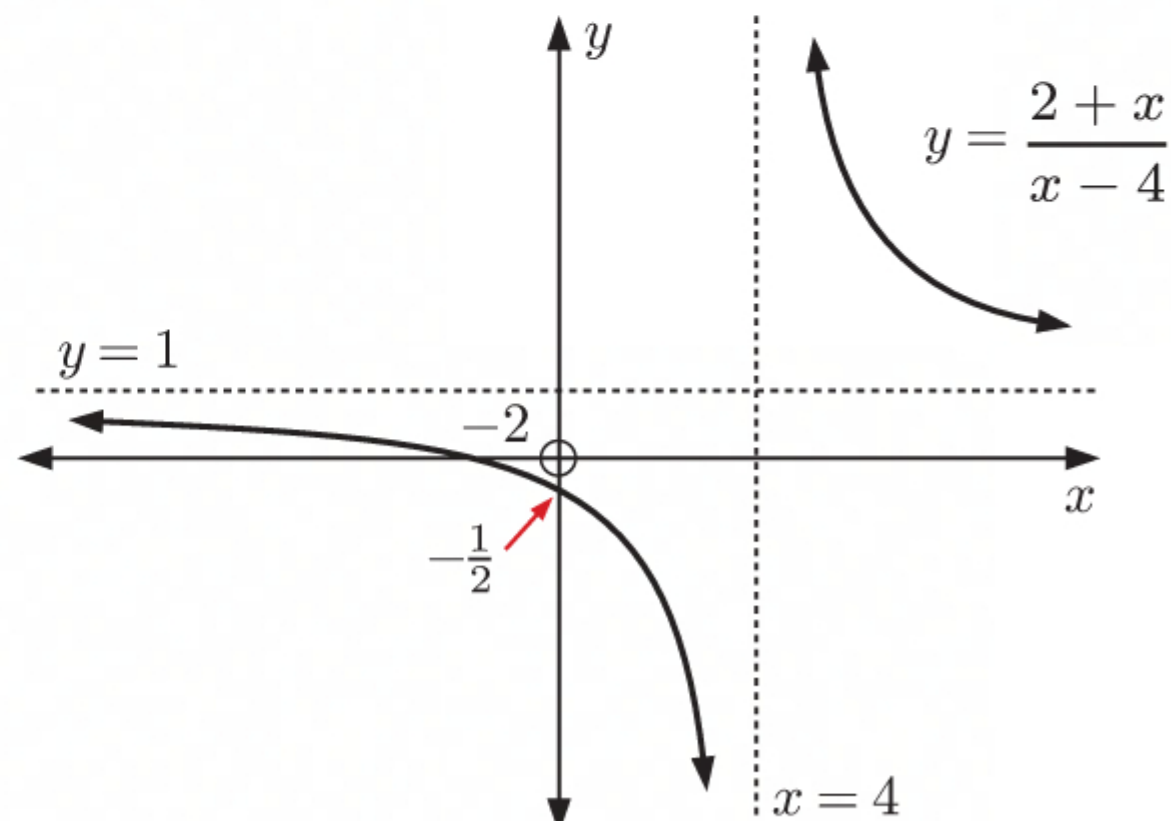
$$\therefore \lim_{x \rightarrow 1} (6x - 7) = -1$$

b $\lim_{h \rightarrow 0} \frac{2h^2 - h}{h} = \lim_{h \rightarrow 0} \frac{\cancel{h}(2h - 1)}{\cancel{h}}$
 $= \lim_{h \rightarrow 0} (2h - 1) \quad \{\text{as } h \neq 0\}$
 $= -1$

c $\lim_{x \rightarrow 4} \frac{x^2 - 16}{x - 4} = \lim_{x \rightarrow 4} \frac{(x + 4)\cancel{(x - 4)}}{\cancel{(x - 4)}}$
 $= \lim_{x \rightarrow 4} (x + 4) \quad \{\text{as } x \neq 4\}$
 $= 8$

4

a



b

As $x \rightarrow 4^-$, $y \rightarrow -\infty$

As $x \rightarrow 4^+$, $y \rightarrow \infty$

As $x \rightarrow -\infty$, $y \rightarrow 1^-$

As $x \rightarrow \infty$, $y \rightarrow 1^+$

The vertical asymptote is $x = 4$.

The horizontal asymptote is $y = 1$.

c $\lim_{x \rightarrow -\infty} \frac{2+x}{x-4} = 1, \quad \lim_{x \rightarrow \infty} \frac{2+x}{x-4} = 1$

5 a

$$\begin{aligned}
 f(x) &= 2x^2 \\
 \frac{f(x+h) - f(x)}{h} &= \frac{2(x+h)^2 - 2x^2}{h} \\
 &= \frac{2(x^2 + 2xh + h^2) - 2x^2}{h} \\
 &= \frac{\cancel{2x^2} + 4xh + 2h^2 - \cancel{2x^2}}{h} \\
 &= \frac{\cancel{h}(4x + 2h)}{\cancel{h}} \\
 &= 4x + 2h \quad \text{provided } h \neq 0
 \end{aligned}$$

b If $x = 3$ then $\frac{f(3+h) - f(3)}{h} = 4(3) + 2h$ {using a}

$$= 12 + 2h$$

When $h = 0.1$,

$$\begin{aligned}
 \frac{f(3+h) - f(3)}{h} &= 12 + 2(0.1) \\
 &= 12 + 0.2 \\
 &= 12.2
 \end{aligned}$$

When $h = 0.001$,

$$\begin{aligned}
 \frac{f(3+h) - f(3)}{h} &= 12 + 2(0.001) \\
 &= 12 + 0.002 \\
 &= 12.002
 \end{aligned}$$

When $h = 0.01$,

$$\begin{aligned}
 \frac{f(3+h) - f(3)}{h} &= 12 + 2(0.01) \\
 &= 12 + 0.02 \\
 &= 12.02
 \end{aligned}$$

When $h = 0.0001$,

$$\begin{aligned}
 \frac{f(3+h) - f(3)}{h} &= 12 + 2(0.0001) \\
 &= 12 + 0.0002 \\
 &= 12.0002
 \end{aligned}$$

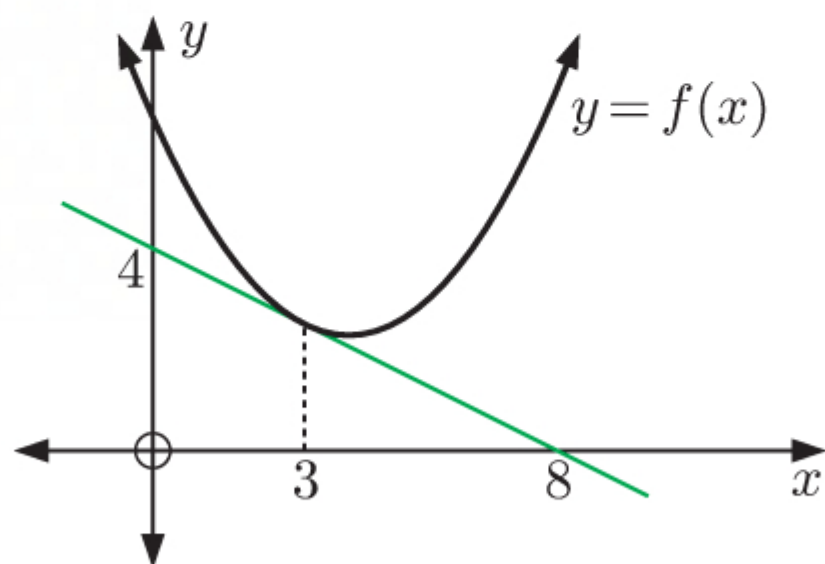
h	$\frac{f(3+h) - f(3)}{h}$
0.1	12.2
0.01	12.02
0.001	12.002
0.0001	12.0002

c $\lim_{h \rightarrow 0} \frac{f(3+h) - f(3)}{h} = \lim_{h \rightarrow 0} (12 + 2h)$

$$= 12$$

The gradient of the tangent to $y = 2x^2$ at the point $(3, 18)$ is 12.

6



The tangent to $y = f(x)$ at the point where $x = 3$ has

gradient $\frac{4 - 0}{0 - 8} = -\frac{1}{2}$.

$\therefore f'(3) = -\frac{1}{2}$

7 a $f(x) = x^2 + 2x$

$$\begin{aligned}
 f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{[(x+h)^2 + 2(x+h)] - [x^2 + 2x]}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\cancel{x^2} + 2xh + h^2 + \cancel{2x} + 2h - \cancel{x^2} - \cancel{2x}}{h} \\
 &= \lim_{h \rightarrow 0} \frac{2xh + h^2 + 2h}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\cancel{h}(2x + h + 2)}{\cancel{h}} \\
 &= \lim_{h \rightarrow 0} (2x + h + 2) \quad \{\text{as } h \neq 0\} \\
 &= 2x + 2
 \end{aligned}$$

b $y = f(x) = 4 - 3x^2$

$$\begin{aligned}
 \frac{dy}{dx} &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{[4 - 3(x+h)^2] - [4 - 3x^2]}{h} \\
 &= \lim_{h \rightarrow 0} \frac{4 - 3(x^2 + 2xh + h^2) - 4 + 3x^2}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\cancel{4} - \cancel{3x^2} - 6xh - 3h^2 - \cancel{4} + \cancel{3x^2}}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\cancel{h}(-6x - 3h)}{\cancel{h}} \\
 &= \lim_{h \rightarrow 0} (-6x - 3h) \quad \{\text{as } h \neq 0\} \\
 &= -6x
 \end{aligned}$$

8 a $y = f(x) = 2x^2 - 1$

$$\begin{aligned}
 \frac{dy}{dx} &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{[2(x+h)^2 - 1] - [2x^2 - 1]}{h} \\
 &= \lim_{h \rightarrow 0} \frac{2(x^2 + 2xh + h^2) - 1 - 2x^2 + 1}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\cancel{2x^2} + 4xh + 2h^2 - \cancel{1} - \cancel{2x^2} + \cancel{1}}{h} \\
 &= \lim_{h \rightarrow 0} \frac{4xh + 2h^2}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\cancel{h}(4x + 2h)}{\cancel{h}} \\
 &= \lim_{h \rightarrow 0} (4x + 2h) \quad \{\text{as } h \neq 0\} \\
 &= 4x
 \end{aligned}$$

b The gradient of the tangent to $y = 2x^2 - 1$ at the point where $x = 4$ is $4 \times 4 = 16$.

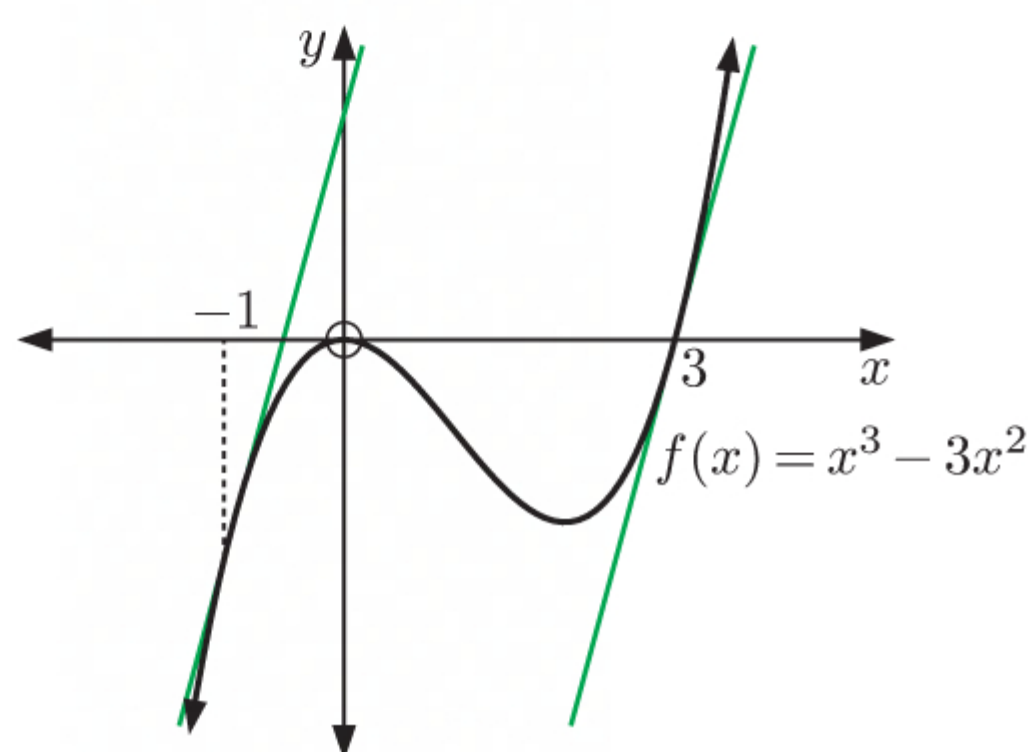
c The gradient of the tangent is equal to -12 when $4x = -12$

$$\therefore x = -3$$

9 a $f(x) = x^3 - 3x^2$

$$\begin{aligned}
 f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{[(x+h)^3 - 3(x+h)^2] - [x^3 - 3x^2]}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\cancel{x^3} + 3x^2h + 3xh^2 + h^3 - \cancel{3x^2} - 6xh - 3h^2 - \cancel{x^3} + \cancel{3x^2}}{h} \\
 &= \lim_{h \rightarrow 0} \frac{3x^2h + 3xh^2 + h^3 - 6xh - 3h^2}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\cancel{h}(3x^2 + 3xh + h^2 - 6x - 3h)}{\cancel{h}} \\
 &= \lim_{h \rightarrow 0} (3x^2 + 3xh + h^2 - 6x - 3h) \quad \{\text{as } h \neq 0\} \\
 &= 3x^2 - 6x
 \end{aligned}$$

b



The illustrated tangents pass through the point where $x = -1$ and the point where $x = 3$.

The gradient of the tangent at $x = -1$ is

$$\begin{aligned}
 f'(-1) &= 3(-1)^2 - 6(-1) \\
 &= 9
 \end{aligned}$$

and the gradient of the tangent at $x = 3$ is

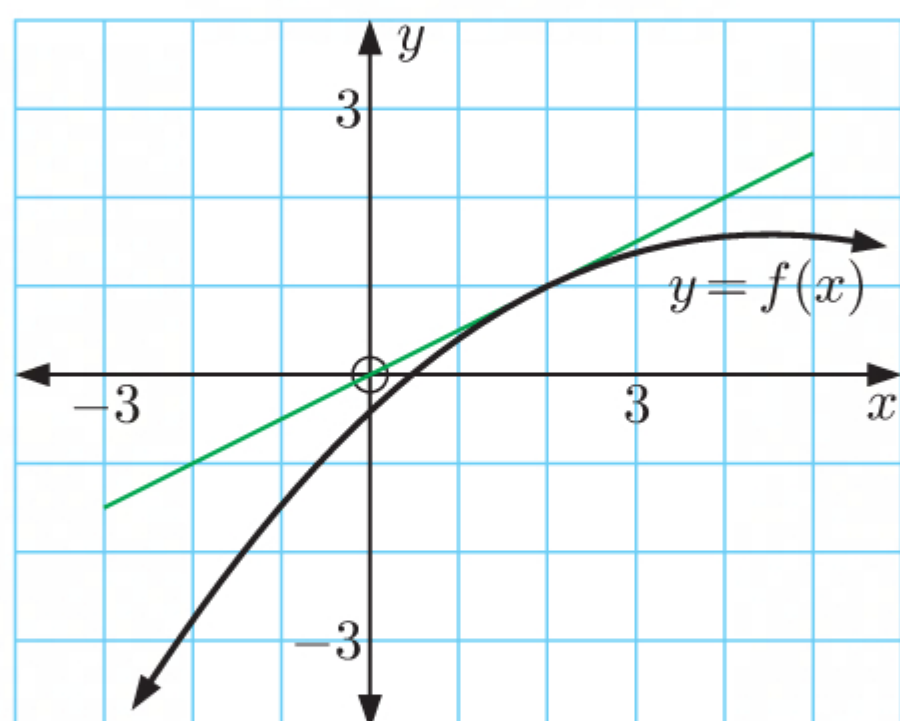
$$\begin{aligned}
 f'(3) &= 3(3)^2 - 6(3) \\
 &= 9
 \end{aligned}$$

Since $f'(-1) = f'(3)$, the gradients of the tangents are equal.

This means the tangents are parallel.

REVIEW SET 11B

1



The tangent at $x = 2$ has gradient $\frac{1-0}{2-0} = \frac{1}{2}$.

\therefore the instantaneous rate of change in $f(x)$ at $x = 2$ is $\frac{1}{2}$.

- 2 a** average rate of change in temperature from 7 am to noon

$$= \frac{(20 - 10)^\circ\text{C}}{(6 - 1) \text{ h}}$$

$$= \frac{10}{5}^\circ\text{C per h}$$

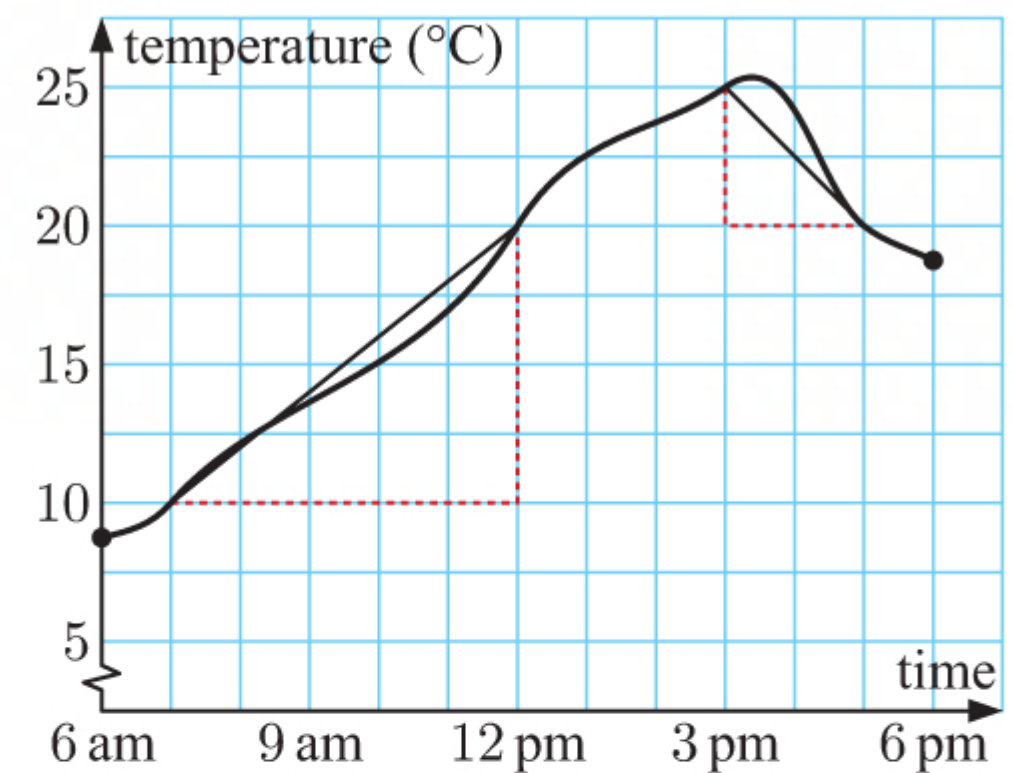
$$= 2^\circ\text{C per h}$$

- b** average rate of change in temperature from 3 pm to 5 pm

$$= \frac{(20 - 25)^\circ\text{C}}{(11 - 9) \text{ h}}$$

$$= \frac{-5}{2}^\circ\text{C per h}$$

$$= -2.5^\circ\text{C per h} \quad \text{or} \quad -2\frac{1}{2}^\circ\text{C per h}$$



3 a $\lim_{h \rightarrow 0} \frac{h^3 - 3h}{h} = \lim_{h \rightarrow 0} \frac{\cancel{h}(h^2 - 3)}{\cancel{h}}$

$$= \lim_{h \rightarrow 0} (h^2 - 3) \quad \{\text{as } h \neq 0\}$$

$$= -3$$

b $\lim_{x \rightarrow 1} \frac{3x^2 - 3x}{x - 1} = \lim_{x \rightarrow 1} \frac{3x\cancel{(x - 1)}}{\cancel{(x - 1)}}$

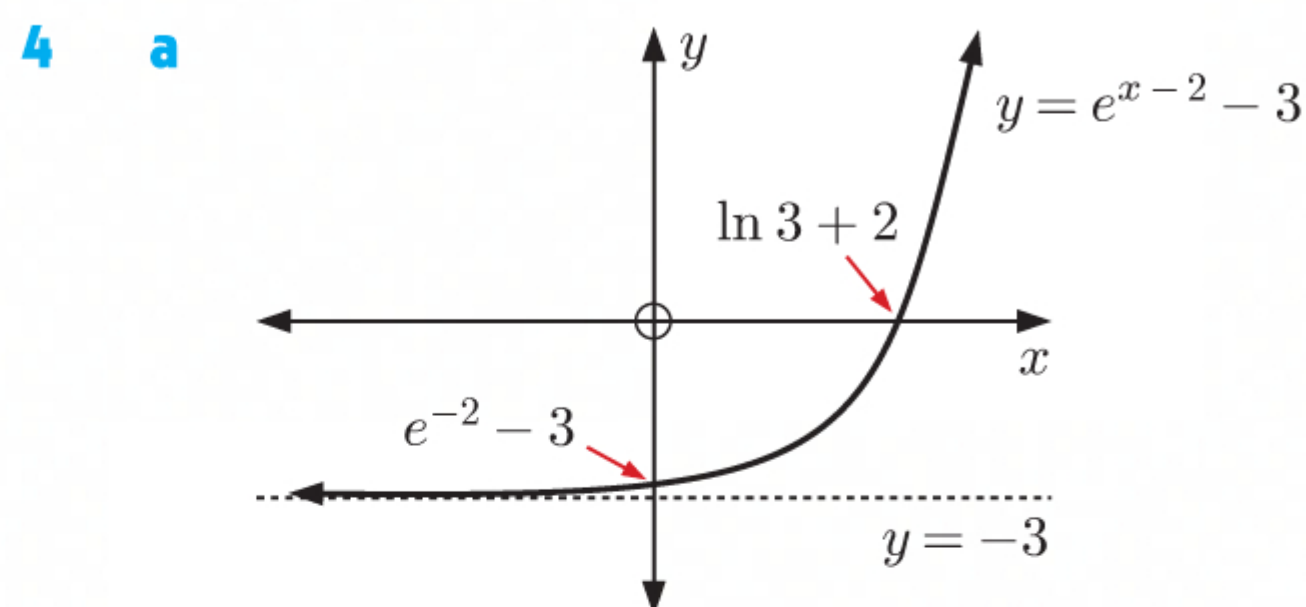
$$= \lim_{x \rightarrow 1} 3x \quad \{\text{as } x \neq 1\}$$

$$= 3$$

c $\lim_{x \rightarrow 2} \frac{x^2 - 3x + 2}{2 - x} = \lim_{x \rightarrow 2} \frac{(x - 1)\cancel{(x - 2)}}{-(\cancel{x - 2})}$

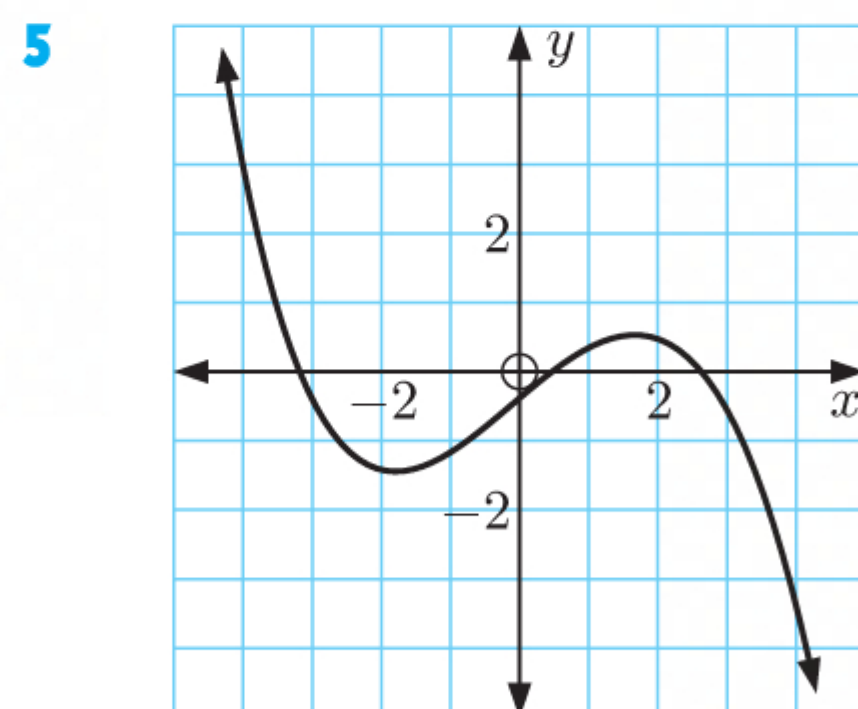
$$= \lim_{x \rightarrow 2} -(x - 1) \quad \{\text{as } x \neq 2\}$$

$$= -1$$



b i $\lim_{x \rightarrow -\infty} (e^{x-2} - 3) = -3$

ii $\lim_{x \rightarrow \infty} (e^{x-2} - 3)$ does not exist



- a** $f(-1)$ is below the x -axis, so $f(-1)$ is negative.

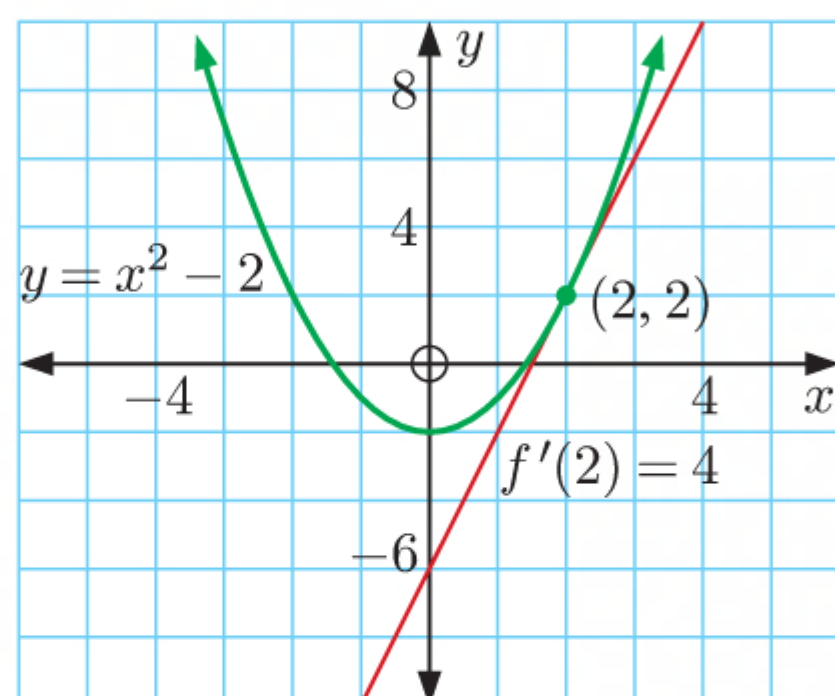
- b** $f'(0)$ is the gradient of the tangent to $f(x)$ at the point where $x = 0$. Since the curve is increasing at $x = 0$, $f'(0)$ is positive.

- c** $f(2)$ is above the x -axis, so $f(2)$ is positive.

- d** $f'(3)$ is the gradient of the tangent to $f(x)$ at the point where $x = 3$. Since the curve is decreasing at $x = 3$, $f'(3)$ is negative.

6 a, b

x	-3	-2	-1	0	1	2	3
$f(x) = x^2 - 2$	7	2	-1	-2	-1	2	7



- c** The tangent to $f(x) = x^2 - 2$ at the point where $x = 2$ has gradient $\frac{6 - (-2)}{3 - 1} = \frac{8}{2} = 4$.
 \therefore the instantaneous rate of change in $f(x) = x^2 - 2$ when $x = 2$ is 4.

d

$$f'(a) = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$$

$$\therefore f'(2) = \lim_{h \rightarrow 0} \frac{f(2+h) - f(2)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{[(2+h)^2 - 2] - [2^2 - 2]}{h}$$

$$= \lim_{h \rightarrow 0} \frac{4 + 4h + h^2 - 2 - 2}{h}$$

$$= \lim_{h \rightarrow 0} \frac{4h + h^2}{h}$$

$$= \lim_{h \rightarrow 0} \frac{\cancel{h}(4+h)}{\cancel{h}}$$

$$= \lim_{h \rightarrow 0} (4+h) \quad \{\text{as } h \neq 0\}$$

$$= 4 \quad \checkmark$$

7 a $f(x) = x^4 - 2x$

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{[(x+h)^4 - 2(x+h)] - [x^4 - 2x]}{h}$$

$$= \lim_{h \rightarrow 0} \frac{\cancel{x^4} + 4x^3h + 6x^2h^2 + 4xh^3 + h^4 - \cancel{2x} - 2h - \cancel{x^4} + \cancel{2x}}{h}$$

$$= \lim_{h \rightarrow 0} \frac{4x^3h + 6x^2h^2 + 4xh^3 + h^4 - 2h}{h}$$

$$= \lim_{h \rightarrow 0} \frac{\cancel{h}(4x^3 + 6x^2h + 4xh^2 + h^3 - 2)}{\cancel{h}}$$

$$= \lim_{h \rightarrow 0} (4x^3 + 6x^2h + 4xh^2 + h^3 - 2) \quad \{\text{as } h \neq 0\}$$

$$= 4x^3 - 2$$

$$\begin{aligned} \text{b } f'(-2) &= 4(-2)^3 - 2 \\ &= -34 \end{aligned}$$

The gradient of the tangent to $y = f(x)$ at the point where $x = -2$ is -34 .

$$8 \quad \text{a } y = f(x) = x^2 + 5x - 2$$

$$\begin{aligned} \frac{dy}{dx} &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{[(x+h)^2 + 5(x+h) - 2] - [x^2 + 5x - 2]}{h} \\ &= \lim_{h \rightarrow 0} \frac{\cancel{x^2} + 2xh + h^2 + \cancel{5x} + 5h - \cancel{2} - \cancel{x^2} - \cancel{5x} + \cancel{2}}{h} \\ &= \lim_{h \rightarrow 0} \frac{2xh + h^2 + 5h}{h} \\ &= \lim_{h \rightarrow 0} \frac{\cancel{h}(2x + h + 5)}{\cancel{h}} \\ &= \lim_{h \rightarrow 0} (2x + h + 5) \quad \{\text{as } h \neq 0\} \\ &= 2x + 5 \end{aligned}$$

$$\text{b } \text{The tangent to } f(x) = x^2 + 5x - 2 \text{ has gradient } -3 \text{ when } f'(x) = \frac{dy}{dx} = -3$$

$$\therefore 2x + 5 = -3$$

$$\therefore 2x = -8$$

$$\therefore x = -4$$

$$\begin{aligned} \text{Now, } f(-4) &= (-4)^2 + 5(-4) - 2 \\ &= -6 \end{aligned}$$

So, the tangent has gradient -3 at the point $(-4, -6)$.

$$9 \quad f(t) = 452 - 4.8t^2 \text{ metres, } 0 \leq t \leq 3 \text{ seconds}$$

$$\begin{aligned} \text{a } \quad \text{i } f(1) &= 452 - 4.8(1)^2 \\ &= 447.2 \end{aligned}$$

The jumper is 447.2 m above ground level after 1 second.

$$\begin{aligned} \text{ii } f(2) &= 452 - 4.8(2)^2 \\ &= 432.8 \end{aligned}$$

The jumper is 432.8 m above ground level after 2 seconds.

$$\begin{aligned} \text{b } f'(t) &= \lim_{h \rightarrow 0} \frac{f(t+h) - f(t)}{h} \\ &= \lim_{h \rightarrow 0} \frac{[452 - 4.8(t+h)^2] - [452 - 4.8t^2]}{h} \\ &= \lim_{h \rightarrow 0} \frac{\cancel{452} - \cancel{4.8t^2} - 9.6th - 4.8h^2 - \cancel{452} + \cancel{4.8t^2}}{h} \\ &= \lim_{h \rightarrow 0} \frac{-9.6th - 4.8h^2}{h} \\ &= \lim_{h \rightarrow 0} \frac{\cancel{h}(-9.6t - 4.8h)}{\cancel{h}} \\ &= \lim_{h \rightarrow 0} (-9.6t - 4.8h) \quad \{\text{as } h \neq 0\} \\ &= -9.6t \end{aligned}$$

- The speed of the jumper is equal to the rate of change in the jumper's altitude which is given by $f'(t)$.

i $f'(1) = -9.6(1)$
 $= -9.6$

The jumper's speed was 9.6 m s^{-1} after 1 second.

(The negative sign indicates the jumper is moving downwards.)

ii $f'(2) = -9.6(2)$
 $= -19.2$

The jumper's speed was 19.2 m s^{-1} after 2 seconds.

(The negative sign indicates the jumper is moving downwards.)

Chapter 12

RULES OF DIFFERENTIATION

INVESTIGATION 1

SIMPLE RULES OF DIFFERENTIATION

1 $f(x) = x^n$

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{(x+h)^n - x^n}{h} \\ &= \lim_{h \rightarrow 0} \frac{\cancel{\binom{n}{0} x^n} + \binom{n}{1} x^{n-1} h + \binom{n}{2} x^{n-2} h^2 + \dots + \binom{n}{n} h^n - \cancel{x^n}}{h} \\ &= \lim_{h \rightarrow 0} \frac{\cancel{h} \left[\binom{n}{1} x^{n-1} + \binom{n}{2} x^{n-2} h + \dots + \binom{n}{n} h^{n-1} \right]}{\cancel{h}} \\ &= \lim_{h \rightarrow 0} \left[\binom{n}{1} x^{n-1} + \binom{n}{2} x^{n-2} h + \dots + \binom{n}{n} h^{n-1} \right] \quad \{\text{as } h \neq 0\} \\ &= \binom{n}{1} x^{n-1} \\ &= nx^{n-1} \end{aligned}$$

2 a i $f(x) = 4x^2$

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{4(x+h)^2 - 4x^2}{h} \\ &= \lim_{h \rightarrow 0} \frac{4(x^2 + 2xh + h^2) - 4x^2}{h} \\ &= \lim_{h \rightarrow 0} \frac{\cancel{4x^2} + 8xh + 4h^2 - \cancel{4x^2}}{h} \\ &= \lim_{h \rightarrow 0} \frac{\cancel{h}(8x + 4h)}{\cancel{h}} \\ &= \lim_{h \rightarrow 0} (8x + 4h) \quad \{\text{as } h \neq 0\} \\ &= 8x \end{aligned}$$

ii $f(x) = 2x^3$

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{2(x+h)^3 - 2x^3}{h} \\ &= \lim_{h \rightarrow 0} \frac{2(x^3 + 3x^2h + 3xh^2 + h^3) - 2x^3}{h} \\ &= \lim_{h \rightarrow 0} \frac{\cancel{2x^3} + 6x^2h + 6xh^2 + 2h^3 - \cancel{2x^3}}{h} \\ &= \lim_{h \rightarrow 0} \frac{\cancel{h}(6x^2 + 6xh + 2h^2)}{\cancel{h}} \\ &= \lim_{h \rightarrow 0} (6x^2 + 6xh + 2h^2) \quad \{\text{as } h \neq 0\} \\ &= 6x^2 \end{aligned}$$

iii $f(x) = 7x^4$

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{7(x+h)^4 - 7x^4}{h} \\ &= \lim_{h \rightarrow 0} \frac{7(x^4 + 4x^3h + 6x^2h^2 + 4xh^3 + h^4) - 7x^4}{h} \\ &= \lim_{h \rightarrow 0} \frac{\cancel{7x^4} + 28x^3h + 42x^2h^2 + 28xh^3 + 7h^4 - \cancel{7x^4}}{h} \\ &= \lim_{h \rightarrow 0} \frac{\cancel{h}(28x^3 + 42x^2h + 28xh^2 + 7h^3)}{\cancel{h}} \\ &= \lim_{h \rightarrow 0} (28x^3 + 42x^2h + 28xh^2 + 7h^3) \quad \{\text{as } h \neq 0\} \\ &= 28x^3 \end{aligned}$$

b If $f(x) = cx^n$, then $f'(x) = cnx^{n-1}$.

3 a i $f(x) = x^2 + 3x$

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{(x+h)^2 + 3(x+h) - (x^2 + 3x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{\cancel{x^2} + 2xh + h^2 + \cancel{3x} + 3h - \cancel{x^2} - \cancel{3x}}{h} \\ &= \lim_{h \rightarrow 0} \frac{\cancel{h}(2x + h + 3)}{\cancel{h}} \\ &= \lim_{h \rightarrow 0} (2x + h + 3) \quad \{\text{as } h \neq 0\} \\ &= 2x + 3 \end{aligned}$$

ii $f(x) = x^3 - 2x^2$

$$\begin{aligned}
 f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{(x+h)^3 - 2(x+h)^2 - (x^3 - 2x^2)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 - 2(x^2 + 2xh + h^2) - x^3 + 2x^2}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\cancel{x^3} + 3x^2h + 3xh^2 + h^3 - \cancel{2x^2} - 4xh - 2h^2 - \cancel{x^3} + \cancel{2x^2}}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\cancel{h}(3x^2 + 3xh + h^2 - 4x - 2h)}{\cancel{h}} \quad \{\text{as } h \neq 0\} \\
 &= \lim_{h \rightarrow 0} (3x^2 + 3xh + h^2 - 4x - 2h) \\
 &= 3x^2 - 4x
 \end{aligned}$$

b If $f(x) = u(x) + v(x)$, then $f'(x) = u'(x) + v'(x)$.

EXERCISE 12A

1 a $f(x) = x^3$
 $\therefore f'(x) = 3x^2$

d $f(x) = 6x$
 $\therefore f'(x) = 6(1)$
 $= 6$

g $f(x) = 3x^5$
 $\therefore f'(x) = 3(5x^4)$
 $= 15x^4$

j $f(x) = x^2 - 3$
 $\therefore f'(x) = 2x$

m $f(x) = 2x^2 + x - 1$
 $\therefore f'(x) = 2(2x) + 1$
 $= 4x + 1$

o $f(x) = 4 - 2x^2$
 $\therefore f'(x) = -2(2x)$
 $= -4x$

q $f(x) = x^3 - 4x^2 + 6x$
 $\therefore f'(x) = 3x^2 - 4(2x) + 6(1)$
 $= 3x^2 - 8x + 6$

s $f(x) = 7 - x - 4x^3$
 $\therefore f'(x) = -1 - 4(3x^2)$
 $= -1 - 12x^2$

b $f(x) = x^8$
 $\therefore f'(x) = 8x^7$

e $f(x) = 2x^3$
 $\therefore f'(x) = 2(3x^2)$
 $= 6x^2$

h $f(x) = 5x^6$
 $\therefore f'(x) = 5(6x^5)$
 $= 30x^5$

k $f(x) = x^2 + x$
 $\therefore f'(x) = 2x + 1$

n $f(x) = 3x^2 - 7x + 8$
 $\therefore f'(x) = 3(2x) - 7(1)$
 $= 6x - 7$

p $f(x) = \frac{1}{2}x^4 - 6x^2$
 $\therefore f'(x) = \frac{1}{2}(4x^3) - 6(2x)$
 $= 2x^3 - 12x$

r $f(x) = 2x^3 + x - 1$
 $\therefore f'(x) = 2(3x^2) + (1)$
 $= 6x^2 + 1$

t $f(x) = \frac{1}{5}x^3 - \frac{7}{2}x^2 - 2$
 $\therefore f'(x) = \frac{1}{5}(3x^2) - \frac{7}{2}(2x)$
 $= \frac{3}{5}x^2 - 7x$

c $f(x) = x^{11}$
 $\therefore f'(x) = 11x^{10}$

f $f(x) = 7x^2$
 $\therefore f'(x) = 7(2x)$
 $= 14x$

i $f(x) = 5x - 2$
 $\therefore f'(x) = 5(1)$
 $= 5$

l $f(x) = x^2 + 3x - 5$
 $\therefore f'(x) = 2x + 3(1)$
 $= 2x + 3$

$$\mathbf{2} \quad \mathbf{a} \quad \text{Let } f(x) = \frac{1}{x^2}$$

$$= x^{-2}$$

$$\therefore f'(x) = -2x^{-3}$$

$$= -\frac{2}{x^3}$$

$$\mathbf{b} \quad \text{Let } f(x) = \frac{1}{x^5}$$

$$= x^{-5}$$

$$\therefore f'(x) = -5x^{-6}$$

$$= -\frac{5}{x^6}$$

$$\mathbf{c} \quad \text{Let } f(x) = \frac{3}{x}$$

$$= 3x^{-1}$$

$$\therefore f'(x) = 3(-1x^{-2})$$

$$= -3x^{-2}$$

$$= -\frac{3}{x^2}$$

$$\mathbf{d} \quad \text{Let } f(x) = \frac{4}{x^3}$$

$$= 4x^{-3}$$

$$\therefore f'(x) = 4(-3x^{-4})$$

$$= -12x^{-4}$$

$$= -\frac{12}{x^4}$$

$$\mathbf{e} \quad \text{Let } f(x) = -\frac{7}{x^4}$$

$$= -7x^{-4}$$

$$\therefore f'(x) = -7(-4x^{-5})$$

$$= 28x^{-5}$$

$$= \frac{28}{x^5}$$

$$\mathbf{f} \quad \text{Let } f(x) = 2x + \frac{3}{x^2}$$

$$= 2x + 3x^{-2}$$

$$\therefore f'(x) = 2(1) + 3(-2x^{-3})$$

$$= 2 - 6x^{-3}$$

$$= 2 - \frac{6}{x^3}$$

$$\mathbf{g} \quad \text{Let } f(x) = x^2 - \frac{6}{x}$$

$$= x^2 - 6x^{-1}$$

$$\therefore f'(x) = 2x - 6(-1x^{-2})$$

$$= 2x + 6x^{-2}$$

$$= 2x + \frac{6}{x^2}$$

$$\mathbf{h} \quad \text{Let } f(x) = 9 - \frac{2}{x^3}$$

$$= 9 - 2x^{-3}$$

$$\therefore f'(x) = -2(-3x^{-4})$$

$$= 6x^{-4}$$

$$= \frac{6}{x^4}$$

$$\mathbf{i} \quad \text{Let } f(x) = \frac{2}{x^2} + \frac{9}{x^4}$$

$$= 2x^{-2} + 9x^{-4}$$

$$\therefore f'(x) = 2(-2x^{-3}) + 9(-4x^{-5})$$

$$= -4x^{-3} - 36x^{-5}$$

$$= -\frac{4}{x^3} - \frac{36}{x^5}$$

$$\mathbf{j} \quad \text{Let } f(x) = 3x - \frac{1}{x} + \frac{2}{x^2}$$

$$= 3x - x^{-1} + 2x^{-2}$$

$$\therefore f'(x) = 3(1) - (-1x^{-2}) + 2(-2x^{-3})$$

$$= 3 + x^{-2} - 4x^{-3}$$

$$= 3 + \frac{1}{x^2} - \frac{4}{x^3}$$

$$\mathbf{k} \quad \text{Let } f(x) = 5 - \frac{8}{x^2} + \frac{4}{x^3}$$

$$= 5 - 8x^{-2} + 4x^{-3}$$

$$\therefore f'(x) = -8(-2x^{-3}) + 4(-3x^{-4})$$

$$= 16x^{-3} - 12x^{-4}$$

$$= \frac{16}{x^3} - \frac{12}{x^4}$$

$$\mathbf{l} \quad \text{Let } f(x) = \frac{1}{5x^2}$$

$$= \frac{1}{5}x^{-2}$$

$$\therefore f'(x) = \frac{1}{5}(-2x^{-3})$$

$$= -\frac{2}{5}x^{-3}$$

$$= -\frac{2}{5x^3}$$

$$\mathbf{m} \quad \text{Let } f(x) = 4x - \frac{1}{4x}$$

$$= 4x - \frac{1}{4}x^{-1}$$

$$\therefore f'(x) = 4(1) - \frac{1}{4}(-1x^{-2})$$

$$= 4 + \frac{1}{4}x^{-2}$$

$$= 4 + \frac{1}{4x^2}$$

$$\begin{aligned}
 \text{n Let } f(x) &= \frac{x^3 + 4}{x} \\
 &= \frac{x^3}{x} + \frac{4}{x} \\
 &= x^2 + 4x^{-1} \\
 \therefore f'(x) &= 2x + 4(-1x^{-2}) \\
 &= 2x - 4x^{-2} \\
 &= 2x - \frac{4}{x^2}
 \end{aligned}$$

$$\begin{aligned}
 \bullet \text{ Let } f(x) &= \frac{2x - 5}{x^2} \\
 &= \frac{2x}{x^2} - \frac{5}{x^2} \\
 &= \frac{2}{x} - \frac{5}{x^2} \\
 &= 2x^{-1} - 5x^{-2} \\
 \therefore f'(x) &= 2(-1x^{-2}) - 5(-2x^{-3}) \\
 &= -2x^{-2} + 10x^{-3} \\
 &= -\frac{2}{x^2} + \frac{10}{x^3}
 \end{aligned}$$

$$\begin{aligned}
 \text{3 a } f(x) &= \sqrt{x} \\
 &= x^{\frac{1}{2}} \\
 \therefore f'(x) &= \frac{1}{2}x^{-\frac{1}{2}} \\
 &= \frac{1}{2\sqrt{x}}
 \end{aligned}$$

$$\begin{aligned}
 \text{b } f(x) &= \sqrt[3]{x} \\
 &= x^{\frac{1}{3}} \\
 \therefore f'(x) &= \frac{1}{3}x^{-\frac{2}{3}} \\
 &= \frac{1}{3\sqrt[3]{x^2}}
 \end{aligned}$$

$$\begin{aligned}
 \text{c } f(x) &= \frac{1}{\sqrt{x}} \\
 &= x^{-\frac{1}{2}} \\
 \therefore f'(x) &= -\frac{1}{2}x^{-\frac{3}{2}} \\
 &= -\frac{1}{2x\sqrt{x}}
 \end{aligned}$$

$$\begin{aligned}
 \text{d } f(x) &= x^3 - \frac{1}{2}\sqrt{x} \\
 &= x^3 - \frac{1}{2}x^{\frac{1}{2}} \\
 \therefore f'(x) &= 3x^2 - \frac{1}{2}(\frac{1}{2}x^{-\frac{1}{2}}) \\
 &= 3x^2 - \frac{1}{4}x^{-\frac{1}{2}} \\
 &= 3x^2 - \frac{1}{4\sqrt{x}}
 \end{aligned}$$

$$\begin{aligned}
 \text{e } f(x) &= \frac{1}{x^2} + 6\sqrt{x} \\
 &= x^{-2} + 6x^{\frac{1}{2}} \\
 \therefore f'(x) &= -2x^{-3} + 6(\frac{1}{2}x^{-\frac{1}{2}}) \\
 &= -2x^{-3} + 3x^{-\frac{1}{2}} \\
 &= -\frac{2}{x^3} + \frac{3}{\sqrt{x}}
 \end{aligned}$$

$$\begin{aligned}
 \text{f } f(x) &= 2x - \sqrt{x} \\
 &= 2x - x^{\frac{1}{2}} \\
 \therefore f'(x) &= 2(1) - \frac{1}{2}x^{-\frac{1}{2}} \\
 &= 2 - \frac{1}{2\sqrt{x}}
 \end{aligned}$$

$$\begin{aligned}
 \text{g } f(x) &= x\sqrt{x} \\
 &= x^{\frac{3}{2}} \\
 \therefore f'(x) &= \frac{3}{2}x^{\frac{1}{2}} \\
 &= \frac{3}{2}\sqrt{x}
 \end{aligned}$$

$$\begin{aligned}
 \text{h } f(x) &= \frac{1}{x\sqrt{x}} \\
 &= x^{-\frac{3}{2}} \\
 \therefore f'(x) &= -\frac{3}{2}x^{-\frac{5}{2}} \\
 &= \frac{-3}{2x^2\sqrt{x}}
 \end{aligned}$$

$$\begin{aligned}
 \text{i} \quad f(x) &= 2x^2 - \frac{3}{\sqrt{x}} \\
 &= 2x^2 - 3x^{-\frac{1}{2}} \\
 \therefore f'(x) &= 2(2x) - 3\left(-\frac{1}{2}x^{-\frac{3}{2}}\right) \\
 &= 4x + \frac{3}{2}x^{-\frac{3}{2}} \\
 &= 4x + \frac{3}{2x\sqrt{x}}
 \end{aligned}$$

$$\begin{aligned}
 \text{k} \quad f(x) &= \frac{x+5}{\sqrt{x}} \\
 &= \frac{x}{\sqrt{x}} + \frac{5}{\sqrt{x}} \\
 &= \frac{x}{x^{\frac{1}{2}}} + \frac{5}{x^{\frac{1}{2}}} \\
 &= x^{\frac{1}{2}} + 5x^{-\frac{1}{2}} \\
 \therefore f'(x) &= \frac{1}{2}x^{-\frac{1}{2}} + 5\left(-\frac{1}{2}x^{-\frac{3}{2}}\right) \\
 &= \frac{1}{2\sqrt{x}} - \frac{5}{2x\sqrt{x}}
 \end{aligned}$$

$$\begin{aligned}
 \text{m} \quad f(x) &= 3x^2 - x\sqrt{x} \\
 &= 3x^2 - x^{\frac{3}{2}} \\
 \therefore f'(x) &= 3(2x) - \frac{3}{2}x^{\frac{1}{2}} \\
 &= 6x - \frac{3\sqrt{x}}{2}
 \end{aligned}$$

$$\begin{aligned}
 \text{j} \quad f(x) &= \frac{\sqrt{x}-4}{x} \\
 &= \frac{\sqrt{x}}{x} - \frac{4}{x} \\
 &= \frac{x^{\frac{1}{2}}}{x} - \frac{4}{x} \\
 &= x^{-\frac{1}{2}} - 4x^{-1} \\
 \therefore f'(x) &= -\frac{1}{2}x^{-\frac{3}{2}} - 4(-x^{-2}) \\
 &= -\frac{1}{2x\sqrt{x}} + \frac{4}{x^2}
 \end{aligned}$$

$$\begin{aligned}
 \text{l} \quad f(x) &= \frac{7-x^2}{\sqrt{x}} \\
 &= \frac{7}{\sqrt{x}} - \frac{x^2}{\sqrt{x}} \\
 &= 7x^{-\frac{1}{2}} - \frac{x^2}{x^{\frac{1}{2}}} \\
 &= 7x^{-\frac{1}{2}} - x^{\frac{3}{2}} \\
 \therefore f'(x) &= 7\left(-\frac{1}{2}x^{-\frac{3}{2}}\right) - \frac{3}{2}x^{\frac{1}{2}} \\
 &= -\frac{7}{2x\sqrt{x}} - \frac{3\sqrt{x}}{2}
 \end{aligned}$$

$$\begin{aligned}
 \text{n} \quad f(x) &= \frac{4}{x^2\sqrt{x}} \\
 &= \frac{4}{x^{\frac{5}{2}}} \\
 &= 4x^{-\frac{5}{2}} \\
 \therefore f'(x) &= 4\left(-\frac{5}{2}x^{-\frac{7}{2}}\right) \\
 &= -\frac{10}{x^3\sqrt{x}}
 \end{aligned}$$

$$\begin{aligned}
 \bullet \quad f(x) &= 2x - \frac{3}{x\sqrt{x}} \\
 &= 2x - \frac{3}{x^{\frac{3}{2}}} \\
 &= 2x - 3x^{-\frac{3}{2}} \\
 \therefore f'(x) &= 2(1) - 3(-\frac{3}{2}x^{-\frac{5}{2}}) \\
 &= 2 + \frac{9}{2}x^{-\frac{5}{2}} \\
 &= 2 + \frac{9}{2x^2\sqrt{x}}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{p} \quad f(x) &= \frac{x^2 - x + 2}{\sqrt[3]{x}} \\
 &= \frac{x^2}{\sqrt[3]{x}} - \frac{x}{\sqrt[3]{x}} + \frac{2}{\sqrt[3]{x}} \\
 &= \frac{x^2}{x^{\frac{1}{3}}} - \frac{x}{x^{\frac{1}{3}}} + \frac{2}{x^{\frac{1}{3}}} \\
 &= x^{\frac{5}{3}} - x^{\frac{2}{3}} + 2x^{-\frac{1}{3}} \\
 \therefore f'(x) &= \frac{5}{3}x^{\frac{2}{3}} - \frac{2}{3}x^{-\frac{1}{3}} + 2(-\frac{1}{3}x^{-\frac{4}{3}}) \\
 &= \frac{5}{3}x^{\frac{2}{3}} - \frac{2}{3}x^{-\frac{1}{3}} - \frac{2}{3}x^{-\frac{4}{3}} \\
 &= \frac{5\sqrt[3]{x^2}}{3} - \frac{2}{3\sqrt[3]{x}} - \frac{2}{3x\sqrt[3]{x}}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{4} \quad \mathbf{a} \quad y &= \pi x^2 \\
 \therefore \frac{dy}{dx} &= \pi(2x) \\
 &= 2\pi x
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad y &= 3x^2 - \frac{8}{x^2} \\
 &= 3x^2 - 8x^{-2} \\
 \therefore \frac{dy}{dx} &= 3(2x) - 8(-2x^{-3}) \\
 &= 6x + 16x^{-3} \\
 &= 6x + \frac{16}{x^3}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{c} \quad y &= 6\sqrt{x} + \frac{5}{x} \\
 &= 6x^{\frac{1}{2}} + 5x^{-1} \\
 \therefore \frac{dy}{dx} &= 6(\frac{1}{2}x^{-\frac{1}{2}}) + 5(-1x^{-2}) \\
 &= 3x^{-\frac{1}{2}} - 5x^{-2} \\
 &= \frac{3}{\sqrt{x}} - \frac{5}{x^2}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{d} \quad y &= 4\pi x^3 \\
 \therefore \frac{dy}{dx} &= 4\pi(3x^2) \\
 &= 12\pi x^2
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{e} \quad y &= 2.5x^3 - 1.4x^2 - 1.3 \\
 \therefore \frac{dy}{dx} &= 2.5(3x^2) - 1.4(2x) \\
 &= 7.5x^2 - 2.8x
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{f} \quad y &= 10(x + 1) \\
 &= 10x + 10 \\
 \therefore \frac{dy}{dx} &= 10(1) \\
 &= 10
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{g} \quad y &= (x + 1)(x - 2) \\
 &= x^2 - x - 2 \\
 \therefore \frac{dy}{dx} &= 2x - 1
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{h} \quad y &= (2x + 1)(3x - 2) \\
 &= 6x^2 - x - 2 \\
 \therefore \frac{dy}{dx} &= 12x - 1
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{i} \quad y &= (5 - x)^2 \\
 &= 25 - 10x + x^2 \\
 \therefore \frac{dy}{dx} &= -10(1) + 2x \\
 &= 2x - 10
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{j} \quad y &= (2x - 1)^2 \\
 &= 4x^2 - 4x + 1 \\
 \therefore \frac{dy}{dx} &= 4(2x) - 4(1) \\
 &= 8x - 4
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{k} \quad y &= x(x+1)(2x-5) \\
 &= x(2x^2 - 3x - 5) \\
 &= 2x^3 - 3x^2 - 5x \\
 \therefore \frac{dy}{dx} &= 2(3x^2) - 3(2x) - 5(1) \\
 &= 6x^2 - 6x - 5
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{l} \quad y &= \frac{(x-3)^2}{\sqrt{x}} \\
 &= \frac{x^2 - 6x + 9}{\sqrt{x}} \\
 &= \frac{x^2}{\sqrt{x}} - \frac{6x}{\sqrt{x}} + \frac{9}{\sqrt{x}} \\
 &= \frac{x^2}{x^{\frac{1}{2}}} - \frac{6x}{x^{\frac{1}{2}}} + \frac{9}{x^{\frac{1}{2}}} \\
 &= x^{\frac{3}{2}} - 6x^{\frac{1}{2}} + 9x^{-\frac{1}{2}} \\
 \therefore \frac{dy}{dx} &= \frac{3}{2}x^{\frac{1}{2}} - 6\left(\frac{1}{2}x^{-\frac{1}{2}}\right) + 9\left(-\frac{1}{2}x^{-\frac{3}{2}}\right) \\
 &= \frac{3}{2}x^{\frac{1}{2}} - 3x^{-\frac{1}{2}} - \frac{9}{2}x^{-\frac{3}{2}} \\
 &= \frac{3}{2}\sqrt{x} - \frac{3}{\sqrt{x}} - \frac{9}{2x\sqrt{x}}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{5} \quad \mathbf{a} \quad y &= \frac{1}{2}t^4 - \frac{1}{3}t \\
 \therefore \frac{dy}{dt} &= \frac{1}{2}(4t^3) - \frac{1}{3}(1) \\
 &= 2t^3 - \frac{1}{3}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad y &= 7 - \frac{6}{\sqrt{t}} \\
 &= 7 - 6t^{-\frac{1}{2}} \\
 \therefore \frac{dy}{dt} &= -6\left(-\frac{1}{2}t^{-\frac{3}{2}}\right) \\
 &= 3t^{-\frac{3}{2}} \\
 &= \frac{3}{t\sqrt{t}}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{c} \quad T &= \sqrt[3]{t} - \frac{2}{t^2} \\
 &= t^{\frac{1}{3}} - 2t^{-2} \\
 \therefore \frac{dT}{dt} &= \frac{1}{3}t^{-\frac{2}{3}} - 2(-2t^{-3}) \\
 &= \frac{1}{3}t^{-\frac{2}{3}} + 4t^{-3} \\
 &= \frac{1}{3\sqrt[3]{t^2}} + \frac{4}{t^3}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{d} \quad P &= \frac{5}{u} - 10u\sqrt{u} \\
 &= 5u^{-1} - 10u^{\frac{3}{2}} \\
 \therefore \frac{dP}{du} &= 5(-u^{-2}) - 10\left(\frac{3}{2}u^{\frac{1}{2}}\right) \\
 &= -5u^{-2} - 15u^{\frac{1}{2}} \\
 &= -\frac{5}{u^2} - 15\sqrt{u}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{6} \quad \mathbf{a} \quad y &= x^2 \\
 \therefore \frac{dy}{dx} &= 2x \\
 \text{When } x &= 2, \quad \frac{dy}{dx} = 2(2) = 4 \\
 \text{So, the tangent has gradient } &4.
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad y &= x^3 - 5x + 2 \\
 \therefore \frac{dy}{dx} &= 3x^2 - 5(1) \\
 &= 3x^2 - 5 \\
 \text{At the point } (3, 14), \\
 \frac{dy}{dx} &= 3(3)^2 - 5 = 22 \\
 \text{So, the tangent has gradient } &22.
 \end{aligned}$$

$$\begin{aligned}
 \text{c} \quad y &= \frac{8}{x^2} \\
 &= 8x^{-2} \\
 \therefore \frac{dy}{dx} &= 8(-2x^{-3}) \\
 &= -16x^{-3} \\
 &= -\frac{16}{x^3}
 \end{aligned}$$

At the point $(9, \frac{8}{81})$,

$$\frac{dy}{dx} = -\frac{16}{9^3} = -\frac{16}{729}.$$

So, the tangent has gradient $-\frac{16}{729}$.

$$\begin{aligned}
 \text{e} \quad y &= 3\sqrt{x} \\
 &= 3x^{\frac{1}{2}} \\
 \therefore \frac{dy}{dx} &= 3(\frac{1}{2}x^{-\frac{1}{2}}) \\
 &= \frac{3}{2\sqrt{x}}
 \end{aligned}$$

At the point $(1, 3)$,

$$\frac{dy}{dx} = \frac{3}{2\sqrt{1}} = \frac{3}{2}.$$

So, the tangent has gradient $\frac{3}{2}$.

$$\begin{aligned}
 \text{g} \quad y &= \frac{x^2 - 4}{x^2} \\
 &= \frac{x^2}{x^2} - \frac{4}{x^2} \\
 &= 1 - 4x^{-2} \\
 \therefore \frac{dy}{dx} &= -4(-2x^{-3}) \\
 &= \frac{8}{x^3}
 \end{aligned}$$

At the point $(4, \frac{3}{4})$,

$$\frac{dy}{dx} = \frac{8}{4^3} = \frac{8}{64} = \frac{1}{8}.$$

So, the tangent has gradient $\frac{1}{8}$.

$$\begin{aligned}
 \text{d} \quad y &= 2x^2 - 3x + 7 \\
 \therefore \frac{dy}{dx} &= 2(2x) - 3(1) \\
 &= 4x - 3 \\
 \text{When } x &= -1, \\
 \frac{dy}{dx} &= 4(-1) - 3 \\
 &= -7
 \end{aligned}$$

So, the tangent has gradient -7 .

$$\begin{aligned}
 \text{f} \quad y &= 2x - \frac{5}{x} \\
 &= 2x - 5x^{-1} \\
 \therefore \frac{dy}{dx} &= 2(1) - 5(-1x^{-2}) \\
 &= 2 + \frac{5}{x^2}
 \end{aligned}$$

At the point $(2, \frac{3}{2})$,

$$\begin{aligned}
 \frac{dy}{dx} &= 2 + \frac{5}{2^2} \\
 &= 2 + \frac{5}{4} \\
 &= \frac{13}{4}
 \end{aligned}$$

So, the tangent has gradient $\frac{13}{4}$.

$$\begin{aligned}
 \text{h} \quad y &= \frac{x^3 - 4x - 8}{x^2} \\
 &= \frac{x^3}{x^2} - \frac{4x}{x^2} - \frac{8}{x^2} \\
 &= x - 4x^{-1} - 8x^{-2} \\
 \therefore \frac{dy}{dx} &= 1 - 4(-1x^{-2}) - 8(-2x^{-3}) \\
 &= 1 + \frac{4}{x^2} + \frac{16}{x^3}
 \end{aligned}$$

When $x = -1$,

$$\begin{aligned}
 \frac{dy}{dx} &= 1 + \frac{4}{(-1)^2} + \frac{16}{(-1)^3} \\
 &= -11
 \end{aligned}$$

So, the tangent has gradient -11 .

7 a $f(x) = x^2 + (b+1)x + 2c$, $f(2) = 4$, and $f'(-1) = 2$
 $\therefore f'(x) = 2x + (b+1)$

But $f'(-1) = 2$, so $2(-1) + b + 1 = 2$
 $\therefore -1 + b = 2$
 $\therefore b = 3$

So, $f(x) = x^2 + (3+1)x + 2c$
 $= x^2 + 4x + 2c$

But $f(2) = 4$, so $2^2 + 4(2) + 2c = 4$
 $\therefore 2c = -8$
 $\therefore c = -4$

b $f(x) = bx + \frac{c}{x}$, $f(3) = 5$, and $f'(1) = 5$
 $= bx + cx^{-1}$
 $\therefore f'(x) = b + c(-x^{-2})$
 $= b - \frac{c}{x^2}$

But $f'(1) = 5$, so $b - \frac{c}{(1)^2} = 5$
 $\therefore b - c = 5$
 $\therefore b = c + 5 \quad \dots (*)$

and $f(3) = 5$, so $b(3) + \frac{c}{3} = 5$
 $\therefore 3b + \frac{c}{3} = 5$
 $\therefore 3(c+5) + \frac{c}{3} = 5 \quad \{\text{using } (*)\}$
 $\therefore 3c + 15 + \frac{c}{3} = 5$
 $\therefore \frac{10}{3}c = -10$
 $\therefore c = -3$

and so $b = -3 + 5$
 $= 2$

8 $y = 4x - \frac{3}{x}$
 $= 4x - 3x^{-1}$
 $\therefore \frac{dy}{dx} = 4 + 3x^{-2}$
 $= 4 + \frac{3}{x^2}$

$\frac{dy}{dx}$ is the gradient function of $y = 4x - \frac{3}{x}$ from which the gradient of the tangent at any point can be found. It is also the instantaneous rate of change of y with respect to x .

9 $f(x) = \sqrt{x} - \frac{4}{\sqrt{x}}$

a The domain of $f(x)$ is $\{x \mid x > 0\}$.

b $f(x) = \sqrt{x} - \frac{4}{\sqrt{x}}$

$$= x^{\frac{1}{2}} - 4x^{-\frac{1}{2}}$$

$$\therefore f'(x) = \frac{1}{2}x^{-\frac{1}{2}} + 2x^{-\frac{3}{2}}$$

$$= \frac{1}{2\sqrt{x}} + \frac{2}{x\sqrt{x}}$$

c The domain of $f'(x)$ is $\{x \mid x > 0\}$.

d $f'(1) = \frac{1}{2\sqrt{1}} + \frac{2}{1\sqrt{1}}$

$$= \frac{1}{2} + 2$$

$$= \frac{5}{2}$$

$$= 2.5$$

The gradient of the tangent to the curve $f(x) = \sqrt{x} - \frac{4}{\sqrt{x}}$ at $x = 1$ is 2.5.

10 a $S = 2t^2 + 4t$ m

$$\therefore \frac{dS}{dt} = 4t + 4 \text{ m s}^{-1}$$

$\frac{dS}{dt}$ is the instantaneous rate of change in position at time t . It is the velocity function of the car.

b When $t = 3$, $\frac{dS}{dt} = 4(3) + 4$
 $= 16$

The instantaneous rate of change in position at time $t = 3$ seconds is 16 m s^{-1} , or the velocity of the car at $t = 3$ seconds is 16 m s^{-1} .

11 $C = 1785 + 3x + 0.002x^2$ pounds

$$\therefore \frac{dC}{dx} = 3 + 0.002(2x)$$

$$= 3 + 0.004x \text{ pounds per toaster}$$

When $x = 1000$, $\frac{dC}{dx} = 3 + 0.004(1000)$
 $= 7$

When 1000 toasters are being produced each week, the cost of production is increasing by £7 per toaster.

ACTIVITY

THE INCREMENTS FORMULA

- 1 a Suppose $y = x^2$, and therefore $\frac{dy}{dx} = 2x$.

To estimate the value of 5.01^2 , we let $x = 5$ and $\delta x = 0.01$.

$$\begin{aligned}\text{Now } \delta y &\approx \frac{dy}{dx} \times \delta x \\ &\approx 2x \times \delta x \\ &\approx 2 \times 5 \times 0.01 \\ &\approx 0.1\end{aligned}$$

Since $5^2 = 25$, we estimate that $5.01^2 \approx 25 + 0.1 \approx 25.1$.

Using technology, $5.01^2 = 25.1001$.

- b Suppose $y = x^6$, and therefore $\frac{dy}{dx} = 6x^5$.

To estimate the value of 2.01^6 , we let $x = 2$ and $\delta x = 0.01$.

$$\begin{aligned}\text{Now } \delta y &\approx \frac{dy}{dx} \times \delta x \\ &\approx 6x^5 \times \delta x \\ &\approx 6 \times 2^5 \times 0.01 \\ &\approx 1.92\end{aligned}$$

Since $2^6 = 64$, we estimate that $2.01^6 \approx 64 + 1.92 \approx 65.92$.

Using technology, $2.01^6 \approx 65.9442$.

- c Suppose $y = x^3$, and therefore $\frac{dy}{dx} = 3x^2$.

To estimate the value of 2.98^3 , we let $x = 3$ and $\delta x = -0.02$.

$$\begin{aligned}\text{Now } \delta y &\approx \frac{dy}{dx} \times \delta x \\ &\approx 3x^2 \times \delta x \\ &\approx 3 \times 3^2 \times (-0.02) \\ &\approx -0.54\end{aligned}$$

Since $3^3 = 27$, we estimate that $2.98^3 \approx 27 - 0.54 \approx 26.46$.

Using technology, $2.98^3 \approx 26.4636$.

- d Suppose $y = x^4$, and therefore $\frac{dy}{dx} = 4x^3$.

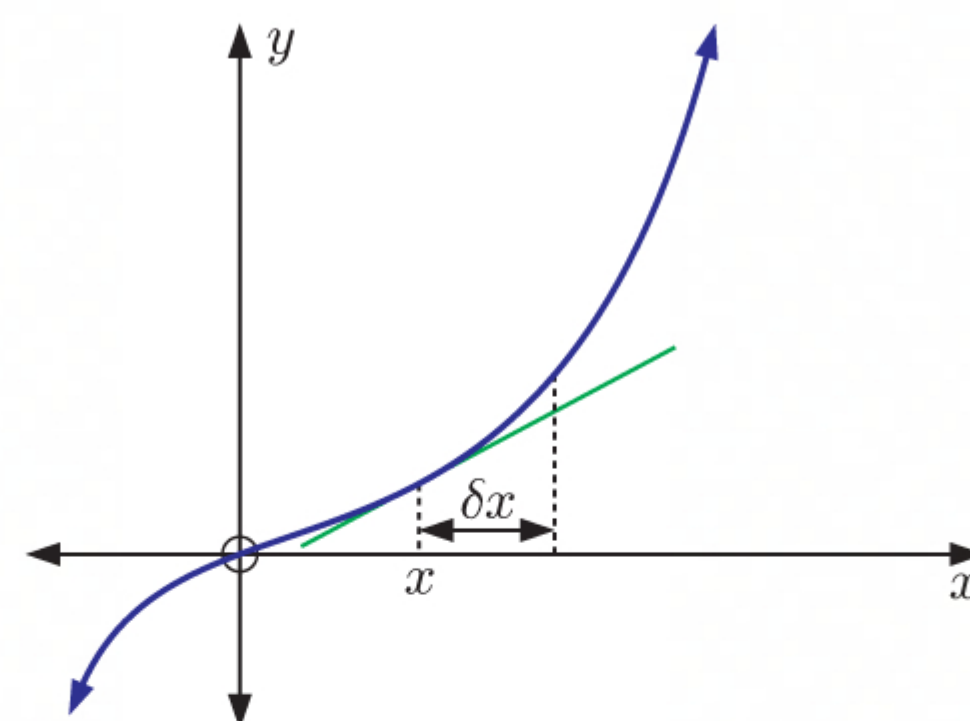
To estimate the value of 1.95^4 , we let $x = 2$ and $\delta x = -0.05$.

$$\begin{aligned}\text{Now } \delta y &\approx \frac{dy}{dx} \times \delta x \\ &\approx 4x^3 \times \delta x \\ &\approx 4 \times 2^3 \times (-0.05) \\ &\approx -1.6\end{aligned}$$

Since $2^4 = 16$, we estimate that $1.95^4 \approx 16 - 1.6 \approx 14.4$.

Using technology, $1.95^4 \approx 14.4590$.

- 2 a** We are estimating δy by multiplying the small increment δx by the gradient of the tangent at x .
- b** When δx is small, the value of the graph at $x + \delta x$ is close to the tangent drawn at x . Our estimate for δy will hence be more accurate.



EXERCISE 12B.1

1 a $g(x) = x^2$ and $f(x) = 2x + 7$

$$\begin{aligned}\therefore g(f(x)) &= g(2x + 7) \\ &= (2x + 7)^2\end{aligned}$$

c $g(x) = \sqrt{x}$ and $f(x) = 3 - 4x$

$$\begin{aligned}\therefore g(f(x)) &= g(3 - 4x) \\ &= \sqrt{3 - 4x}\end{aligned}$$

e $g(x) = \frac{2}{x}$ and $f(x) = x^2 + 3$

$$\begin{aligned}\therefore g(f(x)) &= g(x^2 + 3) \\ &= \frac{2}{x^2 + 3}\end{aligned}$$

b $g(x) = 2x + 7$ and $f(x) = x^2$

$$\begin{aligned}\therefore g(f(x)) &= g(x^2) \\ &= 2(x^2) + 7 \\ &= 2x^2 + 7\end{aligned}$$

d $g(x) = 3 - 4x$ and $f(x) = \sqrt{x}$

$$\begin{aligned}\therefore g(f(x)) &= g(\sqrt{x}) \\ &= 3 - 4\sqrt{x}\end{aligned}$$

f $g(x) = x^2 + 3$ and $f(x) = \frac{2}{x}$

$$\begin{aligned}\therefore g(f(x)) &= g\left(\frac{2}{x}\right) \\ &= \left(\frac{2}{x}\right)^2 + 3 \\ &= \frac{4}{x^2} + 3\end{aligned}$$

2 Note: There may be other answers.

a $g(f(x)) = (3x + 10)^3$

If we let $f(x) = 3x + 10$ then

$$g(f(x)) = (f(x))^3$$

$$\therefore g(x) = x^3 \text{ and } f(x) = 3x + 10$$

c $g(f(x)) = \frac{1}{2x + 4}$

If we let $f(x) = 2x + 4$ then

$$g(f(x)) = \frac{1}{f(x)}$$

$$\therefore g(x) = \frac{1}{x} \text{ and } f(x) = 2x + 4$$

b $g(f(x)) = (7 - 2x)^5$

If we let $f(x) = 7 - 2x$ then

$$g(f(x)) = (f(x))^5$$

$$\therefore g(x) = x^5 \text{ and } f(x) = 7 - 2x$$

d $g(f(x)) = \sqrt{x^2 - 3x}$

If we let $f(x) = x^2 - 3x$ then

$$g(f(x)) = \sqrt{f(x)}$$

$$\therefore g(x) = \sqrt{x} \text{ and } f(x) = x^2 - 3x$$

$$\text{e } g(f(x)) = \frac{1}{(5x-1)^4}$$

If we let $f(x) = 5x - 1$ then

$$g(f(x)) = \frac{1}{(f(x))^4}$$

$$\therefore g(x) = \frac{1}{x^4} \text{ and } f(x) = 5x - 1$$

$$\text{f } g(f(x)) = \frac{10}{(3x-x^2)^3}$$

If we let $f(x) = 3x - x^2$ then

$$g(f(x)) = \frac{10}{(f(x))^3}$$

$$\therefore g(x) = \frac{10}{x^3} \text{ and } f(x) = 3x - x^2$$

INVESTIGATION 2

DIFFERENTIATING COMPOSITE FUNCTIONS

$$\begin{aligned} \text{1 } y &= (2x+1)^2 \\ &= 4x^2 + 4x + 1 \end{aligned}$$

$$\begin{aligned} \therefore \frac{dy}{dx} &= 4(2x) + 4(1) \\ &= 8x + 4 \\ &= 2 \times 2(2x+1)^1 \end{aligned}$$

$$\begin{aligned} \text{2 } y &= (3x+1)^2 \\ &= 9x^2 + 6x + 1 \end{aligned}$$

$$\begin{aligned} \therefore \frac{dy}{dx} &= 9(2x) + 6(1) \\ &= 18x + 6 \\ &= 3 \times 2(3x+1)^1 \end{aligned}$$

$$\begin{aligned} \text{3 } y &= (ax+1)^2 \\ &= a^2x^2 + 2ax + 1 \end{aligned}$$

$$\begin{aligned} \therefore \frac{dy}{dx} &= a^2(2x) + 2a(1) \\ &= 2a^2x + 2a \\ &= a \times 2(ax+1)^1 \end{aligned}$$

$$\begin{aligned} \text{4 a } y &= u^2 \\ \therefore \frac{dy}{du} &= 2u \end{aligned}$$

$$\text{b } u = ax + 1, \quad y = (ax+1)^2$$

$$\text{i } \frac{du}{dx} = a$$

$$\begin{aligned} \text{ii } \frac{dy}{du} &= 2u \\ &= 2(ax+1) \\ &= 2ax + 2 \end{aligned}$$

$$\begin{aligned} \text{iii } \frac{dy}{du} \times \frac{du}{dx} &= (2ax+2) \times a \\ &= a \times 2(ax+1)^1 \end{aligned}$$

iv Our answer to **iii** is the same as the result in **3**.

$$\text{c } \text{If } y = u^2 \text{ where } u \text{ is a function of } x, \text{ we suspect that } \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}.$$

$$\begin{aligned} \text{5 } y &= (x^2+3x)^2 \\ &= (x^2)^2 + 2(x^2)(3x) + (3x)^2 \\ &= x^4 + 6x^3 + 9x^2 \\ \therefore \frac{dy}{dx} &= 4x^3 + 6(3x^2) + 9(2x) \\ &= 4x^3 + 18x^2 + 18x \\ &= 2(2x^3 + 9x^2 + 9x) \\ &= 2(x^2+3x)(2x+3) \quad \dots (*) \end{aligned}$$

Now, consider $y = u^2$ where $u = x^2 + 3x$. $\therefore \frac{du}{dx} = 2x + 3$

Comparing with (*), $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

Our answer agrees with the rule we suggested in **4 c**.

$$\begin{aligned} \text{6 a } y &= (2x+1)^3 \\ &= (2x)^3 + 3(2x)^2(1) + 3(2x)(1)^2 + 1^3 \\ &= 8x^3 + 12x^2 + 6x + 1 \\ \therefore \frac{dy}{dx} &= 24x^2 + 24x + 6 \end{aligned}$$

b $u = 2x + 1, \quad y = u^3$

i $\frac{du}{dx} = 2$

ii $\frac{dy}{du} = 3u^2$
 $= 3(2x + 1)^2$

iii $\frac{dy}{du} \times \frac{du}{dx} = 3(2x + 1)^2 \times 2$
 $= 3(4x^2 + 4x + 1) \times 2$
 $= (12x^2 + 12x + 3) \times 2$
 $= 24x^2 + 24x + 6$

iv Our answer to **iii** is the same as the result in **a**.

7 If y is a function of u , and u is a function of x , then $\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx}$.

EXERCISE 12B.2

1 a $\frac{1}{(2x - 1)^2} = u^{-2}$ where $u = 2x - 1$

c $\frac{2}{\sqrt{2 - x^2}} = 2u^{-\frac{1}{2}}$ where $u = 2 - x^2$

e $\frac{4}{(3 - x)^3} = 4u^{-3}$ where $u = 3 - x$

b $\sqrt{x^2 - 3x} = u^{\frac{1}{2}}$ where $u = x^2 - 3x$

d $\sqrt[3]{x^3 - x^2} = u^{\frac{1}{3}}$ where $u = x^3 - x^2$

f $\frac{10}{x^2 - 3} = 10u^{-1}$ where $u = x^2 - 3$

2 a $y = (2x + 3)^2$
 $\therefore y = u^2$ where $u = 2x + 3$

Now $\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx}$ {chain rule}
 $= 2u(2)$
 $= 4u$
 $= 4(2x + 3)$
 $= 8x + 12$

b $y = (2x + 3)^2$
 $= 4x^2 + 12x + 9$
 $\therefore \frac{dy}{dx} = 4(2x) + 12(1)$
 $= 8x + 12$

3 a $y = (4x - 5)^2$
 $\therefore y = u^2$ where $u = 4x - 5$

Now $\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx}$ {chain rule}
 $= 2u(4)$
 $= 8u$
 $= 8(4x - 5)$

b $y = \frac{1}{5 - 2x}$
 $\therefore y = u^{-1}$ where $u = 5 - 2x$

Now $\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx}$ {chain rule}
 $= -u^{-2}(-2)$
 $= 2u^{-2}$
 $= 2(5 - 2x)^{-2}$

c $y = \sqrt{3x - x^2}$
 $\therefore y = u^{\frac{1}{2}}$ where $u = 3x - x^2$

Now $\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx}$ {chain rule}
 $= \frac{1}{2}u^{-\frac{1}{2}}(3 - 2x)$
 $= \frac{1}{2}(3x - x^2)^{-\frac{1}{2}}(3 - 2x)$

d $y = (1 - 3x)^4$
 $\therefore y = u^4$ where $u = 1 - 3x$

Now $\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx}$ {chain rule}
 $= 4u^3(-3)$
 $= -12u^3$
 $= -12(1 - 3x)^3$

e $y = 6(5 - x)^3$
 $\therefore y = 6u^3$ where $u = 5 - x$
 Now $\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx}$ {chain rule}
 $= 6(3u^2)(-1)$
 $= -18u^2$
 $= -18(5 - x)^2$

g $y = \frac{6}{(5x - 4)^2}$
 $\therefore y = 6u^{-2}$ where $u = 5x - 4$
 Now $\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx}$ {chain rule}
 $= 6(-2u^{-3})(5)$
 $= -60u^{-3}$
 $= -60(5x - 4)^{-3}$

i $y = 2\left(x^2 - \frac{2}{x}\right)^3$
 $\therefore y = 2u^3$ where $u = x^2 - 2x^{-1}$
 Now $\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx}$ {chain rule}
 $= 2(3u^2)(2x - 2(-1x^{-2}))$
 $= 6u^2(2x + 2x^{-2})$
 $= 6(x^2 - 2x^{-1})^2(2x + 2x^{-2})$
 $= 6\left(x^2 - \frac{2}{x}\right)^2\left(2x + \frac{2}{x^2}\right)$

4 a $y = \sqrt{1 - x^2}$
 $\therefore y = u^{\frac{1}{2}}$ where $u = 1 - x^2$
 Now $\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx}$ {chain rule}
 $= \frac{1}{2}u^{-\frac{1}{2}}(-2x)$
 $= \frac{1}{2}(1 - x^2)^{-\frac{1}{2}}(-2x)$
 $= -x(1 - x^2)^{-\frac{1}{2}}$
 At $x = \frac{1}{2}$, $\frac{dy}{dx} = -\frac{1}{2}(1 - (\frac{1}{2})^2)^{-\frac{1}{2}}$
 $= -\frac{1}{2}(\frac{3}{4})^{-\frac{1}{2}}$
 $= -\frac{1}{\sqrt{3}}$

\therefore the gradient of the tangent to
 $y = \sqrt{1 - x^2}$ at $x = \frac{1}{2}$ is $-\frac{1}{\sqrt{3}}$.

f $y = \sqrt[3]{2x^3 - x^2}$
 $\therefore y = u^{\frac{1}{3}}$ where $u = 2x^3 - x^2$
 Now $\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx}$ {chain rule}
 $= \frac{1}{3}u^{-\frac{2}{3}}(2(3x^2) - 2x)$
 $= \frac{1}{3}(2x^3 - x^2)^{-\frac{2}{3}}(6x^2 - 2x)$

h $y = (x^2 - 5x + 8)^5$
 $\therefore y = u^5$ where $u = x^2 - 5x + 8$
 Now $\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx}$ {chain rule}
 $= 5u^4(2x - 5)$
 $= 5(x^2 - 5x + 8)^4(2x - 5)$

b $y = (3x + 2)^6$
 $\therefore y = u^6$ where $u = 3x + 2$
 Now $\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx}$ {chain rule}
 $= 6u^5(3)$
 $= 18u^5$
 $= 18(3x + 2)^5$
 At $x = -1$, $\frac{dy}{dx} = 18(3(-1) + 2)^5$
 $= 18(-1)^5$
 $= -18$
 \therefore the gradient of the tangent to
 $y = (3x + 2)^6$ at $x = -1$ is -18 .

$$\text{c} \quad y = \frac{1}{(2x-1)^4}$$

$$\therefore y = u^{-4} \quad \text{where } u = 2x - 1$$

$$\begin{aligned} \text{Now } \frac{dy}{dx} &= \frac{dy}{du} \frac{du}{dx} \quad \{\text{chain rule}\} \\ &= -4u^{-5}(2) \\ &= -8u^{-5} \\ &= -8(2x-1)^{-5} \end{aligned}$$

$$\text{At } x = 1, \quad \frac{dy}{dx} = -8(2(1)-1)^{-5}$$

$$= -8$$

\therefore the gradient of the tangent to

$$y = \frac{1}{(2x-1)^4} \quad \text{at } x = 1 \text{ is } -8.$$

$$\text{e} \quad y = \frac{4}{x+2\sqrt{x}}$$

$$\therefore y = 4u^{-1} \quad \text{where } u = x + 2x^{\frac{1}{2}}$$

$$\begin{aligned} \text{Now } \frac{dy}{dx} &= \frac{dy}{du} \frac{du}{dx} \quad \{\text{chain rule}\} \\ &= 4(-1u^{-2})(1 + 2(\frac{1}{2}x^{-\frac{1}{2}})) \\ &= -4u^{-2}(1 + x^{-\frac{1}{2}}) \\ &= -4\left(x + 2x^{\frac{1}{2}}\right)^{-2}(1 + x^{-\frac{1}{2}}) \end{aligned}$$

$$\begin{aligned} \text{At } x = 4, \quad \frac{dy}{dx} &= -4\left(4 + 2(4)^{\frac{1}{2}}\right)^{-2}(1 + 4^{-\frac{1}{2}}) \\ &= -4(8)^{-2}\left(\frac{3}{2}\right) \\ &= -\frac{3}{32} \end{aligned}$$

\therefore the gradient of the tangent to $y = \frac{4}{x+2\sqrt{x}}$ at $x = 4$ is $-\frac{3}{32}$.

$$\text{f} \quad y = \left(x + \frac{1}{x}\right)^3$$

$$\therefore y = u^3 \quad \text{where } u = x + x^{-1}$$

$$\begin{aligned} \text{Now } \frac{dy}{dx} &= \frac{dy}{du} \frac{du}{dx} \quad \{\text{chain rule}\} \\ &= 3u^2(1 - x^{-2}) \\ &= 3(x + x^{-1})^2(1 - x^{-2}) \end{aligned}$$

$$\begin{aligned} \text{At } x = 1, \quad \frac{dy}{dx} &= 3(1 + 1^{-1})^2(1 - 1^{-2}) \\ &= 3(4)(0) \\ &= 0 \end{aligned}$$

\therefore the gradient of the tangent to $y = \left(x + \frac{1}{x}\right)^3$ at $x = 1$ is 0.

$$\text{d} \quad y = 6 \times \sqrt[3]{1-2x}$$

$$\therefore y = 6u^{\frac{1}{3}} \quad \text{where } u = 1 - 2x$$

$$\begin{aligned} \text{Now } \frac{dy}{dx} &= \frac{dy}{du} \frac{du}{dx} \quad \{\text{chain rule}\} \\ &= 6\left(\frac{1}{3}u^{-\frac{2}{3}}\right)(-2) \\ &= -4u^{-\frac{2}{3}} \\ &= -4(1-2x)^{-\frac{2}{3}} \end{aligned}$$

$$\begin{aligned} \text{At } x = 0, \quad \frac{dy}{dx} &= -4(1-2(0))^{-\frac{2}{3}} \\ &= -4 \end{aligned}$$

\therefore the gradient of the tangent to

$$y = 6 \times \sqrt[3]{1-2x} \quad \text{at } x = 0 \text{ is } -4.$$

$$5 \quad y = f(x) = (2x - b)^a$$

$$\therefore y = u^a \quad \text{where} \quad u = 2x - b$$

$$\begin{aligned} \text{Now} \quad \frac{dy}{dx} &= \frac{dy}{du} \frac{du}{dx} \quad \{\text{chain rule}\} \\ &= au^{a-1}(2) \\ &= 2au^{a-1} \\ &= 2a(2x - b)^{a-1} \end{aligned}$$

$$\text{But} \quad f'(x) = 24x^2 - 24x + 6 \quad \{\text{given}\}$$

$$\therefore 2a(2x - b)^{a-1} = 24x^2 - 24x + 6$$

$$\begin{aligned} \therefore a(2x - b)^{a-1} &= 12x^2 - 12x + 3 \\ &= 3(4x^2 - 4x + 1) \\ &= 3(2x - 1)^2 \end{aligned}$$

Solving by inspection:

$$\begin{aligned} \therefore a = 3 \quad 2x - b &= 2x - 1 \quad a - 1 = 2 \\ \therefore b &= 1 \quad \therefore a = 3 \end{aligned}$$

So, $a = 3$ and $b = 1$.

$$6 \quad y = \frac{a}{\sqrt{1 + bx}}$$

$$\therefore y = au^{-\frac{1}{2}} \quad \text{where} \quad u = 1 + bx$$

$$\begin{aligned} \text{Now} \quad \frac{dy}{dx} &= \frac{dy}{du} \frac{du}{dx} \quad \{\text{chain rule}\} \\ &= a(-\frac{1}{2}u^{-\frac{3}{2}})(b) \\ &= -\frac{1}{2}abu^{-\frac{3}{2}} \\ &= -\frac{1}{2}ab(1 + bx)^{-\frac{3}{2}} \end{aligned}$$

$$\text{When } x = 3, y = 1, \text{ and } \frac{dy}{dx} = -\frac{1}{8}$$

$$\therefore 1 = \frac{a}{\sqrt{1 + b(3)}} \quad \text{and} \quad -\frac{1}{8} = -\frac{1}{2}ab(1 + b(3))^{-\frac{3}{2}}$$

$$\therefore a = \sqrt{1 + 3b} \quad \dots (*) \quad \therefore \frac{1}{4} = ab(1 + 3b)^{-\frac{3}{2}}$$

$$= \frac{\cancel{\sqrt{1 + 3b}}(b)}{\cancel{\sqrt{1 + 3b}}(1 + 3b)} \quad \{\text{using } (*)\}$$

$$\therefore \frac{1}{4} = \frac{b}{1 + 3b}$$

$$\therefore 1 + 3b = 4b$$

$$\therefore b = 1$$

$$\begin{aligned} \therefore a &= \sqrt{1 + 3(1)} \quad \{\text{substituting } b = 1 \text{ into } (*)\} \\ &= 2 \end{aligned}$$

So, $a = 2$ and $b = 1$.

$$7 \quad y = f(x) = 3\left(ax - \frac{b}{x}\right)^3$$

$$\therefore y = 3u^3 \quad \text{where} \quad u = ax - bx^{-1}$$

$$\begin{aligned} \text{Now} \quad \frac{dy}{dx} &= \frac{dy}{du} \frac{du}{dx} \quad \{\text{chain rule}\} \\ &= 3(3u^2)(a - b(-1x^{-2})) \\ &= 9u^2(a + bx^{-2}) \\ &= 9(ax - bx^{-1})^2(a + bx^{-2}) \end{aligned}$$

$$f\left(\frac{3}{2}\right) = 3 \quad \{\text{given}\}$$

$$\therefore 3\left(a\left(\frac{3}{2}\right) - \frac{b}{\frac{3}{2}}\right)^3 = 3$$

$$\therefore \left(\frac{3}{2}a - \frac{2}{3}b\right)^3 = 1$$

$$\therefore \frac{3}{2}a - \frac{2}{3}b = 1$$

$$\therefore \frac{3}{2}a = \frac{2}{3}b + 1$$

$$\therefore a = \frac{2}{3}\left(\frac{2}{3}b + 1\right)$$

$$\therefore a = \frac{4}{9}b + \frac{2}{3} \quad \dots (*)$$

$$\text{Also, } f'\left(\frac{3}{2}\right) = 30$$

$$\therefore \frac{dy}{dx} = 30 \quad \text{when } x = \frac{3}{2}$$

$$\therefore 9\left(a\left(\frac{3}{2}\right) - b\left(\frac{3}{2}\right)^{-1}\right)^2 \left(a + b\left(\frac{3}{2}\right)^{-2}\right) = 30$$

$$\therefore \left(\frac{3}{2}a - \frac{2}{3}b\right)^2 \left(a + \frac{4}{9}b\right) = \frac{10}{3}$$

$$\therefore \left(\frac{3}{2}\left(\frac{4}{9}b + \frac{2}{3}\right) - \frac{2}{3}b\right)^2 \left(\left(\frac{4}{9}b + \frac{2}{3}\right) + \frac{4}{9}b\right) = \frac{10}{3} \quad \{\text{using } (*)\}$$

$$\therefore \left(\frac{2}{3}b + 1 - \frac{2}{3}b\right)^2 \left(\frac{8}{9}b + \frac{2}{3}\right) = \frac{10}{3}$$

$$\therefore \frac{8}{9}b + \frac{2}{3} = \frac{10}{3}$$

$$\therefore \frac{8}{9}b = \frac{8}{3}$$

$$\therefore b = 3$$

$$\begin{aligned} \therefore a &= \frac{4}{9}(3) + \frac{2}{3} \quad \{\text{substituting } b = 3 \text{ into } (*)\} \\ &= 2 \end{aligned}$$

So, $a = 2$ and $b = 3$.

8 $y = x^3$ and $x = y^{\frac{1}{3}}$

a $\frac{dy}{dx} = 3x^2$ and $\frac{dx}{dy} = \frac{1}{3}y^{-\frac{2}{3}}$

$$\begin{aligned}\frac{dy}{dx} \times \frac{dx}{dy} &= 3x^2 \left(\frac{1}{3}y^{-\frac{2}{3}} \right) \\ &= 3x^2 \left(\frac{1}{3}(x^3)^{-\frac{2}{3}} \right) \\ &= 3x^2 \left(\frac{1}{3}x^{-2} \right) \\ &= 3x^2 \times \frac{1}{3x^2} \\ &= 1 \quad \text{as required}\end{aligned}$$

b We know that

$$\frac{dy}{du} \frac{du}{dx} = \frac{dy}{dx} \quad \{\text{chain rule}\}$$

Letting $x = y$, $\frac{dy}{du} \frac{du}{dy} = \frac{dy}{dy}$

$$\therefore \frac{dy}{du} \frac{du}{dy} = 1$$

Letting $u = x$, $\frac{dy}{dx} \frac{dx}{dy} = 1$

INVESTIGATION 3

THE PRODUCT RULE

1 $u(x) = x$, $v(x) = x$, $f(x) = u(x)v(x) = x^2$

a $f'(x) = 2x$

b $u'(x) = 1$, $v'(x) = 1$

c $f'(x) \neq u'(x)v'(x)$

2 $u(x) = x$, $v(x) = \sqrt{x} = x^{\frac{1}{2}}$, $f(x) = x\sqrt{x} = x^{\frac{3}{2}}$

a $f'(x) = \frac{3}{2}x^{\frac{1}{2}}$
 $= \frac{3}{2\sqrt{x}}$

b $u'(x) = 1$, $v'(x) = \frac{1}{2}x^{-\frac{1}{2}}$
 $= \frac{1}{2\sqrt{x}}$

c $f'(x) \neq u'(x)v'(x)$

3

$f(x)$	$f'(x)$	$u(x)$	$v(x)$	$u'(x)$	$v'(x)$	$u'(x)v(x) + u(x)v'(x)$
x^2	$2x$	x	x	1	1	$2x$
$x^{\frac{3}{2}}$	$\frac{3}{2}\sqrt{x}$	x	\sqrt{x}	1	$\frac{1}{2\sqrt{x}}$	$\frac{3}{2}\sqrt{x}$
$x(x+1)$	$2x+1$	x	$x+1$	1	1	$2x+1$
$(x-1)(2-x^2)$	$-3x^2+2x+2$	$x-1$	$2-x^2$	1	$-2x$	$-3x^2+2x+2$

4 If $f(x) = u(x)v(x)$ then $f'(x) = u'(x)v(x) + u(x)v'(x)$.

EXERCISE 12C

1 a $f(x) = x(x-1)$ is the product of $u(x) = x$ and $v(x) = x-1$
 $\therefore u'(x) = 1$ and $v'(x) = 1$

Now $f'(x) = u'(x)v(x) + u(x)v'(x)$ {product rule}
 $= 1(x-1) + x(1)$
 $= x-1+x$
 $= 2x-1$

b $f(x) = 2x(x+1)$ is the product of $u(x) = 2x$ and $v(x) = x+1$
 $\therefore u'(x) = 2$ and $v'(x) = 1$

Now $f'(x) = u'(x)v(x) + u(x)v'(x)$ {product rule}
 $= 2(x+1) + 2x(1)$
 $= 2x + 2 + 2x$
 $= 4x + 2$

c $f(x) = x^2\sqrt{x+1}$ is the product of $u(x) = x^2$ and $v(x) = (x+1)^{\frac{1}{2}}$
 $\therefore u'(x) = 2x$ and $v'(x) = \frac{1}{2}(x+1)^{-\frac{1}{2}}(1)$ {chain rule}
 $= \frac{1}{2}(x+1)^{-\frac{1}{2}}$

Now $f'(x) = u'(x)v(x) + u(x)v'(x)$ {product rule}
 $= 2x(x+1)^{\frac{1}{2}} + x^2(\frac{1}{2}(x+1)^{-\frac{1}{2}})$
 $= 2x(x+1)^{\frac{1}{2}} + \frac{1}{2}x^2(x+1)^{-\frac{1}{2}}$

d $f(x) = (x+3)(x-1)$ is the product of $u(x) = x+3$ and $v(x) = x-1$
 $\therefore u'(x) = 1$ and $v'(x) = 1$

Now $f'(x) = u'(x)v(x) + u(x)v'(x)$ {product rule}
 $= 1(x-1) + (x+3)(1)$
 $= x-1 + x+3$
 $= 2x+2$

e $f(x) = x\sqrt{x^2-1}$ is the product of $u(x) = x$ and $v(x) = (x^2-1)^{\frac{1}{2}}$
 $\therefore u'(x) = 1$ and $v'(x) = \frac{1}{2}(x^2-1)^{-\frac{1}{2}}(2x)$ {chain rule}
 $= x(x^2-1)^{-\frac{1}{2}}$

Now $f'(x) = u'(x)v(x) + u(x)v'(x)$ {product rule}
 $= 1(x^2-1)^{\frac{1}{2}} + x(x(x^2-1)^{-\frac{1}{2}})$
 $= (x^2-1)^{\frac{1}{2}} + x^2(x^2-1)^{-\frac{1}{2}}$

f $f(x) = x(x+1)^2$ is the product of $u(x) = x$ and $v(x) = (x+1)^2$
 $\therefore u'(x) = 1$ and $v'(x) = 2(x+1)(1)$ {chain rule}
 $= 2x+2$

Now $f'(x) = u'(x)v(x) + u(x)v'(x)$ {product rule}
 $= 1(x+1)^2 + x(2x+2)$
 $= (x+1)^2 + 2x(x+1)$

2 a $y = x^2(2x-1)$ is the product of $u = x^2$ and $v = 2x-1$
 $\therefore u' = 2x$ and $v' = 2$

Now $\frac{dy}{dx} = u'v + uv'$ {product rule}
 $= 2x(2x-1) + x^2(2)$
 $= 2x(2x-1) + 2x^2$

$$\begin{aligned} \text{b } y = 4x(2x+1)^3 \text{ is the product of } u = 4x \text{ and } v = (2x+1)^3 \\ \therefore u' = 4 \text{ and } v' = 3(2x+1)^2(2) \quad \{\text{chain rule}\} \\ = 6(2x+1)^2 \end{aligned}$$

$$\begin{aligned} \text{Now } \frac{dy}{dx} &= u'v + uv' \quad \{\text{product rule}\} \\ &= 4(2x+1)^3 + 4x(6(2x+1)^2) \\ &= 4(2x+1)^3 + 24x(2x+1)^2 \end{aligned}$$

$$\begin{aligned} \text{c } y = x^2\sqrt{3-x} \text{ is the product of } u = x^2 \text{ and } v = (3-x)^{\frac{1}{2}} \\ \therefore u' = 2x \text{ and } v' = \frac{1}{2}(3-x)^{-\frac{1}{2}}(-1) \quad \{\text{chain rule}\} \\ = -\frac{1}{2}(3-x)^{-\frac{1}{2}} \end{aligned}$$

$$\begin{aligned} \text{Now } \frac{dy}{dx} &= u'v + uv' \quad \{\text{product rule}\} \\ &= 2x(3-x)^{\frac{1}{2}} + x^2(-\frac{1}{2}(3-x)^{-\frac{1}{2}}) \\ &= 2x(3-x)^{\frac{1}{2}} - \frac{1}{2}x^2(3-x)^{-\frac{1}{2}} \end{aligned}$$

$$\begin{aligned} \text{d } y = \sqrt{x}(x-3)^2 \text{ is the product of } u = x^{\frac{1}{2}} \text{ and } v = (x-3)^2 \\ \therefore u' = \frac{1}{2}x^{-\frac{1}{2}} \text{ and } v' = 2(x-3)(1) \quad \{\text{chain rule}\} \\ = 2(x-3) \end{aligned}$$

$$\begin{aligned} \text{Now } \frac{dy}{dx} &= u'v + uv' \quad \{\text{product rule}\} \\ &= \frac{1}{2}x^{-\frac{1}{2}}(x-3)^2 + x^{\frac{1}{2}}(2(x-3)) \\ &= \frac{1}{2}x^{-\frac{1}{2}}(x-3)^2 + 2\sqrt{x}(x-3) \end{aligned}$$

$$\begin{aligned} \text{e } y = 5x^2(3x^2-1)^2 \text{ is the product of } u = 5x^2 \text{ and } v = (3x^2-1)^2 \\ \therefore u' = 5(2x) \text{ and } v' = 2(3x^2-1)(3(2x)) \quad \{\text{chain rule}\} \\ = 10x \quad = 12x(3x^2-1) \end{aligned}$$

$$\begin{aligned} \text{Now } \frac{dy}{dx} &= u'v + uv' \quad \{\text{product rule}\} \\ &= 10x(3x^2-1)^2 + 5x^2(12x(3x^2-1)) \\ &= 10x(3x^2-1)^2 + 60x^3(3x^2-1) \end{aligned}$$

$$\begin{aligned} \text{f } y = \sqrt{x}(x-x^2)^3 \text{ is the product of } u = x^{\frac{1}{2}} \text{ and } v = (x-x^2)^3 \\ \therefore u' = \frac{1}{2}x^{-\frac{1}{2}} \text{ and } v' = 3(x-x^2)^2(1-2x) \quad \{\text{chain rule}\} \end{aligned}$$

$$\begin{aligned} \text{Now } \frac{dy}{dx} &= u'v + uv' \quad \{\text{product rule}\} \\ &= \frac{1}{2}x^{-\frac{1}{2}}(x-x^2)^3 + x^{\frac{1}{2}}(3(x-x^2)^2(1-2x)) \\ &= \frac{1}{2}x^{-\frac{1}{2}}(x-x^2)^3 + 3\sqrt{x}(x-x^2)^2(1-2x) \end{aligned}$$

$$\begin{aligned}
 \text{3 a } y = x^4(1-2x)^2 \text{ is the product of } u = x^4 \text{ and } v = (1-2x)^2 \\
 \therefore u' = 4x^3 \text{ and } v' = 2(1-2x)(-2) \quad \{\text{chain rule}\} \\
 = -4(1-2x)
 \end{aligned}$$

$$\begin{aligned}
 \text{Now } \frac{dy}{dx} &= u'v + uv' \quad \{\text{product rule}\} \\
 &= 4x^3(1-2x)^2 + x^4(-4(1-2x)) \\
 &= 4x^3(1-2x)^2 - 4x^4(1-2x)
 \end{aligned}$$

$$\begin{aligned}
 \text{At } x = -1, \quad \frac{dy}{dx} &= 4(-1)^3(1-2(-1))^2 - 4(-1)^4(1-2(-1)) \\
 &= -4(9) - 4(3) \\
 &= -48
 \end{aligned}$$

\therefore the gradient of the tangent to $y = x^4(1-2x)^2$ at $x = -1$ is -48 .

$$\text{b } y = \sqrt{x}(x^2 - x + 1)^2 \text{ is the product of}$$

$$u = x^{\frac{1}{2}} \quad \text{and} \quad v = (x^2 - x + 1)^2$$

$$\therefore u' = \frac{1}{2}x^{-\frac{1}{2}} \quad \text{and} \quad v' = 2(x^2 - x + 1)(2x - 1) \quad \{\text{chain rule}\}$$

$$\begin{aligned}
 \text{Now } \frac{dy}{dx} &= u'v + uv' \quad \{\text{product rule}\} \\
 &= \frac{1}{2}x^{-\frac{1}{2}}(x^2 - x + 1)^2 + x^{\frac{1}{2}}(2(x^2 - x + 1)(2x - 1))
 \end{aligned}$$

$$\begin{aligned}
 \text{At } x = 4, \quad \frac{dy}{dx} &= \frac{1}{2}(4)^{-\frac{1}{2}}(4^2 - 4 + 1)^2 + 4^{\frac{1}{2}}(2(4^2 - 4 + 1)(2(4) - 1)) \\
 &= \frac{1}{4}(169) + 2(2(13)(7)) \\
 &= \frac{169}{4} + 364 \\
 &= 406\frac{1}{4}
 \end{aligned}$$

\therefore the gradient of the tangent to $y = \sqrt{x}(x^2 - x + 1)^2$ at $x = 4$ is $406\frac{1}{4}$.

$$\begin{aligned}
 \text{c } y = x\sqrt{1-2x} \text{ is the product of } u = x \text{ and } v = (1-2x)^{\frac{1}{2}} \\
 \therefore u' = 1 \quad \text{and} \quad v' = \frac{1}{2}(1-2x)^{-\frac{1}{2}}(-2) \quad \{\text{chain rule}\} \\
 = -(1-2x)^{-\frac{1}{2}}
 \end{aligned}$$

$$\begin{aligned}
 \text{Now } \frac{dy}{dx} &= u'v + uv' \quad \{\text{product rule}\} \\
 &= 1(1-2x)^{\frac{1}{2}} + x(-(1-2x)^{-\frac{1}{2}}) \\
 &= (1-2x)^{\frac{1}{2}} - x(1-2x)^{-\frac{1}{2}}
 \end{aligned}$$

$$\begin{aligned}
 \text{At } x = -4, \quad \frac{dy}{dx} &= (1-2(-4))^{\frac{1}{2}} - (-4)(1-2(-4))^{-\frac{1}{2}} \\
 &= 3 + 4\left(\frac{1}{3}\right) \\
 &= \frac{13}{3}
 \end{aligned}$$

\therefore the gradient of the tangent to $y = x\sqrt{1-2x}$ at $x = -4$ is $\frac{13}{3}$.

d $y = x^3\sqrt{5-x^2}$ is the product of $u = x^3$ and $v = (5-x^2)^{\frac{1}{2}}$

$$\therefore u' = 3x^2 \quad \text{and} \quad v' = \frac{1}{2}(5-x^2)^{-\frac{1}{2}}(-2x) \quad \{\text{chain rule}\}$$

$$= -x(5-x^2)^{-\frac{1}{2}}$$

Now $\frac{dy}{dx} = u'v + uv' \quad \{\text{product rule}\}$

$$= 3x^2(5-x^2)^{\frac{1}{2}} + x^3(-x(5-x^2)^{-\frac{1}{2}})$$

$$= 3x^2(5-x^2)^{\frac{1}{2}} - x^4(5-x^2)^{-\frac{1}{2}}$$

At $x = 1$, $\frac{dy}{dx} = 3(1)^2(5-1^2)^{\frac{1}{2}} - 1^4(5-1^2)^{-\frac{1}{2}}$

$$= 3(2) - 1\left(\frac{1}{2}\right)$$

$$= \frac{11}{2}$$

\therefore the gradient of the tangent to $y = x^3\sqrt{5-x^2}$ at $x = 1$ is $\frac{11}{2}$.

4 a $y = \sqrt{x}(3-x)^2$ is the product of $u = x^{\frac{1}{2}}$ and $v = (3-x)^2$

$$\therefore u' = \frac{1}{2}x^{-\frac{1}{2}} \quad \text{and} \quad v' = 2(3-x)(-1) \quad \{\text{chain rule}\}$$

$$= -2(3-x)$$

Now $\frac{dy}{dx} = u'v + uv' \quad \{\text{product rule}\}$

$$= \frac{1}{2}x^{-\frac{1}{2}}(3-x)^2 + x^{\frac{1}{2}}(-2(3-x))$$

$$= \frac{1}{2}x^{-\frac{1}{2}}(3-x)^2 - 2x^{\frac{1}{2}}(3-x)$$

$$= (3-x) \left[\frac{1}{2\sqrt{x}}(3-x) - 2\sqrt{x} \right]$$

$$= (3-x) \left[\frac{3-x}{2\sqrt{x}} - 2\sqrt{x} \times \frac{2\sqrt{x}}{2\sqrt{x}} \right]$$

$$= (3-x) \left(\frac{3-x-4x}{2\sqrt{x}} \right)$$

$$= \frac{(3-x)(3-5x)}{2\sqrt{x}} \quad \text{as required}$$

b The tangent is horizontal when its gradient is zero.

$$\therefore \frac{dy}{dx} = \frac{(3-x)(3-5x)}{2\sqrt{x}} = 0$$

$$\therefore (3-x)(3-5x) = 0$$

$$\therefore x = 3 \text{ or } \frac{3}{5}$$

c $\frac{dy}{dx}$ is defined if its denominator is greater than zero.

$$\therefore 2\sqrt{x} > 0$$

$$\therefore x > 0$$

\therefore the domain of $\frac{dy}{dx}$ is $\{x \mid x > 0\}$ and the domain of the original function is $\{x \mid x \geq 0\}$.

$\frac{dy}{dx}$ is undefined when $x = 0$.

$$\begin{aligned}
 \text{5 } y = -2x^2(x+4) \text{ is the product of } u = -2x^2 \quad \text{and} \quad v = x+4 \\
 \therefore u' = -2(2x) \quad \text{and} \quad v' = 1 \\
 = -4x
 \end{aligned}$$

$$\begin{aligned}
 \text{Now } \frac{dy}{dx} &= u'v + uv' \quad \{\text{product rule}\} \\
 &= -4x(x+4) + (-2x^2)(1) \\
 &= -4x^2 - 16x - 2x^2 \\
 &= -6x^2 - 16x
 \end{aligned}$$

$$\begin{aligned}
 \frac{dy}{dx} = 10 \quad \text{when} \quad -6x^2 - 16x = 10 \\
 \therefore 6x^2 + 16x + 10 = 0 \\
 \therefore 3x^2 + 8x + 5 = 0 \\
 \therefore (3x+5)(x+1) = 0 \\
 \therefore x = -1 \quad \text{or} \quad -\frac{5}{3}
 \end{aligned}$$

$$\begin{aligned}
 \text{6 } y = (x+3)(x-2)^2 \text{ is the product of } u = x+3 \quad \text{and} \quad v = (x-2)^2 \\
 \therefore u' = 1 \quad \text{and} \quad v' = 2(x-2) \\
 = 2x-4
 \end{aligned}$$

$$\begin{aligned}
 \text{Now } \frac{dy}{dx} &= u'v + uv' \quad \{\text{product rule}\} \\
 &= 1(x-2)^2 + (x+3)(2x-4) \\
 &= x^2 - 4x + 4 + 2x^2 - 4x + 6x - 12 \\
 &= 3x^2 - 2x - 8
 \end{aligned}$$

$$\begin{aligned}
 \frac{dy}{dx} = -7 \quad \text{when} \quad 3x^2 - 2x - 8 = -7 \\
 \therefore 3x^2 - 2x - 1 = 0 \\
 \therefore (3x+1)(x-1) = 0 \\
 \therefore x = 1 \quad \text{or} \quad x = -\frac{1}{3}
 \end{aligned}$$

$$\begin{aligned}
 \text{7 } f(x) = ax\sqrt{1-x} \text{ is the product of } u(x) = ax \quad \text{and} \quad v(x) = (1-x)^{\frac{1}{2}} \\
 \therefore u'(x) = a \quad \text{and} \quad v'(x) = \frac{1}{2}(1-x)^{-\frac{1}{2}}(-1) \quad \{\text{chain rule}\} \\
 = -\frac{1}{2}(1-x)^{-\frac{1}{2}}
 \end{aligned}$$

$$\begin{aligned}
 \text{Now } f'(x) &= u'(x)v(x) + u(x)v'(x) \quad \{\text{product rule}\} \\
 &= a(1-x)^{\frac{1}{2}} + ax\left(-\frac{1}{2}(1-x)^{-\frac{1}{2}}\right) \\
 &= a(1-x)^{\frac{1}{2}} - \frac{1}{2}ax(1-x)^{-\frac{1}{2}}
 \end{aligned}$$

a The tangent to $f(x) = ax\sqrt{1-x}$ has gradient 0 when $f'(x) = 0$

$$\therefore a(1-x)^{\frac{1}{2}} - \frac{1}{2}ax(1-x)^{-\frac{1}{2}} = 0$$

$$\therefore a\sqrt{1-x} - \frac{ax}{2\sqrt{1-x}} = 0$$

$$\therefore a\sqrt{1-x} = \frac{ax}{2\sqrt{1-x}}$$

$$\therefore 2a(1-x) = ax$$

$$\therefore 2 - 2x = x$$

$$\therefore 3x = 2$$

$$\therefore x = \frac{2}{3}$$

b The tangent to $f(x) = ax\sqrt{1-x}$ has gradient a when $f'(x) = a$

$$\therefore a(1-x)^{\frac{1}{2}} - \frac{1}{2}ax(1-x)^{-\frac{1}{2}} = a$$

$$\therefore a\sqrt{1-x} - \frac{ax}{2\sqrt{1-x}} = a$$

$$\therefore \sqrt{1-x} - \frac{x}{2\sqrt{1-x}} = 1$$

$$\therefore \frac{2(1-x) - x}{2\sqrt{1-x}} = 1$$

$$\therefore 2 - 3x = 2\sqrt{1-x}$$

$$\therefore (2 - 3x)^2 = 4(1 - x)$$

{squaring both sides assuming both sides are ≥ 0 }

$$\therefore 4 - 12x + 9x^2 = 4 - 4x$$

$$\therefore 9x^2 - 8x = 0$$

$$\therefore x(9x - 8) = 0$$

$$\therefore x = 0 \text{ or } \frac{8}{9}$$

Check: If $x = 0$, $f'(0) = a(1-0)^{\frac{1}{2}} - \frac{1}{2}a(0)(1-0)^{-\frac{1}{2}} = a$ ✓

If $x = \frac{8}{9}$, $f'(\frac{8}{9}) = a(1-\frac{8}{9})^{\frac{1}{2}} - \frac{1}{2}a(\frac{8}{9})(1-\frac{8}{9})^{-\frac{1}{2}} = \frac{1}{3}a - \frac{4}{3}a = -a$ ✗

So, $x = 0$.

8 $f(x) = x^2\sqrt{x^2+a}$ is the product of $u(x) = x^2$ and $v(x) = (x^2+a)^{\frac{1}{2}}$

$$\begin{aligned} \therefore u'(x) &= 2x \text{ and } v'(x) = \frac{1}{2}(x^2+a)^{-\frac{1}{2}}(2x) \quad \{\text{chain rule}\} \\ &= x(x^2+a)^{-\frac{1}{2}} \end{aligned}$$

Now $f'(x) = u'(x)v(x) + u(x)v'(x)$ {product rule}

$$= 2x(x^2+a)^{\frac{1}{2}} + x^2(x(x^2+a)^{-\frac{1}{2}})$$

$$= 2x\sqrt{x^2+a} + \frac{x^3}{\sqrt{x^2+a}}$$

$$\begin{aligned}
& \text{and } f'(-2) = -\frac{34}{3} \\
\therefore 2(-2)\sqrt{(-2)^2 + a} + \frac{(-2)^3}{\sqrt{(-2)^2 + a}} &= -\frac{34}{3} \\
\therefore -4\sqrt{a+4} - \frac{8}{\sqrt{a+4}} &= -\frac{34}{3} \\
\therefore -4(a+4) - 8 &= -\frac{34}{3}\sqrt{a+4} \\
\therefore -4(a+4) + \frac{34}{3}\sqrt{a+4} - 8 &= 0 \\
\therefore -12(a+4) + 34\sqrt{a+4} - 24 &= 0 \\
\therefore -12X^2 + 34X - 24 &= 0 \quad \{\text{letting } X = \sqrt{a+4}\} \\
\therefore X &= \frac{-34 \pm \sqrt{(34)^2 - 4(-12)(-24)}}{2(-12)} \\
&= \frac{-34 \pm 2}{-24} \\
&= \frac{-32}{-24} \quad \text{or} \quad \frac{-36}{-24} \\
&= \frac{4}{3} \quad \text{or} \quad \frac{3}{2} \\
\therefore \sqrt{a+4} &= \frac{4}{3} \quad \text{or} \quad \sqrt{a+4} = \frac{3}{2} \\
\therefore a+4 &= \frac{16}{9} \quad \text{or} \quad a+4 = \frac{9}{4} \\
\therefore a &= -\frac{20}{9} \quad \text{or} \quad a = -\frac{7}{4}
\end{aligned}$$

EXERCISE 12D

1 a $y = \frac{1+3x}{2-x}$ is a quotient with

$$u = 1+3x \quad \text{and} \quad v = 2-x$$

$$\therefore u' = 3 \quad \text{and} \quad v' = -1$$

$$\begin{aligned}
\text{Now } \frac{dy}{dx} &= \frac{u'v - uv'}{v^2} \quad \{\text{quotient rule}\} \\
&= \frac{3(2-x) - (1+3x)(-1)}{(2-x)^2} \\
&= \frac{6-3x+1+3x}{(2-x)^2} \\
&= \frac{7}{(2-x)^2}
\end{aligned}$$

b $y = \frac{x^2}{2x+1}$ is a quotient with

$$u = x^2 \quad \text{and} \quad v = 2x+1$$

$$\therefore u' = 2x \quad \text{and} \quad v' = 2$$

$$\begin{aligned}
\text{Now } \frac{dy}{dx} &= \frac{u'v - uv'}{v^2} \quad \{\text{quotient rule}\} \\
&= \frac{2x(2x+1) - x^2(2)}{(2x+1)^2} \\
&= \frac{4x^2 + 2x - 2x^2}{(2x+1)^2} \\
&= \frac{2x^2 + 2x}{(2x+1)^2}
\end{aligned}$$

c $y = \frac{x}{x^2 - 3}$ is a quotient with

$$u = x \quad \text{and} \quad v = x^2 - 3$$

$$\therefore u' = 1 \quad \text{and} \quad v' = 2x$$

$$\begin{aligned} \text{Now } \frac{dy}{dx} &= \frac{u'v - uv'}{v^2} \quad \{\text{quotient rule}\} \\ &= \frac{1(x^2 - 3) - x(2x)}{(x^2 - 3)^2} \\ &= \frac{x^2 - 3 - 2x^2}{(x^2 - 3)^2} \\ &= \frac{-x^2 - 3}{(x^2 - 3)^2} \end{aligned}$$

e $y = \frac{x^2 - 3}{3x - x^2}$ is a quotient with

$$u = x^2 - 3 \quad \text{and} \quad v = 3x - x^2$$

$$\therefore u' = 2x \quad \text{and} \quad v' = 3 - 2x$$

$$\begin{aligned} \text{Now } \frac{dy}{dx} &= \frac{u'v - uv'}{v^2} \quad \{\text{quotient rule}\} \\ &= \frac{2x(3x - x^2) - (x^2 - 3)(3 - 2x)}{(3x - x^2)^2} \\ &= \frac{6x^2 - 2x^3 - (3x^2 - 2x^3 - 9 + 6x)}{(3x - x^2)^2} \\ &= \frac{3x^2 - 6x + 9}{(3x - x^2)^2} \end{aligned}$$

f $y = \frac{x}{\sqrt{1-3x}}$ is a quotient with $u = x$ and $v = (1-3x)^{\frac{1}{2}}$

$$\therefore u' = 1 \quad \text{and} \quad v' = \frac{1}{2}(1-3x)^{-\frac{1}{2}}(-3) \quad \{\text{chain rule}\}$$

$$= -\frac{3}{2}(1-3x)^{-\frac{1}{2}}$$

$$\begin{aligned} \text{Now } \frac{dy}{dx} &= \frac{u'v - uv'}{v^2} \quad \{\text{quotient rule}\} \\ &= \frac{1(1-3x)^{\frac{1}{2}} - x(-\frac{3}{2}(1-3x)^{-\frac{1}{2}})}{(\sqrt{1-3x})^2} \\ &= \frac{(1-3x)^{\frac{1}{2}} + \frac{3}{2}x(1-3x)^{-\frac{1}{2}}}{1-3x} \\ &= \frac{\sqrt{1-3x} + \frac{3x}{2\sqrt{1-3x}}}{1-3x} \\ &= \frac{\sqrt{1-3x} \times \frac{2\sqrt{1-3x}}{2\sqrt{1-3x}} + \frac{3x}{2\sqrt{1-3x}}}{1-3x} \\ &= \frac{2(1-3x) + 3x}{2\sqrt{1-3x}(1-3x)} \\ &= \frac{2-3x}{2(1-3x)^{\frac{3}{2}}} \end{aligned}$$

d $y = \frac{\sqrt{x}}{1-2x}$ is a quotient with

$$u = x^{\frac{1}{2}} \quad \text{and} \quad v = 1 - 2x$$

$$\therefore u' = \frac{1}{2}x^{-\frac{1}{2}} \quad \text{and} \quad v' = -2$$

$$\begin{aligned} \text{Now } \frac{dy}{dx} &= \frac{u'v - uv'}{v^2} \quad \{\text{quotient rule}\} \\ &= \frac{\frac{1}{2}x^{-\frac{1}{2}}(1-2x) - \sqrt{x}(-2)}{(1-2x)^2} \\ &= \frac{\frac{1-2x}{2\sqrt{x}} + 2\sqrt{x} \times \frac{2\sqrt{x}}{2\sqrt{x}}}{(1-2x)^2} \\ &= \frac{1-2x+4x}{2\sqrt{x}(1-2x)^2} \\ &= \frac{2x+1}{2\sqrt{x}(1-2x)^2} \end{aligned}$$

2 a $\frac{x+1}{3-x}$ is a quotient with $u = x+1$ and $v = 3-x$
 $\therefore u' = 1$ and $v' = -1$

$$\begin{aligned}\text{Now } \frac{d}{dx} \left(\frac{x+1}{3-x} \right) &= \frac{u'v - uv'}{v^2} \quad \{\text{quotient rule}\} \\ &= \frac{1(3-x) - (x+1)(-1)}{(3-x)^2} \\ &= \frac{3 - \cancel{x} + \cancel{x} + 1}{(3-x)^2} \\ &= \frac{4}{(3-x)^2}\end{aligned}$$

b $\frac{3x}{x^2-1}$ is a quotient with $u = 3x$ and $v = x^2-1$
 $\therefore u' = 3$ and $v' = 2x$

$$\begin{aligned}\text{Now } \frac{d}{dx} \left(\frac{3x}{x^2-1} \right) &= \frac{u'v - uv'}{v^2} \quad \{\text{quotient rule}\} \\ &= \frac{3(x^2-1) - (3x)(2x)}{(x^2-1)^2} \\ &= \frac{3x^2 - 3 - 6x^2}{(x^2-1)^2} \\ &= \frac{-3x^2 - 3}{(x^2-1)^2}\end{aligned}$$

c $\frac{x^3}{2x-1}$ is a quotient with $u = x^3$ and $v = 2x-1$
 $\therefore u' = 3x^2$ and $v' = 2$

$$\begin{aligned}\text{Now } \frac{d}{dx} \left(\frac{x^3}{2x-1} \right) &= \frac{u'v - uv'}{v^2} \\ &= \frac{3x^2(2x-1) - x^3(2)}{(2x-1)^2} \\ &= \frac{6x^3 - 3x^2 - 2x^3}{(2x-1)^2} \\ &= \frac{4x^3 - 3x^2}{(2x-1)^2}\end{aligned}$$

d $\frac{4x}{\sqrt{x-5}}$ is a quotient with $u = 4x$ and $v = (x-5)^{\frac{1}{2}}$

$$\therefore u' = 4 \quad \text{and} \quad v' = \frac{1}{2}(x-5)^{-\frac{1}{2}}(1) \quad \{\text{chain rule}\}$$

$$= \frac{1}{2}(x-5)^{-\frac{1}{2}}$$

Now $\frac{d}{dx} \left(\frac{4x}{\sqrt{x-5}} \right) = \frac{u'v - uv'}{v^2}$

$$= \frac{4(x-5)^{\frac{1}{2}} - 4x(\frac{1}{2}(x-5)^{-\frac{1}{2}})}{\left((x-5)^{\frac{1}{2}} \right)^2}$$

$$= \frac{4(x-5)^{\frac{1}{2}} - 2x(x-5)^{-\frac{1}{2}}}{x-5} \times \frac{(x-5)^{\frac{1}{2}}}{(x-5)^{\frac{1}{2}}}$$

$$= \frac{4(x-5) - 2x}{(x-5)^{\frac{3}{2}}}$$

$$= \frac{4x - 20 - 2x}{(x-5)^{\frac{3}{2}}}$$

$$= \frac{2x - 20}{(x-5)^{\frac{3}{2}}}$$

e $\frac{\sqrt{x}}{3-x^2}$ is a quotient with $u = x^{\frac{1}{2}}$ and $v = 3-x^2$

$$\therefore u' = \frac{1}{2}x^{-\frac{1}{2}} \quad \text{and} \quad v' = -2x$$

Now $\frac{d}{dx} \left(\frac{\sqrt{x}}{3-x^2} \right) = \frac{u'v - uv'}{v^2}$

$$= \frac{\frac{1}{2}x^{-\frac{1}{2}}(3-x^2) - x^{\frac{1}{2}}(-2x)}{(3-x^2)^2}$$

$$= \frac{\frac{3}{2}x^{-\frac{1}{2}} - \frac{1}{2}x^{\frac{3}{2}} + 2x^{\frac{3}{2}}}{(3-x^2)^2}$$

$$= \frac{\frac{3}{2\sqrt{x}} + \frac{3x\sqrt{x}}{2}}{(3-x^2)^2} \times \frac{2\sqrt{x}}{2\sqrt{x}}$$

$$= \frac{3x^2 + 3}{2\sqrt{x}(3-x^2)^2}$$

f $-\frac{x^2}{\sqrt{x^2+3}}$ is a quotient with $u = -x^2$ and $v = (x^2+3)^{\frac{1}{2}}$

$$\therefore u' = -2x \quad \text{and} \quad v' = \frac{1}{2}(x^2+3)^{-\frac{1}{2}} \times (2x) \quad \{\text{chain rule}\}$$

$$= x(x^2+3)^{-\frac{1}{2}}$$

Now $\frac{d}{dx} \left(-\frac{x^2}{\sqrt{x^2+3}} \right) = \frac{u'v - uv'}{v^2}$

$$= \frac{(-2x)(x^2+3)^{\frac{1}{2}} - (-x^2)(x(x^2+3)^{-\frac{1}{2}})}{\left((x^2+3)^{\frac{1}{2}} \right)^2}$$

$$= \frac{(-2x)\sqrt{x^2+3} + \frac{x^3}{\sqrt{x^2+3}}}{x^2+3} \times \frac{\sqrt{x^2+3}}{\sqrt{x^2+3}}$$

$$= \frac{(-2x)(x^2+3) + x^3}{(x^2+3)^{\frac{3}{2}}}$$

$$= \frac{-2x^3 - 6x + x^3}{(x^2+3)^{\frac{3}{2}}}$$

$$= \frac{-x^3 - 6x}{(x^2+3)^{\frac{3}{2}}}$$

3 a $y = \frac{x}{1-2x}$ is a quotient with

$$u = x \quad \text{and} \quad v = 1 - 2x$$

$$\therefore u' = 1 \quad \text{and} \quad v' = -2$$

Now $\frac{dy}{dx} = \frac{u'v - uv'}{v^2} \quad \{\text{quotient rule}\}$

$$= \frac{1(1-2x) - x(-2)}{(1-2x)^2}$$

$$= \frac{1}{(1-2x)^2}$$

At $x = 1$, $\frac{dy}{dx} = \frac{1}{(1-2(1))^2}$

$$= \frac{1}{(-1)^2}$$

$$= 1$$

\therefore the gradient of the tangent = 1

b $y = \frac{x^3}{x^2+1}$ is a quotient with

$$u = x^3 \quad \text{and} \quad v = x^2 + 1$$

$$\therefore u' = 3x^2 \quad \text{and} \quad v' = 2x$$

Now $\frac{dy}{dx} = \frac{u'v - uv'}{v^2} \quad \{\text{quotient rule}\}$

$$= \frac{3x^2(x^2+1) - x^3(2x)}{(x^2+1)^2}$$

$$= \frac{3x^4 + 3x^2 - 2x^4}{(x^2+1)^2}$$

$$= \frac{x^4 + 3x^2}{(x^2+1)^2}$$

At $x = -1$, $\frac{dy}{dx} = \frac{(-1)^4 + 3(-1)^2}{((-1)^2 + 1)^2}$

$$= \frac{4}{4}$$

$$= 1$$

\therefore the gradient of the tangent = 1

$$\begin{aligned}
 \text{c } y &= \frac{\sqrt{x}}{2x+1} \text{ is a quotient with} \\
 u &= x^{\frac{1}{2}} \quad \text{and} \quad v = 2x+1 \\
 \therefore u' &= \frac{1}{2}x^{-\frac{1}{2}} \quad \text{and} \quad v' = 2 \\
 \text{Now } \frac{dy}{dx} &= \frac{u'v - uv'}{v^2} \quad \{\text{quotient rule}\} \\
 &= \frac{\frac{1}{2\sqrt{x}}(2x+1) - \sqrt{x}(2)}{(2x+1)^2}
 \end{aligned}$$

$$\begin{aligned}
 \text{At } x = 4, \quad \frac{dy}{dx} &= \frac{\frac{1}{2\sqrt{4}}(2(4)+1) - 2\sqrt{4}}{(2(4)+1)^2} \\
 &= \frac{\frac{9}{4} - 4}{81} \\
 &= \frac{-\frac{7}{4}}{81} \\
 &= -\frac{7}{324}
 \end{aligned}$$

\therefore the gradient of the tangent $= -\frac{7}{324}$

$$\begin{aligned}
 \text{d } y &= \frac{x^2}{\sqrt{x^2+5}} \text{ is a quotient with} \\
 u &= x^2 \quad \text{and} \quad v = (x^2+5)^{\frac{1}{2}} \\
 \therefore u' &= 2x \quad \text{and} \quad v' = \frac{1}{2}(x^2+5)^{-\frac{1}{2}}(2x) \\
 &= x(x^2+5)^{-\frac{1}{2}}
 \end{aligned}$$

$$\begin{aligned}
 \text{Now } \frac{dy}{dx} &= \frac{u'v - uv'}{v^2} \quad \{\text{quotient rule}\} \\
 &= \frac{2x\sqrt{x^2+5} - x^2\left(\frac{x}{\sqrt{x^2+5}}\right)}{(\sqrt{x^2+5})^2} \\
 &= \frac{2x\sqrt{x^2+5} - \frac{x^3}{\sqrt{x^2+5}}}{x^2+5}
 \end{aligned}$$

$$\begin{aligned}
 \text{At } x = -2, \\
 \frac{dy}{dx} &= \frac{2(-2)\sqrt{(-2)^2+5} - \frac{(-2)^3}{\sqrt{(-2)^2+5}}}{(-2)^2+5} \\
 &= \frac{-4(3) - \frac{-8}{3}}{9} \\
 &= \frac{-\frac{28}{3}}{9} \\
 &= -\frac{28}{27}
 \end{aligned}$$

\therefore the gradient of the tangent $= -\frac{28}{27}$

$$\begin{aligned}
 \text{4 } f(x) &= \frac{x}{\sqrt{x-1}} \text{ is a quotient with} \\
 u(x) &= x \quad \text{and} \quad v(x) = (x-1)^{\frac{1}{2}} \\
 \therefore u'(x) &= 1 \quad \text{and} \quad v'(x) = \frac{1}{2}(x-1)^{-\frac{1}{2}}(1) \quad \{\text{chain rule}\} \\
 &= \frac{1}{2\sqrt{x-1}}
 \end{aligned}$$

$$\begin{aligned}
 \text{Now } f'(x) &= \frac{u'(x)v(x) + u(x)v'(x)}{[v(x)]^2} \quad \{\text{quotient rule}\} \\
 &= \frac{1\sqrt{x-1} - x\left(\frac{1}{2\sqrt{x-1}}\right)}{(\sqrt{x-1})^2} \\
 &= \frac{\sqrt{x-1} - \frac{x}{2\sqrt{x-1}}}{x-1} \\
 &= \frac{\sqrt{x-1} \times \frac{2\sqrt{x-1}}{2\sqrt{x-1}} - \frac{x}{2\sqrt{x-1}}}{x-1} \\
 &= \frac{2(x-1) - x}{2\sqrt{x-1}(x-1)} \\
 &= \frac{x-2}{2(x-1)^{\frac{3}{2}}}
 \end{aligned}$$

Check: $f(x) = \frac{x}{\sqrt{x-1}}$

$$= \frac{x-1}{\sqrt{x-1}} + \frac{1}{\sqrt{x-1}}$$

$$= (x-1)^{\frac{1}{2}} + (x-1)^{-\frac{1}{2}}$$

$$\therefore f'(x) = \frac{1}{2}(x-1)^{-\frac{1}{2}}(1) + \left(-\frac{1}{2}(x-1)^{-\frac{3}{2}}(1)\right) \quad \{\text{chain rule}\}$$

$$= \frac{1}{2\sqrt{x-1}} - \frac{1}{2(x-1)^{\frac{3}{2}}}$$

$$= \frac{1}{2\sqrt{x-1}} \times \frac{(x-1)}{(x-1)} - \frac{1}{2(x-1)^{\frac{3}{2}}}$$

$$= \frac{x-1-1}{2(x-1)^{\frac{3}{2}}}$$

$$= \frac{x-2}{2(x-1)^{\frac{3}{2}}} \quad \checkmark$$

5 a $y = \frac{2x+3}{x+1}$ is a quotient with $u = 2x+3$ and $v = x+1$
 $\therefore u' = 2$ and $v' = 1$

Now $\frac{dy}{dx} = \frac{u'v - uv'}{v^2} \quad \{\text{quotient rule}\}$

$$= \frac{2(x+1) - (2x+3)(1)}{(x+1)^2}$$

$$= \frac{2x+2-2x-3}{(x+1)^2}$$

$$= -\frac{1}{(x+1)^2}$$

b The illustrated tangents to the graph of $y = \frac{2x+3}{x+1}$ meet the graph at the points where $x = -2$ and $x = 0$.

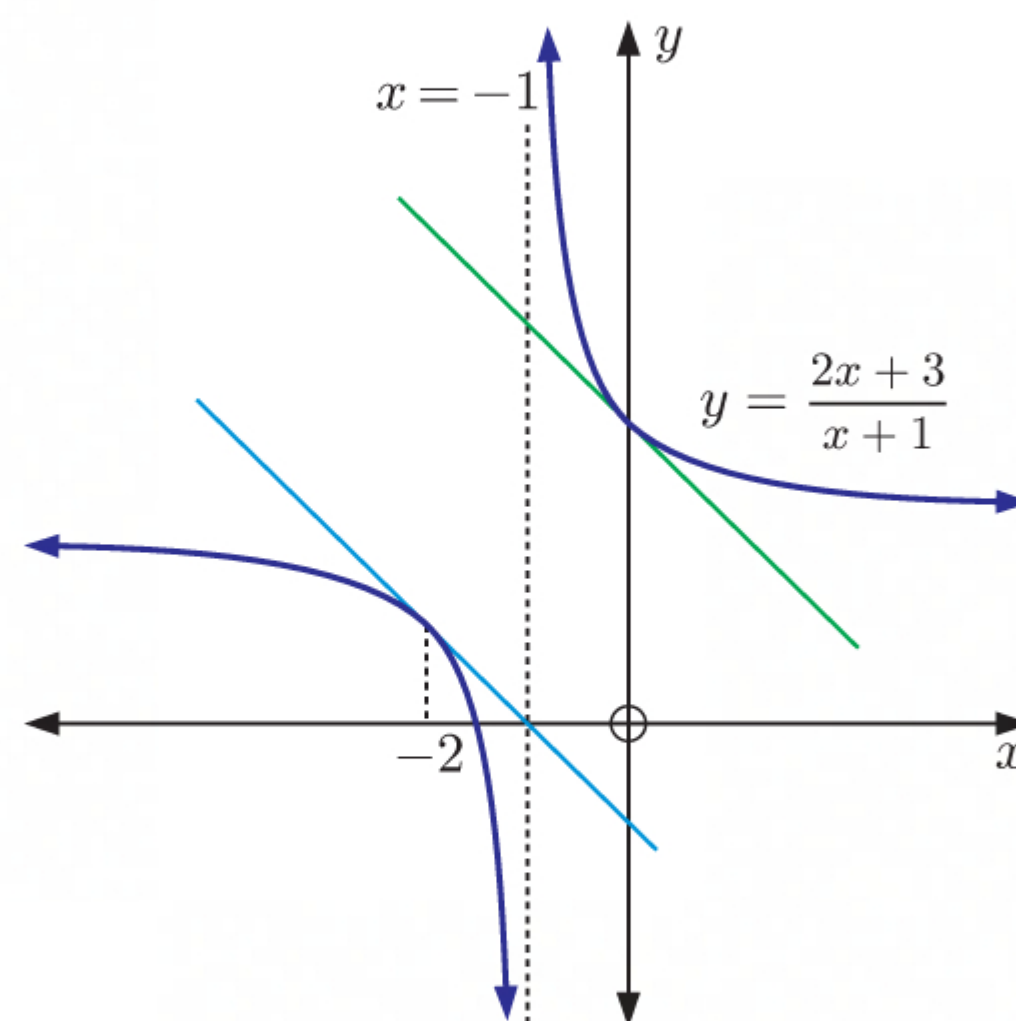
Now $\frac{dy}{dx}$ is the gradient of the tangent to the graph at any point.

At $x = -2$, $\frac{dy}{dx} = -\frac{1}{(-2+1)^2} = -1$

At $x = 0$, $\frac{dy}{dx} = -\frac{1}{(0+1)^2} = -1$

So, the gradients of the two tangents to the graph of $y = \frac{2x+3}{x+1}$ are equal.

\therefore the tangents are parallel.



6 a $y = \frac{2\sqrt{x}}{1-x}$ is a quotient with $u = 2x^{\frac{1}{2}}$ and $v = 1 - x$
 $\therefore u' = x^{-\frac{1}{2}}$ and $v' = -1$

Now $\frac{dy}{dx} = \frac{u'v - uv'}{v^2}$ {quotient rule}
 $= \frac{\frac{1}{\sqrt{x}}(1-x) - 2\sqrt{x}(-1)}{(1-x)^2} \times \left(\frac{\sqrt{x}}{\sqrt{x}} \right)$
 $= \frac{(1-x) + 2x}{\sqrt{x}(1-x)^2}$
 $= \frac{x+1}{\sqrt{x}(1-x)^2}$ as required

b i $\frac{dy}{dx} = 0$ when $x+1=0 \therefore x=-1$.

However $\frac{dy}{dx}$ is not defined for $x \leq 0$ because of the \sqrt{x} term. Hence $\frac{dy}{dx}$ never equals 0.

ii $\frac{dy}{dx}$ is undefined when \sqrt{x} is undefined or $\sqrt{x}(1-x)^2 = 0$
 \therefore when $x \leq 0$ and when $x = 1$

7 a $y = \frac{x^2+6}{2x+1}$ is a quotient with $u = x^2+6$ and $v = 2x+1$
 $\therefore u' = 2x$ and $v' = 2$

Now $\frac{dy}{dx} = \frac{u'v - uv'}{v^2}$ {quotient rule}
 $= \frac{2x(2x+1) - (x^2+6)(2)}{(2x+1)^2}$
 $= \frac{4x^2 + 2x - 2x^2 - 12}{(2x+1)^2}$
 $= \frac{2x^2 + 2x - 12}{(2x+1)^2}$ as required

b i $\frac{dy}{dx} = 0$ when $2x^2 + 2x - 12 = 0$
 $\therefore x^2 + x - 6 = 0$
 $\therefore (x+3)(x-2) = 0$
 $\therefore x = -3$ or $x = 2$

ii $\frac{dy}{dx}$ is undefined when $(2x+1)^2 = 0$
 $\therefore 2x+1 = 0$
 $\therefore x = -\frac{1}{2}$

8 a $y = \frac{x^2 - 3x + 1}{x + 2}$ is a quotient with $u = x^2 - 3x + 1$ and $v = x + 2$
 $\therefore u' = 2x - 3$ and $v' = 1$

Now $\frac{dy}{dx} = \frac{u'v - uv'}{v^2}$ {quotient rule}
 $= \frac{(2x - 3)(x + 2) - (x^2 - 3x + 1)(1)}{(x + 2)^2}$
 $= \frac{2x^2 + 4x - 3x - 6 - x^2 + 3x - 1}{(x + 2)^2}$
 $= \frac{x^2 + 4x - 7}{(x + 2)^2}$ as required

b i $\frac{dy}{dx} = 0$ when $x^2 + 4x - 7 = 0$
 $\therefore x = \frac{-4 \pm \sqrt{4^2 - 4(1)(-7)}}{2(1)}$
 $= \frac{-4 \pm \sqrt{44}}{2}$
 $= -2 \pm \sqrt{11}$

ii $\frac{dy}{dx}$ is undefined when $(x + 2)^2 = 0$
 $\therefore x = -2$

INVESTIGATION 4**THE DERIVATIVE OF b^x**

1 $y = 2^x$

x	y	$\frac{dy}{dx}$	$\frac{dy}{dx} \div y$
0	1	0.6931	0.6931
0.5	1.4142	0.9803	0.6931
1	2	1.3863	0.6931
1.5	2.8284	1.9605	0.6931
2	4	2.7726	0.6931

2 a $y = 3^x$

x	y	$\frac{dy}{dx}$	$\frac{dy}{dx} \div y$
0	1	1.0986	1.0986
0.5	1.7321	1.9029	1.0986
1	3	3.2958	1.0986
1.5	5.1962	5.7086	1.0986
2	9	9.8875	1.0986

b $y = 5^x$

x	y	$\frac{dy}{dx}$	$\frac{dy}{dx} \div y$
0	1	1.6094	1.6094
0.5	2.2361	3.5988	1.6094
1	5	8.0472	1.6094
1.5	11.1803	17.9941	1.6094
2	25	40.2359	1.6094

c $y = (0.5)^x$

x	y	$\frac{dy}{dx}$	$\frac{dy}{dx} \div y$
0	1	-0.6931	-0.6931
0.5	0.7071	-0.4901	-0.6931
1	0.5	-0.3466	-0.6931
1.5	0.3536	-0.2451	-0.6931
2	0.25	-0.1733	-0.6931

3 From **2 a**, **b**, and **c**, we can see that $\frac{dy}{dx} \div y$ is always equal to the value of $\frac{dy}{dx}$ at $x = 0$.

So, if $f(x) = b^x$, then $\frac{f'(x)}{b^x} = f'(0)$
 $\therefore f'(x) = f'(0) \times b^x$

EXERCISE 12E

1 a If $f(x) = e^{4x}$
 then $f'(x) = e^{4x}(4)$
 $= 4e^{4x}$

c If $f(x) = e^{-2x}$
 then $f'(x) = e^{-2x}(-2)$
 $= -2e^{-2x}$

e If $f(x) = 2e^{-\frac{x}{2}}$
 then $f'(x) = 2e^{-\frac{x}{2}}(-\frac{1}{2})$
 $= -e^{-\frac{x}{2}}$

g If $f(x) = 4e^{\frac{x}{2}} - 3e^{-x}$
 then $f'(x) = 4e^{\frac{x}{2}}(\frac{1}{2}) - 3e^{-x}(-1)$
 {addition rule}
 $= 2e^{\frac{x}{2}} + 3e^{-x}$

i If $f(x) = e^{-x^2}$
 then $f'(x) = e^{-x^2}(-2x)$
 $= -2xe^{-x^2}$

b If $f(x) = e^x + 3$
 then $f'(x) = e^x + 0$ {addition rule}
 $= e^x$

d If $f(x) = e^{\frac{x}{2}}$
 then $f'(x) = e^{\frac{x}{2}}(\frac{1}{2})$
 $= \frac{1}{2}e^{\frac{x}{2}}$

f If $f(x) = 1 - 2e^{-x}$
 then $f'(x) = 0 - 2e^{-x}(-1)$
 {addition rule}
 $= 2e^{-x}$

h If $f(x) = \frac{e^x + e^{-x}}{2}$
 $= \frac{1}{2}e^x + \frac{1}{2}e^{-x}$
 then $f'(x) = \frac{1}{2}e^x + \frac{1}{2}e^{-x}(-1)$
 $= \frac{e^x - e^{-x}}{2}$

j If $f(x) = e^{\frac{1}{x}}$
 then $f'(x) = e^{\frac{1}{x}}\left(-\frac{1}{x^2}\right)$
 $= -\frac{e^{\frac{1}{x}}}{x^2}$

$$\begin{aligned}
 \mathbf{k} \quad & \text{If } f(x) = 10(1 + e^{2x}) \\
 & = 10 + 10e^{2x} \\
 \text{then } & f'(x) = 0 + 10e^{2x}(2) \\
 & \quad \{\text{addition rule}\} \\
 & = 20e^{2x}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{m} \quad & \text{If } f(x) = e^{2x+1} \\
 \text{then } & f'(x) = e^{2x+1}(2) \\
 & = 2e^{2x+1}
 \end{aligned}$$

$$\begin{aligned}
 \circ \quad & \text{If } f(x) = e^{1-2x^2} \\
 \text{then } & f'(x) = e^{1-2x^2}(-4x) \\
 & = -4xe^{1-2x^2}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{2} \quad \mathbf{a} \quad & \text{If } f(x) = xe^x \\
 \text{then } & f'(x) = 1e^x + xe^x \\
 & \quad \{\text{product rule}\} \\
 & = e^x + xe^x
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{c} \quad & \text{If } f(x) = \frac{e^x}{x} \\
 \text{then } & f'(x) = \frac{e^x x - e^x(1)}{x^2} \\
 & \quad \{\text{quotient rule}\} \\
 & = \frac{xe^x - e^x}{x^2}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{e} \quad & \text{If } f(x) = x^2 e^{3x} \\
 \text{then } & f'(x) = 2xe^{3x} + x^2 e^{3x}(3) \\
 & \quad \{\text{product rule}\} \\
 & = 2xe^{3x} + 3x^2 e^{3x}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{g} \quad & \text{If } f(x) = 20xe^{-0.5x} \\
 \text{then } & f'(x) = 20e^{-0.5x} + 20xe^{-0.5x}(-0.5) \quad \{\text{product rule}\} \\
 & = 20e^{-0.5x} - 10xe^{-0.5x}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{l} \quad & \text{If } f(x) = 20(1 - e^{-2x}) \\
 & = 20 - 20e^{-2x} \\
 \text{then } & f'(x) = 0 - 20e^{-2x}(-2) \\
 & \quad \{\text{addition rule}\} \\
 & = 40e^{-2x}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{n} \quad & \text{If } f(x) = e^{\frac{x}{4}} \\
 \text{then } & f'(x) = e^{\frac{x}{4}}\left(\frac{1}{4}\right) \\
 & = \frac{1}{4}e^{\frac{x}{4}}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{p} \quad & \text{If } f(x) = e^{-0.02x} \\
 \text{then } & f'(x) = e^{-0.02x} \times (-0.02) \\
 & = -0.02e^{-0.02x}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad & \text{If } f(x) = x^3 e^{-x} \\
 \text{then } & f'(x) = 3x^2 e^{-x} + x^3(e^{-x})(-1) \\
 & \quad \{\text{product rule}\} \\
 & = 3x^2 e^{-x} - x^3 e^{-x}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{d} \quad & \text{If } f(x) = \frac{x}{e^x} \\
 \text{then } & f'(x) = \frac{1e^x - xe^x}{(e^x)^2} \\
 & \quad \{\text{quotient rule}\} \\
 & = \frac{e^x(1-x)}{(e^x)^2} \\
 & = \frac{1-x}{e^x}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{f} \quad & \text{If } f(x) = \frac{e^x}{\sqrt{x}} \\
 \text{then } & f'(x) = \frac{e^x \sqrt{x} - \frac{e^x}{2\sqrt{x}}}{(\sqrt{x})^2} \\
 & \quad \{\text{quotient rule}\} \\
 & = \frac{e^x \sqrt{x} \times \left(\frac{\sqrt{x}}{\sqrt{x}}\right) - \frac{e^x}{2\sqrt{x}}}{x} \\
 & = \frac{xe^x - \frac{1}{2}e^x}{x\sqrt{x}}
 \end{aligned}$$

h If $f(x) = \frac{e^x + 2}{e^{-x} + 1}$

then $f'(x) = \frac{e^x(e^{-x} + 1) - (e^x + 2)(e^{-x})(-1)}{(e^{-x} + 1)^2}$ {quotient rule}

$$= \frac{1 + e^x + 1 + 2e^{-x}}{(e^{-x} + 1)^2}$$

$$= \frac{e^x + 2 + 2e^{-x}}{(e^{-x} + 1)^2}$$

3 a $y = (2 + e^x)^4$
 $= u^4$ where $u = 2 + e^x$

$$\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx} \quad \{\text{chain rule}\}$$

$$= 4u^3 \frac{du}{dx}$$

$$= 4(2 + e^x)^3 \times e^x$$

$$= 4e^x(2 + e^x)^3$$

c $y = (e^x + e^{-x})^{\frac{3}{2}}$
 $= u^{\frac{3}{2}}$ where $u = e^x + e^{-x}$

$$\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx} \quad \{\text{chain rule}\}$$

$$= \frac{3}{2}u^{\frac{1}{2}} \frac{du}{dx}$$

$$= \frac{3}{2}(e^x + e^{-x})^{\frac{1}{2}} \times (e^x + e^{-x}(-1))$$

$$= \frac{3}{2}(e^x + e^{-x})^{\frac{1}{2}}(e^x - e^{-x})$$

4 a $y = (e^x + 2)^4$
 $= u^4$ where $u = e^x + 2$

$$\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx} \quad \{\text{chain rule}\}$$

$$= 4u^3 \frac{du}{dx}$$

$$= 4(e^x + 2)^3 \times e^x$$

$$= 4e^x(e^x + 2)^3$$

At $x = 0$, $\frac{dy}{dx} = 4e^0(e^0 + 2)^3$
 $= 108$

\therefore gradient of tangent = 108

b $y = \sqrt{e^x - 1}$
 $= u^{\frac{1}{2}}$ where $u = e^x - 1$

$$\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx} \quad \{\text{chain rule}\}$$

$$= \frac{1}{2}u^{-\frac{1}{2}} \frac{du}{dx}$$

$$= \frac{1}{2}(e^x - 1)^{-\frac{1}{2}} \times e^x$$

$$= \frac{e^x}{2\sqrt{e^x - 1}}$$

d $y = \frac{1}{\sqrt{e^{2x} + 2}}$
 $= u^{-\frac{1}{2}}$ where $u = e^{2x} + 2$

$$\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx} \quad \{\text{chain rule}\}$$

$$= -\frac{1}{2}u^{-\frac{3}{2}} \frac{du}{dx}$$

$$= -\frac{1}{2}(e^{2x} + 2)^{-\frac{3}{2}} \times e^{2x}(2)$$

$$= -e^{2x}(e^{2x} + 2)^{-\frac{3}{2}}$$

b $y = \frac{1}{2 - e^{-x}}$
 $= u^{-1}$ where $u = 2 - e^{-x}$

$$\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx} \quad \{\text{chain rule}\}$$

$$= -u^{-2} \frac{du}{dx}$$

$$= -\frac{1}{(2 - e^{-x})^2} \times (-e^{-x}(-1))$$

$$= -\frac{e^{-x}}{(2 - e^{-x})^2}$$

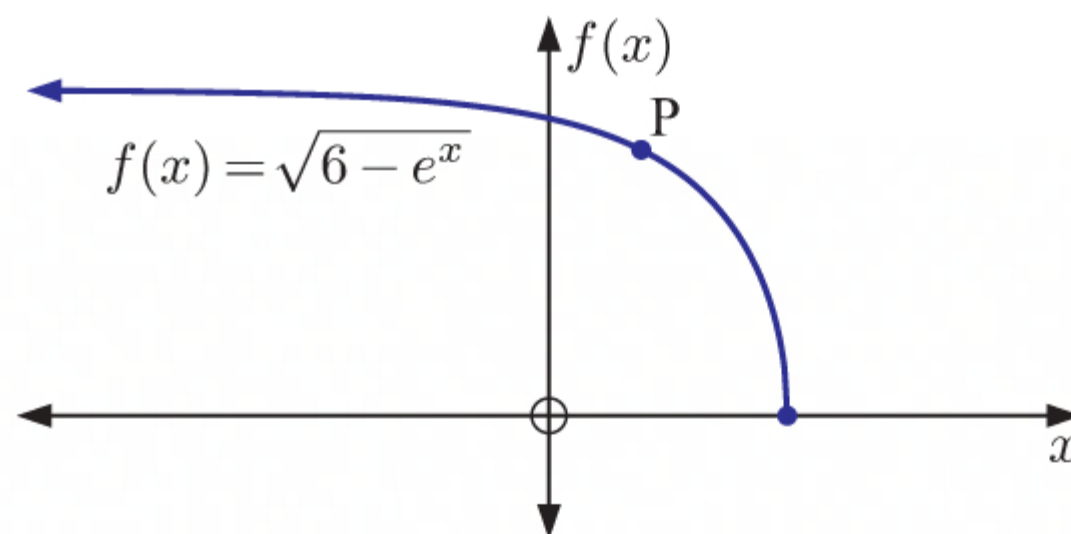
At $x = 0$, $\frac{dy}{dx} = -\frac{e^0}{(2 - e^0)^2} = -1$

\therefore gradient of tangent = -1

$$\begin{aligned}
 \text{c} \quad y &= \sqrt{e^{2x} + 10} \\
 &= u^{\frac{1}{2}} \quad \text{where } u = e^{2x} + 10 \\
 \frac{dy}{dx} &= \frac{dy}{du} \frac{du}{dx} \quad \{\text{chain rule}\} \\
 &= \frac{1}{2} u^{-\frac{1}{2}} \frac{du}{dx} \\
 &= \frac{1}{2\sqrt{e^{2x} + 10}} \times e^{2x}(2) \\
 &= \frac{e^{2x}}{\sqrt{e^{2x} + 10}} \\
 \text{At } x = \ln 3, \quad \frac{dy}{dx} &= \frac{e^{2 \ln 3}}{\sqrt{e^{2 \ln 3} + 10}} \\
 &= \frac{e^{\ln 3^2}}{\sqrt{e^{\ln 3^2} + 10}} \\
 &= \frac{3^2}{\sqrt{3^2 + 10}} = \frac{9}{\sqrt{19}} \\
 \therefore \text{gradient of tangent} &= \frac{9}{\sqrt{19}}
 \end{aligned}$$

$$\begin{aligned}
 \text{d} \quad y &= \frac{2-x}{e^{3x}} \\
 \therefore \frac{dy}{dx} &= \frac{(-1)e^{3x} - (2-x)e^{3x}(3)}{(e^{3x})^2} \quad \{\text{quotient rule}\} \\
 &= \frac{-e^{3x} - 3(2-x)e^{3x}}{e^{6x}} \\
 \text{At } x = 1, \quad \frac{dy}{dx} &= \frac{-e^{3(1)} - 3(2-1)e^{3(1)}}{e^{6(1)}} \\
 &= \frac{-e^3 - 3e^3}{e^6} \\
 &= \frac{-4e^3}{e^6} \\
 &= -\frac{4}{e^3} \\
 \therefore \text{gradient of tangent} &= -\frac{4}{e^3}
 \end{aligned}$$

$$\begin{aligned}
 \text{5 a} \quad f(x) &= \sqrt{6 - e^x} \text{ is defined when} \\
 6 - e^x &\geq 0 \\
 \therefore e^x &\leq 6 \\
 \therefore x &\leq \ln 6 \\
 \text{So, the domain is } &\{x \mid x \leq \ln 6\}.
 \end{aligned}$$



$$\begin{aligned}
 \text{b i} \quad \text{P has } y\text{-coordinate } 2. \\
 \therefore 2 &= \sqrt{6 - e^x} \\
 \therefore 6 - e^x &= 4 \\
 \therefore e^x &= 2 \\
 \therefore x &= \ln 2 \\
 \text{So, P is at } &(\ln 2, 2).
 \end{aligned}$$

$$\begin{aligned}
 \text{ii} \quad f(x) &= \sqrt{6 - e^x} = (6 - e^x)^{\frac{1}{2}} \\
 \therefore f'(x) &= \frac{1}{2}(6 - e^x)^{-\frac{1}{2}} \times (-e^x) \quad \{\text{chain rule}\} \\
 &= -\frac{1}{2}e^x(6 - e^x)^{-\frac{1}{2}} \\
 \text{Now } f'(\ln 2) &= -\frac{1}{2}e^{\ln 2}(6 - e^{\ln 2})^{-\frac{1}{2}} \\
 &= -\frac{1}{2} \times 2 \times \frac{1}{\sqrt{6-2}} \\
 &= -1 \times \frac{1}{\sqrt{4}} \\
 &= -1 \times \frac{1}{2} \\
 &= -\frac{1}{2}
 \end{aligned}$$

\therefore the gradient of the tangent at P is $-\frac{1}{2}$.

$$\begin{aligned}
 \text{6} \quad f(x) &= e^{kx} + x \quad \therefore f'(x) = ke^{kx} + 1 \\
 \text{Now } f'(0) &= -8, \text{ so } ke^0 + 1 = -8 \\
 \therefore k &= -9
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{7 \quad a} \quad y &= 2^x \\
 &= (e^{\ln 2})^x \\
 &= e^{x \ln 2} \\
 \therefore \frac{dy}{dx} &= e^{x \ln 2} \times \ln 2 \\
 &= e^{\ln 2^x} \times \ln 2 \\
 &= 2^x \ln 2
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad y &= b^x \\
 &= (e^{\ln b})^x \\
 &= e^{x \ln b} \\
 \therefore \frac{dy}{dx} &= e^{x \ln b} \times \ln b \\
 &= e^{\ln b^x} \times \ln b \\
 &= b^x \times \ln b
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{c \quad i} \quad y &= 5^x \\
 \therefore \frac{dy}{dx} &= 5^x \times \ln 5 \quad \{\text{using } \mathbf{b}\} \\
 &= 5^x \ln 5
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{ii} \quad y &= 8 \times 10^x \\
 \therefore \frac{dy}{dx} &= 8 \times 10^x \times \ln 10 \quad \{\text{using } \mathbf{b}\} \\
 &= 8 \times 10^x \ln 10
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{8} \quad f(x) = x^2 e^{-x} \text{ is the product of } u(x) = x^2 \text{ and } v(x) = e^{-x} \\
 \therefore u'(x) = 2x \text{ and } v'(x) = -e^{-x}
 \end{aligned}$$

$$\begin{aligned}
 \therefore f'(x) &= u'(x)v(x) + u(x)v'(x) \quad \{\text{product rule}\} \\
 &= 2x(e^{-x}) + x^2(-e^{-x}) \\
 &= 2xe^{-x} - x^2e^{-x}
 \end{aligned}$$

The tangent to $f(x) = x^2 e^{-x}$ is horizontal at point P.

\therefore the gradient of the tangent at point P is zero.

$\therefore f'(x) = 0$ at point P.

$$\begin{aligned}
 \text{Now } f'(x) = 0 \text{ when } 2xe^{-x} - x^2e^{-x} &= 0 \\
 \therefore xe^{-x}(2 - x) &= 0 \\
 \therefore x = 0 \text{ or } 2 - x = 0 &\quad \{\text{as } e^{-x} > 0 \text{ for all } x\} \\
 \therefore x = 0 \text{ or } x = 2
 \end{aligned}$$

$$\begin{aligned}
 f(0) &= 0^2 e^0 \text{ and } f(2) = 2^2 e^{-2} \\
 &= 0 &= 4e^{-2} \\
 & &= \frac{4}{e^2}
 \end{aligned}$$

So, the possible coordinates of P are $(0, 0)$ and $(2, \frac{4}{e^2})$.

$$\mathbf{9} \quad S(x) = \frac{1}{2}(e^x - e^{-x}), \quad C(x) = \frac{1}{2}(e^x + e^{-x})$$

$$\begin{aligned}
 \mathbf{a} \quad [C(x)]^2 - [S(x)]^2 &= \left[\frac{1}{2}(e^x + e^{-x})\right]^2 - \left[\frac{1}{2}(e^x - e^{-x})\right]^2 \\
 &= \frac{1}{4}(e^x + e^{-x})^2 - \frac{1}{4}(e^x - e^{-x})^2 \\
 &= \frac{1}{4}(e^{2x} + 2e^0 + e^{-2x}) - \frac{1}{4}(e^{2x} - 2e^0 + e^{-2x}) \\
 &= \frac{1}{4}(e^{2x} + 2 + e^{-2x}) - \frac{1}{4}(e^{2x} - 2 + e^{-2x}) \\
 &= \cancel{\frac{1}{4}e^{2x}} + \frac{1}{2} + \cancel{\frac{1}{4}e^{-2x}} - \cancel{\frac{1}{4}e^{2x}} + \frac{1}{2} - \cancel{\frac{1}{4}e^{-2x}} \\
 &= 1
 \end{aligned}$$

$$\begin{aligned}
 \text{b } \frac{d}{dx} [S(x)] &= \frac{d}{dx} \left[\frac{1}{2}(e^x - e^{-x}) \right] \\
 &= \frac{d}{dx} \left[\frac{1}{2}e^x - \frac{1}{2}e^{-x} \right] \\
 &= \frac{1}{2}e^x + \frac{1}{2}e^{-x} \\
 &= \frac{1}{2}(e^x + e^{-x}) \\
 &= C(x)
 \end{aligned}$$

$$\begin{aligned}
 \text{c } \frac{d}{dx} [C(x)] &= \frac{d}{dx} \left[\frac{1}{2}(e^x + e^{-x}) \right] \\
 &= \frac{d}{dx} \left[\frac{1}{2}e^x + \frac{1}{2}e^{-x} \right] \\
 &= \frac{1}{2}e^x - \frac{1}{2}e^{-x} \\
 &= \frac{1}{2}(e^x - e^{-x}) \\
 &= S(x)
 \end{aligned}$$

$$\begin{aligned}
 \text{d } \frac{d}{dx} [T(x)] &= \frac{d}{dx} \left[\frac{S(x)}{C(x)} \right] \\
 &= \frac{S'(x)C(x) - S(x)C'(x)}{[C(x)]^2} && \{\text{quotient rule}\} \\
 &= \frac{C(x)C(x) - S(x)S(x)}{[C(x)]^2} && \{\text{using b and c}\} \\
 &= \frac{[C(x)]^2 - [S(x)]^2}{[C(x)]^2} \\
 &= \frac{1}{[C(x)]^2} && \{\text{using a}\}
 \end{aligned}$$

INVESTIGATION 6

THE DERIVATIVE OF $\ln x$

1 The gradient function has a vertical asymptote $x = 0$, and as x increases, it approaches its horizontal asymptote $y = 0$.

2 We predict that for $y = \ln x$, $\frac{dy}{dx} = \frac{1}{x}$.

3 It appears that $f'(x) = \frac{1}{x}$, which agrees with our prediction in 2.

x	$f'(x)$
0.25	4
0.5	2
1	1
2	0.5
3	0.3333
4	0.25
5	0.2

EXERCISE 12F

$$\begin{aligned}
 \text{1 a } y &= \ln(7x) && \text{or } y = \ln(7x) \\
 &= \ln 7 + \ln x && \therefore \frac{dy}{dx} = \frac{7}{7x} \\
 \therefore \frac{dy}{dx} &= 0 + \frac{1}{x} = \frac{1}{x} && = \frac{1}{x}
 \end{aligned}$$

$$\mathbf{b} \quad y = \ln(2x + 1)$$

$$\therefore \frac{dy}{dx} = \frac{2}{2x + 1}$$

$$\mathbf{d} \quad y = 3 - 2 \ln x$$

$$\begin{aligned} \therefore \frac{dy}{dx} &= 0 - 2 \left(\frac{1}{x} \right) \\ &= -\frac{2}{x} \end{aligned}$$

$$\mathbf{f} \quad y = \frac{\ln x}{2x}$$

$$\begin{aligned} \therefore \frac{dy}{dx} &= \frac{\left(\frac{1}{x} \right) 2x - \ln x \times 2}{(2x)^2} \\ &\quad \{\text{quotient rule}\} \\ &= \frac{2 - 2 \ln x}{4x^2} \\ &= \frac{1 - \ln x}{2x^2} \end{aligned}$$

$$\mathbf{i} \quad y = \sqrt{\ln x} = (\ln x)^{\frac{1}{2}}$$

$$\begin{aligned} \therefore \frac{dy}{dx} &= \frac{1}{2} (\ln x)^{-\frac{1}{2}} \left(\frac{1}{x} \right) \quad \{\text{chain rule}\} \\ &= \frac{1}{2x\sqrt{\ln x}} \end{aligned}$$

$$\mathbf{k} \quad y = \sqrt{x} \ln(2x) = x^{\frac{1}{2}} \ln(2x)$$

$$\begin{aligned} \therefore \frac{dy}{dx} &= \frac{1}{2} x^{-\frac{1}{2}} \ln(2x) + x^{\frac{1}{2}} \left(\frac{2}{2x} \right) \\ &\quad \{\text{product rule}\} \\ &= \frac{1}{2\sqrt{x}} \ln(2x) + \sqrt{x} \left(\frac{1}{x} \right) \\ &= \frac{\ln(2x)}{2\sqrt{x}} + \frac{1}{\sqrt{x}} \end{aligned}$$

$$\mathbf{c} \quad y = \ln(x - x^2)$$

$$\therefore \frac{dy}{dx} = \frac{1 - 2x}{x - x^2}$$

$$\mathbf{e} \quad y = x^2 \ln x$$

$$\begin{aligned} \therefore \frac{dy}{dx} &= 2x \ln x + x^2 \left(\frac{1}{x} \right) \\ &\quad \{\text{product rule}\} \\ &= 2x \ln x + x \end{aligned}$$

$$\mathbf{g} \quad y = e^x \ln x$$

$$\begin{aligned} \therefore \frac{dy}{dx} &= e^x \ln x + e^x \left(\frac{1}{x} \right) \\ &\quad \{\text{product rule}\} \\ &= e^x \ln x + \frac{e^x}{x} \end{aligned}$$

$$\mathbf{h} \quad y = (\ln x)^2$$

$$\begin{aligned} \therefore \frac{dy}{dx} &= 2(\ln x)^1 \left(\frac{1}{x} \right) \quad \{\text{chain rule}\} \\ &= \frac{2 \ln x}{x} \end{aligned}$$

$$\mathbf{j} \quad y = e^{-x} \ln x$$

$$\begin{aligned} \therefore \frac{dy}{dx} &= e^{-x} (-1) \ln x + e^{-x} \left(\frac{1}{x} \right) \\ &\quad \{\text{product rule}\} \\ &= \frac{e^{-x}}{x} - e^{-x} \ln x \end{aligned}$$

$$\mathbf{l} \quad y = \frac{2\sqrt{x}}{\ln x} = \frac{2x^{\frac{1}{2}}}{\ln x}$$

$$\begin{aligned} \therefore \frac{dy}{dx} &= \frac{x^{-\frac{1}{2}} \ln x - 2x^{\frac{1}{2}} \left(\frac{1}{x} \right)}{(\ln x)^2} \\ &\quad \{\text{quotient rule}\} \\ &= \frac{\frac{1}{\sqrt{x}} \ln x - 2\sqrt{x} \left(\frac{1}{x} \right)}{(\ln x)^2} \\ &= \frac{\frac{1}{\sqrt{x}} \ln x - \frac{2}{\sqrt{x}}}{(\ln x)^2} \\ &= \frac{\ln x - 2}{\sqrt{x}(\ln x)^2} \end{aligned}$$

$$\text{m} \quad y = 3 - 4 \ln(1 - x)$$

$$\begin{aligned} \therefore \frac{dy}{dx} &= -4 \left(\frac{-1}{1-x} \right) \\ &= \frac{4}{1-x} \end{aligned}$$

$$\text{o} \quad y = \frac{\ln x}{x^2}$$

$$\begin{aligned} \therefore \frac{dy}{dx} &= \frac{\left(\frac{1}{x}\right)x^2 - \ln x(2x)}{(x^2)^2} \quad \{\text{quotient rule}\} \\ &= \frac{x - 2x \ln x}{x^4} \\ &= \frac{1 - 2 \ln x}{x^3} \end{aligned}$$

$$\text{2} \quad \begin{aligned} f(x) &= \ln(kx) \\ &= \ln x + \ln k \end{aligned}$$

$$\begin{aligned} \therefore f'(x) &= \frac{1}{x} + 0 \quad \{\ln k \text{ is constant}\} \\ &= \frac{1}{x} \end{aligned}$$

$$\text{3 a} \quad y = x \ln 5$$

$$\therefore \frac{dy}{dx} = \ln 5$$

$$\text{c} \quad y = \ln(x^4 + x)$$

$$\therefore \frac{dy}{dx} = \frac{4x^3 + 1}{x^4 + x}$$

$$\text{e} \quad y = [\ln(2x + 1)]^3$$

$$\begin{aligned} \therefore \frac{dy}{dx} &= 3[\ln(2x + 1)]^2 \times \frac{2}{2x + 1} \\ &\quad \{\text{chain rule}\} \\ &= \frac{6}{2x + 1} [\ln(2x + 1)]^2 \end{aligned}$$

$$\begin{aligned} \text{g} \quad y &= \ln\left(\frac{1}{x}\right) = \ln(x^{-1}) \\ &= -\ln x \quad \{\ln(a^n) = n \ln a\} \end{aligned}$$

$$\therefore \frac{dy}{dx} = -\frac{1}{x}$$

$$\text{i} \quad y = \frac{1}{\ln x} = (\ln x)^{-1}$$

$$\begin{aligned} \therefore \frac{dy}{dx} &= -1(\ln x)^{-2} \times \frac{1}{x} \quad \{\text{chain rule}\} \\ &= \frac{-1}{x(\ln x)^2} \end{aligned}$$

$$\text{n} \quad y = x \ln(x^2 + 1)$$

$$\begin{aligned} \therefore \frac{dy}{dx} &= 1(\ln(x^2 + 1)) + x \left(\frac{2x}{x^2 + 1} \right) \\ &\quad \{\text{product rule}\} \\ &= \ln(x^2 + 1) + \frac{2x^2}{x^2 + 1} \end{aligned}$$

$$\text{b} \quad y = \ln(x^3) = 3 \ln x$$

$$\{\ln(a^n) = n \ln a\}$$

$$\therefore \frac{dy}{dx} = 3 \left(\frac{1}{x} \right) = \frac{3}{x}$$

$$\text{d} \quad y = \ln(10 - 5x)$$

$$\therefore \frac{dy}{dx} = \frac{-5}{10 - 5x} = \frac{1}{x - 2}$$

$$\text{f} \quad y = \frac{\ln(4x)}{x}$$

$$\begin{aligned} \therefore \frac{dy}{dx} &= \frac{\left(\frac{4}{4x}\right)x - \ln(4x) \times 1}{x^2} \\ &\quad \{\text{quotient rule}\} \\ &= \frac{1 - \ln(4x)}{x^2} \end{aligned}$$

$$\text{h} \quad y = \ln(\ln x)$$

$$\begin{aligned} \therefore \frac{dy}{dx} &= \frac{\frac{1}{x}}{\ln x} \\ &= \frac{1}{x \ln x} \end{aligned}$$

$$\begin{aligned}
 \text{4 a} \quad y &= \ln \sqrt{1-2x} \\
 &= \ln \left((1-2x)^{\frac{1}{2}} \right) \\
 &= \frac{1}{2} \ln(1-2x) \quad \{ \ln(a^n) = n \ln a \} \\
 \therefore \frac{dy}{dx} &= \frac{1}{2} \times \frac{-2}{1-2x} \\
 &= \frac{-1}{1-2x} \\
 &= \frac{1}{2x-1}
 \end{aligned}$$

$$\begin{aligned}
 \text{b} \quad y &= \ln \left(\frac{1}{2x+3} \right) \\
 &= \ln \left((2x+3)^{-1} \right) \\
 &= -\ln(2x+3) \quad \{ \ln(a^n) = n \ln a \} \\
 \therefore \frac{dy}{dx} &= -\frac{2}{2x+3}
 \end{aligned}$$

$$\begin{aligned}
 \text{c} \quad y &= \ln(e^x \sqrt{x}) \\
 &= \ln(e^x) + \ln(x^{\frac{1}{2}}) \quad \{ \ln(ab) = \ln a + \ln b \} \\
 &= \ln(e^x) + \frac{1}{2} \ln x \quad \{ \ln(a^n) = n \ln a \} \\
 &= x + \frac{1}{2} \ln x \quad \{ \ln e^a = a \} \\
 \therefore \frac{dy}{dx} &= 1 + \frac{1}{2} \left(\frac{1}{x} \right) \\
 &= 1 + \frac{1}{2x}
 \end{aligned}$$

$$\begin{aligned}
 \text{d} \quad y &= \ln(x\sqrt{2-x}) \\
 &= \ln x + \ln \left((2-x)^{\frac{1}{2}} \right) \\
 &= \ln x + \frac{1}{2} \ln(2-x) \\
 &\quad \{ \ln(a^n) = n \ln a \} \\
 \therefore \frac{dy}{dx} &= \frac{1}{x} + \frac{1}{2} \left(\frac{-1}{2-x} \right) \\
 &= \frac{1}{x} - \frac{1}{2(2-x)}
 \end{aligned}$$

$$\begin{aligned}
 \text{e} \quad y &= \ln \left(\frac{x+3}{x-1} \right) \\
 &= \ln(x+3) - \ln(x-1) \\
 &\quad \{ \ln \left(\frac{a}{b} \right) = \ln a - \ln b \} \\
 \therefore \frac{dy}{dx} &= \frac{1}{x+3} - \frac{1}{x-1}
 \end{aligned}$$

$$\begin{aligned}
 \text{f} \quad y &= \ln \left(\frac{x^2}{3-x} \right) \\
 &= \ln(x^2) - \ln(3-x) \quad \{ \ln \left(\frac{a}{b} \right) = \ln a - \ln b \} \\
 &= 2 \ln x - \ln(3-x) \quad \{ \ln(a^n) = n \ln a \} \\
 \therefore \frac{dy}{dx} &= 2 \left(\frac{1}{x} \right) - \frac{-1}{3-x} \\
 &= \frac{2}{x} + \frac{1}{3-x}
 \end{aligned}$$

$$\begin{aligned}
 \text{5 a} \quad f(x) &= \ln((3x-4)^3) \\
 &= 3 \ln(3x-4) \\
 &\quad \{ \ln(a^n) = n \ln a \} \\
 \therefore f'(x) &= 3 \times \frac{3}{3x-4} = \frac{9}{3x-4}
 \end{aligned}$$

$$\begin{aligned}
 \text{b} \quad f(x) &= \ln(x(x^2+1)) \\
 &= \ln x + \ln(x^2+1) \\
 &\quad \{ \ln(ab) = \ln a + \ln b \} \\
 \therefore f'(x) &= \frac{1}{x} + \frac{2x}{x^2+1}
 \end{aligned}$$

$$\begin{aligned}
 \text{c} \quad f(x) &= \ln\left(\frac{x^2 + 2x}{x - 5}\right) \\
 &= \ln(x^2 + 2x) - \ln(x - 5) \quad \left\{ \ln\left(\frac{a}{b}\right) = \ln a - \ln b \right\} \\
 \therefore f'(x) &= \frac{2x + 2}{x^2 + 2x} - \frac{1}{x - 5}
 \end{aligned}$$

$$\begin{aligned}
 \text{d} \quad f(x) &= \ln\left(\frac{x^3}{(x + 4)(x - 1)}\right) \\
 &= \ln(x^3) - \ln[(x + 4)(x - 1)] \quad \left\{ \ln\left(\frac{a}{b}\right) = \ln a - \ln b \right\} \\
 &= \ln(x^3) - [\ln(x + 4) + \ln(x - 1)] \quad \left\{ \ln(ab) = \ln a + \ln b \right\} \\
 &= 3 \ln x - \ln(x + 4) - \ln(x - 1) \quad \left\{ \ln(a^n) = n \ln a \right\} \\
 \therefore f'(x) &= 3\left(\frac{1}{x}\right) - \frac{1}{x + 4} - \frac{1}{x - 1} \\
 &= \frac{3}{x} - \frac{1}{x + 4} - \frac{1}{x - 1}
 \end{aligned}$$

$$\begin{aligned}
 \text{6} \quad y &= x \ln x \\
 \therefore \frac{dy}{dx} &= 1 \ln x + x \times \frac{1}{x} \quad \{\text{product rule}\} \\
 &= \ln x + 1
 \end{aligned}$$

$$\begin{aligned}
 \text{At } x = e, \quad \frac{dy}{dx} &= \ln e + 1 \\
 &= 1 + 1 \\
 &= 2
 \end{aligned}$$

\therefore gradient of tangent = 2

$$\begin{aligned}
 \text{7} \quad f(x) &= a \ln(bx^2) \\
 \text{Now } f(e) &= 3, \quad \therefore 3 = a \ln(be^2) \\
 \therefore a &= \frac{3}{\ln(be^2)} \\
 &= \frac{3}{\ln b + \ln(e^2)} \\
 &= \frac{3}{\ln b + 2} \quad \dots (1)
 \end{aligned}$$

$$\begin{aligned}
 \text{Now } f'(x) &= a \times \frac{2bx}{bx^2} \\
 &= \frac{2a}{x} \quad \text{and} \quad f'(1) = 6
 \end{aligned}$$

$$\therefore 6 = 2a$$

$$\therefore a = 3 \quad \dots (2)$$

$$\therefore 3 = \frac{3}{\ln b + 2} \quad \{\text{equating (1) and (2)}\}$$

$$\therefore \ln b + 2 = 1$$

$$\therefore \ln b = -1$$

$$\therefore b = e^{-1}$$

$$\text{So, } a = 3, \quad b = \frac{1}{e}.$$

8 $f(x) = ax \ln(bx)$

Now $f(1) = 12$, $\therefore 12 = a \ln b \dots (*)$

Now $f'(x) = a \ln(bx) + ax \times \frac{b}{bx}$ {product rule}

$= a \ln(bx) + a$ and $f'(1) = 16$

$\therefore 16 = a \ln b + a$

$\therefore 16 = 12 + a$ {using $(*)$ }

$\therefore a = 4$

Substituting $a = 4$ into $(*)$ gives $12 = 4 \ln b$

$\therefore \ln b = 3$

$\therefore b = e^3$

So, $a = 4$, $b = e^3$.

9 $y = \ln(15 - x^2)$

$\therefore \frac{dy}{dx} = \frac{-2x}{15 - x^2}$

Now $\frac{dy}{dx} = 1$ when $\frac{-2x}{15 - x^2} = 1$

$\therefore -2x = 15 - x^2$

$\therefore x^2 - 2x - 15 = 0$

$\therefore (x + 3)(x - 5) = 0$

$\therefore x = -3$ or 5

But $y = \ln(15 - x^2)$ is undefined for $x = 5$.

When $x = -3$, $y = \ln(15 - (-3)^2)$

$= \ln(15 - 9)$

$= \ln 6$

\therefore the tangent at $(-3, \ln 6)$ has gradient 1.

INVESTIGATION 7

DERIVATIVES OF $\sin x$ AND $\cos x$

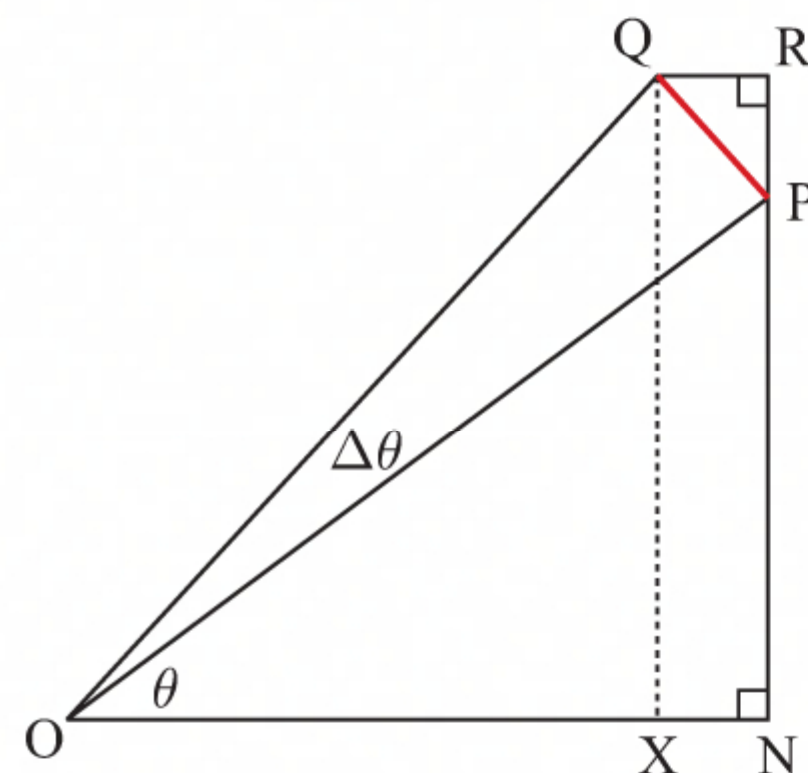
1 We predict that if $y = \sin x$, then $\frac{dy}{dx} = \cos x$.

2 We predict that if $y = \cos x$, then $\frac{dy}{dx} = -\sin x$.

3 a $\sin(\theta + \Delta\theta) = \frac{QX}{OQ} = NR$ { $OQ = 1$, $QX = NR$ }

$\sin \theta = \frac{NP}{OP} = NP$ { $OP = 1$ }

$\therefore \sin(\theta + \Delta\theta) - \sin \theta = NR - NP$
 $= PR$



- b i** For very small $\Delta\theta$, the line segment [PQ] is a good approximation of the arc PQ.
 \therefore as Q approaches P, the arc PQ resembles line segment [PQ].
- ii** $\text{arc PQ} = 1 \times \Delta\theta \quad \{l = r\theta\}$
 $= \Delta\theta$
 \therefore as Q approaches P, $PQ \approx \Delta\theta$.
- iii** $\widehat{QPO} = \frac{\pi}{2} - \frac{\Delta\theta}{2} \quad \{\text{isosceles triangle}\}$
As $\Delta\theta \rightarrow 0$, $\widehat{QPO} \rightarrow \frac{\pi}{2}$
 \therefore as Q approaches P, \widehat{QPO} approaches a right angle.
- iv** As Q approaches P, \widehat{QPO} approaches a right angle $\{\text{from iii}\}$
 $\therefore \widehat{QPR} \approx \pi - \frac{\pi}{2} - \widehat{OPN} \quad \{\text{angles on a line}\}$
 $\approx \pi - \frac{\pi}{2} - (\pi - \frac{\pi}{2} - \theta)$
 $\approx \cancel{\pi} - \cancel{\frac{\pi}{2}} - \cancel{\pi} + \cancel{\frac{\pi}{2}} + \theta$
 $\approx \theta$
- c** For small $\Delta\theta$, $\cos \theta \approx \frac{PR}{PQ}$
 $\approx \frac{\sin(\theta + \Delta\theta) - \sin \theta}{\Delta\theta} \quad \{\text{using a, b ii}\}$
 $\therefore \lim_{\Delta\theta \rightarrow 0} \cos \theta = \lim_{\Delta\theta \rightarrow 0} \frac{\sin(\theta + \Delta\theta) - \sin \theta}{\Delta\theta}$
 $\therefore \cos \theta = \frac{d}{d\theta}(\sin \theta)$

EXERCISE 12G

- 1 a** $y = \sin 2x$
 $\therefore \frac{dy}{dx} = (\cos 2x) \times 2$
 $= 2 \cos 2x$
- c** $y = \cos 3x - \sin x$
 $\therefore \frac{dy}{dx} = (-\sin 3x) \times 3 - \cos x$
 $= -3 \sin 3x - \cos x$
- e** $y = \cos(3 - 2x)$
 $\therefore \frac{dy}{dx} = (-\sin(3 - 2x)) \times (-2)$
 $= 2 \sin(3 - 2x)$
- g** $y = \sin \frac{x}{2} - 3 \cos x$
 $\therefore \frac{dy}{dx} = \left(\cos \frac{x}{2}\right) \left(\frac{1}{2}\right) - 3(-\sin x)$
 $= \frac{1}{2} \cos \frac{x}{2} + 3 \sin x$
- i** $y = \frac{1}{2} \cos 6x - 5 \sin 4x$
 $\therefore \frac{dy}{dx} = \frac{1}{2}(-\sin 6x) \times 6 - 5(\cos 4x) \times 4$
 $= -3 \sin 6x - 20 \cos 4x$
- b** $y = \sin x + \cos x$
 $\therefore \frac{dy}{dx} = \cos x - \sin x$
- d** $y = \sin(x + 1)$
 $\therefore \frac{dy}{dx} = (\cos(x + 1)) \times 1$
 $= \cos(x + 1)$
- f** $y = 3 - 2 \cos 3x$
 $\therefore \frac{dy}{dx} = -2(-3 \sin 3x)$
 $= 6 \sin 3x$
- h** $y = 4 \sin x - \cos 2x$
 $\therefore \frac{dy}{dx} = 4 \cos x + (\sin 2x) \times 2$
 $= 4 \cos x + 2 \sin 2x$

$$\begin{aligned} \mathbf{2} \quad \mathbf{a} \quad y &= x^2 + \cos x \\ \therefore \frac{dy}{dx} &= 2x - \sin x \end{aligned}$$

$$\begin{aligned} \mathbf{c} \quad y &= e^{-x} \sin x \\ \therefore \frac{dy}{dx} &= -e^{-x} \sin x + e^{-x} \cos x \\ &\quad \{\text{product rule}\} \end{aligned}$$

$$\begin{aligned} \mathbf{e} \quad y &= e^{\cos 5x} \\ \therefore \frac{dy}{dx} &= e^{\cos 5x} \times (-\sin 5x) \times 5 \\ &\quad \{\text{chain rule}\} \\ &= -5 \sin 5x \times e^{\cos 5x} \end{aligned}$$

$$\begin{aligned} \mathbf{g} \quad y &= x \cos x \\ \therefore \frac{dy}{dx} &= 1 \times \cos x + x(-\sin x) \\ &\quad \{\text{product rule}\} \\ &= \cos x - x \sin x \end{aligned}$$

$$\begin{aligned} \mathbf{3} \quad \mathbf{a} \quad y &= \sin(x^2) \\ \therefore \frac{dy}{dx} &= (\cos(x^2)) \times 2x \quad \{\text{chain rule}\} \\ &= 2x \cos(x^2) \end{aligned}$$

$$\begin{aligned} \mathbf{c} \quad y &= \sqrt{\cos x} = (\cos x)^{\frac{1}{2}} \\ \therefore \frac{dy}{dx} &= \frac{1}{2}(\cos x)^{-\frac{1}{2}} \times (-\sin x) \\ &\quad \{\text{chain rule}\} \\ &= -\frac{\sin x}{2\sqrt{\cos x}} \end{aligned}$$

$$\begin{aligned} \mathbf{e} \quad y &= \cos^3 x = (\cos x)^3 \\ \therefore \frac{dy}{dx} &= 3(\cos x)^2 \times (-\sin x) \\ &\quad \{\text{chain rule}\} \\ &= -3 \sin x \cos^2 x \end{aligned}$$

$$\begin{aligned} \mathbf{g} \quad y &= \cos^3 4x = (\cos 4x)^3 \\ \therefore \frac{dy}{dx} &= 3(\cos 4x)^2 \times (-4 \sin 4x) \\ &\quad \{\text{chain rule}\} \\ &= -12 \sin 4x \cos^2 4x \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad y &= e^x \cos x \\ \therefore \frac{dy}{dx} &= e^x \cos x + e^x(-\sin x) \\ &\quad \{\text{product rule}\} \\ &= e^x \cos x - e^x \sin x \end{aligned}$$

$$\begin{aligned} \mathbf{d} \quad y &= \ln(\sin x) \\ \therefore \frac{dy}{dx} &= \frac{\cos x}{\sin x} \end{aligned}$$

$$\begin{aligned} \mathbf{f} \quad y &= \cos \frac{x}{2} \\ \therefore \frac{dy}{dx} &= \left(-\sin \frac{x}{2}\right) \times \left(\frac{1}{2}\right) \quad \{\text{chain rule}\} \\ &= -\frac{1}{2} \sin \frac{x}{2} \end{aligned}$$

$$\begin{aligned} \mathbf{h} \quad y &= \frac{\sin x}{x} \\ \therefore \frac{dy}{dx} &= \frac{(\cos x)(x) - \sin x \times 1}{x^2} \\ &\quad \{\text{quotient rule}\} \\ &= \frac{x \cos x - \sin x}{x^2} \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad y &= \cos(\sqrt{x}) = \cos(x^{\frac{1}{2}}) \\ \therefore \frac{dy}{dx} &= -\sin(x^{\frac{1}{2}}) \times \frac{1}{2}x^{-\frac{1}{2}} \quad \{\text{chain rule}\} \\ &= -\frac{1}{2\sqrt{x}} \sin(\sqrt{x}) \end{aligned}$$

$$\begin{aligned} \mathbf{d} \quad y &= \sin^2 x = (\sin x)^2 \\ \therefore \frac{dy}{dx} &= 2 \sin x \cos x \quad \{\text{chain rule}\} \end{aligned}$$

$$\begin{aligned} \mathbf{f} \quad y &= \cos x \sin 2x \\ \therefore \frac{dy}{dx} &= (-\sin x) \sin 2x + \cos x(2 \cos 2x) \\ &\quad \{\text{product rule}\} \\ &= -\sin x \sin 2x + 2 \cos x \cos 2x \end{aligned}$$

$$\begin{aligned} \mathbf{h} \quad y &= \frac{2}{\sin^2 2x} = 2(\sin 2x)^{-2} \\ \therefore \frac{dy}{dx} &= -4(\sin 2x)^{-3} \times 2 \cos 2x \\ &\quad \{\text{chain rule}\} \\ &= -\frac{8 \cos 2x}{\sin^3 2x} \end{aligned}$$

$$\begin{aligned}
4 \quad \mathbf{a} \quad \frac{d}{dx}(\tan x) &= \frac{d}{dx} \left(\frac{\sin x}{\cos x} \right) \\
&= \frac{\cos x \times \cos x - \sin x \times (-\sin x)}{\cos^2 x} \quad \{\text{quotient rule}\} \\
&= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} \\
&= \frac{1}{\cos^2 x}
\end{aligned}$$

$$\begin{aligned}
\mathbf{b} \quad \mathbf{i} \quad y &= \tan 5x \\
\therefore \frac{dy}{dx} &= \frac{5}{\cos^2 5x}
\end{aligned}$$

$$\begin{aligned}
\mathbf{iii} \quad y &= e^{2x} \tan x \\
\therefore \frac{dy}{dx} &= 2e^{2x} \tan x + e^{2x} \left(\frac{1}{\cos^2 x} \right) \\
&\quad \{\text{product rule}\} \\
&= 2e^{2x} \tan x + \frac{e^{2x}}{\cos^2 x}
\end{aligned}$$

$$\begin{aligned}
\mathbf{ii} \quad y &= \tan x - 3 \sin x \\
\therefore \frac{dy}{dx} &= \frac{1}{\cos^2 x} - 3 \cos x
\end{aligned}$$

$$\begin{aligned}
\mathbf{iv} \quad y &= x \tan x \\
\therefore \frac{dy}{dx} &= 1(\tan x) + x \left(\frac{1}{\cos^2 x} \right) \\
&\quad \{\text{product rule}\} \\
&= \tan x + \frac{x}{\cos^2 x}
\end{aligned}$$

$$\begin{aligned}
5 \quad \mathbf{a} \quad f(x) &= \sin^3 x \\
&= (\sin x)^3 \\
\therefore f'(x) &= 3(\sin x)^2(\cos x) \quad \{\text{chain rule}\} \\
&= 3 \sin^2 x \cos x \\
\therefore f' \left(\frac{2\pi}{3} \right) &= 3 \sin^2 \left(\frac{2\pi}{3} \right) \cos \frac{2\pi}{3} \\
&= 3 \left(\frac{\sqrt{3}}{2} \right)^2 \left(-\frac{1}{2} \right) \\
&= -\frac{9}{8} \\
\therefore \text{gradient of tangent} &= -\frac{9}{8}
\end{aligned}$$

$$\begin{aligned}
\mathbf{b} \quad f(x) &= \cos x \sin x \\
\therefore f'(x) &= -\sin x \sin x + \cos x \cos x \quad \{\text{product rule}\} \\
&= \cos^2 x - \sin^2 x \\
\therefore f' \left(\frac{\pi}{4} \right) &= \cos^2 \left(\frac{\pi}{4} \right) - \sin^2 \left(\frac{\pi}{4} \right) \\
&= 0 \\
\therefore \text{gradient of tangent} &= 0
\end{aligned}$$

$$\begin{aligned}
6 \quad \mathbf{a} \quad f(x) &= 2 \cos^2 x + 2 \sin^2 x + 1 \\
&= 2(\cos x)^2 + 2(\sin x)^2 + 1 \\
\therefore f'(x) &= 2(2 \cos x)(-\sin x) + 2(2 \sin x)(\cos x) \\
&= -4 \cos x \sin x + 4 \sin x \cos x \\
&= 0
\end{aligned}$$

$$\begin{aligned}
\mathbf{b} \quad f(x) &= 2 \cos^2 x + 2 \sin^2 x + 1 \\
&= 2(\cos^2 x + \sin^2 x) + 1 \\
&= 2(1) + 1 \quad \{\cos^2 x + \sin^2 x = 1\} \\
&= 3 \quad \text{which is a constant and the derivative of a constant is zero.}
\end{aligned}$$

7 a Tangent **B** appears to have the steeper gradient.

b $y = \cos x + 2 \sin 2x$

$$\begin{aligned}\therefore \frac{dy}{dx} &= -\sin x + 2 \cos 2x \times 2 \\ &= 4 \cos 2x - \sin x\end{aligned}$$

Tangent **A** meets the graph at $x = \frac{\pi}{6}$.

$$\begin{aligned}\text{When } x = \frac{\pi}{6}, \quad \frac{dy}{dx} &= 4 \cos\left(2 \times \frac{\pi}{6}\right) - \sin \frac{\pi}{6} \\ &= 4 \cos \frac{\pi}{3} - \sin \frac{\pi}{6} \\ &= 2 - \frac{1}{2} \\ &= \frac{3}{2}\end{aligned}$$

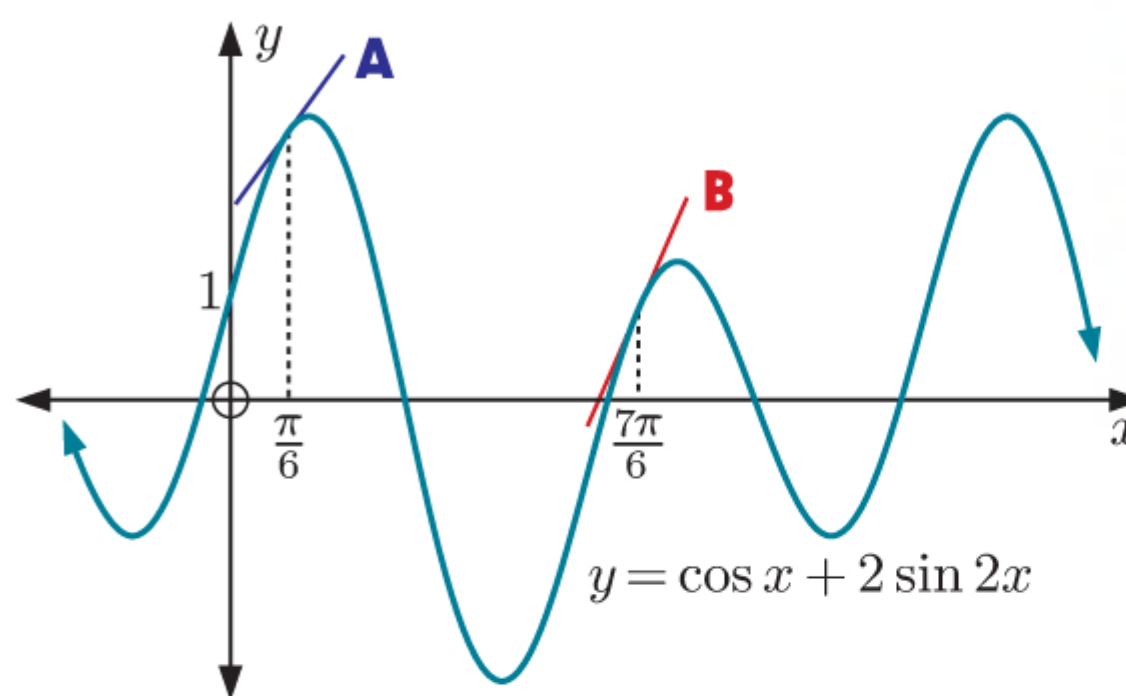
\therefore gradient of tangent **A** is $\frac{3}{2}$.

Tangent **B** meets the graph at $x = \frac{7\pi}{6}$.

$$\begin{aligned}\text{When } x = \frac{7\pi}{6}, \quad \frac{dy}{dx} &= 4 \cos\left(2 \times \frac{7\pi}{6}\right) - \sin \frac{7\pi}{6} \\ &= 4 \cos \frac{7\pi}{3} - \sin \frac{7\pi}{6} \\ &= 2 - \left(-\frac{1}{2}\right) \\ &= \frac{5}{2}\end{aligned}$$

\therefore gradient of tangent **B** is $\frac{5}{2}$.

Since $\frac{5}{2} > \frac{3}{2}$, tangent **B** has the steeper gradient. ✓



EXERCISE 12H

1 a $f(x) = 3x^2 - 6x + 2$
 $\therefore f'(x) = 6x - 6$
 $\therefore f''(x) = 6$

c $f(x) = 2x^3 - 3x^2 - x + 5$
 $\therefore f'(x) = 6x^2 - 6x - 1$
 $\therefore f''(x) = 12x - 6$

e $f(x) = (1 - 2x)^2$
 $\therefore f'(x) = 2(1 - 2x)(-2) \quad \{\text{chain rule}\}$
 $= -4(1 - 2x)$
 $= -4 + 8x$
 $\therefore f''(x) = 8$

b $f(x) = \frac{2}{\sqrt{x}} - 1 = 2x^{-\frac{1}{2}} - 1$
 $\therefore f'(x) = -x^{-\frac{3}{2}}$
 $f''(x) = \frac{3}{2}x^{-\frac{5}{2}}$
 $= \frac{3}{2x^{\frac{5}{2}}} = \frac{3}{2x^2\sqrt{x}}$

d $f(x) = \frac{2 - 3x}{x^2} = 2x^{-2} - 3x^{-1}$
 $\therefore f'(x) = -4x^{-3} + 3x^{-2}$
 $\therefore f''(x) = 12x^{-4} - 6x^{-3}$
 $= \frac{12 - 6x}{x^4}$

$$\begin{aligned}
 \mathbf{f} \quad f(x) &= \frac{x+2}{2x-1} \\
 \therefore f'(x) &= \frac{1(2x-1) - (x+2)(2)}{(2x-1)^2} && \{\text{quotient rule}\} \\
 &= \frac{2x-1-2x-4}{(2x-1)^2} \\
 &= \frac{-5}{(2x-1)^2} \\
 &= -5(2x-1)^{-2} \\
 \therefore f''(x) &= 10(2x-1)^{-3}(2) && \{\text{chain rule}\} \\
 &= \frac{20}{(2x-1)^3}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{2} \quad \mathbf{a} \quad y &= x - x^3 \\
 \therefore \frac{dy}{dx} &= 1 - 3x^2 \\
 \therefore \frac{d^2y}{dx^2} &= -6x
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad y &= x^2 - \frac{5}{x^2} \\
 &= x^2 - 5x^{-2} \\
 \therefore \frac{dy}{dx} &= 2x + 10x^{-3} \\
 \therefore \frac{d^2y}{dx^2} &= 2 - 30x^{-4} \\
 &= 2 - \frac{30}{x^4}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{c} \quad y &= 2 - \frac{3}{\sqrt{x}} \\
 &= 2 - 3x^{-\frac{1}{2}} \\
 \therefore \frac{dy}{dx} &= \frac{3}{2}x^{-\frac{3}{2}} \\
 \therefore \frac{d^2y}{dx^2} &= -\frac{9}{4}x^{-\frac{5}{2}} = -\frac{9}{4x^2\sqrt{x}}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{d} \quad y &= \frac{4-x}{x} \\
 &= 4x^{-1} - 1 \\
 \therefore \frac{dy}{dx} &= -4x^{-2} \\
 \therefore \frac{d^2y}{dx^2} &= 8x^{-3} \\
 &= \frac{8}{x^3}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{e} \quad y &= (x^2 - 3x)^2 \\
 &= x^4 - 2(x^2)(3x) + (3x)^2 \\
 &= x^4 - 6x^3 + 9x^2 \\
 \therefore \frac{dy}{dx} &= 4x^3 - 18x^2 + 18x \\
 \therefore \frac{d^2y}{dx^2} &= 12x^2 - 36x + 18
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{f} \quad y &= x^2 - x + \frac{1}{1-x} \\
 &= x^2 - x + (1-x)^{-1} \\
 \therefore \frac{dy}{dx} &= 2x - 1 + (-1)(1-x)^{-2}(-1) && \{\text{chain rule}\} \\
 &= 2x - 1 + (1-x)^{-2} \\
 \therefore \frac{d^2y}{dx^2} &= 2 - 2(1-x)^{-3}(-1) \\
 &= 2 + \frac{2}{(1-x)^3}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{g} \quad y &= e^{3x} + 2x \\
 \therefore \frac{dy}{dx} &= 3e^{3x} + 2 \\
 \therefore \frac{d^2y}{dx^2} &= 3(3e^{3x}) \\
 &= 9e^{3x}
 \end{aligned}$$

h $y = \frac{1 - e^{-x}}{x}$

$$\begin{aligned} \therefore \frac{dy}{dx} &= \frac{(e^{-x})(x) - (1 - e^{-x})(1)}{x^2} && \{\text{quotient rule}\} \\ &= \frac{xe^{-x} - 1 + e^{-x}}{x^2} \\ &= \frac{(x+1)e^{-x} - 1}{x^2} \end{aligned}$$

$$\begin{aligned} \therefore \frac{d^2y}{dx^2} &= \frac{[e^{-x} - (x+1)e^{-x}](x^2) - [(x+1)e^{-x} - 1](2x)}{x^4} && \{\text{quotient rule}\} \\ &= \frac{\cancel{x^2e^{-x}} - x^3e^{-x} - \cancel{2x^2e^{-x}} - 2x^2e^{-x} - 2xe^{-x} + 2x}{x^4} \\ &= \frac{x[-x^2e^{-x} - 2xe^{-x} + 2 - 2e^{-x}]}{x^4} \\ &= \frac{-x^2e^{-x} - 2xe^{-x} + 2 - 2e^{-x}}{x^3} \end{aligned}$$

i $y = \frac{3-x}{xe^x}$

$$\begin{aligned} \therefore \frac{dy}{dx} &= \frac{(-1)(xe^x) - (3-x)e^x(x+1)}{x^2e^{2x}} && \{\text{quotient rule}\} \\ &= \frac{-xe^x - e^x(3x+3-x^2-x)}{x^2e^{2x}} \\ &= \frac{-xe^x - 2xe^x - 3e^x + x^2e^x}{x^2e^{2x}} \\ &= \frac{e^x(x^2 - 3x - 3)}{x^2e^{2x}} \\ &= \frac{x^2 - 3x - 3}{x^2e^x} \end{aligned}$$

$$\begin{aligned} \therefore \frac{d^2y}{dx^2} &= \frac{(2x-3)(x^2e^x) - (x^2-3x-3)(2xe^x + x^2e^x)}{x^4e^{2x}} && \{\text{quotient rule}\} \\ &= \frac{2x^3e^x - 3x^2e^x - (2x^3e^x + x^4e^x - 6x^2e^x - 3x^3e^x - 6xe^x - 3x^2e^x)}{x^4e^{2x}} \\ &= \frac{\cancel{2x^3e^x} - \cancel{3x^2e^x} - \cancel{2x^3e^x} - x^4e^x + 6x^2e^x + 3x^3e^x + 6xe^x + \cancel{3x^2e^x}}{x^4e^{2x}} \\ &= \frac{-xe^x(x^3 - 3x^2 - 6x - 6)}{x^4e^{2x}} \\ &= -\frac{x^3 - 3x^2 - 6x - 6}{x^3e^x} \end{aligned}$$

3 a $f(x) = x^3 - 2x + 5$

$$\begin{aligned} \therefore f(2) &= (2)^3 - 2(2) + 5 \\ &= 9 \end{aligned}$$

b $f(x) = x^3 - 2x + 5$

$$\begin{aligned} \therefore f'(x) &= 3x^2 - 2 \\ \therefore f'(2) &= 3(2)^2 - 2 \\ &= 10 \end{aligned}$$

$$\begin{aligned}
 \text{c} \quad f'(x) &= 3x^2 - 2 \quad \{\text{from b}\} \\
 \therefore f''(x) &= 6x \\
 \therefore f''(2) &= 6(2) \\
 &= 12
 \end{aligned}$$

$$\begin{aligned}
 4 \quad \text{a} \quad f(x) &= 2x^3 - 6x^2 + 5x + 1 \\
 \therefore f'(x) &= 6x^2 - 12x + 5 \\
 \therefore f''(x) &= 12x - 12 \\
 f''(x) = 0 \quad \text{when} \quad 12x - 12 &= 0 \\
 \therefore 12x &= 12 \\
 \therefore x &= 1
 \end{aligned}$$

$$\begin{aligned}
 \text{b} \quad f(x) &= x^4 - 10x^3 + 36x^2 - 72x + 108 \\
 \therefore f'(x) &= 4x^3 - 30x^2 + 72x - 72 \\
 \therefore f''(x) &= 12x^2 - 60x + 72 \\
 f''(x) = 0 \quad \text{when} \quad 12x^2 - 60x + 72 &= 0 \\
 \therefore x^2 - 5x + 6 &= 0 \\
 \therefore (x - 2)(x - 3) &= 0 \\
 \therefore x &= 2 \quad \text{or} \quad x = 3
 \end{aligned}$$

$$\begin{aligned}
 5 \quad f(x) &= 2x^3 - x & f(0) &= 2(0)^3 - 0 = 0 & \therefore 0 \\
 \therefore f'(x) &= 6x^2 - 1 & f'(0) &= 6(0)^2 - 1 = -1 & \therefore - \\
 \therefore f''(x) &= 12x & f''(0) &= 12(0) = 0 & \therefore 0 \\
 f(-1) &= 2(-1)^3 - (-1) = -1 & \therefore - & f(1) &= 2(1)^3 - 1 = 1 & \therefore + \\
 f'(-1) &= 6(-1)^2 - 1 = 5 & \therefore + & f'(1) &= 6(1)^2 - 1 = 5 & \therefore + \\
 f''(-1) &= 12(-1) = -12 & \therefore - & f''(1) &= 12(1) = 12 & \therefore +
 \end{aligned}$$

We can fill in the table as follows:

x	-1	0	1
$f(x)$	-	0	+
$f'(x)$	+	-	+
$f''(x)$	-	0	+

$$\begin{aligned}
 6 \quad \text{a} \quad f(x) &= x^2 - \frac{1}{x} & \text{b} \quad f(x) &= x^2 - \frac{1}{x} & \text{c} \quad f'(x) &= 2x + x^{-2} \quad \{\text{from b}\} \\
 \therefore f(1) &= (1)^2 - \frac{1}{1} & &= x^2 - x^{-1} & \therefore f''(x) &= 2 - 2x^{-3} \\
 &= 1 - 1 & \therefore f'(x) &= 2x - (-x^{-2}) & &= 2 - \frac{2}{x^3} \\
 &= 0 & &= 2x + x^{-2} & \therefore f''(1) &= 2 - \frac{2}{1^3} \\
 & & &= 2x + \frac{1}{x^2} & &= 2 - 2 \\
 & & \therefore f'(1) &= 2(1) + \frac{1}{1^2} & &= 0 \\
 & & &= 2 + 1 & & \\
 & & &= 3 & &
 \end{aligned}$$

$$\begin{array}{lll}
 \text{7 a} & f(x) = 3e^x - 2x & \text{b} \quad f(x) = 3e^x - 2x \quad \text{c} \quad f'(x) = 3e^x - 2 \quad \{\text{from b}\} \\
 & \therefore f(1) = 3e^1 - 2(1) & \therefore f'(x) = 3e^x - 2 \quad \therefore f''(x) = 3e^x \\
 & = 3e - 2 & \therefore f'(1) = 3e - 2 \quad \therefore f''(1) = 3e
 \end{array}$$

$$\begin{array}{ll}
 \text{8 a} & y = x \sin x \\
 & \therefore \frac{dy}{dx} = \sin x + x(\cos x) \quad \{\text{product rule}\} \\
 & = \sin x + x \cos x \\
 & \therefore \frac{d^2y}{dx^2} = \cos x + \cos x + x(-\sin x) \quad \{\text{product rule}\} \\
 & = 2 \cos x - x \sin x
 \end{array}$$

$$\begin{array}{ll}
 \text{b} & y = \frac{\cos^2 x - x}{x^2} \text{ is a quotient with } u = \cos^2 x - x \quad \text{and } v = x^2 \\
 & \therefore u' = 2 \cos x(-\sin x) - 1 \quad \text{and } v' = 2x \\
 & = -2 \cos x \sin x - 1 \\
 & = -\sin 2x - 1
 \end{array}$$

$$\begin{array}{ll}
 \therefore \frac{dy}{dx} = \frac{(-\sin 2x - 1)(x^2) - (\cos^2 x - x)(2x)}{x^4} & \{\text{quotient rule}\} \\
 = \frac{-x^2 \sin 2x - x^2 - 2x \cos^2 x + 2x^2}{x^4} \\
 = \frac{x(-2 \cos^2 x - x \sin 2x + x)}{x^4} \\
 = \frac{-2 \cos^2 x - x \sin 2x + x}{x^3}
 \end{array}$$

which is a quotient with $u = -2 \cos^2 x - x \sin 2x + x$

$$\begin{array}{l}
 \therefore u' = -4 \cos x(-\sin x) - \sin 2x - 2x \cos 2x + 1 \\
 = 4 \sin x \cos x - \sin 2x - 2x \cos 2x + 1 \\
 = 2 \sin 2x - \sin 2x - 2x \cos 2x + 1 \\
 = \sin 2x - 2x \cos 2x + 1
 \end{array}$$

$$\begin{array}{l}
 \text{and } v = x^3 \\
 \therefore v' = 3x^2
 \end{array}$$

$$\begin{array}{ll}
 \therefore \frac{d^2y}{dx^2} = \frac{(\sin 2x - 2x \cos 2x + 1)(x^3) - (-2 \cos^2 x - x \sin 2x + x)(3x^2)}{x^6} & \{\text{quotient rule}\} \\
 = \frac{x^3 \sin 2x - 2x^4 \cos 2x + x^3 - (-6x^2 \cos^2 x - 3x^3 \sin 2x + 3x^3)}{x^6} \\
 = \frac{x^3 \sin 2x - 2x^4 \cos 2x + x^3 + 6x^2 \cos^2 x + 3x^3 \sin 2x - 3x^3}{x^6} \\
 = \frac{6x^2 \cos^2 x + 4x^3 \sin 2x - 2x^3 - 2x^4 \cos 2x}{x^6} \\
 = \frac{6 \cos^2 x}{x^4} + \frac{4 \sin 2x - 2}{x^3} - \frac{2 \cos 2x}{x^2}
 \end{array}$$

$$\begin{aligned}
 \text{c} \quad y &= e^{-x} \sin x \\
 \therefore \frac{dy}{dx} &= (-e^{-x}) \sin x + e^{-x}(\cos x) \quad \{\text{product rule}\} \\
 &= -e^{-x} \sin x + e^{-x} \cos x \\
 \therefore \frac{d^2y}{dx^2} &= -(-e^{-x} \sin x + e^{-x} \cos x) + (-e^{-x}) \cos x + e^{-x}(-\sin x) \quad \{\text{product rule}\} \\
 &= \cancel{e^{-x} \sin x} - e^{-x} \cos x - e^{-x} \cos x - \cancel{e^{-x} \sin x} \\
 &= -2e^{-x} \cos x
 \end{aligned}$$

$$9 \quad y = Ae^{kx}$$

$$\begin{aligned}
 \text{a} \quad \frac{dy}{dx} &= Ae^{kx}(k) \\
 &= k(Ae^{kx}) \\
 &= ky \\
 \text{b} \quad \frac{d^2y}{dx^2} &= \frac{d}{dx}(kAe^{kx}) \quad \{\text{from a}\} \\
 &= kAe^{kx}(k) \\
 &= k^2 Ae^{kx} \\
 &= k^2 y
 \end{aligned}$$

$$\begin{aligned}
 10 \quad \text{a} \quad f(x) &= 2 \sin^3 x - 3 \sin x \\
 &= 2(\sin x)^3 - 3 \sin x \\
 \therefore f'(x) &= 2 \times 3(\sin x)^2 \times (\cos x) - 3 \cos x \quad \{\text{chain rule}\} \\
 &= 3 \cos x(2 \sin^2 x - 1) \\
 &= -3 \cos x(1 - 2 \sin^2 x) \\
 &= -3 \cos x \cos 2x \quad \{\text{using } 1 - 2 \sin^2 x = \cos 2x\}
 \end{aligned}$$

$$\begin{aligned}
 \text{b} \quad f''(x) &= -3(-\sin x \times \cos 2x + \cos x \times (-\sin 2x) \times 2) \\
 &= 3 \sin x \cos 2x + 6 \cos x \sin 2x
 \end{aligned}$$

$$\begin{aligned}
 11 \quad f(x) &= \frac{2}{3} \sin 3x \\
 \therefore f'(x) &= \frac{2}{3}(\cos 3x) \times 3 \\
 &= 2 \cos 3x \\
 \therefore f''(x) &= 2(-\sin 3x) \times 3 \\
 &= -6 \sin 3x \\
 \therefore f''\left(\frac{2\pi}{9}\right) &= -6 \times \sin\left(3 \times \frac{2\pi}{9}\right) \\
 &= -6 \times \sin \frac{2\pi}{3} \\
 &= -6 \times \frac{\sqrt{3}}{2} \\
 &= -3\sqrt{3}
 \end{aligned}$$

$$\begin{aligned}
 12 \quad \text{a} \quad y &= -\ln x \\
 \therefore \frac{dy}{dx} &= -1 \times \frac{1}{x} \\
 &= -x^{-1} \\
 \therefore \frac{d^2y}{dx^2} &= -(-x^{-2}) \\
 &= x^{-2} = \frac{1}{x^2} \\
 \text{b} \quad y &= x \ln x \\
 \therefore \frac{dy}{dx} &= 1 \times \ln x + x \times \frac{1}{x} \quad \{\text{product rule}\} \\
 &= \ln x + 1 \\
 \therefore \frac{d^2y}{dx^2} &= \frac{1}{x}
 \end{aligned}$$

$$\begin{aligned} \text{c} \quad y &= (\ln x)^2 \\ \therefore \frac{dy}{dx} &= 2(\ln x) \left(\frac{1}{x} \right) = \frac{2 \ln x}{x} \quad \text{which is a quotient with } u = 2 \ln x \quad \text{and} \quad v = x \\ &\qquad\qquad\qquad \therefore u' = \frac{2}{x} \quad \text{and} \quad v' = 1 \end{aligned}$$

$$\begin{aligned} \text{Now } \frac{d^2y}{dx^2} &= \frac{\frac{2}{x} \times x - 2 \ln x \times 1}{x^2} \quad \{\text{quotient rule}\} \\ &= \frac{2 - 2 \ln x}{x^2} = \frac{2}{x^2} (1 - \ln x) \end{aligned}$$

$$\text{13 } y = 2e^{3x} + 5e^{4x}$$

$$\begin{aligned} \therefore \frac{dy}{dx} &= 2e^{3x}(3) + 5e^{4x}(4) \quad \text{and} \quad \frac{d^2y}{dx^2} = 6e^{3x}(3) + 20e^{4x}(4) \\ &= 6e^{3x} + 20e^{4x} \qquad\qquad\qquad = 18e^{3x} + 80e^{4x} \end{aligned}$$

$$\begin{aligned} \text{Now } \frac{d^2y}{dx^2} - 7 \frac{dy}{dx} + 12y &= (18e^{3x} + 80e^{4x}) - 7(6e^{3x} + 20e^{4x}) + 12(2e^{3x} + 5e^{4x}) \\ &= 18e^{3x} + 80e^{4x} - 42e^{3x} - 140e^{4x} + 24e^{3x} + 60e^{4x} \\ &= e^{3x}(18 - 42 + 24) + e^{4x}(80 - 140 + 60) \\ &= e^{3x}(0) + e^{4x}(0) \\ &= 0 \end{aligned}$$

$$\therefore \frac{d^2y}{dx^2} - 7 \frac{dy}{dx} + 12y = 0 \quad \text{as required}$$

$$\begin{aligned} \text{14 } \text{If } y &= \sin(2x + 3), \text{ then } \frac{dy}{dx} = (\cos(2x + 3))(2) \quad \text{and} \quad \frac{d^2y}{dx^2} = (-2 \sin(2x + 3))(2) \\ &\qquad\qquad\qquad = 2 \cos(2x + 3) \qquad\qquad\qquad = -4 \sin(2x + 3) \end{aligned}$$

$$\therefore \frac{d^2y}{dx^2} + 4y = -4 \sin(2x + 3) + 4 \sin(2x + 3) = 0 \quad \text{as required}$$

$$\text{15 } y = 2 \sin x + 3 \cos x$$

$$\therefore \frac{dy}{dx} = 2 \cos x - 3 \sin x$$

$$\therefore \frac{d^2y}{dx^2} = -2 \sin x - 3 \cos x$$

$$\begin{aligned} \text{Now } \frac{d^2y}{dx^2} + y &= (-2 \sin x - 3 \cos x) + (2 \sin x + 3 \cos x) \\ &= 0 \quad \text{as required} \end{aligned}$$

REVIEW SET 12A

$$\begin{aligned} \text{1 a } f(x) &= 5x^3 \\ \therefore f'(x) &= 15x^2 \end{aligned}$$

$$\begin{aligned} \text{c } f(x) &= 7x^2 - \frac{3}{x} \\ &= 7x^2 - 3x^{-1} \\ \therefore f'(x) &= 7(2x) - 3(-x^{-2}) \\ &= 14x + 3x^{-2} \\ &= 14x + \frac{3}{x^2} \end{aligned}$$

$$\begin{aligned} \text{b } f(x) &= x^6 - 5x \\ \therefore f'(x) &= 6x^5 - 5 \end{aligned}$$

$$\begin{aligned} \text{d } f(x) &= 3x - \frac{4}{x^2} \\ &= 3x - 4x^{-2} \\ \therefore f'(x) &= 3 - 4(-2x^{-3}) \\ &= 3 + 8x^{-3} \\ &= 3 + \frac{8}{x^3} \end{aligned}$$

$$\begin{aligned}
 \text{e} \quad f(x) &= 2x\sqrt{x} = 2x^{\frac{3}{2}} \\
 \therefore f'(x) &= 2\left(\frac{3}{2}x^{\frac{1}{2}}\right) \\
 &= 3x^{\frac{1}{2}} \\
 &= 3\sqrt{x}
 \end{aligned}$$

$$\begin{aligned}
 \text{f} \quad f(x) &= 4\sqrt{x} - \frac{1}{\sqrt{x}} = 4x^{\frac{1}{2}} - x^{-\frac{1}{2}} \\
 \therefore f'(x) &= 4\left(\frac{1}{2}x^{-\frac{1}{2}}\right) - \left(-\frac{1}{2}x^{-\frac{3}{2}}\right) \\
 &= 2x^{-\frac{1}{2}} + \frac{1}{2}x^{-\frac{3}{2}} \\
 &= \frac{2}{\sqrt{x}} + \frac{1}{2x\sqrt{x}}
 \end{aligned}$$

$$\begin{aligned}
 \text{2 a} \quad y &= 3x^2 - x^4 \\
 \therefore \frac{dy}{dx} &= 3(2x) - 4x^3 \\
 &= 6x - 4x^3
 \end{aligned}$$

$$\begin{aligned}
 \text{b} \quad y &= \frac{x^3 - x}{x^2} \\
 &= x - x^{-1} \\
 \therefore \frac{dy}{dx} &= 1 - (-x^{-2}) \\
 &= 1 + x^{-2} \\
 &= 1 + \frac{1}{x^2}
 \end{aligned}$$

$$\begin{aligned}
 \text{c} \quad y &= x^2\sqrt{x-2} \quad \text{is the product of} \\
 u &= x^2 \quad \text{and} \quad v = (x-2)^{\frac{1}{2}} \\
 \therefore u' &= 2x \quad \text{and} \quad v' = \frac{1}{2}(x-2)^{-\frac{1}{2}}(1) \quad \{\text{chain rule}\} \\
 &= \frac{1}{2}(x-2)^{-\frac{1}{2}}
 \end{aligned}$$

$$\begin{aligned}
 \text{Now} \quad \frac{dy}{dx} &= u'v + uv' \quad \{\text{product rule}\} \\
 &= 2x(x-2)^{\frac{1}{2}} + x^2\left(\frac{1}{2}(x-2)^{-\frac{1}{2}}\right) \\
 &= 2x(x-2)^{\frac{1}{2}} + \frac{1}{2}x^2(x-2)^{-\frac{1}{2}} \\
 &= 2x\sqrt{x-2} + \frac{x^2}{2\sqrt{x-2}}
 \end{aligned}$$

$$\begin{aligned}
 \text{3 a} \quad f(x) &= \frac{x}{\sqrt{x^2+1}} \quad \text{is a quotient with} \quad u = x \quad \text{and} \quad v = (x^2+1)^{\frac{1}{2}} \\
 \therefore u' &= 1 \quad \text{and} \quad v' = \frac{1}{2}(x^2+1)^{-\frac{1}{2}} \times 2x \\
 &= x(x^2+1)^{-\frac{1}{2}}
 \end{aligned}$$

$$\begin{aligned}
 \text{Now} \quad f'(x) &= \frac{u'v - uv'}{v^2} \quad \{\text{quotient rule}\} \\
 &= \frac{1 \times (x^2+1)^{\frac{1}{2}} - x \times x(x^2+1)^{-\frac{1}{2}}}{\left((x^2+1)^{\frac{1}{2}}\right)^2} \\
 &= \frac{\sqrt{x^2+1} - \frac{x^2}{\sqrt{x^2+1}}}{x^2+1} \\
 &= \frac{\sqrt{x^2+1} \times \frac{\sqrt{x^2+1}}{\sqrt{x^2+1}} - \frac{x^2}{\sqrt{x^2+1}}}{x^2+1} \\
 &= \frac{(x^2+1) - x^2}{(x^2+1)\sqrt{x^2+1}} \\
 &= \frac{1}{(x^2+1)\sqrt{x^2+1}} \\
 &= (x^2+1)^{-\frac{3}{2}}
 \end{aligned}$$

- b** The tangent to $f(x)$ has gradient 1 when $f'(x) = 1$

$$\therefore (x^2 + 1)^{-\frac{3}{2}} = 1$$

$$\therefore x^2 + 1 = 1$$

$$\therefore x^2 = 0$$

$$\therefore x = 0$$

$$\text{and } f(0) = \frac{0}{\sqrt{0^2 + 1}} = 0$$

\therefore the tangent to $f(x) = \frac{x}{\sqrt{x^2 + 1}}$ has gradient 1 at the point $(0, 0)$.

4 a $y = e^{x^3+2}$
 $= e^u$ where $u = x^3 + 2$

Now $\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx}$ {chain rule}

$$= e^u \frac{du}{dx}$$

$$= e^{x^3+2} \times 3x^2$$

$$= 3x^2 e^{x^3+2}$$

b $y = \ln\left(\frac{x+3}{x^2}\right)$
 $= \ln(x+3) - \ln(x^2)$
 $\{\ln\left(\frac{a}{b}\right) = \ln a - \ln b\}$

$$\therefore \frac{dy}{dx} = \frac{1}{x+3} - \frac{2x}{x^2}$$

$$= \frac{1}{x+3} - \frac{2}{x}$$

c $y = x^3 e^{2x}$
 $\therefore \frac{dy}{dx} = 3x^2 e^{2x} + x^3 e^{2x}(2)$ {product rule}
 $= 3x^2 e^{2x} + 2x^3 e^{2x}$

5 a $f(x) = -x^2 + 4x - 2$
 $\therefore f'(x) = -2x + 4$
 At the point $(-3, -23)$,
 $f'(-3) = -2(-3) + 4$
 $= 10$

So, the gradient of the tangent is 10.

b $y = (2 - 3x)^5$
 $\therefore \frac{dy}{dx} = 5(2 - 3x)^4 \times (-3)$ {chain rule}
 $= -15(2 - 3x)^4$
 When $x = 1$, $\frac{dy}{dx} = -15(2 - 3(1))^4$
 $= -15(-1)^4$
 $= -15$

So, the gradient of the tangent is -15 .

6 a $y = 5x - 3x^{-1}$ **b** $y = (3x^2 + \sqrt{x})^4 = \left(3x^2 + x^{\frac{1}{2}}\right)^4$
 $\therefore \frac{dy}{dx} = 5 + 3x^{-2}$ $\therefore \frac{dy}{dx} = 4\left(3x^2 + x^{\frac{1}{2}}\right)^3 \left(6x + \frac{1}{2}x^{-\frac{1}{2}}\right)$ {chain rule}

c $y = (x^2 + 1)(1 - x^2)^3$ is a product with $u = x^2 + 1$ and $v = (1 - x^2)^3$
 $\therefore u' = 2x$ and $v' = 3(1 - x^2)^2(-2x)$
 $= -6x(1 - x^2)^2$

$$\therefore \frac{dy}{dx} = 2x(1 - x^2)^3 + (x^2 + 1)[-6x(1 - x^2)^2]$$
 {product rule}
 $= 2x(1 - x^2)^3 - 6x(x^2 + 1)(1 - x^2)^2$

$$7 \quad y = 2x^3 + 3x^2 - 10x + 3$$

$$\therefore \frac{dy}{dx} = 6x^2 + 6x - 10$$

The gradient of the tangent is 2 when $6x^2 + 6x - 10 = 2$

$$\therefore 6x^2 + 6x - 12 = 0$$

$$\therefore x^2 + x - 2 = 0$$

$$\therefore (x+2)(x-1) = 0$$

$$\therefore x = -2 \text{ or } 1$$

$$\begin{aligned} \text{When } x = -2, \quad y &= 2(-2)^3 + 3(-2)^2 - 10(-2) + 3 \\ &= 19 \end{aligned}$$

$$\begin{aligned} \text{When } x = 1, \quad y &= 2(1)^3 + 3(1)^2 - 10(1) + 3 \\ &= -2 \end{aligned}$$

So, the gradient of the tangent to $y = 2x^3 + 3x^2 - 10x + 3$ is 2 at the points $(-2, 19)$ and $(1, -2)$.

$$\begin{aligned} 8 \quad a \quad & \frac{d}{dx}(\sin 5x \ln x) \\ &= (\cos 5x)(5) \ln x + \sin 5x \left(\frac{1}{x} \right) \\ & \quad \text{\{product rule\}} \\ &= (5 \cos 5x) \ln x + \frac{\sin 5x}{x} \end{aligned}$$

$$\begin{aligned} b \quad & \frac{d}{dx}(\sin x \cos 2x) \\ &= \cos x \cos 2x + \sin x(-\sin 2x)(2) \\ & \quad \text{\{product rule\}} \\ &= \cos x \cos 2x - 2 \sin x \sin 2x \end{aligned}$$

$$\begin{aligned} c \quad & \frac{d}{dx}(e^{-x} \cos x) \\ &= (-e^{-x})(\cos x) + (e^{-x})(-\sin x) \quad \text{\{product rule\}} \\ &= -e^{-x} \cos x - e^{-x} \sin x \end{aligned}$$

$$\begin{aligned} 9 \quad & y = \sin^2 x \\ &= (\sin x)^2 \\ \therefore \frac{dy}{dx} &= 2 \sin x \cos x \quad \text{\{chain rule\}} \end{aligned}$$

$$\begin{aligned} \text{When } x = \frac{\pi}{3}, \quad \frac{dy}{dx} &= 2 \sin \frac{\pi}{3} \cos \frac{\pi}{3} \\ &= \frac{\sqrt{3}}{2} \end{aligned}$$

$$\therefore \text{gradient of tangent} = \frac{\sqrt{3}}{2}$$

$$\begin{aligned} 10 \quad & y = 3e^x - e^{-x} \\ \therefore \frac{dy}{dx} &= 3e^x + e^{-x} \\ \therefore \frac{d^2y}{dx^2} &= 3e^x - e^{-x} = y \quad \text{as required} \end{aligned}$$

11 a $f(x) = \frac{x^2 - 4x - 1}{e^x}$

$$\begin{aligned}\therefore f'(x) &= \frac{(2x - 4)e^x - (x^2 - 4x - 1)e^x}{(e^x)^2} && \{\text{quotient rule}\} \\ &= \frac{e^x(2x - 4 - x^2 + 4x + 1)}{(e^x)^2} \\ &= \frac{-x^2 + 6x - 3}{e^x}\end{aligned}$$

b $f'(1) = \frac{-1^2 + 6(1) - 3}{e^1}$
 $= \frac{2}{e}$

\therefore gradient of tangent $= \frac{2}{e}$

c The tangent to $y = f(x)$ is horizontal when $f'(x) = 0$.

$$\therefore -x^2 + 6x - 3 = 0$$

$$\therefore x^2 - 6x + 3 = 0$$

$$\begin{aligned}\therefore x &= \frac{6 \pm \sqrt{(-6)^2 - 4(1)(3)}}{2(1)} \\ &= \frac{6 \pm \sqrt{24}}{2} \\ &= 3 \pm \sqrt{6}\end{aligned}$$

12 a $f(x) = (x^2 + 3)^4$

$$\begin{aligned}\therefore f'(x) &= 4(x^2 + 3)^3(2x) && \{\text{chain rule}\} \\ &= 8x(x^2 + 3)^3\end{aligned}$$

b $g(x) = \frac{\sqrt{x+5}}{x^2}$ is a quotient with $u = (x+5)^{\frac{1}{2}}$ and $v = x^2$

$$\therefore u' = \frac{1}{2}(x+5)^{-\frac{1}{2}} \quad \text{and} \quad v' = 2x$$

$$\begin{aligned}\therefore g'(x) &= \frac{\frac{1}{2}(x+5)^{-\frac{1}{2}}(x^2) - (x+5)^{\frac{1}{2}}(2x)}{(x^2)^2} && \{\text{quotient rule}\} \\ &= \frac{\frac{1}{2}x(x+5)^{-\frac{1}{2}} - 2(x+5)^{\frac{1}{2}}}{x^3}\end{aligned}$$

c $h(x) = \frac{e^{4x}}{1-2x}$ is a quotient with $u = e^{4x}$ and $v = 1-2x$
 $\therefore u' = 4e^{4x}$ and $v' = -2$

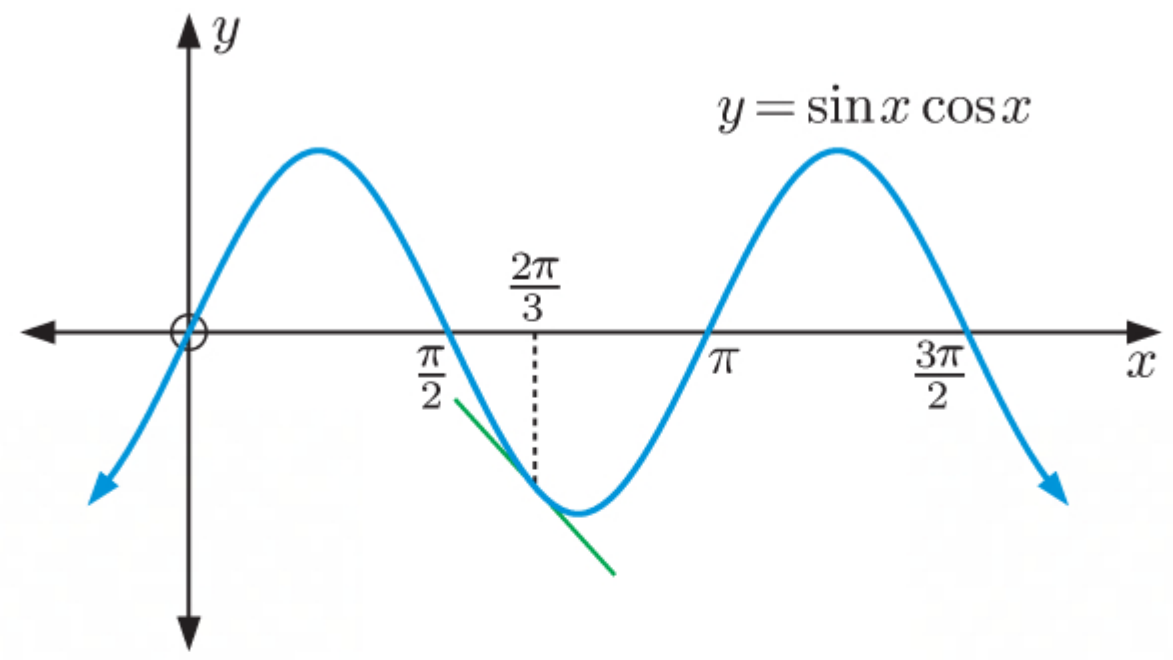
$$\begin{aligned}\therefore h'(x) &= \frac{4e^{4x}(1-2x) - e^{4x}(-2)}{(1-2x)^2} && \{\text{quotient rule}\} \\ &= \frac{4e^{4x} - 8xe^{4x} + 2e^{4x}}{(1-2x)^2} \\ &= \frac{6e^{4x} - 8xe^{4x}}{(1-2x)^2}\end{aligned}$$

13 a $y = \sin x \cos x$

$$\begin{aligned}\therefore \frac{dy}{dx} &= \cos x \cos x + \sin x(-\sin x) \\ &\quad \{\text{product rule}\} \\ &= \cos^2 x - \sin^2 x\end{aligned}$$

b $y = \frac{1}{2} \sin 2x$

$$\begin{aligned}\therefore \frac{dy}{dx} &= \left(\frac{1}{2} \cos 2x\right)(2) \\ &= \cos 2x \\ &= \cos^2 x - \sin^2 x \quad \{\text{double angle formula}\} \\ &\text{which is the same derivative as in a } \checkmark\end{aligned}$$



- c** The tangent meets the graph of $y = \sin x \cos x$ at the point where $x = \frac{2\pi}{3}$.

$$\begin{aligned}\text{When } x = \frac{2\pi}{3}, \quad \frac{dy}{dx} &= \cos^2\left(\frac{2\pi}{3}\right) - \sin^2\left(\frac{2\pi}{3}\right) \\ &= \frac{1}{4} - \frac{3}{4} \\ &= -\frac{1}{2} \\ \therefore \text{gradient of tangent} &= -\frac{1}{2}\end{aligned}$$

14 a $f(x) = 2 \sin x + \cos 2x$

$$\begin{aligned}\therefore f\left(\frac{\pi}{2}\right) &= 2 \sin \frac{\pi}{2} + \cos\left(2 \times \frac{\pi}{2}\right) \\ &= 2(1) + (-1) \\ &= 1\end{aligned}$$

b $f'(x) = 2 \cos x - (\sin 2x)(2)$

$$\begin{aligned}&= 2 \cos x - 2 \sin 2x \\ \therefore f'\left(\frac{\pi}{2}\right) &= 2 \cos \frac{\pi}{2} - 2 \sin\left(2 \times \frac{\pi}{2}\right) \\ &= 2(0) - 2(0) \\ &= 0\end{aligned}$$

c $f'(x) = 2 \cos x - 2 \sin 2x$ {from **b**}

$$\begin{aligned}\therefore f''(x) &= -2 \sin x - (2 \cos 2x)(2) \\ &= -2 \sin x - 4 \cos 2x \\ \therefore f''\left(\frac{\pi}{2}\right) &= -2 \sin \frac{\pi}{2} - 4 \cos\left(2 \times \frac{\pi}{2}\right) \\ &= -2(1) - 4(-1) \\ &= 2\end{aligned}$$

15 a $y = \frac{1}{8}x^4 + \frac{1}{6}x^3 - \frac{1}{4}x^2$

$$\begin{aligned}\therefore \frac{dy}{dx} &= \frac{1}{2}x^3 + \frac{1}{2}x^2 - \frac{1}{2}x \\ \therefore \frac{d^2y}{dx^2} &= \frac{3}{2}x^2 + x - \frac{1}{2}\end{aligned}$$

b $y = xe^{-x}$

$$\begin{aligned}\therefore \frac{dy}{dx} &= (1)e^{-x} + x(e^{-x})(-1) \quad \{\text{product rule}\} \\ &= e^{-x} - xe^{-x} \\ \therefore \frac{d^2y}{dx^2} &= e^{-x}(-1) - [(1)e^{-x} + xe^{-x}(-1)] \\ &= -e^{-x} - e^{-x} + xe^{-x} \\ &= -2e^{-x} + xe^{-x}\end{aligned}$$

16 $f(x) = \sqrt{x} \cos 4x = x^{\frac{1}{2}} \cos 4x$

a $f'(x) = \frac{1}{2}x^{-\frac{1}{2}} \cos 4x + x^{\frac{1}{2}} \times (-\sin 4x) \times 4 \quad \{\text{product rule}\}$

$$= \frac{1}{2}x^{-\frac{1}{2}} \cos 4x - 4x^{\frac{1}{2}} \sin 4x$$

$$= \frac{1}{2\sqrt{x}} \cos 4x - 4\sqrt{x} \sin 4x$$

$\therefore f''(x) = -\frac{1}{4}x^{-\frac{3}{2}} \cos 4x + \frac{1}{2}x^{-\frac{1}{2}} \times (-\sin 4x) \times 4 - [2x^{-\frac{1}{2}} \sin 4x + 4x^{\frac{1}{2}} \times (\cos 4x) \times 4]$
{product rule}

$$= -\frac{1}{4}x^{-\frac{3}{2}} \cos 4x - 2x^{-\frac{1}{2}} \sin 4x - 2x^{-\frac{1}{2}} \sin 4x - 16x^{\frac{1}{2}} \cos 4x$$

$$= -\frac{1}{4}x^{-\frac{3}{2}} \cos 4x - 4x^{-\frac{1}{2}} \sin 4x - 16x^{\frac{1}{2}} \cos 4x$$

$$= -\frac{1}{4x\sqrt{x}} \cos 4x - \frac{4}{\sqrt{x}} \sin 4x - 16\sqrt{x} \cos 4x$$

b i $f'\left(\frac{\pi}{16}\right) = \frac{1}{2\sqrt{\frac{\pi}{16}}} \cos\left(4 \times \frac{\pi}{16}\right) - 4\sqrt{\frac{\pi}{16}} \sin\left(4 \times \frac{\pi}{16}\right)$

$$= \frac{1}{2 \times \frac{\sqrt{\pi}}{4}} \cos \frac{\pi}{4} - 4 \times \frac{\sqrt{\pi}}{4} \sin \frac{\pi}{4}$$

$$= \frac{1}{\frac{\sqrt{\pi}}{2}} \times \frac{1}{\sqrt{2}} - \sqrt{\pi} \times \frac{1}{\sqrt{2}}$$

$$= \frac{2}{\sqrt{\pi}} \times \frac{1}{\sqrt{2}} - \sqrt{\pi} \times \frac{1}{\sqrt{2}}$$

$$= \frac{1}{\sqrt{2}} \left(\frac{2}{\sqrt{\pi}} - \sqrt{\pi} \right)$$

ii $f''\left(\frac{\pi}{8}\right) = -\frac{1}{4 \times \frac{\pi}{8} \times \sqrt{\frac{\pi}{8}}} \cos\left(4 \times \frac{\pi}{8}\right) - \frac{4}{\sqrt{\frac{\pi}{8}}} \sin\left(4 \times \frac{\pi}{8}\right) - 16\sqrt{\frac{\pi}{8}} \cos\left(4 \times \frac{\pi}{8}\right)$

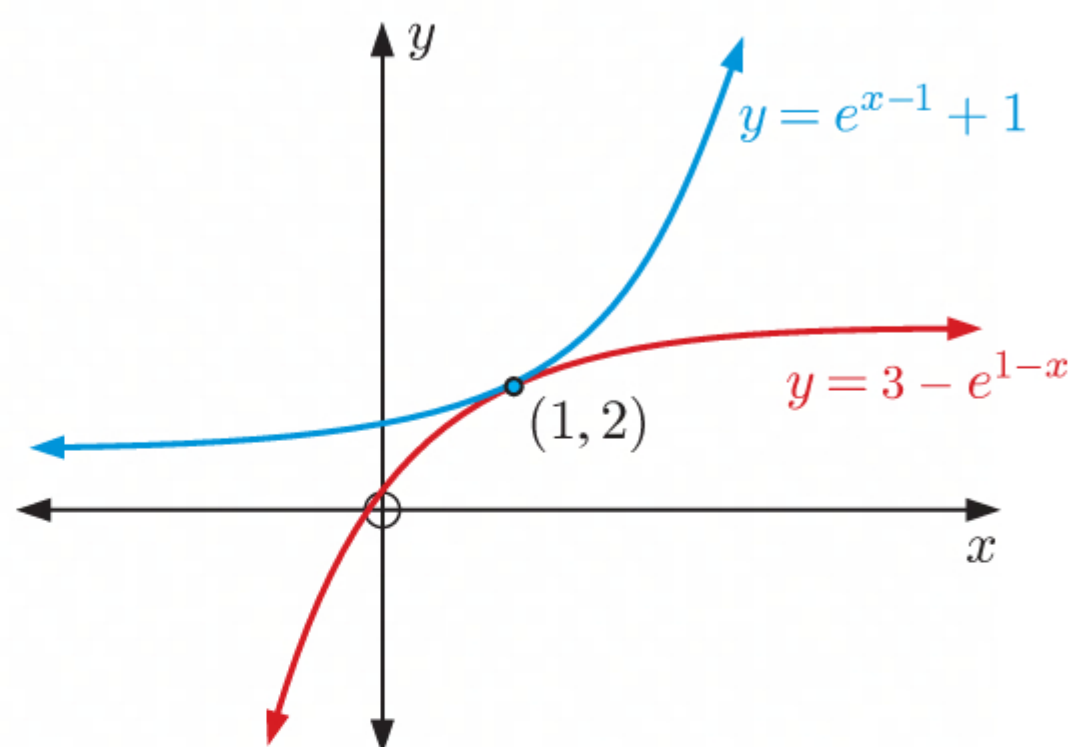
$$= -\frac{1}{\frac{\pi}{2} \times \frac{\sqrt{\pi}}{\sqrt{8}}} \cos \frac{\pi}{2} - \frac{4}{\frac{\sqrt{\pi}}{\sqrt{8}}} \sin \frac{\pi}{2} - 16\frac{\sqrt{\pi}}{\sqrt{8}} \cos \frac{\pi}{2}$$

$$= -\frac{1}{\frac{\pi}{2} \times \frac{\sqrt{\pi}}{\sqrt{8}}} (0) - 4 \times \frac{\sqrt{8}}{\sqrt{\pi}} (1) - 16\frac{\sqrt{\pi}}{\sqrt{8}} (0)$$

$$= -4 \times \frac{2\sqrt{2}}{\sqrt{\pi}}$$

$$= -\frac{8\sqrt{2}}{\sqrt{\pi}}$$

17 a

b The graphs intersect when $e^{x-1} + 1 = 3 - e^{1-x}$

$$\therefore e^{x-1} - 2 + e^{1-x} = 0$$

$$\therefore \frac{1}{e} \times e^x - 2 + e^1 \times e^{-x} = 0$$

$$\therefore e^x - 2e + e^2 \times e^{-x} = 0$$

$$\therefore e^{2x} - 2e \times e^x + e^2 = 0$$

$$\therefore (e^x - e)^2 = 0$$

$$\therefore e^x = e$$

$$\therefore x = 1$$

Substituting $x = 1$ into $y = e^{x-1} + 1$ gives

$$= e^0 + 1$$

$$= 1 + 1$$

$$= 2$$

So, the curves intersect at $(1, 2)$.c If $y = e^{x-1} + 1$,

$$\begin{aligned} \frac{dy}{dx} &= e^{x-1} \times 1 \quad \{\text{chain rule}\} \\ &= e^{x-1} \end{aligned}$$

$$\begin{aligned} \text{When } x = 1, \quad \frac{dy}{dx} &= e^0 \\ &= 1 \end{aligned}$$

If $y = 3 - e^{1-x}$,

$$\begin{aligned} \frac{dy}{dx} &= -e^{1-x} \times (-1) \quad \{\text{chain rule}\} \\ &= e^{1-x} \end{aligned}$$

$$\begin{aligned} \text{When } x = 1, \quad \frac{dy}{dx} &= e^0 \\ &= 1 \end{aligned}$$

The gradient of both curves at $(1, 2)$ is $e^0 = 1$. \therefore the tangents to each of the curves at this point are the same line.

REVIEW SET 12B

$$\begin{aligned} \mathbf{1} \quad \mathbf{a} \quad f(x) &= 3x^2 - 7x + 4 \\ \therefore f'(x) &= 3(2x) - 7(1) \\ &= 6x - 7 \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad f(x) &= (x + 5)^2 \\ &= x^2 + 10x + 25 \\ \therefore f'(x) &= 2x + 10(1) \\ &= 2x + 10 \end{aligned}$$

$$\begin{aligned} \text{c} \quad f(x) &= 2\sqrt{x} - \frac{3}{x} \\ &= 2x^{\frac{1}{2}} - 3x^{-1} \end{aligned}$$

$$\begin{aligned} \therefore f'(x) &= 2\left(\frac{1}{2}x^{-\frac{1}{2}}\right) - 3(-x^{-2}) \\ &= x^{-\frac{1}{2}} + 3x^{-2} \\ &= \frac{1}{\sqrt{x}} + \frac{3}{x^2} \end{aligned}$$

$$\begin{aligned} \text{d} \quad f(x) &= 6x^2\sqrt{x} \\ &= 6x^{\frac{5}{2}} \end{aligned}$$

$$\begin{aligned} \therefore f'(x) &= 6\left(\frac{5}{2}x^{\frac{3}{2}}\right) \\ &= 15x^{\frac{3}{2}} \\ &= 15x\sqrt{x} \end{aligned}$$

$$\begin{aligned} \text{2 a} \quad y &= 2x^3 - 6x^2 + 7x - 4 \\ \therefore \frac{dy}{dx} &= 2(3x^2) - 6(2x) + 7(1) \\ &= 6x^2 - 12x + 7 \end{aligned}$$

$$\begin{aligned} \text{c} \quad y &= \frac{15}{\sqrt[3]{x}} \\ &= 15x^{-\frac{1}{3}} \\ \therefore \frac{dy}{dx} &= 15\left(-\frac{1}{3}x^{-\frac{4}{3}}\right) \\ &= -5x^{-\frac{4}{3}} \\ &= -\frac{5}{x^{\frac{4}{3}}} \end{aligned}$$

$$\begin{aligned} \text{b} \quad y &= \frac{3}{x} - \frac{5}{x^3} \\ &= 3x^{-1} - 5x^{-3} \\ \therefore \frac{dy}{dx} &= 3(-x^{-2}) - 5(-3x^{-4}) \\ &= -3x^{-2} + 15x^{-4} \\ &= -\frac{3}{x^2} + \frac{15}{x^4} \end{aligned}$$

$$\begin{aligned} \text{3 a} \quad f(x) &= 7 + x - 3x^2 \\ \therefore f(3) &= 7 + 3 - 3(3)^2 \\ &= 7 + 3 - 27 \\ &= -17 \end{aligned}$$

$$\begin{aligned} \text{b} \quad f(x) &= 7 + x - 3x^2 \\ \therefore f'(x) &= 1 - 6x \\ \therefore f'(3) &= 1 - 6(3) \\ &= 1 - 18 \\ &= -17 \end{aligned}$$

$$\begin{aligned} \text{c} \quad f'(x) &= 1 - 6x \quad \{\text{from b}\} \\ f''(x) &= -6 \\ \therefore f''(3) &= -6 \end{aligned}$$

$$\begin{aligned} \text{4 a} \quad y &= x^3\sqrt{1-x^2} \text{ is the product of} \\ u &= x^3 \quad \text{and} \quad v = (1-x^2)^{\frac{1}{2}} \\ \therefore u' &= 3x^2 \quad \text{and} \quad v' = \frac{1}{2}(1-x^2)^{-\frac{1}{2}} \times (-2x) \quad \{\text{chain rule}\} \\ &= -x(1-x^2)^{-\frac{1}{2}} \end{aligned}$$

$$\begin{aligned} \text{Now } \frac{dy}{dx} &= u'v + uv' \quad \{\text{product rule}\} \\ &= 3x^2(1-x^2)^{\frac{1}{2}} + x^3 \times \left[-x(1-x^2)^{-\frac{1}{2}}\right] \\ &= 3x^2(1-x^2)^{\frac{1}{2}} - x^4(1-x^2)^{-\frac{1}{2}} \end{aligned}$$

b $y = \frac{x^2 - 3x}{\sqrt{x+1}}$ is a quotient with $u = x^2 - 3x$ and $v = (x+1)^{\frac{1}{2}}$
 $\therefore u' = 2x - 3$ and $v' = \frac{1}{2}(x+1)^{-\frac{1}{2}}$

Now $\frac{dy}{dx} = \frac{u'v - uv'}{v^2}$ {quotient rule}

$$= \frac{(2x-3)(x+1)^{\frac{1}{2}} - (x^2-3x) \times \frac{1}{2}(x+1)^{-\frac{1}{2}}}{(\sqrt{x+1})^2}$$

$$= \frac{(2x-3)(x+1)^{\frac{1}{2}} - \frac{1}{2}(x^2-3x)(x+1)^{-\frac{1}{2}}}{x+1}$$

5 a $y = xe^x$ is the product of $u = x$ and $v = e^x$
 $\therefore u' = 1$ and $v' = e^x$

Now $\frac{dy}{dx} = u'v + uv'$ {product rule}

$$= 1 \times e^x + x \times e^x$$

$$= e^x + xe^x$$

b $\frac{dy}{dx} = e^x + xe^x = (1+x)e^x$ {from **a**}

$\frac{dy}{dx} = 2e$ when $(1+x)e^x = 2e$

Solving by inspection, we find $x = 1$.

When $x = 1$, $y = 1 \times e^1 = e$.

\therefore the gradient of $y = xe^x$ is $2e$ at the point $(1, e)$.

6 a $f(x) = \ln(e^x + 3)$

$\therefore f'(x) = \frac{e^x}{e^x + 3}$

b $f(x) = \ln \left[\frac{(x+2)^3}{x} \right]$

$= \ln(x+2)^3 - \ln x$ { $\ln \left(\frac{a}{b} \right) = \ln a - \ln b$ }

$= 3 \ln(x+2) - \ln x$ { $\ln a^n = n \ln a$ }

$\therefore f'(x) = \frac{3}{x+2} - \frac{1}{x}$

7 a $f(x) = \frac{x^2 + 2}{x}$

$= x + 2x^{-1}$

$\therefore f'(x) = 1 - 2x^{-2}$

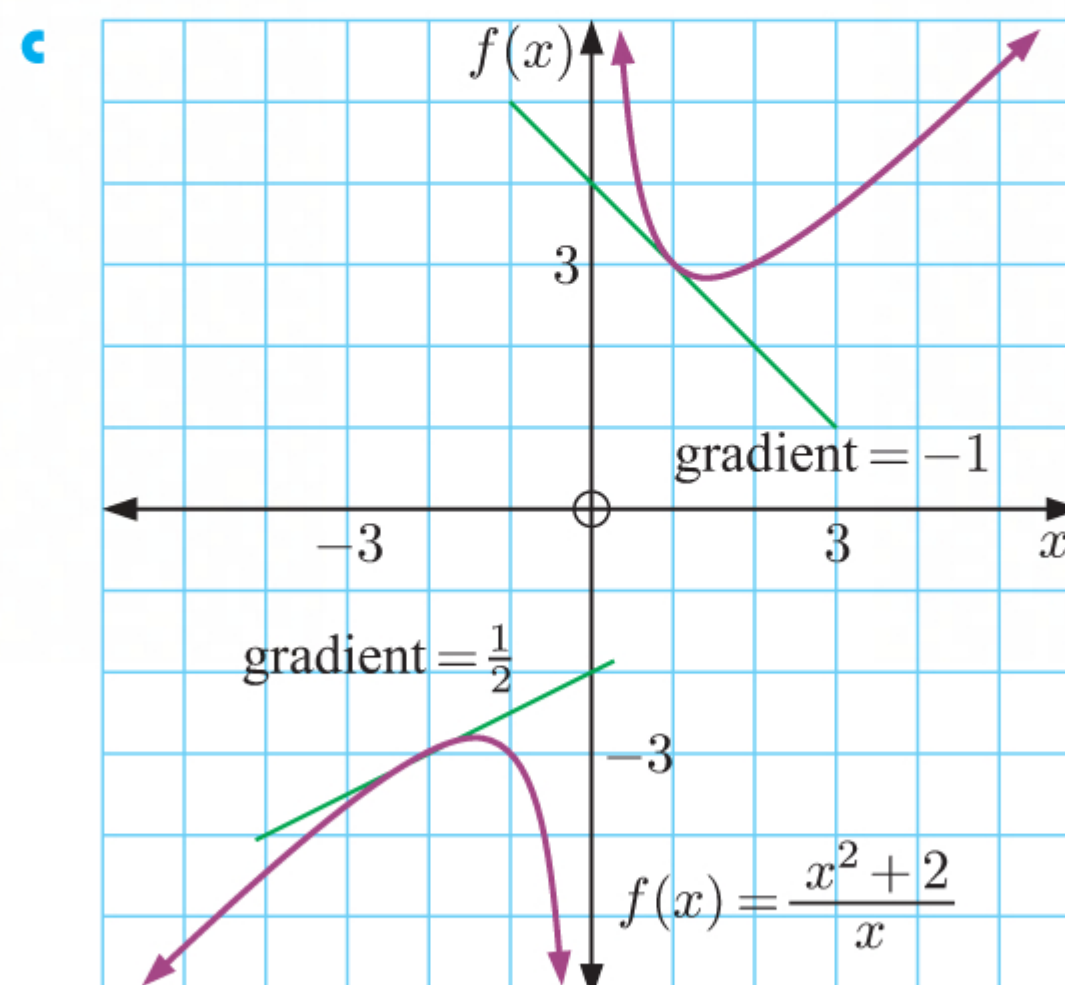
$= 1 - \frac{2}{x^2}$

b i $f'(1) = 1 - \frac{2}{1^2}$
 $= -1$

\therefore gradient of tangent $= -1$

ii $f'(-2) = 1 - \frac{2}{(-2)^2}$
 $= 1 - \frac{2}{4}$
 $= \frac{1}{2}$

\therefore gradient of tangent $= \frac{1}{2}$



$$\begin{aligned}
 8 \quad y &= \left(x - \frac{1}{x}\right)^4 \\
 &= (x - x^{-1})^4 \\
 \therefore \frac{dy}{dx} &= 4(x - x^{-1})^3(1 + x^{-2}) \quad \{\text{chain rule}\} \\
 &= 4\left(x - \frac{1}{x}\right)^3 \left(1 + \frac{1}{x^2}\right)
 \end{aligned}$$

$$\begin{aligned}
 \text{When } x = 1, \quad \frac{dy}{dx} &= 4\left(1 - \frac{1}{1}\right)^3 \left(1 + \frac{1}{1^2}\right) \\
 &= 4 \times 0 \times 2 \\
 &= 0
 \end{aligned}$$

$$\begin{array}{ll}
 9 \quad \mathbf{a} \quad y = \ln(x^3 - 3x) & \mathbf{b} \quad y = \frac{e^x}{x^2} \\
 \therefore \frac{dy}{dx} = \frac{3x^2 - 3}{x^3 - 3x} & \therefore \frac{dy}{dx} = \frac{e^x(x^2) - e^x(2x)}{(x^2)^2} \quad \{\text{quotient rule}\} \\
 & = \frac{e^x(x - 2)}{x^3} \\
 \mathbf{c} \quad y = e^{2x} \sin x & \\
 \therefore \frac{dy}{dx} = 2e^{2x} \sin x + e^{2x} \cos x & \\
 \quad \quad \quad \{\text{product rule}\} &
 \end{array}$$

$$\begin{array}{ll}
 10 \quad \mathbf{a} \quad f(x) = 2x^4 - 4x^3 - 9x^2 + 4x + 7 & \mathbf{b} \quad f''(x) = 0 \text{ when} \\
 \therefore f'(x) = 8x^3 - 12x^2 - 18x + 4 & \quad 24x^2 - 24x - 18 = 0 \\
 \therefore f''(x) = 24x^2 - 24x - 18 & \quad \therefore 4x^2 - 4x - 3 = 0 \\
 & \quad \therefore (2x + 1)(2x - 3) = 0 \\
 & \quad \therefore x = -\frac{1}{2} \text{ or } x = \frac{3}{2}
 \end{array}$$

$$\begin{array}{ll}
 11 \quad \mathbf{a} \quad y = 10x - \sin 10x & \mathbf{b} \quad y = \ln\left(\frac{1}{\cos x}\right) \\
 \therefore \frac{dy}{dx} = 10 - 10 \cos 10x & \quad = \ln[(\cos x)^{-1}] \\
 & \therefore \frac{dy}{dx} = \frac{-(\cos x)^{-2}(-\sin x)}{(\cos x)^{-1}} \quad \{\text{chain rule}\} \\
 & \quad = \frac{\sin x \cos x}{\cos^2 x} \\
 & \quad = \frac{\sin x}{\cos x} \\
 & \quad = \tan x
 \end{array}$$

$$\begin{aligned}
 \mathbf{c} \quad y &= \sin 5x \ln(2x) \\
 \therefore \frac{dy}{dx} &= (5 \cos 5x) \ln(2x) + \sin 5x \times \frac{2}{2x} \\
 &\quad \quad \quad \{\text{product rule}\} \\
 &= (5 \cos 5x) \ln(2x) + \frac{\sin 5x}{x}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{12} \quad y &= \frac{x^3}{x+1} \\
 \therefore \frac{dy}{dx} &= \frac{3x^2(x+1) - x^3(1)}{(x+1)^2} \quad \{\text{quotient rule}\} \\
 &= \frac{3x^3 + 3x^2 - x^3}{(x+1)^2} \\
 &= \frac{2x^3 + 3x^2}{(x+1)^2}
 \end{aligned}$$

$$\begin{aligned}
 \text{At } x = 2, \quad \frac{dy}{dx} &= \frac{2(2)^3 + 3(2)^2}{(2+1)^2} \\
 &= \frac{16 + 12}{9} \\
 &= \frac{28}{9}
 \end{aligned}$$

$$\therefore \text{gradient of tangent} = \frac{28}{9}$$

$$\begin{aligned}
 \mathbf{13} \quad f(x) &= a \ln(bx) \\
 \text{Now } f(e) &= 12 \\
 \therefore 12 &= a \ln(be) \\
 &= a(\ln b + \ln e) \quad \{\ln(ab) = \ln a + \ln b\} \\
 &= a(\ln b + 1) \\
 \therefore a &= \frac{12}{\ln b + 1} \quad \dots (*) \\
 f(x) &= a \ln(bx) \\
 &= a(\ln b + \ln x) \quad \{\ln(ab) = \ln a + \ln b\} \\
 \therefore f'(x) &= a\left(0 + \frac{1}{x}\right) \\
 &= \frac{a}{x}
 \end{aligned}$$

$$\text{Now } f'(2) = 2$$

$$\therefore \frac{a}{2} = 2$$

$$\therefore a = 4$$

Substituting $a = 4$ into $(*)$ gives:

$$4 = \frac{12}{\ln b + 1}$$

$$\therefore 4 \ln b + 4 = 12$$

$$\therefore 4 \ln b = 8$$

$$\therefore \ln b = 2$$

$$\therefore b = e^2$$

So, $a = 4$ and $b = e^2$.

14 a $y = \frac{\cos x}{\sin x + 2}$

$$\therefore \frac{dy}{dx} = \frac{(-\sin x)(\sin x + 2) - \cos x(\cos x)}{(\sin x + 2)^2}$$

{quotient rule}

The tangent meets the graph at $x = 0$.

At $x = 0$,

$$\begin{aligned} \frac{dy}{dx} &= \frac{(-\sin 0)(\sin 0 + 2) - \cos 0(\cos 0)}{(\sin 0 + 2)^2} \\ &= \frac{0 - 1}{4} \\ &= -\frac{1}{4} \end{aligned}$$

$$\therefore \text{gradient of tangent} = -\frac{1}{4}$$

b $\frac{dy}{dx} = \frac{(-\sin x)(\sin x + 2) - \cos x(\cos x)}{(\sin x + 2)^2}$ {from a}

$$\begin{aligned} &= \frac{-\sin^2 x - 2\sin x - \cos^2 x}{(\sin x + 2)^2} \\ &= \frac{-(\sin^2 x + \cos^2 x) - 2\sin x}{(\sin x + 2)^2} \\ &= -\frac{2\sin x + 1}{(\sin x + 2)^2} \end{aligned}$$

A tangent of gradient $-\frac{1}{2}$ occurs when $\frac{dy}{dx} = -\frac{1}{2}$.

$$\therefore -\frac{2\sin x + 1}{(\sin x + 2)^2} = -\frac{1}{2}$$

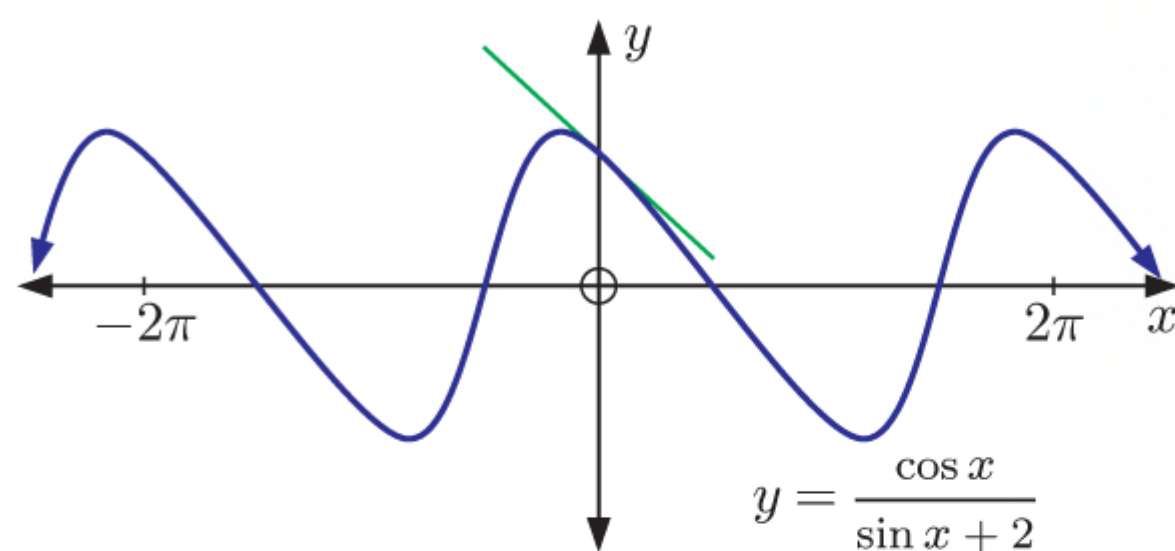
$$\therefore \frac{2\sin x + 1}{(\sin x + 2)^2} = \frac{1}{2}$$

$$\therefore 2(2\sin x + 1) = (\sin x + 2)^2$$

$$\therefore \cancel{4\sin x} + 2 = \sin^2 x + \cancel{4\sin x} + 4$$

$$\therefore \sin^2 x = -2 \quad \text{which has no real solutions}$$

\therefore it is impossible to draw a tangent to the graph with gradient $-\frac{1}{2}$.



15 a $y = \frac{e^x}{\sqrt{x}} = \frac{e^x}{x^{\frac{1}{2}}}$

$$\therefore \frac{dy}{dx} = \frac{e^x x^{\frac{1}{2}} - e^x (\frac{1}{2} x^{-\frac{1}{2}})}{\left(x^{\frac{1}{2}}\right)^2}$$

{quotient rule}

$$= \frac{e^x \sqrt{x} \times \frac{2\sqrt{x}}{2\sqrt{x}} - \frac{e^x}{2\sqrt{x}}}{x}$$

$$= \frac{2xe^x - e^x}{2x\sqrt{x}}$$

$$= \frac{e^x(2x - 1)}{2x\sqrt{x}} \quad \text{as required}$$

- b** **i** $\frac{dy}{dx} = 0$ when $e^x(2x - 1) = 0$
 $\therefore e^x = 0$ or $2x - 1 = 0$
 $\therefore x = \frac{1}{2}$ {as $e^x > 0$ for all x }
- ii** $\frac{dy}{dx}$ is undefined when $2x\sqrt{x} = 0$ or \sqrt{x} is undefined
 $\therefore x \leq 0$

16 a $y = \frac{3x^2 - 2}{1 - 2x}$

$$\therefore \frac{dy}{dx} = \frac{6x(1 - 2x) - (3x^2 - 2)(-2)}{(1 - 2x)^2} \quad \{\text{quotient rule}\}$$

$$= \frac{6x - 12x^2 + 6x^2 - 4}{(1 - 2x)^2}$$

$$= \frac{-6x^2 + 6x - 4}{(1 - 2x)^2}$$

$$\therefore \frac{d^2y}{dx^2} = \frac{(-12x + 6)(1 - 2x)^2 - (-6x^2 + 6x - 4) \times 2(1 - 2x) \times (-2)}{(1 - 2x)^4}$$

{quotient rule, chain rule}

$$= \frac{(1 - 2x)[(-12x + 6)(1 - 2x) + 4(-6x^2 + 6x - 4)]}{(1 - 2x)^4}$$

$$= \frac{\cancel{-12x} + \cancel{24x^2} + 6 - \cancel{12x} - \cancel{24x^2} + \cancel{24x} - 16}{(1 - 2x)^3}$$

$$= \frac{-10}{(1 - 2x)^3}$$

$$= -\frac{10}{(1 - 2x)^3}$$

b $y = x^3 - x + \frac{1}{\sqrt{x}} = x^3 - x + x^{-\frac{1}{2}}$

$$\therefore \frac{dy}{dx} = 3x^2 - 1 - \frac{1}{2}x^{-\frac{3}{2}}$$

$$\therefore \frac{d^2y}{dx^2} = 6x + \frac{3}{4}x^{-\frac{5}{2}}$$

17 $y = 3 \sin 2x + 2 \cos 2x$

$$\therefore \frac{dy}{dx} = 3 \times (\cos 2x) \times 2 + 2 \times (-\sin 2x) \times 2$$

$$= 6 \cos 2x - 4 \sin 2x$$

$$\therefore \frac{d^2y}{dx^2} = 6 \times (-\sin 2x) \times 2 - 4 \times (\cos 2x) \times 2$$

$$= -12 \sin 2x - 8 \cos 2x$$

$$\therefore 4y + \frac{d^2y}{dx^2} = 4(3 \sin 2x + 2 \cos 2x) + (-12 \sin 2x - 8 \cos 2x)$$

$$= 12 \sin 2x + 8 \cos 2x - 12 \sin 2x - 8 \cos 2x$$

$$= 0 \quad \text{as required}$$

$$18 \quad \mathbf{a} \quad f(x) = -\frac{1}{2} \quad \text{when} \quad \frac{6x}{3+x^2} = -\frac{1}{2}$$

$$\therefore 12x = -(3+x^2)$$

$$\therefore 12x = -3 - x^2$$

$$\therefore x^2 + 12x + 3 = 0$$

$$\therefore x = \frac{-12 \pm \sqrt{(12)^2 - 4(1)(3)}}{2(1)}$$

$$= \frac{-12 \pm \sqrt{144 - 12}}{2}$$

$$= \frac{-12 \pm \sqrt{132}}{2}$$

$$= \frac{-12 \pm 2\sqrt{33}}{2}$$

$$= -6 \pm \sqrt{33}$$

$$\mathbf{b} \quad f(x) = \frac{6x}{3+x^2}$$

$$\therefore f'(x) = \frac{6(3+x^2) - 6x(2x)}{(3+x^2)^2} \quad \{\text{quotient rule}\}$$

$$= \frac{18 + 6x^2 - 12x^2}{(3+x^2)^2}$$

$$= \frac{18 - 6x^2}{(3+x^2)^2}$$

$$f'(x) = 0 \quad \text{when} \quad \frac{18 - 6x^2}{(3+x^2)^2} = 0$$

$$\therefore 18 - 6x^2 = 0$$

$$\therefore 6x^2 = 18$$

$$\therefore x^2 = 3$$

$$\therefore x = \pm\sqrt{3}$$

$$\text{c} \quad f'(x) = \frac{18 - 6x^2}{(3 + x^2)^2} \quad \{\text{from b}\}$$

$$\begin{aligned} \therefore f''(x) &= \frac{(-12x)(3 + x^2)^2 - (18 - 6x^2) \times 2(3 + x^2) \times (2x)}{(3 + x^2)^4} \\ &= \frac{-12x(9 + 6x^2 + x^4) - 4x(18 - 6x^2)(3 + x^2)}{(3 + x^2)^4} \\ &= \frac{-108x - 72x^3 - 12x^5 - 4x(54 + 18x^2 - 18x^2 - 6x^4)}{(3 + x^2)^4} \\ &= \frac{-12x^5 - 72x^3 - 108x - 4x(-6x^4 + 54)}{(3 + x^2)^4} \\ &= \frac{-12x^5 - 72x^3 - 108x + 24x^5 - 216x}{(3 + x^2)^4} \\ &= \frac{12x^5 - 72x^3 - 324x}{(3 + x^2)^4} \end{aligned}$$

$$f''(x) = 0 \quad \text{when} \quad \frac{12x^5 - 72x^3 - 324x}{(3 + x^2)^4} = 0$$

$$\therefore 12x^5 - 72x^3 - 324x = 0$$

$$\therefore x^5 - 6x^3 - 27x = 0$$

$$\therefore x(x^4 - 6x^2 - 27) = 0$$

$$\therefore x(x^2 - 9)(x^2 + 3) = 0$$

$$\therefore x = 0 \quad \text{or} \quad x^2 - 9 = 0$$

$$\therefore x^2 = 9$$

$$\therefore x = \pm 3$$

$$\therefore x = -3, 0, \text{ or } 3$$

Chapter 13

PROPERTIES OF CURVES

EXERCISE 13A

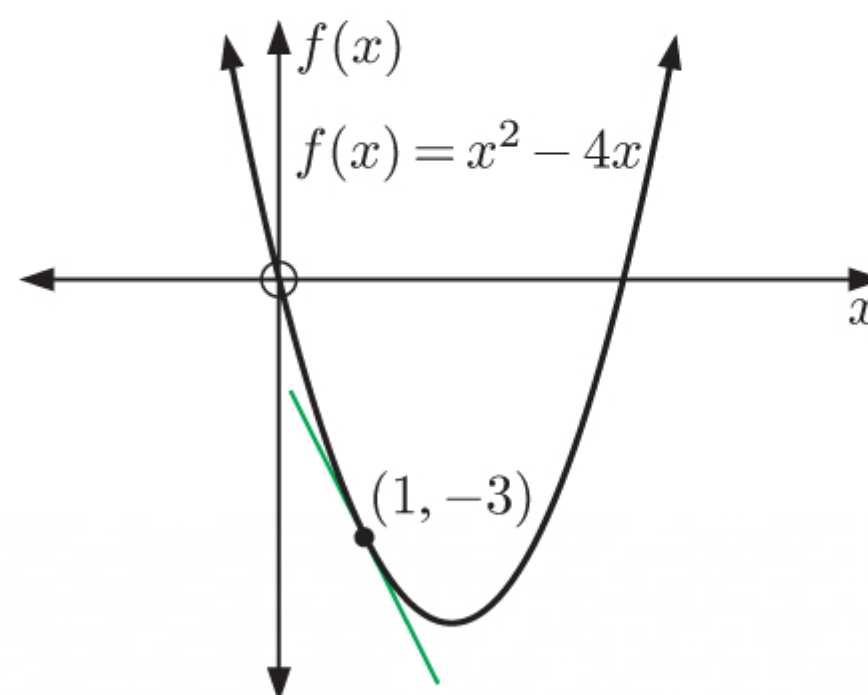
1 a $f(x) = x^2 - 4x$

$$\therefore f'(x) = 2x - 4$$

b The point of contact is $(1, -3)$.

$$\begin{aligned} f'(1) &= 2(1) - 4 \\ &= -2 \end{aligned}$$

So, the tangent has equation $y = -2(x - 1) - 3$
 $\therefore y = -2x - 1$



2 a $y = x - 2x^2 + 3$

When $x = 2$,

$$y = 2 - 2(2)^2 + 3 = -3$$

So, the point of contact is $(2, -3)$.

Now $\frac{dy}{dx} = 1 - 4x$, so at $x = 2$,

$$\frac{dy}{dx} = 1 - 4(2) = -7$$

So, the tangent has gradient -7 .

The tangent has equation

$$y = f'(a)(x - a) + f(a)$$

$$\therefore y = -7(x - 2) + (-3)$$

$$\therefore y = -7x + 14 - 3$$

$$\therefore y = -7x + 11$$

c $y = x^3 - 5x$

When $x = 1$,

$$y = 1^3 - 5(1) = -4$$

So, the point of contact is $(1, -4)$.

Now $\frac{dy}{dx} = 3x^2 - 5$, so at $x = 1$,

$$\frac{dy}{dx} = 3(1)^2 - 5 = -2$$

The tangent has equation

$$y = f'(a)(x - a) + f(a)$$

$$\therefore y = -2(x - 1) + (-4)$$

$$\therefore y = -2x - 2$$

b $y = \sqrt{x} + 1$

$$= x^{\frac{1}{2}} + 1$$

When $x = 4$,

$$y = \sqrt{4} + 1 = 3$$

So, the point of contact is $(4, 3)$.

Now $\frac{dy}{dx} = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$, so at $x = 4$,

$$\frac{dy}{dx} = \frac{1}{2\sqrt{4}} = \frac{1}{4}$$

So, the tangent has gradient $\frac{1}{4}$.

The tangent has equation

$$y = f'(a)(x - a) + f(a)$$

$$\therefore y = \frac{1}{4}(x - 4) + 3$$

$$\therefore y = \frac{1}{4}x + 2$$

d $y = \frac{4}{\sqrt{x}} = 4x^{-\frac{1}{2}}$

Now $\frac{dy}{dx} = -2x^{-\frac{3}{2}}$, so at $x = 1$,

$$\frac{dy}{dx} = -2(1^{-\frac{3}{2}}) = -2$$

The tangent has equation

$$y = f'(a)(x - a) + f(a)$$

$$\therefore y = -2(x - 1) + 4$$

$$\therefore y = -2x + 6$$

$$\begin{aligned} \text{e} \quad y &= \frac{3}{x} - \frac{1}{x^2} \\ &= 3x^{-1} - x^{-2} \end{aligned}$$

$$\begin{aligned} \text{Now } \frac{dy}{dx} &= -3x^{-2} + 2x^{-3} \\ &= -\frac{3}{x^2} + \frac{2}{x^3}, \text{ so at } (-1, -4), \\ \frac{dy}{dx} &= -\frac{3}{(-1)^2} + \frac{2}{(-1)^3} \\ &= -3 - 2 \\ &= -5 \end{aligned}$$

The tangent has equation

$$\begin{aligned} y &= f'(a)(x - a) + f(a) \\ \therefore y &= -5(x + 1) + (-4) \\ \therefore y &= -5x - 9 \end{aligned}$$

$$\text{3 a} \quad y = 2x^3 + 3x^2 - 12x + 1$$

$$\therefore \frac{dy}{dx} = 6x^2 + 6x - 12$$

Horizontal tangents have gradient 0,

$$\begin{aligned} \text{so } 6x^2 + 6x - 12 &= 0 \\ \therefore 6(x^2 + x - 2) &= 0 \\ \therefore 6(x + 2)(x - 1) &= 0 \\ \therefore x &= -2 \text{ or } 1 \end{aligned}$$

When $x = -2$,

$$\begin{aligned} y &= 2(-2)^3 + 3(-2)^2 - 12(-2) + 1 \\ &= 21 \end{aligned}$$

When $x = 1$,

$$\begin{aligned} y &= 2(1)^3 + 3(1)^2 - 12(1) + 1 \\ &= -6 \end{aligned}$$

\therefore the points of contact are $(-2, 21)$ and $(1, -6)$.

\therefore the tangents are $y = 21$ and $y = -6$.

$$\begin{aligned} \text{f} \quad y &= 3x^2 - \frac{1}{x} \\ &= 3x^2 - x^{-1} \end{aligned}$$

When $x = -1$,

$$y = 3(-1)^2 - \frac{1}{(-1)} = 4$$

So, the point of contact is $(-1, 4)$.

$$\begin{aligned} \text{Now } \frac{dy}{dx} &= 6x + x^{-2} \\ &= 6x + \frac{1}{x^2}, \text{ so at } x = -1, \\ \frac{dy}{dx} &= 6(-1) + \frac{1}{(-1)^2} = -5 \end{aligned}$$

The tangent has equation

$$\begin{aligned} y &= f'(a)(x - a) + f(a) \\ \therefore y &= -5(x + 1) + 4 \\ \therefore y &= -5x - 1 \end{aligned}$$

$$\text{b} \quad y = -x^3 + 3x^2 + 9x - 4$$

$$\therefore \frac{dy}{dx} = -3x^2 + 6x + 9$$

Horizontal tangents have gradient 0,

$$\begin{aligned} \text{so } -3x^2 + 6x + 9 &= 0 \\ \therefore -3(x^2 - 2x - 3) &= 0 \\ \therefore -3(x + 1)(x - 3) &= 0 \\ \therefore x &= -1 \text{ or } 3 \end{aligned}$$

When $x = -1$,

$$\begin{aligned} y &= -(-1)^3 + 3(-1)^2 + 9(-1) - 4 \\ &= -9 \end{aligned}$$

When $x = 3$,

$$\begin{aligned} y &= -3^3 + 3(3)^2 + 9(3) - 4 \\ &= 23 \end{aligned}$$

\therefore the points of contact are $(-1, -9)$ and $(3, 23)$.

\therefore the tangents are $y = -9$ and $y = 23$.

$$\begin{aligned}
 \text{c} \quad y &= \sqrt{x} + \frac{1}{\sqrt{x}} \\
 &= x^{\frac{1}{2}} + x^{-\frac{1}{2}} \\
 \therefore \frac{dy}{dx} &= \frac{1}{2}x^{-\frac{1}{2}} - \frac{1}{2}x^{-\frac{3}{2}} \\
 &= \frac{1}{2\sqrt{x}} - \frac{1}{2x\sqrt{x}}
 \end{aligned}$$

Horizontal tangents have gradient 0, so $\frac{1}{2\sqrt{x}} - \frac{1}{2x\sqrt{x}} = 0$

$$\begin{aligned}
 \therefore \frac{x-1}{2x\sqrt{x}} &= 0 \\
 \therefore x-1 &= 0 \\
 \therefore x &= 1
 \end{aligned}$$

When $x = 1$, $y = \sqrt{1} + \frac{1}{\sqrt{1}}$
 $= 2$

\therefore the point of contact is $(1, 2)$.
 \therefore the tangent is $y = 2$.

4 $y = 2x^3 + kx^2 - 3$

a $\frac{dy}{dx} = 6x^2 + 2kx$

When $x = 2$, $\frac{dy}{dx} = 4$

$$\begin{aligned}
 \therefore 6(2)^2 + 2k(2) &= 4 \\
 \therefore 24 + 4k &= 4 \\
 \therefore 4k &= -20 \\
 \therefore k &= -5
 \end{aligned}$$

b Since $k = -5$,
 $y = 2x^3 - 5x^2 - 3$

When $x = 2$,
 $y = 2(2)^3 - 5(2)^2 - 3$
 $= -7$

So, the point of contact is $(2, -7)$.

The tangent has equation

$$\begin{aligned}
 y &= 4(x - 2) + (-7) \\
 \therefore y &= 4x - 15
 \end{aligned}$$

5 $y = 1 - 3x + 12x^2 - 8x^3$

$$\therefore \frac{dy}{dx} = -3 + 24x - 24x^2$$

When $x = 1$, $\frac{dy}{dx} = -3 + 24(1) - 24(1)^2$
 $= -3$

So, the tangent at $(1, 2)$ has gradient -3 .

The tangents to the curve have gradient -3 when $-3 + 24x - 24x^2 = -3$

$$\begin{aligned}
 \therefore 24x^2 - 24x &= 0 \\
 \therefore 24x(x-1) &= 0 \\
 \therefore x &= 0 \text{ or } 1
 \end{aligned}$$

So the other x -value for which the tangent to the curve has gradient -3 is $x = 0$,
and when $x = 0$, $y = 1 - 3(0) + 12(0)^2 - 8(0)^3 = 1$.

\therefore the tangent to the curve at $(0, 1)$ is parallel to the tangent at $(1, 2)$.

This tangent has equation $y = -3(x - 0) + 1$
 $\therefore y = -3x + 1$

- 6** The tangent to the curve $y = x^2 + ax + b$ at the point where $x = 1$ is $2x + y = 6$ or $y = -2x + 6$.
 \therefore the tangent has gradient -2 , and the point of contact is $(1, -2(1) + 6)$ which is $(1, 4)$.

Now, $y = x^2 + ax + b$

$$\therefore \frac{dy}{dx} = 2x + a$$

When $x = 1$, $\frac{dy}{dx} = -2$

$$\therefore 2(1) + a = -2$$

$$\therefore 2 + a = -2$$

$$\therefore a = -4 \quad \dots (*)$$

and $y = 4$

$$\therefore 1^2 + a(1) + b = 4$$

$$\therefore 1 + (-4) + b = 4 \quad \{\text{using } (*)\}$$

$$\therefore b = 7$$

So, $a = -4$ and $b = 7$.

- 7** The tangent to the curve $y = a\sqrt{x} + bx$ at the point where $x = 4$ is $y = x + 2$.
 \therefore the tangent has gradient 1 , and the point of contact is $(4, 4 + 2)$ which is $(4, 6)$.

Now, $y = a\sqrt{x} + bx$

$$= ax^{\frac{1}{2}} + bx$$

$$\therefore \frac{dy}{dx} = \frac{1}{2}ax^{-\frac{1}{2}} + b$$

$$= \frac{a}{2\sqrt{x}} + b$$

When $x = 4$, $\frac{dy}{dx} = 1$

$$\therefore \frac{a}{2\sqrt{4}} + b = 1$$

$$\therefore \frac{a}{4} + b = 1$$

$$\therefore a + 4b = 4$$

$$\therefore a = 4 - 4b \quad \dots (*)$$

and $y = 6$

$$\therefore a\sqrt{4} + b(4) = 6$$

$$\therefore 2(4 - 4b) + 4b = 6 \quad \{\text{using } (*)\}$$

$$\therefore 4 - 4b + 2b = 3$$

$$\therefore -2b = -1$$

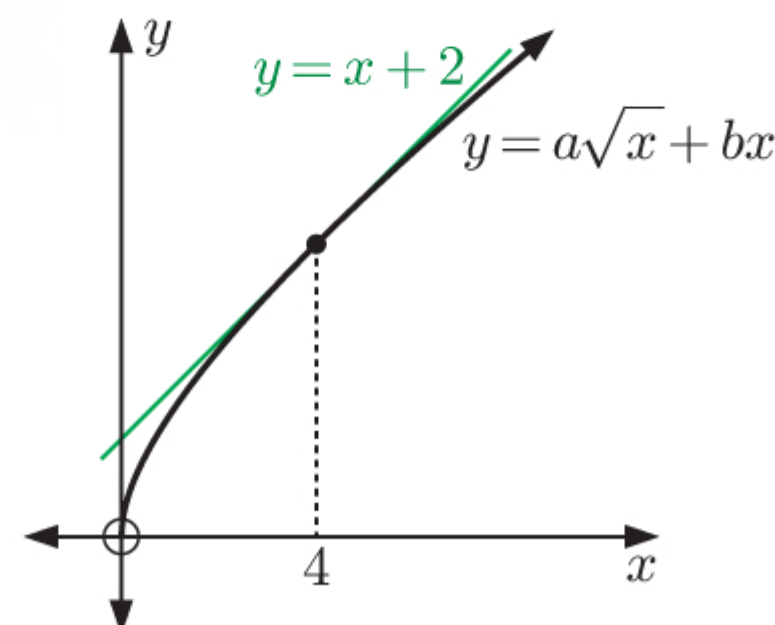
$$\therefore b = \frac{1}{2}$$

Substituting $b = \frac{1}{2}$ into $(*)$ gives $a = 4 - 4\left(\frac{1}{2}\right)$

$$= 4 - 2$$

$$= 2$$

So, $a = 2$ and $b = \frac{1}{2}$.



8 $y = 2x^2 - 1$

$$\therefore \frac{dy}{dx} = 4x$$

$$\therefore \text{at the point where } x = a, \frac{dy}{dx} = 4a$$

$$\therefore \text{the gradient of the tangent at the point where } x = a \text{ is } 4a.$$

Also, at $x = a$, $y = 2a^2 - 1$.

$$\therefore \text{the tangent has equation } y = 4a(x - a) + (2a^2 - 1)$$

$$\therefore y = 4ax - 4a^2 + 2a^2 - 1$$

$$\therefore 4ax - y = 2a^2 + 1$$

$$\begin{aligned}
 9 \quad a \quad f(x) &= x^2 + \frac{4}{x^2} \\
 &= x^2 + 4x^{-2} \\
 \therefore f'(x) &= 2x - 8x^{-3} \\
 &= 2x - \frac{8}{x^3}
 \end{aligned}$$

b Horizontal tangents have gradient 0, so

$$\begin{aligned}
 2x - \frac{8}{x^3} &= 0 \\
 \therefore \frac{2x^4 - 8}{x^3} &= 0 \\
 \therefore 2x^4 - 8 &= 0 \\
 \therefore 2x^4 &= 8 \\
 \therefore x^4 &= 4 \\
 \therefore x &= \pm\sqrt{2}
 \end{aligned}$$

$$\begin{aligned}
 c \quad \text{When } x = \sqrt{2}, \quad f(\sqrt{2}) &= (\sqrt{2})^2 + \frac{4}{(\sqrt{2})^2} \\
 &= 2 + \frac{4}{2} \\
 &= 4
 \end{aligned}$$

\therefore the horizontal tangent at $(\sqrt{2}, 4)$ is $y = 4$.

$$\begin{aligned}
 \text{When } x = -\sqrt{2}, \quad f(-\sqrt{2}) &= (-\sqrt{2})^2 + \frac{4}{(-\sqrt{2})^2} \\
 &= 2 + \frac{4}{2} \\
 &= 4
 \end{aligned}$$

\therefore the horizontal tangent at $(-\sqrt{2}, 4)$ is $y = 4$.

The tangents are the same line, $y = 4$.

10 The tangent to the curve $y = a\sqrt{1-bx}$ at the point where $x = -1$ is $3x + y = 5$ or $y = -3x + 5$.

\therefore the tangent has gradient -3 , and the point of contact is $(-1, -3(-1) + 5)$ which is $(-1, 8)$.

$$\begin{aligned}
 \text{Now } y &= a\sqrt{1-bx} \\
 &= a(1-bx)^{\frac{1}{2}} \\
 \therefore \frac{dy}{dx} &= \frac{1}{2}a(1-bx)^{-\frac{1}{2}}(-b) \\
 &= -\frac{ab}{2\sqrt{1-bx}}
 \end{aligned}$$

$$\text{When } x = -1, \quad \frac{dy}{dx} = -3$$

and

$$y = 8$$

$$\therefore -\frac{ab}{2\sqrt{1-b(-1)}} = -3$$

$$\therefore ab = 6\sqrt{1+b}$$

$$\therefore a = \frac{6\sqrt{1+b}}{b} \quad \dots (*)$$

$$\therefore a\sqrt{1-b(-1)} = 8$$

$$\therefore \left(\frac{6\sqrt{1+b}}{b}\right)\sqrt{1+b} = 8 \quad \{\text{using } (*)\}$$

$$\therefore \frac{6(1+b)}{b} = 8$$

$$\therefore 6 + 6b = 8b$$

$$\therefore -2b = -6$$

$$\therefore b = 3$$

$$\text{Substituting } b = 3 \text{ into } (*) \text{ gives } a = \frac{6\sqrt{1+3}}{3}$$

$$= 2\sqrt{4}$$

$$= 4$$

So, $a = 4$ and $b = 3$.

11 a $f(x) = e^{-x}$

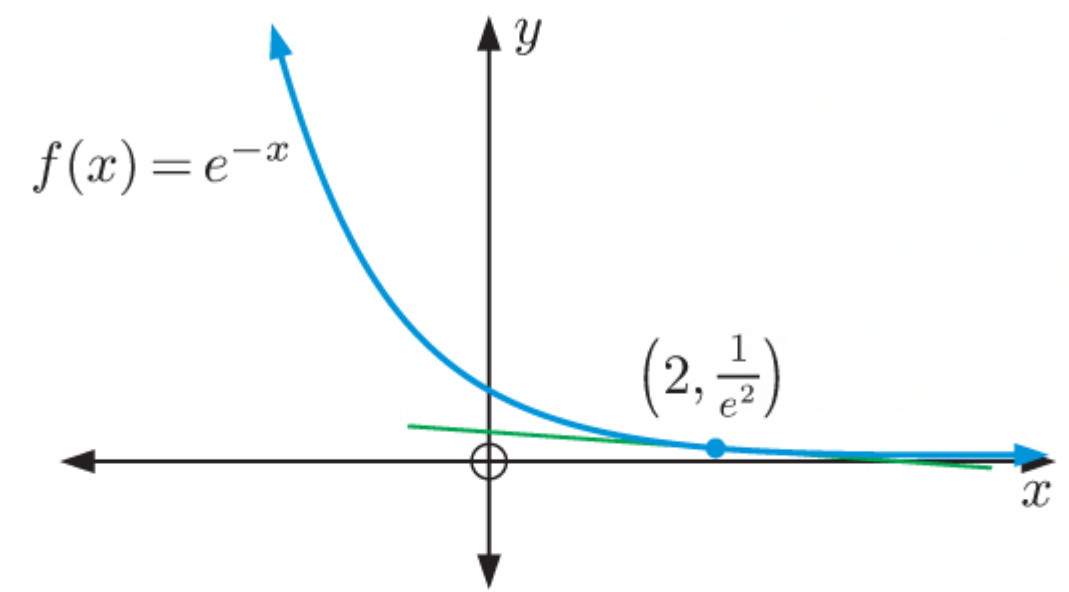
$$\therefore f(2) = e^{-2} \\ = \frac{1}{e^2}$$

\therefore the point of contact is $\left(2, \frac{1}{e^2}\right)$.

Now $f(x) = e^{-x}$ has derivative $f'(x) = -e^{-x}$
 $= -\frac{1}{e^x}$

\therefore the tangent at $\left(2, \frac{1}{e^2}\right)$ has gradient $-\frac{1}{e^2}$.

\therefore the tangent has equation $y = -\frac{1}{e^2}(x - 2) + \frac{1}{e^2}$
 $= -\frac{x}{e^2} + \frac{2}{e^2} + \frac{1}{e^2}$
 $= -\frac{x}{e^2} + \frac{3}{e^2}$
 $\therefore y = -e^{-2}x + 3e^{-2}$



b $y = \ln(2 - x)$

When $x = -1$, $y = \ln(2 - (-1))$
 $= \ln 3$

\therefore the point of contact is $(-1, \ln 3)$.

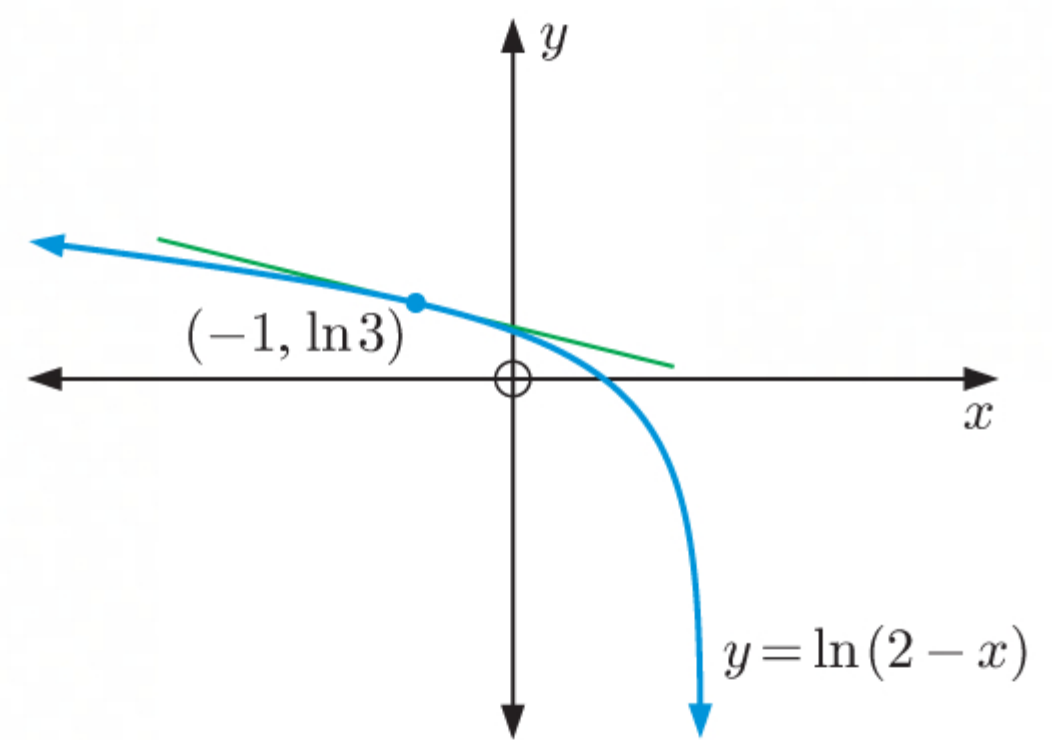
Now $y = \ln(2 - x)$ has derivative

$$\frac{dy}{dx} = \frac{-1}{2 - x} = \frac{1}{x - 2}$$

\therefore the tangent at $(-1, \ln 3)$ has gradient

$$\frac{1}{-1 - 2} = -\frac{1}{3}$$

\therefore the tangent has equation $y = -\frac{1}{3}(x + 1) + \ln 3$
 which is $y = -\frac{1}{3}x - \frac{1}{3} + \ln 3$



c $y = (x + 2)e^x$

When $x = 1$, $y = (1 + 2)e^1$
 $= 3e$

\therefore the point of contact is $(1, 3e)$.

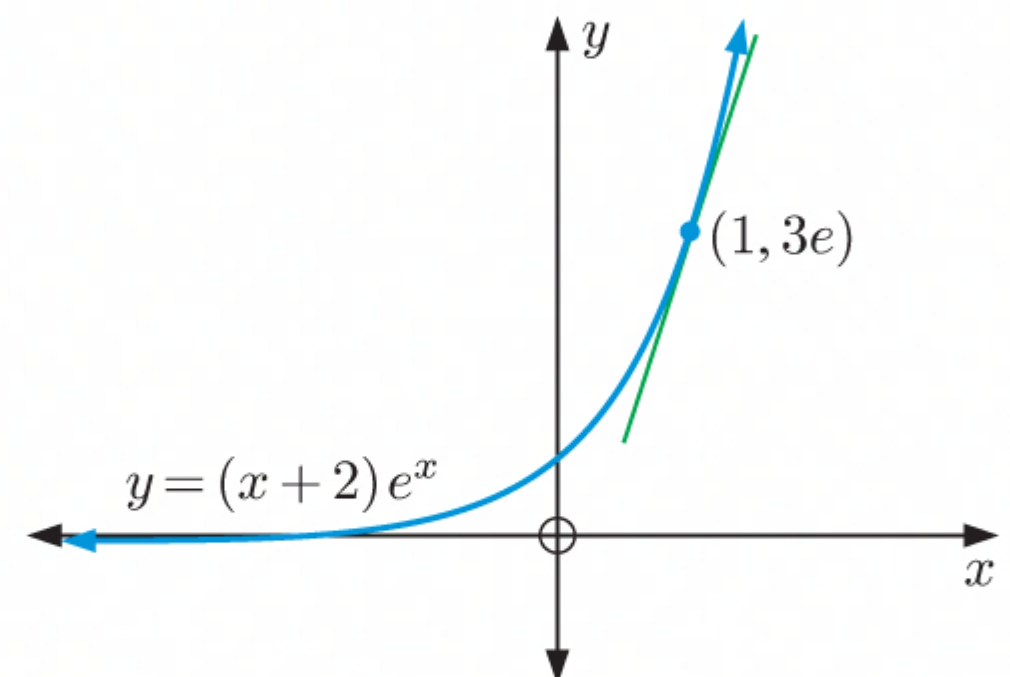
Now $y = (x + 2)e^x$ has derivative

$$\frac{dy}{dx} = (1)e^x + (x + 2)e^x \quad \{\text{product rule}\}$$

\therefore the tangent at $(1, 3e)$ has gradient

$$e^1 + (1 + 2)e^1 = 4e$$

\therefore the tangent has equation $y = 4e(x - 1) + 3e$
 which is $y = 4ex - e$



d $y = \ln \sqrt{x}$

When $y = -1$, $\ln \sqrt{x} = -1$

$$\therefore \sqrt{x} = e^{-1}$$

$$\therefore x = e^{-2} = \frac{1}{e^2}$$

\therefore the point of contact is $\left(\frac{1}{e^2}, -1\right)$.

Now $y = \ln \sqrt{x} = \ln(x^{\frac{1}{2}})$ has derivative

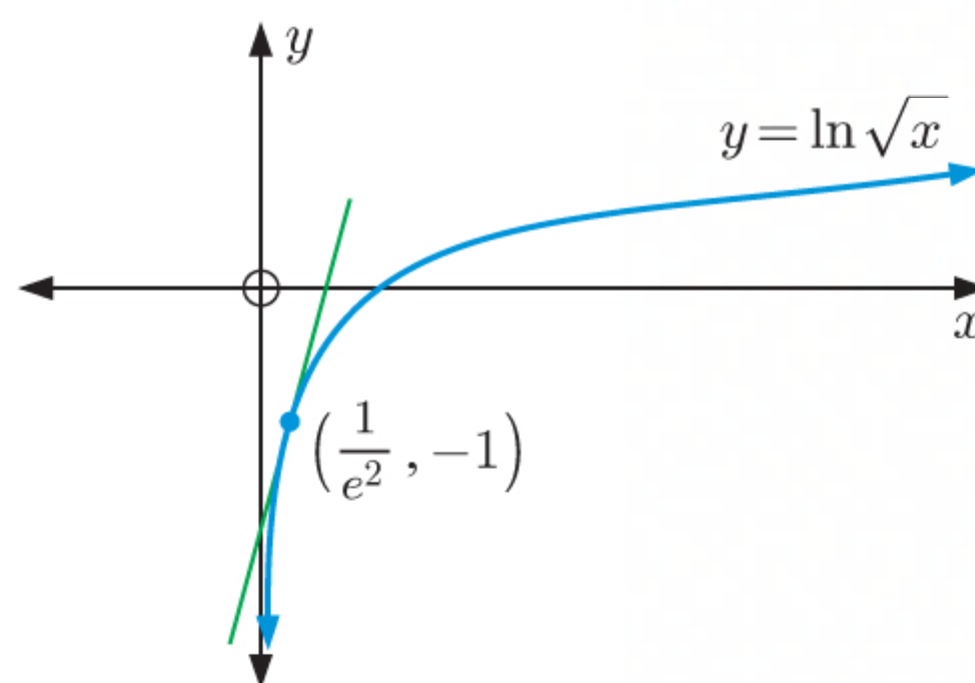
$$\frac{dy}{dx} = \frac{\frac{1}{2}x^{-\frac{1}{2}}}{x^{\frac{1}{2}}} = \frac{1}{2x}$$

\therefore the tangent at $\left(\frac{1}{e^2}, -1\right)$ has gradient $\frac{1}{\frac{2}{e^2}} = \frac{e^2}{2}$

\therefore the tangent has equation $y = \frac{e^2}{2}\left(x - \frac{1}{e^2}\right) - 1$

$$= \frac{e^2}{2}x - \frac{1}{2} - 1$$

$$\therefore y = \frac{e^2}{2}x - \frac{3}{2}$$



e $y = e^{3x-5}$

When $y = e$, $e^{3x-5} = e$

$$\therefore 3x - 5 = 1$$

$$\therefore 3x = 6$$

$$\therefore x = 2$$

\therefore the point of contact is $(2, e)$.

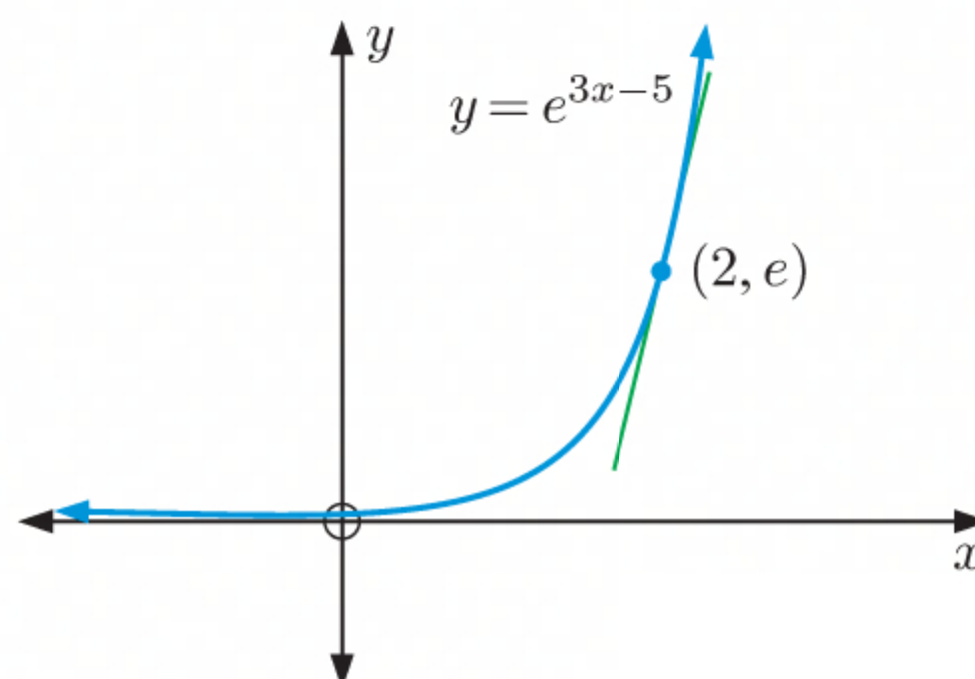
Now $y = e^{3x-5}$ has derivative $\frac{dy}{dx} = 3e^{3x-5}$

\therefore the tangent at $(2, e)$ has gradient

$$3e^{3(2)-5} = 3e$$

\therefore the tangent has equation $y = 3e(x - 2) + e$

which is $y = 3ex - 5e$

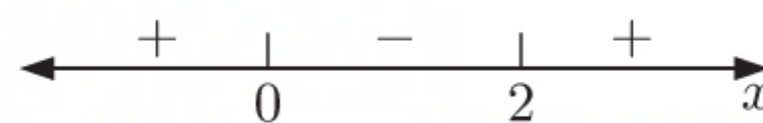


12 a $f(x) = \ln(x(x-2))$ is defined when $x(x-2) > 0$
 $\therefore x < 0$ or $x > 2$

\therefore the domain of $f(x)$ is $\{x \mid x < 0 \text{ or } x > 2\}$

b $f(x) = \ln(x(x-2))$
 $= \ln x + \ln(x-2)$ $\{\ln(ab) = \ln a + \ln b\}$

$$\therefore f'(x) = \frac{1}{x} + \frac{1}{x-2}$$



$$\begin{aligned} \bullet \quad f(3) &= \ln(3(3-2)) \\ &= \ln 3 \end{aligned}$$

\therefore the point of contact is $(3, \ln 3)$.

$$\begin{aligned} \text{Now } f'(3) &= \frac{1}{3} + \frac{1}{3-2} \\ &= \frac{1}{3} + 1 \\ &= \frac{4}{3} \end{aligned}$$

\therefore the tangent at $(3, \ln 3)$ has gradient $\frac{4}{3}$.

$$\begin{aligned} \therefore \text{ the tangent has equation } y &= \frac{4}{3}(x-3) + \ln 3 \\ \text{which is } y &= \frac{4}{3}x - 4 + \ln 3 \end{aligned}$$

13 $y = x^2 e^x$

When $x = 1$, $y = (1)^2 e^1 = e$

\therefore the point of contact is $(1, e)$.

Now $y = x^2 e^x$

$$\therefore \frac{dy}{dx} = 2xe^x + x^2 e^x \quad \{\text{product rule}\}$$

When $x = 1$, $\frac{dy}{dx} = 2e + e = 3e$

$$\begin{aligned} \text{So, the tangent has equation } y &= 3e(x-1) + e \\ \therefore y &= 3ex - 2e \end{aligned}$$

When $y = 0$, $0 = 3ex - 2e$

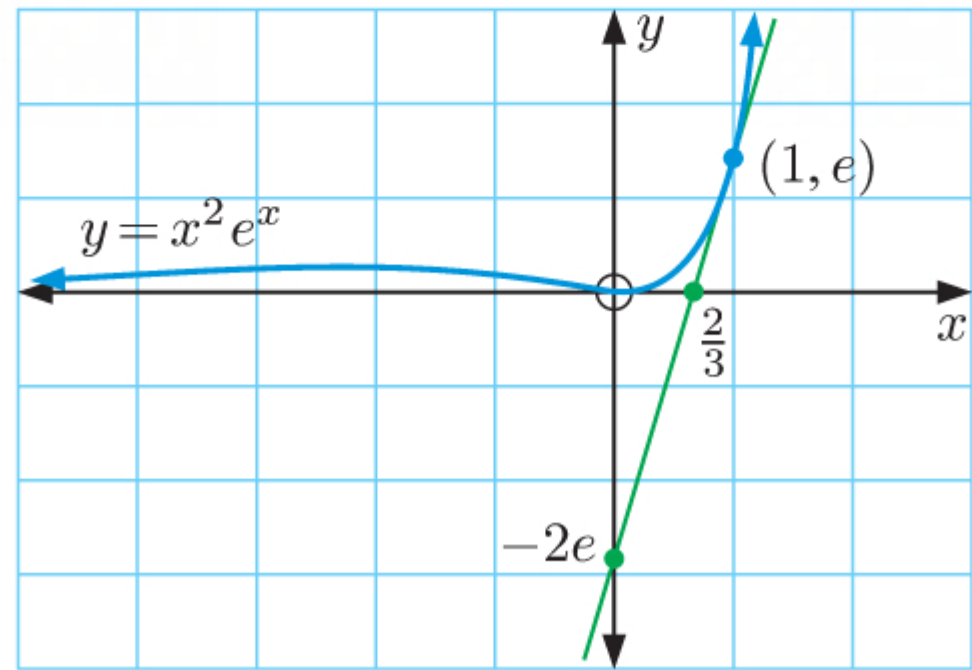
$$\therefore 3ex = 2e$$

$$\therefore x = \frac{2}{3}$$

\therefore the x -intercept is $\frac{2}{3}$.

When $x = 0$, $y = -2e$

\therefore the y -intercept is $-2e$.



14 $y = 3xe^{\frac{x}{2}}$

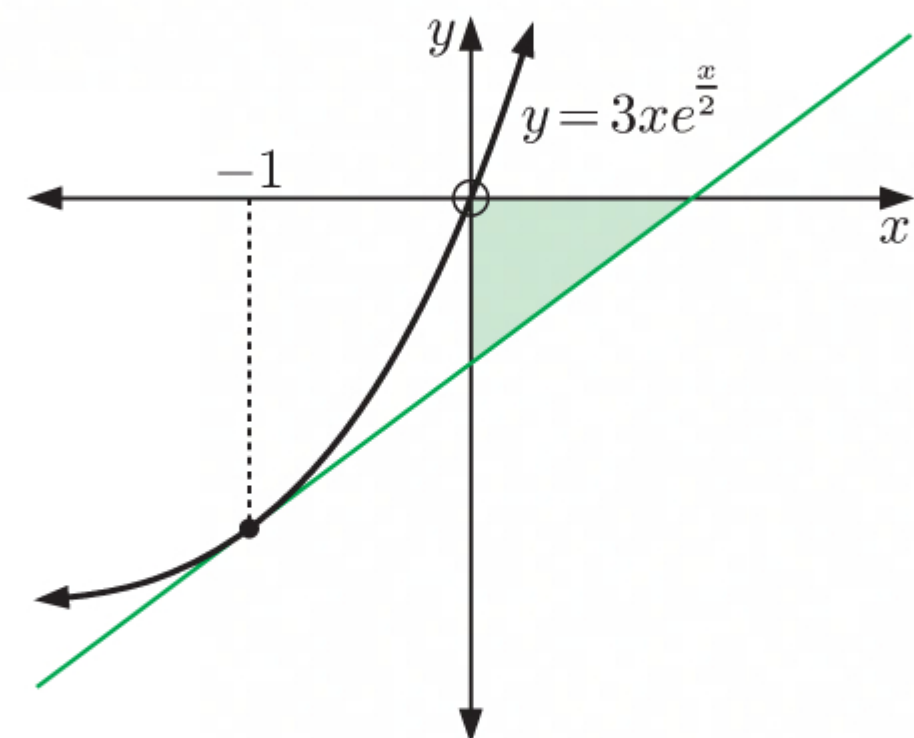
$$\begin{aligned} \text{When } x = -1, \quad y &= 3(-1)e^{\frac{-1}{2}} \\ &= -\frac{3}{\sqrt{e}} \end{aligned}$$

\therefore the point of contact is $\left(-1, -\frac{3}{\sqrt{e}}\right)$.

Now $y = 3xe^{\frac{x}{2}}$

$$\begin{aligned} \therefore \frac{dy}{dx} &= 3e^{\frac{x}{2}} + 3x\left(\frac{1}{2}e^{\frac{x}{2}}\right) \quad \{\text{product rule}\} \\ &= 3e^{\frac{x}{2}} + \frac{3}{2}xe^{\frac{x}{2}} \end{aligned}$$

$$\begin{aligned} \text{When } x = -1, \quad \frac{dy}{dx} &= 3e^{-\frac{1}{2}} - \frac{3}{2}e^{-\frac{1}{2}} \\ &= \frac{3}{\sqrt{e}} - \frac{3}{2\sqrt{e}} \\ &= \frac{3}{2\sqrt{e}} \end{aligned}$$



$$\begin{aligned}
 \text{So, the tangent has equation } y &= \frac{3}{2\sqrt{e}}(x+1) - \frac{3}{\sqrt{e}} \\
 &= \frac{3}{2\sqrt{e}}x + \frac{3}{2\sqrt{e}} - \frac{3}{\sqrt{e}} \\
 &= \frac{3}{2\sqrt{e}}x - \frac{3}{2\sqrt{e}} \\
 &= \frac{3}{2\sqrt{e}}(x-1)
 \end{aligned}$$

$$\text{When } y = 0, \quad 0 = \frac{3}{2\sqrt{e}}(x-1)$$

$$\therefore x = 1$$

\therefore the x -intercept is 1.

$$\text{When } x = 0, \quad y = \frac{3}{2\sqrt{e}}(0-1)$$

$$= -\frac{3}{2\sqrt{e}}$$

$$\therefore \text{ the } y\text{-intercept is } -\frac{3}{2\sqrt{e}}.$$

$$\therefore \text{ the shaded triangle has area } = \frac{1}{2} \times \text{base} \times \text{height}$$

$$= \frac{1}{2} \times 1 \times \frac{3}{2\sqrt{e}}$$

$$= \frac{3}{4\sqrt{e}} \text{ units}^2$$

15 a $y = \sin x$ has derivative $\frac{dy}{dx} = \cos x$

$$\therefore \text{ the tangent at } (0, 0) \text{ has gradient } \cos 0 = 1$$

$$\therefore \text{ the tangent has equation } y = 1(x-0) + 0$$

$$\text{which is } y = x$$

b $y = \cos x$

$$\text{When } x = \frac{\pi}{6}, \quad y = \cos \frac{\pi}{6} = \frac{\sqrt{3}}{2}$$

$$\therefore \text{ the point of contact is } \left(\frac{\pi}{6}, \frac{\sqrt{3}}{2}\right).$$

$$\text{Now } y = \cos x \text{ has derivative } \frac{dy}{dx} = -\sin x$$

$$\therefore \text{ the tangent at } \left(\frac{\pi}{6}, \frac{\sqrt{3}}{2}\right) \text{ has gradient } -\sin \frac{\pi}{6} = -\frac{1}{2}$$

$$\therefore \text{ the tangent has equation } y = -\frac{1}{2}\left(x - \frac{\pi}{6}\right) + \frac{\sqrt{3}}{2}$$

$$\text{which is } y = -\frac{1}{2}x + \frac{\pi}{12} + \frac{\sqrt{3}}{2}$$

c $y = \frac{1}{\sin 2x}$

When $x = \frac{\pi}{4}$, $y = \frac{1}{\sin \frac{\pi}{2}} = 1$

\therefore the point of contact is $(\frac{\pi}{4}, 1)$.

Now $y = \frac{1}{\sin 2x} = (\sin 2x)^{-1}$ has derivative

$$\begin{aligned} \frac{dy}{dx} &= -(\sin 2x)^{-2}(2 \cos 2x) \quad \{\text{chain rule}\} \\ &= -\frac{2 \cos 2x}{\sin^2 2x} \end{aligned}$$

\therefore the tangent at $(\frac{\pi}{4}, 1)$ has gradient $-\frac{2 \cos \frac{\pi}{2}}{\sin^2(\frac{\pi}{2})} = 0$

\therefore the tangent has equation $y = 0(x - \frac{\pi}{4}) + 1$
which is $y = 1$

d $y = \cos 2x + 3 \sin x$

When $x = \frac{\pi}{2}$, $y = \cos \pi + 3 \sin \frac{\pi}{2}$
 $= -1 + 3$
 $= 2$

\therefore the point of contact is $(\frac{\pi}{2}, 2)$.

Now $y = \cos 2x + 3 \sin x$ has derivative $\frac{dy}{dx} = -2 \sin 2x + 3 \cos x$

\therefore the tangent at $(\frac{\pi}{2}, 2)$ has gradient $-2 \sin \pi + 3 \cos \frac{\pi}{2} = 0$

\therefore the tangent has equation $y = 0(x - \frac{\pi}{2}) + 2$
which is $y = 2$

16

$y = \frac{\cos x}{1 + \sin x}$

$$\begin{aligned} \therefore \frac{dy}{dx} &= \frac{-\sin x(1 + \sin x) - \cos x(\cos x)}{(1 + \sin x)^2} \quad \{\text{quotient rule}\} \\ &= \frac{-\sin x - \sin^2 x - \cos^2 x}{(1 + \sin x)^2} \\ &= \frac{-(\sin^2 x + \cos^2 x + \sin x)}{(1 + \sin x)^2} \\ &= \frac{-(1 + \sin x)}{(1 + \sin x)^2} \\ &= -\frac{1}{1 + \sin x} \neq 0 \end{aligned}$$

\therefore there are no tangents which have gradient 0

\therefore there are no horizontal tangents to $y = \frac{\cos x}{1 + \sin x}$

- 17** Consider the tangent to $y = x^3$ at $x = 2$.

When $x = 2$, $y = 2^3 = 8$ so the point of contact is $(2, 8)$.

Now $\frac{dy}{dx} = 3x^2$ and so at $x = 2$,

$$\frac{dy}{dx} = 3(2)^2 = 12$$

So, the tangent at $(2, 8)$ has gradient 12 and its equation is

$$12x - y = 12(2) - 8$$

$$\therefore 12x - y = 16$$

$$\therefore y = 12x - 16$$

The tangent meets the curve where

$$12x - 16 = x^3$$

$$\therefore x^3 - 12x + 16 = 0$$

Because the tangent touches the curve at $x = 2$, there must be a repeated solution at this point.

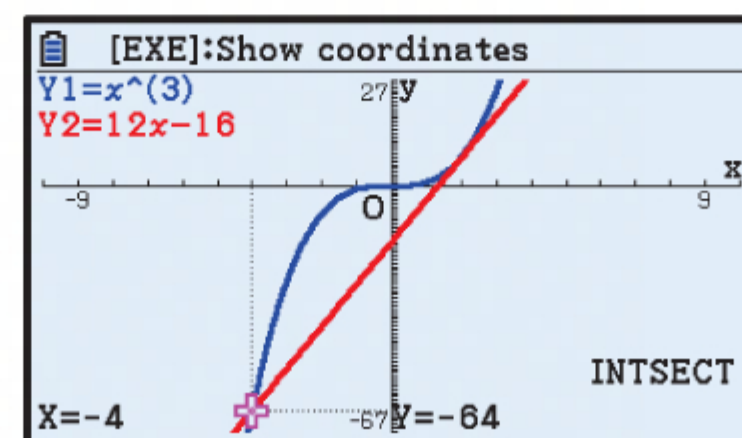
$\therefore (x - 2)^2$ must be a factor of this cubic

$$\therefore (x - 2)^2(x + 4) = 0$$

\therefore the tangent meets the curve again when $x = -4$.

When $x = -4$, $y = (-4)^3 = -64$

\therefore the tangent meets the curve again at $(-4, -64)$.



- 18** Consider the tangent to $y = -x^3 + 2x^2 + 1$ at $x = -1$.

When $x = -1$, $y = -(-1)^3 + 2(-1)^2 + 1 = 4$ and so the point of contact is $(-1, 4)$.

Now $\frac{dy}{dx} = -3x^2 + 4x$ and so at $x = -1$,

$$\frac{dy}{dx} = -3(-1)^2 + 4(-1) = -7$$

So, the tangent at $(-1, 4)$ has gradient -7 and its equation is

$$-7x - y = -7(-1) - 4$$

$$\therefore 7x + y = -3$$

$$\therefore y = -7x - 3$$

The tangent meets the curve where $-7x - 3 = -x^3 + 2x^2 + 1$

$$\therefore x^3 - 2x^2 - 7x - 4 = 0$$

Because the tangent touches the curve at $x = -1$, there must be a repeated solution at this point.

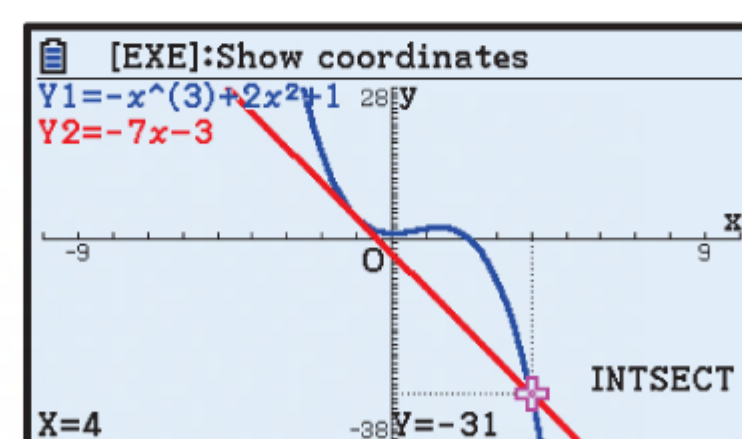
$\therefore (x + 1)^2$ must be a factor of this cubic

$$\therefore (x + 1)^2(x - 4) = 0$$

\therefore the tangent meets the curve again when $x = 4$.

When $x = 4$, $y = -(4)^3 + 2(4)^2 + 1$
 $= -64 + 32 + 1$
 $= -31$

\therefore the tangent meets the curve again at $(4, -31)$.



- 19** Consider the tangent to $y = \frac{1}{x} - \frac{1}{x^2}$ at $x = 1$.

When $x = 1$, $y = \frac{1}{1} - \frac{1}{1^2} = 0$ and so the point of contact is $(1, 0)$.

Now $y = x^{-1} - x^{-2}$

$$\begin{aligned}\therefore \frac{dy}{dx} &= -x^{-2} + 2x^{-3} \\ &= -\frac{1}{x^2} + \frac{2}{x^3} \quad \text{and so at } x = 1,\end{aligned}$$

$$\frac{dy}{dx} = -\frac{1}{1^2} + \frac{2}{1^3} = 1$$

So, the tangent at $(1, 0)$ has gradient 1 and its equation is

$$y = 1 \times (x - 1) + 0$$

$$\therefore y = x - 1$$

The tangent meets the curve where $x - 1 = \frac{1}{x} - \frac{1}{x^2}$

$$\therefore x^3 - x^2 = x - 1$$

$$\therefore x^3 - x^2 - x + 1 = 0$$

Because the tangent touches the curve at $x = 1$, there must be a repeated solution at this point.

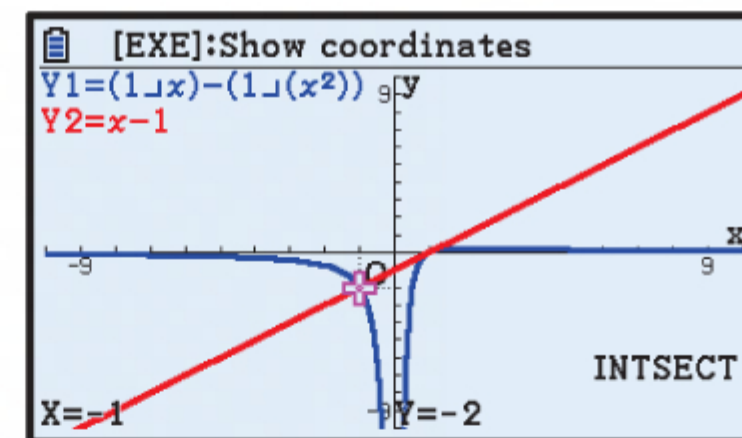
$\therefore (x - 1)^2$ must be a factor of this cubic

$$\therefore (x - 1)^2(x + 1) = 0$$

\therefore the tangent meets the curve again when $x = -1$.

$$\begin{aligned}\text{When } x = -1, \quad y &= \frac{1}{(-1)} - \frac{1}{(-1)^2} \\ &= -1 - 1 \\ &= -2\end{aligned}$$

\therefore the tangent meets the curve again at $(-1, -2)$.



- 20 a** Consider the tangent to $y = x^2 - x + 9$ at $x = a$.

When $x = a$, $y = a^2 - a + 9$, so the point of contact is $(a, a^2 - a + 9)$.

Now $\frac{dy}{dx} = 2x - 1$ and so at $x = a$, $\frac{dy}{dx} = 2a - 1$

So, the gradient of the tangent at $(a, a^2 - a + 9)$ is $2a - 1$

$$\begin{aligned}\therefore \text{the equation of the tangent is } y &= (2a - 1)(x - a) + (a^2 - a + 9) \\ &= 2ax - 2a^2 - x + a + a^2 - a + 9 \\ &= 2ax - a^2 - x + 9 \\ \therefore y &= (2a - 1)x - a^2 + 9 \quad \dots (*)\end{aligned}$$

- b** This tangent passes through $(0, 0)$, so $0 = -a^2 + 9$
 $\therefore a^2 = 9$
 $\therefore a = \pm 3$

When $a = 3$, $y = (2(3) - 1)x - 3^2 + 9$ {from (*)}
 $\therefore y = 5x$

The tangent is $y = 5x$ with point of contact $(3, 15)$.

When $a = -3$, $y = (2(-3) - 1)x - (-3)^2 + 9$ {from (*)}
 $\therefore y = -7x$

The tangent is $y = -7x$ with point of contact $(-3, 21)$.

- 21 a** Consider the tangent to $y = x^2 + 4x$ at the point where $x = a$.

When $x = a$, $y = a^2 + 4a$, so the point of contact is $(a, a^2 + 4a)$.

Now $\frac{dy}{dx} = 2x + 4$, and so at $x = a$, $\frac{dy}{dx} = 2a + 4$

So, the gradient of the tangent at $(a, a^2 + 4a)$ is $2a + 4$

\therefore the equation of the tangent is $y = (2a + 4)(x - a) + (a^2 + 4a)$
 $= 2ax - 2a^2 + 4x - \cancel{4a} + a^2 + \cancel{4a}$
 $= 2ax + 4x - a^2$
 $\therefore y = (2a + 4)x - a^2 \dots (*)$

- b** This tangent passes through $(1, -4)$, so $-4 = (2a + 4)(1) - a^2$
 $\therefore -4 = 2a + 4 - a^2$
 $\therefore a^2 - 2a - 8 = 0$
 $\therefore (a + 2)(a - 4) = 0$
 $\therefore a = -2 \text{ or } 4$

When $a = -2$, $y = (2(-2) + 4)x - (-2)^2$ {from (*)}
 $\therefore y = (-4 + 4)x - 4$
 $\therefore y = -4$

The tangent is $y = -4$ with point of contact $(-2, -4)$.

When $a = 4$, $y = (2(4) + 4)x - (4)^2$
 $\therefore y = (8 + 4)x - 16$
 $\therefore y = 12x - 16$

When $x = 4$, $y = 12(4) - 16$
 $= 48 - 16$
 $= 32$

The tangent is $y = 12x - 16$ with point of contact $(4, 32)$.

22 Consider the tangent to $y = x^2 - 3x + 1$ at the point where $x = a$.

When $x = a$, $y = a^2 - 3a + 1$, so the point of contact is $(a, a^2 - 3a + 1)$.

Now $\frac{dy}{dx} = 2x - 3$, and so at $x = a$, $\frac{dy}{dx} = 2a - 3$

So, the gradient of the tangent at $(a, a^2 - 3a + 1)$ is $2a - 3$

$$\begin{aligned}\therefore \text{ the equation of the tangent is } y &= (2a - 3)(x - a) + a^2 - 3a + 1 \\ &= 2ax - 2a^2 - 3x + \cancel{3a} + a^2 - \cancel{3a} + 1 \\ &= 2ax - 3x - a^2 + 1 \\ \therefore y &= (2a - 3)x - a^2 + 1 \quad \dots (*)\end{aligned}$$

This tangent passes through $(1, -10)$, so $-10 = (2a - 3)(1) - a^2 + 1$

$$\therefore -10 = 2a - 3 - a^2 + 1$$

$$\therefore a^2 - 2a - 8 = 0$$

$$\therefore (a + 2)(a - 4) = 0$$

$$\therefore a = -2 \text{ or } 4$$

When $a = -2$, $y = (2(-2) - 3)x - (-2)^2 + 1$ {from (*)}

$$\therefore y = (-4 - 3)x - 4 + 1$$

$$\therefore y = -7x - 3$$

The tangent is $y = -7x - 3$.

When $a = 4$, $y = (2(4) - 3)x - (4)^2 + 1$

$$\therefore y = (8 - 3)x - 16 + 1$$

$$\therefore y = 5x - 15$$

The tangent is $y = 5x - 15$.

23 a $y = e^x$

When $x = a$, $y = e^a$

\therefore the point of contact is (a, e^a) .

Now $y = e^x$

$$\therefore \frac{dy}{dx} = e^x$$

When $x = a$, $\frac{dy}{dx} = e^a$

So, the tangent has equation

$$\begin{aligned}y &= e^a(x - a) + e^a \\ &= e^a x - ae^a + e^a\end{aligned}$$

$$\therefore y = e^a x + e^a(1 - a)$$

b The tangent passes through the origin when $0 = e^a(1 - a)$

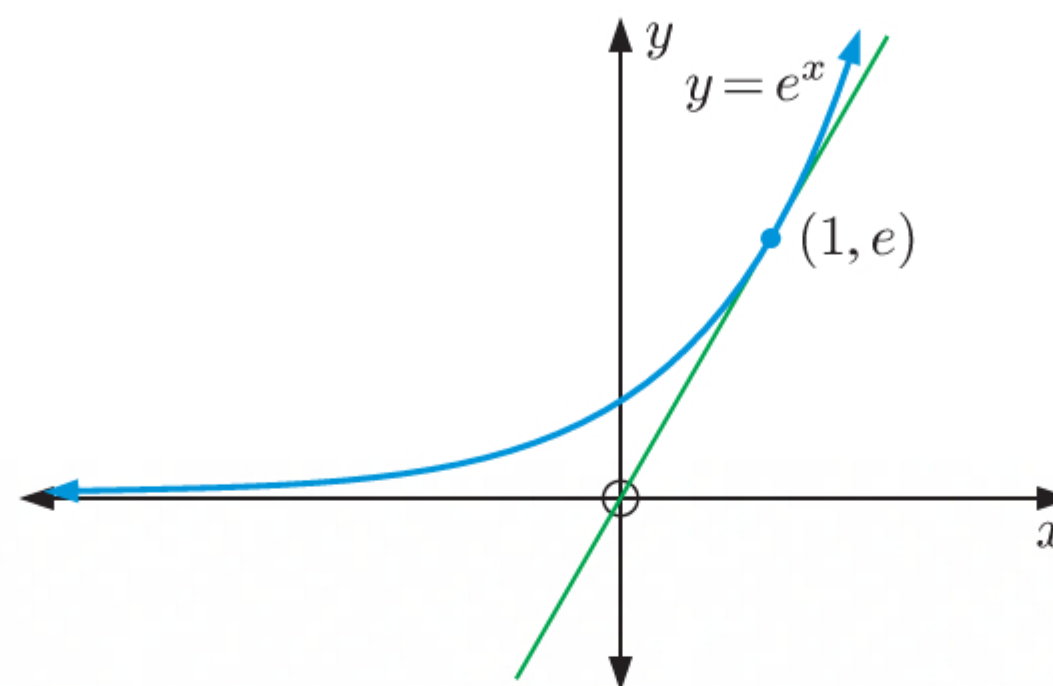
$$\therefore 1 - a = 0 \quad \{\text{as } e^a > 0 \text{ for all } a\}$$

$$\therefore a = 1$$

When $a = 1$, $y = e^1 x + e^1(1 - 1)$

$$= ex$$

\therefore the tangent to $y = e^x$ which passes through the origin is $y = ex$.



24 a $y = 2x^2$

When $x = a$, $y = 2a^2$

\therefore the point of contact is $(a, 2a^2)$.

Now $y = 2x^2$

$$\therefore \frac{dy}{dx} = 4x$$

When $x = a$, $\frac{dy}{dx} = 4a$

So, the tangent has equation

$$\begin{aligned} y &= 4a(x - a) + 2a^2 \\ &= 4ax - 4a^2 + 2a^2 \end{aligned}$$

$$\therefore y = 4ax - 2a^2$$

The tangent passes through $(1, -6)$ when $-6 = 4a(1) - 2a^2$

$$\therefore 2a^2 - 4a - 6 = 0$$

$$\therefore a^2 - 2a - 3 = 0$$

$$\therefore (a + 1)(a - 3) = 0$$

$$\therefore a = -1 \text{ or } 3$$

When $a = -1$, $y = 4(-1)x - 2(-1)^2$
 $= -4x - 2$

When $a = 3$, $y = 4(3)x - 2(3)^2$
 $= 12x - 18$

\therefore the tangents to $y = 2x^2$ which passes through $(1, -6)$ are $y = -4x - 2$ and $y = 12x - 18$.

b $y = 12x - 18$ meets $y = 2x^2$

where $12x - 18 = 2x^2$

$$\therefore 2x^2 - 12x + 18 = 0$$

$$\therefore x^2 - 6x + 9 = 0$$

$$\therefore (x - 3)^2 = 0$$

$$\therefore x = 3$$

When $x = 3$, $y = 2(3)^2$
 $= 18$

So $y = 12x - 18$ has point of contact $(3, 18)$.

$y = -4x - 2$ meets $y = 2x^2$

where $-4x - 2 = 2x^2$

$$\therefore 2x^2 + 4x + 2 = 0$$

$$\therefore x^2 + 2x + 1 = 0$$

$$\therefore (x + 1)^2 = 0$$

$$\therefore x = -1$$

When $x = -1$, $y = 2(-1)^2$
 $= 2$

So $y = -4x - 2$ has point of contact $(-1, 2)$.

c The tangent to $y = 2x^2$ at a general point $(a, 2a^2)$ has equation $y = 4ax - 2a^2$.

The tangent passes through $(1, 4)$ when $4 = 4a(1) - 2a^2$

$$\therefore 4 = 4a - 2a^2$$

$$\therefore 2a^2 - 4a + 4 = 0$$

$$\therefore a^2 - 2a + 2 = 0$$

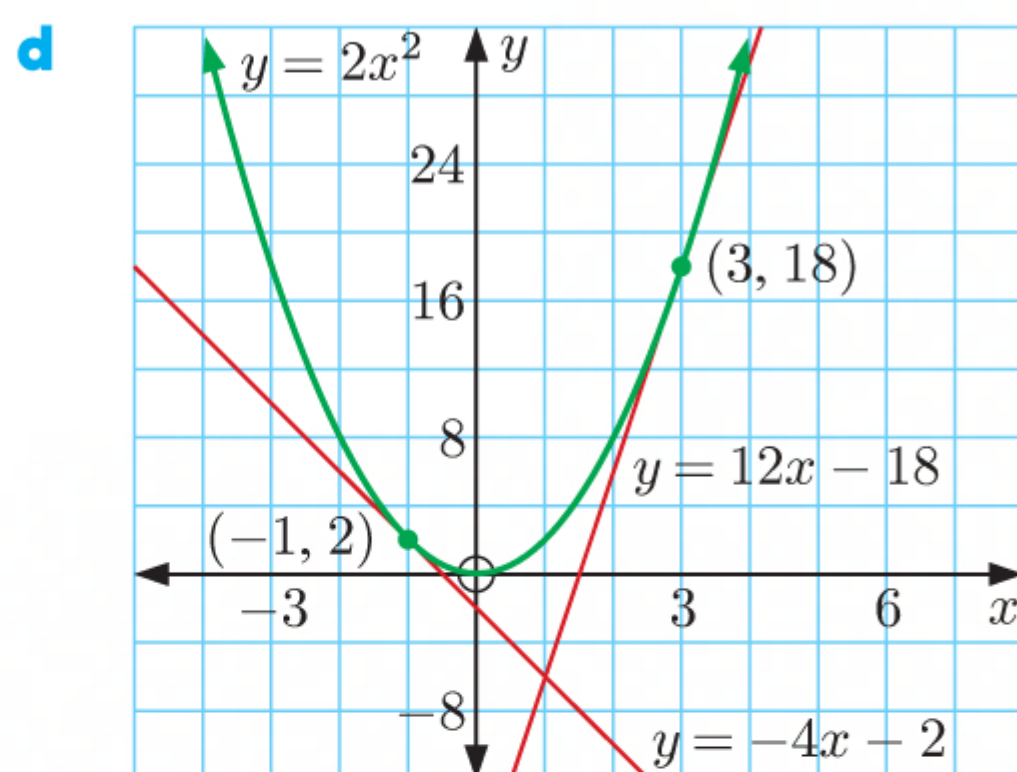
which has $\Delta = (-2)^2 - 4(1)(2)$

$$= 4 - 8$$

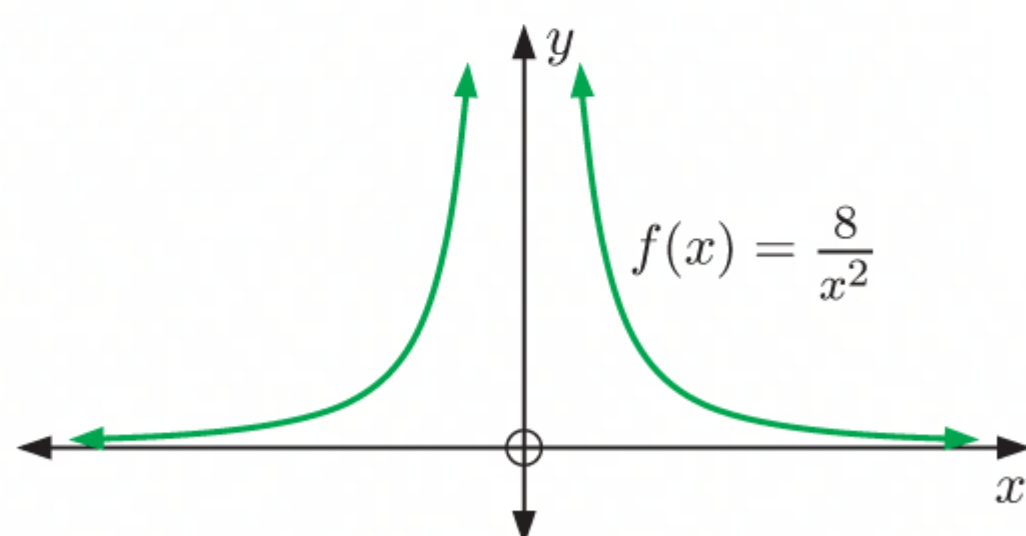
$$= -4$$

\therefore there are no real solutions as $\Delta < 0$.

\therefore there are no tangents to the function that pass through the point $(1, 4)$.



25 a



b $f(x) = \frac{8}{x^2} = 8x^{-2}$

$$\therefore f(a) = \frac{8}{a^2}$$

So, the point of contact is $\left(a, \frac{8}{a^2}\right)$.

$$\text{Now } f'(x) = -16x^{-3} = -\frac{16}{x^3}$$

$$\therefore f'(a) = -\frac{16}{a^3}$$

So, the gradient of the tangent at

$$\left(a, \frac{8}{a^2}\right) \text{ is } -\frac{16}{a^3}$$

\therefore the equation of the tangent is

$$-16x - a^3y = -16a - a^3\left(\frac{8}{a^2}\right)$$

$$\therefore -16x - a^3y = -16a - 8a$$

$$\therefore 16x + a^3y = 24a$$

c The tangent cuts the x -axis when $y = 0$

$$\therefore 16x = 24a$$

$$\therefore x = \frac{3}{2}a$$

\therefore A is $\left(\frac{3}{2}a, 0\right)$.

The tangent cuts the y -axis when $x = 0$

$$\therefore a^3y = 24a$$

$$\therefore y = \frac{24}{a^2}$$

\therefore B is $\left(0, \frac{24}{a^2}\right)$.

d Area of triangle OAB = $\left| \frac{1}{2} \times \left(\frac{3}{2}a\right) \times \left(\frac{24}{a^2}\right) \right|$

$$= \frac{18}{|a|} \text{ units}^2$$

$$\text{As } a \rightarrow \infty, \frac{18}{|a|} \rightarrow 0$$

$$\therefore \text{area} \rightarrow 0$$

26 $y = 3e^{-x}$ and $y = 2 + e^x$ meet where $3e^{-x} = 2 + e^x$
 $\therefore 3 = 2e^x + e^{2x}$ {multiplying both sides by e^x }
 $\therefore e^{2x} + 2e^x - 3 = 0$
 $\therefore (e^x + 3)(e^x - 1) = 0$
 $\therefore e^x = 1$ {as $e^x > 0$ }
 $\therefore x = 0$

Now when $x = 0$, $y = 3e^0 = 3$, so the graphs intersect at $(0, 3)$.

For $y = 2 + e^x$, $\frac{dy}{dx} = e^x$

When $x = 0$, $\frac{dy}{dx} = e^0 = 1$

\therefore the gradient of the tangent at $(0, 3)$ is 1.

\therefore the equation of the tangent is $y = x + 3$.

For $y = 3e^{-x}$, $\frac{dy}{dx} = -3e^{-x}$

When $x = 0$, $\frac{dy}{dx} = -3$

\therefore the gradient of the tangent at $(0, 3)$ is -3 .

\therefore the equation of the tangent is

$$y = -3(x - 0) + 3$$

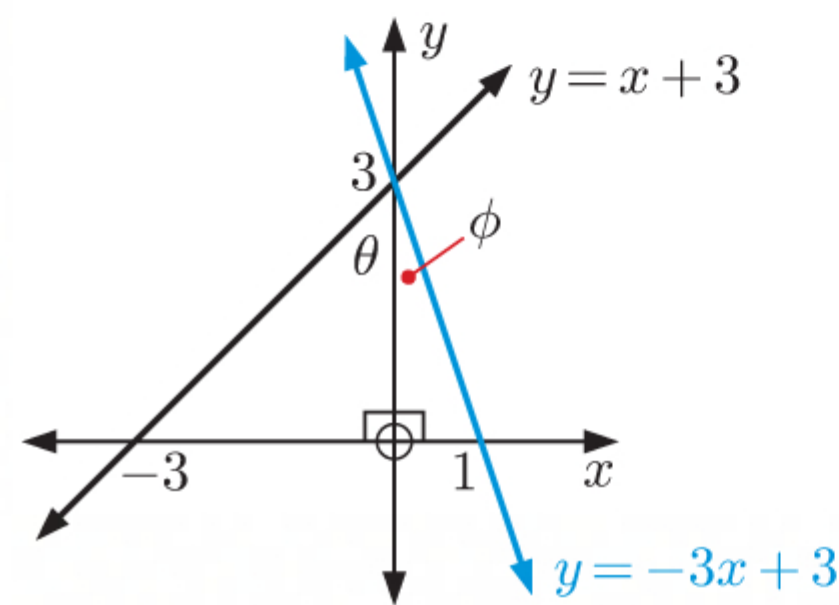
$$= -3x + 3$$

We graph the tangents on the same set of axes.

$$\tan \theta = \frac{3}{3} \quad \text{and} \quad \tan \phi = \frac{1}{3}$$

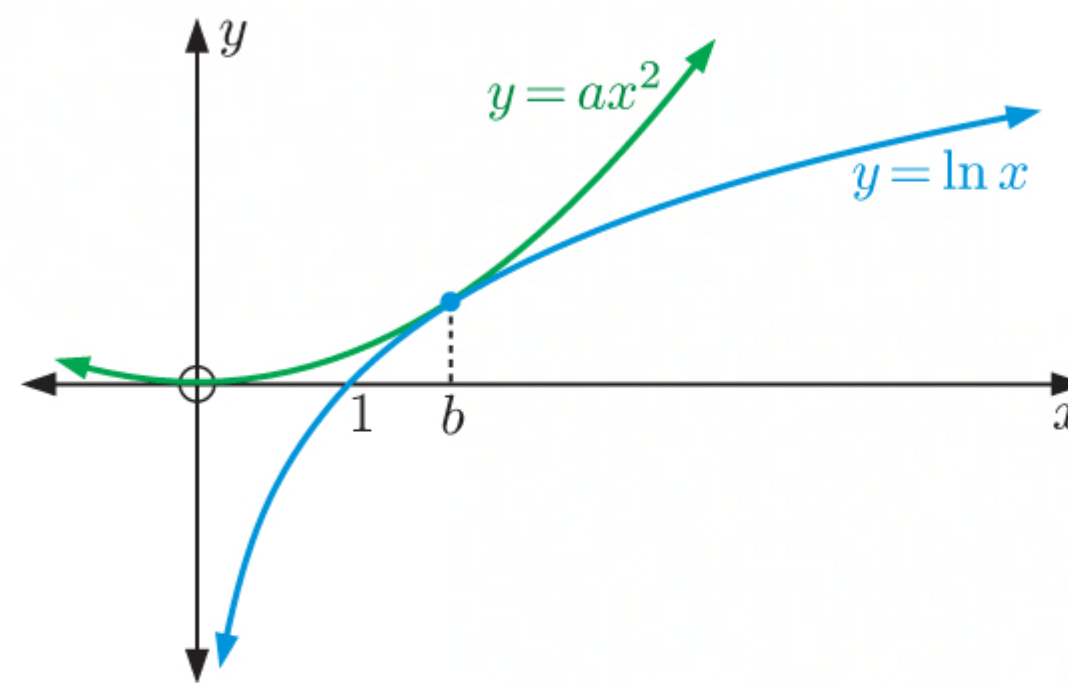
$$= 1 \quad \therefore \phi \approx 18.43^\circ$$

$$\therefore \theta = 45^\circ$$



So, the acute angle between the tangents to $y = 3e^{-x}$ and $y = 2 + e^x$ at their point of intersection is about $45^\circ + 18.43^\circ \approx 63.43^\circ$.

27 a $y = ax^2$, $a > 0$ touches $y = \ln x$ when
 $ax^2 = \ln x$
 If the curves touch when $x = b$ then
 $ab^2 = \ln b$ (1)



Now for $y = ax^2$, $\frac{dy}{dx} = 2ax$ and for $y = \ln x$, $\frac{dy}{dx} = \frac{1}{x}$

\therefore when $x = b$, $\frac{dy}{dx} = 2ab$ \therefore when $x = b$, $\frac{dy}{dx} = \frac{1}{b}$

Since the curves touch each other, they share a common tangent.

$\therefore 2ab = \frac{1}{b}$ (2)

b Now $ab^2 = \frac{1}{2}$ {from (2)}
 and $ab^2 = \ln b$ {from (1)}
 $\therefore \ln b = \frac{1}{2}$
 $\therefore b = e^{\frac{1}{2}} = \sqrt{e}$

When $x = b = \sqrt{e}$, $y = \ln x = \ln e^{\frac{1}{2}} = \frac{1}{2}$
 \therefore the point of contact is $(\sqrt{e}, \frac{1}{2})$.

d The tangent has gradient $\frac{1}{b} = \frac{1}{\sqrt{e}}$ and passes through $(\sqrt{e}, \frac{1}{2})$

\therefore the tangent is $\frac{y - \frac{1}{2}}{x - \sqrt{e}} = \frac{1}{\sqrt{e}}$ $\therefore y - \frac{1}{2} = \frac{1}{\sqrt{e}}(x - \sqrt{e})$
 $\therefore y - \frac{1}{2} = \frac{1}{\sqrt{e}}x - 1$
 $\therefore y = e^{-\frac{1}{2}}x - \frac{1}{2}$

28 $p(x) = ax^2$, $a \neq 0$

a $p(s) = a(s)^2$
 $= as^2$

So the point on the curve where $x = s$ is (s, as^2) .

Now $p'(x) = 2ax$
 $\therefore p'(s) = 2as$

\therefore the equation of the tangent at (s, as^2) is $y = 2as(x - s) + as^2$
 $= 2asx - 2as^2 + as^2$
 $= 2asx - as^2$

Similarly, the equation of the tangent when $x = t$, at (t, at^2) , is $y = 2atx - at^2$.

b $y = 2asx - as^2$ and $y = 2atx - at^2$ meet where $2asx - as^2 = 2atx - at^2$
 $\therefore 2asx - 2atx = as^2 - at^2$
 $\therefore 2ax(s - t) = a(s^2 - t^2)$
 $\therefore 2x(\cancel{s - t}) = (s + t)(\cancel{s - t}) \quad \{a \neq 0\}$
 $\therefore 2x = s + t$
 $\therefore x = \frac{s + t}{2}$

c If the tangent lines $y = 2asx - as^2$ and $y = 2atx - at^2$ are perpendicular, then their gradients are negative reciprocals of each other.

$\therefore 2as = -\frac{1}{2at}$

$\therefore 2ast = -\frac{1}{2a}$

$\therefore ast = -\frac{1}{4a} \dots (*)$

The tangents $y = 2asx - as^2$ and $y = 2atx - at^2$ intersect at $x = \frac{s + t}{2}$. {from **b**}

$$\begin{aligned}
\text{When } x = \frac{s+t}{2}, \quad y &= 2as\left(\frac{s+t}{2}\right) - as^2 \\
&= as(s+t) - as^2 \\
&= as^2 + ast - as^2 \\
&= ast \\
&= -\frac{1}{4a} \quad \{\text{from } (*)\}
\end{aligned}$$

\therefore if the tangent lines are perpendicular then they intersect at $y = -\frac{1}{4a}$.

EXERCISE 13B

1 a $y = x^2$

Now $\frac{dy}{dx} = 2x$, so at $(4, 16)$,

$$\frac{dy}{dx} = 2(4) = 8 = \frac{8}{1}$$

\therefore the normal at $(4, 16)$ has gradient $-\frac{1}{8}$.

\therefore the equation of the normal is

$$-x - 8y = -(4) - 8(16)$$

$$\therefore x + 8y = 132$$

c $y = \frac{5}{\sqrt{x}} - \sqrt{x}$

$$= 5x^{-\frac{1}{2}} - x^{\frac{1}{2}}$$

$\therefore \frac{dy}{dx} = -\frac{5}{2}x^{-\frac{3}{2}} - \frac{1}{2}x^{-\frac{1}{2}}$, so at $(1, 4)$,

$$\begin{aligned}
\frac{dy}{dx} &= -\frac{5}{2}(1^{-\frac{3}{2}}) - \frac{1}{2}(1^{-\frac{1}{2}}) \\
&= -\frac{5}{2} - \frac{1}{2} \\
&= -3
\end{aligned}$$

\therefore the normal at $(1, 4)$ has gradient $\frac{1}{3}$.

\therefore the equation of the normal is

$$x - 3y = 1 - 3(4)$$

$$\therefore x - 3y = -11$$

b $y = x^3 - 5x + 2$

When $x = -2$,

$$\begin{aligned}
y &= (-2)^3 - 5(-2) + 2 \\
&= 4
\end{aligned}$$

So, the point of contact is $(-2, 4)$.

Now $\frac{dy}{dx} = 3x^2 - 5$, so at $x = -2$,

$$\frac{dy}{dx} = 3(-2)^2 - 5 = 7 = \frac{7}{1}$$

\therefore the normal at $(-2, 4)$ has gradient $-\frac{1}{7}$.

\therefore the equation of the normal is

$$-x - 7y = -(-2) - 7(4)$$

$$\therefore x + 7y = 26$$

d $y = 8\sqrt{x} - \frac{1}{x^2}$

When $x = 1$,

$$y = 8\sqrt{1} - \frac{1}{1^2} = 7$$

So, the point of contact is $(1, 7)$.

Now $y = 8\sqrt{x} - \frac{1}{x^2}$

$$= 8x^{\frac{1}{2}} - x^{-2}$$

$\therefore \frac{dy}{dx} = 4x^{-\frac{1}{2}} + 2x^{-3}$, so at $x = 1$,

$$\begin{aligned}
\frac{dy}{dx} &= 4(1^{-\frac{1}{2}}) + 2(1^{-3}) \\
&= 4 + 2 \\
&= 6
\end{aligned}$$

\therefore the normal at $(1, 7)$ has gradient $-\frac{1}{6}$.

\therefore the equation of the normal is

$$-x - 6y = -(1) - 6(7)$$

$$\therefore x + 6y = 43$$

$$\begin{aligned}
 \text{e} \quad f(x) &= \frac{x}{1-3x} \\
 &= x(1-3x)^{-1} \\
 \therefore f'(x) &= (1-3x)^{-1} - x(1-3x)^{-2}(-3) \quad \{\text{product rule}\} \\
 &= \frac{1}{1-3x} + \frac{3x}{(1-3x)^2} \\
 \therefore f'(-1) &= \frac{1}{1-3(-1)} + \frac{3(-1)}{(1-3(-1))^2} \\
 &= \frac{1}{1+3} - \frac{3}{(1+3)^2} \\
 &= \frac{1}{4} - \frac{3}{16} \\
 &= \frac{1}{16}
 \end{aligned}$$

\therefore the normal at $(-1, -\frac{1}{4})$ has gradient -16 .

$$\begin{aligned}
 \therefore \text{ the equation of the normal is } 16x + y &= 16(-1) + (-\frac{1}{4}) \\
 &= -16 - \frac{1}{4} \\
 \therefore 16x + y &= -\frac{65}{4} \\
 \therefore 64x + 4y &= -65
 \end{aligned}$$

$$\begin{aligned}
 \text{f} \quad f(x) &= \frac{x^2}{1-x} \\
 &= x^2(1-x)^{-1} \\
 \therefore f'(x) &= 2x(1-x)^{-1} - x^2(1-x)^{-2}(-1) \quad \{\text{product rule}\} \\
 &= \frac{2x}{1-x} + \frac{x^2}{(1-x)^2} \\
 \therefore f'(2) &= \frac{2(2)}{1-2} + \frac{(2)^2}{(1-2)^2} \\
 &= \frac{4}{-1} + \frac{4}{1} \\
 &= 0
 \end{aligned}$$

\therefore the tangent at $(2, -4)$ is a horizontal line.

\therefore the normal at $(2, -4)$ is a vertical line passing through $(2, -4)$.

\therefore the equation of the normal is $x = 2$.

2 a $f(x) = x^2 - \frac{8}{x}$
 $\therefore f(-2) = (-2)^2 - \frac{8}{(-2)}$
 $= 4 + 4$
 $= 8$

So, the point of contact is $(-2, 8)$.

Now $f(x) = x^2 - 8x^{-1}$
 $\therefore f'(x) = 2x + 8x^{-2}$
 $= 2x + \frac{8}{x^2}$
 $\therefore f'(-2) = 2(-2) + \frac{8}{(-2)^2}$
 $= -4 + 2$
 $= -2$

\therefore the tangent at $(-2, 8)$ has gradient -2 .

\therefore the equation of the tangent is

$$y = -2(x + 2) + 8$$

$$\therefore y = -2x - 4 + 8$$

$$\therefore y = 4 - 2x$$

3 a When $x = 0$, $y = e^0 = 1$. So, the point of contact is $(0, 1)$.

Now as $y = e^{-x}$, $\frac{dy}{dx} = -e^{-x}$
 \therefore when $x = 0$, $\frac{dy}{dx} = -e^0 = -1$
 \therefore the normal at $(0, 1)$ has gradient 1 .
 \therefore the equation of the normal is
 $y = x + 1$.

c When $x = 1$, $y = e^{2(1)-1} = e$. So, the point of contact is $(1, e)$.

Now as $y = e^{2x-1}$, $\frac{dy}{dx} = 2e^{2x-1}$
 \therefore when $x = 1$, $\frac{dy}{dx} = 2e^{2(1)-1} = 2e$
 \therefore the normal at $(1, e)$ has gradient $-\frac{1}{2e}$.
 \therefore the equation of the normal is
 $y = -\frac{1}{2e}(x - 1) + e$
 $\therefore 2ey = -(x - 1) + 2e^2$
 $\therefore 2ey = -x + 1 + 2e^2$
 $\therefore x + 2ey = 1 + 2e^2$

b $f(x) = x^2 - \frac{8}{x}$
 $\therefore f(3) = (3)^2 - \frac{8}{3}$
 $= 9 - \frac{8}{3}$
 $= \frac{19}{3}$

So, the point of contact is $(3, \frac{19}{3})$.

Now $f'(x) = 2x + \frac{8}{x^2}$ {from **a**}
 $\therefore f'(3) = 2(3) + \frac{8}{(3)^2}$
 $= 6 + \frac{8}{9}$
 $= \frac{62}{9}$

\therefore the normal at $(3, \frac{19}{3})$ has gradient $-\frac{9}{62}$.

\therefore the equation of the normal is

$$y = -\frac{9}{62}(x - 3) + \frac{19}{3}$$

$$\therefore y = -\frac{9}{62}x + \frac{27}{62} + \frac{19}{3}$$

$$\therefore y = -\frac{9}{62}x + \frac{1259}{186}$$

b When $x = e$, $y = \ln e = 1$. So, the point of contact is $(e, 1)$.

Now as $y = \ln x$, $\frac{dy}{dx} = \frac{1}{x}$
 \therefore when $x = e$, $\frac{dy}{dx} = \frac{1}{e}$
 \therefore the normal at $(e, 1)$ has gradient $-e$.
 \therefore the equation of the normal is
 $y = -e(x - e) + 1$
 $\therefore y = -ex + e^2 + 1$
 $\therefore ex + y = e^2 + 1$

d When $x = \frac{\pi}{3}$, $y = \sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}$. So, the point of contact is $(\frac{\pi}{3}, \frac{\sqrt{3}}{2})$.

Now as $y = \sin x$, $\frac{dy}{dx} = \cos x$
 \therefore when $x = \frac{\pi}{3}$, $\frac{dy}{dx} = \cos \frac{\pi}{3} = \frac{1}{2}$
 \therefore the normal at $(\frac{\pi}{3}, \frac{\sqrt{3}}{2})$ has gradient -2 .
 \therefore the equation of the normal is
 $y = -2(x - \frac{\pi}{3}) + \frac{\sqrt{3}}{2}$
 $\therefore y = -2x + \frac{2\pi}{3} + \frac{\sqrt{3}}{2}$
 $\therefore 2x + y = \frac{2\pi}{3} + \frac{\sqrt{3}}{2}$

$$\begin{aligned}
 4 \quad y &= a\sqrt{x} + \frac{b}{\sqrt{x}} \\
 &= ax^{\frac{1}{2}} + bx^{-\frac{1}{2}}
 \end{aligned}$$

$$\therefore \frac{dy}{dx} = \frac{a}{2}x^{-\frac{1}{2}} - \frac{b}{2}x^{-\frac{3}{2}}$$

$$\begin{aligned}
 \text{When } x = 4, \quad \frac{dy}{dx} &= \frac{a}{2}\left(4^{-\frac{1}{2}}\right) - \frac{b}{2}\left(4^{-\frac{3}{2}}\right) \\
 &= \frac{a}{2}\left(\frac{1}{2}\right) - \frac{b}{2}\left(\frac{1}{8}\right) \\
 &= \frac{a}{4} - \frac{b}{16} \\
 &= \frac{4a - b}{16}
 \end{aligned}$$

The gradient of the normal to the curve at $x = 4$ will be $\frac{16}{b - 4a}$.

However, the equation of the normal is $4x + y = 22$ or $y = -4x + 22$ which has gradient -4 .

$$\therefore \frac{16}{b - 4a} = -4$$

$$\therefore b - 4a = -4$$

$$\therefore b = 4a - 4 \quad \dots (*)$$

Also, at $x = 4$ the normal line intersects the curve.

$$\therefore a\sqrt{4} + \frac{b}{\sqrt{4}} = -4(4) + 22$$

$$\therefore 2a + \frac{b}{2} = 6$$

$$\text{Consequently, } 2a + \frac{4a - 4}{2} = 6 \quad \{\text{using } (*)\}$$

$$\therefore 2a + 2a - 2 = 6$$

$$\therefore 4a = 8$$

$$\therefore a = 2$$

$$\text{and so } b = 4(2) - 4 = 4 \quad \{\text{from } (*)\}$$

5 When $x = 1$, $y = (1)^3 - 2(1)^2 + 1 = 0$. So the point of contact is $(1, 0)$.

$$\text{Now as } y = x^3 - 2x^2 + 1, \quad \frac{dy}{dx} = 3x^2 - 4x$$

$$\therefore \text{ when } x = 1, \quad \frac{dy}{dx} = 3(1)^2 - 4(1) = -1$$

\therefore the normal at $(1, 0)$ has gradient 1.

\therefore the equation of the normal is $y = x - 1$.

This line meets the curve where $x - 1 = x^3 - 2x^2 + 1$

$$\therefore x^3 - 2x^2 - x + 2 = 0$$

$$\therefore x = 2, 1, \text{ or } -1 \quad \{\text{using technology}\}$$

$$\begin{aligned}
 \text{When } x = 2, \quad y &= (2)^3 - 2(2)^2 + 1 \\
 &= 8 - 8 + 1 \\
 &= 1
 \end{aligned}$$

$$\begin{aligned}
 \text{When } x = -1, \quad y &= (-1)^3 - 2(-1)^2 + 1 \\
 &= -1 - 2 + 1 \\
 &= -2
 \end{aligned}$$

\therefore the normal meets the curve again at $(2, 1)$ and $(-1, -2)$.

- 6 Let $(a, \cos a)$ be a general point on $f(x) = \cos x$.

Now $f'(x) = -\sin x$, so $f'(a) = -\sin a$

\therefore the normal at $(a, \cos a)$ has gradient $\frac{1}{\sin a}$.

\therefore the equation of the normal is $y = \frac{1}{\sin a}(x - a) + \cos a$.

The normal passes through the origin when

$$0 = \frac{1}{\sin a}(0 - a) + \cos a$$

$$\therefore 0 = -\frac{a}{\sin a} + \cos a$$

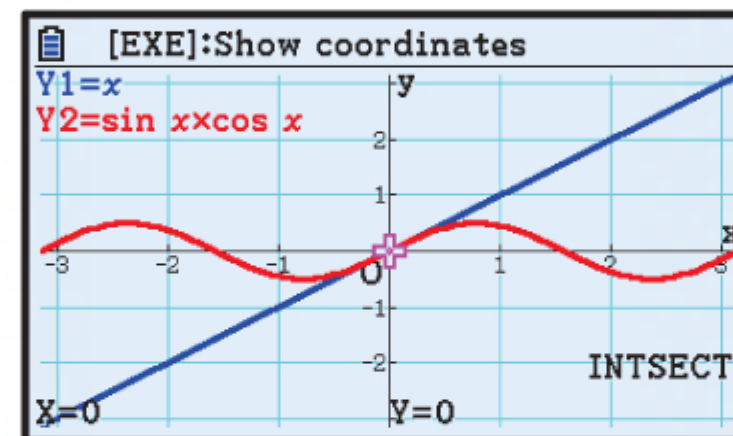
$$\therefore \frac{a}{\sin a} = \cos a$$

$$\therefore a = \sin a \cos a$$

$$\therefore a = 0$$

So, the normal at $(0, \cos 0)$, or $(0, 1)$, has gradient $\frac{1}{\sin 0}$ which is undefined. The normal is a vertical line.

\therefore the equation of the normal to $f(x) = \cos x$ which passes through the origin is the vertical line $x = 0$.



- 7 Let (a, \sqrt{a}) be a general point on $y = \sqrt{x}$.

$$\begin{aligned} \text{Now } \frac{dy}{dx} &= \frac{1}{2}x^{-\frac{1}{2}} \\ &= \frac{1}{2\sqrt{x}} \end{aligned}$$

$$\text{so at } x = a, \quad \frac{dy}{dx} = \frac{1}{2\sqrt{a}}$$

So, the gradient of the normal at this point is $-2\sqrt{a}$.

\therefore the normal has equation $y = -2\sqrt{a}(x - a) + \sqrt{a}$.

But this normal passes through $(4, 0)$, so $0 = -2\sqrt{a}(4 - a) + \sqrt{a}$

$$\therefore 2\sqrt{a}(4 - a) - \sqrt{a} = 0$$

$$\therefore \sqrt{a}(8 - 2a - 1) = 0$$

$$\therefore \sqrt{a}(7 - 2a) = 0$$

$$\therefore a = 0 \text{ or } \frac{7}{2}$$

But $a = 0$ is the end point of the function, so there is no normal here.

$$\text{When } a = \frac{7}{2}, \quad y = -2\sqrt{\frac{7}{2}}\left(x - \frac{7}{2}\right) + \sqrt{\frac{7}{2}}$$

$$\therefore y + 2\sqrt{\frac{7}{2}}\left(x - \frac{7}{2}\right) = \sqrt{\frac{7}{2}}$$

$$\therefore \sqrt{2}y + 2\sqrt{7}\left(x - \frac{7}{2}\right) = \sqrt{7}$$

$$\therefore \sqrt{2}y + 2\sqrt{7}x - 7\sqrt{2} = \sqrt{7}$$

$$\therefore \sqrt{2}y + 2\sqrt{7}x = 8\sqrt{7}$$

$$\therefore 2y + 2\sqrt{14}x = 8\sqrt{14}$$

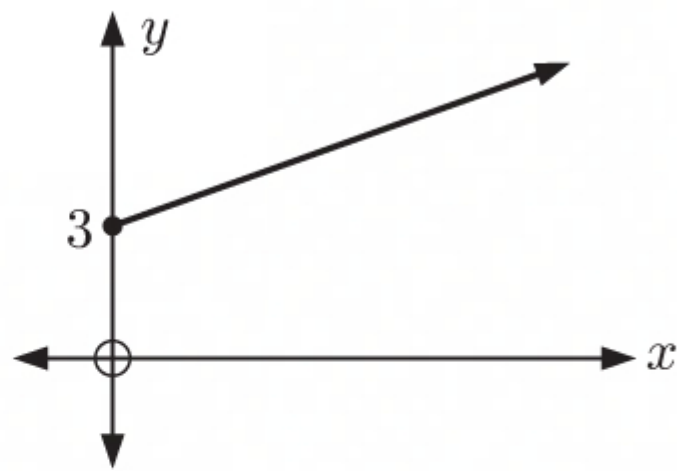
$$\therefore y + \sqrt{14}x = 4\sqrt{14}$$

$$\therefore y = -\sqrt{14}x + 4\sqrt{14} \text{ is the normal to } y = \sqrt{x} \text{ from } (4, 0), \text{ with}$$

$$\text{contact point } \left(\frac{7}{2}, \sqrt{\frac{7}{2}}\right).$$

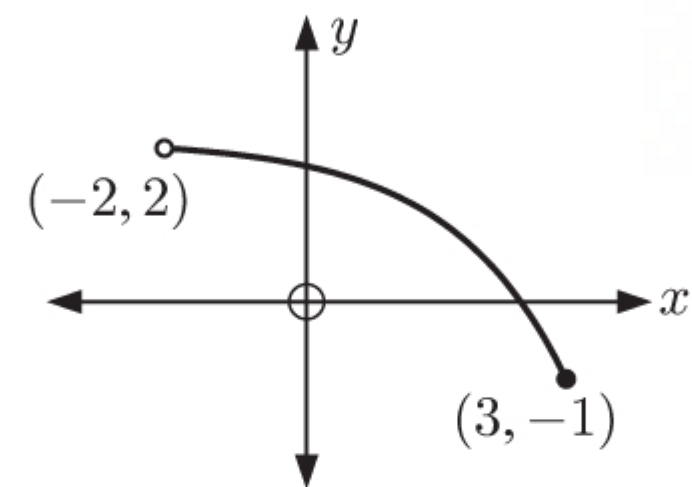
EXERCISE 13C

1 a



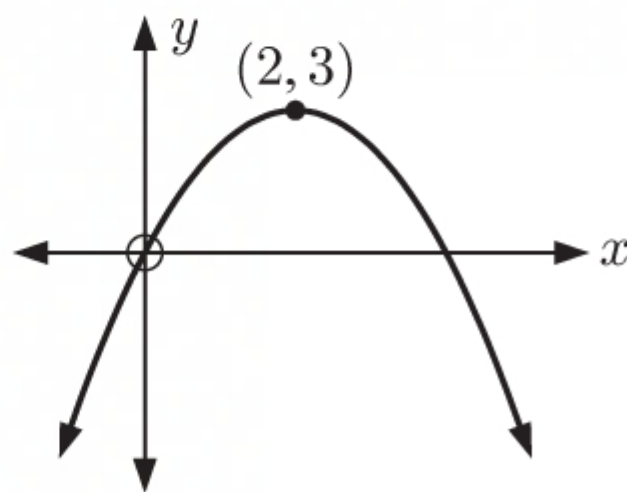
- i The graph is increasing for $x \geq 0$.
- ii The graph is never decreasing.

b



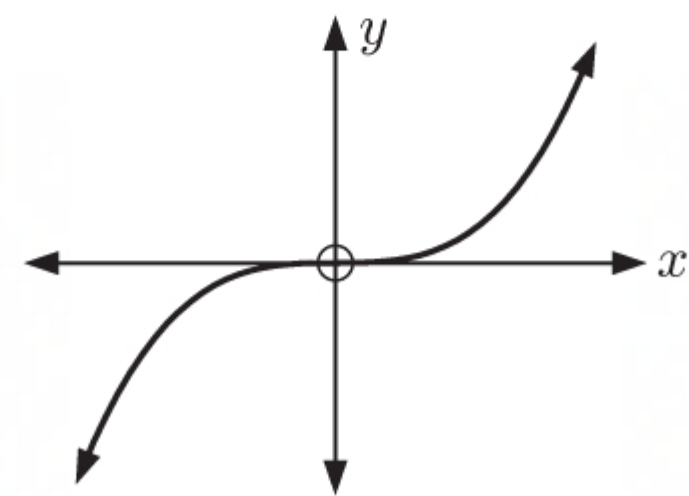
- i The graph is never increasing.
- ii The graph is decreasing for $-2 < x \leq 3$.

c



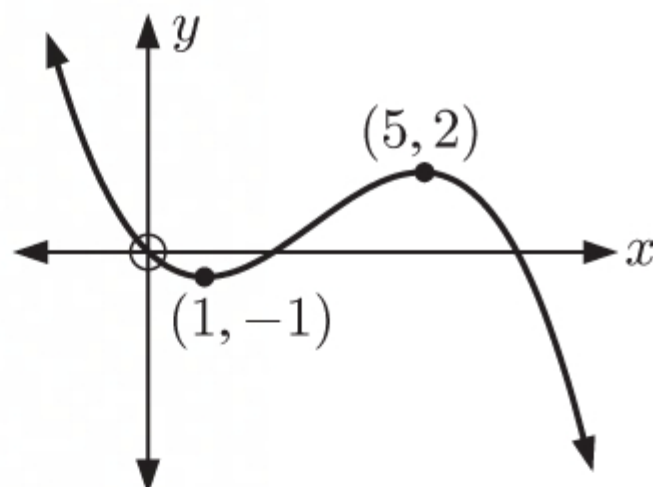
- i The graph is increasing for $x \leq 2$.
- ii The graph is decreasing for $x \geq 2$.

d



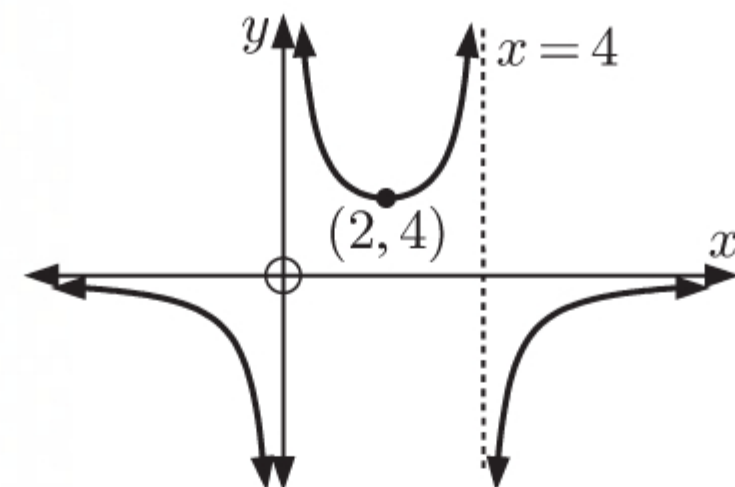
- i The graph is increasing for all real x .
- ii The graph is never decreasing.

e



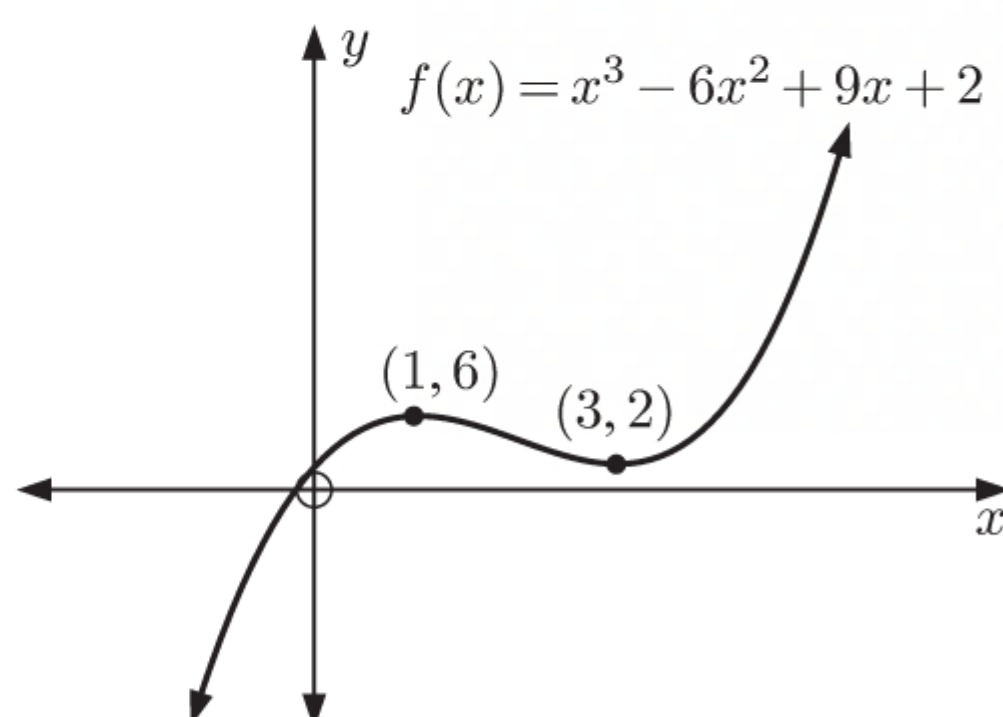
- i The graph is increasing for $1 \leq x \leq 5$.
- ii The graph is decreasing for $x \leq 1$, $x \geq 5$.

f



- i The graph is increasing for $2 \leq x < 4$, $x > 4$.
- ii The graph is decreasing for $x < 0$, $0 < x \leq 2$.

2 a



- i The function is increasing for $x \leq 1$ and $x \geq 3$.
- ii The function is decreasing for $1 \leq x \leq 3$.

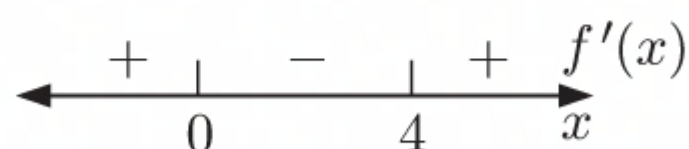
b $f(x) = x^3 - 6x^2 + 9x + 2$
 $\therefore f'(x) = 3x^2 - 12x + 9$
 $= 3(x^2 - 4x + 3)$
 $= 3(x - 1)(x - 3)$

which has sign diagram: 

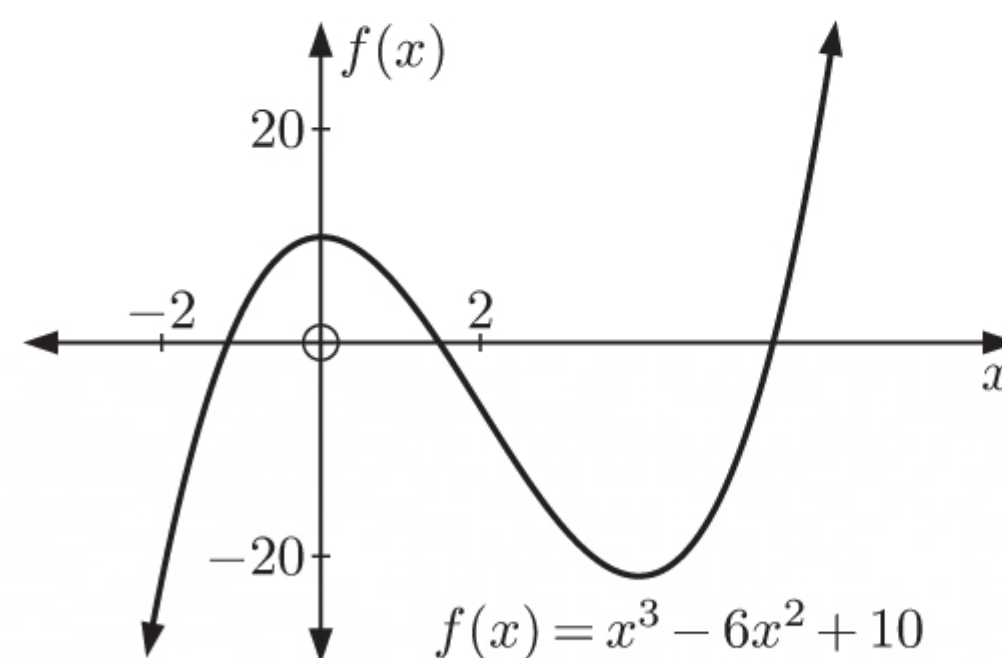
From the sign diagram, $f(x)$ is increasing for $x \leq 1$ and $x \geq 3$, and decreasing for $1 \leq x \leq 3$.

3 a $f(x) = x^3 - 6x^2 + 10$
 $\therefore f'(x) = 3x^2 - 12x$
 $= 3x(x - 4)$

which has sign diagram:

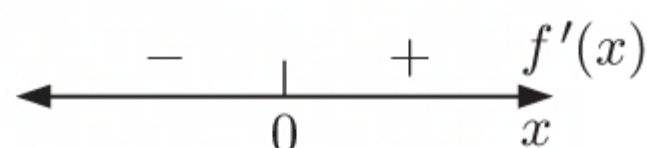


b $f(x)$ is increasing for $x \leq 0$ and for $x \geq 4$.
 $f(x)$ is decreasing for $0 \leq x \leq 4$.

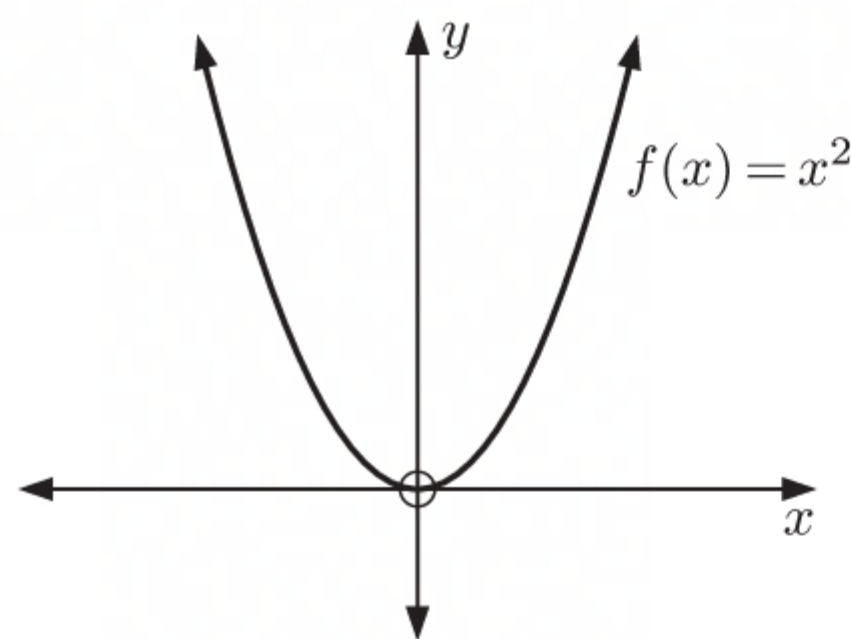


4 a $f(x) = x^2$
 $\therefore f'(x) = 2x$

which has sign diagram:

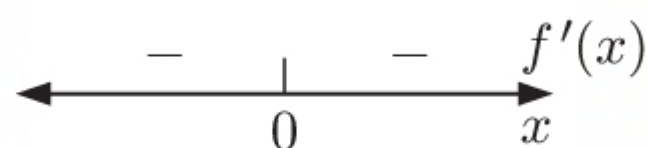


So, $f(x)$ is increasing for $x \geq 0$,
and decreasing for $x \leq 0$.

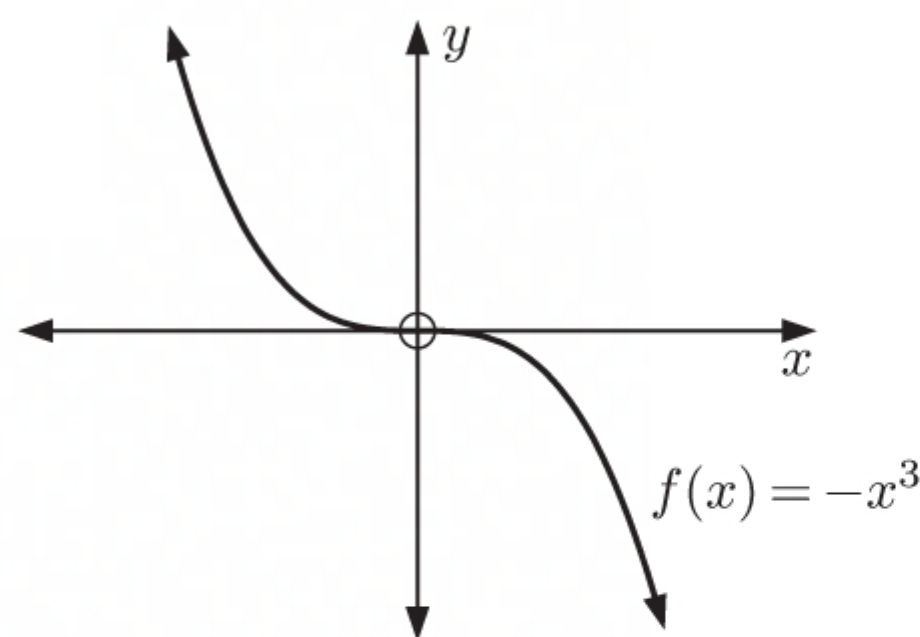


b $f(x) = -x^3$
 $\therefore f'(x) = -3x^2$

which has sign diagram:

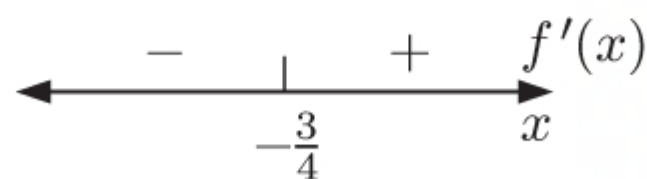


So, $f(x)$ is decreasing for all $x \in \mathbb{R}$.

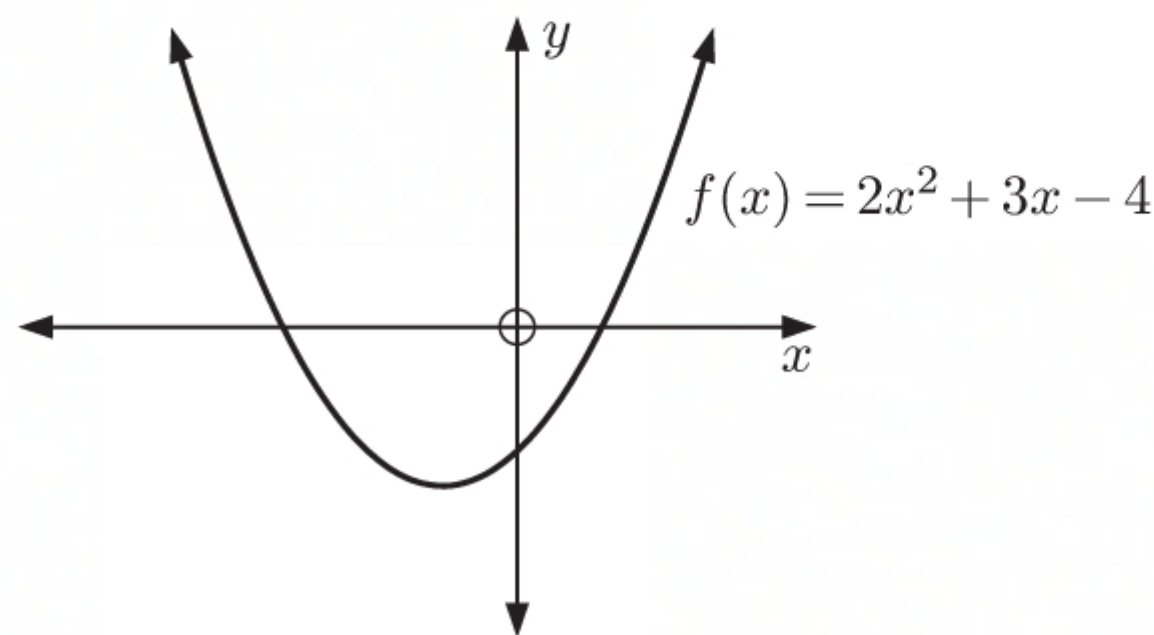


c $f(x) = 2x^2 + 3x - 4$
 $\therefore f'(x) = 4x + 3$

which has sign diagram:

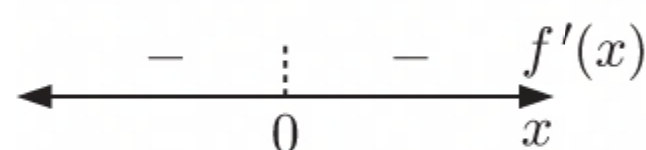


So, $f(x)$ is increasing for $x \geq -\frac{3}{4}$,
and decreasing for $x \leq -\frac{3}{4}$.

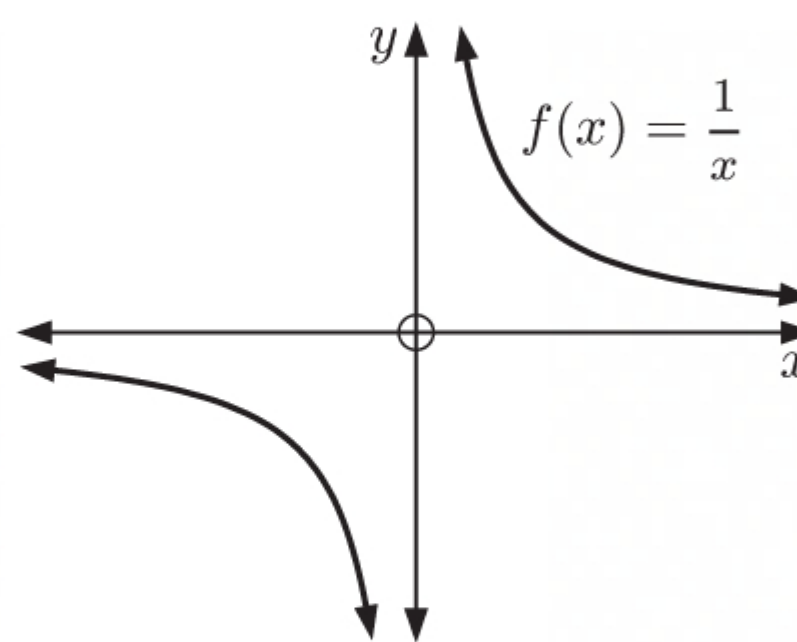


d $f(x) = \frac{1}{x} = x^{-1}$
 $\therefore f'(x) = -x^{-2} = -\frac{1}{x^2}$

which has sign diagram:

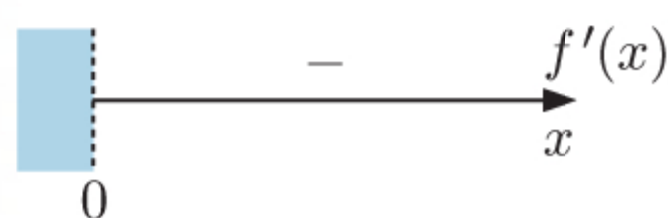


So, $f(x)$ is decreasing for all $x \neq 0$.



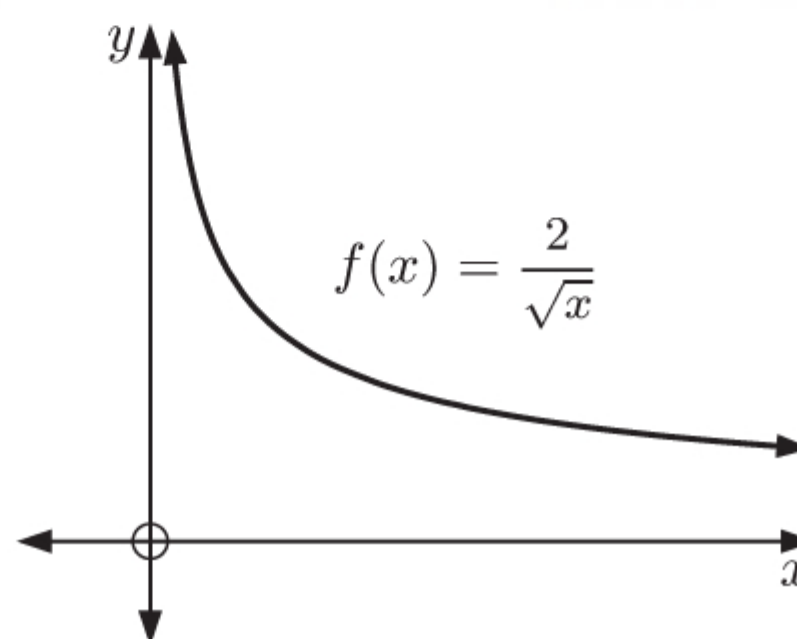
e $f(x) = \frac{2}{\sqrt{x}} = 2x^{-\frac{1}{2}}$
 $\therefore f'(x) = -x^{-\frac{3}{2}} = -\frac{1}{x\sqrt{x}}$

which has sign diagram:



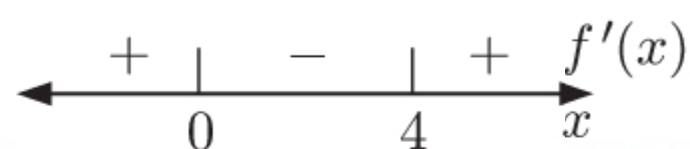
So, $f(x)$ is only defined for $x > 0$.

$f(x)$ is never increasing, but is decreasing for $x > 0$.

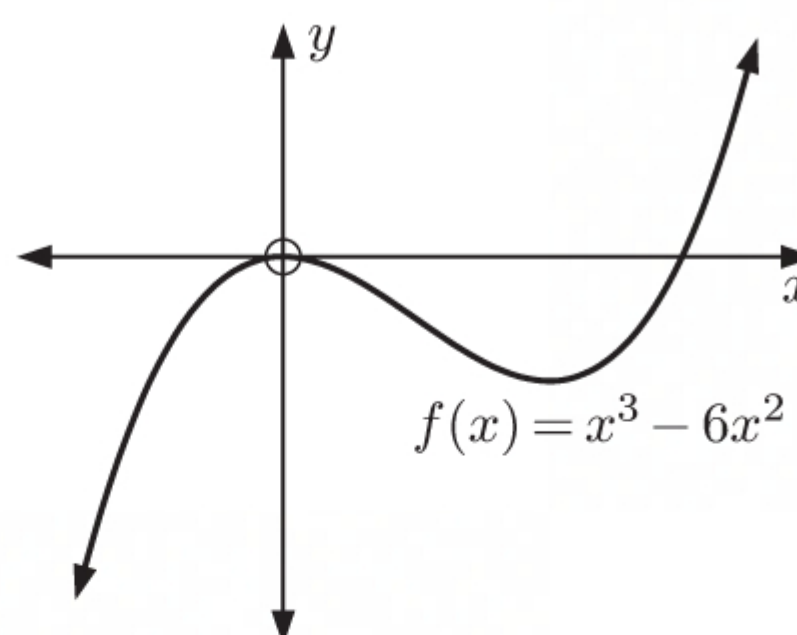


f $f(x) = x^3 - 6x^2$
 $\therefore f'(x) = 3x^2 - 12x$
 $= 3x(x - 4)$

which has sign diagram:

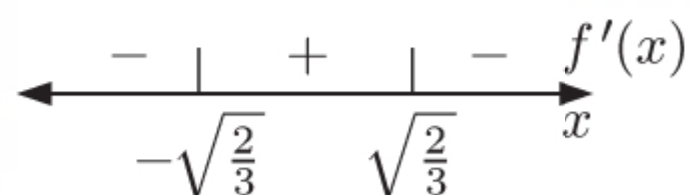


So, $f(x)$ is increasing for $x \leq 0$ and $x \geq 4$, and decreasing for $0 \leq x \leq 4$.



g $f(x) = -2x^3 + 4x$
 $\therefore f'(x) = -6x^2 + 4$
 $= -2(3x^2 - 2)$
 $= -2(\sqrt{3}x + \sqrt{2})(\sqrt{3}x - \sqrt{2})$

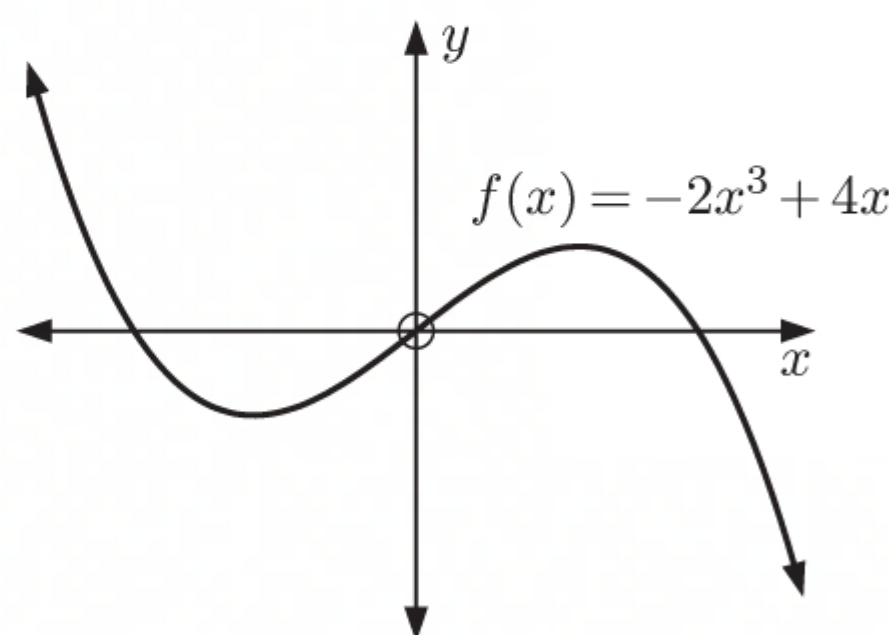
which has sign diagram:



So, $f(x)$ is increasing for

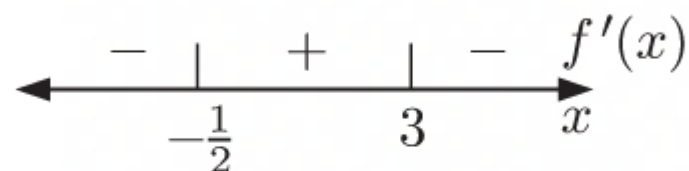
$-\sqrt{\frac{2}{3}} \leq x \leq \sqrt{\frac{2}{3}}$, and decreasing for

$x \leq -\sqrt{\frac{2}{3}}$ and $x \geq \sqrt{\frac{2}{3}}$.

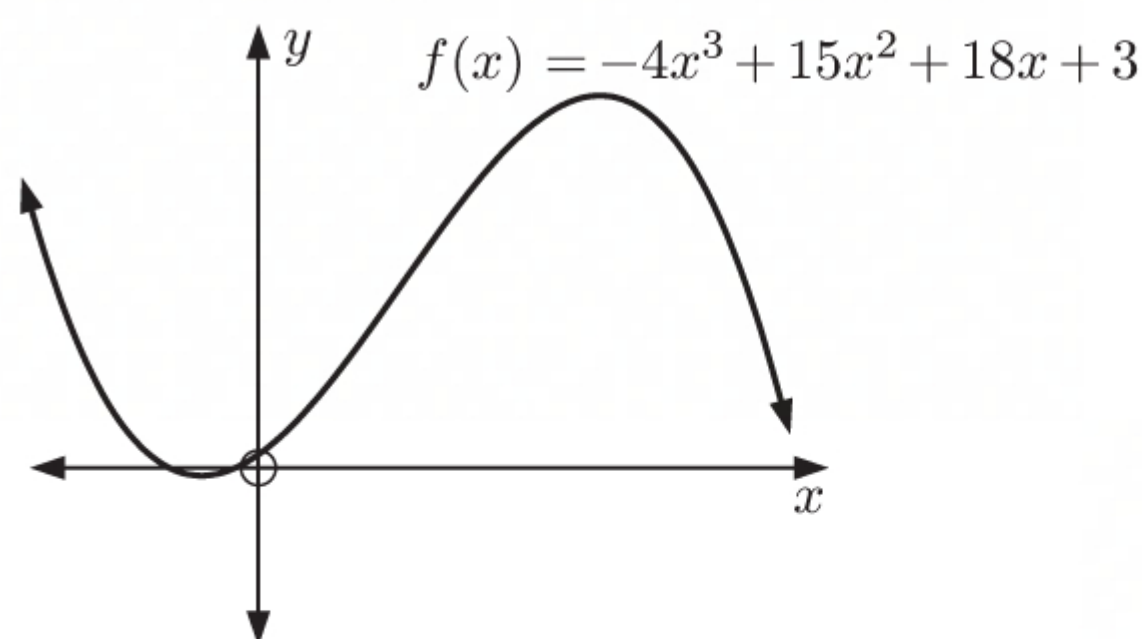


h $f(x) = -4x^3 + 15x^2 + 18x + 3$
 $\therefore f'(x) = -12x^2 + 30x + 18$
 $= -6(2x^2 - 5x - 3)$
 $= -6(2x + 1)(x - 3)$

which has sign diagram:

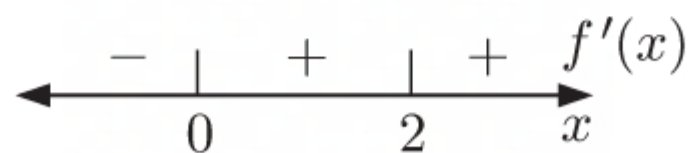


So, $f(x)$ is increasing for $-\frac{1}{2} \leq x \leq 3$,
and decreasing for $x \leq -\frac{1}{2}$ or $x \geq 3$.

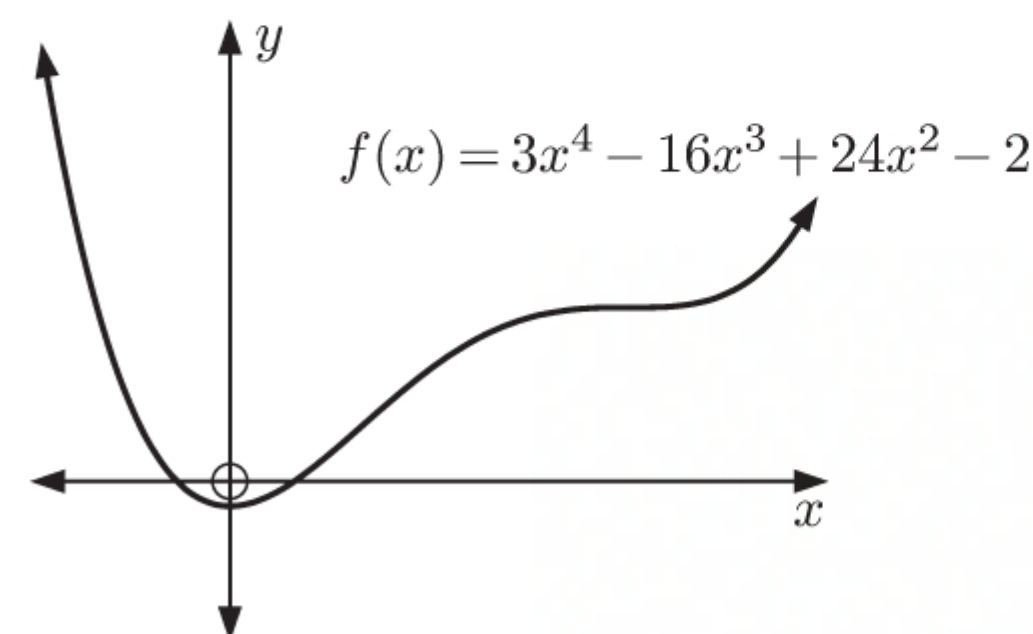


i $f(x) = 3x^4 - 16x^3 + 24x^2 - 2$
 $\therefore f'(x) = 12x^3 - 48x^2 + 48x$
 $= 12x(x^2 - 4x + 4)$
 $= 12x(x - 2)^2$

which has sign diagram:

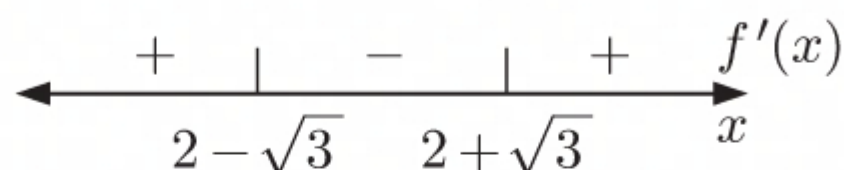


So, $f(x)$ is increasing for $x \geq 0$,
and decreasing for $x \leq 0$.

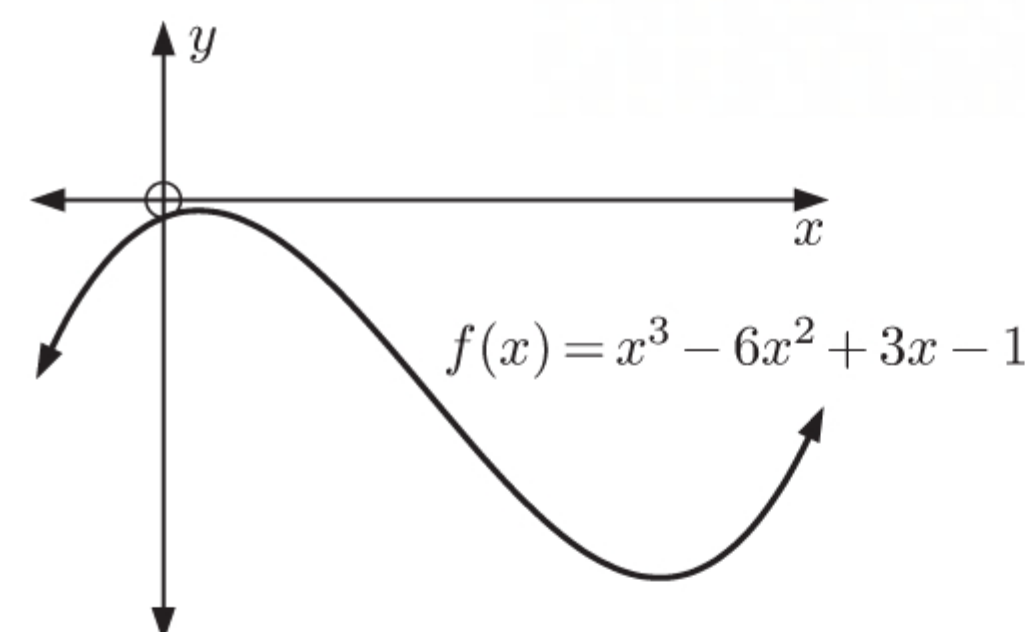


j $f(x) = x^3 - 6x^2 + 3x - 1$
 $\therefore f'(x) = 3x^2 - 12x + 3$
 $= 3(x^2 - 4x + 1)$
 $f'(x) = 0$ when $x = \frac{4 \pm \sqrt{16 - 4}}{2}$
 $= 2 \pm \sqrt{3}$

Sign diagram of $f'(x)$:



So, $f(x)$ is increasing for $x \leq 2 - \sqrt{3}$ and $x \geq 2 + \sqrt{3}$,
and decreasing for $2 - \sqrt{3} \leq x \leq 2 + \sqrt{3}$.



5 a $f(x) = x^3 - 3x^2 + 5x + 2$
 $\therefore f'(x) = 3x^2 - 6x + 5$

b $\Delta = b^2 - 4ac$
 $= (-6)^2 - 4(3)(5)$
 $= 36 - 60$
 $= -24$ which is < 0

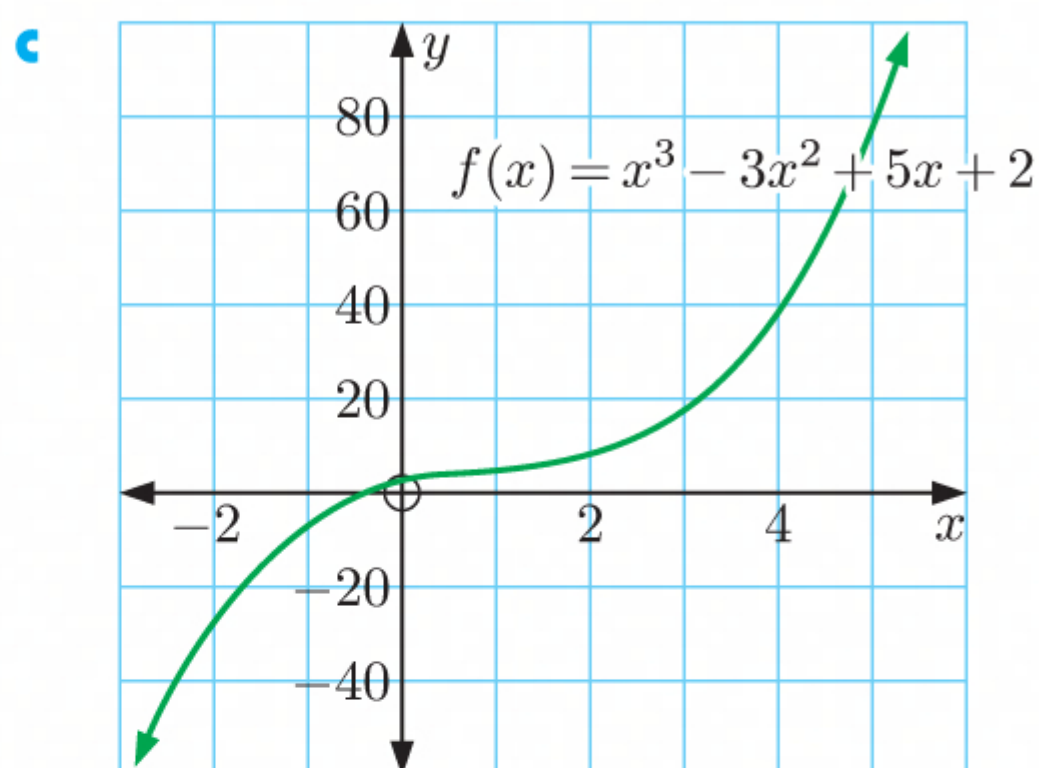
$\therefore f'(x)$ has no real roots.

Also, $a > 0$ which means $f'(x)$ is concave up .

$\therefore f'(x)$ lies entirely above the x -axis.

$\therefore f'(x) > 0$ for all x .

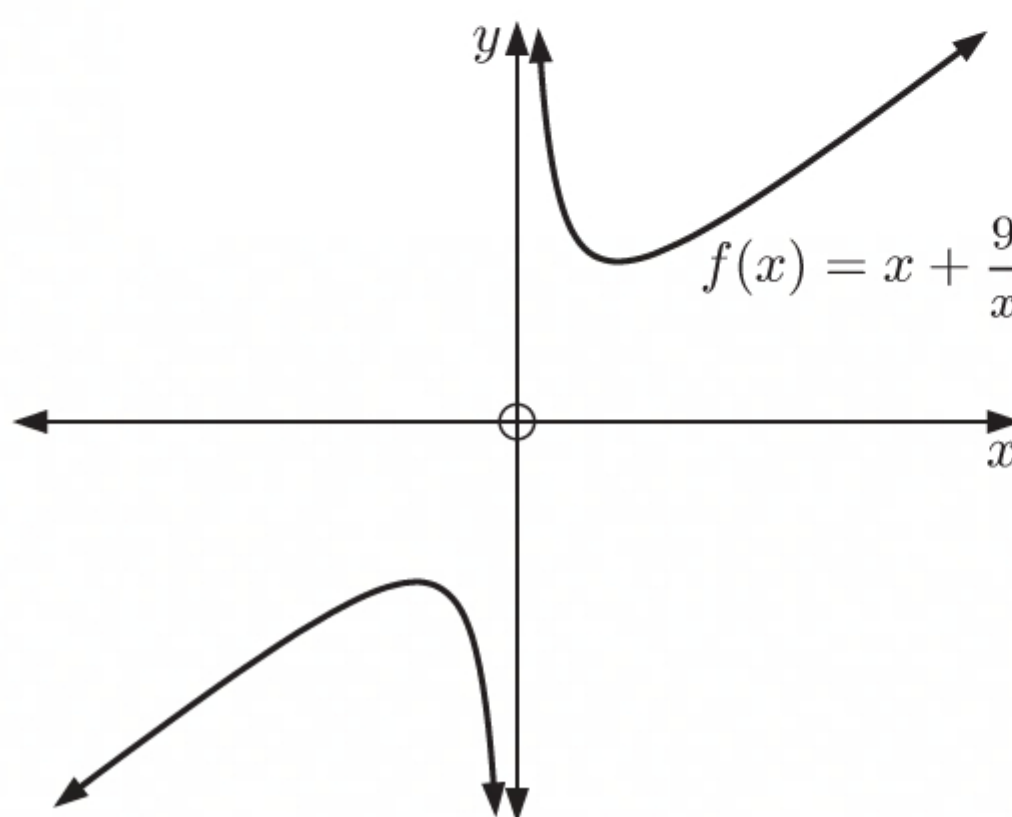
$\therefore f(x)$ is increasing for all x .



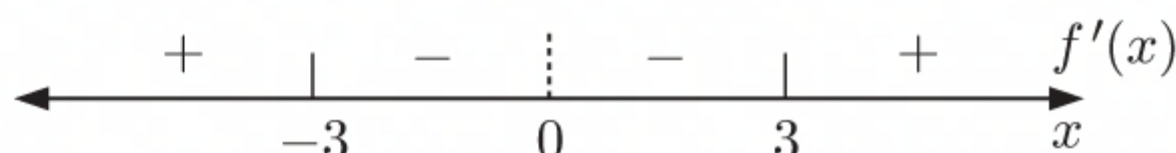
As we can see from the graph of $y = f(x)$, $f(x)$ is increasing for all x .

6 a

$$\begin{aligned} f(x) &= x + \frac{9}{x} \\ &= x + 9x^{-1} \\ \therefore f'(x) &= 1 - 9x^{-2} \\ &= 1 - \frac{9}{x^2} \\ &= \frac{x^2 - 9}{x^2} \\ &= \frac{(x+3)(x-3)}{x^2} \end{aligned}$$



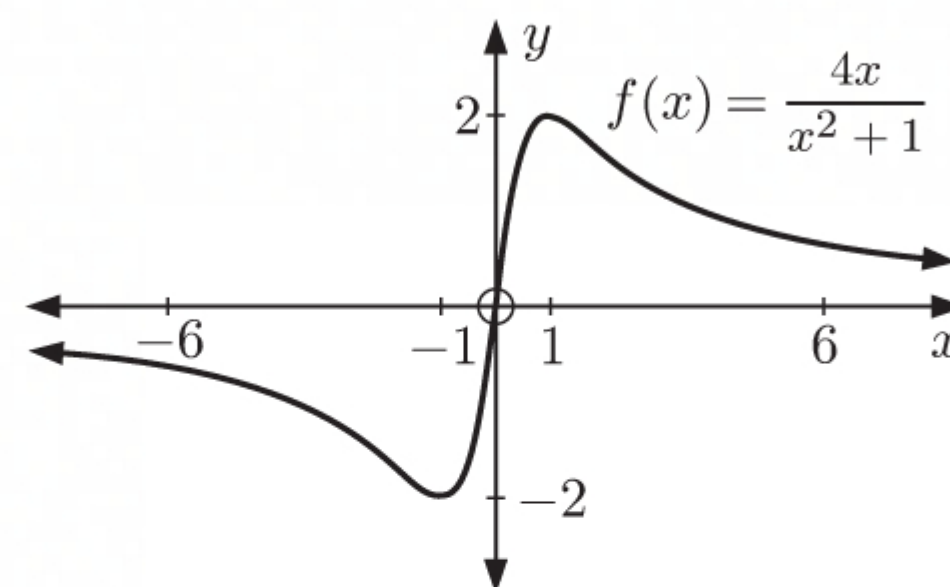
which has sign diagram:



b $y = f(x)$ is increasing for $x \leq -3$ and $x \geq 3$, and decreasing for $-3 \leq x < 0$ and $0 < x \leq 3$.

7 a

$$\begin{aligned} f(x) &= \frac{4x}{x^2 + 1} \\ \therefore f'(x) &= \frac{4(x^2 + 1) - 4x(2x)}{(x^2 + 1)^2} \quad \{\text{quotient rule}\} \\ &= \frac{4x^2 + 4 - 8x^2}{(x^2 + 1)^2} \\ &= \frac{-4x^2 + 4}{(x^2 + 1)^2} \\ &= \frac{-4(x^2 - 1)}{(x^2 + 1)^2} \\ &= \frac{-4(x+1)(x-1)}{(x^2 + 1)^2} \end{aligned}$$



which has sign diagram:

$f'(x)$

b $y = f(x)$ is increasing for $-1 \leq x \leq 1$, and decreasing for $x \leq -1$ and for $x \geq 1$.

8 a $f(x) = \frac{4x}{(x-1)^2}$

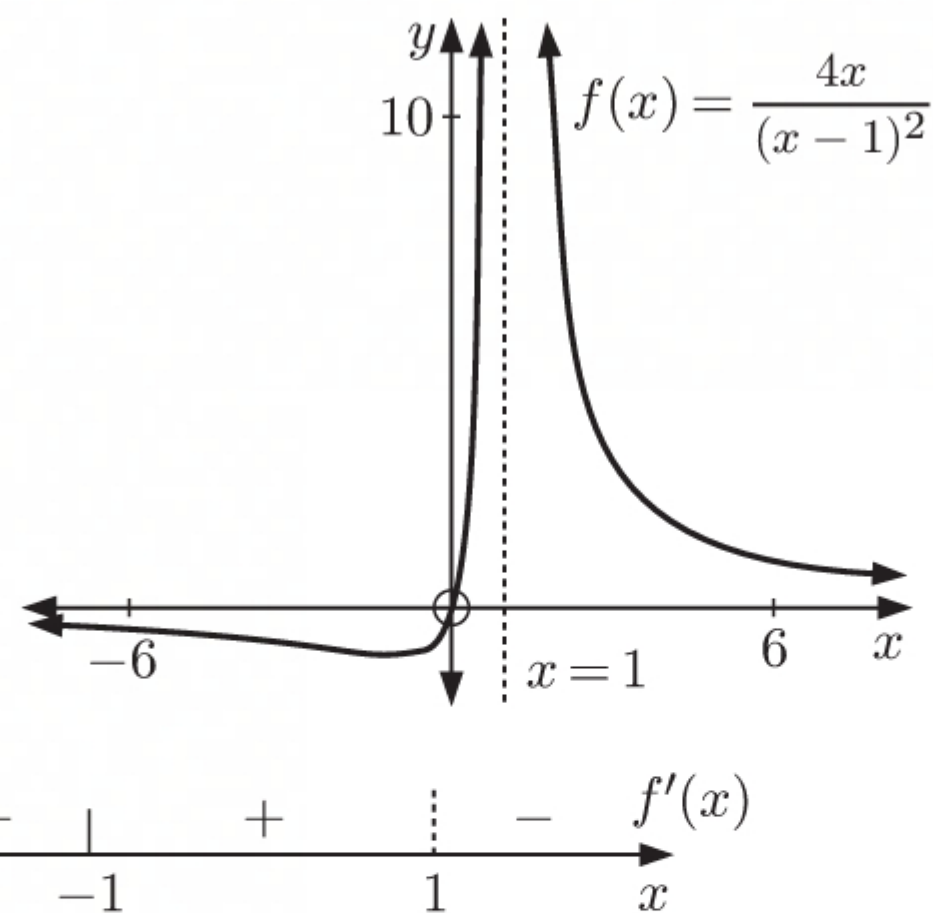
$$\therefore f'(x) = \frac{4(x-1)^2 - 4x \times 2(x-1)}{(x-1)^4} \quad \{\text{quotient rule}\}$$

$$= \frac{(x-1)[4(x-1) - 8x]}{(x-1)^4}$$

$$= \frac{4x - 4 - 8x}{(x-1)^3}$$

$$= \frac{-4x - 4}{(x-1)^3}$$

$$= \frac{-4(x+1)}{(x-1)^3} \quad \text{which has sign diagram:}$$



b $y = f(x)$ is increasing for $-1 \leq x < 1$, and decreasing for $x \leq -1$ and for $x > 1$.

9 a $f(x) = \frac{-x^2 + 4x - 7}{x-1}$

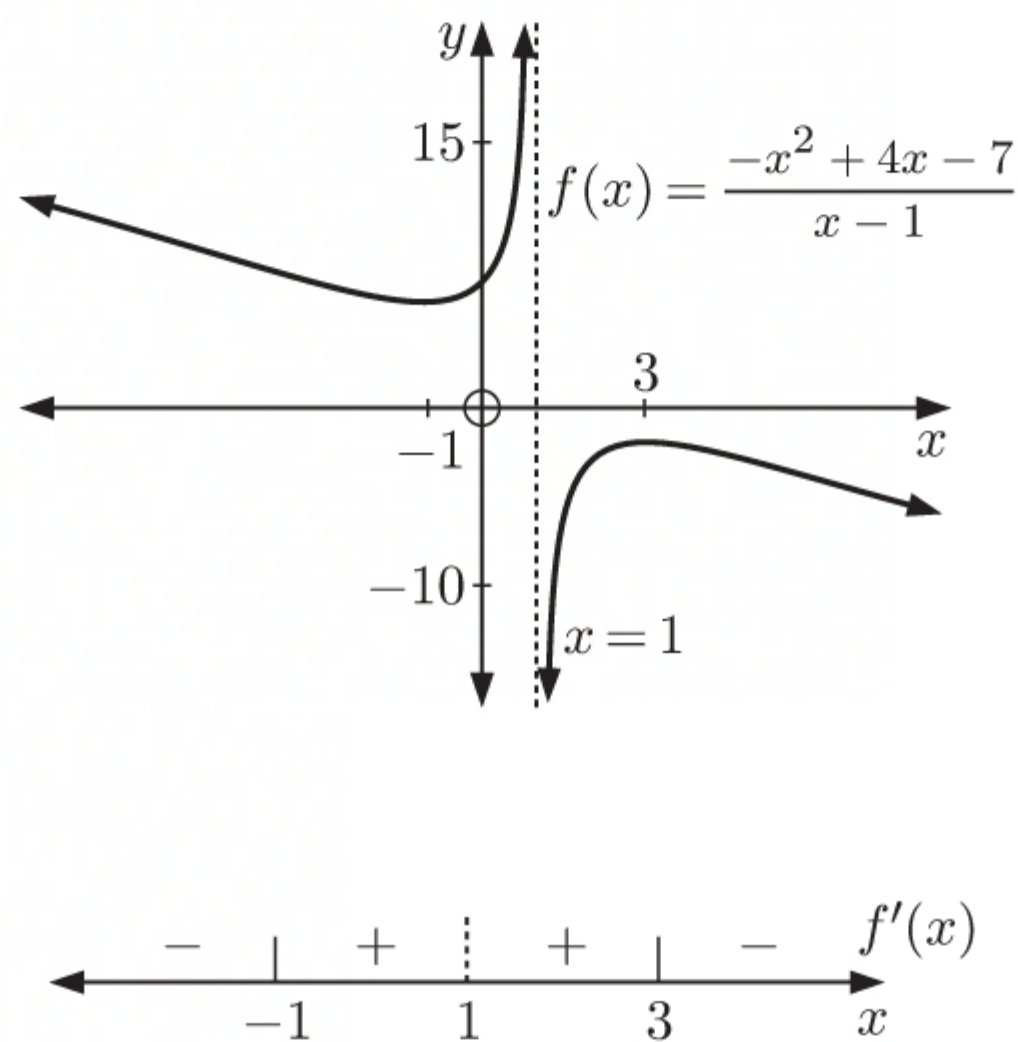
$$\therefore f'(x) = \frac{(-2x+4)(x-1) - (-x^2+4x-7)(1)}{(x-1)^2} \quad \{\text{quotient rule}\}$$

$$= \frac{-2x^2 + 2x + 4x - 4 + x^2 - 4x + 7}{(x-1)^2}$$

$$= \frac{-x^2 + 2x + 3}{(x-1)^2}$$

$$= \frac{-(x^2 - 2x - 3)}{(x-1)^2}$$

$$= \frac{-(x+1)(x-3)}{(x-1)^2} \quad \text{which has sign diagram:}$$



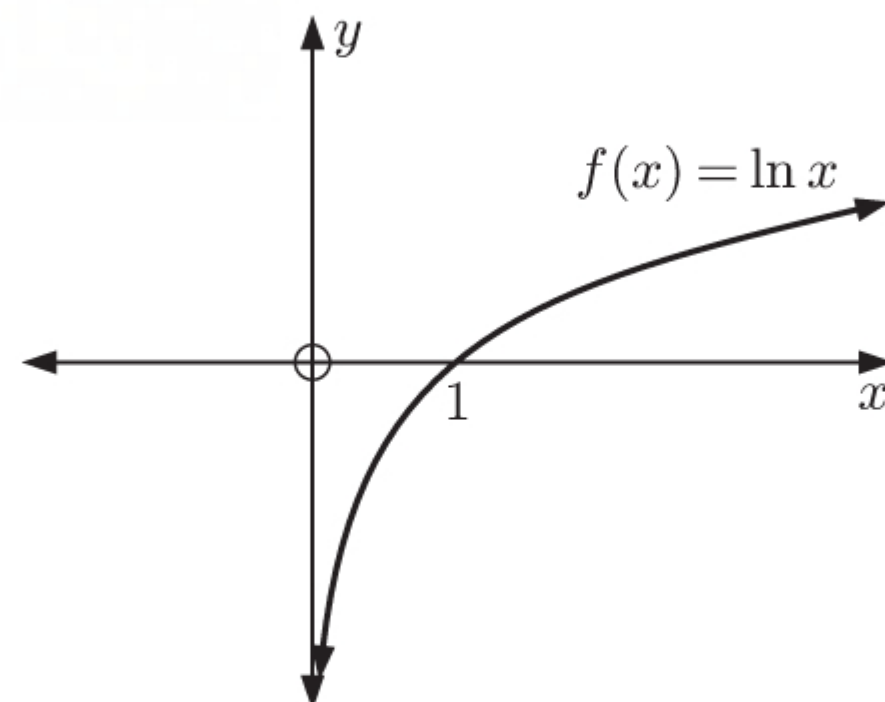
b $y = f(x)$ is increasing for $-1 \leq x < 1$ and for $1 < x \leq 3$, and decreasing for $x \leq -1$ and for $x \geq 3$.

10 Kenneth's conclusion that $f(x) = \ln x$ is decreasing for $x < 0$ is incorrect.

In this case, $f(x)$ is only defined when $x > 0$.

$\therefore f'(x)$ is only defined when $x > 0$.

As we can see from the graph alongside, $f(x) = \ln x$ is increasing for all $x > 0$, and never decreasing.



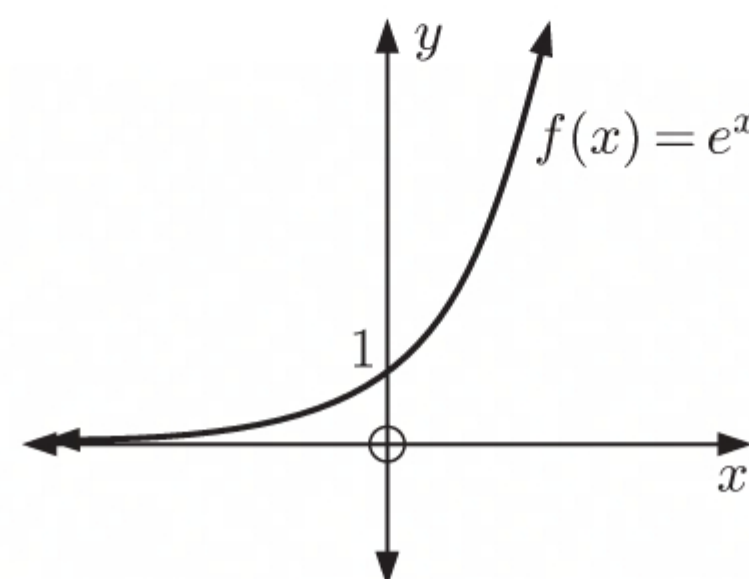
11 a $f(x) = e^x$

$$\therefore f'(x) = e^x$$

which has sign diagram:



$\therefore f(x)$ is increasing for all $x \in \mathbb{R}$, and never decreasing.



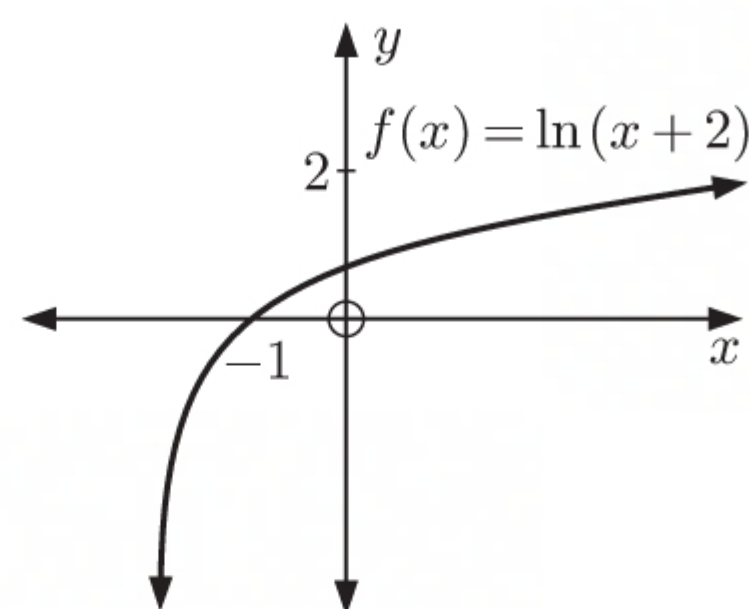
b $f(x) = \ln(x+2)$

$$\therefore f'(x) = \frac{1}{x+2}$$

which has sign diagram:



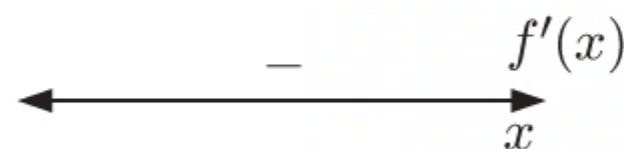
$\therefore f(x)$ is increasing for $x > -2$,
and never decreasing.



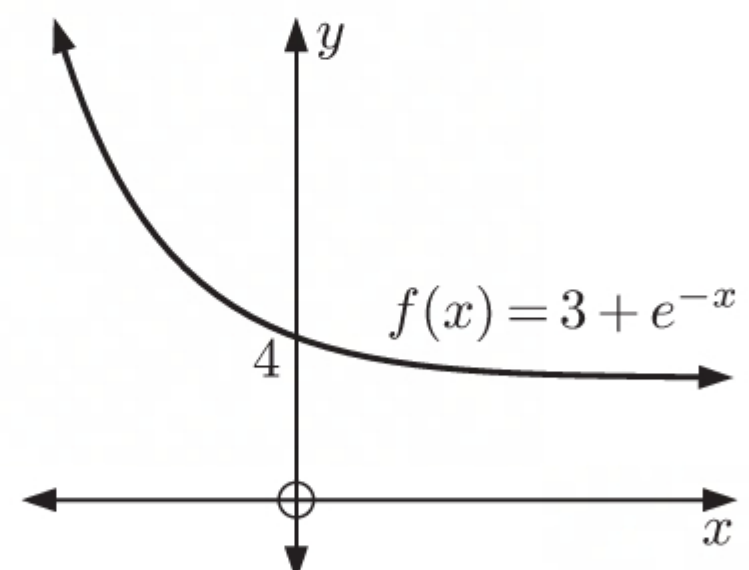
c $f(x) = 3 + e^{-x}$

$$\therefore f'(x) = -e^{-x}$$

which has sign diagram:



$\therefore f(x)$ is never increasing,
and decreasing for all $x \in \mathbb{R}$.



d $f(x) = xe^x$

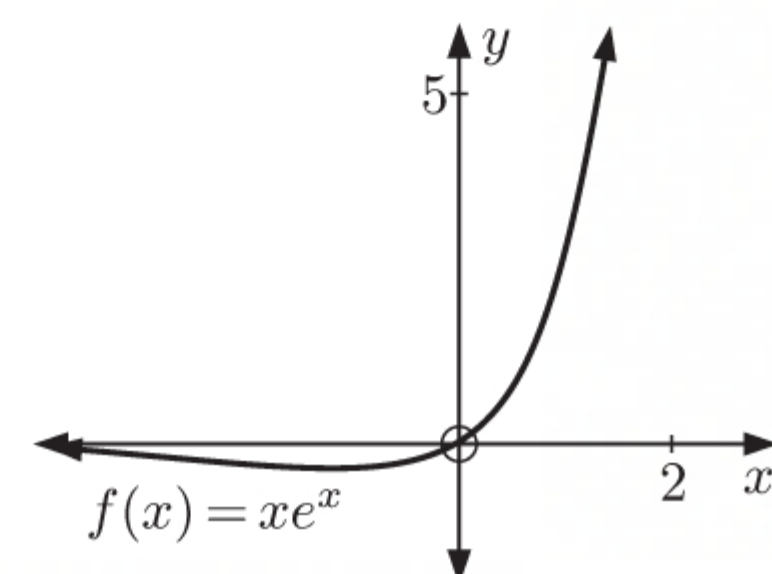
$$\therefore f'(x) = e^x + xe^x \quad \{\text{product rule}\}$$

$$= e^x(1+x)$$

which has sign diagram:



$\therefore f(x)$ is increasing for $x \geq -1$,
and decreasing for $x \leq -1$.

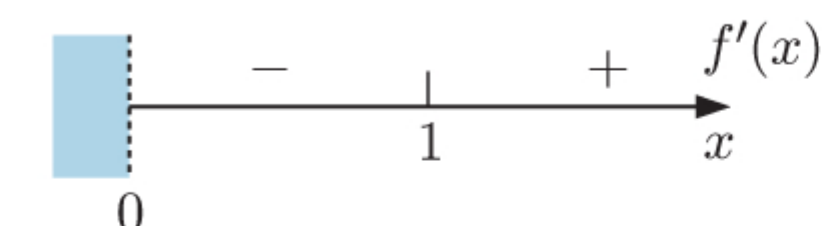


e $f(x) = x - 2\sqrt{x} = x - 2x^{\frac{1}{2}}$

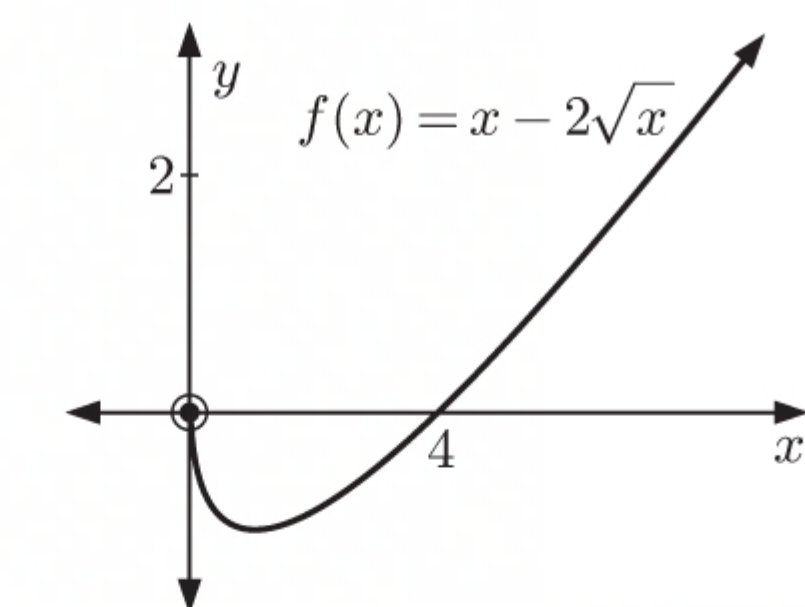
$$\therefore f'(x) = 1 - x^{-\frac{1}{2}}$$

$$= 1 - \frac{1}{\sqrt{x}}$$

which has sign diagram:



$\therefore f(x)$ is increasing for $x \geq 1$,
and decreasing for $0 < x \leq 1$.



f $f(x) = x^3 \ln x$

$$\therefore f'(x) = 3x^2 \ln x + x^3 \left(\frac{1}{x}\right) \quad \{\text{product rule}\}$$

$$= 3x^2 \ln x + x^2$$

$$= x^2(3 \ln x + 1)$$

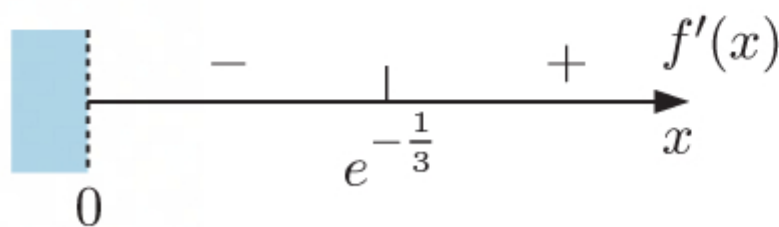
$$f'(x) = 0 \text{ when } x = 0 \text{ or } 3 \ln x + 1 = 0$$

$$\therefore 3 \ln x = -1$$

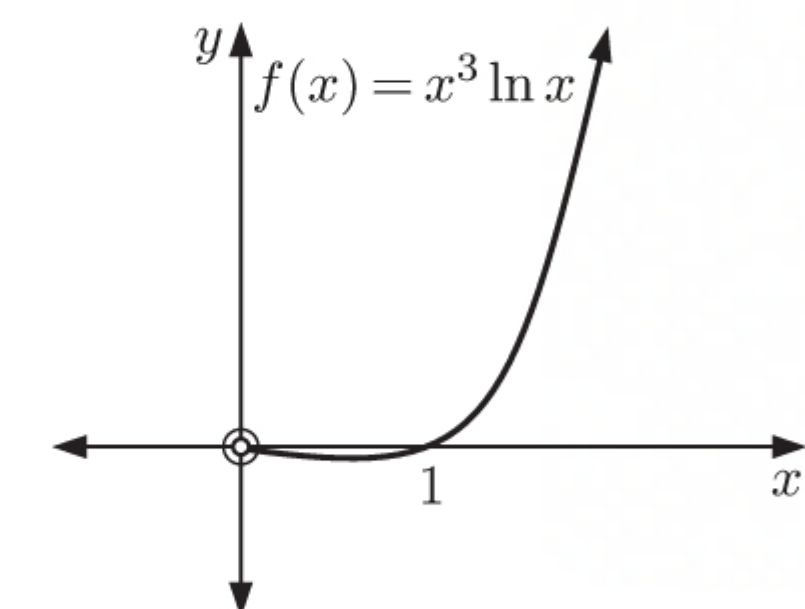
$$\therefore \ln x = -\frac{1}{3}$$

$$\therefore x = e^{-\frac{1}{3}}$$

$\therefore f'(x)$ has sign diagram:



$\therefore f(x)$ is increasing for $x \geq e^{-\frac{1}{3}}$, and decreasing for $0 < x \leq e^{-\frac{1}{3}}$.



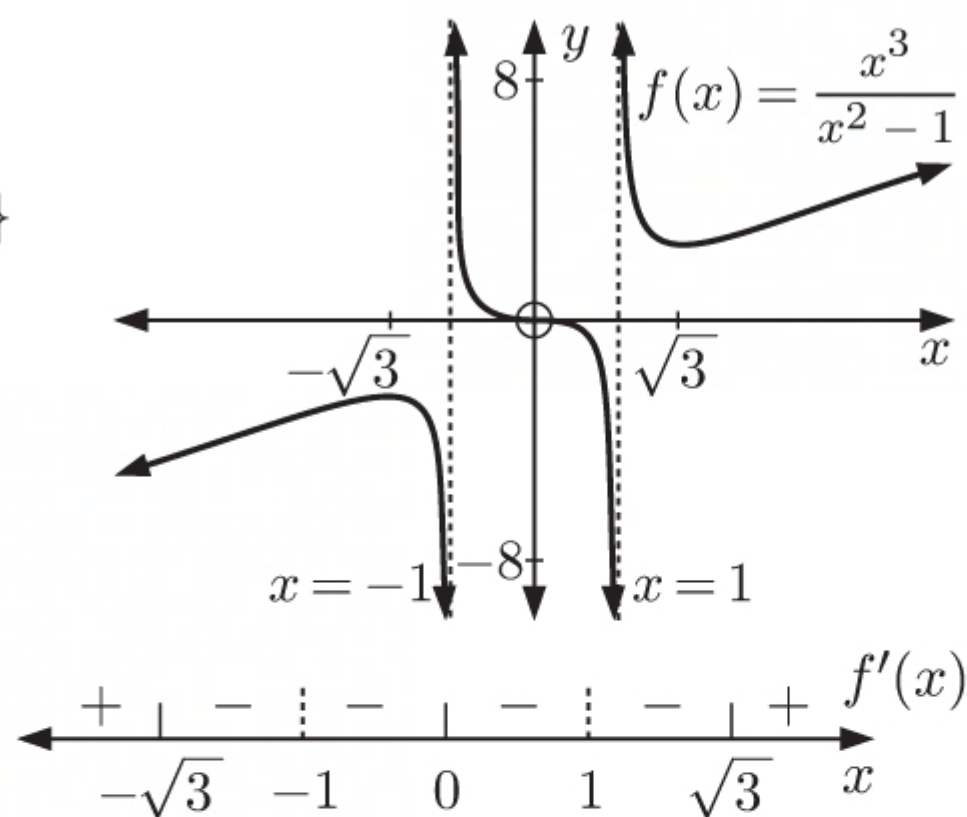
g $f(x) = \frac{x^3}{x^2 - 1}$

$$\therefore f'(x) = \frac{3x^2(x^2 - 1) - x^3(2x)}{(x^2 - 1)^2} \quad \{\text{quotient rule}\}$$

$$= \frac{3x^4 - 3x^2 - 2x^4}{(x^2 - 1)^2}$$

$$= \frac{x^4 - 3x^2}{(x^2 - 1)^2}$$

$$= \frac{x^2(x^2 - 3)}{(x^2 - 1)^2} \quad \text{which has sign diagram:}$$

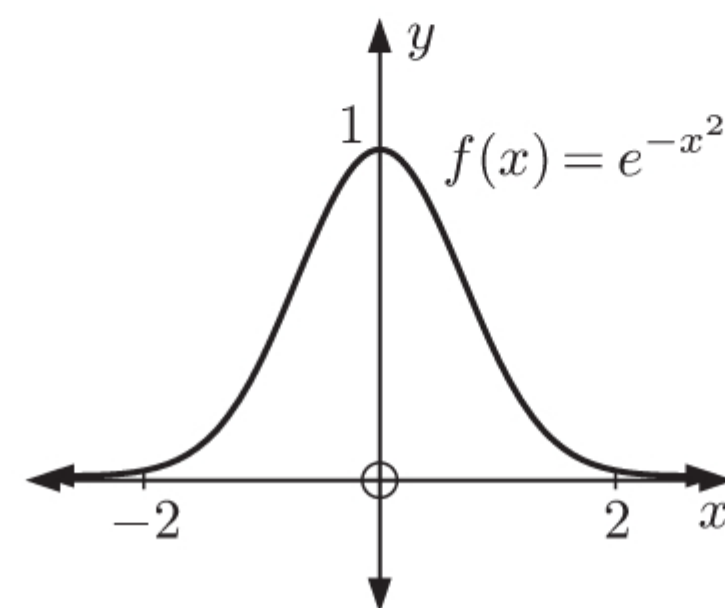


$\therefore f(x)$ is increasing for $x \leq -\sqrt{3}$ and for $x \geq \sqrt{3}$, and decreasing for $-\sqrt{3} \leq x < -1$, $-1 < x < 1$, and for $1 < x \leq \sqrt{3}$.

h $f(x) = e^{-x^2}$

$$\therefore f'(x) = -2xe^{-x^2}$$

which has sign diagram:



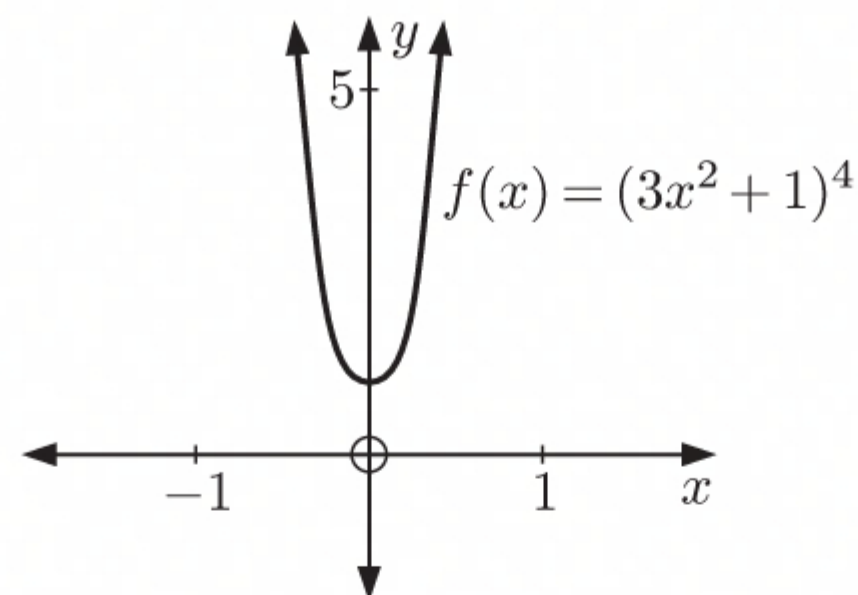
$\therefore f(x)$ is increasing for $x \leq 0$,
and decreasing for $x \geq 0$.

i $f(x) = (3x^2 + 1)^4$

$$\therefore f'(x) = 4(3x^2 + 1)^3(6x) \quad \{\text{chain rule}\}$$

$$= 24x(3x^2 + 1)^3$$

which has sign diagram:



$\therefore f(x)$ is increasing for $x \geq 0$,
and decreasing for $x \leq 0$.

j $f(x) = x^2 + \frac{4}{x-1} = x^2 + 4(x-1)^{-1}$

$$\therefore f'(x) = 2x - 4(x-1)^{-2}(1) \quad \{\text{chain rule}\}$$

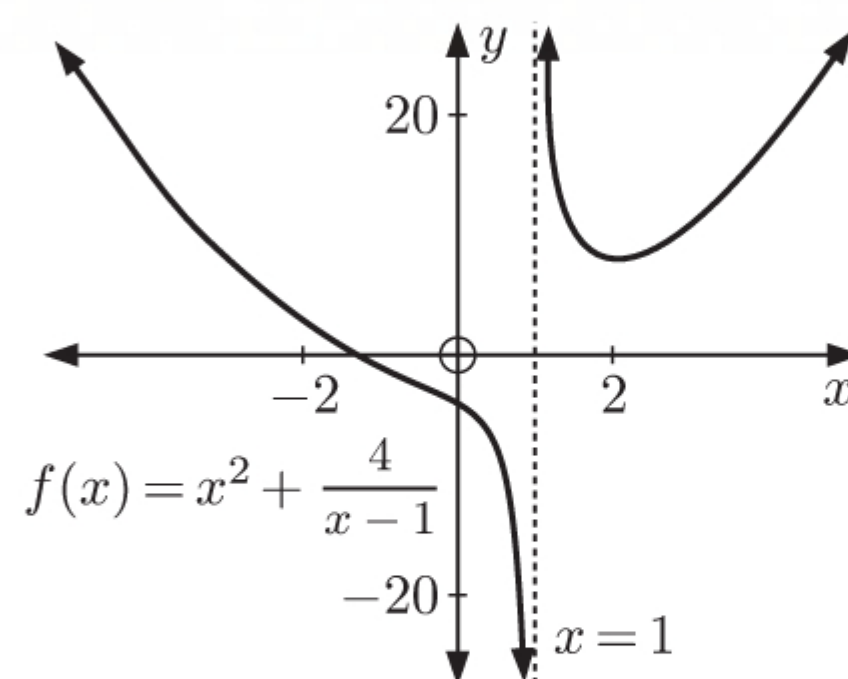
$$= 2x - \frac{4}{(x-1)^2}$$

$$f'(x) = 0 \quad \text{when} \quad 2x = \frac{4}{(x-1)^2}$$

$$\therefore x = \frac{2}{(x-1)^2}$$

$$\therefore x(x-1)^2 = 2$$

$$\therefore x = 2 \quad \{\text{using technology}\}$$



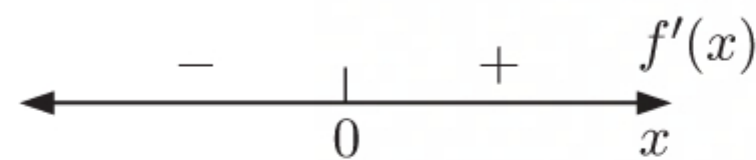
$\therefore f'(x)$ has sign diagram:

$\therefore f(x)$ is increasing for $x \geq 2$, and decreasing for $x < 1$ and for $1 < x \leq 2$.

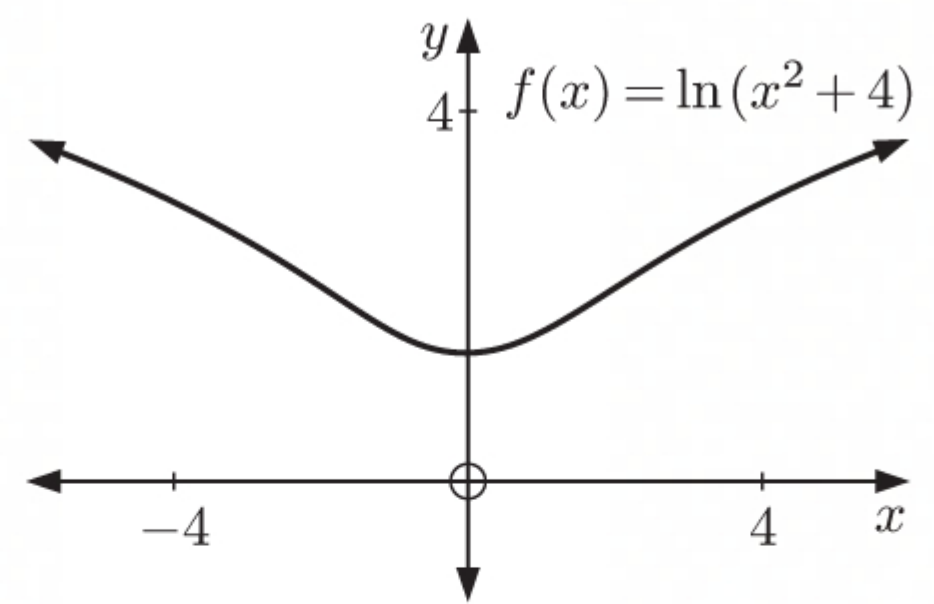
k $f(x) = \ln(x^2 + 4)$

$$\therefore f'(x) = \frac{2x}{x^2 + 4}$$

which has sign diagram:



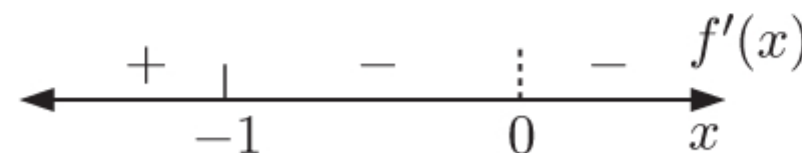
$\therefore f(x)$ is increasing for $x \geq 0$
and decreasing for $x \leq 0$.



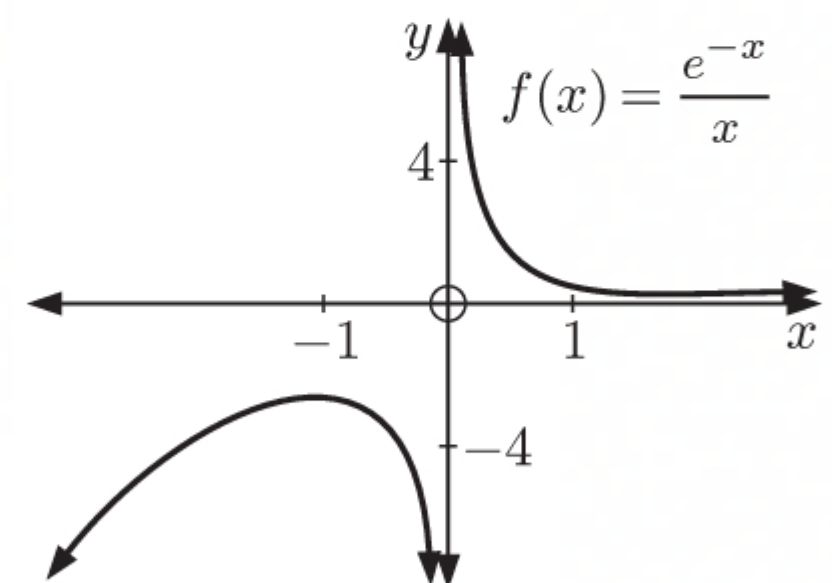
l $f(x) = \frac{e^{-x}}{x}$

$$\begin{aligned} \therefore f'(x) &= \frac{(-e^{-x})x - e^{-x}(1)}{x^2} \quad \{\text{quotient rule}\} \\ &= \frac{-xe^{-x} - e^{-x}}{x^2} \\ &= \frac{-e^{-x}(x+1)}{x^2} \end{aligned}$$

which has sign diagram:



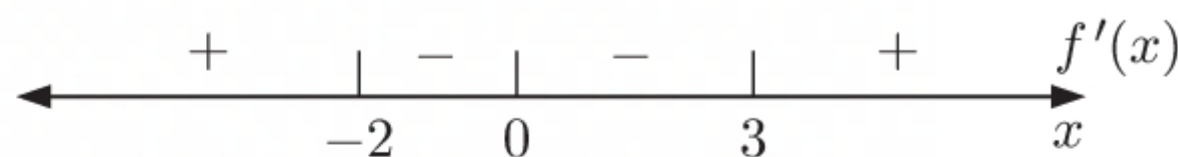
$\therefore f(x)$ is increasing for $x \leq -1$ and decreasing for $-1 \leq x < 0$ and for $x > 0$.



EXERCISE 13D

- 1 a** A is a local maximum, O is a stationary inflection, B is a local minimum.

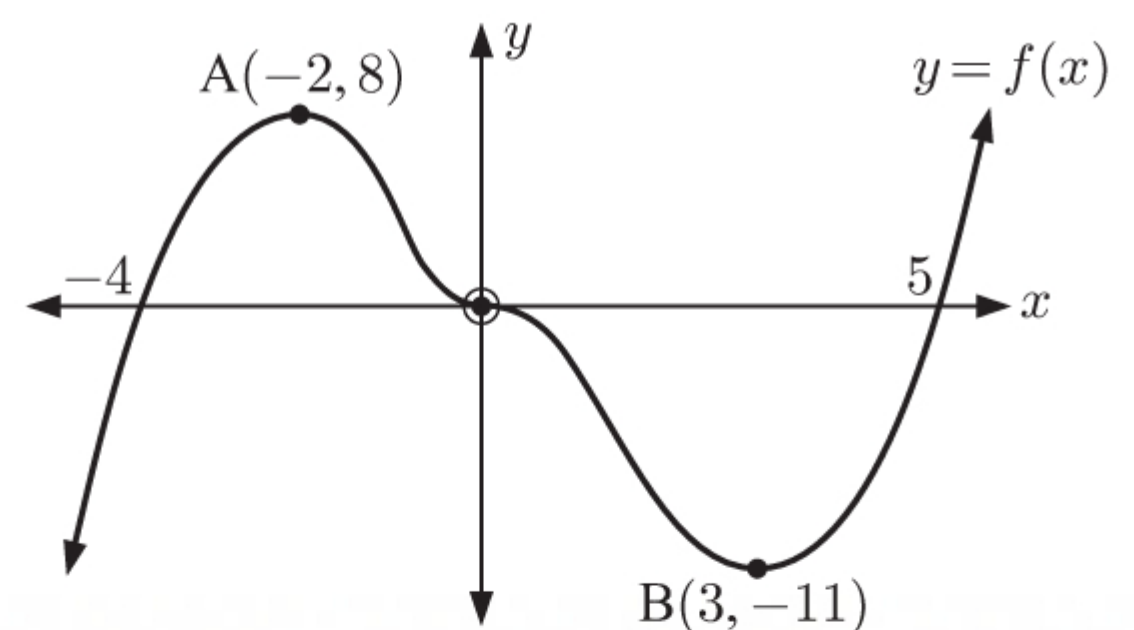
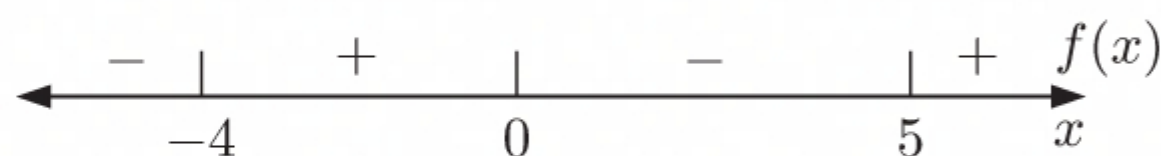
- b** $f'(x)$ has sign diagram:



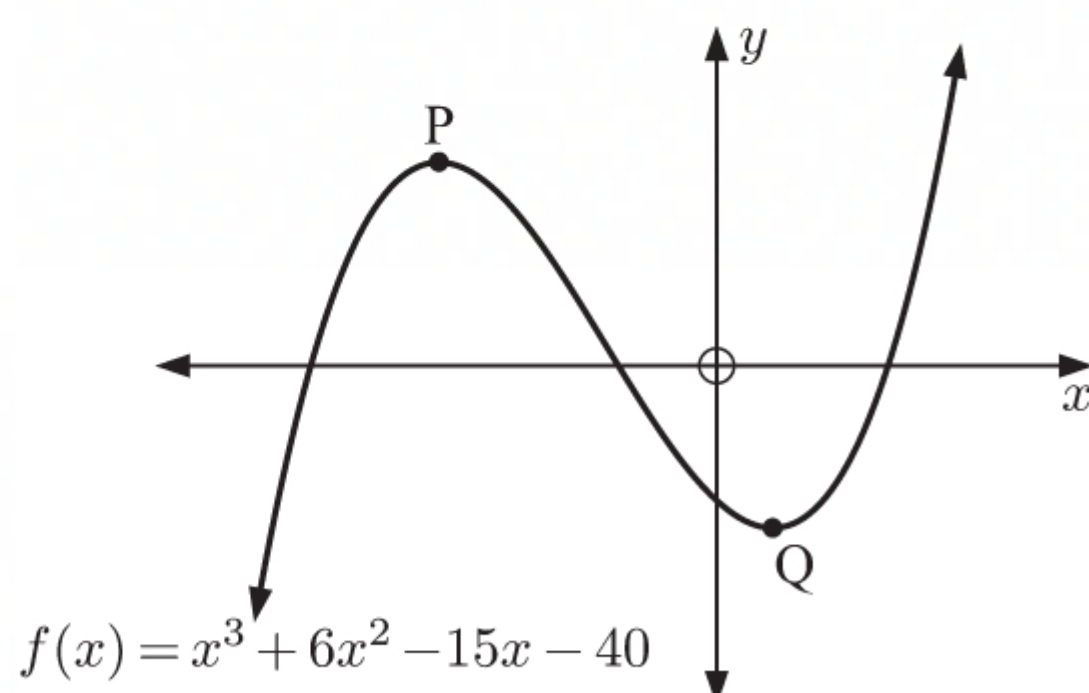
- c i** $f(x)$ is increasing for $x \leq -2$ and $x \geq 3$.

- ii** $f(x)$ is decreasing for $-2 \leq x \leq 3$.

- d** $f(x)$ has sign diagram:



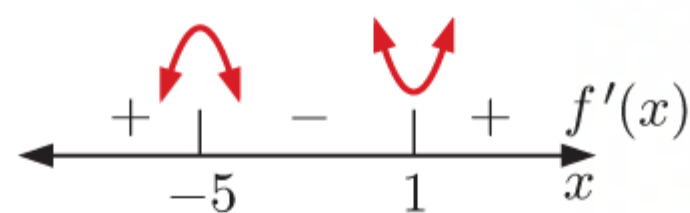
2



- a** P is a local maximum.
Q is a local minimum.

b
$$\begin{aligned} f(x) &= x^3 + 6x^2 - 15x - 40 \\ \therefore f'(x) &= 3x^2 + 12x - 15 \\ &= 3(x^2 + 4x - 5) \\ &= 3(x-1)(x+5) \end{aligned}$$

- c $f'(x)$ has sign diagram:



\therefore there is a local maximum at $x = -5$
and a local minimum at $x = 1$.

So, P has x -coordinate -5 , and Q has x -coordinate 1 .

$$f(-5) = (-5)^3 + 6(-5)^2 - 15(-5) - 40 \\ = 60$$

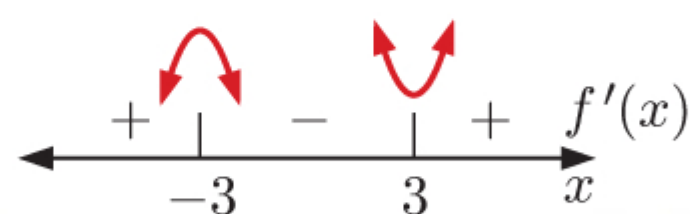
$$f(1) = 1^3 + 6(1)^2 - 15(1) - 40 \\ = -48$$

So, P is $(-5, 60)$ and Q is $(1, -48)$.

3 a $f(x) = \frac{1}{3}x^3 - 9x + 4$

$$\therefore f'(x) = x^2 - 9 \\ = (x + 3)(x - 3)$$

which has sign diagram:



- b $f(x)$ is increasing for $x \leq -3$ and $x \geq 3$, and decreasing for $-3 \leq x \leq 3$.

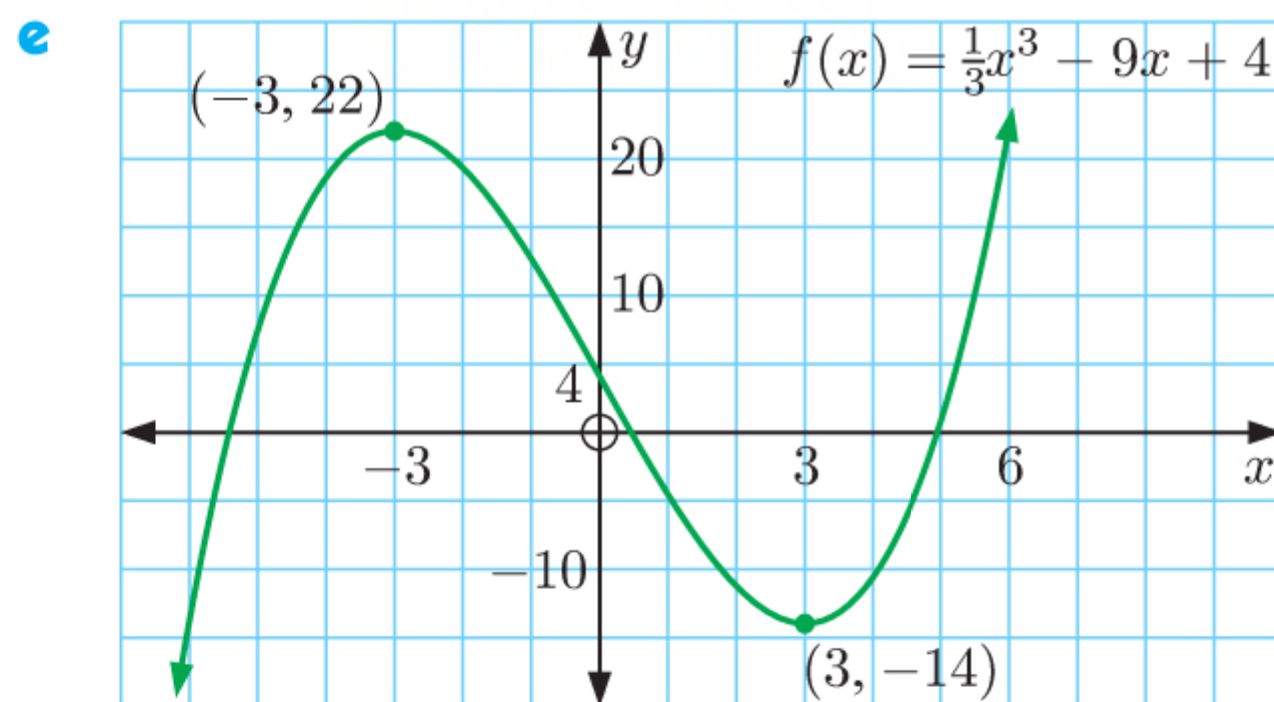
- c From the sign diagram in a, there is a local maximum at $x = -3$, and a local minimum at $x = 3$.

$$f(-3) = \frac{1}{3}(-3)^3 - 9(-3) + 4 \\ = 22$$

$$f(3) = \frac{1}{3}(3)^3 - 9(3) + 4 \\ = -14$$

So, there is a local maximum at $(-3, 22)$, and a local minimum at $(3, -14)$.

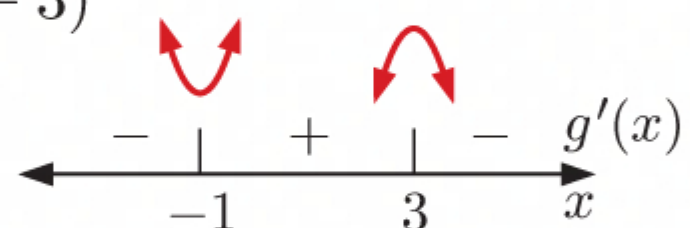
- d As $x \rightarrow \infty$, $f(x) \rightarrow \infty$,
as $x \rightarrow -\infty$, $f(x) \rightarrow -\infty$.



4 a $g(x) = -2x^3 + 6x^2 + 18x - 7$

$$\therefore g'(x) = -6x^2 + 12x + 18 \\ = -6(x^2 - 2x - 3) \\ = -6(x + 1)(x - 3)$$

which has sign diagram:



- b $g(x)$ is increasing for $-1 \leq x \leq 3$, and decreasing for $x \leq -1$ and $x \geq 3$.

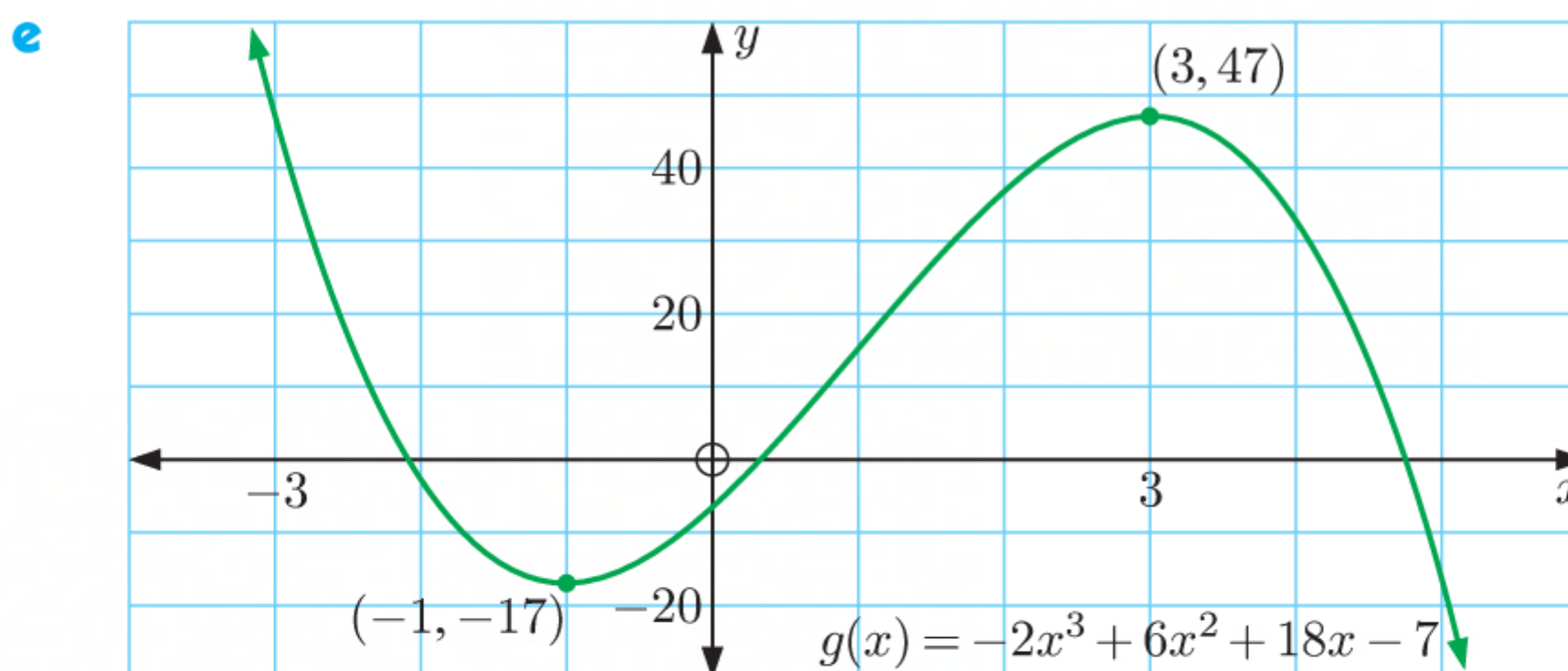
- c From the sign diagram, there is a local minimum at $x = -1$, and a local maximum at $x = 3$.

$$g(-1) = -2(-1)^3 + 6(-1)^2 + 18(-1) - 7 \\ = -17$$

$$g(3) = -2(3)^3 + 6(3)^2 + 18(3) - 7 \\ = 47$$

So, there is a local minimum at $(-1, -17)$, and a local maximum at $(3, 47)$.

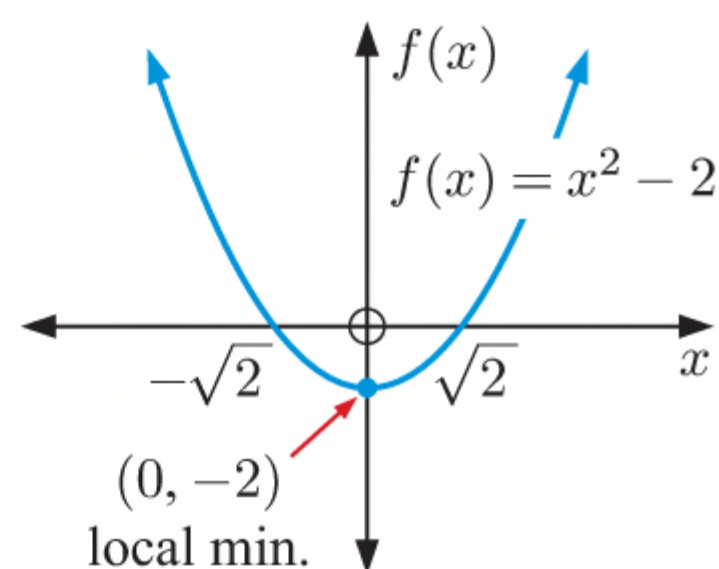
- d As $x \rightarrow \infty$, $g(x) \rightarrow -\infty$, as $x \rightarrow -\infty$, $g(x) \rightarrow \infty$.



5 a $f(x) = x^2 - 2$
 $\therefore f'(x) = 2x$

which has
 sign diagram:

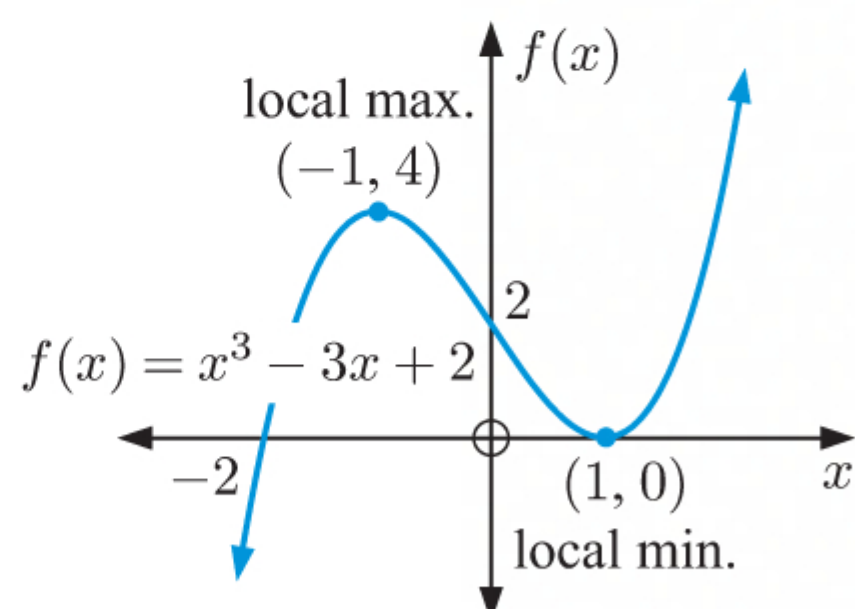
Now $f(0) = -2$, so there is a local minimum at $(0, -2)$.



c $f(x) = x^3 - 3x + 2$
 $\therefore f'(x) = 3x^2 - 3$
 $= 3(x^2 - 1)$
 $= 3(x + 1)(x - 1)$

which has
 sign diagram:

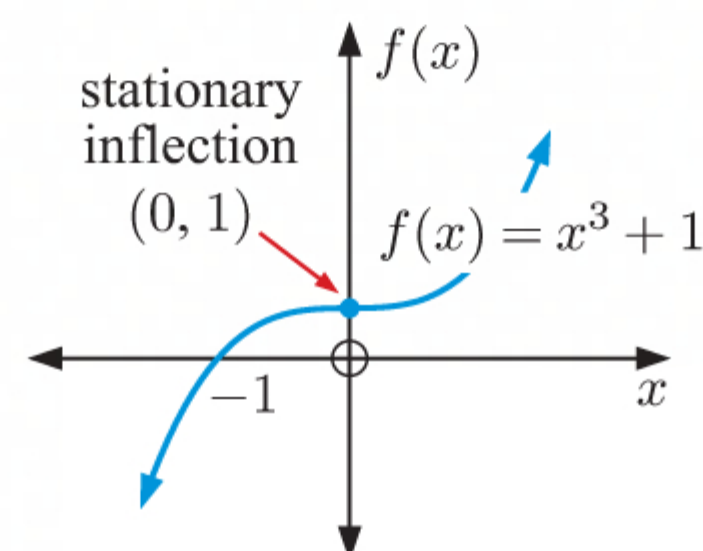
Now $f(-1) = 4$, $f(1) = 0$, so there is a local maximum at $(-1, 4)$, and a local minimum at $(1, 0)$.



b $f(x) = x^3 + 1$
 $\therefore f'(x) = 3x^2$

which has
 sign diagram:

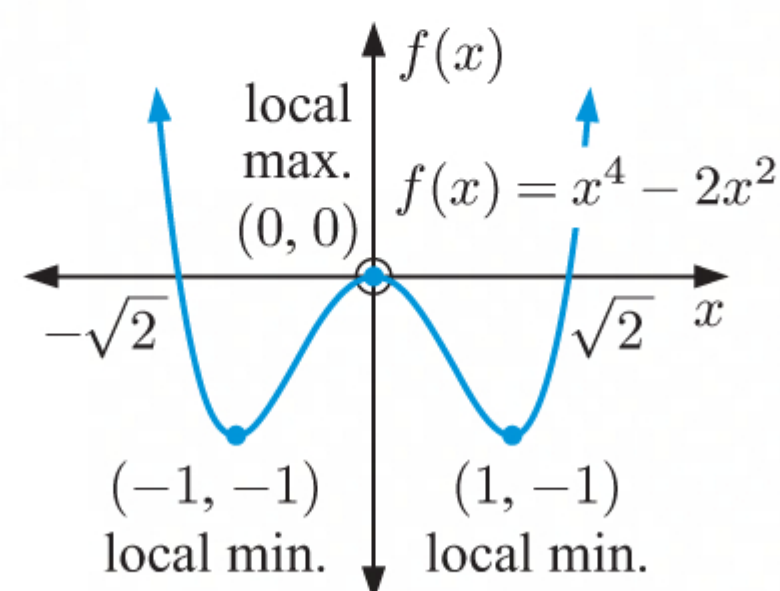
Now $f(0) = 1$, so there is a stationary inflection at $(0, 1)$.



d $f(x) = x^4 - 2x^2$
 $\therefore f'(x) = 4x^3 - 4x$
 $= 4x(x^2 - 1)$
 $= 4x(x + 1)(x - 1)$

which has
 sign diagram:

Now $f(-1) = -1$, $f(1) = -1$, $f(0) = 0$, so there are local minima at $(-1, -1)$ and $(1, -1)$, and a local maximum at $(0, 0)$.

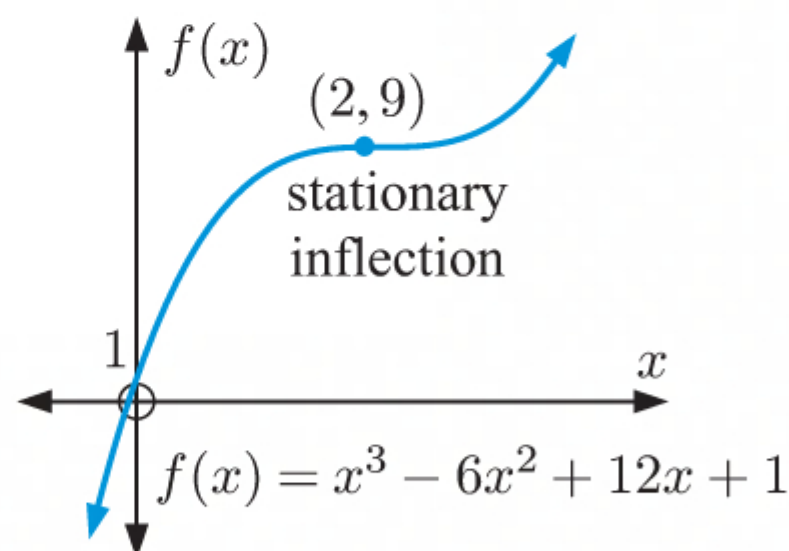


$$\begin{aligned}
 \text{e} \quad f(x) &= x^3 - 6x^2 + 12x + 1 \\
 \therefore f'(x) &= 3x^2 - 12x + 12 \\
 &= 3(x^2 - 4x + 4) \\
 &= 3(x - 2)^2
 \end{aligned}$$

which has
sign diagram:

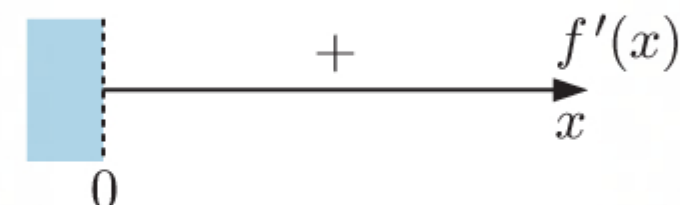


Now $f(2) = 9$, so there is a stationary inflection at $(2, 9)$.

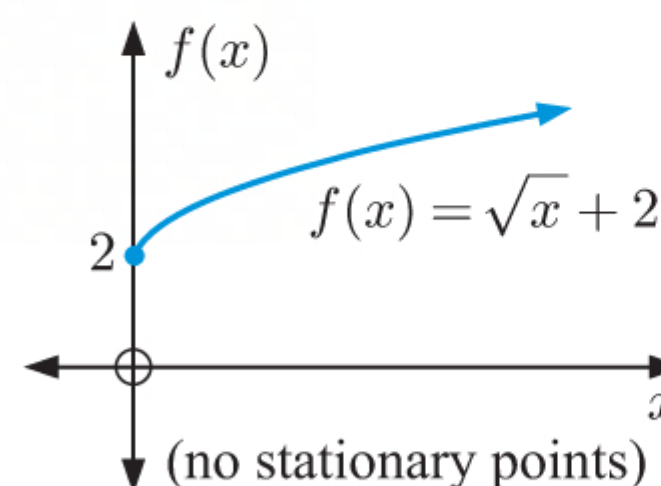


$$\begin{aligned}
 \text{f} \quad f(x) &= \sqrt{x} + 2 \\
 &= x^{\frac{1}{2}} + 2 \\
 \therefore f'(x) &= \frac{1}{2}x^{-\frac{1}{2}} \\
 &= \frac{1}{2\sqrt{x}} \neq 0
 \end{aligned}$$

which has
sign diagram:

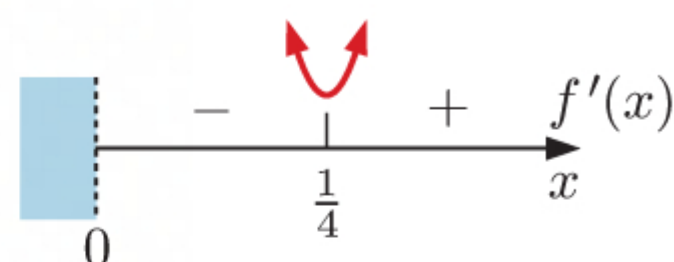


\therefore there are no stationary points.



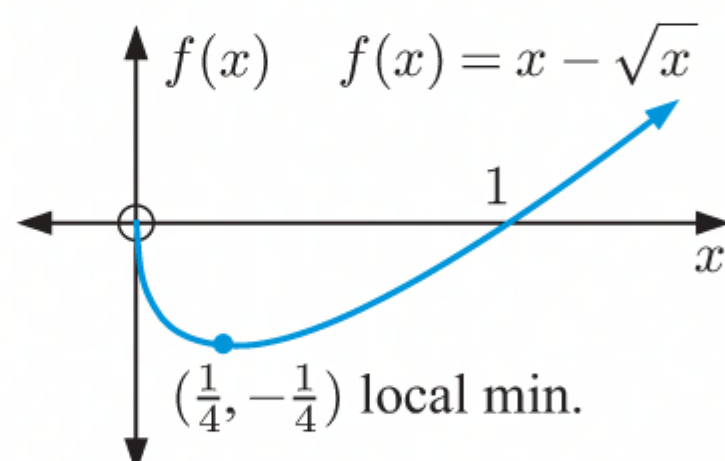
$$\begin{aligned}
 \text{g} \quad f(x) &= x - \sqrt{x} \\
 &= x - x^{\frac{1}{2}} \\
 \therefore f'(x) &= 1 - \frac{1}{2}x^{-\frac{1}{2}} \\
 &= 1 - \frac{1}{2\sqrt{x}} \\
 &= \frac{2\sqrt{x} - 1}{2\sqrt{x}}
 \end{aligned}$$

which has
sign diagram:



$f(x)$ is defined for all $x \geq 0$

Now $f(\frac{1}{4}) = -\frac{1}{4}$, so there is a local minimum at $(\frac{1}{4}, -\frac{1}{4})$.

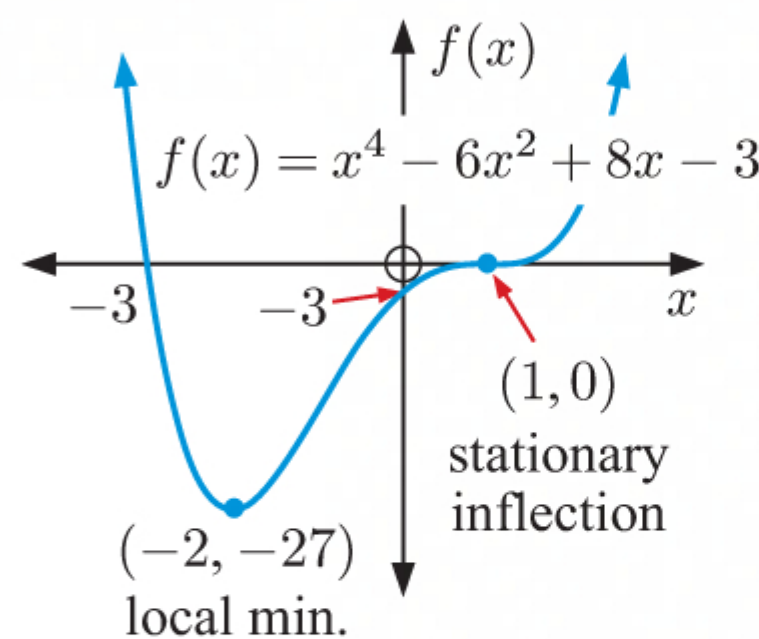


$$\begin{aligned}
 \text{h} \quad f(x) &= x^4 - 6x^2 + 8x - 3 \\
 \therefore f'(x) &= 4x^3 - 12x + 8 \\
 &= 4(x^3 - 3x + 2) \\
 &= 4(x - 1)(x^2 + x - 2) \\
 &= 4(x - 1)(x + 2)(x - 1)
 \end{aligned}$$


which has
sign diagram:



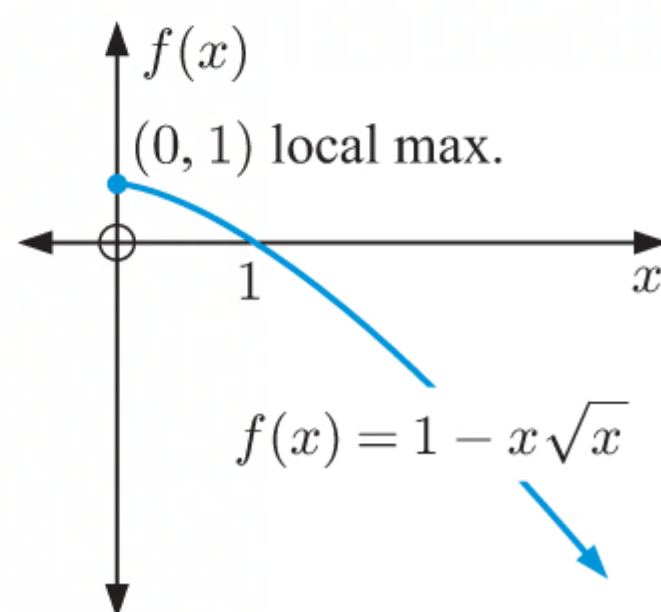
Now $f(-2) = -27$, $f(1) = 0$, so there is a local minimum at $(-2, -27)$, and a stationary inflection at $(1, 0)$.



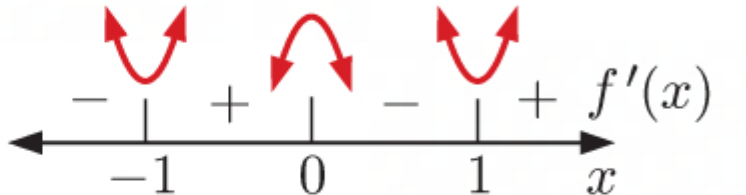
$$\begin{aligned}
 \text{i} \quad f(x) &= 1 - x\sqrt{x} \\
 &= 1 - x^{\frac{3}{2}} \\
 \therefore f'(x) &= -\frac{3}{2}x^{\frac{1}{2}} \\
 &= -\frac{3\sqrt{x}}{2}
 \end{aligned}$$

which has
sign diagram: 

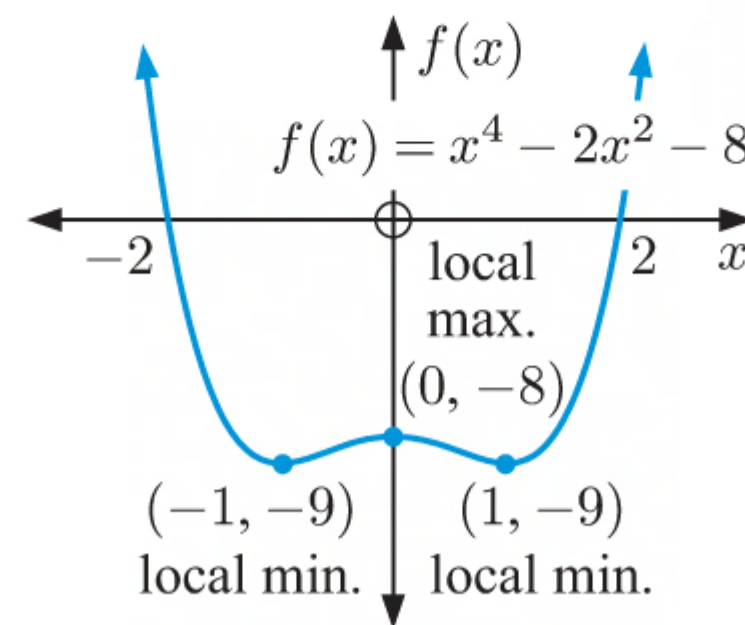
$f(x)$ is only defined when $x \geq 0$.
Now $f(0) = 1$, so there is a local maximum at $(0, 1)$.



$$\begin{aligned}
 \text{j} \quad f(x) &= x^4 - 2x^2 - 8 \\
 \therefore f'(x) &= 4x^3 - 4x \\
 &= 4x(x^2 - 1) \\
 &= 4x(x+1)(x-1)
 \end{aligned}$$

which has
sign diagram: 

Now $f(-1) = -9$, $f(1) = -9$,
 $f(0) = -8$, so there are local minima
at $(-1, -9)$ and $(1, -9)$, and a local
maximum at $(0, -8)$.



$$\begin{aligned}
 \text{6 a} \quad f(x) &= ax^2 + bx + c, \quad a \neq 0 \\
 \therefore f'(x) &= 2ax + b
 \end{aligned}$$

$f(x)$ has a stationary point when $f'(x) = 0$

$$\therefore 2ax + b = 0$$

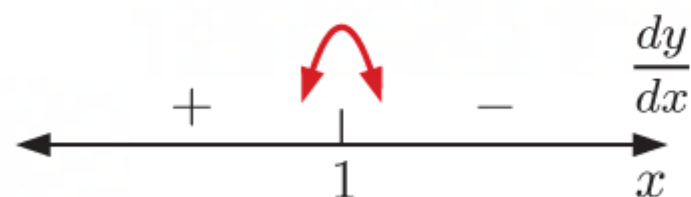
$$\therefore x = -\frac{b}{2a}$$

b When $a < 0$, $f(x)$ is concave down , so there is a local maximum when $a < 0$.

When $a > 0$, $f(x)$ is concave up , so there is a local minimum when $a > 0$.

$$\begin{aligned}
 \text{7 a} \quad y &= xe^{-x} \\
 \therefore \frac{dy}{dx} &= (1)e^{-x} + x(e^{-x})(-1) \quad \{\text{product rule}\} \\
 &= e^{-x} - xe^{-x} \\
 &= e^{-x}(1 - x) \quad \text{where } e^{-x} \text{ is positive for all } x
 \end{aligned}$$

So, $\frac{dy}{dx} = 0$ when $x = 1$.

The sign diagram of $\frac{dy}{dx}$ is: 

When $x = 1$, $y = (1)e^{-1} = \frac{1}{e}$

\therefore there is a local maximum at $\left(1, \frac{1}{e}\right)$.

b $y = x^2 e^x$

$$\begin{aligned}\therefore \frac{dy}{dx} &= (2x)e^x + x^2(e^x) \quad \{\text{product rule}\} \\ &= 2xe^x + x^2e^x \\ &= xe^x(2+x) \quad \text{where } e^x \text{ is positive for all } x\end{aligned}$$

So, $\frac{dy}{dx} = 0$ when $x = 0$ or -2 .

The sign diagram of $\frac{dy}{dx}$ is:

When $x = -2$, $y = (-2)^2 e^{-2} = \frac{4}{e^2}$

When $x = 0$, $y = 0^2 e^0 = 0$

\therefore there is a local maximum at $(-2, \frac{4}{e^2})$ and a local minimum at $(0, 0)$.

c $y = \frac{e^x}{x}$

$$\begin{aligned}\therefore \frac{dy}{dx} &= \frac{e^x x - e^x(1)}{x^2} \quad \{\text{quotient rule}\} \\ &= \frac{e^x(x-1)}{x^2} \quad \text{where } e^x \text{ is positive for all } x\end{aligned}$$

So, $\frac{dy}{dx} = 0$ when $x = 1$.

The sign diagram of $\frac{dy}{dx}$ is:

When $x = 1$, $y = \frac{e^1}{1} = e$

\therefore there is a local minimum at $(1, e)$.

d $y = e^{-x}(x+2)$

$$\begin{aligned}\therefore \frac{dy}{dx} &= e^{-x}(-1)(x+2) + e^{-x}(1) \quad \{\text{product rule}\} \\ &= -e^{-x}(x+2-1) \\ &= -e^{-x}(x+1) \quad \text{where } -e^{-x} \text{ is negative for all } x\end{aligned}$$

So, $\frac{dy}{dx} = 0$ when $x = -1$.

The sign diagram of $\frac{dy}{dx}$ is:

When $x = -1$, $y = e^{-(-1)}(-1+2) = e$

\therefore there is a local maximum at $(-1, e)$.

8 a $f(x) = 2x^3 + ax^2 - 24x + 1$

$\therefore f'(x) = 6x^2 + 2ax - 24$

But $f'(-4) = 0$, so $6(-4)^2 + 2a(-4) - 24 = 0$

$\therefore 96 - 8a - 24 = 0$

$\therefore 72 = 8a$

$\therefore a = 9$

b Since $a = 9$, then $f(x) = 2x^3 + 9x^2 - 24x + 1$
 $\therefore f(-4) = 2(-4)^3 + 9(-4)^2 - 24(-4) + 1$
 $= 113$

\therefore the local maximum is at $(-4, 113)$.

9 a $f(x) = x^3 + ax + b$

$\therefore f'(x) = 3x^2 + a$

But $f'(-2) = 0$

$\therefore 3(-2)^2 + a = 0$

$\therefore 12 + a = 0$

$\therefore a = -12$

Also, $f(-2) = 3$

$\therefore (-2)^3 - 12(-2) + b = 3$

$\therefore -8 + 24 + b = 3$

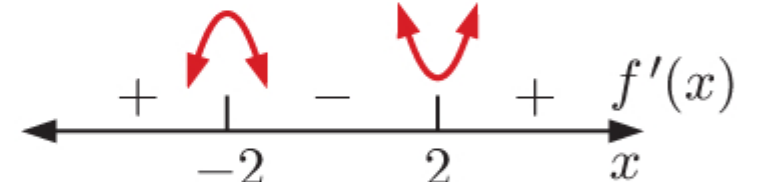
$\therefore b = -13$

b Now $f(x) = x^3 - 12x - 13$

$\therefore f'(x) = 3x^2 - 12$

$= 3(x^2 - 4)$

$= 3(x + 2)(x - 2)$

which has
sign diagram: 

Now $f(2) = -29$, $f(-2) = 3$, so there is a local minimum at $(2, -29)$ and a local maximum at $(-2, 3)$.

10 a $y = \frac{e^{ax}}{bx}$
 $\therefore \frac{dy}{dx} = \frac{e^{ax}(a)(bx) - e^{ax}(b)}{(bx)^2}$ {quotient rule}
 $= \frac{abxe^{ax} - be^{ax}}{b^2x^2}$
 $= \frac{be^{ax}(ax - 1)}{b^2x^2}$
 $= \frac{e^{ax}(ax - 1)}{bx^2}$ (*)

Since $\left(\frac{1}{3}, \frac{e}{2}\right)$ is a stationary point, then

when $x = \frac{1}{3}$, $\frac{dy}{dx} = 0$

Substituting $x = \frac{1}{3}$ into (*) gives:

$\therefore \frac{e^{\frac{a}{3}}\left(\frac{a}{3} - 1\right)}{b\left(\frac{1}{3}\right)^2} = 0$

$\therefore e^{\frac{a}{3}}\left(\frac{a}{3} - 1\right) = 0$ {as $b \neq 0$ }

$\therefore \frac{a}{3} - 1 = 0$ {as $e^{\frac{a}{3}} > 0$ }

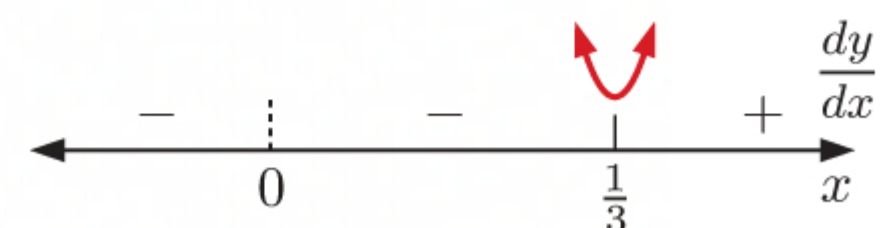
$\therefore \frac{a}{3} = 1$

$\therefore a = 3$

So, $a = 3$ and $b = 6$.

b Since $a = 3$ and $b = 6$, then $\frac{dy}{dx} = \frac{e^{3x}(3x - 1)}{6x^2}$

which has sign diagram:



\therefore there is a local minimum at $\left(\frac{1}{3}, \frac{e}{2}\right)$.

- 11 a** x is defined for all $x \in \mathbb{R}$, but $\ln x$ is only defined for $x > 0$.
 $\therefore f(x) = x \ln x$ is only defined for $x > 0$.

b $f'(x) = (1) \ln x + x \left(\frac{1}{x} \right)$ {product rule}
 $= \ln x + 1$

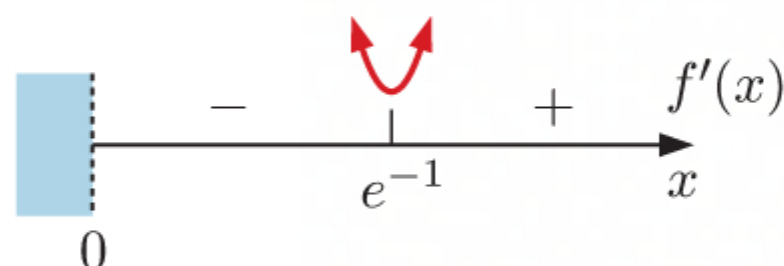
$f'(x) = 0$ when $\ln x + 1 = 0$
 $\therefore \ln x = -1$
 $\therefore x = e^{-1}$

$\therefore f'(x)$ has sign diagram:

$f(e^{-1}) = e^{-1} \ln(e^{-1})$
 $= -\frac{1}{e}$

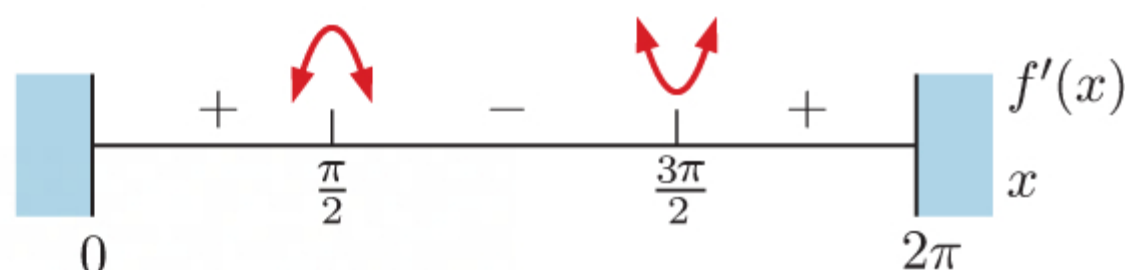
\therefore there is a local minimum at $\left(\frac{1}{e}, -\frac{1}{e} \right)$.

\therefore the minimum value of $f(x)$ is $-\frac{1}{e}$.



- 12 a** $f(x) = \sin x$, $0 \leq x \leq 2\pi$
 $\therefore f'(x) = \cos x$

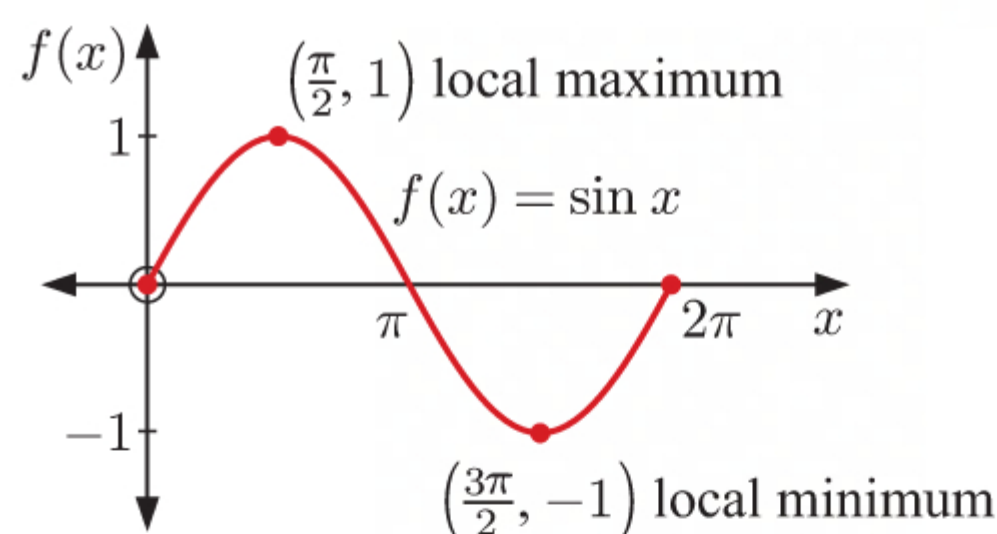
which has sign diagram:



$f\left(\frac{\pi}{2}\right) = \sin \frac{\pi}{2} = 1$

$f\left(\frac{3\pi}{2}\right) = \sin \frac{3\pi}{2} = -1$

\therefore there is a local maximum at $\left(\frac{\pi}{2}, 1\right)$,
 and a local minimum at $\left(\frac{3\pi}{2}, -1\right)$.

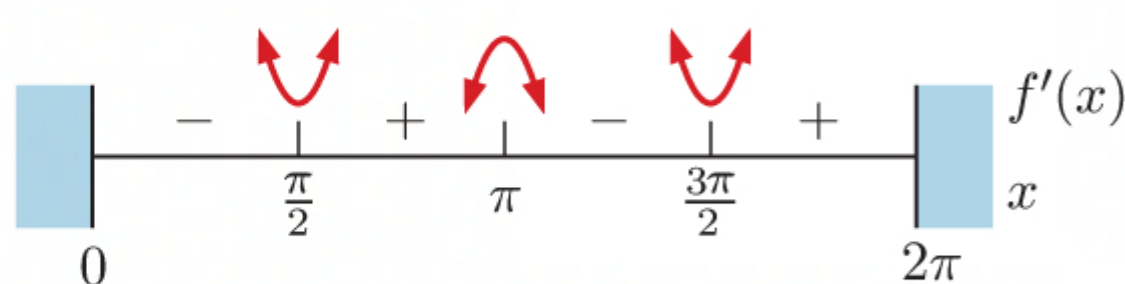


- b** $f(x) = \cos 2x$, $0 \leq x \leq 2\pi$

$\therefore f'(x) = -2 \sin 2x$

$f'(x) = 0$ when $\sin 2x = 0$
 $\therefore 2x = 0, \pi, 2\pi, 3\pi, 4\pi$
 $\therefore x = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$

So, $f'(x)$ has sign diagram:

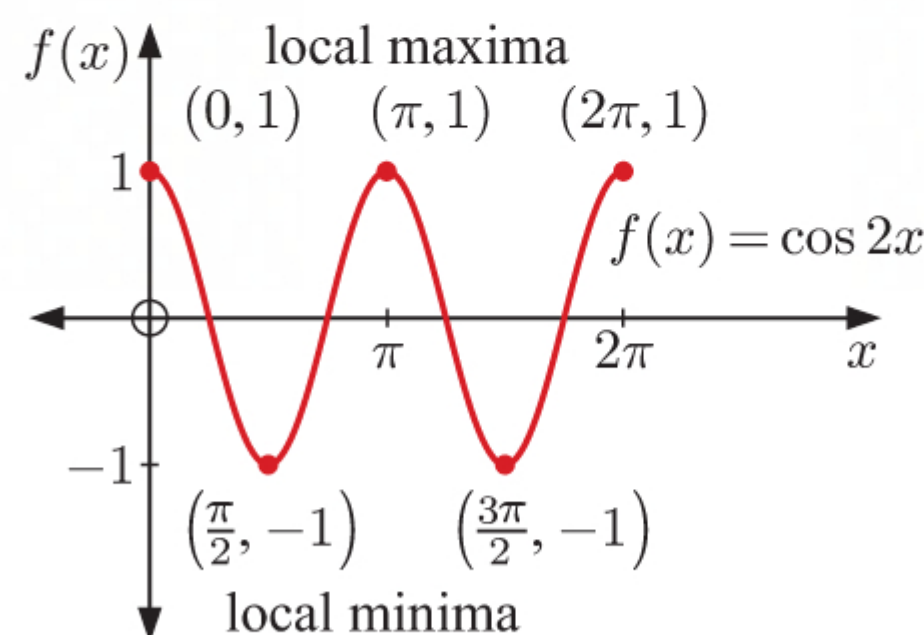


$f(0) = \cos 0 = 1$, $f\left(\frac{\pi}{2}\right) = \cos \pi = -1$,

$f(\pi) = \cos 2\pi = 1$, $f\left(\frac{3\pi}{2}\right) = \cos 3\pi = -1$,

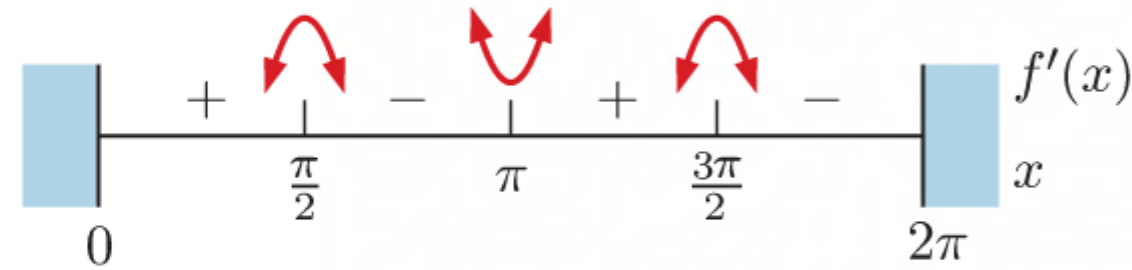
$f(2\pi) = \cos 4\pi = 1$

\therefore there are local maxima at $(0, 1)$, $(\pi, 1)$, and $(2\pi, 1)$,
 and local minima at $\left(\frac{\pi}{2}, -1\right)$ and $\left(\frac{3\pi}{2}, -1\right)$.



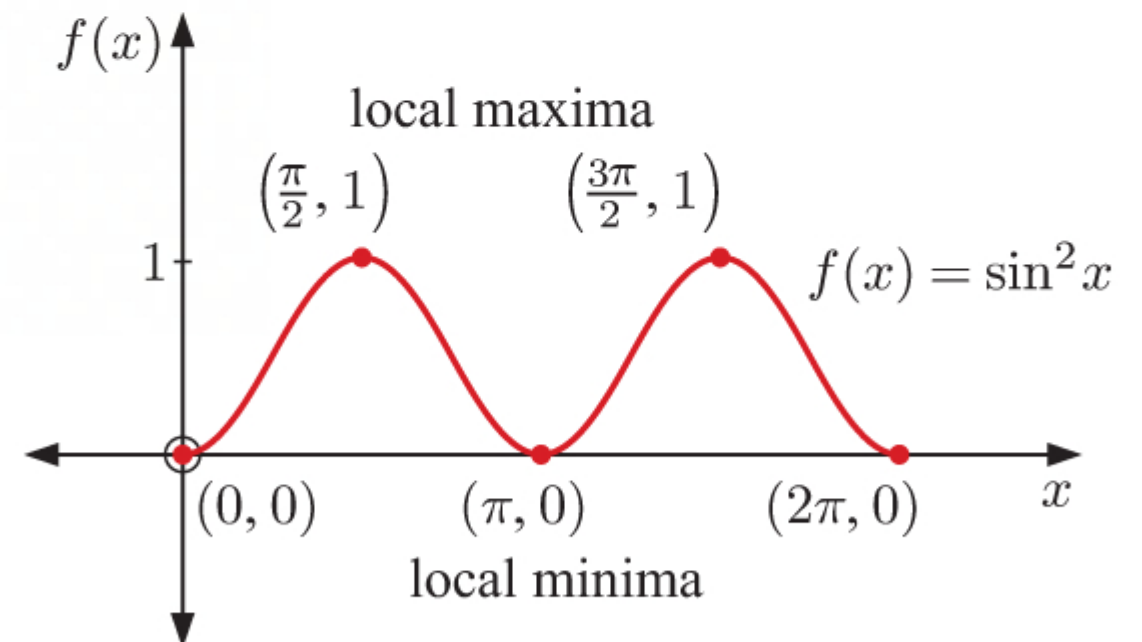
$$\begin{aligned}
 \text{c} \quad f(x) &= \sin^2 x = (\sin x)^2, \quad 0 \leq x \leq 2\pi \\
 \therefore f'(x) &= 2 \sin x (\cos x) \quad \{\text{chain rule}\} \\
 &= \sin 2x \quad \{\sin 2x = 2 \sin x \cos x\} \\
 f'(x) &= 0 \quad \text{when} \quad \sin 2x = 0 \\
 &\therefore 2x = 0, \pi, 2\pi, 3\pi, 4\pi \\
 &\therefore x = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi
 \end{aligned}$$

So, $f'(x)$ has sign diagram:



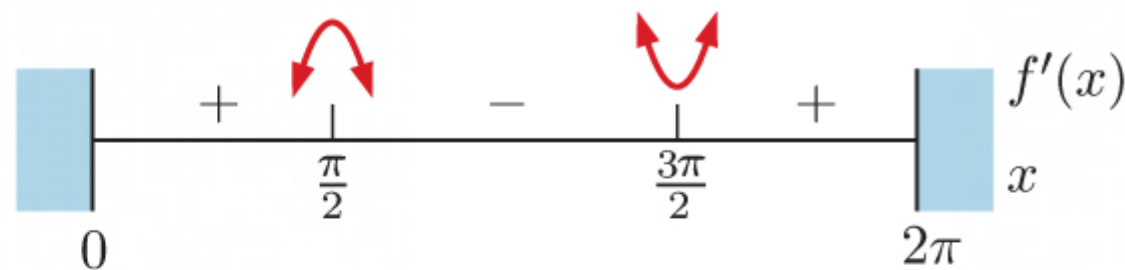
$$\begin{aligned}
 f(0) &= (\sin 0)^2 = 0^2 = 0, \\
 f\left(\frac{\pi}{2}\right) &= \left(\sin \frac{\pi}{2}\right)^2 = 1^2 = 1, \\
 f(\pi) &= (\sin \pi)^2 = 0^2 = 0, \\
 f\left(\frac{3\pi}{2}\right) &= \left(\sin \frac{3\pi}{2}\right)^2 = (-1)^2 = 1, \\
 f(2\pi) &= (\sin 2\pi)^2 = 0^2 = 0
 \end{aligned}$$

\therefore there are local minima at $(0, 0)$, $(\pi, 0)$, and $(2\pi, 0)$, and local maxima at $(\frac{\pi}{2}, 1)$ and $(\frac{3\pi}{2}, 1)$.



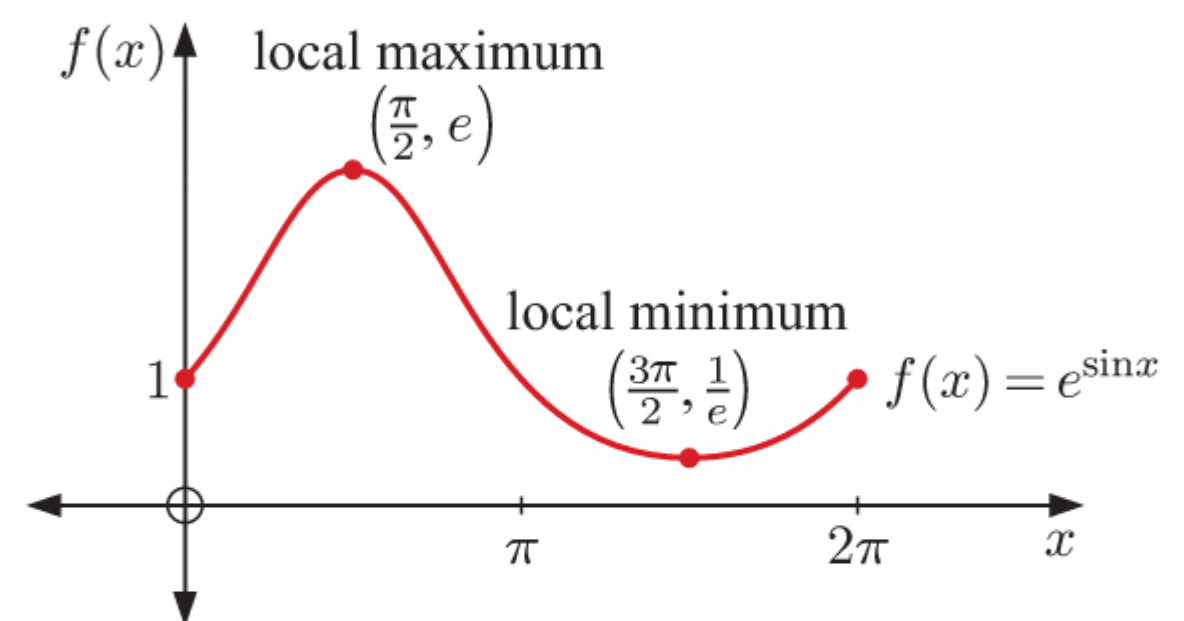
$$\begin{aligned}
 \text{d} \quad f(x) &= e^{\sin x}, \quad 0 \leq x \leq 2\pi \\
 \therefore f'(x) &= e^{\sin x} \cos x \quad \text{where } e^{\sin x} \text{ is positive for all } x \\
 \text{So, } f'(x) &= 0 \quad \text{when} \quad \cos x = 0 \\
 &\therefore x = \frac{\pi}{2}, \frac{3\pi}{2}
 \end{aligned}$$

So, $f'(x)$ has sign diagram:



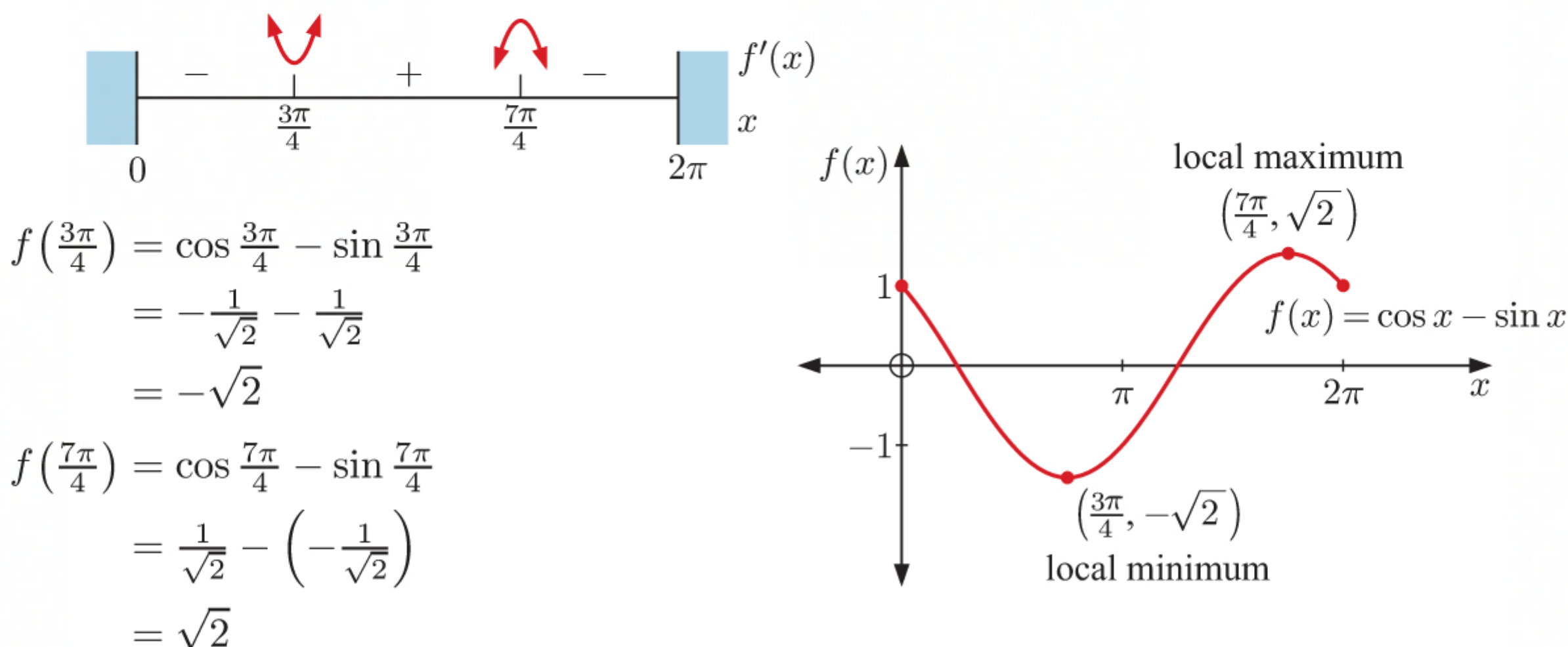
$$\begin{aligned}
 f\left(\frac{\pi}{2}\right) &= e^{\sin \frac{\pi}{2}} = e^1 = e, \\
 f\left(\frac{3\pi}{2}\right) &= e^{\sin \frac{3\pi}{2}} = e^{-1} = \frac{1}{e}
 \end{aligned}$$

\therefore there is a local maximum at $(\frac{\pi}{2}, e)$,
and a local minimum at $(\frac{3\pi}{2}, \frac{1}{e})$.



$$\begin{aligned}
 \text{e} \quad f(x) &= \cos x - \sin x, \quad 0 \leq x \leq 2\pi \\
 \therefore f'(x) &= -\sin x - \cos x \\
 &= -(\sin x + \cos x) \\
 f'(x) = 0 &\text{ when } \sin x + \cos x = 0 \\
 &\therefore \sin x = -\cos x \\
 &\therefore \tan x = -1 \\
 &\therefore x = \frac{3\pi}{4}, \frac{7\pi}{4}
 \end{aligned}$$

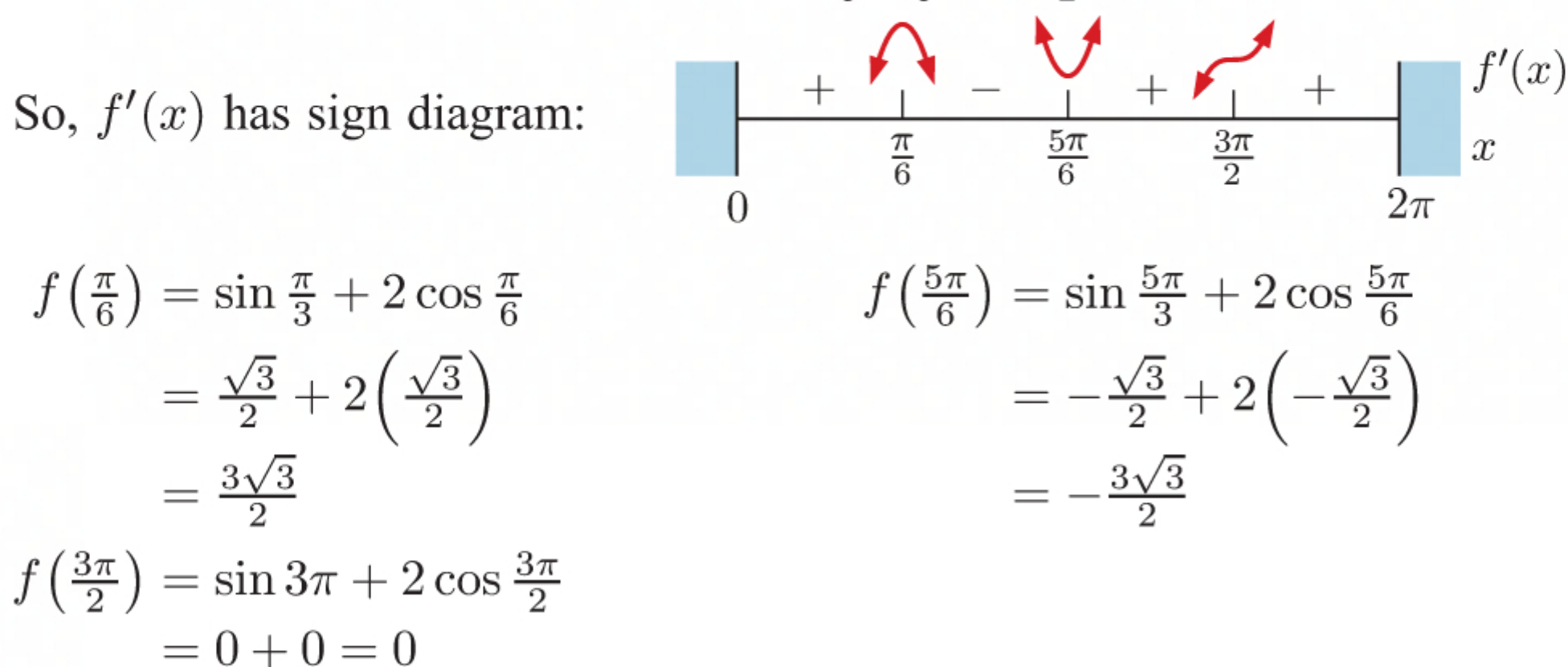
So, $f'(x)$ has sign diagram:



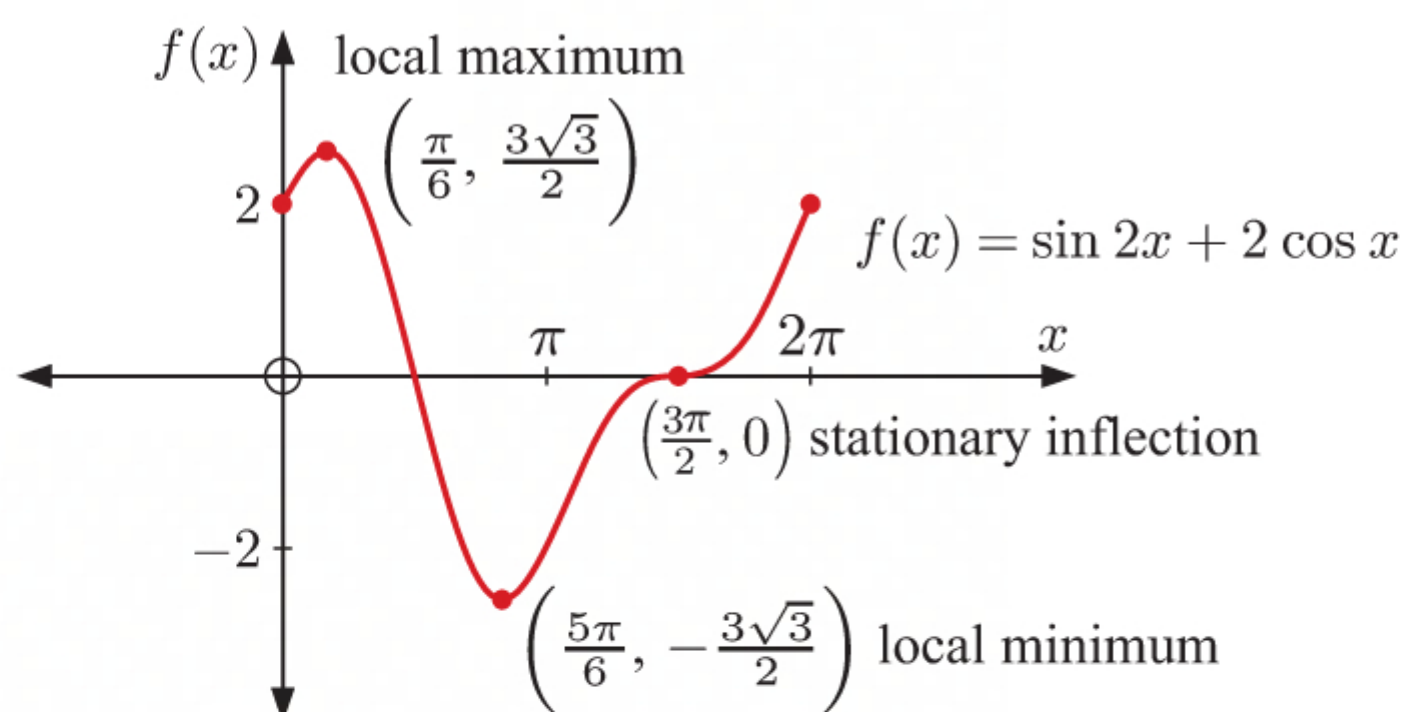
\therefore there is a local minimum at $(\frac{3\pi}{4}, -\sqrt{2})$, and a local maximum at $(\frac{7\pi}{4}, \sqrt{2})$.

$$\begin{aligned}
 \text{f} \quad f(x) &= \sin 2x + 2 \cos x, \quad 0 \leq x \leq 2\pi \\
 \therefore f'(x) &= 2 \cos 2x - 2 \sin x \\
 f'(x) = 0 &\text{ when } 2 \cos 2x - 2 \sin x = 0 \\
 &\therefore 2 \sin x = 2 \cos 2x \\
 &\therefore \sin x = \cos 2x \\
 &\therefore \sin x = 1 - 2 \sin^2 x \quad \{\text{double angle formula}\} \\
 &\therefore 2 \sin^2 x + \sin x - 1 = 0 \\
 &\therefore (2 \sin x - 1)(\sin x + 1) = 0 \quad \{\text{compare } 2a^2 + a - 1 = (2a - 1)(a + 1)\} \\
 &\therefore \sin x = \frac{1}{2} \text{ or } -1 \\
 &\therefore x = \frac{\pi}{6}, \frac{5\pi}{6}, \text{ or } \frac{3\pi}{2}
 \end{aligned}$$

So, $f'(x)$ has sign diagram:



\therefore there is a local maximum at $(\frac{\pi}{6}, \frac{3\sqrt{3}}{2})$, a local minimum at $(\frac{5\pi}{6}, -\frac{3\sqrt{3}}{2})$, and a stationary inflection at $(\frac{3\pi}{2}, 0)$.



13 $P(x) = ax^3 + bx^2 + cx + d$
 $\therefore P'(x) = 3ax^2 + 2bx + c \quad \dots (1)$

Now $(0, 2)$ lies on $P(x)$, so $P(0) = 2$

$$\therefore a(0) + b(0) + c(0) + d = 2$$

$$\therefore d = 2$$

The tangent at $(0, 2)$ is $y = 9x + 2$, so $P'(0) = 9$

$$\therefore 3a(0) + 2b(0) + c = 9$$

$$\therefore c = 9 \quad \dots (2)$$

There is a stationary point at $(-1, -7)$, so $P'(-1) = 0$.

$$\therefore 3a(-1)^2 + 2b(-1) + c = 0 \quad \{\text{using (1)}\}$$

$$\therefore 3a - 2b + c = 0$$

$$\text{So, using (2), } 3a - 2b = -9 \quad \dots (3)$$

Finally, $(-1, -7)$ lies on $P(x)$

$$\therefore a(-1)^3 + b(-1)^2 + c(-1) + d = -7$$

$$\therefore -a + b - 9 + 2 = -7$$

$$\therefore b - a = 0$$

$$\therefore a = b$$

$$\text{So, using (3), } 3a - 2a = -9$$

$$\therefore a = -9$$

$$\therefore a = b = -9$$

$$\therefore P(x) = -9x^3 - 9x^2 + 9x + 2$$

14 a Let $f(x) = x^3 - 12x - 2$, for $-3 \leq x \leq 5$

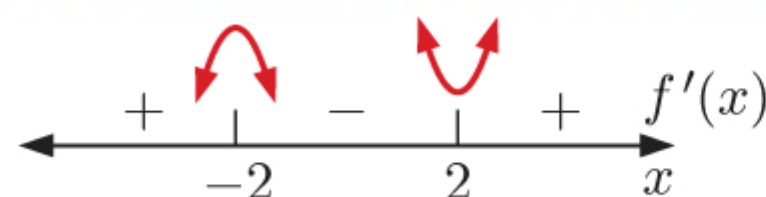
$$\therefore f'(x) = 3x^2 - 12$$

$$= 3(x^2 - 4)$$

$$= 3(x + 2)(x - 2)$$

which is 0 when $x = -2$ or 2

The sign diagram of $f'(x)$ is:



\therefore there is a local maximum at $x = -2$, and a local minimum at $x = 2$.

Critical value (x)	$f(x)$
-3 (end point)	7
-2 (local maximum)	14
2 (local minimum)	-18
5 (end point)	63

The greatest of these values is 63 when $x = 5$.

The least of these values is -18 when $x = 2$.

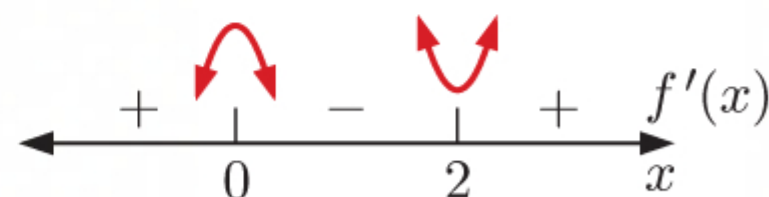
b Let $f(x) = 4 - 3x^2 + x^3$, for $-2 \leq x \leq 3$

$$\therefore f'(x) = -6x + 3x^2$$

$$= 3x(x - 2)$$

which is 0 when $x = 0$ or 2

The sign diagram of $f'(x)$ is:



\therefore there is a local maximum at $x = 0$, and a local minimum at $x = 2$.

Critical value (x)	$f(x)$
-2 (end point)	-16
0 (local maximum)	4
2 (local minimum)	0
3 (end point)	4

The greatest of these values is 4 when $x = 0$ or $x = 3$.

The least of these values is -16 when $x = -2$.

c Let $f(x) = x^2 + \frac{16}{x} = x^2 + 16x^{-1}$, for $1 \leq x \leq 4$

$$\therefore f'(x) = 2x - 16x^{-2}$$

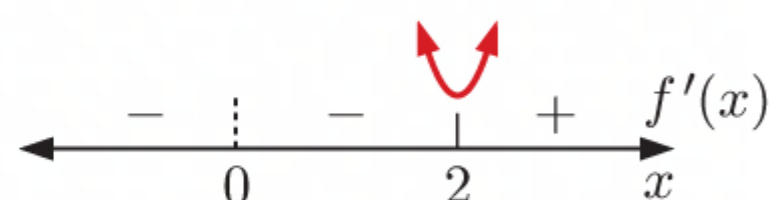
$$= 2x - \frac{16}{x^2}$$

$$= \frac{2x^3 - 16}{x^2}$$

$$= \frac{2(x^3 - 8)}{x^2}$$

which is 0 when $x = 2$

The sign diagram of $f'(x)$ is:



\therefore there is a local minimum at $x = 2$.

Critical value (x)	$f(x)$
1 (end point)	17
2 (local minimum)	12
4 (end point)	20

The greatest of these values is 20 when $x = 4$.

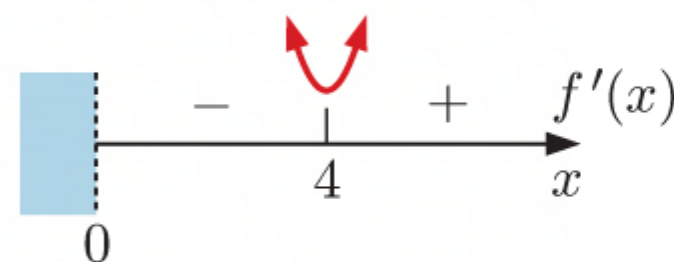
The least of these values is 12 when $x = 2$.

d Let $f(x) = x - 4\sqrt{x} = x - 4x^{\frac{1}{2}}$, for $0 \leq x \leq 5$

$$\begin{aligned}\therefore f'(x) &= 1 - 2x^{-\frac{1}{2}} \\ &= 1 - \frac{2}{\sqrt{x}} \\ &= \frac{\sqrt{x} - 2}{\sqrt{x}}\end{aligned}$$

which is 0 when $x = 4$

The sign diagram of $f'(x)$ is:



\therefore there is a local minimum at $x = 4$.

Critical value (x)	$f(x)$
0 (end point)	0
4 (local minimum)	-4
5 (end point)	≈ -3.94

The greatest of these values is 0 when $x = 0$.

The least of these values is -4 when $x = 4$.

15 $y = 4e^{-x} \sin x$

$$\begin{aligned}\therefore \frac{dy}{dx} &= -4e^{-x} \sin x + 4e^{-x} \cos x \quad \{\text{product rule}\} \\ &= 4e^{-x}(\cos x - \sin x) \quad \text{where } 4e^{-x} \text{ is positive for all } x\end{aligned}$$

So, $\frac{dy}{dx} = 0$ when $\cos x - \sin x = 0$

$$\therefore \cos x = \sin x$$

$$\therefore \tan x = 1$$

$$\therefore x = \frac{\pi}{4} + k\pi, \quad k \in \mathbb{Z}$$



$\therefore y = 4e^{-x} \sin x$ has a local maximum when $x = \frac{\pi}{4}$.

16 a $f(x) = \sin x \cos 2x$, for $0 \leq x \leq \pi$

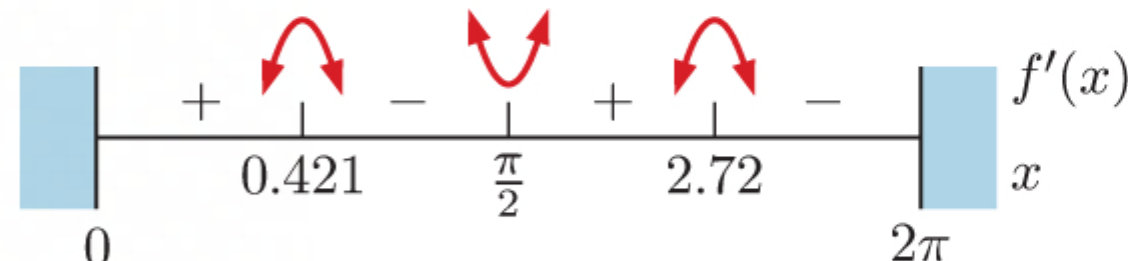
$$\begin{aligned}\therefore f'(x) &= \cos x \cos 2x + \sin x(-2 \sin 2x) \quad \{\text{product rule}\} \\ &= \cos x \cos 2x - 2 \sin x \sin 2x \\ &= \cos x(2 \cos^2 x - 1) - 2 \sin x(2 \sin x \cos x) \\ &= 2 \cos^3 x - \cos x - 4 \sin^2 x \cos x \\ &= 2 \cos^3 x - \cos x - 4(1 - \cos^2 x) \cos x \\ &= 2 \cos^3 x - \cos x - 4 \cos x + 4 \cos^3 x \\ &= 6 \cos^3 x - 5 \cos x\end{aligned}$$

$$\begin{aligned} \text{b } f'(x) &= 6 \cos^3 x - 5 \cos x \\ &= \cos x(6 \cos^2 x - 5) \end{aligned}$$

$$\begin{aligned} \text{So } f'(x) = 0 \text{ when } \cos x = 0 \text{ or } 6 \cos^2 x - 5 &= 0 \\ \therefore 6 \cos^2 x &= 5 \\ \therefore \cos^2 x &= \frac{5}{6} \\ \therefore \cos x &= \pm \sqrt{\frac{5}{6}} \end{aligned}$$

$$\begin{aligned} \text{c } \cos x &= 0 \text{ or } \pm \sqrt{\frac{5}{6}} \\ \therefore x &= \frac{\pi}{2}, x \approx 0.421, 2.72 \end{aligned}$$

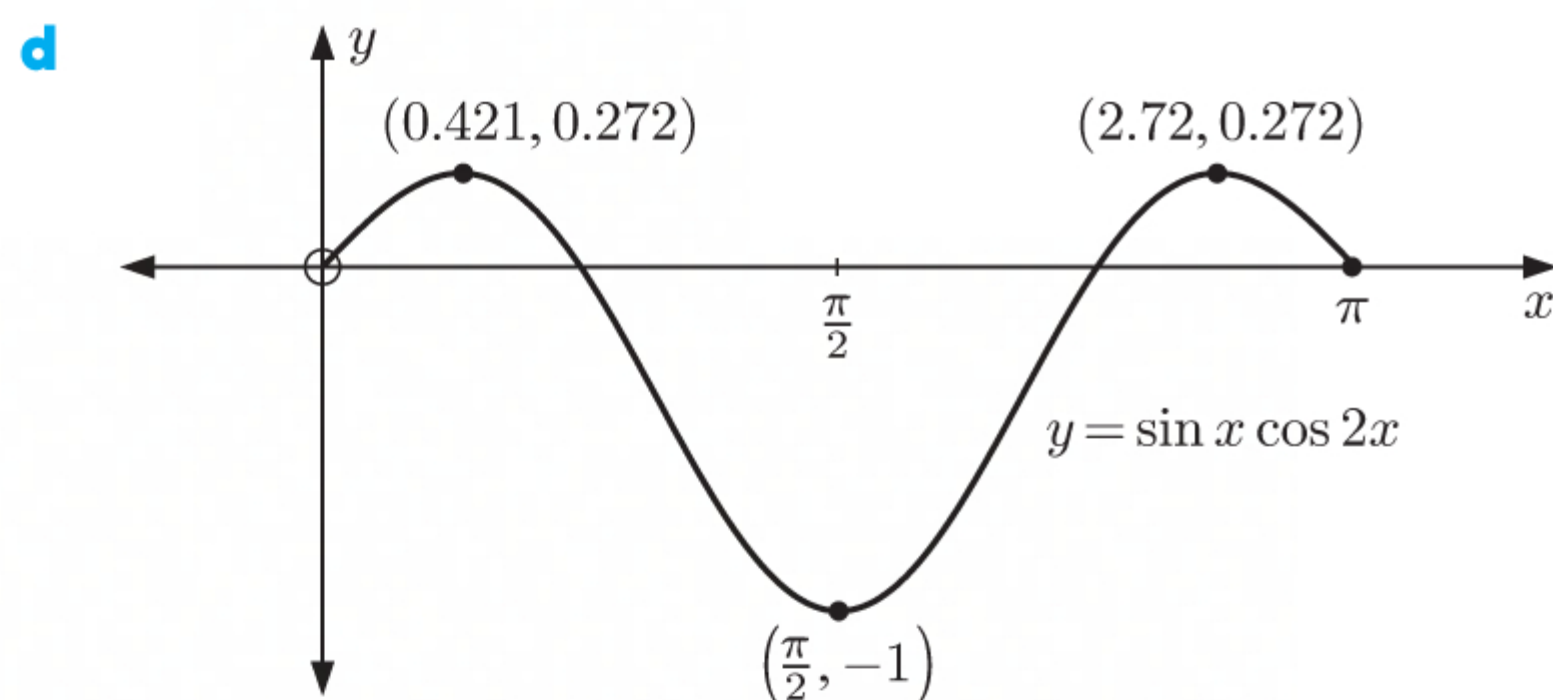
The sign diagram of $f'(x)$ is:



$$\begin{aligned} f\left(\frac{\pi}{2}\right) &= \sin \frac{\pi}{2} \cos\left(2\left(\frac{\pi}{2}\right)\right) \\ &= \sin \frac{\pi}{2} \cos \pi \\ &= 1 \times (-1) \\ &= -1 \end{aligned}$$

$$\begin{aligned} f(0.421) &= \sin 0.421 \cos(2 \times 0.421) \\ &\approx 0.272 \\ f(2.72) &= \sin 2.72 \cos(2 \times 2.72) \\ &\approx 0.272 \end{aligned}$$

\therefore there are local maxima at $(0.421, 0.272)$, $(2.72, 0.272)$, and a local minimum at $(\frac{\pi}{2}, -1)$.



$$\begin{aligned} \text{17 } f(t) &= ate^{bt^2} \\ \therefore f'(t) &= ae^{bt^2} + ate^{bt^2}(2bt) \quad \{\text{product rule}\} \\ &= ae^{bt^2} + 2abt^2e^{bt^2} \\ &= ae^{bt^2}(1 + 2bt^2) \end{aligned}$$

If $f(t)$ has maximum value 1 when $t = 2$, then

$$\begin{aligned} f(2) &= 1 & \text{and} & & f'(2) &= 0 \\ \therefore a(2)e^{b(2)^2} &= 1 & \therefore & & ae^{b(2)^2}(1 + 2b(2)^2) &= 0 \\ \therefore 2ae^{4b} &= 1 & \therefore & & ae^{4b}(1 + 8b) &= 0 \quad \dots (2) \\ \therefore ae^{4b} &= \frac{1}{2} \quad \dots (1) \end{aligned}$$

$$\begin{aligned} \text{Substituting (1) into (2) gives:} & \quad \frac{1}{2}(1 + 8b) = 0 \\ \therefore 1 + 8b &= 0 \\ \therefore 8b &= -1 \\ \therefore b &= -\frac{1}{8} \end{aligned}$$

Substituting $b = -\frac{1}{8}$ into (1) gives: $ae^{4(-\frac{1}{8})} = \frac{1}{2}$

$$\therefore ae^{-\frac{1}{2}} = \frac{1}{2}$$

$$\therefore a = \frac{1}{2}e^{\frac{1}{2}} = \frac{\sqrt{e}}{2}$$

So, $a = \frac{\sqrt{e}}{2}$ and $b = -\frac{1}{8}$.

18 Let $f(x) = \frac{\ln x}{x}$

$$\begin{aligned}\therefore f'(x) &= \frac{\left(\frac{1}{x}\right)x - \ln x(1)}{x^2} && \{\text{quotient rule}\} \\ &= \frac{1 - \ln x}{x^2}\end{aligned}$$

$$f'(x) = 0 \text{ when } 1 - \ln x = 0$$

$$\therefore \ln x = 1$$

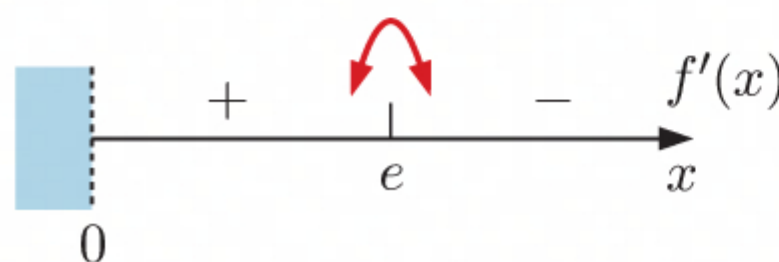
$$\therefore x = e$$

So, the sign diagram of $f'(x)$ is:

$$f(e) = \frac{\ln e}{e} = \frac{1}{e}$$

\therefore there is a local maximum at $\left(e, \frac{1}{e}\right)$.

$\therefore f(x) \leq \frac{1}{e}$ for all $x > 0$, and so $\frac{\ln x}{x} \leq \frac{1}{e}$ for all $x > 0$.



19 a $f(x) = x - \ln x$

$$\therefore f'(x) = 1 - \frac{1}{x} = \frac{x-1}{x}$$

So, the sign diagram of $f'(x)$ is:

$$f(1) = 1 - \ln 1$$

$$= 1 - 0$$

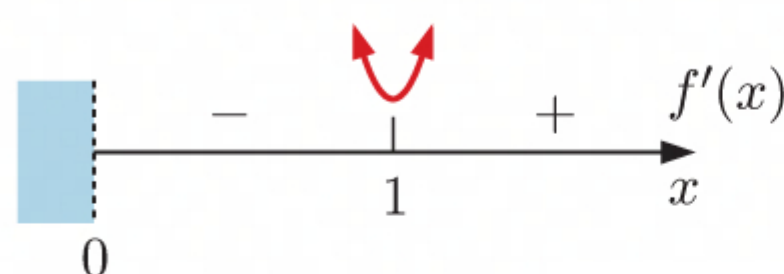
$$= 1$$

$\therefore f(x)$ has a local minimum at $(1, 1)$. This is the only turning point.

b $f(x) \geq 1$ for all $x > 0$

$$\therefore x - \ln x \geq 1$$

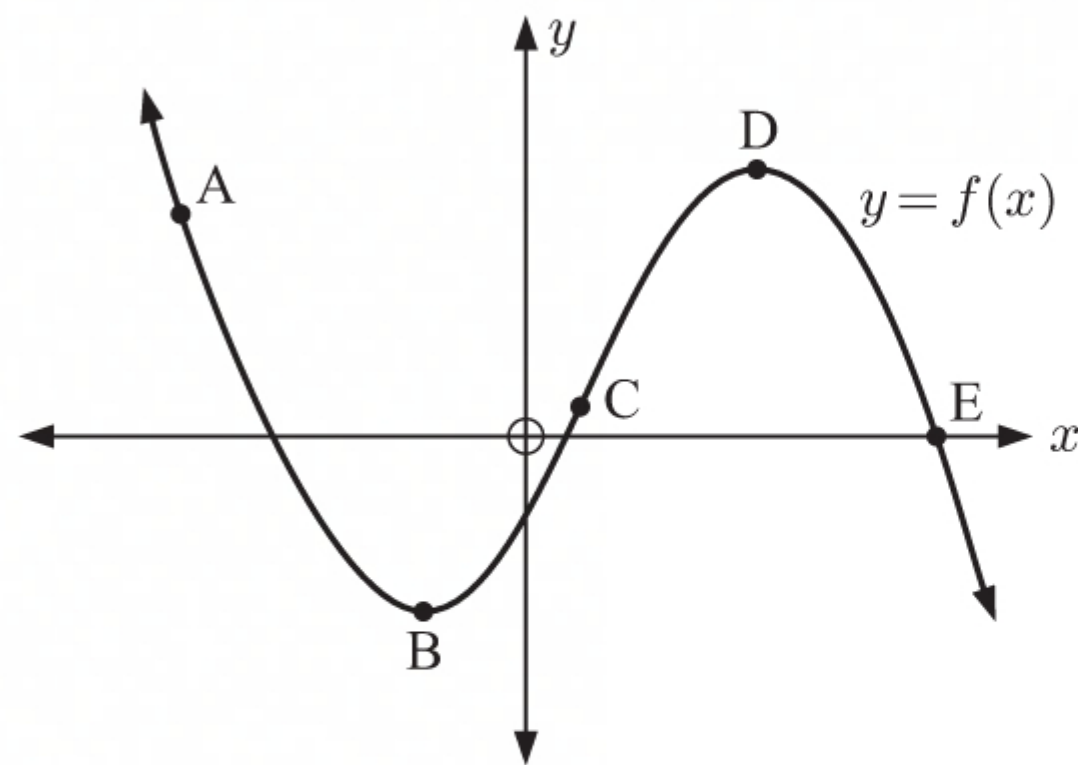
$$\therefore \ln x \leq x - 1 \text{ for all } x > 0$$



EXERCISE 13E

1 a

Point	$f(x)$	$f'(x)$	$f''(x)$
A	+	-	+
B	-	0	+
C	+	+	0
D	+	0	-
E	0	-	-



b B is a local minimum, D is a local maximum.

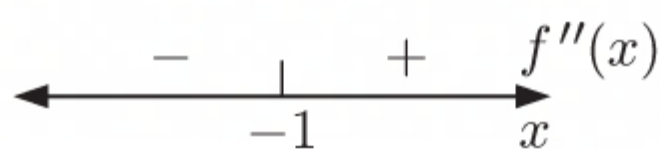
c The shape of $y = f(x)$ changes at C.

2 a

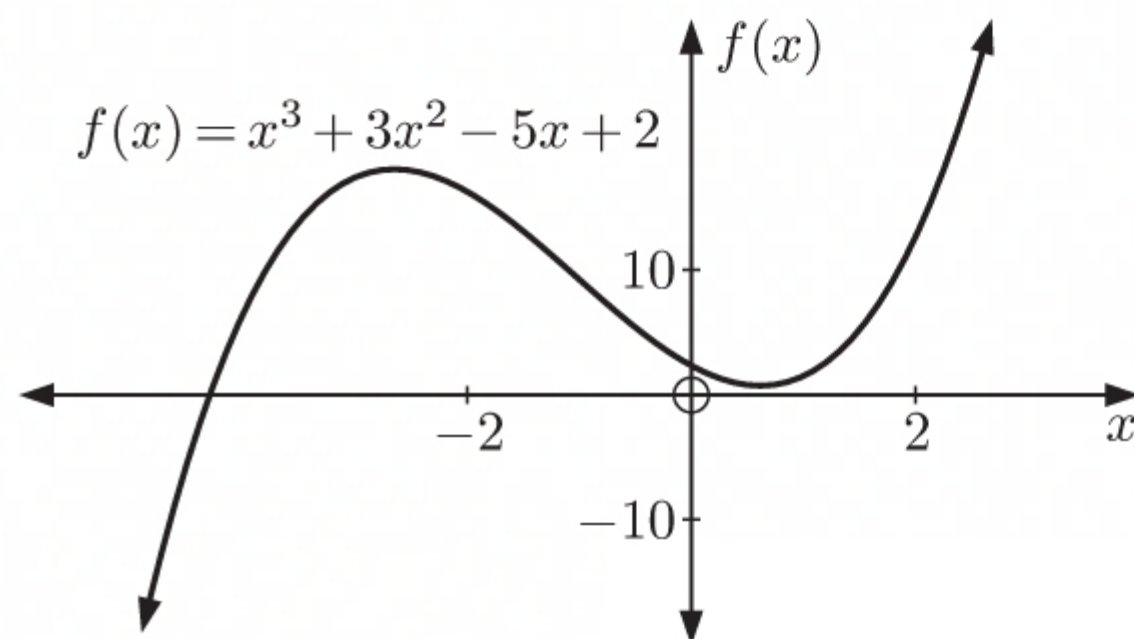
$$f(x) = x^3 + 3x^2 - 5x + 2$$

$$\therefore f'(x) = 3x^2 + 6x - 5$$

$$\therefore f''(x) = 6x + 6$$


b $f''(x)$ has sign diagram:

- c i The curve is concave up for $x \geq -1$.
 ii The curve is concave down for $x \leq -1$.




3 a

$y = 2x^2 - 3x + 4$ is a quadratic with $a = 2 > 0$.

\therefore the quadratic has shape 
 \therefore the quadratic is concave up.


c

$y = -4 - x^2 + 6x$ is a quadratic with $a = -1 < 0$.

\therefore the quadratic has shape 
 \therefore the quadratic is concave down.

b

$y = -2(x - 3)(x + 1)$ is a quadratic with $a = -2 < 0$.

\therefore the quadratic has shape 
 \therefore the quadratic is concave down.


d

$$y = (5 - x)(1 - 2x)$$

$$= 5 - 10x - x + 2x^2$$

$$= 2x^2 - 11x + 5$$

is a quadratic with $a = 2 > 0$.

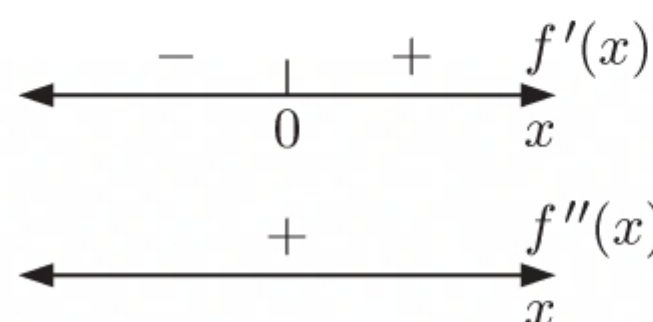
\therefore the quadratic has shape 
 \therefore the quadratic is concave up.

4 a

$$f(x) = x^2 + 1$$

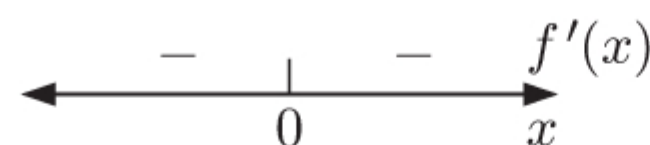
$$\therefore f'(x) = 2x \quad \text{which has sign diagram:}$$

$$\therefore f''(x) = 2 \quad \text{which has sign diagram:}$$

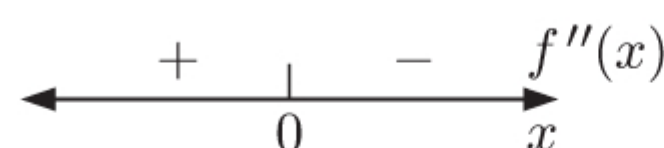
i $f(x)$ is increasing for $x \geq 0$.iii $f(x)$ is concave upwards for all $x \in \mathbb{R}$.ii $f(x)$ is decreasing for $x \leq 0$.iv $f(x)$ is never concave downwards.

b $f(x) = -x^3$

$\therefore f'(x) = -3x^2$ which has sign diagram:



$\therefore f''(x) = -6x$ which has sign diagram:



i $f(x)$ is never increasing.

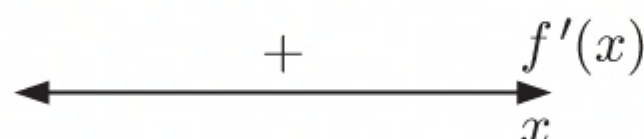
ii $f(x)$ is decreasing for all $x \in \mathbb{R}$.

iii $f(x)$ is concave upwards for $x \leq 0$.

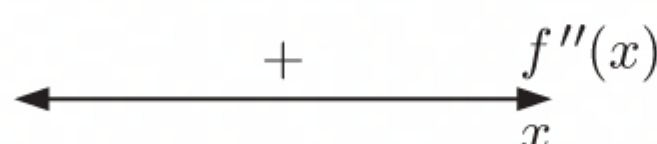
iv $f(x)$ is concave downwards for $x \geq 0$.

c $f(x) = e^x$

$\therefore f'(x) = e^x$ which has sign diagram:



$\therefore f''(x) = e^x$ which has sign diagram:



i $f(x)$ is increasing for all $x \in \mathbb{R}$.

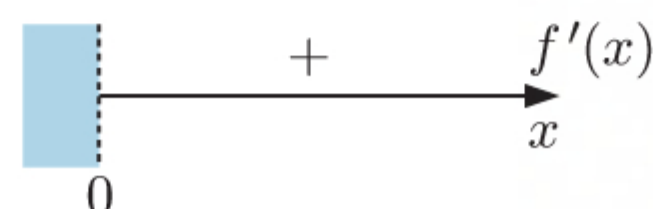
ii $f(x)$ is never decreasing.

iii $f(x)$ is concave upwards for all $x \in \mathbb{R}$.

iv $f(x)$ is never concave downwards.

d $f(x) = \sqrt{x} - 2 = x^{\frac{1}{2}} - 2$

$\therefore f'(x) = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$ which has sign diagram:



$\therefore f''(x) = -\frac{1}{4}x^{-\frac{3}{2}} = -\frac{1}{4x\sqrt{x}}$ which has sign diagram:



i $f(x)$ is increasing for $x > 0$.

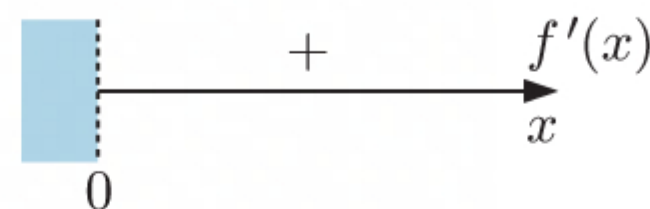
ii $f(x)$ is never decreasing.

iii $f(x)$ is never concave upwards.

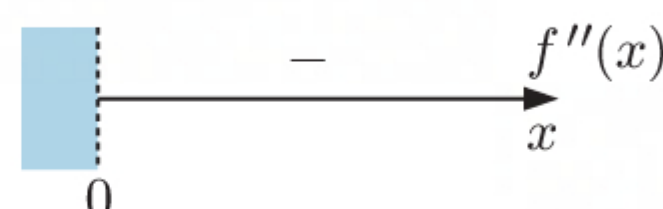
iv $f(x)$ is concave downwards for $x > 0$.

e $f(x) = -\frac{1}{\sqrt{x}} = -x^{-\frac{1}{2}}$

$\therefore f'(x) = \frac{1}{2}x^{-\frac{3}{2}} = \frac{1}{2x\sqrt{x}}$ which has sign diagram:



$\therefore f''(x) = -\frac{3}{4}x^{-\frac{5}{2}} = -\frac{3}{4x^2\sqrt{x}}$ which has sign diagram:



i $f(x)$ is increasing for $x > 0$.

ii $f(x)$ is never decreasing.

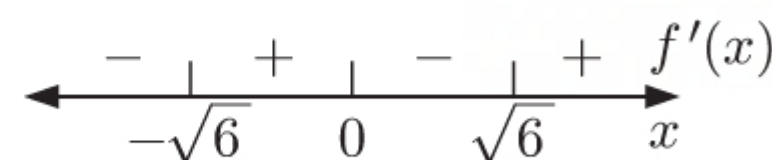
iii $f(x)$ is never concave upwards.

iv $f(x)$ is concave downwards for $x > 0$.

f $f(x) = x^4 - 12x^2$

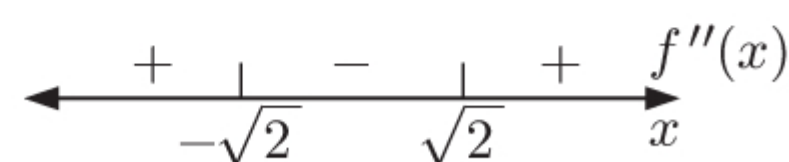
$\therefore f'(x) = 4x^3 - 24x$
 $= 4x(x^2 - 6)$

$= 4x(x + \sqrt{6})(x - \sqrt{6})$ which has sign diagram:



$\therefore f''(x) = 12x^2 - 24$
 $= 12(x^2 - 2)$

$= 12(x + \sqrt{2})(x - \sqrt{2})$ which has sign diagram:



i $f(x)$ is increasing for $-\sqrt{6} \leq x \leq 0$ and $x \geq \sqrt{6}$.

- ii $f(x)$ is decreasing for $x \leq -\sqrt{6}$ and $0 \leq x \leq \sqrt{6}$.
- iii $f(x)$ is concave upwards for all $x \leq -\sqrt{2}$ and $x \geq \sqrt{2}$.
- iv $f(x)$ is concave downwards for $-\sqrt{2} \leq x \leq \sqrt{2}$.

5 $f(x) = \ln(2x - 1) - 3$

a The x -intercept occurs when $y = 0$

$$\therefore \ln(2x - 1) - 3 = 0$$

$$\therefore \ln(2x - 1) = 3$$

$$\therefore 2x - 1 = e^3$$

$$\therefore 2x = e^3 + 1$$

$$\therefore x = \frac{e^3 + 1}{2} \approx 10.5$$

\therefore the x -intercept is $\frac{e^3 + 1}{2}$.

b $f(0)$ cannot be found as $\ln(-1)$ is not defined.

\therefore there is no y -intercept.

c $f(x) = \ln(2x - 1) - 3$ is defined when $2x - 1 > 0$

$$\therefore 2x > 1$$

$$\therefore x > \frac{1}{2}$$

\therefore domain of $f = \{x \mid x > \frac{1}{2}\}$

d $f'(x) = \frac{2}{2x - 1}$

$$\therefore f'(1) = \frac{2}{2(1) - 1} = 2$$


\therefore the tangent to the curve at $x = 1$ has gradient 2.

e $f'(x) = 2(2x - 1)^{-1}$

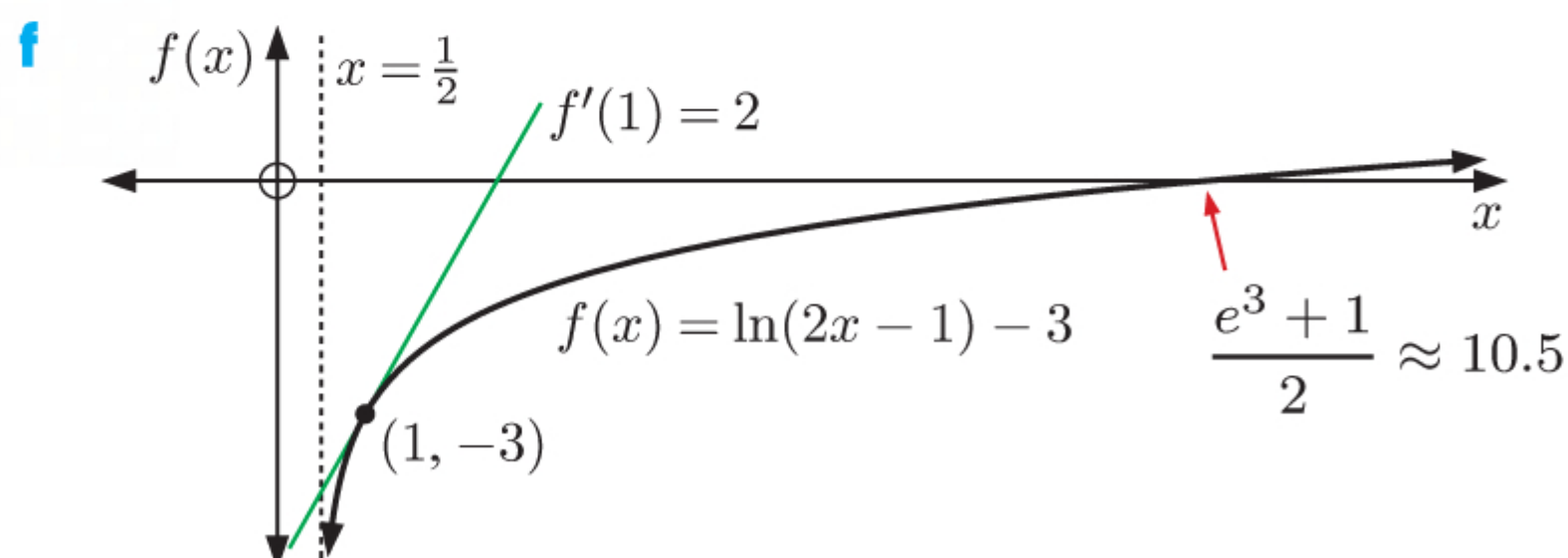
$$\therefore f''(x) = -2(2x - 1)^{-2}(2) \quad \{\text{chain rule}\}$$

$$= -\frac{4}{(2x - 1)^2}, \quad x > \frac{1}{2}$$

$\therefore f''(x)$ has sign diagram:

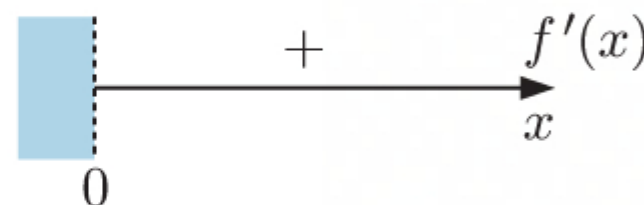


$\therefore f''(x) < 0$ for all $x > \frac{1}{2}$, so $f(x)$ is concave down for all x in the domain of f .



6 a $f(x) = \ln x$ is defined when $x > 0$.

b $f'(x) = \frac{1}{x}$ which has sign diagram:



$\therefore f(x)$ is increasing for $x > 0$.

$$f'(x) = x^{-1}$$

$\therefore f''(x) = -x^{-2}$

$= -\frac{1}{x^2}$ which has sign diagram:



$\therefore f(x)$ is concave down for $x > 0$.

c When $y = 1$, $\ln x = 1$

$$\therefore x = e$$

So, the point of contact is $(e, 1)$.

$$\text{Now } f'(e) = \frac{1}{e}.$$

\therefore the normal at $(e, 1)$ has gradient $-e$.

\therefore the normal has equation $y = -e(x - e) + 1$

$$\therefore y = -ex + e^2 + 1$$

$$\therefore ex + y = e^2 + 1$$

7 a $f(x) = \frac{e^x}{x} \neq 0$ since $e^x \neq 0$ for all x \therefore there is no x -intercept.

Also, $f(x)$ is not defined when $x = 0$ \therefore there is no y -intercept.

\therefore the graph of $y = f(x)$ does not have any x or y -intercepts.

b As $x \rightarrow \infty$, $e^x \rightarrow \infty$ (at a much faster rate than x)

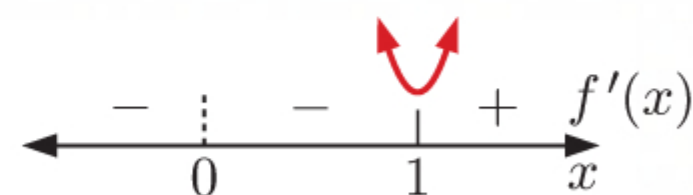
$$\therefore \text{ as } x \rightarrow \infty, f(x) = \frac{e^x}{x} \rightarrow \infty$$

$$\text{As } x \rightarrow -\infty, e^x \rightarrow 0^+$$

$$\therefore \text{ as } x \rightarrow -\infty, f(x) = \frac{e^x}{x} \rightarrow 0^- \quad \{f(x) < 0 \text{ for } x < 0\}$$

c $f'(x) = \frac{e^x x - e^x(1)}{x^2}$ {quotient rule}

$$= \frac{e^x(x-1)}{x^2} \text{ which has sign diagram:}$$



$$\text{Now } f(1) = \frac{e^1}{1} = e$$

$\therefore (1, e)$ is a local minimum.

$$\begin{aligned}
 \text{d} \quad f'(x) &= \frac{e^x x - e^x}{x^2} \\
 &= \frac{e^x}{x} - \frac{e^x}{x^2} \\
 \therefore f''(x) &= \frac{e^x x - e^x(1)}{x^2} - \left(\frac{e^x x^2 - e^x(2x)}{(x^2)^2} \right) \quad \{\text{quotient rule twice}\} \\
 &= \frac{e^x x - e^x}{x^2} - \left(\frac{e^x x^2 - 2e^x x}{x^4} \right) \\
 &= \frac{e^x x - e^x}{x^2} - \left(\frac{e^x x - 2e^x}{x^3} \right) \\
 &= \frac{e^x x^2 - e^x x - (e^x x - 2e^x)}{x^3} \\
 &= \frac{e^x x^2 - e^x x - e^x x + 2e^x}{x^3} \\
 &= \frac{e^x x^2 - 2e^x x + 2e^x}{x^3} \\
 &= \frac{e^x(x^2 - 2x + 2)}{x^3}
 \end{aligned}$$

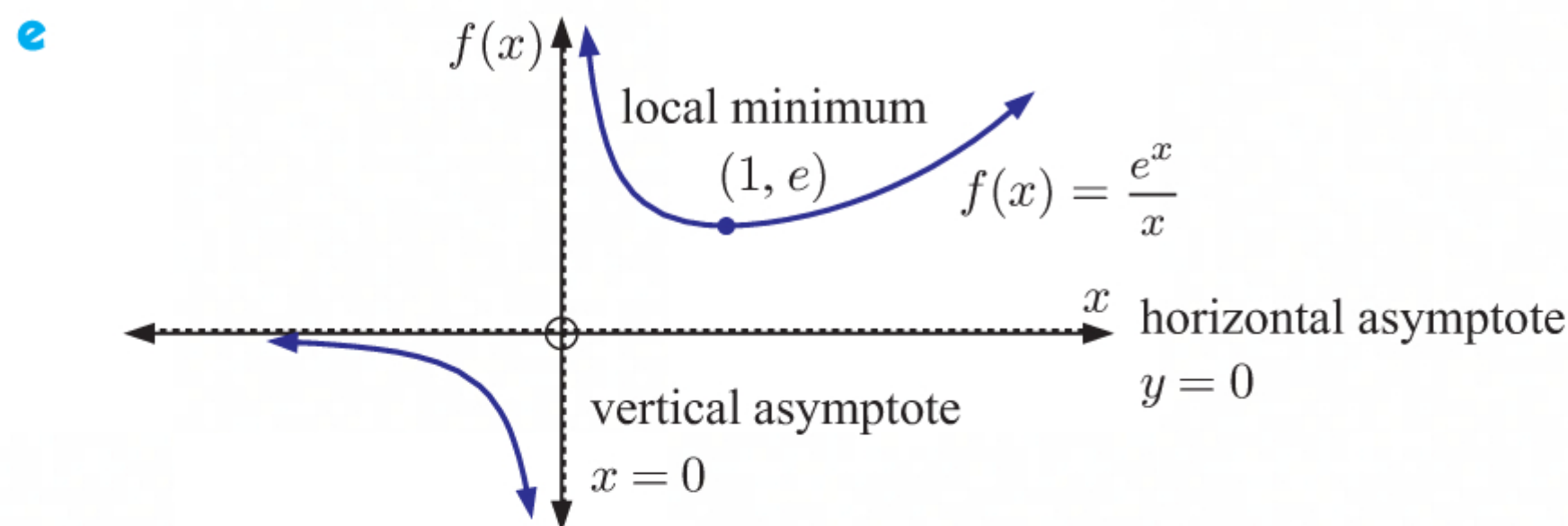
Now consider $x^2 - 2x + 2$: $\Delta = (-2)^2 - 4(1)(2)$
 $= -4 < 0$

$\therefore x^2 - 2x + 2$ has no real roots.

$\therefore f''(x)$ has sign diagram: $\begin{array}{ccc} & - & + \\ & \vdots & \\ & 0 & \end{array} \xrightarrow{f''(x)} x$

i $f(x)$ is concave up for $x > 0$.

ii $f(x)$ is concave down for $x < 0$.



f $f(-1) = \frac{e^{-1}}{-1} = -\frac{1}{e}$

\therefore the point of contact is $\left(-1, -\frac{1}{e}\right)$.

$$\begin{aligned}
 \text{Now } f'(-1) &= \frac{e^{-1}(-1-1)}{(-1)^2} \\
 &= -\frac{2}{e}
 \end{aligned}$$

So, the tangent has equation $y = -\frac{2}{e}(x+1) - \frac{1}{e}$

$$\therefore ey = -2(x+1) - 1$$

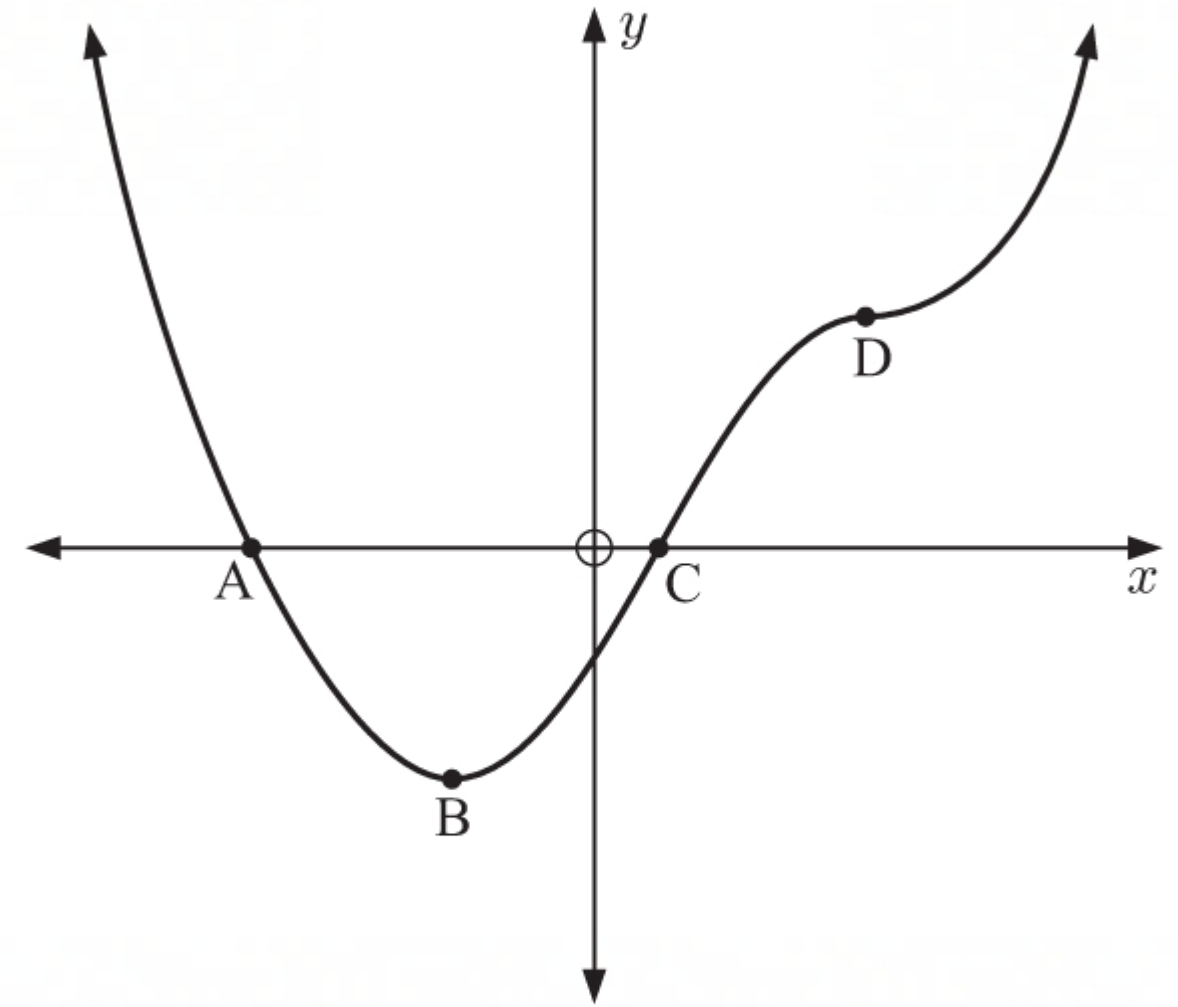
$$\therefore ey = -2x - 2 - 1$$

$$\therefore 2x + ey = -3$$

EXERCISE 13F

1 a

Point	$f(x)$	$f'(x)$	$f''(x)$
A	0	−	+
B	−	0	+
C	0	+	0
D	+	0	0

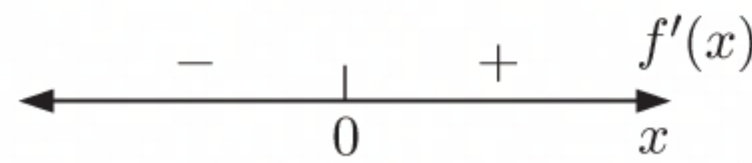


- b The turning point B is a local minimum.
 c C is a non-stationary inflection point, and D is a stationary inflection point.

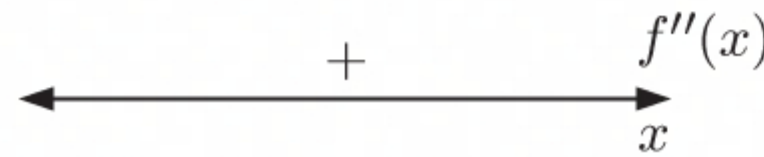
2 a

$$f(x) = x^2 + 3$$

$$\therefore f'(x) = 2x$$



$$\therefore f''(x) = 2$$

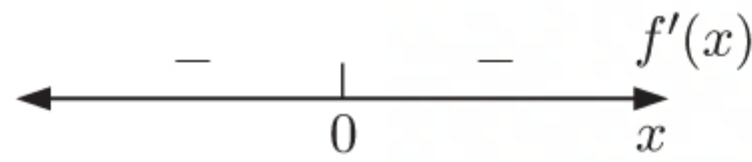


$f''(x) \neq 0$, so there are no points of inflection.

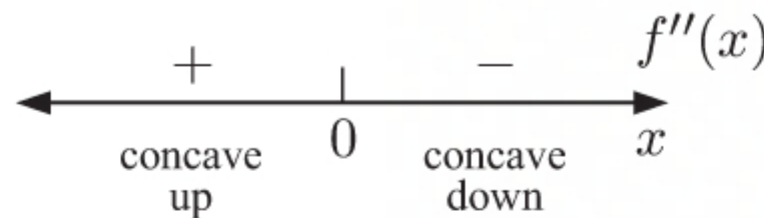
b

$$f(x) = 2 - x^3$$

$$\therefore f'(x) = -3x^2$$



$$\therefore f''(x) = -6x$$



Since the sign of $f''(x)$ changes at $x = 0$, this is a point of inflection.

$$f(0) = 2 \text{ and } f'(0) = 0$$

$\therefore (0, 2)$ is a stationary inflection.

c

$$f(x) = x^3 - 6x^2 + 9x + 1$$

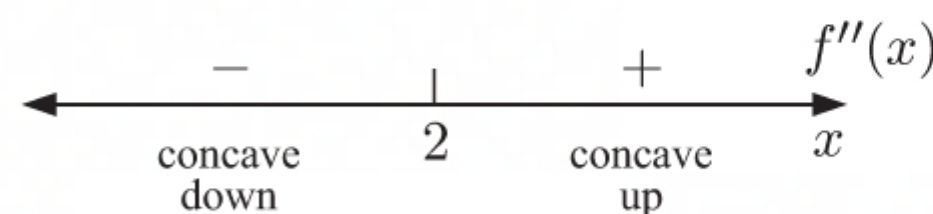
$$\therefore f'(x) = 3x^2 - 12x + 9$$

$$= 3(x^2 - 4x + 3)$$

$$= 3(x - 1)(x - 3)$$

$$\therefore f''(x) = 6x - 12$$

$$= 6(x - 2)$$



Since the sign of $f''(x)$ changes about $x = 2$, this is a point of inflection.

$$f(2) = 2^3 - 6(2)^2 + 9(2) + 1 \quad \text{and} \quad f'(2) = 3(1)(-1) \neq 0$$

$$= 8 - 24 + 18 + 1$$

$$= 3$$

$\therefore (2, 3)$ is a non-stationary inflection.

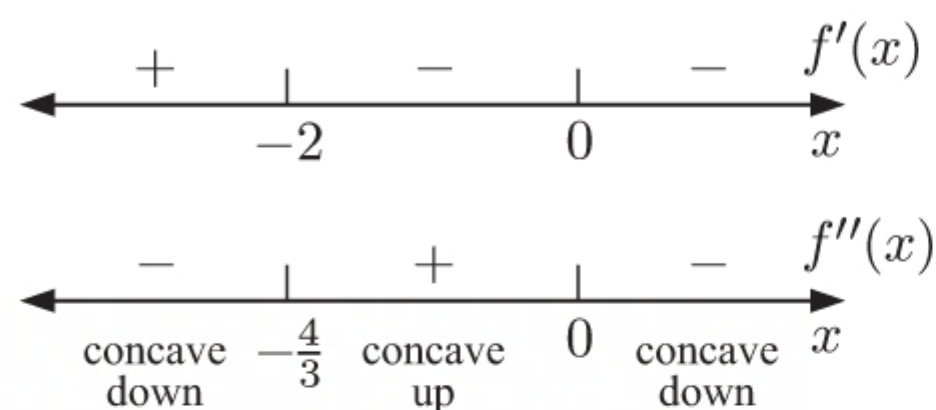
d $f(x) = -3x^4 - 8x^3 + 2$

$$\therefore f'(x) = -12x^3 - 24x^2$$

$$= -12x^2(x + 2)$$

$$\therefore f''(x) = -36x^2 - 48x$$

$$= -12x(3x + 4)$$



Since the signs of $f''(x)$ change about $x = -\frac{4}{3}$ and $x = 0$, both of these points are points of inflection.

$$f\left(-\frac{4}{3}\right) = -3\left(-\frac{4}{3}\right)^4 - 8\left(-\frac{4}{3}\right)^3 + 2 = \frac{310}{27}$$

and $f'\left(-\frac{4}{3}\right) = -12\left(-\frac{4}{3}\right)^3 - 24\left(-\frac{4}{3}\right)^2 \neq 0$

$\therefore \left(-1\frac{1}{3}, 11\frac{13}{27}\right)$ is a non-stationary inflection.

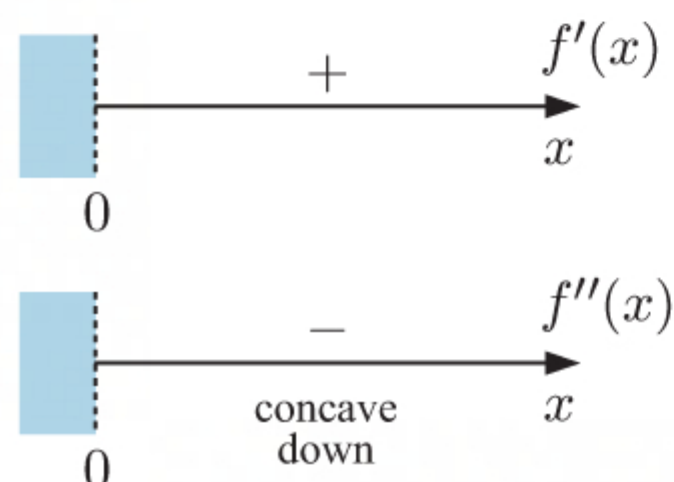
$$f(0) = 2 \text{ and } f'(0) = 0$$

$\therefore (0, 2)$ is a stationary inflection.

e $f(x) = 3 - \frac{1}{\sqrt{x}} = 3 - x^{-\frac{1}{2}}$

$$\therefore f'(x) = \frac{1}{2}x^{-\frac{3}{2}} = \frac{1}{2x\sqrt{x}}$$

$$\therefore f''(x) = -\frac{3}{4}x^{-\frac{5}{2}} = -\frac{3}{4x^2\sqrt{x}}$$



$f''(x) \neq 0$, so there are no points of inflection.

f $f(x) = x^3 + 6x^2 + 12x + 5$

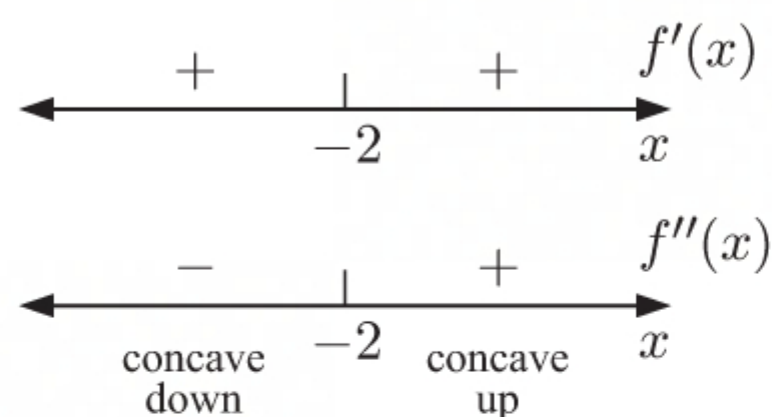
$$\therefore f'(x) = 3x^2 + 12x + 12$$

$$= 3(x^2 + 4x + 4)$$

$$= 3(x + 2)^2$$

$$\therefore f''(x) = 6x + 12$$

$$= 6(x + 2)$$



Since the sign of $f''(x)$ changes at $x = -2$, this is a point of inflection.

$$f(-2) = (-2)^3 + 6(-2)^2 + 12(-2) + 5 \quad \text{and} \quad f'(-2) = 3(0)^2 = 0$$

$$= -8 + 24 - 24 + 5$$

$$= -3$$

$\therefore (-2, -3)$ is a stationary inflection.

g $f(x) = x^2 + 8\sqrt{x} = x^2 + 8x^{\frac{1}{2}}$

$\therefore f'(x) = 2x + 4x^{-\frac{1}{2}} = 2x + \frac{4}{\sqrt{x}}$

$\therefore f''(x) = 2 - 2x^{-\frac{3}{2}} = 2 - \frac{2}{x\sqrt{x}}$

$= 2\left(1 - \frac{1}{x\sqrt{x}}\right)$

Since the sign of $f''(x)$ changes at $x = 1$, this is a point of inflection.

$$\begin{aligned} f(1) &= 1^2 + 8\sqrt{1} & \text{and} & & f'(1) &= 2(1) + \frac{4}{\sqrt{1}} \neq 0 \\ &= 1 + 8 \\ &= 9 \end{aligned}$$

$\therefore (1, 9)$ is a non-stationary inflection.

h $f(x) = x^4 - 6x^2 + 10$

$\therefore f'(x) = 4x^3 - 12x$

$= 4x(x^2 - 3)$

$= 4x(x + \sqrt{3})(x - \sqrt{3})$

$\therefore f''(x) = 12x^2 - 12$

$= 12(x^2 - 1)$

$= 12(x + 1)(x - 1)$

Since the sign of $f''(x)$ changes at $x = -1$ and $x = 1$, both of these points are points of inflection.

$$\begin{aligned} f(-1) &= (-1)^4 - 6(-1)^2 + 10 \\ &= 1 - 6 + 10 \\ &= 5 \end{aligned}$$

$$\begin{aligned} f(1) &= 1^4 - 6(1)^2 + 10 \\ &= 1 - 6 + 10 \\ &= 5 \end{aligned}$$

and $f'(-1) = 4(-1)^3 - 12(-1) \neq 0$

and $f'(1) = 4(1)^3 - 12(1) \neq 0$

$\therefore (-1, 5)$ and $(1, 5)$ are non-stationary inflection points.

3 a i $f(x) = x^2 - 5x + 4$

$\therefore f'(x) = 2x - 5$ $\therefore f'(x)$ has sign diagram:

Now $f\left(\frac{5}{2}\right) = \left(\frac{5}{2}\right)^2 - 5\left(\frac{5}{2}\right) + 4$

$= \frac{25}{4} - \frac{25}{2} + 4$

$= -\frac{9}{4}$

$\therefore \left(\frac{5}{2}, -\frac{9}{4}\right)$ is a local minimum.

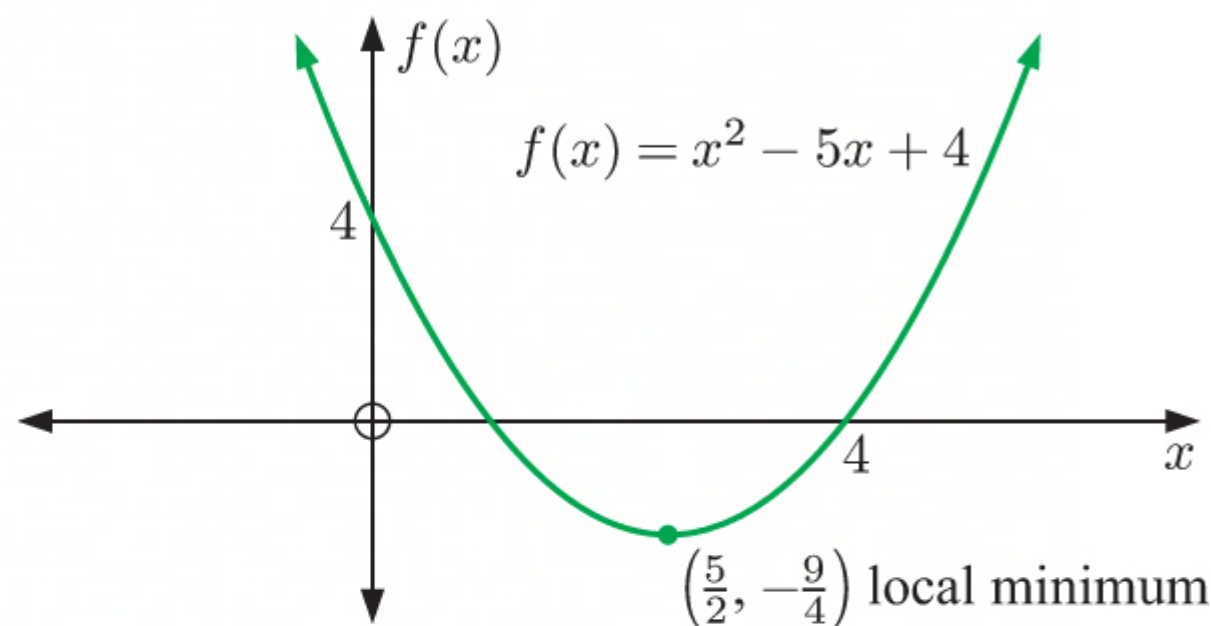
ii $f''(x) = 2$ $\therefore f''(x)$ has sign diagram:

$f''(x) \neq 0$, so there are no points of inflection.

iii $f(x)$ is increasing for $x \geq \frac{5}{2}$, and decreasing for $x \leq \frac{5}{2}$.

iv $f(x)$ is concave up for all $x \in \mathbb{R}$, and never concave down.

v

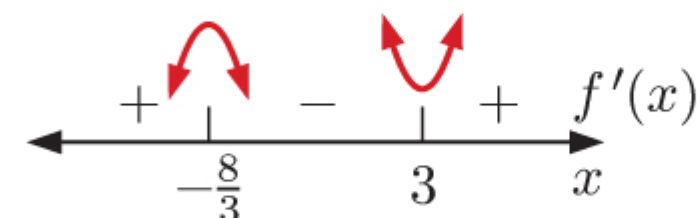


b i

$$f(x) = x^3 + 4x^2$$

$$\therefore f'(x) = 3x^2 + 8x \quad \therefore f'(x) \text{ has sign diagram:}$$

$$= x(3x + 8)$$



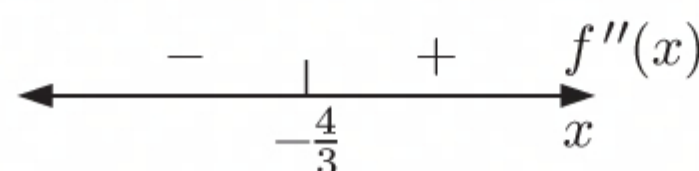
$$\text{Now } f(0) = 0, \quad f(-\frac{8}{3}) = (-\frac{8}{3})^3 + 4(-\frac{8}{3})^2$$

$$= \frac{256}{27}$$

$\therefore (-\frac{8}{3}, \frac{256}{27})$ is a local maximum, $(0, 0)$ is a local minimum.

ii

$$f''(x) = 6x + 8 \quad \therefore f''(x) \text{ has sign diagram:}$$



$$\text{Now } f(-\frac{4}{3}) = (-\frac{4}{3})^3 + 4(-\frac{4}{3})^2 \quad \text{and} \quad f'(-\frac{4}{3}) \neq 0$$

$$= -\frac{64}{27} + \frac{64}{9}$$

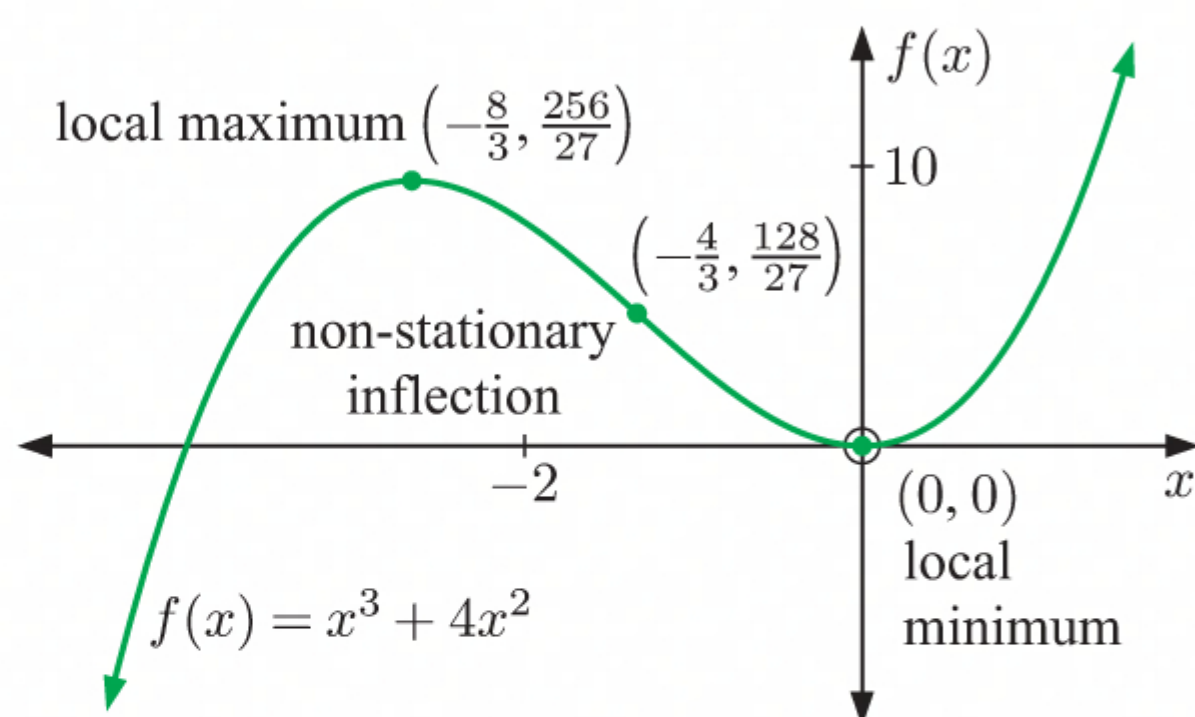
$$= \frac{128}{27}$$

$\therefore (-\frac{4}{3}, \frac{128}{27})$ is a non-stationary inflection.

iii $f(x)$ is increasing for all $x \leq -\frac{8}{3}$ and $x \geq 0$, and decreasing for $-\frac{8}{3} \leq x \leq 0$.

iv $f(x)$ is concave up for all $x \geq -\frac{4}{3}$, and concave down for $x \leq -\frac{4}{3}$.

v

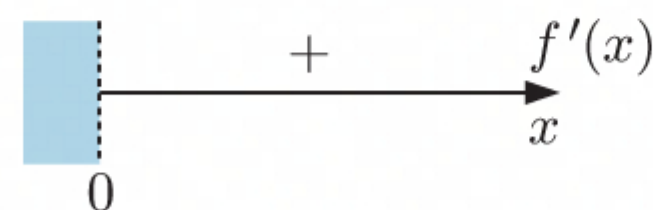


c i

$$f(x) = \sqrt{x} = x^{\frac{1}{2}}$$

$$\therefore f'(x) = \frac{1}{2}x^{-\frac{1}{2}} \quad \therefore f'(x) \text{ has sign diagram:}$$

$$= \frac{1}{2\sqrt{x}}$$



\therefore there are no turning points.

ii $f''(x) = -\frac{1}{4}x^{-\frac{3}{2}} \quad \therefore f''(x) \text{ has sign diagram:}$

$$= -\frac{1}{4x\sqrt{x}}$$

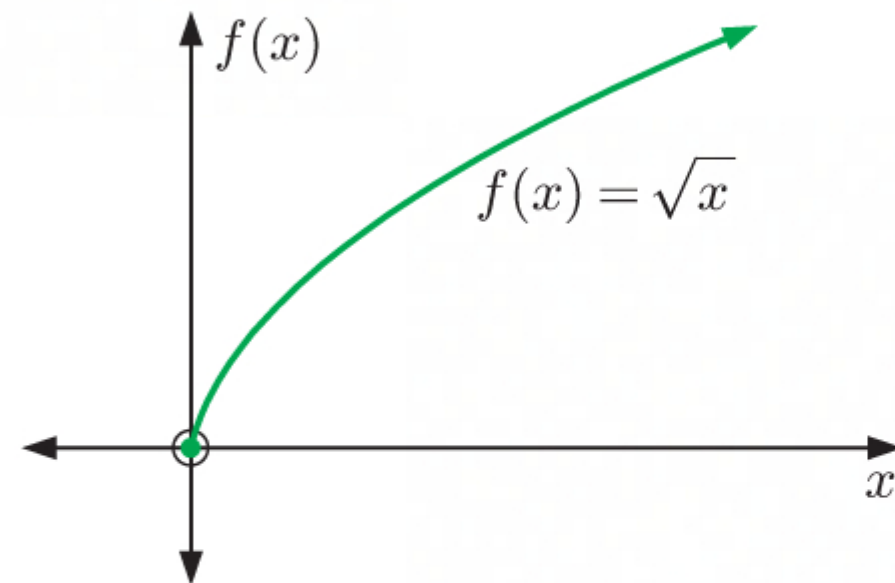


$f''(x) \neq 0$, so there are no points of inflection.

iii $f(x)$ is increasing for $x > 0$, and never decreasing.

iv $f(x)$ is concave down for $x > 0$, and never concave up.

v

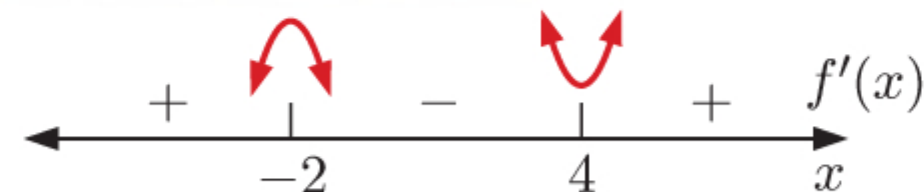


d i $f(x) = x^3 - 3x^2 - 24x + 1$

$\therefore f'(x) = 3x^2 - 6x - 24 \quad \therefore f'(x) \text{ has sign diagram:}$

$$= 3(x^2 - 2x - 8)$$

$$= 3(x+2)(x-4)$$

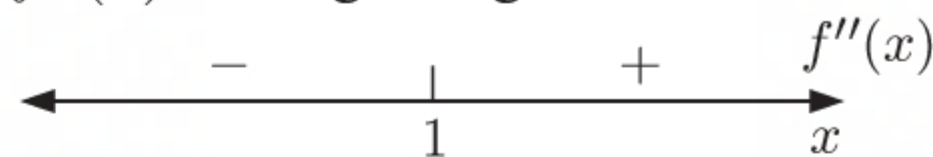


Now $f(-2) = (-2)^3 - 3(-2)^2 - 24(-2) + 1$
 $= -8 - 12 + 48 + 1$
 $= 29$

and $f(4) = 4^3 - 3(4)^2 - 24(4) + 1$
 $= 64 - 48 - 96 + 1$
 $= -79$

$\therefore (-2, 29)$ is a local maximum, and $(4, -79)$ is a local minimum.

ii $f''(x) = 6x - 6 \quad \therefore f''(x) \text{ has sign diagram:}$
 $= 6(x - 1)$

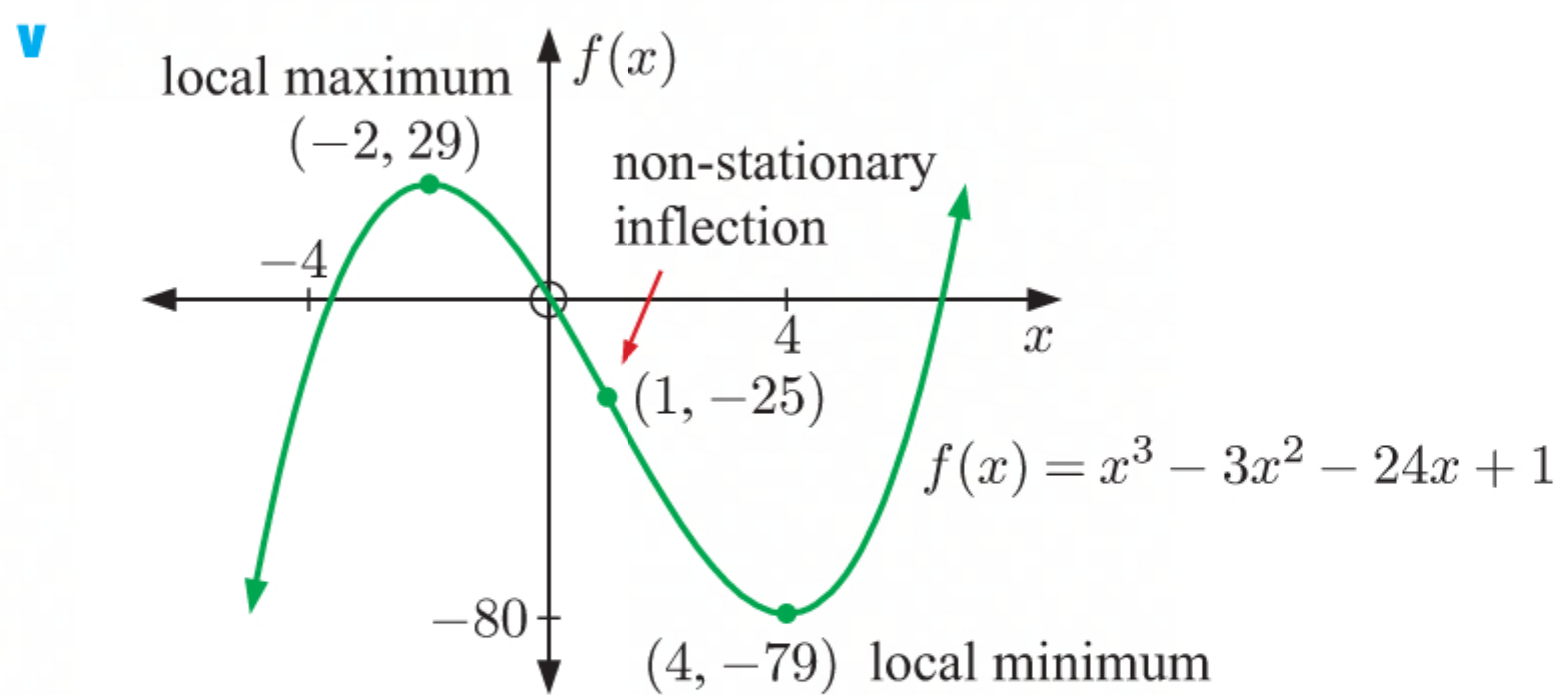


$f(1) = 1^3 - 3(1)^2 - 24(1) + 1 \quad \text{and} \quad f'(1) \neq 0$
 $= 1 - 3 - 24 + 1$
 $= -25$

$\therefore (1, -25)$ is a non-stationary point of inflection.

iii $f(x)$ is increasing for $x \leq -2$ and $x \geq 4$, and decreasing for $-2 \leq x \leq 4$.

iv $f(x)$ is concave down for $x \leq 1$, and concave up for $x \geq 1$.



e i $f(x) = 3x^4 + 4x^3 - 2$
 $\therefore f'(x) = 12x^3 + 12x^2 = 12x^2(x+1)$ $\therefore f'(x)$ has sign diagram:



Now $f(-1) = 3(-1)^4 + 4(-1)^3 - 2 = 3 - 4 - 2 = -3$ and $f(0) = -2$

$\therefore (-1, -3)$ is a local minimum, and $(0, -2)$ is a point of inflection but not a turning point.

ii $f''(x) = 36x^2 + 24x = 12x(3x+2)$ $\therefore f''(x)$ has sign diagram:

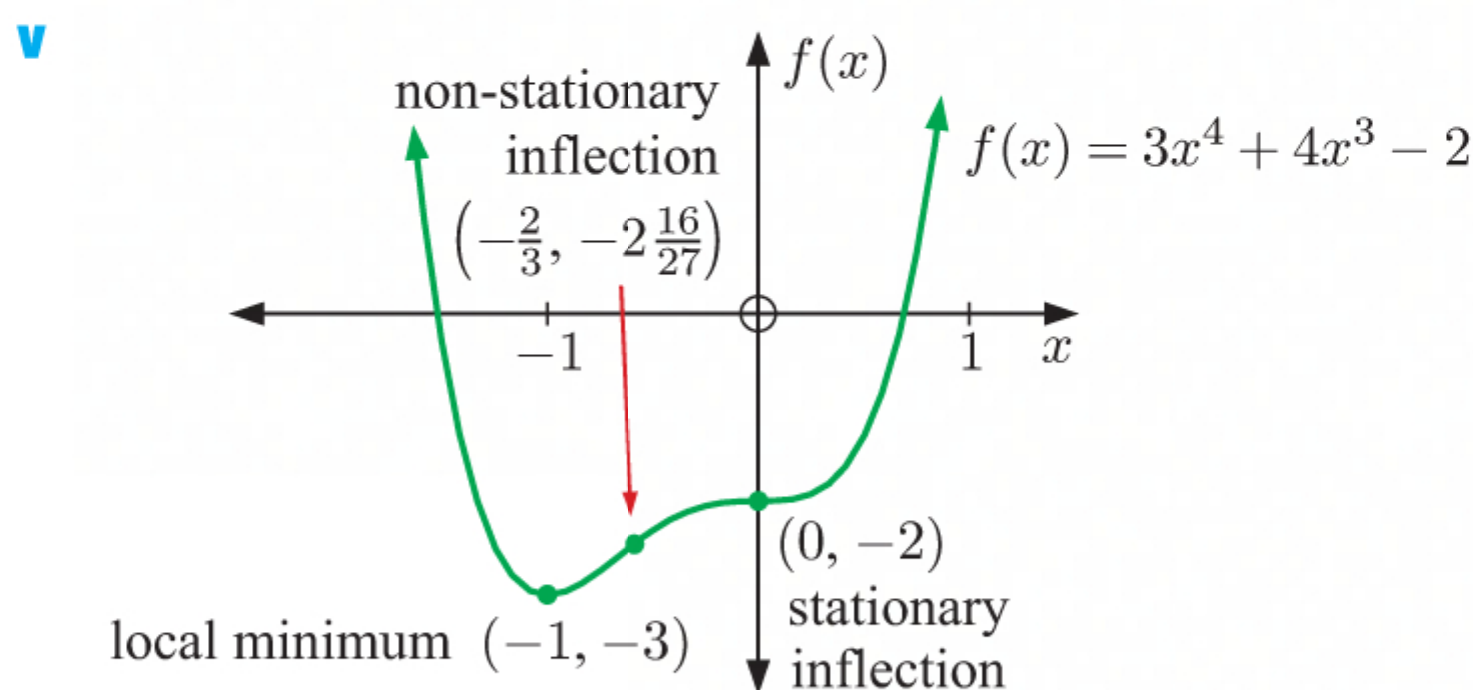


Now $f(-\frac{2}{3}) = 3(-\frac{2}{3})^4 + 4(-\frac{2}{3})^3 - 2 = -\frac{70}{27}$ and $f'(-\frac{2}{3}) \neq 0$

$\therefore (-\frac{2}{3}, -\frac{70}{27})$ is a non-stationary inflection, and $(0, -2)$ is a stationary inflection.

iii $f(x)$ is increasing for $x \geq -1$, and decreasing for $x \leq -1$.

iv $f(x)$ is concave down for $-\frac{2}{3} \leq x \leq 0$, and concave up for $x \leq -\frac{2}{3}$ and $x \geq 0$.



f i $f(x) = (x-1)^4$
 $\therefore f'(x) = 4(x-1)^3(1)$ {chain rule}
 $= 4(x-1)^3$

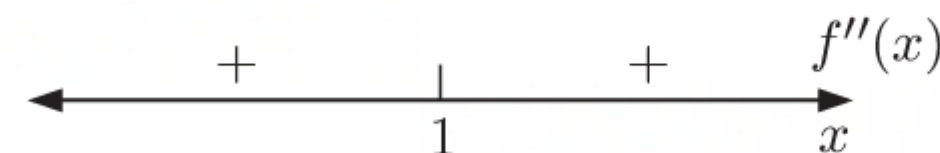
$\therefore f'(x)$ has sign diagram:



Now $f(1) = (1-1)^4 = 0$

$\therefore (1, 0)$ is a local minimum.

ii $f''(x) = 12(x-1)^2(1)$ {chain rule} $\therefore f''(x)$ has sign diagram:

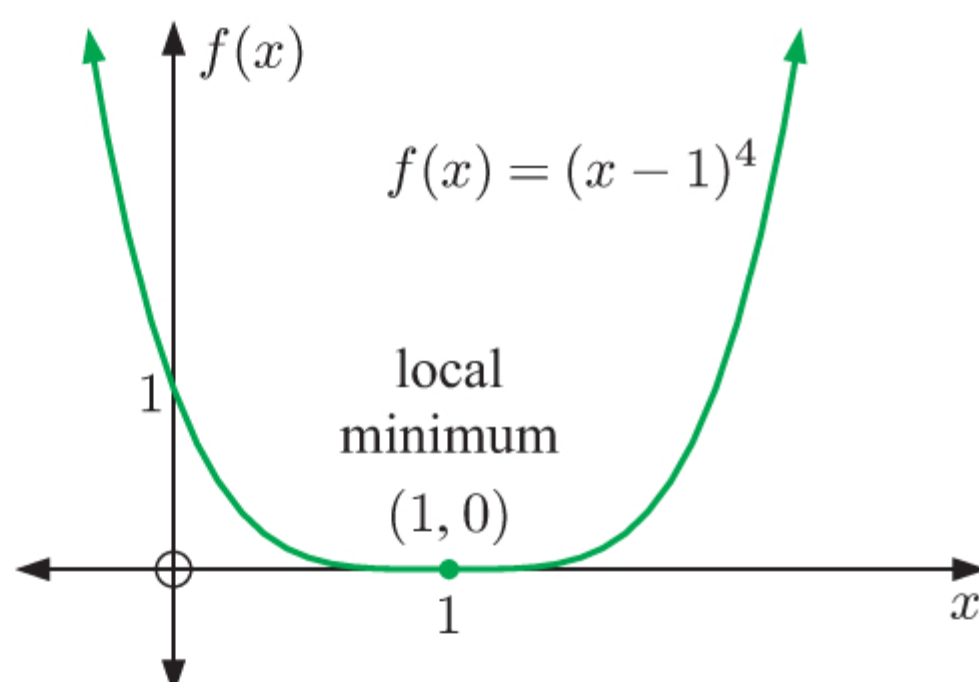


There is no change in sign of $f''(x)$, so there are no points of inflection.

iii $f(x)$ is increasing for $x \geq 1$, and decreasing for $x \leq 1$.

iv $f(x)$ is concave up for all $x \in \mathbb{R}$, and never concave down.

v



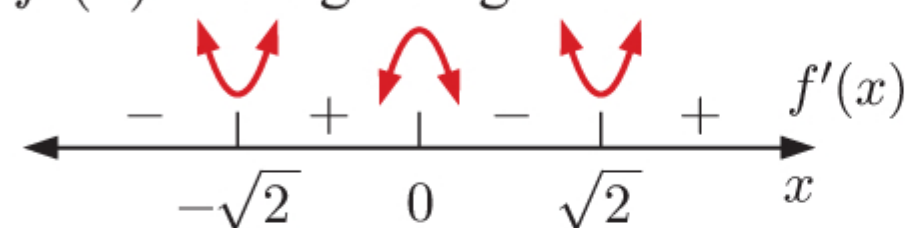
g i $f(x) = x^4 - 4x^2 + 3$

$\therefore f'(x) = 4x^3 - 8x$

$= 4x(x^2 - 2)$

$= 4x(x + \sqrt{2})(x - \sqrt{2})$

$\therefore f'(x)$ has sign diagram:



Now $f(-\sqrt{2}) = (-\sqrt{2})^4 - 4(-\sqrt{2})^2 + 3$

$= 4 - 8 + 3$

$= -1$

$f(0) = 3$

and $f(\sqrt{2}) = (\sqrt{2})^4 - 4(\sqrt{2})^2 + 3$

$= -1$

$\therefore (-\sqrt{2}, -1)$ and $(\sqrt{2}, -1)$ are local minima, and $(0, 3)$ is a local maximum.

ii $f''(x) = 12x^2 - 8$

$= 4(3x^2 - 2)$

$= 4(\sqrt{3}x + \sqrt{2})(\sqrt{3}x - \sqrt{2})$

$\therefore f''(x)$ has sign diagram:



Now $f\left(-\sqrt{\frac{2}{3}}\right) = \left(-\sqrt{\frac{2}{3}}\right)^4 - 4\left(-\sqrt{\frac{2}{3}}\right)^2 + 3$ and $f'\left(-\sqrt{\frac{2}{3}}\right) \neq 0$

$= \frac{4}{9} - \frac{8}{3} + 3$

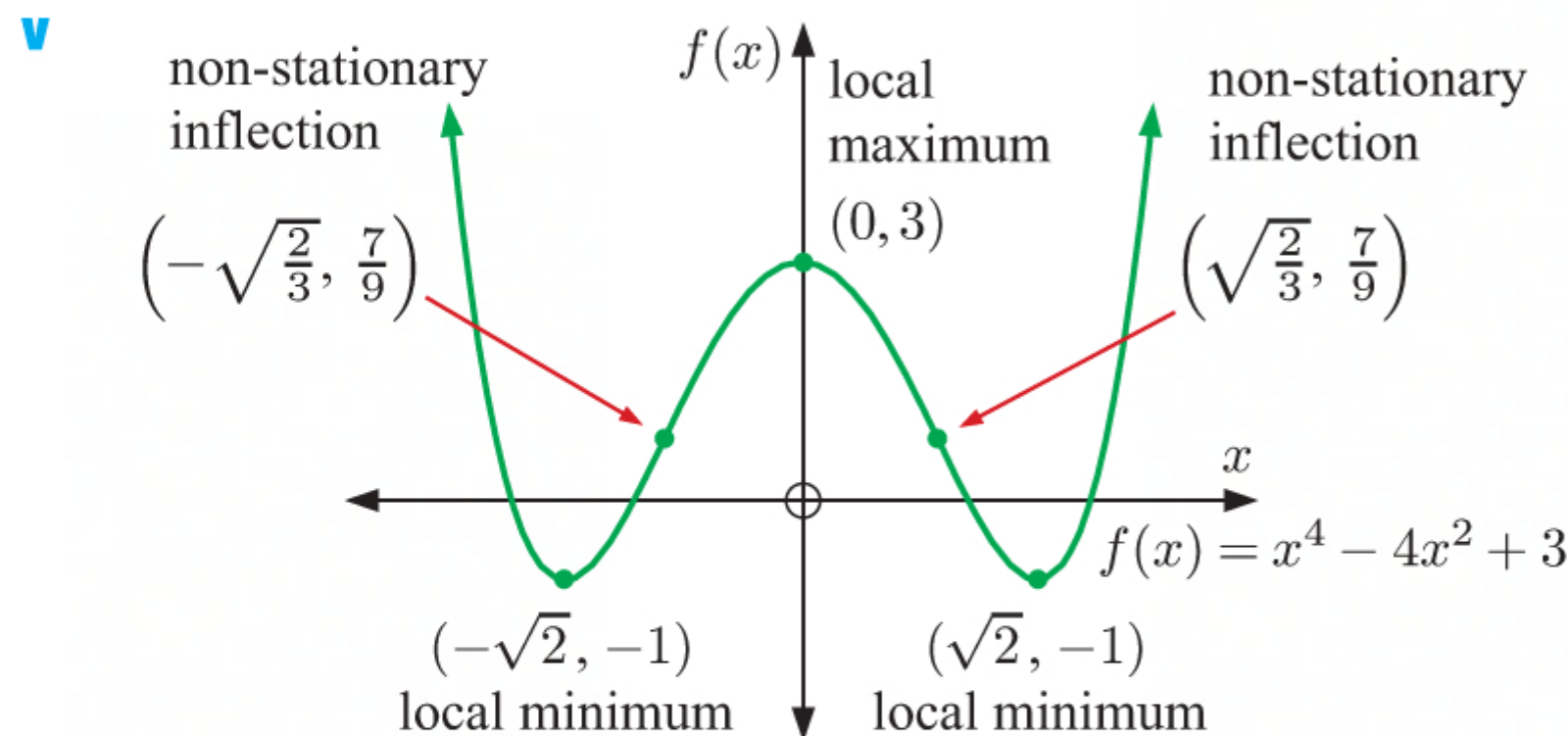
$= \frac{7}{9}$

$f\left(\sqrt{\frac{2}{3}}\right) = \frac{7}{9}$ and $f'\left(\sqrt{\frac{2}{3}}\right) \neq 0$

$\therefore \left(\sqrt{\frac{2}{3}}, \frac{7}{9}\right)$ and $\left(-\sqrt{\frac{2}{3}}, \frac{7}{9}\right)$ are non-stationary inflections.

iii $f(x)$ is increasing for $-\sqrt{2} \leq x \leq 0$ and $x \geq \sqrt{2}$, and decreasing for $x \leq -\sqrt{2}$ and $0 \leq x \leq \sqrt{2}$.

- iv** $f(x)$ is concave down for $-\sqrt{\frac{2}{3}} \leq x \leq \sqrt{\frac{2}{3}}$, and concave up for $x \leq -\sqrt{\frac{2}{3}}$ and $x \geq \sqrt{\frac{2}{3}}$.

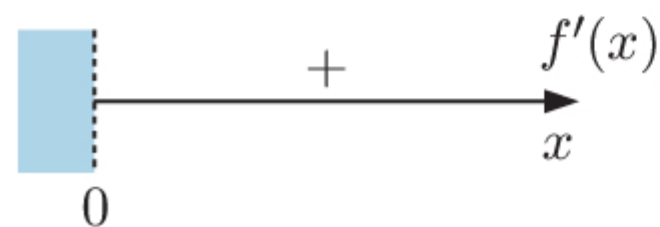


h i $f(x) = 3 - \frac{4}{\sqrt{x}} = 3 - 4x^{-\frac{1}{2}}$

$\therefore f'(x) = -4(-\frac{1}{2})x^{-\frac{3}{2}} \quad \therefore f'(x)$ has sign diagram:

$$= 2x^{-\frac{3}{2}}$$

$$= \frac{2}{x\sqrt{x}}$$

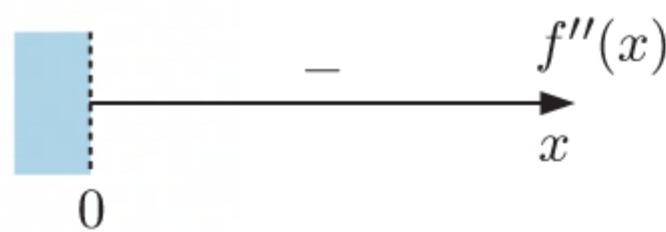


$f'(x) \neq 0$, so there are no turning points.

ii $f''(x) = 2(-\frac{3}{2})x^{-\frac{5}{2}} \quad \therefore f''(x)$ has sign diagram:

$$= -3x^{-\frac{5}{2}}$$

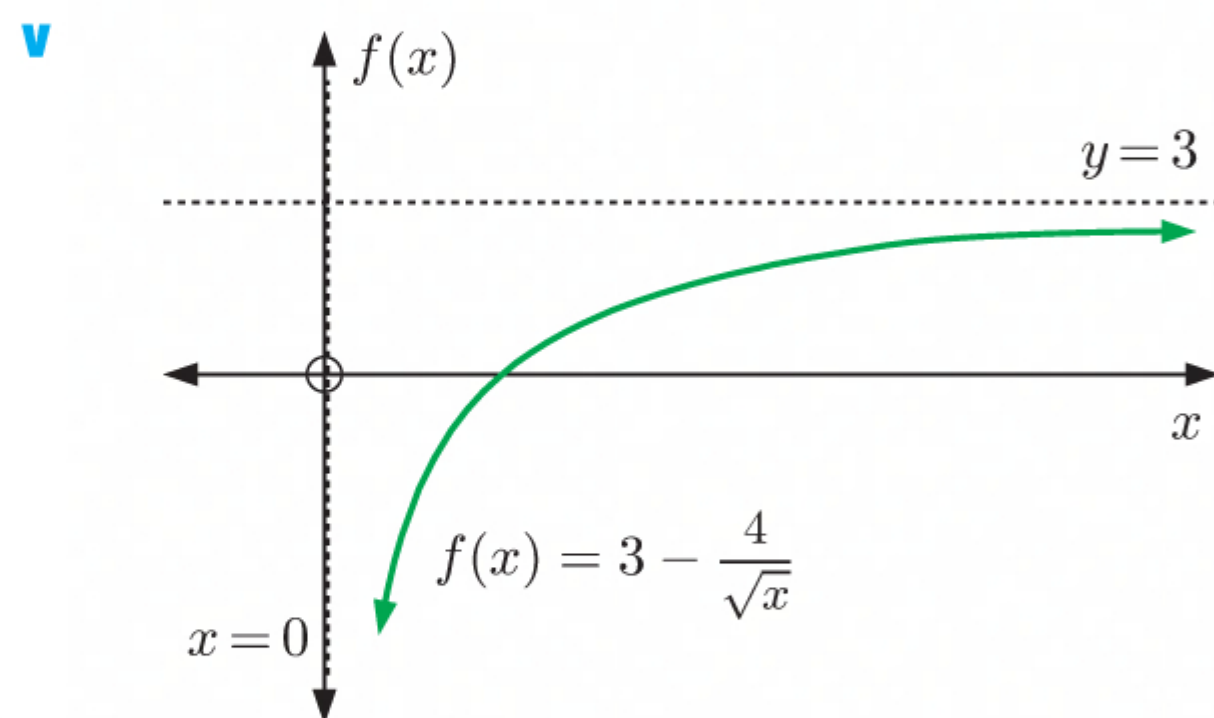
$$= -\frac{3}{x^2\sqrt{x}}$$



$f''(0) \neq 0$, so there are no points of inflection.

- iii** $f(x)$ is increasing for $x > 0$, and never decreasing.

- iv** $f(x)$ is concave down for $x > 0$, and never concave up.



4 a $f(x) = e^{2x} - 3$

The x -intercept occurs when $f(x) = 0$

$$\therefore e^{2x} - 3 = 0$$

$$\therefore e^{2x} = 3$$

$$\therefore 2x = \ln 3$$

$$\therefore x = \frac{\ln 3}{2}$$

$$= \frac{1}{2} \ln 3$$

$$= \ln 3^{\frac{1}{2}}$$

$$= \ln \sqrt{3}$$

\therefore the x -intercept is $\ln \sqrt{3}$ and the y -intercept is -2 .

b $f'(x) = 2e^{2x}$

Now $e^{2x} > 0$ for all x ,

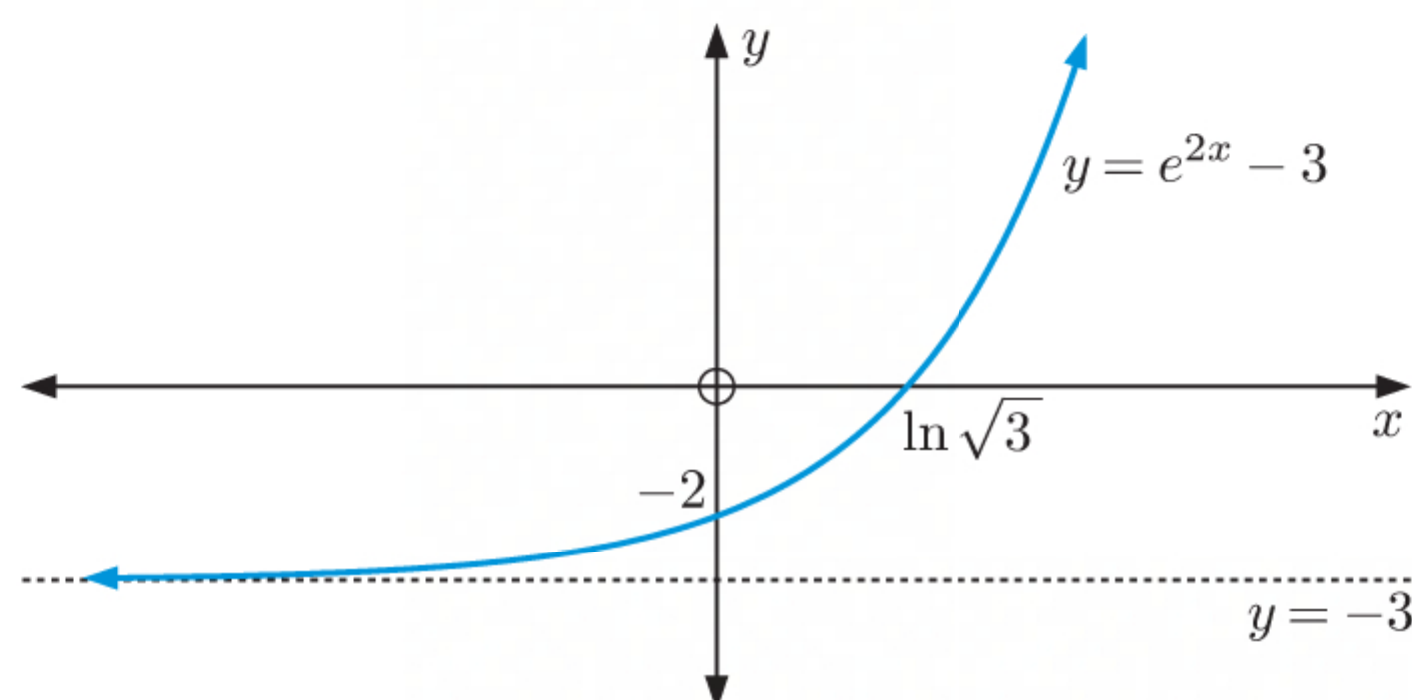
so $f'(x) > 0$ for all x .

\therefore the function is increasing for all x .

d As $x \rightarrow -\infty$, $e^{2x} \rightarrow 0^+$ $\therefore e^{2x} - 3 \rightarrow -3^+$

So, the horizontal asymptote is $y = -3$.

e



5 a $f(x) = e^x - 3$

The x -intercept occurs when $f(x) = 0$

$$\therefore e^x - 3 = 0$$

$$\therefore e^x = 3$$

$$\therefore x = \ln 3$$

\therefore the x -intercept of $f(x)$ is $\ln 3$, and the y -intercept is -2 .

$$g(x) = 3 - 5e^{-x}$$

The x -intercept occurs when $g(x) = 0$

$$\therefore 3 - 5e^{-x} = 0$$

$$\therefore 5e^{-x} = 3$$

$$\therefore e^{-x} = \frac{3}{5}$$

$$\therefore e^x = \frac{5}{3}$$

$$\therefore x = \ln\left(\frac{5}{3}\right)$$

\therefore the x -intercept of $g(x)$ is $\ln\left(\frac{5}{3}\right)$, and the y -intercept is -2 .

The y -intercept occurs when $x = 0$

$$f(0) = e^0 - 3 = -2$$

c $f''(x) = 2e^{2x}(2)$

$$= 4e^{2x} \text{ which is } > 0 \text{ for all } x.$$

$\therefore f(x)$ is concave up for all x .

The y -intercept occurs when $x = 0$

$$f(0) = e^0 - 3$$

$$= -2$$

The y -intercept occurs when $x = 0$

$$g(0) = 3 - 5e^0$$

$$= -2$$

b $f(x)$: as $x \rightarrow \infty$, $e^x \rightarrow \infty$
 $\therefore f(x) = e^x - 3 \rightarrow \infty$
 as $x \rightarrow -\infty$, $e^x \rightarrow 0^+$
 $\therefore f(x) = e^x - 3 \rightarrow -3^+$

$g(x)$: as $x \rightarrow \infty$, $-5e^{-x} \rightarrow 0^-$
 $\therefore g(x) = 3 - 5e^{-x} \rightarrow 3^-$
 as $x \rightarrow -\infty$, $-5e^{-x} \rightarrow -\infty$
 $\therefore g(x) = 3 - 5e^{-x} \rightarrow -\infty$

c $f'(x) = e^x$, $f''(x) = e^x$

$\leftarrow \xrightarrow{+} \xrightarrow{f'(x)} x$

$\leftarrow \xrightarrow{+} \xrightarrow{f''(x)} x$

$\therefore f(x)$ is increasing and concave up for all $x \in \mathbb{R}$.

$g'(x) = 5e^{-x}$, $g''(x) = -5e^{-x}$

$\leftarrow \xrightarrow{+} \xrightarrow{g'(x)} x$

$\leftarrow \xrightarrow{-} \xrightarrow{g''(x)} x$

$\therefore g(x)$ is increasing and concave down for all $x \in \mathbb{R}$.

d The functions intersect when $f(x) = g(x)$

$\therefore e^x - 3 = 3 - 5e^{-x}$

$\therefore e^x - 6 + 5e^{-x} = 0$

$\therefore e^{2x} - 6e^x + 5 = 0$ {multiplying both sides by e^x }

$\therefore (e^x - 1)(e^x - 5) = 0$ {compare $a^2 - 6a + 5 = (a - 1)(a - 5)$ }

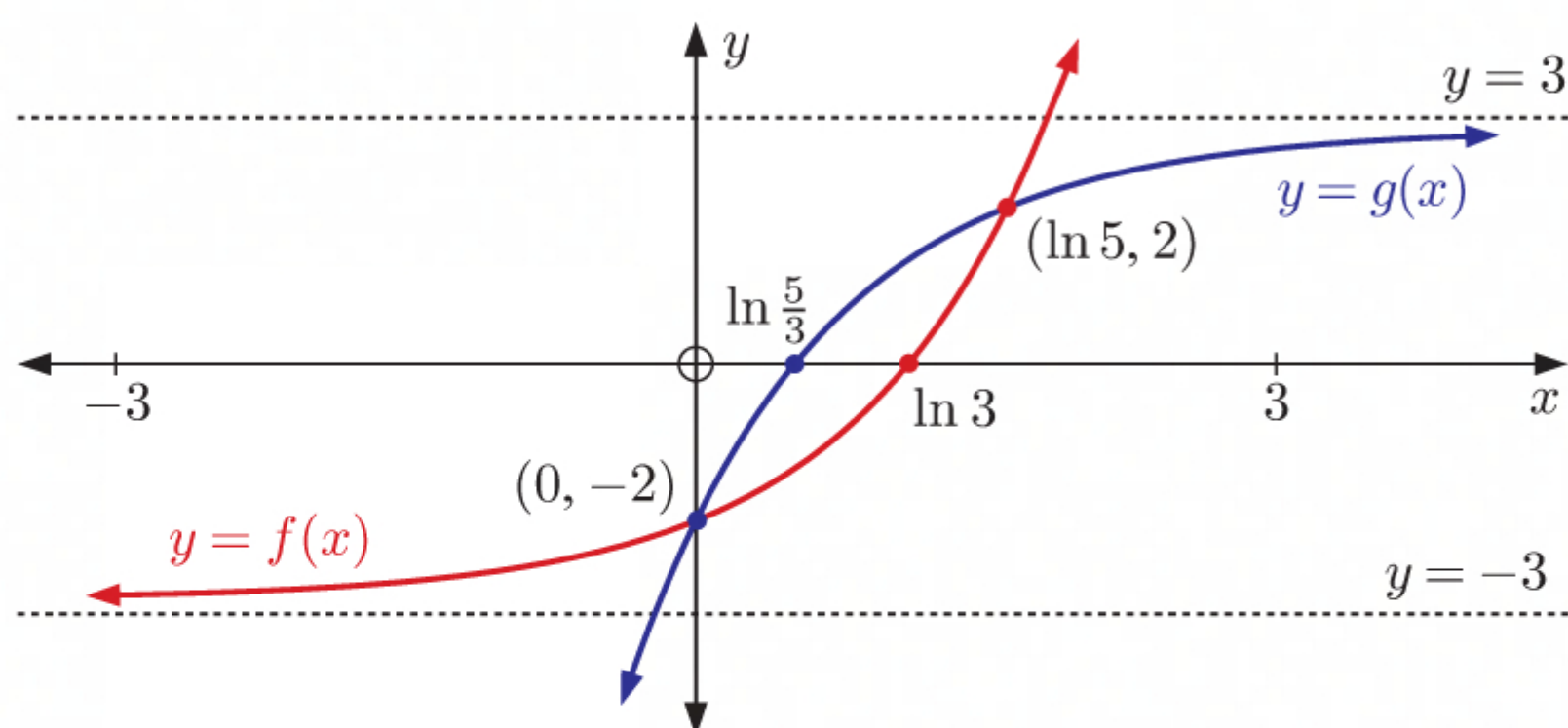
$\therefore e^x = 1$ or 5

$\therefore x = 0$ or $\ln 5$

Now $f(0) = e^0 - 3 = -2$ and $f(\ln 5) = e^{\ln 5} - 3$
 $= 5 - 3$
 $= 2$

\therefore the points of intersection are $(0, -2)$ and $(\ln 5, 2)$.

e



6 a $y = e^x - 3e^{-x}$

The x -intercept occurs when $y = 0$

$\therefore e^x - 3e^{-x} = 0$

$\therefore e^{2x} - 3 = 0$ {multiplying both sides by e^x }

$\therefore e^{2x} = 3$

$\therefore 2x = \ln 3$

$\therefore x = \frac{1}{2} \ln 3 = \ln(3^{\frac{1}{2}}) = \ln \sqrt{3}$

The y -intercept occurs when $x = 0$

$$\begin{aligned}\therefore y &= e^0 - 3e^0 \\ &= 1 - 3 \\ &= -2\end{aligned}$$

\therefore the x -intercept is $\ln \sqrt{3}$ and the y -intercept is -2 .

b
$$\begin{aligned}\frac{dy}{dx} &= e^x + 3e^{-x} \\ &= e^x + \frac{3}{e^x}\end{aligned}$$

Now $e^x > 0$ for all x ,

so $\frac{dy}{dx} > 0$ for all x .

\therefore the function is increasing for all x .

c
$$\begin{aligned}\frac{dy}{dx} &= e^x + 3e^{-x} \\ \therefore \frac{d^2y}{dx^2} &= e^x - 3e^{-x} \\ &= y\end{aligned}$$

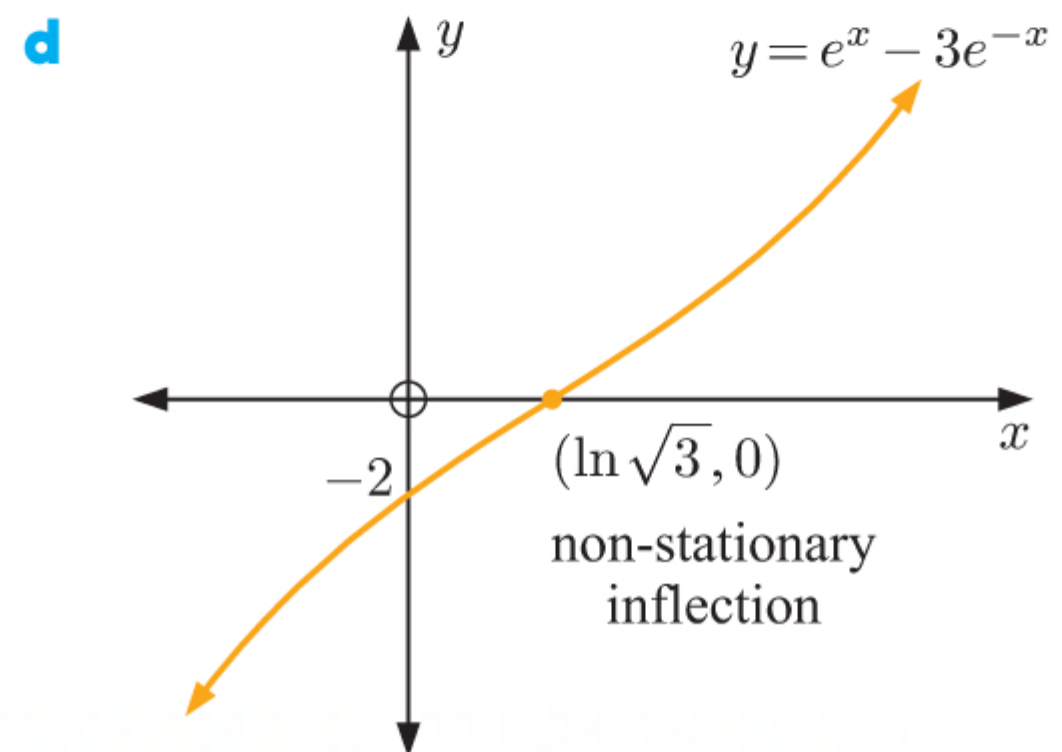
Above the x -axis, $y > 0 \therefore \frac{d^2y}{dx^2} > 0$

\therefore the function is concave up.

Below the x -axis, $y < 0 \therefore \frac{d^2y}{dx^2} < 0$

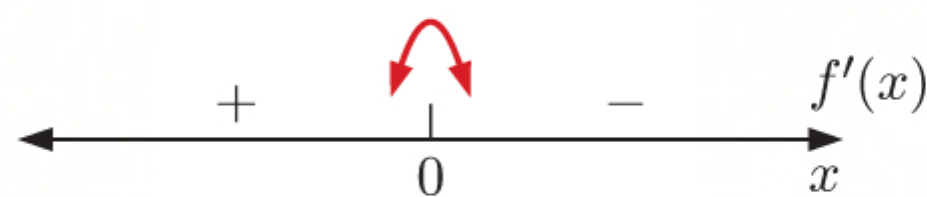
\therefore the function is concave down.

$\therefore y$ is concave down below the x -axis and concave up above the x -axis.



7 a
$$\begin{aligned}f(x) &= \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} \\ \therefore f'(x) &= \frac{1}{\sqrt{2\pi}} (-x)e^{-\frac{1}{2}x^2} \\ &= -\frac{1}{\sqrt{2\pi}} xe^{-\frac{1}{2}x^2} \quad \text{where } e^{-\frac{1}{2}x^2} \text{ is positive for all } x\end{aligned}$$

So, $f'(x)$ has sign diagram:



Now
$$\begin{aligned}f(0) &= \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(0)^2} \\ &= \frac{1}{\sqrt{2\pi}}\end{aligned}$$

$\therefore \left(0, \frac{1}{\sqrt{2\pi}}\right)$ is a local maximum.

$f(x)$ is increasing for $x \leq 0$, and decreasing for $x \geq 0$.

b $f''(x) = -\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} + \left(-\frac{1}{\sqrt{2\pi}} x(-x)e^{-\frac{1}{2}x^2}\right)$ {product rule}

$$= -\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} + \frac{1}{\sqrt{2\pi}} x^2 e^{-\frac{1}{2}x^2}$$

$$= -\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} (1 - x^2)$$

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} (x^2 - 1) \quad \text{which has sign diagram:}$$

Now $f(-1) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(-1)^2}$ and $f(1) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(1)^2}$

$$= \frac{1}{\sqrt{2\pi}\sqrt{e}} \qquad \qquad \qquad = \frac{1}{\sqrt{2\pi}\sqrt{e}}$$

$$= \frac{1}{\sqrt{2e\pi}} \qquad \qquad \qquad = \frac{1}{\sqrt{2e\pi}}$$

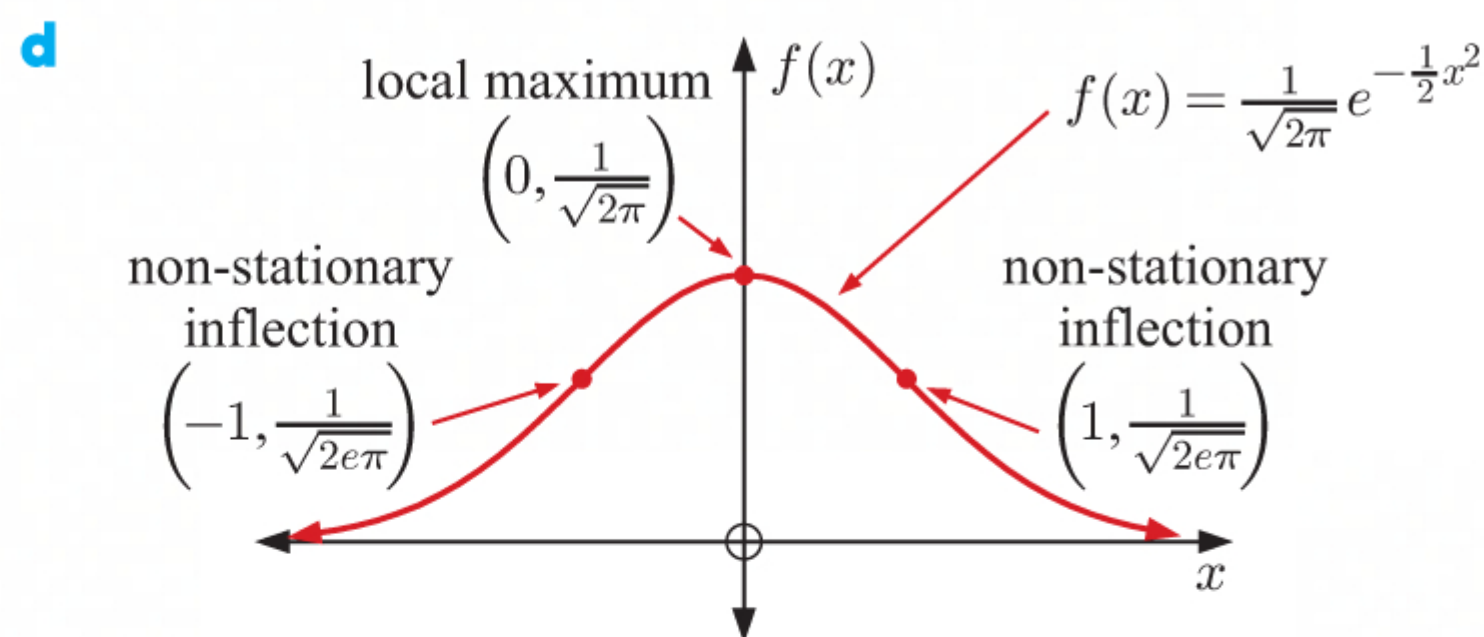
Since $f'(-1) \neq 0$ and $f'(1) \neq 0$, then $\left(-1, \frac{1}{\sqrt{2e\pi}}\right)$ and $\left(1, \frac{1}{\sqrt{2e\pi}}\right)$ are non-stationary inflections.

c As $x \rightarrow \infty$, $e^{-\frac{1}{2}x^2} \rightarrow 0^+$

\therefore as $x \rightarrow \infty$, $f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} \rightarrow 0^+$

As $x \rightarrow -\infty$, $e^{-\frac{1}{2}x^2} \rightarrow 0^+$

\therefore as $x \rightarrow -\infty$, $f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} \rightarrow 0^+$



8 a $f(x) = \cos x$

$\therefore f'(x) = -\sin x$

$\therefore f''(x) = -\cos x = -f(x)$

\therefore the inflection points coincide with the x -intercepts.

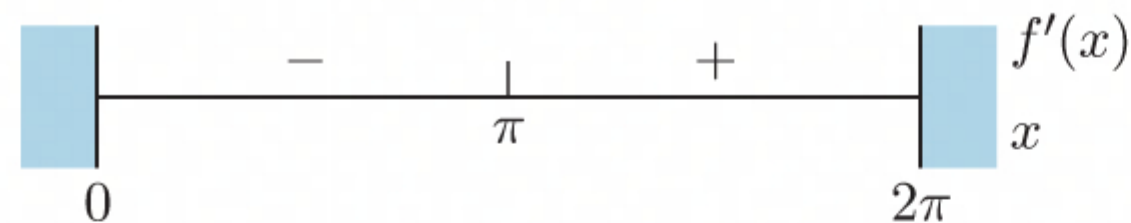
b $f''(x) = -\cos x$, $0 \leq x \leq 2\pi$

The sign diagram of $f''(x)$ is:

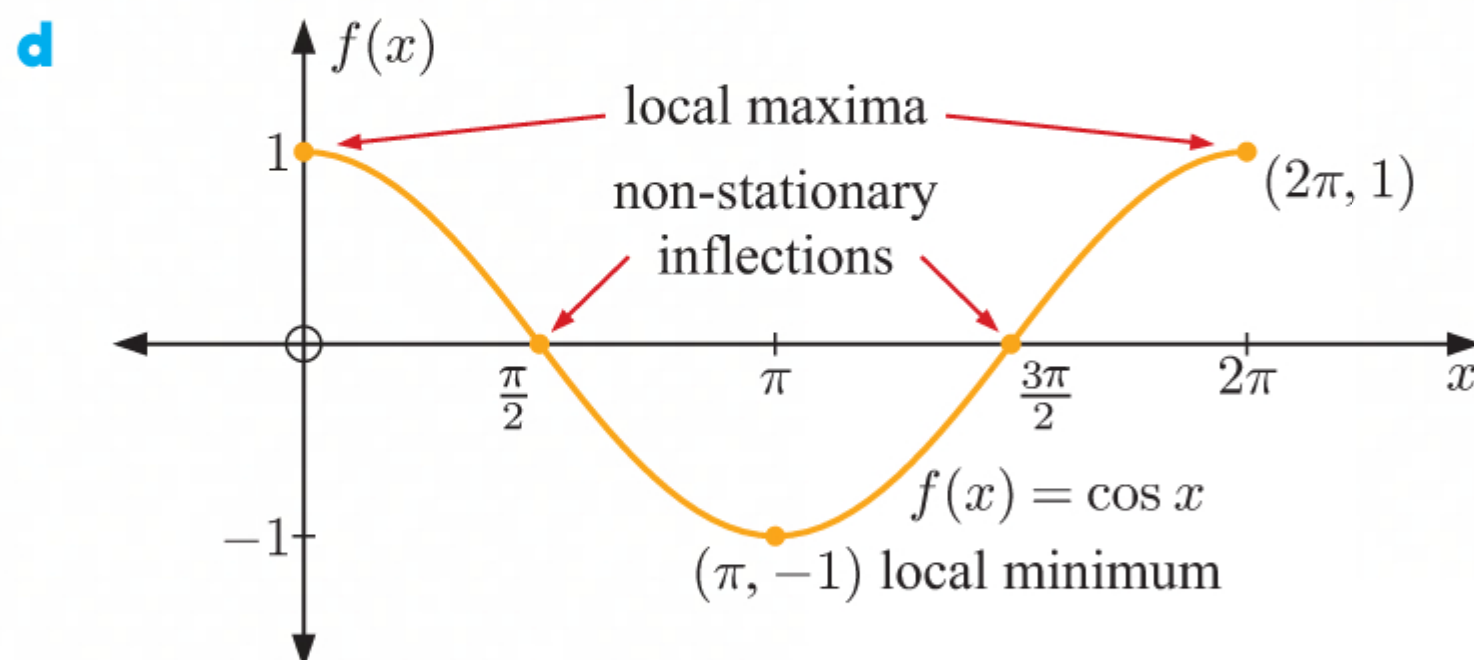
Since $f'(\frac{\pi}{2}) \neq 0$ and $f'(\frac{3\pi}{2}) \neq 0$, then there are non-stationary inflection points at $(\frac{\pi}{2}, 0)$ and $(\frac{3\pi}{2}, 0)$.

c $f'(x) = -\sin x, \quad 0 \leq x \leq 2\pi$

The sign diagram of $f'(x)$ is:



- i $f(x)$ is increasing for $\pi \leq x \leq 2\pi$
- ii $f(x)$ is decreasing for $0 \leq x \leq \pi$
- iii $f(x)$ is concave up for $\frac{\pi}{2} \leq x \leq \frac{3\pi}{2}$
- iv $f(x)$ is concave down for $0 \leq x \leq \frac{\pi}{2}$ and $\frac{3\pi}{2} \leq x \leq 2\pi$



9 $f(t) = Ate^{-bt}, \quad t \geq 0, \quad A, b > 0$

a i $f'(t) = Ae^{-bt} + Ate^{-bt}(-b) \quad \{\text{product rule}\}$
 $= Ae^{-bt} - Abte^{-bt}$
 $= Ae^{-bt}(1 - bt)$

$f'(t) = 0$ when $Ae^{-bt}(1 - bt) = 0$ but $A > 0$ and $e^{-bt} > 0$

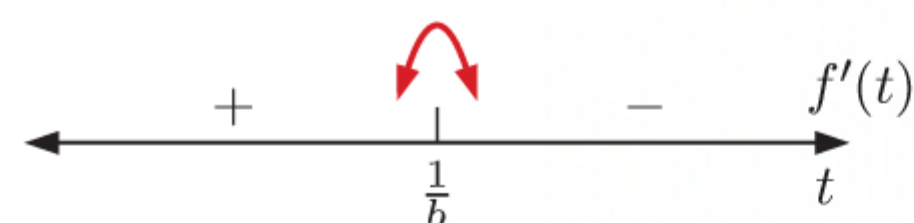
$\therefore 1 - bt = 0$

$\therefore -bt = -1$

$\therefore t = \frac{1}{b} \quad \{\text{where } b > 0\}$

\therefore there is a local maximum at $t = \frac{1}{b}$.

\therefore the sign diagram of $f'(t)$ is:



ii $f''(t) = Ae^{-bt}(-b) - [Abe^{-bt} + Abte^{-bt}(-b)] \quad \{\text{product rule}\}$
 $= -Abe^{-bt} - (Abe^{-bt} - Ab^2te^{-bt})$
 $= -2Abe^{-bt} + Ab^2te^{-bt}$
 $= Abe^{-bt}(bt - 2)$

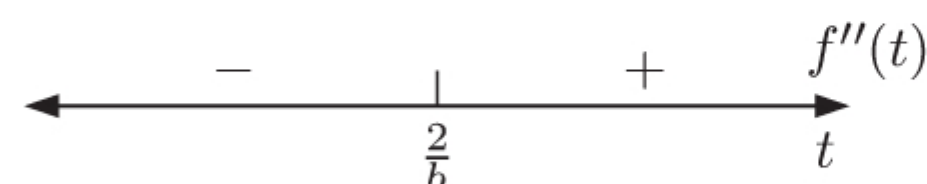
$f''(t) = 0$ when $Abe^{-bt}(bt - 2) = 0$ but $A, b > 0$ and $e^{-bt} > 0$

$\therefore bt - 2 = 0$

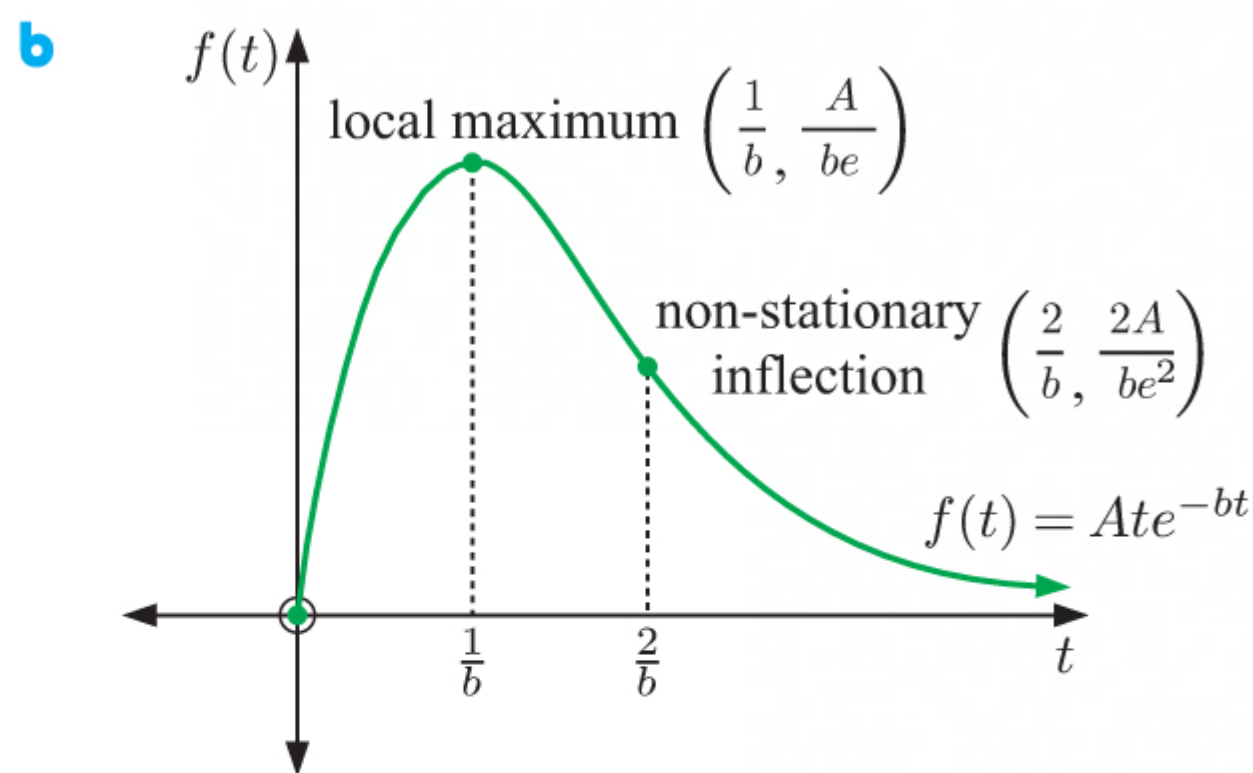
$\therefore bt = 2$

$\therefore t = \frac{2}{b} \quad \{\text{where } b > 0\}$

\therefore the sign diagram of $f''(t)$ is:



Since $f'(\frac{2}{b}) \neq 0$, there is a non-stationary point of inflection at $t = \frac{2}{b}$.



10 $f(t) = \frac{C}{1 + Ae^{-bt}}, \quad t \geq 0, \quad A, b, C > 0$

a The y -intercept occurs when $t = 0$.

$$\text{Now } f(0) = \frac{C}{1 + Ae^{-b(0)}} = \frac{C}{1 + A}$$

So, the y -intercept is $\frac{C}{1 + A}$.

b i As $t \rightarrow \infty$, $Ae^{-bt} \rightarrow 0^+$

$$\therefore f(t) = \frac{C}{1 + Ae^{-bt}} \rightarrow \frac{C}{1 + 0^+} = C^-$$

$\therefore y = C$ is a horizontal asymptote.

ii $f(t) = C(1 + Ae^{-bt})^{-1}$

$$\begin{aligned} \therefore f'(t) &= C(-1)(1 + Ae^{-bt})^{-2}(-bAe^{-bt}) \quad \{\text{chain rule}\} \\ &= AbCe^{-bt}(1 + Ae^{-bt})^{-2} \end{aligned}$$

$$\begin{aligned} \therefore f''(t) &= (-b)AbCe^{-bt}(1 + Ae^{-bt})^{-2} + AbCe^{-bt}(-2)(1 + Ae^{-bt})^{-3}(-bAe^{-bt}) \\ &\quad \{\text{product rule and chain rule}\} \end{aligned}$$

$$= -\frac{Ab^2C}{e^{bt}(1 + Ae^{-bt})^2} + \frac{2A^2b^2C}{e^{2bt}(1 + Ae^{-bt})^3}$$

$$f''(t) = 0 \quad \text{when} \quad \frac{2A^2b^2C}{e^{2bt}(1 + Ae^{-bt})^3} = \frac{Ab^2C}{e^{bt}(1 + Ae^{-bt})^2}$$

$$\therefore \frac{2A(Ab^2C)}{e^{bt}e^{bt}(1 + Ae^{-bt})^3} = \frac{Ab^2C}{e^{bt}(1 + Ae^{-bt})^2}$$

$$\therefore \frac{2A}{e^{bt}(1 + Ae^{-bt})} = 1$$

$$\therefore 2A = e^{bt} + Ae^{-bt}e^{bt}$$

$$\therefore 2A = e^{bt} + A$$

$$\therefore A = e^{bt}$$

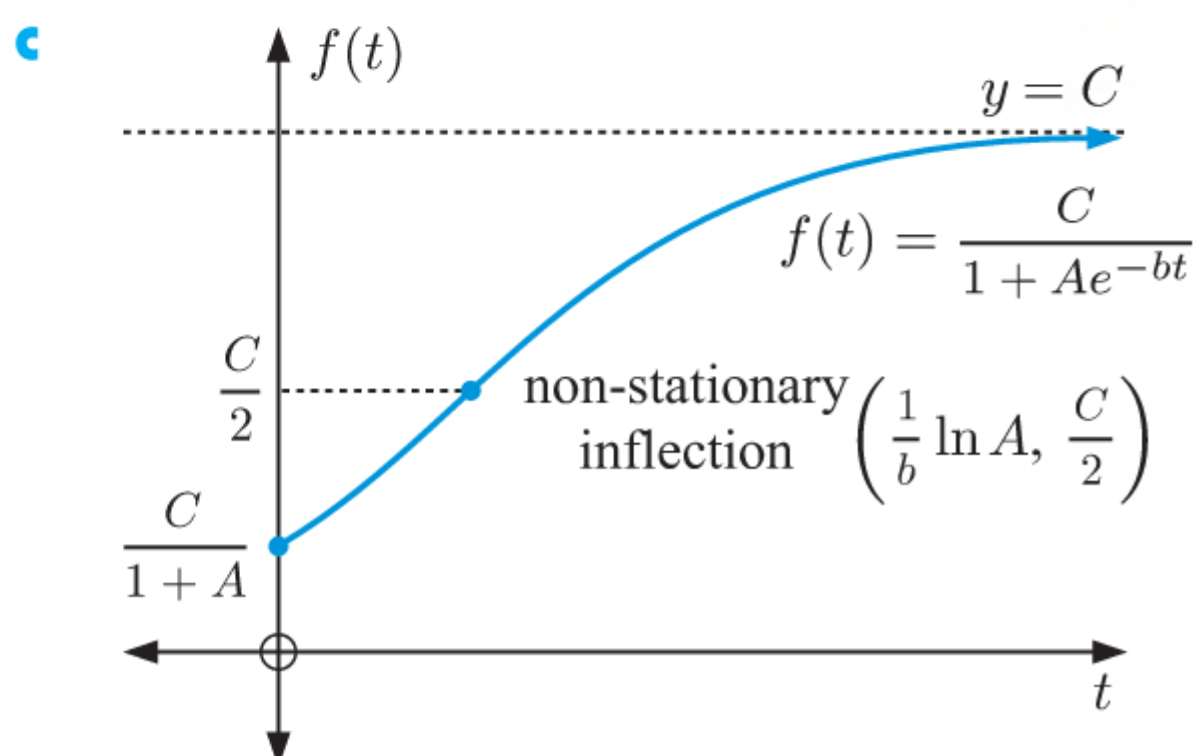
$$\therefore \ln A = bt$$

$$\therefore t = \frac{\ln A}{b}$$

Since $t \geq 0$ and $b > 0$, then $\frac{\ln A}{b} \geq 0 \quad \therefore A > 1$

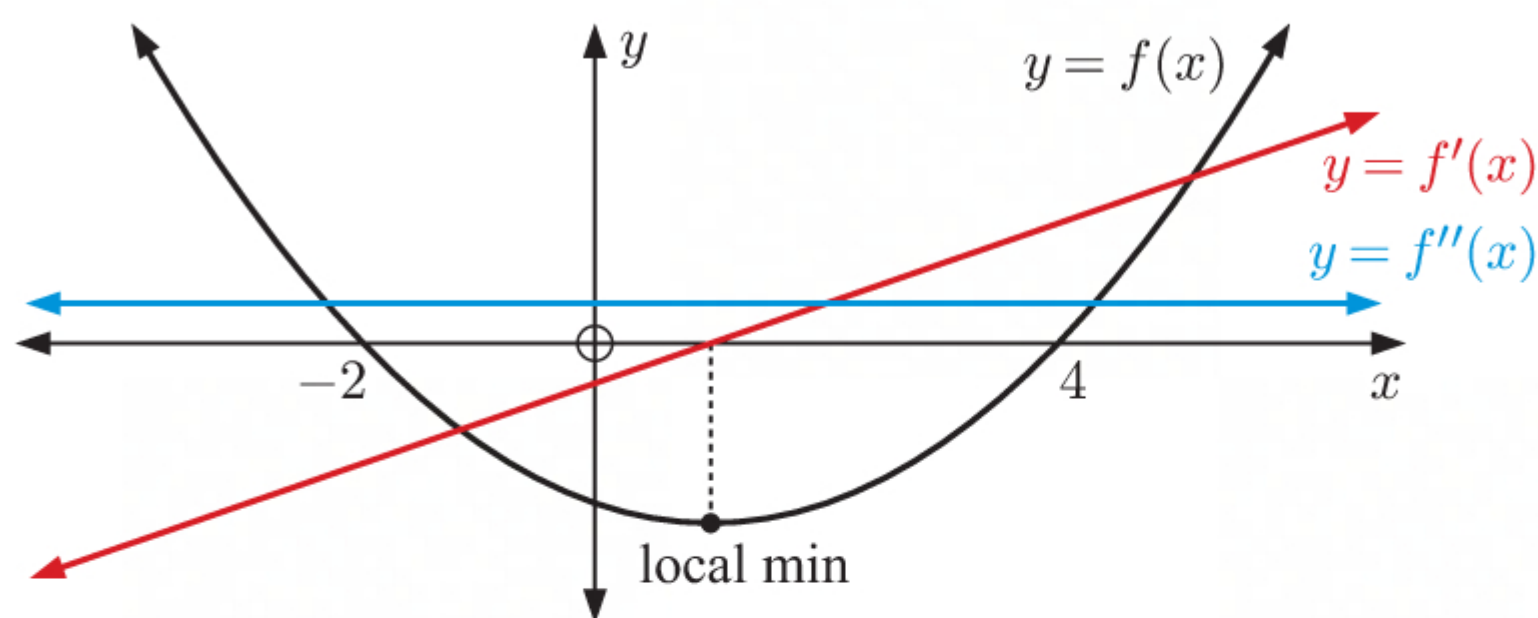
$$\begin{aligned}
 f\left(\frac{\ln A}{b}\right) &= \frac{C}{1 + Ae^{-b\left(\frac{\ln A}{b}\right)}} \\
 &= \frac{C}{1 + Ae^{-\ln A}} \\
 &= \frac{C}{1 + Ae^{\ln A^{-1}}} \\
 &= \frac{C}{1 + AA^{-1}} \\
 &= \frac{C}{1+1} = \frac{C}{2}
 \end{aligned}$$

So, if $A > 1$, there is a point of inflection with y -coordinate $\frac{C}{2}$.

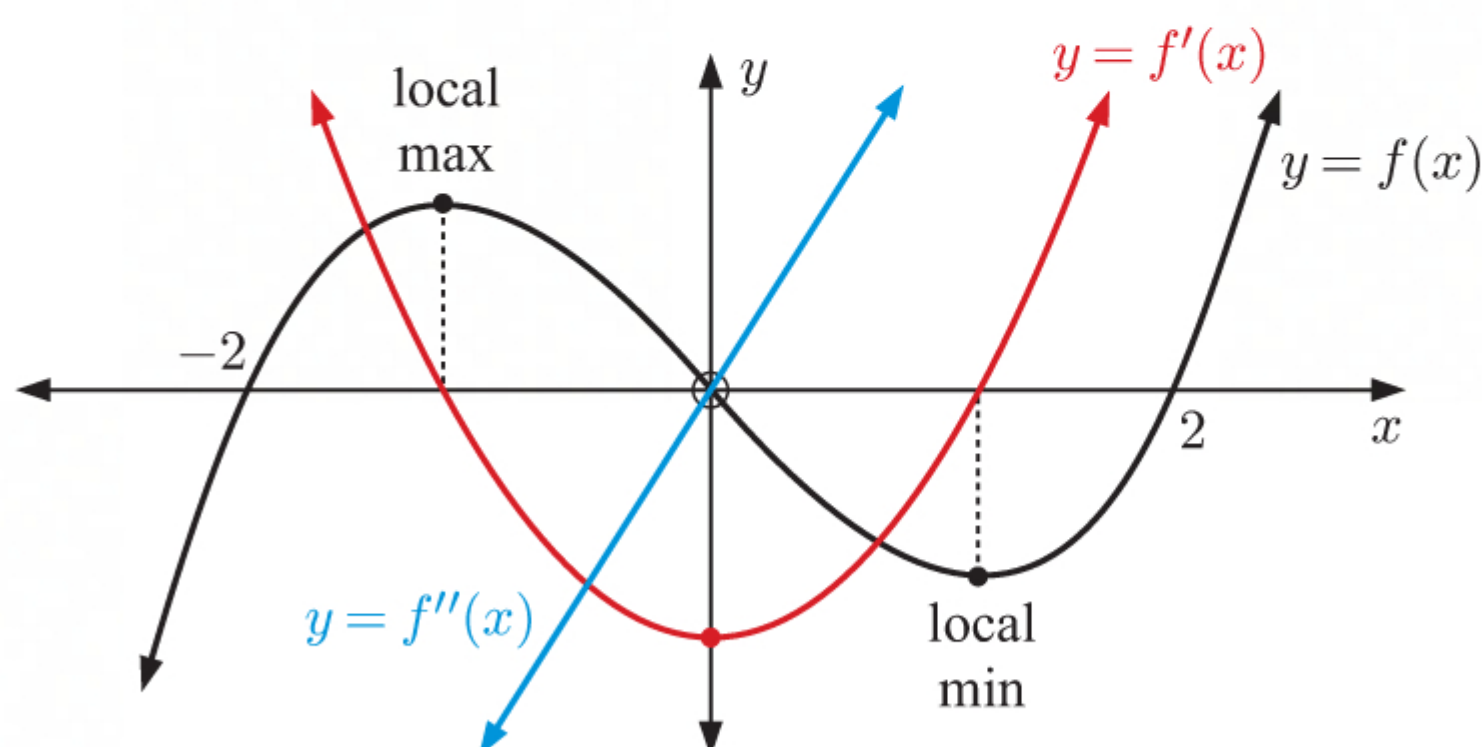


EXERCISE 13G

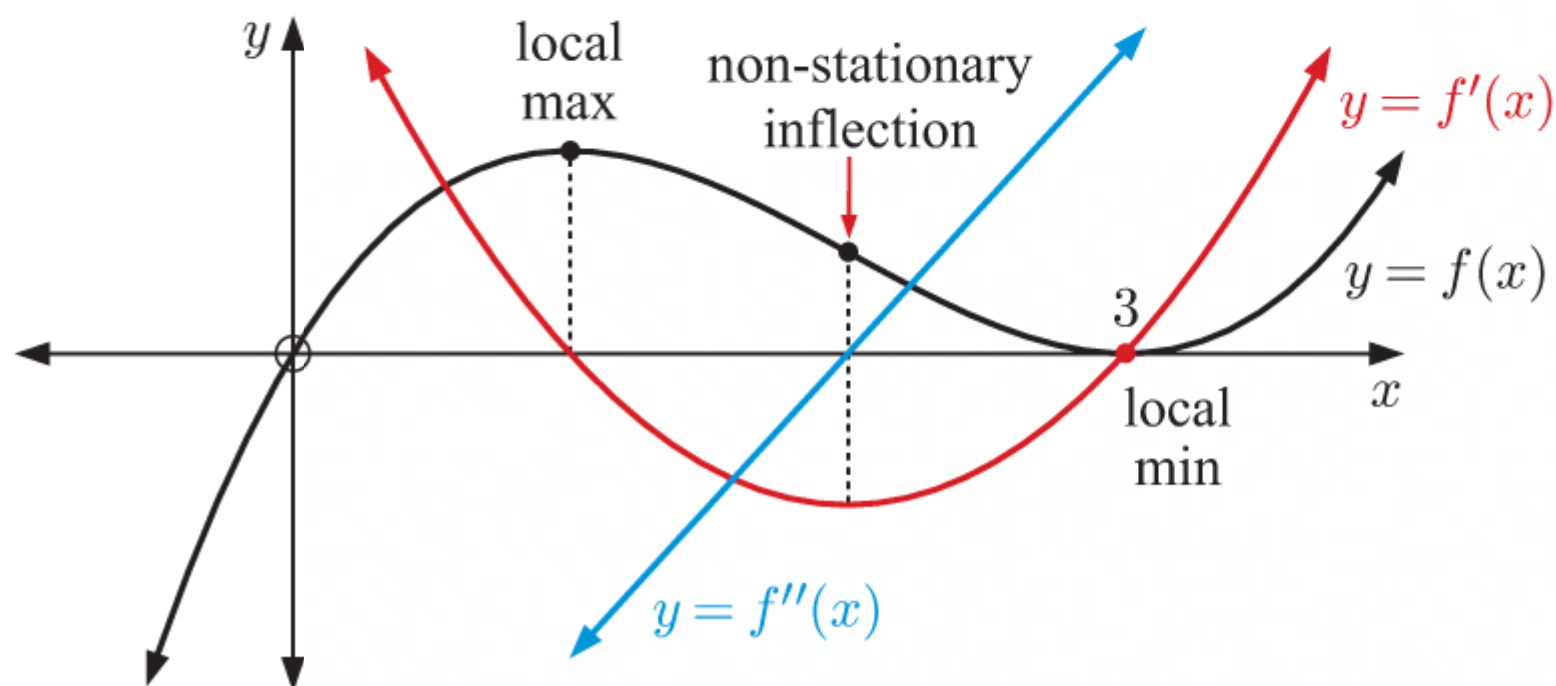
- 1 a** At the local minimum,
 $f'(x) = 0$ and $f''(x) > 0$.
 The graph is concave up for
 all x , so $f''(x) > 0$.



- b** At the local maximum,
 $f'(x) = 0$ and $f''(x) < 0$.
 At the local minimum,
 $f'(x) = 0$ and $f''(x) > 0$.
 At the non-stationary point
 of inflection, $f'(x) \neq 0$
 and $f''(x) = 0$.

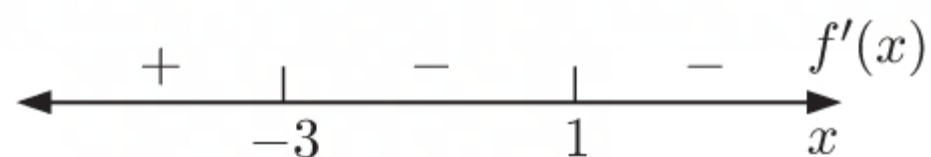


- At the local maximum, $f'(x) = 0$ and $f''(x) < 0$.
 At the local minimum, $f'(x) = 0$ and $f''(x) > 0$.
 At the non-stationary point of inflection, $f'(x) \neq 0$ and $f''(x) = 0$.



2 Note: Other solutions are possible.

- a** The sign diagram of $f'(x)$ is:

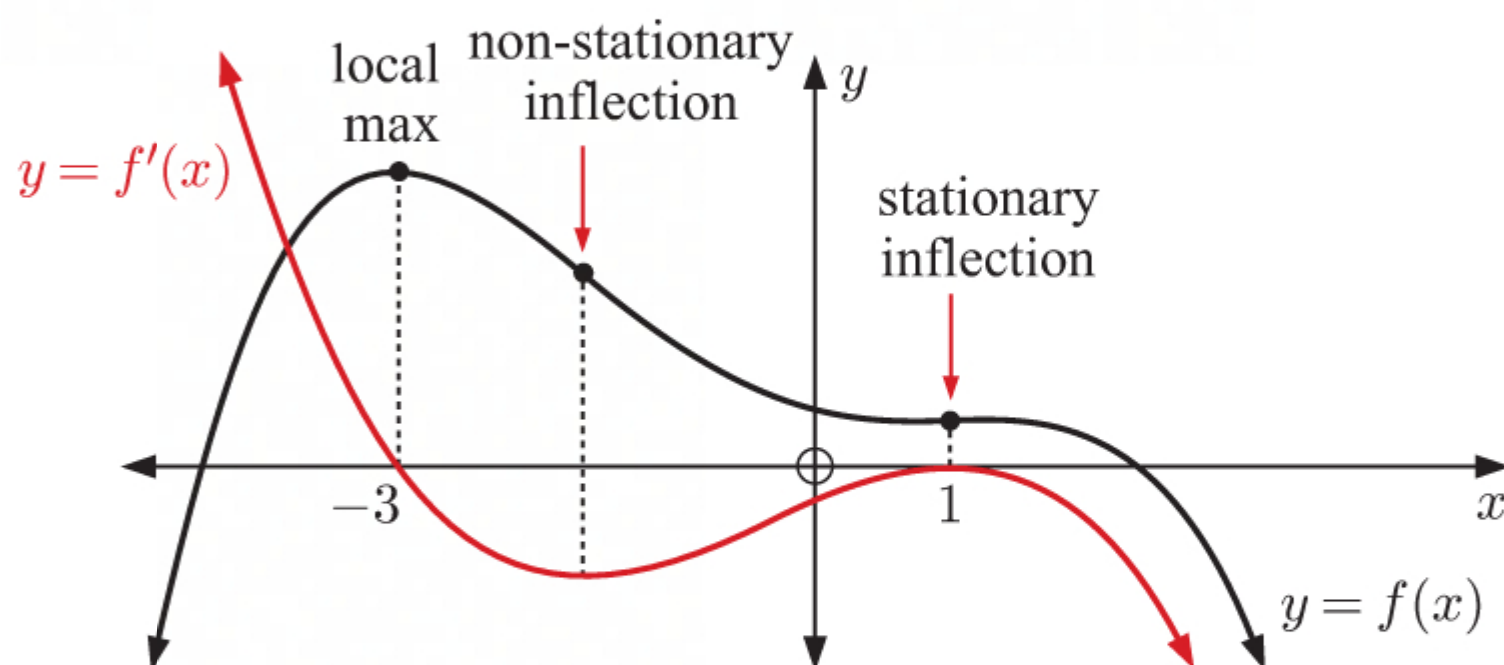


$\therefore y = f(x)$ has a local maximum at $x = -3$, and an inflection point at $x = 1$.

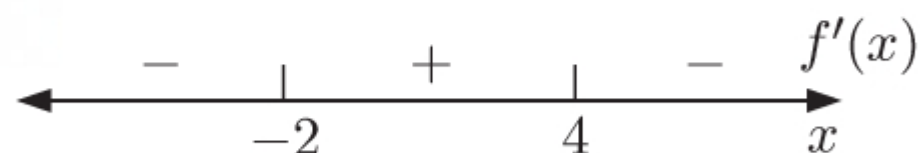
$f'(x)$ is a minimum when $x \approx -\frac{3}{2}$ and a maximum when $x = 1$.

Now $f''(-\frac{3}{2}) = 0$ but $f'(-\frac{3}{2}) \neq 0$, so this point corresponds to a non-stationary point of inflection.

Also $f''(1) = 0$ and $f'(1) = 0$, so this point corresponds to a stationary point of inflection.



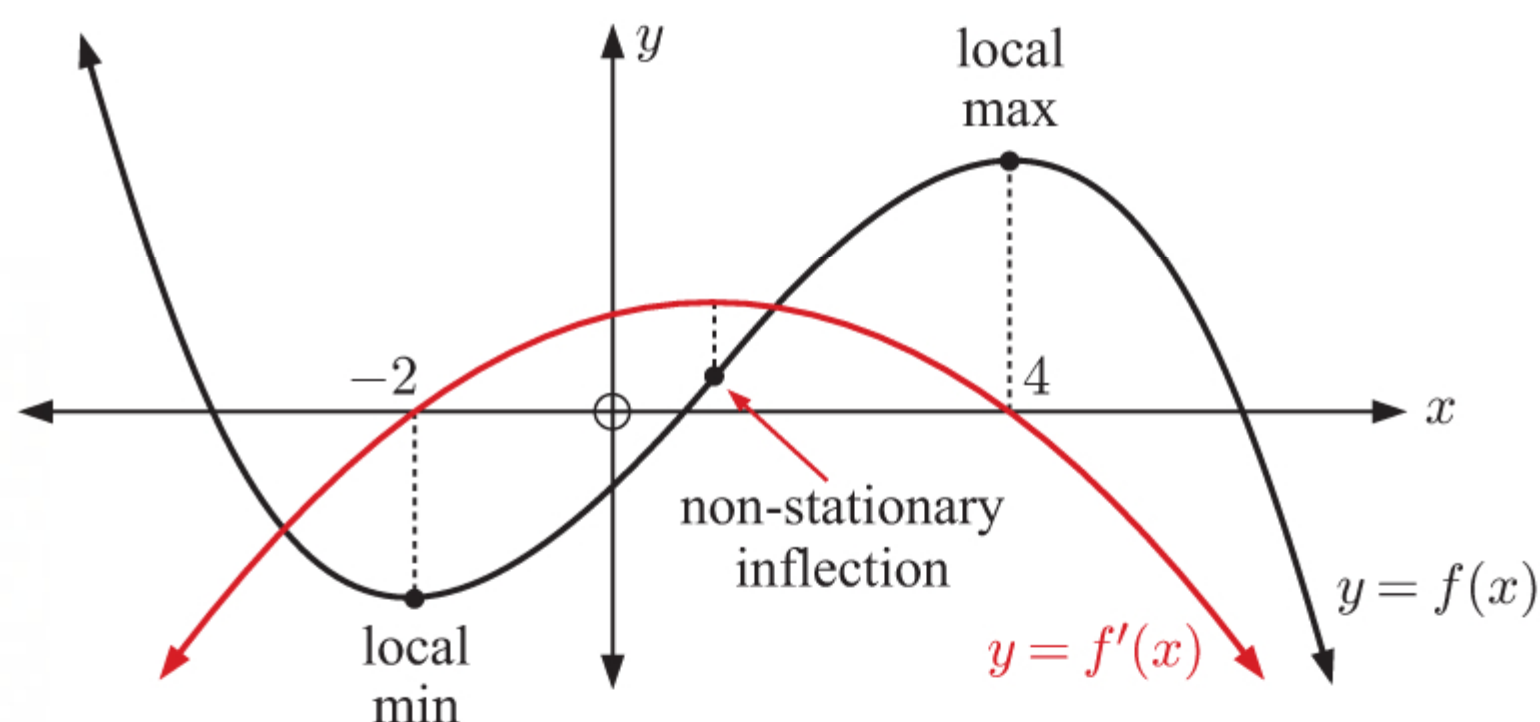
- b** The sign diagram of $f'(x)$ is:



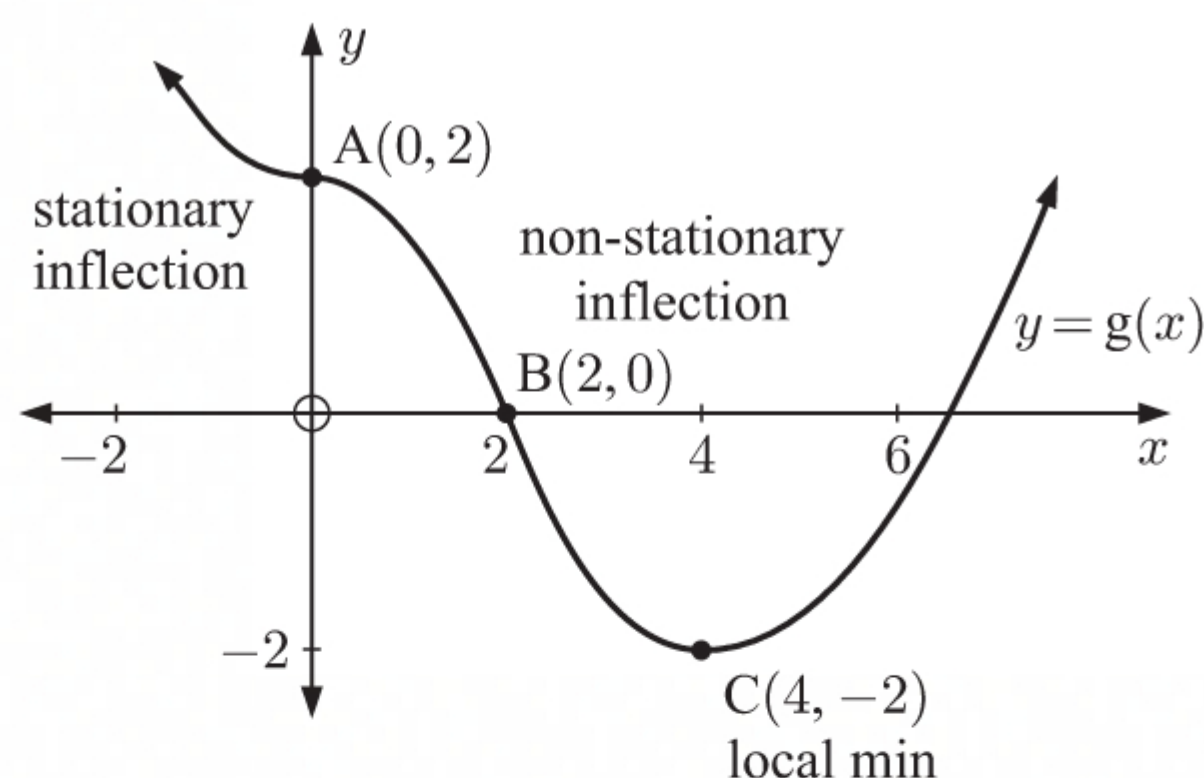
$\therefore y = f(x)$ has a local minimum at $x = -2$, and a local maximum at $x = 4$.

$f'(x)$ is a maximum when $x \approx 1$.

At this point, $f''(1) = 0$ but $f'(1) \neq 0$, so it corresponds to a non-stationary point of inflection.



- 3 $g'(0) = 0$ and $g''(0) = 0$
 \therefore there is a stationary inflection point at $A(0, 2)$.
 $g''(2) = 0$ and $g'(2) \neq 0$
 \therefore there is a non-stationary inflection point at $B(2, 0)$.
 $g'(4) = 0$ and $g''(4) > 0$
 \therefore there is a local minimum at $C(4, -2)$.



REVIEW SET 13A

- 1 a $y = -2x^2$

When $x = -1$, $y = -2(-1)^2 = -2$

\therefore the point of contact is $(-1, -2)$.

Now $\frac{dy}{dx} = -4x$

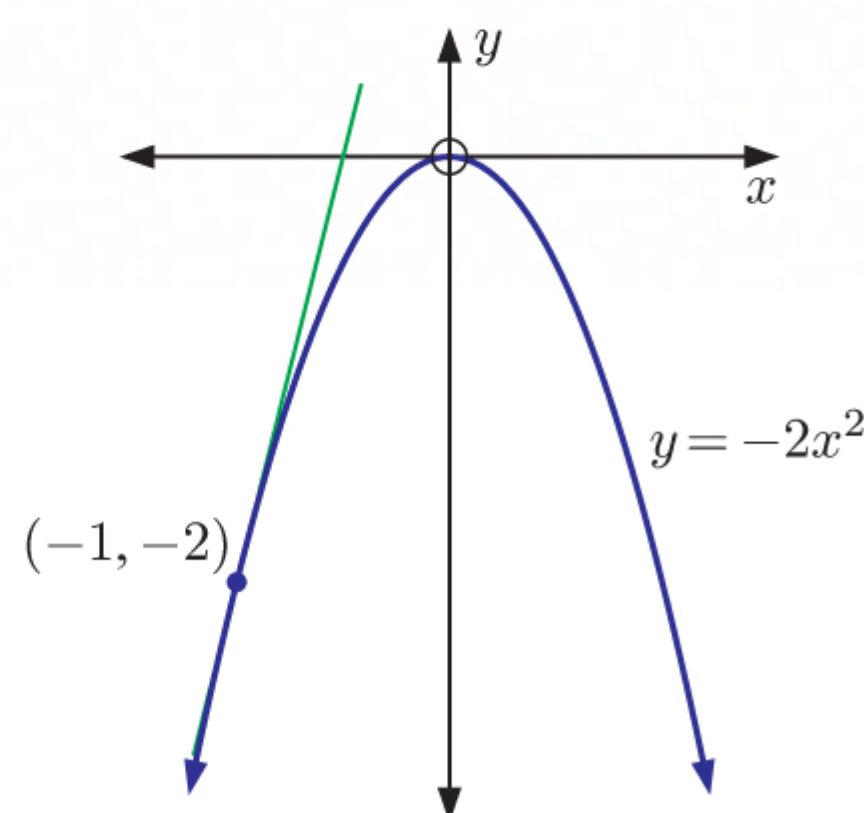
When $x = -1$, $\frac{dy}{dx} = -4(-1) = 4$

So, the tangent has equation $y = 4(x - (-1)) - 2$

$$\therefore y = 4(x + 1) - 2$$

$$\therefore y = 4x + 4 - 2$$

$$\therefore y = 4x + 2$$



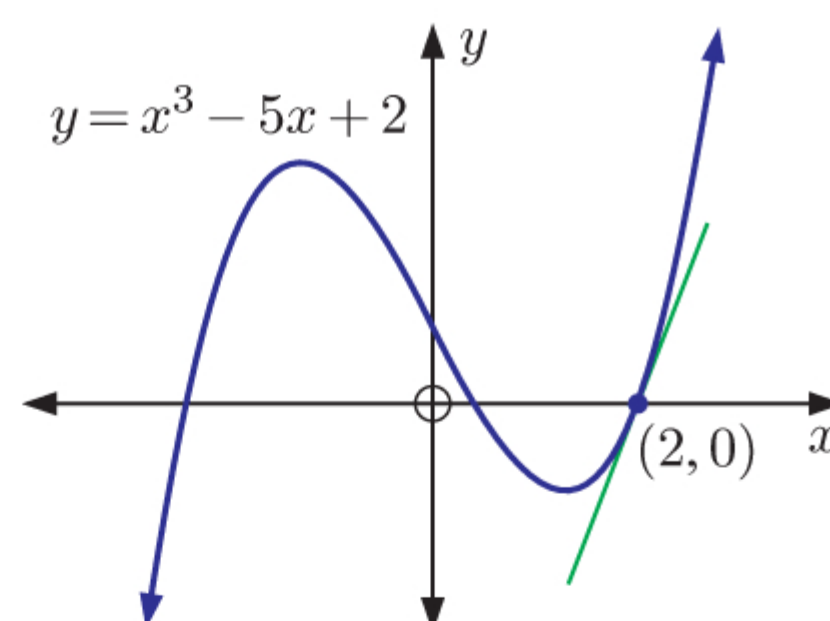
- b $y = x^3 - 5x + 2$

$$\therefore \frac{dy}{dx} = 3x^2 - 5$$

$$\begin{aligned} \text{When } x = 2, \quad \frac{dy}{dx} &= 3(2)^2 - 5 \\ &= 12 - 5 \\ &= 7 \end{aligned}$$

So, the tangent has equation $y = 7(x - 2) + 0$

$$\therefore y = 7x - 14$$



- c $y = \frac{1 - 2x}{x^2}$

$$\therefore \frac{dy}{dx} = \frac{(-2)x^2 - (1 - 2x)(2x)}{x^4} \quad \{\text{quotient rule}\}$$

$$= \frac{-2x^2 - 2x + 4x^2}{x^4}$$

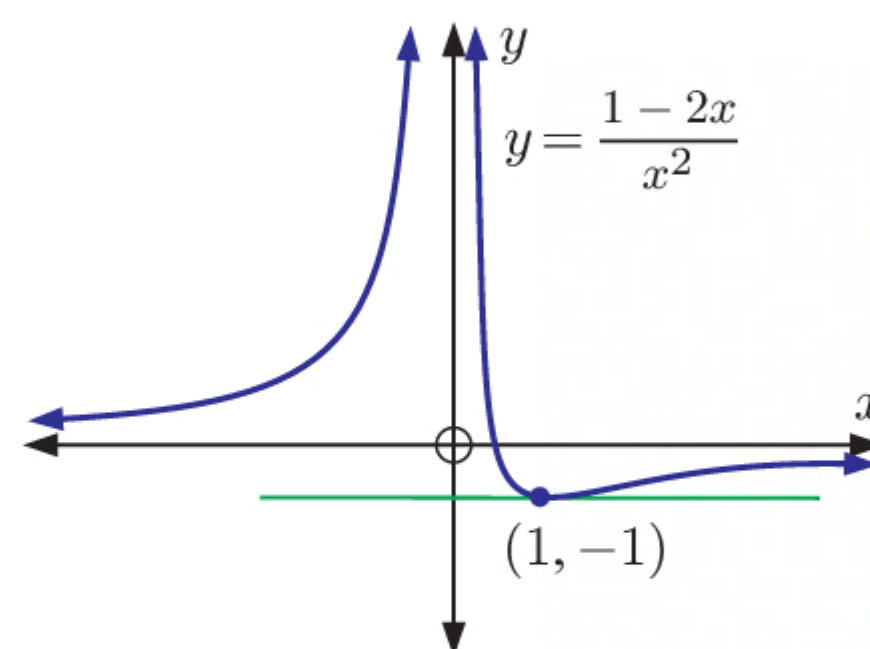
$$= \frac{2x(x - 1)}{x^4}$$

$$= \frac{2(x - 1)}{x^3}$$

$$\text{When } x = 1, \quad \frac{dy}{dx} = \frac{2(1 - 1)}{1^3} = 0$$

So, the tangent has equation $y = 0(x - 1) - 1$

$$\therefore y = -1$$

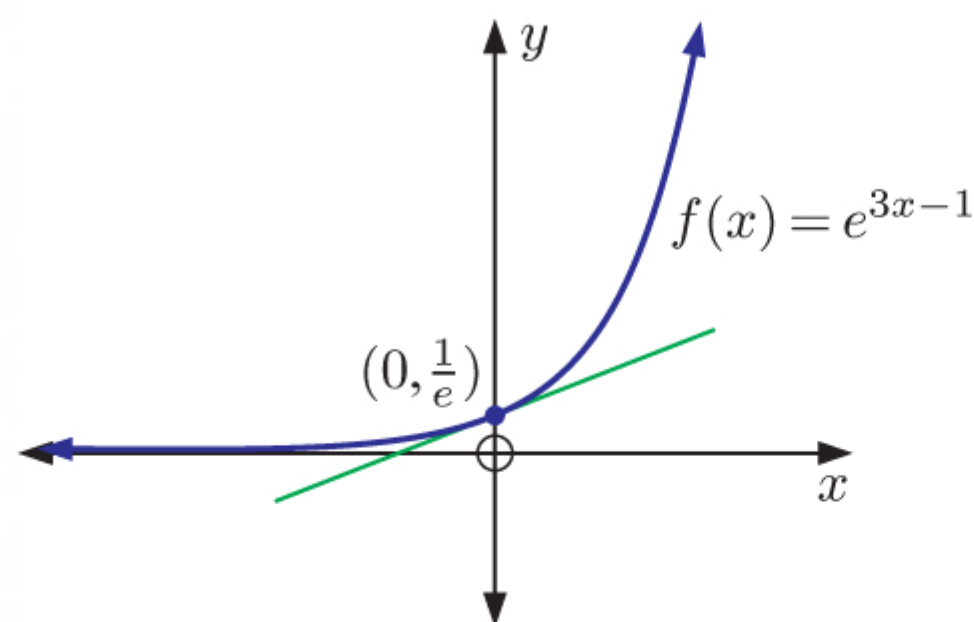


$$\begin{aligned}
 \text{d} \quad f(x) &= e^{3x-1} \\
 \therefore f(0) &= e^{3(0)-1} \\
 &= e^{-1} \\
 &= \frac{1}{e}
 \end{aligned}$$

\therefore the point of contact is $(0, \frac{1}{e})$.

$$\begin{aligned}
 \text{Now } f(x) &= e^{3x-1} \\
 \therefore f'(x) &= 3e^{3x-1} \\
 \therefore f'(0) &= 3e^{3(0)-1} \\
 &= 3e^{-1} \\
 &= \frac{3}{e}
 \end{aligned}$$

So, the tangent has equation $3x - ey = 3(0) - e\left(\frac{1}{e}\right)$
 $\therefore 3x - ey = -1$

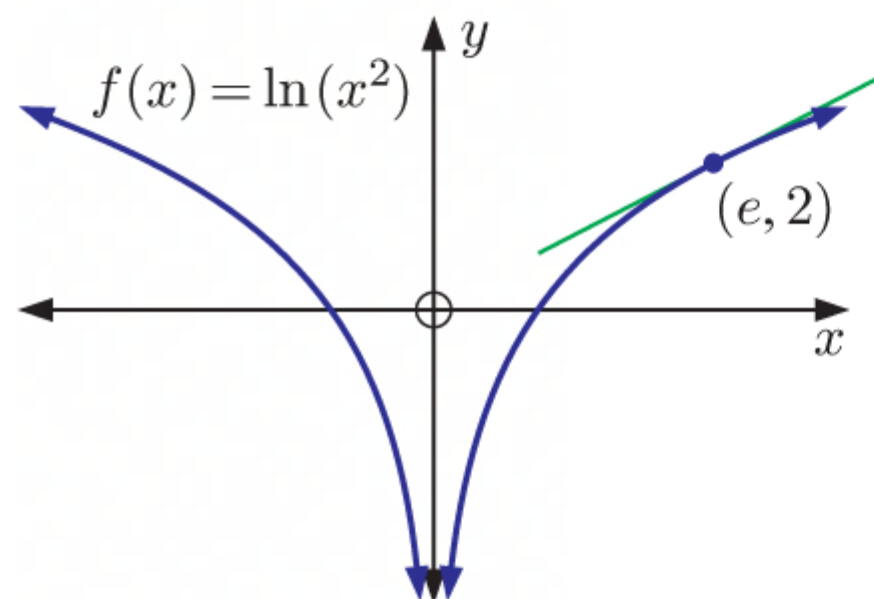


$$\begin{aligned}
 \text{e} \quad f(x) &= \ln(x^2) \\
 \therefore f(e) &= \ln(e^2) \\
 &= 2
 \end{aligned}$$

\therefore the point of contact is $(e, 2)$.

$$\begin{aligned}
 \text{Now } f(x) &= \ln(x^2) \\
 &= 2 \ln x \\
 \therefore f'(x) &= \frac{2}{x} \\
 \therefore f'(e) &= \frac{2}{e}
 \end{aligned}$$

So, the tangent has equation $y = \frac{2}{e}(x - e) + 2$
 $\therefore y = \frac{2}{e}x - 2 + 2$
 $\therefore y = \frac{2}{e}x$



$$\begin{aligned}
 \text{2 a} \quad y &= \sqrt{3x+4} = (3x+4)^{\frac{1}{2}} \\
 \therefore \frac{dy}{dx} &= \frac{1}{2}(3x+4)^{-\frac{1}{2}} \times 3 \quad \{\text{chain rule}\} \\
 &= \frac{3}{2\sqrt{3x+4}}, \quad \text{so at } (4, 4) \\
 \frac{dy}{dx} &= \frac{3}{2\sqrt{3(4)+4}} \\
 &= \frac{3}{2\sqrt{16}} \\
 &= \frac{3}{8}
 \end{aligned}$$

\therefore the normal at $(4, 4)$ has gradient $-\frac{8}{3}$.

\therefore the equation of the normal is $8x + 3y = 8(4) + 3(4)$
 $\therefore 8x + 3y = 44$

b $y = 3e^{2x}$

When $x = 1$, $y = 3e^{2(1)} = 3e^2$

So, the point of contact is $(1, 3e^2)$.

Now $\frac{dy}{dx} = 3e^{2x} \times 2 \quad \{\text{chain rule}\}$
 $= 6e^{2x}$

So at $x = 1$, $\frac{dy}{dx} = 6e^{2(1)}$
 $= 6e^2$

\therefore the normal at $(1, 3e^2)$ has gradient $-\frac{1}{6e^2}$.

\therefore the equation of the normal is $y = -\frac{1}{6e^2}(x - 1) + 3e^2$

$$\therefore y = -\frac{1}{6e^2}x + \frac{1}{6e^2} + 3e^2 \times \frac{6e^2}{6e^2}$$

$$\therefore y = -\frac{1}{6e^2}x + \frac{1}{6e^2} + \frac{18e^4}{6e^2}$$

$$\therefore y = -\frac{1}{6e^2}x + \frac{18e^4 + 1}{6e^2}$$

3 $f(x) = e^{4x} + px + q$

$$f(0) = e^{4(0)} + p(0) + q$$

$$= 1 + q$$

So, the point of contact is $(0, 1 + q)$.

Now $f'(x) = 4e^{4x} + p$
 $\therefore f'(0) = 4e^{4(0)} + p$
 $= 4 + p$

So, the tangent has equation $y = (4 + p)(x - 0) + 1 + q$
 $\therefore y = (4 + p)x + 1 + q$

But we know the tangent has equation $y = 5x - 7$.

$$\therefore 4 + p = 5 \quad \text{and} \quad 1 + q = -7$$

$$\therefore p = 1 \quad \text{and} \quad q = -8$$

4 Note: The first print of this book erroneously repeats **Review set 12A** question **7**.

$$y = 4x^3 + 6x^2 - 13x + 1$$

$$\therefore \frac{dy}{dx} = 12x^2 + 12x - 13$$

The gradient of the tangent is 11 when $12x^2 + 12x - 13 = 11$

$$\therefore 12x^2 + 12x - 24 = 0$$

$$\therefore 12(x^2 + x - 2) = 0$$

$$\therefore 12(x + 2)(x - 1) = 0$$

$$\therefore x = -2 \text{ or } 1$$

When $x = -2$,

$$y = 4(-2)^3 + 6(-2)^2 - 13(-2) + 1$$

$$= 19$$

When $x = 1$,

$$y = 4(1)^3 + 6(1)^2 - 13(1) + 1$$

$$= -2$$

So, the gradient of the tangent to $y = 4x^3 + 6x^2 - 13x + 1$ is 11 at the points $(-2, 19)$ and $(1, -2)$.

$$5 \quad y = \frac{a}{(x+2)^2} = a(x+2)^{-2}, \quad A(2, 4), \quad B(0, 8)$$

$$\text{The gradient of the line (AB)} = \frac{8-4}{0-2} = \frac{4}{-2} = -2$$

$$\therefore \text{ the equation of the tangent is } \frac{y-8}{x-0} = -2 \text{ or } y = -2x + 8$$

$$\text{Now } \frac{dy}{dx} = -2a(x+2)^{-3}, \text{ so for the given tangent, } -2a(x+2)^{-3} = -2$$

$$\begin{aligned} \therefore \frac{a}{(x+2)^3} &= 1 \\ \therefore a &= (x+2)^3 \quad \dots (*) \end{aligned}$$

$$\text{The line (AB) meets the curve where } -2x + 8 = \frac{a}{(x+2)^2}$$

$$\therefore -2x + 8 = \frac{(x+2)^3}{(x+2)^2} \quad \{\text{using } (*)\}$$

$$\therefore -2x + 8 = x + 2$$

$$\therefore -3x = -6$$

$$\therefore x = 2$$

$$\text{and so } a = (2+2)^3 = 64$$

$$6 \quad \text{Let } f(x) = 2x^3 + 4x - 1$$

$$\therefore f'(x) = 6x^2 + 4 \quad \text{and} \quad \therefore f'(1) = 6(1)^2 + 4 = 10$$

$$\therefore \text{ the equation of the tangent at } (1, 5) \text{ is } y = 10(x-1) + 5$$

$$\text{which is } y = 10x - 5$$

$$\text{The curve meets the tangent when } 2x^3 + 4x - 1 = 10x - 5$$

$$\therefore 2x^3 - 6x + 4 = 0$$

$$\therefore x^3 - 3x + 2 = 0$$

$$\therefore (x-1)^2(x+2) = 0 \quad \{\text{tangent at } x = 1\}$$

$$\begin{aligned} f(-2) &= 2(-2)^3 + 4(-2) - 1 \\ &= -16 - 8 - 1 \\ &= -25 \end{aligned}$$

$$\therefore \text{ the tangent meets the curve again at } (-2, -25).$$

$$7 \quad a \quad y = e^{2x}$$

$$\text{When } x = a, \quad y = e^{2a}$$

$$\text{So, the point of contact is } (a, e^{2a}).$$

$$\text{Now } \frac{dy}{dx} = 2e^{2x}$$

$$\text{When } x = a, \quad \frac{dy}{dx} = 2e^{2a}$$

$$\therefore \text{ the normal at } x = a \text{ has gradient } -\frac{1}{2e^{2a}}.$$

$$\therefore \text{ the equation of the normal is } y = -\frac{1}{2e^{2a}}(x-a) + e^{2a}$$

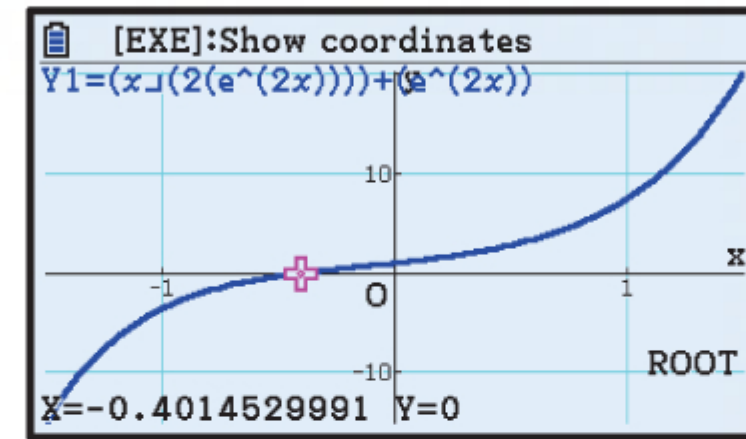
$$\therefore y = -\frac{1}{2e^{2a}}x + \frac{a}{2e^{2a}} + e^{2a}$$

b $y = -\frac{1}{2e^{2a}}x + \frac{a}{2e^{2a}} + e^{2a}$ passes through the origin when $x = 0, y = 0$

$$\therefore 0 = -\frac{1}{2e^{2a}}(0) + \frac{a}{2e^{2a}} + e^{2a}$$

$$\therefore \frac{a}{2e^{2a}} + e^{2a} = 0$$

$$\therefore a \approx -0.40145$$



\therefore the normal to $y = e^{2x}$ which passes through the origin is

$$y = -\frac{1}{2e^{2(-0.40145)}}x \quad \left\{ \frac{a}{2e^{2a}} + e^{2a} = 0 \right\}$$

$$\therefore y \approx -1.12x$$

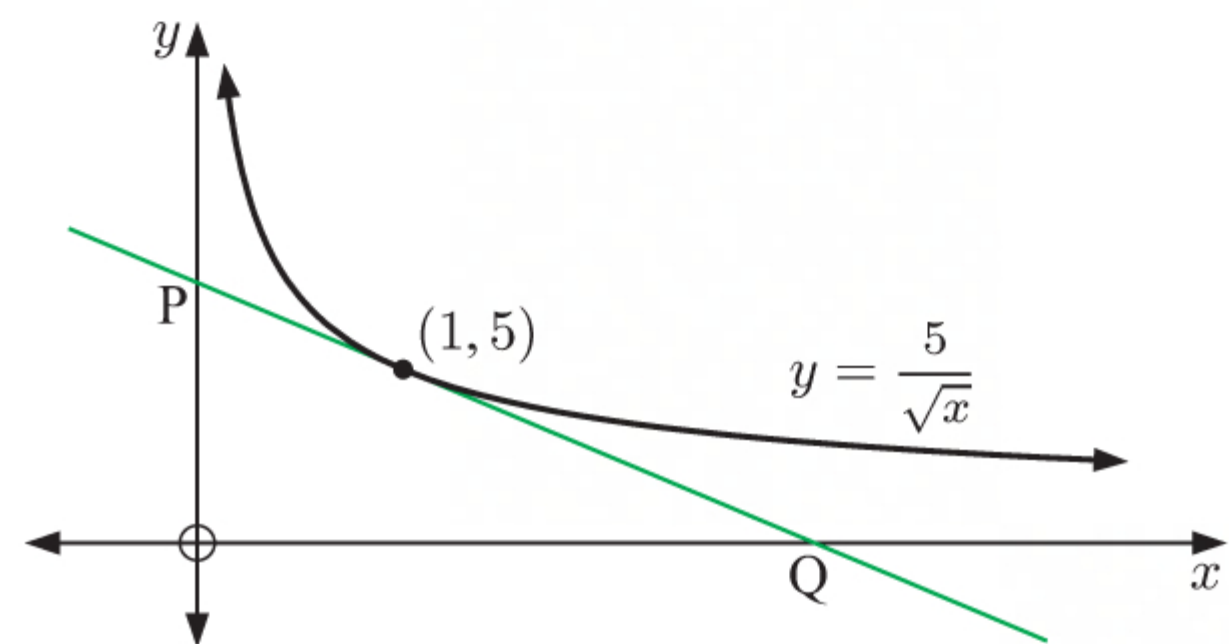
8 $y = \frac{5}{\sqrt{x}} = 5x^{-\frac{1}{2}}$

$$\therefore \frac{dy}{dx} = -\frac{5}{2}x^{-\frac{3}{2}}$$

$$= -\frac{5}{2x\sqrt{x}}$$

At the point $(1, 5)$, $\frac{dy}{dx} = -\frac{5}{2(1)\sqrt{1}}$

$$= -\frac{5}{2}$$



\therefore the gradient of the tangent at $(1, 5)$ is $-\frac{5}{2}$.

\therefore the tangent has equation $y = -\frac{5}{2}(x - 1) + 5$

which is $y = -\frac{5}{2}x + \frac{15}{2}$

At point P, $x = 0 \therefore y = -\frac{5}{2}(0) + \frac{15}{2} = \frac{15}{2}$

At point Q, $y = 0 \therefore 0 = -\frac{5}{2}x + \frac{15}{2}$

$$\therefore \frac{5}{2}x = \frac{15}{2}$$

$$\therefore x = 3$$

So, P is $(0, \frac{15}{2})$ and Q is $(3, 0)$.

9 $y = x^2\sqrt{1-x}$

When $x = -3$, $y = (-3)^2\sqrt{1-(-3)}$

$$= 18$$

\therefore the point of contact is $(-3, 18)$.

Now $y = x^2\sqrt{1-x} = x^2(1-x)^{\frac{1}{2}}$

$$\therefore \frac{dy}{dx} = 2x(1-x)^{\frac{1}{2}} + x^2\left(\frac{1}{2}\right)(1-x)^{-\frac{1}{2}}(-1) \quad \{\text{product rule and chain rule}\}$$

$$= 2x\sqrt{1-x} - \frac{x^2}{2\sqrt{1-x}}$$

$$\begin{aligned}
 \text{When } x = -3, \quad \frac{dy}{dx} &= 2(-3)\sqrt{1 - (-3)} - \frac{(-3)^2}{2\sqrt{1 - (-3)}} \\
 &= -12 - \frac{9}{4} \\
 &= -\frac{57}{4}
 \end{aligned}$$

So, the tangent has equation $y = -\frac{57}{4}(x + 3) + 18$

$$\therefore y = -\frac{57}{4}x - \frac{99}{4}$$

$$\begin{aligned}
 \text{When } y = 0, \quad -\frac{57}{4}x - \frac{99}{4} &= 0 \\
 \therefore 57x &= -99 \\
 \therefore x &= -\frac{99}{57}
 \end{aligned}$$

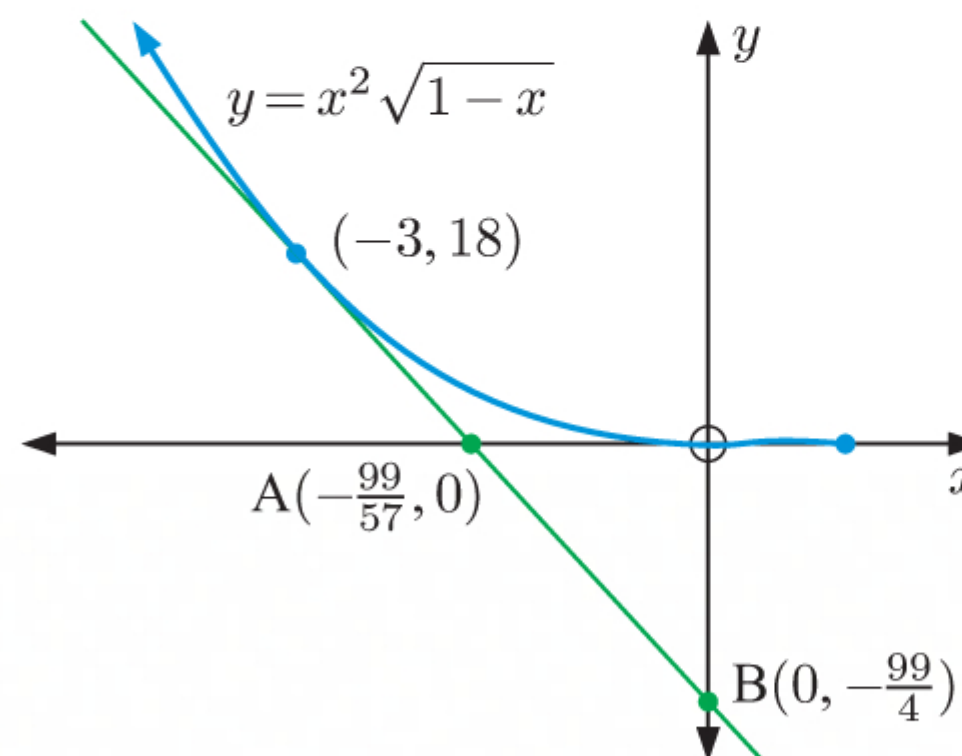
\therefore the x -intercept of the tangent is $-\frac{99}{57}$.

$$\text{When } x = 0, \quad y = -\frac{99}{4}$$

\therefore the y -intercept of the tangent is $-\frac{99}{4}$.

\therefore A is $(-\frac{99}{57}, 0)$ and B is $(0, -\frac{99}{4})$.

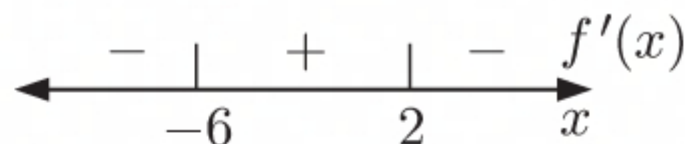
$$\begin{aligned}
 \therefore \text{area of triangle OAB} &= \frac{1}{2} \times \frac{99}{57} \times \frac{99}{4} \\
 &= \frac{3267}{152} \approx 21.5 \text{ units}^2
 \end{aligned}$$



10 $f(x) = -x^3 - 6x^2 + 36x - 17$

$$\begin{aligned}
 \therefore f'(x) &= -3x^2 - 12x + 36 \\
 &= -3(x^2 + 4x - 12) \\
 &= -3(x + 6)(x - 2)
 \end{aligned}$$

which has sign diagram:



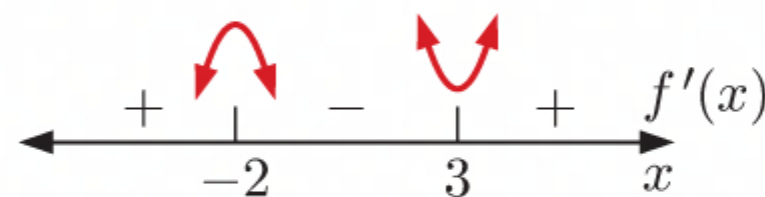
a $f(x)$ is increasing for $-6 \leq x \leq 2$.

b $f(x)$ is decreasing for $x \leq -6$ or $x \geq 2$.

11 $f(x) = 2x^3 - 3x^2 - 36x + 7$

$$\begin{aligned}
 \text{a } f'(x) &= 6x^2 - 6x - 36 \\
 &= 6(x^2 - x - 6) \\
 &= 6(x + 2)(x - 3)
 \end{aligned}$$

which has sign diagram:



So, there is a local maximum at $x = -2$ and a local minimum at $x = 3$.

$$\begin{aligned}
 f(-2) &= 2(-2)^3 - 3(-2)^2 - 36(-2) + 7 \\
 &= 51
 \end{aligned}$$

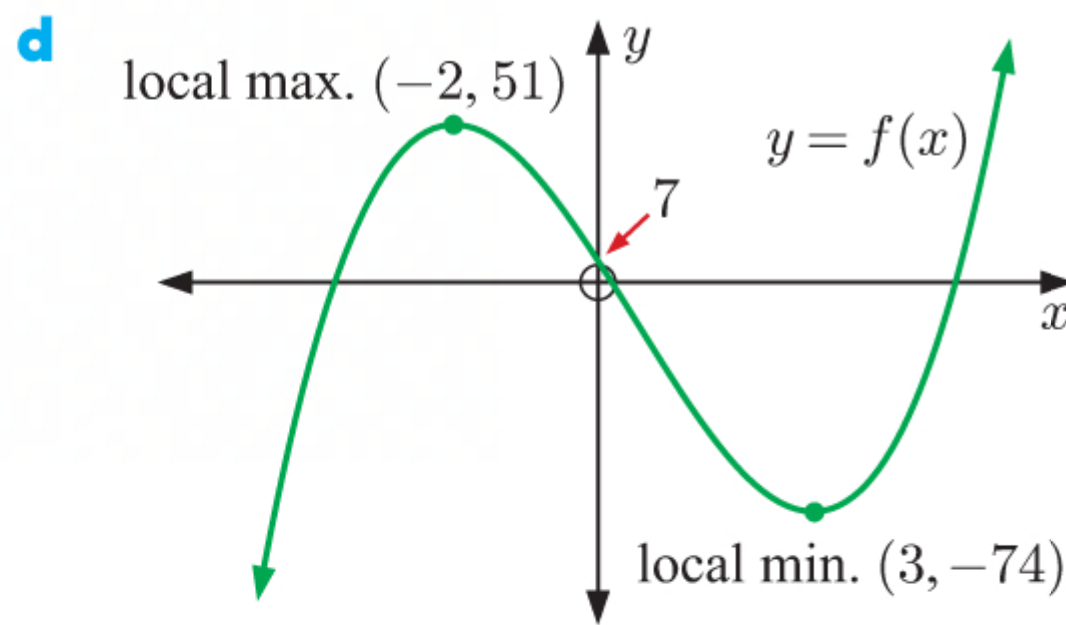
$$\begin{aligned}
 f(3) &= 2(3)^3 - 3(3)^2 - 36(3) + 7 \\
 &= -74
 \end{aligned}$$

There is a local maximum at $(-2, 51)$ and a local minimum at $(3, -74)$.

b $f(x)$ is increasing for $x \leq -2$ and $x \geq 3$.

$f(x)$ is decreasing for $-2 \leq x \leq 3$.

c as $x \rightarrow \infty$, $f(x) \rightarrow \infty$
as $x \rightarrow -\infty$, $f(x) \rightarrow -\infty$



12 a $f(x) = \frac{3x-2}{x+3}$ is defined when $x+3 \neq 0$
 $\therefore x \neq -3$

$\therefore f(x)$ has domain $\{x \mid x \neq -3\}$.

b $f(x) = 0$ when $3x - 2 = 0$
 $\therefore 3x = 2$
 $\therefore x = \frac{2}{3}$

\therefore the x -intercept is $\frac{2}{3}$.

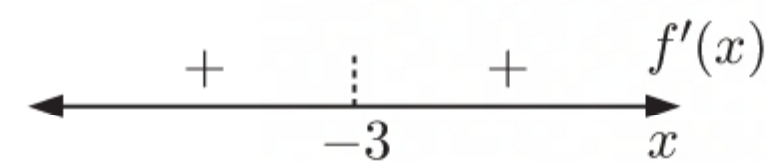
$$f(0) = \frac{3(0) - 2}{0 + 3} = -\frac{2}{3}$$

\therefore the y -intercept is $-\frac{2}{3}$.

c $f'(x) = \frac{3(x+3) - (3x-2)(1)}{(x+3)^2}$ {quotient rule}

$$= \frac{\cancel{3x} + 9 - \cancel{3x} + 2}{(x+3)^2}$$

$$= \frac{11}{(x+3)^2} \text{ which has sign diagram:}$$



d There are no values of x such that $f'(x) = 0$.
 $\therefore f(x)$ does not have any stationary points.

13 Let $y = x + \frac{32}{x^2} = x + 32x^{-2}$, $2 \leq x \leq 10$

$$\therefore \frac{dy}{dx} = 1 - 64x^{-3} = 1 - \frac{64}{x^3}$$

$$\frac{dy}{dx} = 0 \text{ when } 1 - \frac{64}{x^3} = 0$$

$$\therefore \frac{64}{x^3} = 1$$

$$\therefore 64 = x^3$$

$$\therefore x = \sqrt[3]{64} = 4$$

$\frac{dy}{dx}$ has sign diagram:

\therefore there is a local minimum at $x = 4$.

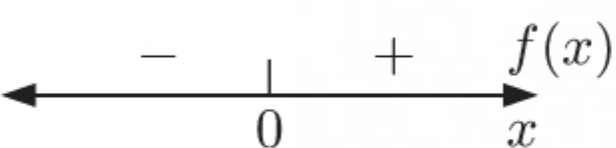
Critical value (x)	y
2 (end point)	10
4 (local minimum)	6
10 (end point)	≈ 10.32

The greatest of these values is about 10.3 when $x = 10$.

The least of these values is 6 when $x = 4$.

14 a $f(x) = xe^{1-2x}$
 $\therefore f'(x) = (1)e^{1-2x} + xe^{1-2x}(-2)$ {product rule, chain rule}
 $= e^{1-2x} - 2xe^{1-2x}$
 $= e^{1-2x}(1 - 2x)$

b i $f(x) = 0$ when $x = 0$ $\{e^{1-2x} > 0\}$

$\therefore f(x)$ has sign diagram: 

$\therefore f(x) > 0$ when $x > 0$.

ii $f'(x) = e^{1-2x}(1 - 2x)$
 $f'(x) = 0$ when $1 - 2x = 0$ $\{e^{1-2x} > 0\}$
 $\therefore x = \frac{1}{2}$

$\therefore f'(x)$ has sign diagram: 


$\therefore f'(x) > 0$ when $x < \frac{1}{2}$.

c Stationary points corresponds to where $f'(x) = 0$.

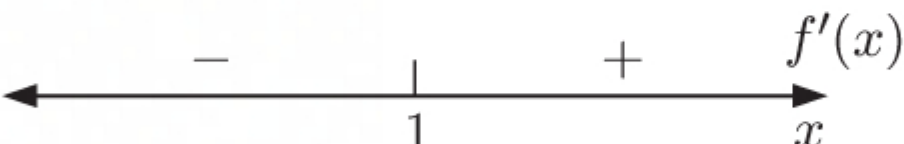
$f'(x) = 0$ when $x = \frac{1}{2}$

and $f\left(\frac{1}{2}\right) = \left(\frac{1}{2}\right)e^{1-2\left(\frac{1}{2}\right)}$
 $= \frac{1}{2}e^0$
 $= \frac{1}{2}$

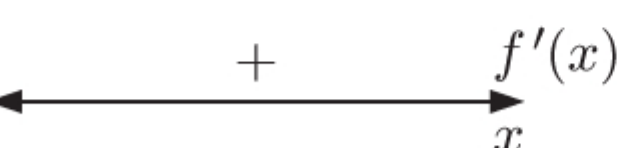
$\therefore \left(\frac{1}{2}, \frac{1}{2}\right)$ is a local maximum.

15 a $f(x) = x^3 - 6x$
 $\therefore f'(x) = 3x^2 - 6$
 $= 3(x^2 - 2)$
 $= 3(x + \sqrt{2})(x - \sqrt{2})$ which has sign diagram: 

$f(x)$ is increasing for $x \leq -\sqrt{2}$ and $x \geq \sqrt{2}$, and decreasing for $-\sqrt{2} \leq x \leq \sqrt{2}$.

b $f(x) = e^x(x - 2)$
 $\therefore f'(x) = e^x(x - 2) + e^x(1)$ {product rule}
 $= e^x(x - 1)$ which has sign diagram: 

$f(x)$ is increasing for $x \geq 1$, and decreasing for $x \leq 1$.

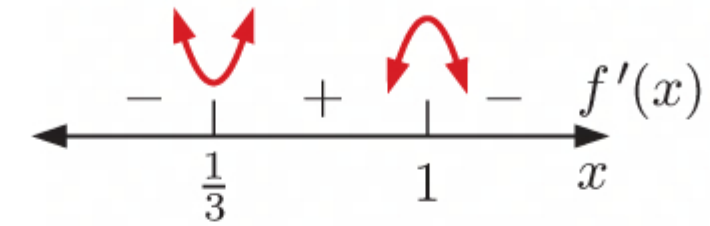
c $f(x) = 2x - \sin x$
 $\therefore f'(x) = 2 - \cos x$ which has sign diagram: 

$f(x)$ is increasing for all $x \in \mathbb{R}$.

16 a $f(x) = -x^3 + 2x^2 - x + 3$

$$\therefore f'(x) = -3x^2 + 4x - 1$$

$$= (1 - 3x)(x - 1) \quad \text{which has sign diagram:}$$



$$\begin{aligned} \text{Now } f\left(\frac{1}{3}\right) &= -\left(\frac{1}{3}\right)^3 + 2\left(\frac{1}{3}\right)^2 - \left(\frac{1}{3}\right) + 3 & \text{and } f(1) &= -(1)^3 + 2(1)^2 - (1) + 3 \\ &= -\frac{1}{27} + \frac{2}{9} - \frac{1}{3} + 3 & &= -1 + 2 - 1 + 3 \\ &= \frac{77}{27} & &= 3 \end{aligned}$$

$\therefore \left(\frac{1}{3}, \frac{77}{27}\right)$ is a local minimum, and $(1, 3)$ is a local maximum.

b $f(x) = \frac{x^2}{x+3}$

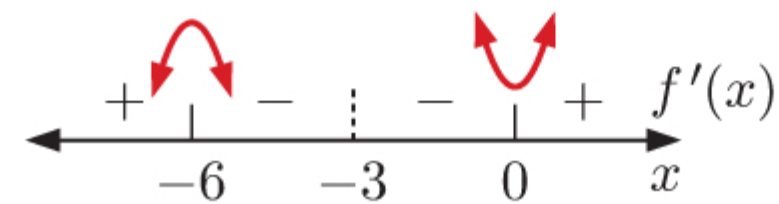
$$\therefore f'(x) = \frac{2x(x+3) - x^2(1)}{(x+3)^2} \quad \{\text{quotient rule}\}$$

$$= \frac{2x^2 + 6x - x^2}{(x+3)^2}$$

$$= \frac{x^2 + 6x}{(x+3)^2}$$

$$= \frac{x(x+6)}{(x+3)^2}$$

which has sign diagram:



$$\begin{aligned} f(-6) &= \frac{(-6)^2}{(-6+3)} & \text{and } f(0) &= \frac{(0)^2}{0+3} \\ &= \frac{36}{-3} & &= 0 \\ &= -12 \end{aligned}$$

$\therefore (-6, -12)$ is a local maximum, and $(0, 0)$ is a local minimum.

17 a $y = \sin \frac{x}{2}, \quad -\pi \leq x \leq \pi$

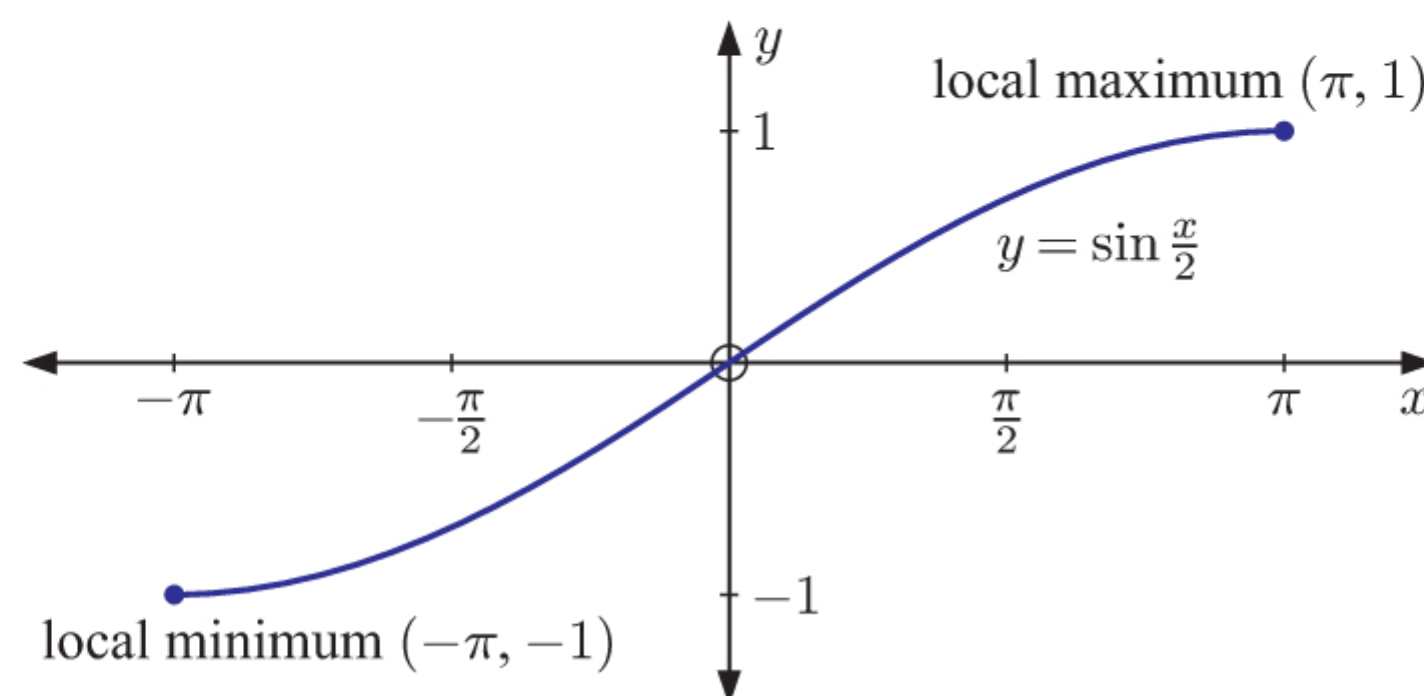
$$\therefore \frac{dy}{dx} = \frac{1}{2} \cos \frac{x}{2}$$

$$= 0 \quad \text{when } x = -\pi \text{ or } \pi$$

$$\begin{aligned} \text{When } x = -\pi, \quad y &= \sin\left(-\frac{\pi}{2}\right) \\ &= -1 \end{aligned}$$

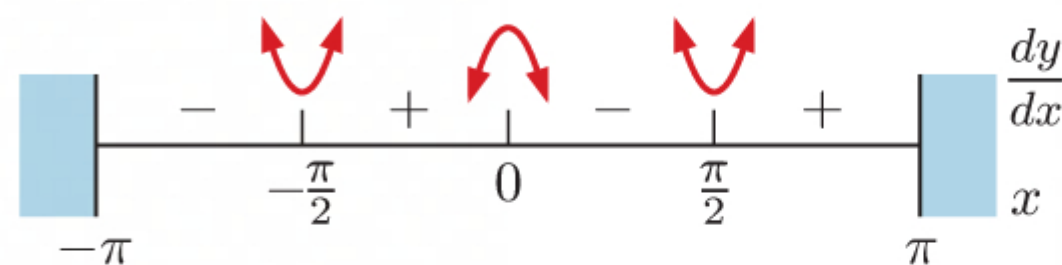
$$\begin{aligned} \text{When } x = \pi, \quad y &= \sin \frac{\pi}{2} \\ &= 1 \end{aligned}$$

$\therefore (-\pi, -1)$ is a local minimum, $(\pi, 1)$ is a local maximum.



b $y = \cos^2 x, \quad -\pi \leq x \leq \pi$

$$\begin{aligned}\therefore \frac{dy}{dx} &= 2 \cos x(-\sin x) \\ &= -2 \sin x \cos x \\ &= -\sin 2x \quad \text{which has sign diagram:}\end{aligned}$$



When $x = -\pi$, $y = \cos^2(-\pi) = 1$

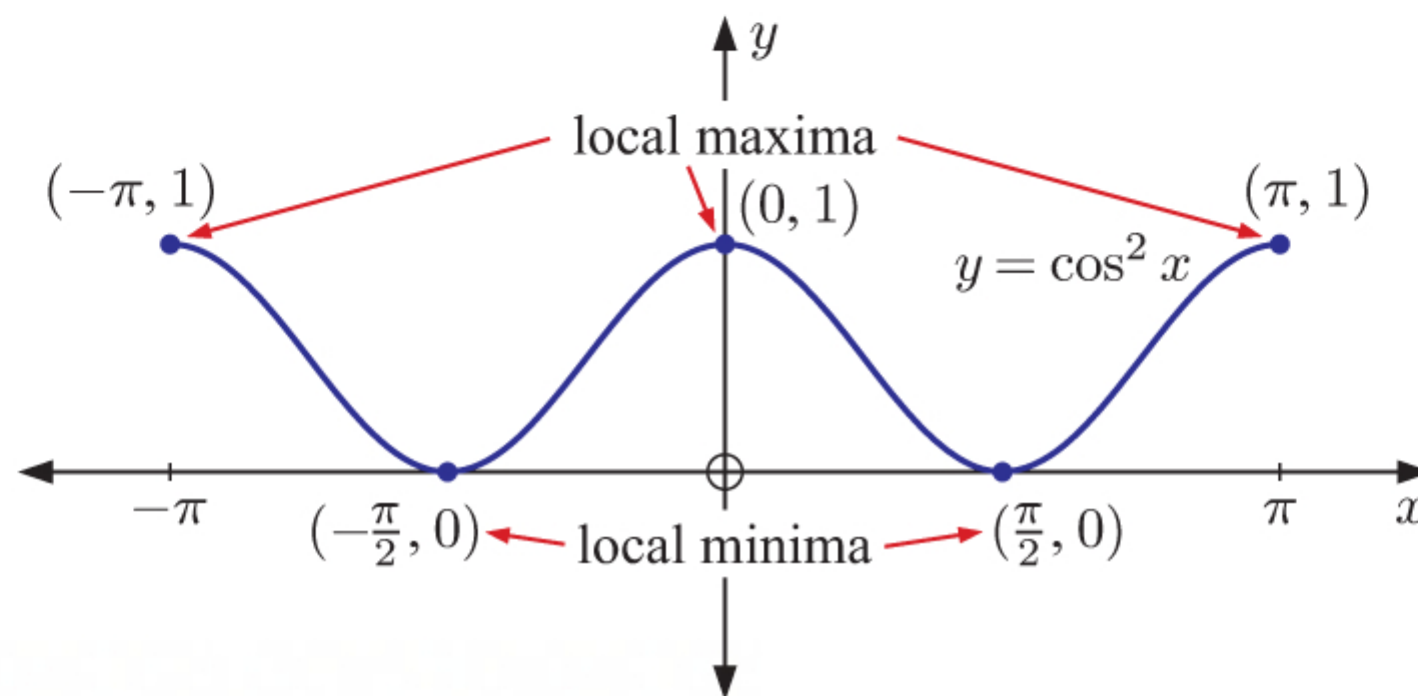
When $x = -\frac{\pi}{2}$, $y = \cos^2(-\frac{\pi}{2}) = 0$

When $x = 0$, $y = \cos^2 0 = 1$

When $x = \frac{\pi}{2}$, $y = \cos^2(\frac{\pi}{2}) = 0$

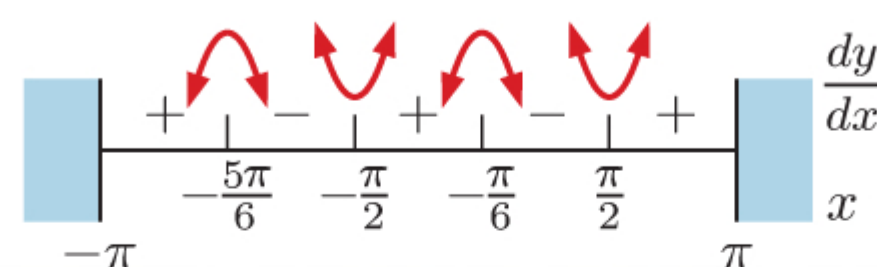
When $x = \pi$, $y = \cos^2 \pi = 1$

$\therefore (-\pi, 1)$, $(0, 1)$, and $(\pi, 1)$ are local maxima, $(-\frac{\pi}{2}, 0)$ and $(\frac{\pi}{2}, 0)$ are local minima.



c $y = \cos 2x - 2 \sin x$

$$\begin{aligned}\therefore \frac{dy}{dx} &= -2 \sin 2x - 2 \cos x \\ &= -2(2 \sin x \cos x) - 2 \cos x \\ &= -4 \sin x \cos x - 2 \cos x \\ &= -2 \cos x(2 \sin x + 1) \quad \text{which has sign diagram:}\end{aligned}$$



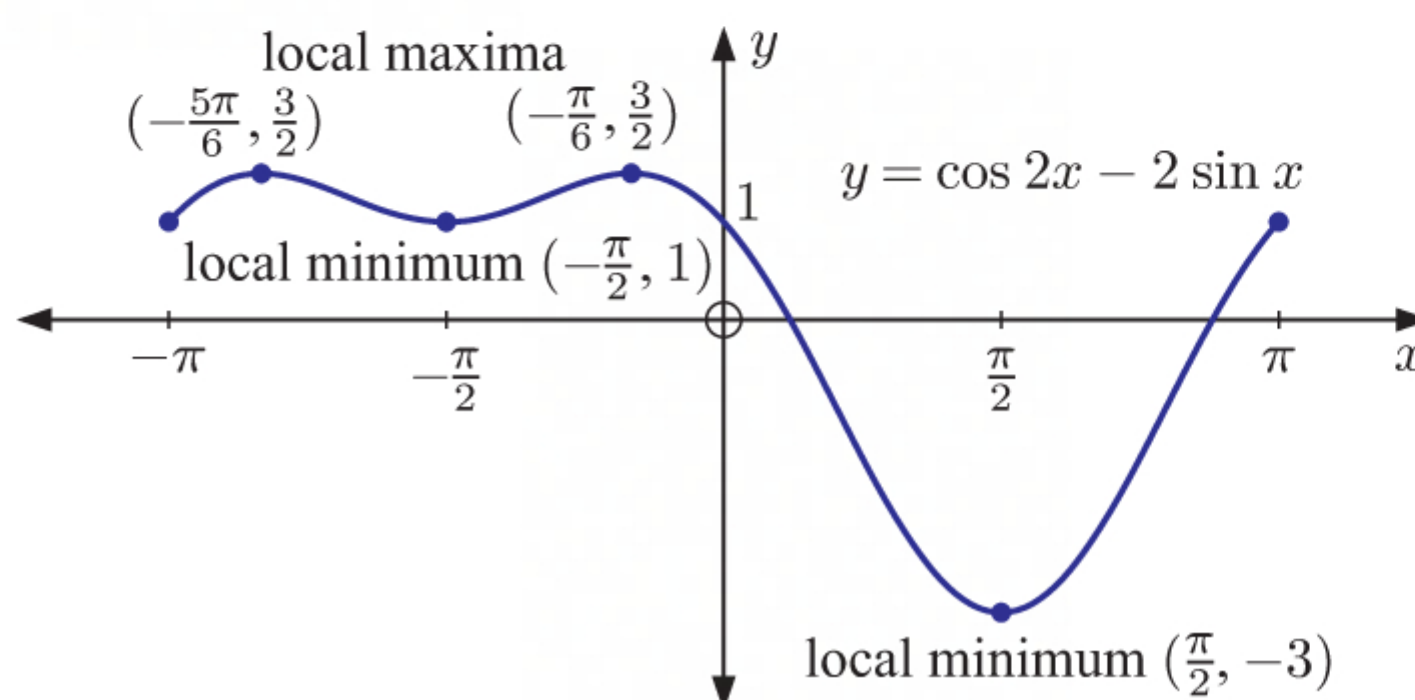
When $x = -\frac{5\pi}{6}$, $y = \cos(2 \times -\frac{5\pi}{6}) - 2 \sin(-\frac{5\pi}{6})$
 $= \cos(-\frac{5\pi}{3}) - 2 \sin(-\frac{5\pi}{6})$
 $= \frac{1}{2} - 2 \times (-\frac{1}{2})$
 $= \frac{1}{2} + 1$
 $= \frac{3}{2}$

When $x = -\frac{\pi}{2}$, $y = \cos(2 \times -\frac{\pi}{2}) - 2 \sin(-\frac{\pi}{2})$
 $= \cos(-\pi) - 2 \sin(-\frac{\pi}{2})$
 $= -1 - 2 \times -1$
 $= -1 + 2$
 $= 1$

$$\begin{aligned}
 \text{When } x = -\frac{\pi}{6}, \quad y &= \cos\left(2 \times -\frac{\pi}{6}\right) - 2 \sin\left(-\frac{\pi}{6}\right) \\
 &= \cos\left(-\frac{\pi}{3}\right) - 2 \sin\left(-\frac{\pi}{6}\right) \\
 &= \frac{1}{2} - 2 \times \left(-\frac{1}{2}\right) \\
 &= \frac{1}{2} + 1 \\
 &= \frac{3}{2}
 \end{aligned}$$

$$\begin{aligned}
 \text{When } x = \frac{\pi}{2}, \quad y &= \cos\left(2 \times \frac{\pi}{2}\right) - 2 \sin \frac{\pi}{2} \\
 &= \cos \pi - 2 \sin \frac{\pi}{2} \\
 &= -1 - 2 \\
 &= -3
 \end{aligned}$$

$\therefore \left(-\frac{5\pi}{6}, \frac{3}{2}\right)$ and $\left(-\frac{\pi}{6}, \frac{3}{2}\right)$ are local maxima, $\left(-\frac{\pi}{2}, 1\right)$ and $\left(\frac{\pi}{2}, -3\right)$ are local minima.

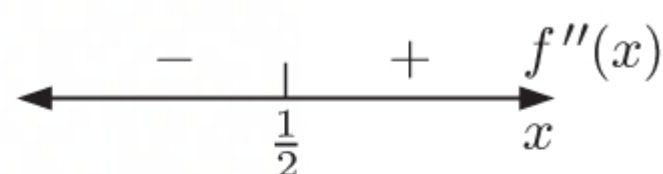


18 a $f(x) = 2x^3 - 3x^2 + x - 12$

$$\therefore f'(x) = 6x^2 - 6x + 1$$

$$\therefore f''(x) = 12x - 6$$

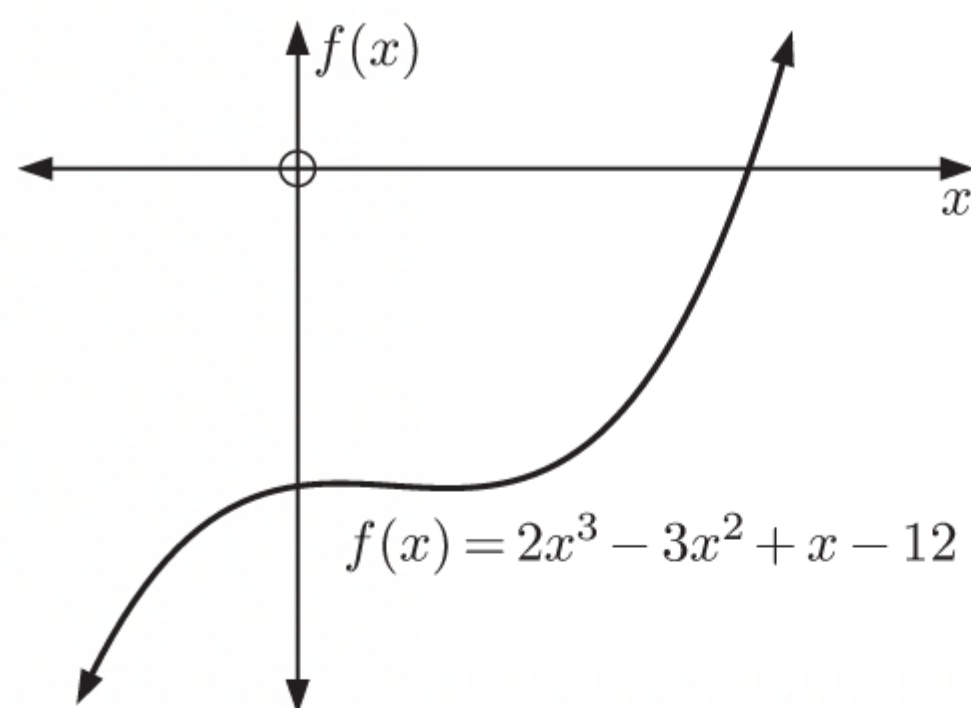
b $f''(x) = 6(2x - 1)$ has sign diagram:



c $f(x)$ is concave down for $x \leq \frac{1}{2}$.

d
$$\begin{aligned}
 f\left(\frac{1}{2}\right) &= 2\left(\frac{1}{2}\right)^3 - 3\left(\frac{1}{2}\right)^2 + \frac{1}{2} - 12 \\
 &= \frac{1}{4} - \frac{3}{4} + \frac{1}{2} - 12 \\
 &= -12
 \end{aligned}$$

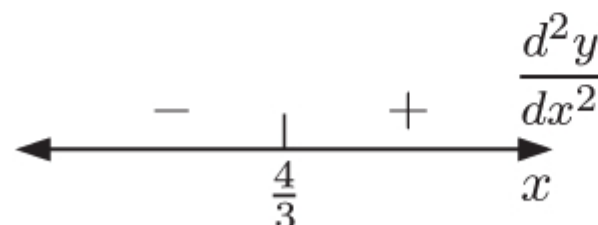
\therefore the shape of $f(x)$ changes at $\left(\frac{1}{2}, -12\right)$.



19 a $y = x^3 - 4x^2 + 11$

$$\therefore \frac{dy}{dx} = 3x^2 - 8x$$

$$\therefore \frac{d^2y}{dx^2} = 6x - 8 \quad \text{which has sign diagram:}$$



The curve is concave up for $x \geq \frac{4}{3}$, and concave down for $x \leq \frac{4}{3}$.

b

$$y = -\frac{x+1}{x^2}$$

$$= \frac{x+1}{-x^2}$$

$$\therefore \frac{dy}{dx} = \frac{(1)(-x^2) - (x+1)(-2x)}{(-x^2)^2} \quad \{\text{quotient rule}\}$$

$$= \frac{-x^2 + 2x^2 + 2x}{x^4}$$

$$= \frac{x^2 + 2x}{x^4}$$

$$= \frac{x(x+2)}{x^4}$$

$$= \frac{x+2}{x^3}$$

$$\therefore \frac{d^2y}{dx^2} = \frac{(1)x^3 - (x+2)(3x^2)}{(x^3)^2}$$

$$= \frac{x^3 - 3x^3 - 6x^2}{x^6}$$

$$= \frac{-2x^3 - 6x^2}{x^6}$$

$$= \frac{-x^2(2x+6)}{x^6}$$

$$= -\frac{2x+6}{x^4} \quad \text{which has sign diagram:}$$

The curve is concave up for $x \leq -3$, and concave down for $-3 \leq x < 0$ and $x > 0$.

c

$$y = \frac{x+2}{x(x+4)}$$

$$= \frac{x+2}{x^2+4x}$$

$$\therefore \frac{dy}{dx} = \frac{(1)(x^2+4x) - (x+2)(2x+4)}{(x^2+4x)^2} \quad \{\text{quotient rule}\}$$

$$= \frac{x^2+4x - (2x^2+8x+8)}{(x^2+4x)^2}$$

$$= \frac{x^2+4x-2x^2-8x-8}{(x^2+4x)^2}$$

$$= \frac{-x^2-4x-8}{(x^2+4x)^2}$$

$$= \frac{-(x^2+4x)}{(x^2+4x)^2} - \frac{8}{(x^2+4x)^2}$$

$$= -(x^2+4x)^{-1} - 8(x^2+4x)^{-2}$$

$$\begin{aligned}
 \therefore \frac{d^2y}{dx^2} &= (x^2 + 4x)^{-2}(2x + 4) + 16(x^2 + 4x)^{-3}(2x + 4) \\
 &= (2x + 4) \left[\frac{1}{(x^2 + 4x)^2} + \frac{16}{(x^2 + 4x)^3} \right] \\
 &= (2x + 4) \left[\frac{x^2 + 4x + 16}{(x^2 + 4x)^3} \right]
 \end{aligned}$$

Now $x^2 + 4x + 16$ has $\Delta = 4^2 - 4(1)(16) = -48 < 0$

and so does not have real roots.

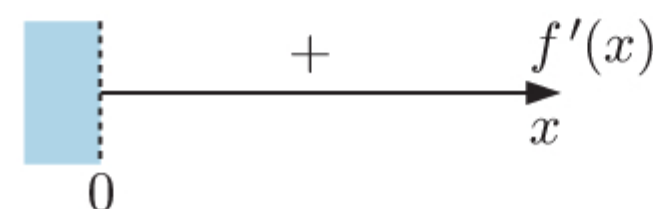
So, $\frac{d^2y}{dx^2} = 0$ when $x = -2$ and is undefined when $x = 0$ or -4

So, $\frac{d^2y}{dx^2}$ has sign diagram:

The curve is concave up for $-4 < x \leq -2$ and $x > 0$, and concave down for $x < -4$ and $-2 \leq x < 0$.

20 a $f(x) = x + \ln x$ is defined for $x > 0$.

b $f'(x) = 1 + \frac{1}{x}$ which has sign diagram:



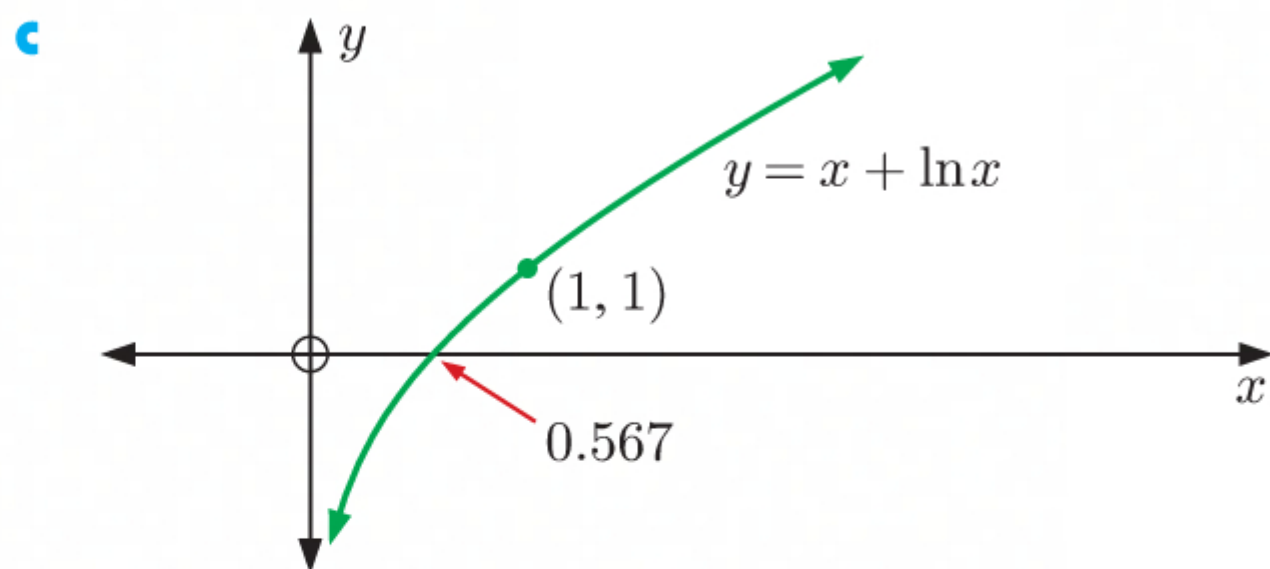
$$f'(x) = 1 + x^{-1}$$

$$\therefore f''(x) = -x^{-2}$$

$= -\frac{1}{x^2}$ which has sign diagram:



So, $f(x)$ is increasing for all $x > 0$ and is concave downwards for all $x > 0$.



21 a $f(x) = e^{x\sqrt{3}} \sin x$

$$\begin{aligned}
 \therefore f'(x) &= e^{x\sqrt{3}}(\sqrt{3}) \sin x + e^{x\sqrt{3}} \cos x \quad \{\text{chain rule and product rule}\} \\
 &= e^{x\sqrt{3}}(\cos x + \sqrt{3} \sin x)
 \end{aligned}$$

b $f'(x) = 0$

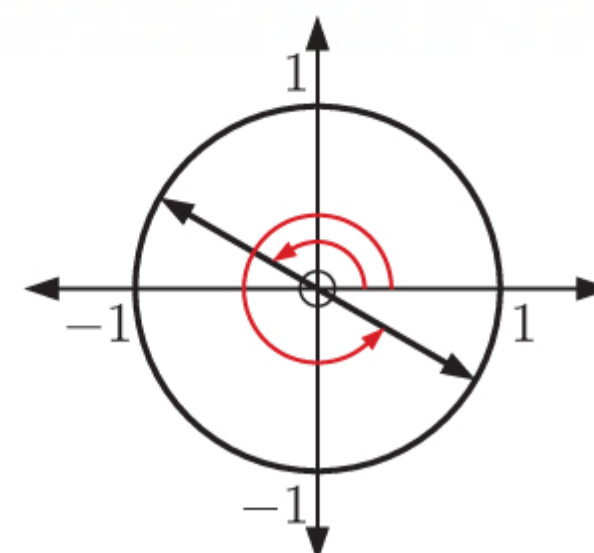
$$\text{when } e^{x\sqrt{3}}(\cos x + \sqrt{3} \sin x) = 0$$

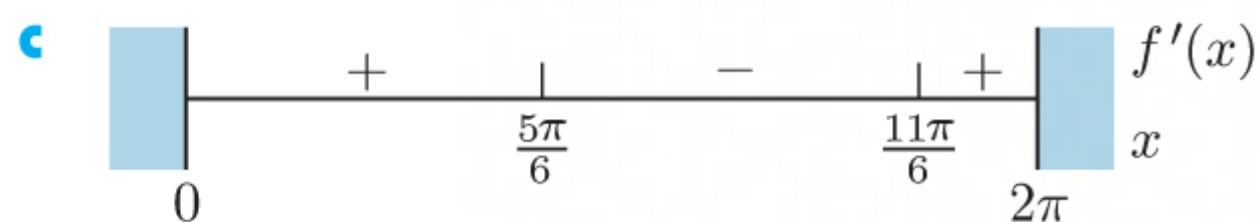
$$\therefore \cos x + \sqrt{3} \sin x = 0 \quad \{\text{as } e^{x\sqrt{3}} > 0 \text{ for all } x\}$$

$$\therefore \sqrt{3} \sin x = -\cos x$$

$$\therefore \tan x = -\frac{1}{\sqrt{3}}$$

$$\therefore x = \frac{5\pi}{6} \text{ or } \frac{11\pi}{6}$$

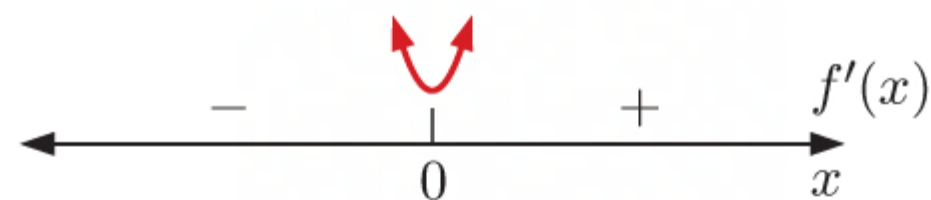




- d**
- i** $f(x)$ is increasing for $0 \leq x \leq \frac{5\pi}{6}$ and $\frac{11\pi}{6} \leq x \leq 2\pi$.
 - ii** $f(x)$ is decreasing for $\frac{5\pi}{6} \leq x \leq \frac{11\pi}{6}$.

22 a $f(x) = \ln(x^2 + 5)$

$\therefore f'(x) = \frac{2x}{x^2 + 5}$ which has sign diagram:



Now $f(0) = \ln(0^2 + 5)$
 $= \ln 5$

$\therefore (0, \ln 5)$ is a local minimum.

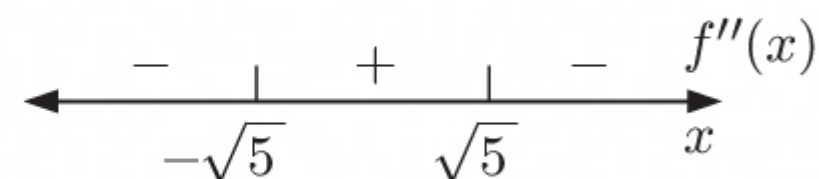
b $f''(x) = \frac{2(x^2 + 5) - 2x(2x)}{(x^2 + 5)^2}$ {quotient rule}

$$= \frac{2x^2 + 10 - 4x^2}{(x^2 + 5)^2}$$

$$= \frac{-2(x^2 - 5)}{(x^2 + 5)^2}$$

$$= \frac{-2(x + \sqrt{5})(x - \sqrt{5})}{(x^2 + 5)^2}$$

which has sign diagram:



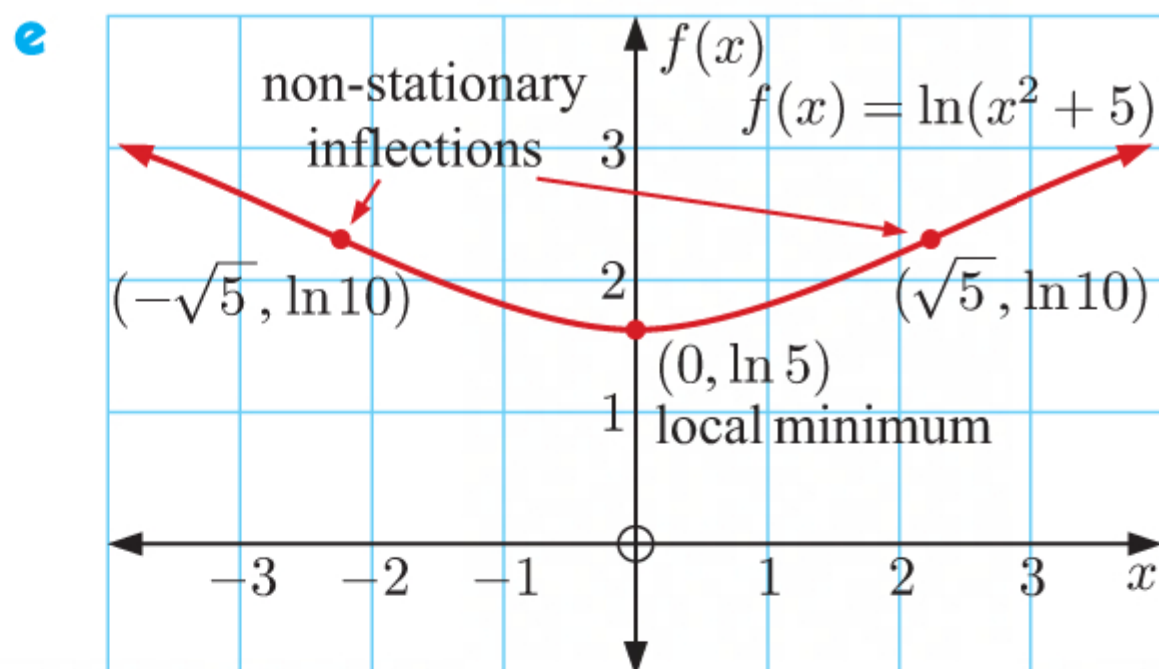
Now $f(-\sqrt{5}) = \ln((- \sqrt{5})^2 + 5)$ and $f(\sqrt{5}) = \ln((\sqrt{5})^2 + 5)$
 $= \ln 10$ $= \ln 10$

also $f'(-\sqrt{5}) = \frac{2(-\sqrt{5})}{(-\sqrt{5})^2 + 5}$ $f'(\sqrt{5}) = \frac{2\sqrt{5}}{(\sqrt{5})^2 + 5}$
 $= -\frac{2\sqrt{5}}{10} \neq 0$ $= \frac{2\sqrt{5}}{10} \neq 0$

$\therefore (-\sqrt{5}, \ln 10)$ and $(\sqrt{5}, \ln 10)$ are non-stationary inflections.

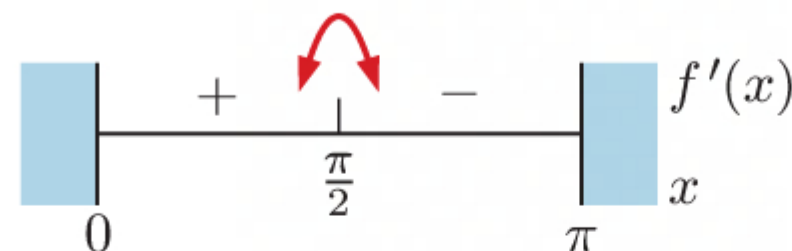
- c** $f(x)$ is increasing for $x \geq 0$, and decreasing for $x \leq 0$.

- d** $f(x)$ is concave up for $-\sqrt{5} \leq x \leq \sqrt{5}$, and concave down for $x \leq -\sqrt{5}$ and $x \geq \sqrt{5}$.



23 $f(x) = e^{\sin^2 x}$, $0 \leq x \leq \pi$

a $f'(x) = e^{\sin^2 x} \times 2 \sin x \cos x$ {chain rule}
 $= e^{\sin^2 x} \sin 2x$ which has sign diagram:



$f'(x) = 0$ when $\sin 2x = 0$

$\therefore 2x = 0 + k\pi, \quad k \in \mathbb{Z}$

$\therefore x = 0 + \frac{k\pi}{2}$

$\therefore x = 0, \frac{\pi}{2}, \text{ or } \pi$

$\therefore f(x)$ has a maximum turning point when $x = \frac{\pi}{2}$, and this maximum value is e .

b $f''(x) = e^{\sin^2 x} \times \sin 2x \sin 2x + e^{\sin^2 x} \times 2 \cos 2x$
 $= e^{\sin^2 x} (\sin^2 2x + 2 \cos 2x)$

$f''(x) = 0$ when $\sin^2 2x + 2 \cos 2x = 0$ on $0 \leq x \leq \pi$

$\therefore \sin^2 2x = -2 \cos 2x$

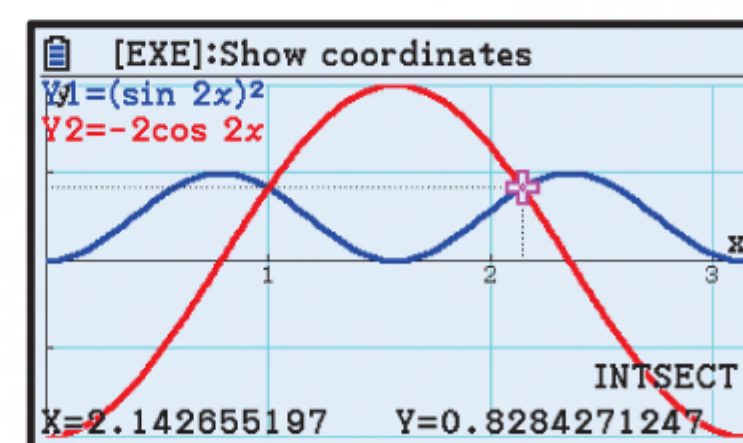
$\therefore x \approx 0.999 \text{ or } 2.14$

{using technology}

So, $f(0.999) \approx 2.03$ and $f(2.14) \approx 2.03$

$f'(0.999) \neq 0$ $f'(2.14) \neq 0$

$\therefore (0.999, 2.03)$ and $(2.14, 2.03)$ are non-stationary points of inflection.



24 $f(x)$ has a turning point at $x = 0$

$\therefore f'(0) = 0$

$f(x)$ is increasing for $x \geq 0$, except at the asymptote.

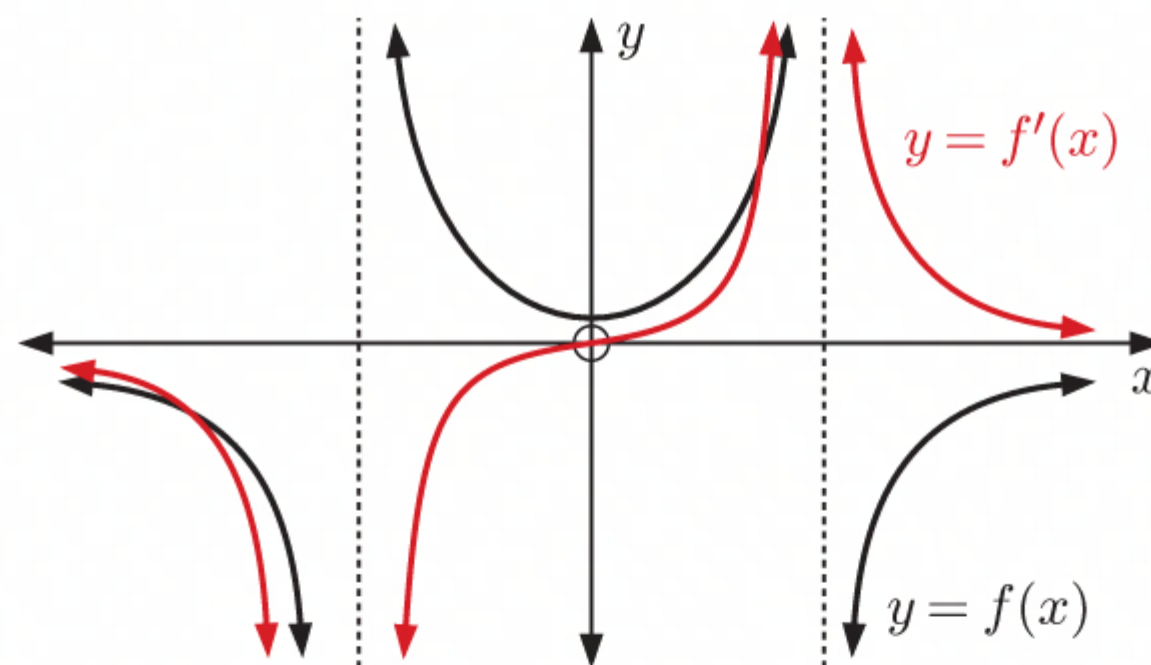
$\therefore f'(x)$ is positive for $x \geq 0$.

$f(x)$ is decreasing for $x \leq 0$, except at the asymptote.

$\therefore f'(x)$ is negative for $x \leq 0$.

As $x \rightarrow \infty$, $f'(x) \rightarrow 0^+$.

As $x \rightarrow -\infty$, $f'(x) \rightarrow 0^-$.



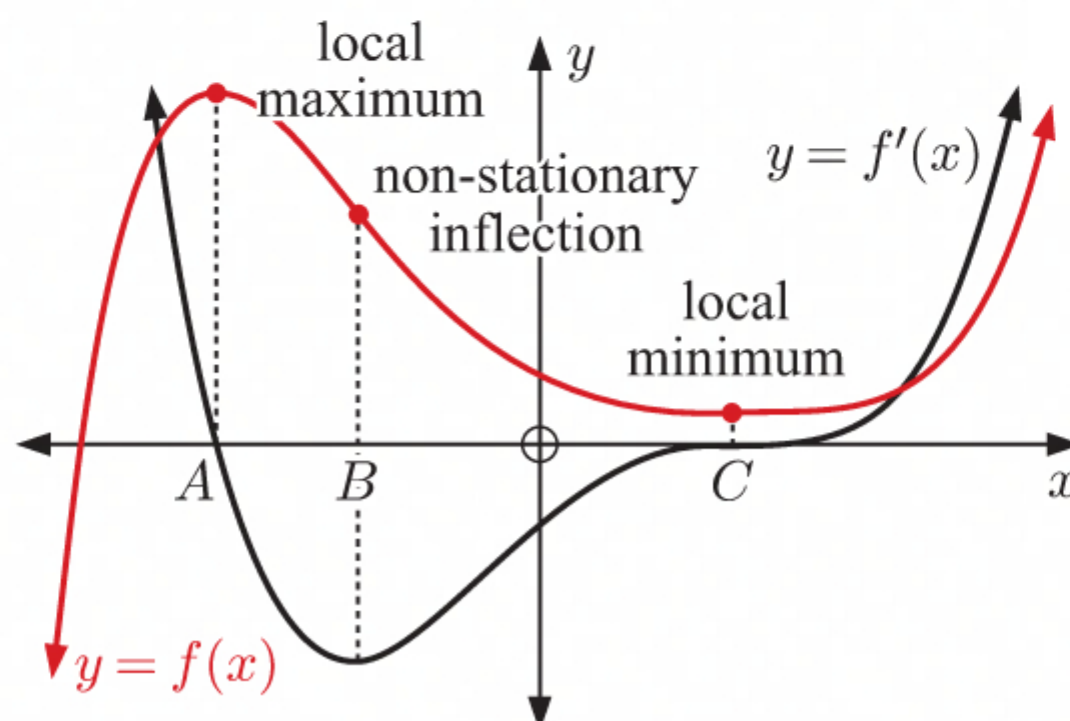
25 At $x = B$, $f''(x) = 0$ but $f'(x) \neq 0$

$\therefore f(x)$ has a non-stationary inflection point at $x = B$.

$f'(x)$ is above the x -axis for $x \leq A$ and $x \geq C$, and below the x -axis for $A \leq x \leq C$

$\therefore f(x)$ is increasing for $x \leq A$, decreasing for $A \leq x \leq C$, then increasing for $x \geq C$

$\therefore f(x)$ has a local maximum at $x = A$ and a local minimum at $x = C$.

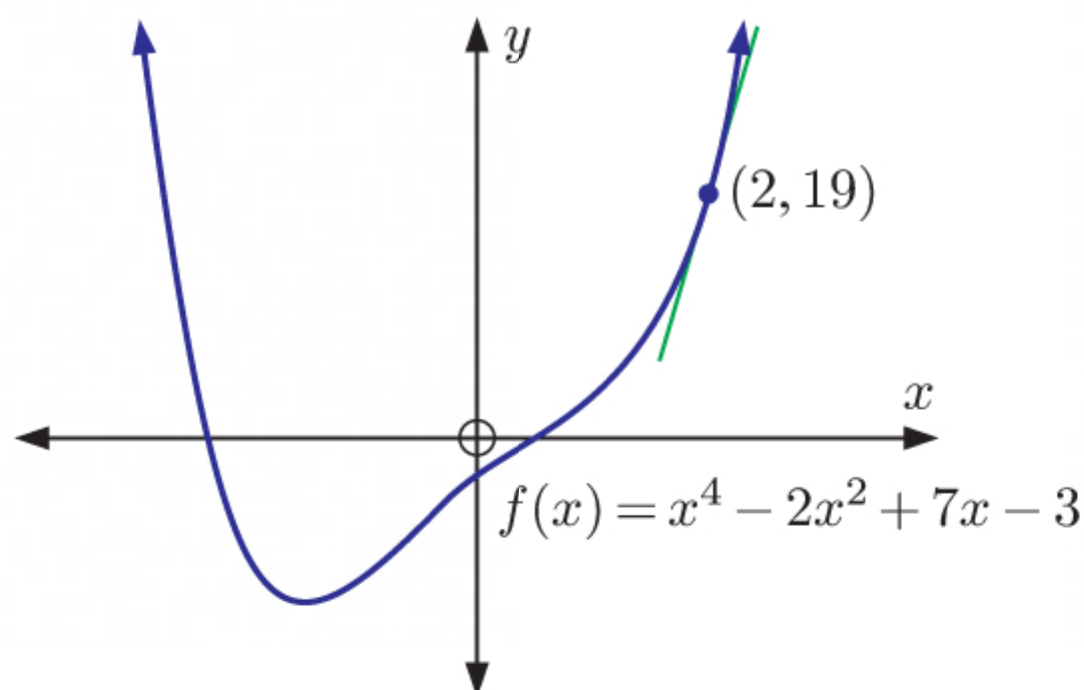


REVIEW SET 13B

$$\begin{aligned}
 \mathbf{1} \quad \mathbf{a} \quad & f(x) = x^4 - 2x^2 + 7x - 3 \\
 \therefore & f'(x) = 4x^3 - 4x + 7 \\
 \therefore & f'(2) = 4(2)^3 - 4(2) + 7 \\
 & = 32 - 8 + 7 \\
 & = 31
 \end{aligned}$$

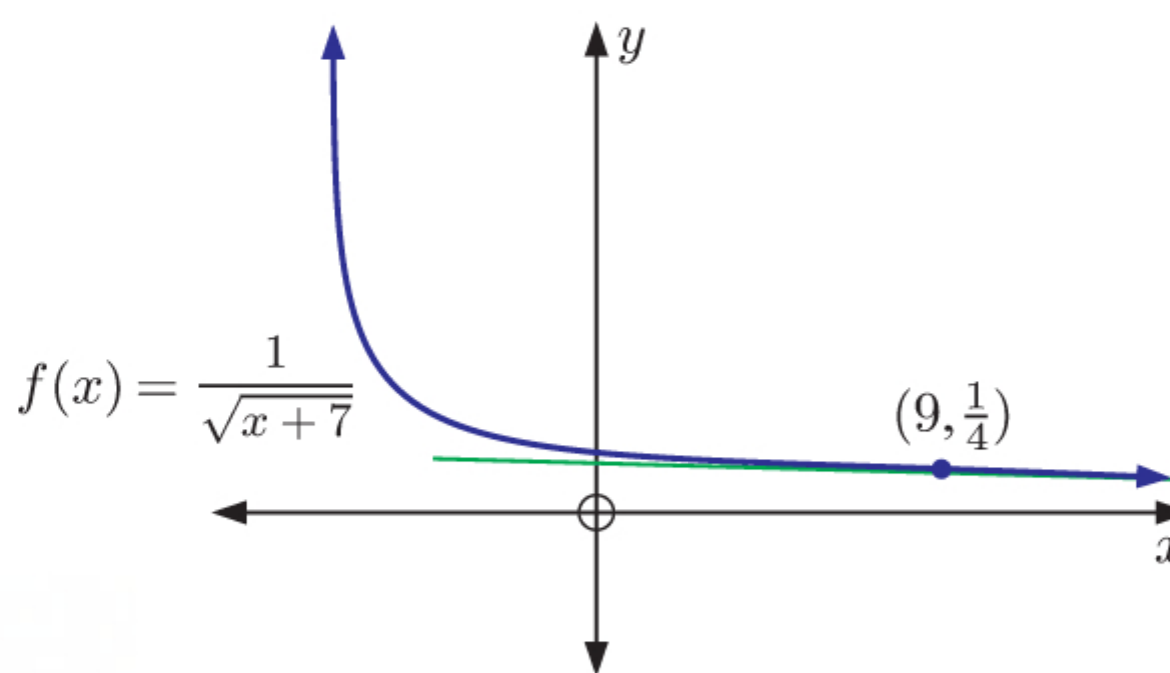
So, the tangent has equation

$$\begin{aligned}
 y &= 31(x - 2) + 19 \\
 \therefore y &= 31x - 43
 \end{aligned}$$



$$\begin{aligned}
 \mathbf{b} \quad & f(x) = \frac{1}{\sqrt{x+7}} = (x+7)^{-\frac{1}{2}} \\
 \therefore & f'(x) = -\frac{1}{2}(x+7)^{-\frac{3}{2}} \\
 & = -\frac{1}{2(x+7)^{\frac{3}{2}}} \\
 \therefore & f'(9) = -\frac{1}{2(9+7)^{\frac{3}{2}}} \\
 & = -\frac{1}{2(16)^{\frac{3}{2}}} \\
 & = -\frac{1}{128}
 \end{aligned}$$

So, the tangent has equation $x + 128y = 9 + 128\left(\frac{1}{4}\right)$
 $\therefore x + 128y = 41$



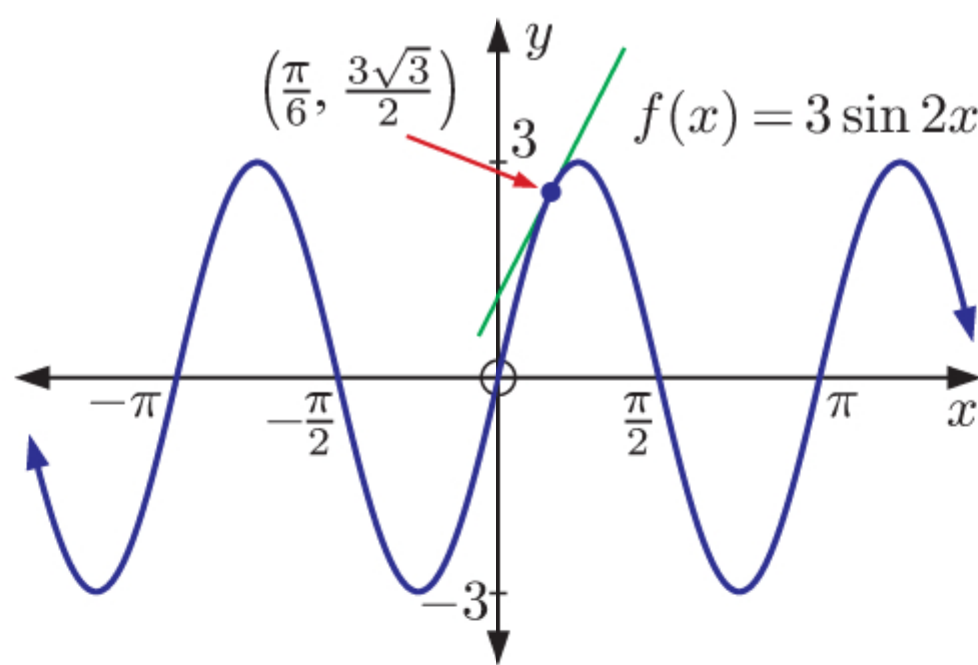
$$\begin{aligned}
 \mathbf{c} \quad & f(x) = 3 \sin 2x \\
 \therefore & f\left(\frac{\pi}{6}\right) = 3 \sin \frac{\pi}{3} \\
 & = 3\left(\frac{\sqrt{3}}{2}\right) \\
 & = \frac{3\sqrt{3}}{2} \\
 \therefore & \text{the point of contact is } \left(\frac{\pi}{6}, \frac{3\sqrt{3}}{2}\right).
 \end{aligned}$$

Now $f(x) = 3 \sin 2x$

$$\therefore f'(x) = 6 \cos 2x$$

$$\begin{aligned}
 \therefore f'\left(\frac{\pi}{6}\right) &= 6 \cos \frac{\pi}{3} \\
 &= 6\left(\frac{1}{2}\right) \\
 &= 3
 \end{aligned}$$

So, the tangent has equation $y = 3\left(x - \frac{\pi}{6}\right) + \frac{3\sqrt{3}}{2}$
 $= 3x + \frac{3\sqrt{3}}{2} - \frac{\pi}{2}$



d $f(x) = \frac{e^x}{2-x}$

$$\therefore f(0) = \frac{e^0}{2-0} = \frac{1}{2}$$

\therefore the point of contact is $(0, \frac{1}{2})$.

Now $f(x) = \frac{e^x}{2-x}$

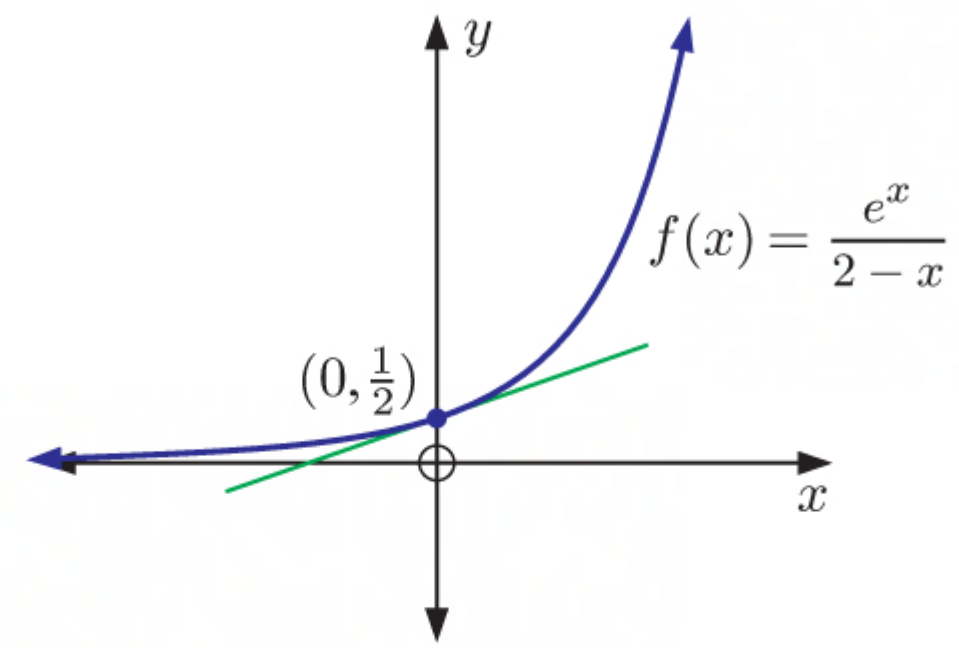
$$\therefore f'(x) = \frac{e^x(2-x) - e^x(-1)}{(2-x)^2} \quad \{\text{quotient rule}\}$$

$$= \frac{e^x(3-x)}{(2-x)^2}$$

$$\begin{aligned} \therefore f'(0) &= \frac{e^0(3-0)}{(2-0)^2} \\ &= \frac{3}{4} \end{aligned}$$

So, the tangent has equation $y = \frac{3}{4}(x-0) + \frac{1}{2}$

$$\therefore y = \frac{3}{4}x + \frac{1}{2}$$



2 a $y = \frac{1}{x^2} - \frac{2}{x} = x^{-2} - 2x^{-1}$

When $x = 1$, $y = \frac{1}{1^2} - \frac{2}{1} = 1 - 2 = -1$

So, the point of contact is $(1, -1)$.

$$\begin{aligned} \text{Now as } y &= x^{-2} - 2x^{-1}, \quad \frac{dy}{dx} = -2x^{-3} + 2x^{-2} \\ &= -\frac{2}{x^3} + \frac{2}{x^2} \end{aligned}$$

$$\therefore \text{ when } x = 1, \quad \frac{dy}{dx} = -\frac{2}{1^3} + \frac{2}{1^2} = -2 + 2 = 0$$

\therefore the normal at $(1, -1)$ has gradient which is undefined.

So, the normal must be a vertical line.

Since the normal passes through $(1, -1)$, then the equation of the normal is $x = 1$.

b $y = x \sin x$

$$\begin{aligned} \therefore \frac{dy}{dx} &= (1) \sin x + x(\cos x) \quad \{\text{product rule}\} \\ &= \sin x + x \cos x \end{aligned}$$

$$\begin{aligned} \therefore \text{ when } x = 0, \quad \frac{dy}{dx} &= \sin 0 + 0 \times \cos x \\ &= 0 \end{aligned}$$

\therefore the normal at $(0, 0)$ has gradient which is undefined.

So, the normal must be a vertical line.

Since the normal passes through $(0, 0)$, then the equation of the normal is $x = 0$.

3 $y = 2x^3 + ax + b$

$$\therefore \frac{dy}{dx} = 6x^2 + a$$

Since the gradient of the tangent at $(-2, 33)$ is 10, then $6(-2)^2 + a = 10$

$$\therefore 24 + a = 10$$

$$\therefore a = -14$$

$$\therefore y = 2x^3 - 14x + b$$

Since the curve passes through $(-2, 33)$, then $33 = 2(-2)^3 - 14(-2) + b$
 $= -16 + 28 + b$

$$\therefore b = 21$$

4 $y = 2 - \frac{7}{1+2x} = 2 - 7(1+2x)^{-1}$

$$\therefore \frac{dy}{dx} = 7(1+2x)^{-2} \times 2 \quad \{\text{chain rule}\}$$

$$= \frac{14}{(1+2x)^2}$$

The tangent is horizontal when the gradient $\frac{dy}{dx} = 0$.

But $\frac{14}{(1+2x)^2}$ is never 0, so $y = 2 - \frac{7}{1+2x}$ has no horizontal tangents.

5 a The tangent shown on the graph passes through $(0, 5)$ and $(5, 0)$.

\therefore the gradient of the tangent is $\frac{0-5}{5-0} = -1$, so

$$f'(3) = -1.$$

Also, since the tangent passes through $(0, 5)$, it has

equation $\frac{y-5}{x-0} = -1$

$$\therefore y - 5 = -x$$

$$\therefore y = -x + 5$$

So when $x = 3$, $y = -3 + 5 = 2$

\therefore the point of contact is $(3, 2)$, and hence $f(3) = 2$.

b $f(x)$ has the form $f(x) = ax^2 + bx + c$

The y -intercept is 14 $\therefore f(0) = 14$

$$\therefore a(0)^2 + b(0) + c = 14$$

$$\therefore c = 14$$

Now $f(3) = 2$ {from **a**}

$$\therefore a(3)^2 + b(3) + 14 = 2$$

$$\therefore 9a + 3b = -12 \quad \dots (1)$$

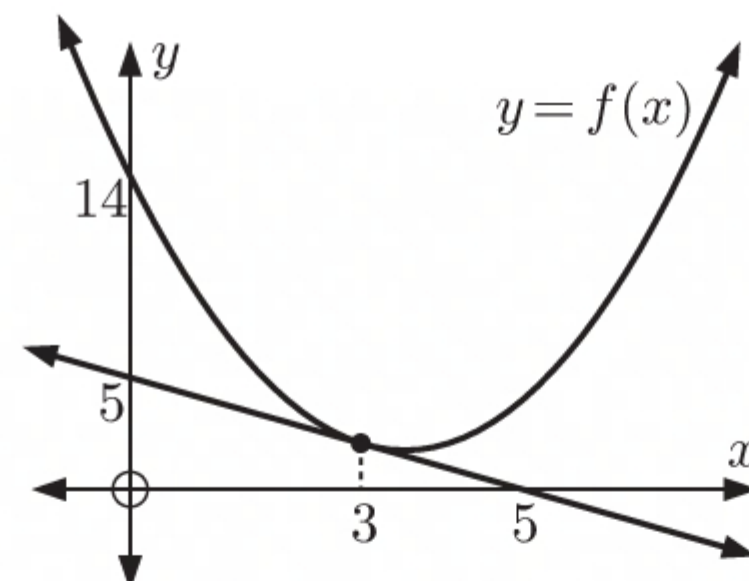
Also $f'(3) = -1$

and $f'(x) = 2ax + b$

$$\therefore 2a(3) + b = -1$$

$$\therefore 6a + b = -1$$

$$\therefore b = -6a - 1 \quad \dots (2)$$



$$\begin{aligned}
 \text{Substituting (2) into (1) gives } 9a + 3(-6a - 1) &= -12 \\
 \therefore 9a - 18a - 3 &= -12 \\
 \therefore -9a &= -9 \\
 \therefore a &= 1
 \end{aligned}$$

$$\begin{aligned}
 \text{Using (2), } b &= -6(1) - 1 \\
 \therefore b &= -7
 \end{aligned}$$

$$\text{So, } f(x) = x^2 - 7x + 14$$

6 $y = x^3 + ax^2 - 4x + 3$

a $\frac{dy}{dx} = 3x^2 + 2ax - 4$

The tangent at $x = 1$ is parallel to the line $y = 3x$, and $y = 3x$ has gradient 3.
 \therefore the tangent at $x = 1$ has gradient 3.

$$\begin{aligned}
 \therefore 3(1)^2 + 2a(1) - 4 &= 3 \\
 \therefore 3 + 2a - 4 &= 3 \\
 \therefore 2a &= 4 \\
 \therefore a &= 2
 \end{aligned}$$

b Since $a = 2$, $y = x^3 + 2x^2 - 4x + 3$ and $\frac{dy}{dx} = 3x^2 + 4x - 4$

$$\text{When } x = 1, y = 1^3 + 2(1)^2 - 4(1) + 3 = 2$$

$$\text{and } \frac{dy}{dx} = 3(1)^2 + 4(1) - 4 = 3$$

So, the point of contact is $(1, 2)$, and the tangent at $(1, 2)$ has gradient 3.

$$\begin{aligned}
 \therefore \text{ the tangent has equation } y &= 3(x - 1) + 2 \\
 \text{which is } y &= 3x - 1
 \end{aligned}$$

c The tangent meets the curve again when $x^3 + 2x^2 - 4x + 3 = 3x - 1$

$$\therefore x^3 + 2x^2 - 7x + 4 = 0$$

$$\therefore (x - 1)^2(x + 4) = 0$$

$$\{(x - 1)^2 \text{ is a factor since the tangent to the curve is at } x = 1\}$$

$$\text{When } x = -4, y = (-4)^3 + 2(-4)^2 - 4(-4) + 3 = -13$$

\therefore the tangent meets the curve again at $(-4, -13)$.

7 $y = x^2 - 4x + 2$

$$\therefore \frac{dy}{dx} = 2x - 4$$

$$\begin{aligned}
 \text{When } x = 3, y &= (3)^2 - 4(3) + 2 & \text{and } \frac{dy}{dx} &= 2(3) - 4 \\
 &= 9 - 12 + 2 & &= 6 - 4 \\
 &= -1 & &= 2
 \end{aligned}$$

So, the point of contact is $(3, -1)$ and the normal at $(3, -1)$ has gradient $-\frac{1}{2}$.

$$\therefore \text{ the normal has equation } y = -\frac{1}{2}(x - 3) - 1$$

$$\therefore y = -\frac{1}{2}x + \frac{3}{2} - 1$$

$$\therefore y = -\frac{1}{2}x + \frac{1}{2}$$

The normal meets the curve again when $-\frac{1}{2}x + \frac{1}{2} = x^2 - 4x + 2$
 $\therefore -x + 1 = 2x^2 - 8x + 4$
 $\therefore 2x^2 - 7x + 3 = 0$
 $\therefore (2x - 1)(x - 3) = 0$

When $x = \frac{1}{2}$, $y = \left(\frac{1}{2}\right)^2 - 4\left(\frac{1}{2}\right) + 2$
 $= \frac{1}{4} - 2 + 2$
 $= \frac{1}{4}$

\therefore the normal meets the curve again at $\left(\frac{1}{2}, \frac{1}{4}\right)$.

8 a

$$y = \frac{1}{\sin x}$$

When $x = \frac{\pi}{3}$, $y = \frac{1}{\sin \frac{\pi}{3}}$
 $= \frac{1}{\frac{\sqrt{3}}{2}} = \frac{2}{\sqrt{3}}$

\therefore the point of contact is $\left(\frac{\pi}{3}, \frac{2}{\sqrt{3}}\right)$.

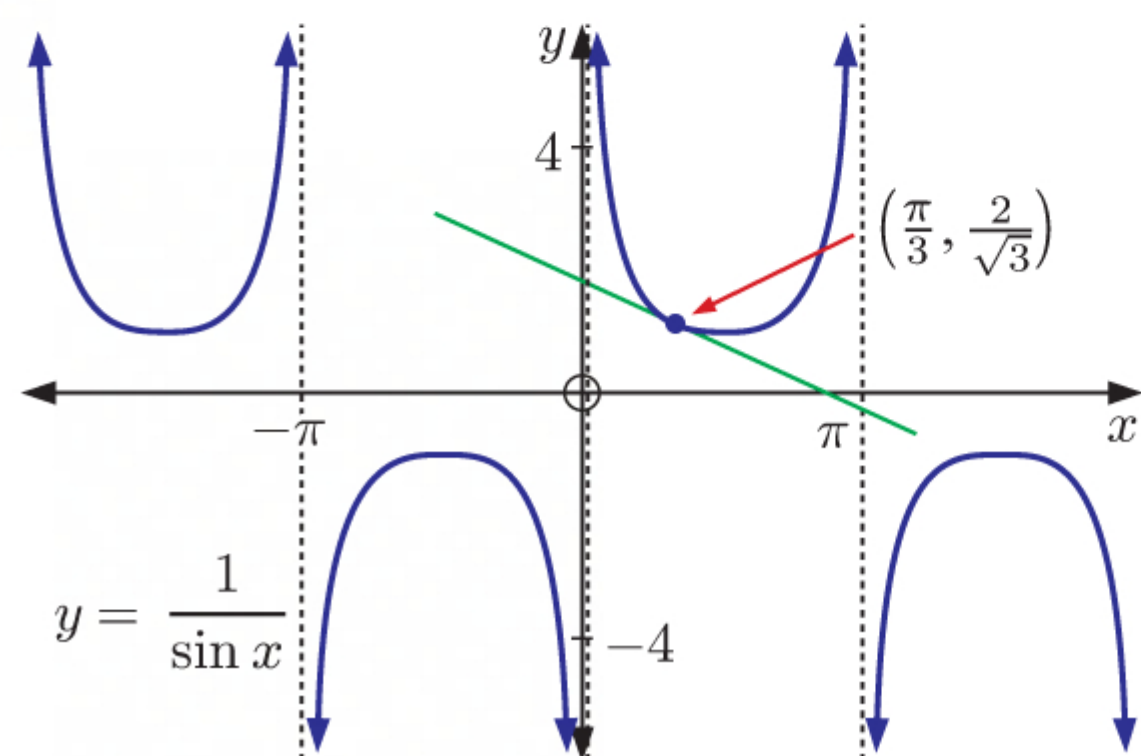
Now $y = \frac{1}{\sin x} = (\sin x)^{-1}$

$$\therefore \frac{dy}{dx} = -(\sin x)^{-2} \cos x \quad \{\text{chain rule}\}$$

$$= -\frac{\cos x}{\sin^2 x}$$

When $x = \frac{\pi}{3}$, $\frac{dy}{dx} = -\frac{\cos \frac{\pi}{3}}{\sin^2(\frac{\pi}{3})}$
 $= -\frac{\frac{1}{2}}{\left(\frac{\sqrt{3}}{2}\right)^2}$
 $= -\frac{\frac{1}{2}}{\frac{3}{4}} = -\frac{2}{3}$

So, the tangent has equation $y = -\frac{2}{3}\left(x - \frac{\pi}{3}\right) + \frac{2}{\sqrt{3}}$
 $\therefore 3y = -2\left(x - \frac{\pi}{3}\right) + 2\sqrt{3}$
 $= -2x + \frac{2\pi}{3} + 2\sqrt{3}$
 $\therefore 2x + 3y = \frac{2\pi}{3} + 2\sqrt{3}$



b

$$y = \cos \frac{x}{2}$$

$$\text{When } x = \frac{\pi}{2}, \quad y = \cos \frac{\pi}{4} = \frac{1}{\sqrt{2}}$$

$$\therefore \text{ the point of contact is } \left(\frac{\pi}{2}, \frac{1}{\sqrt{2}} \right).$$

$$\text{Now } y = \cos \frac{x}{2}$$

$$\therefore \frac{dy}{dx} = -\frac{1}{2} \sin \frac{x}{2}$$

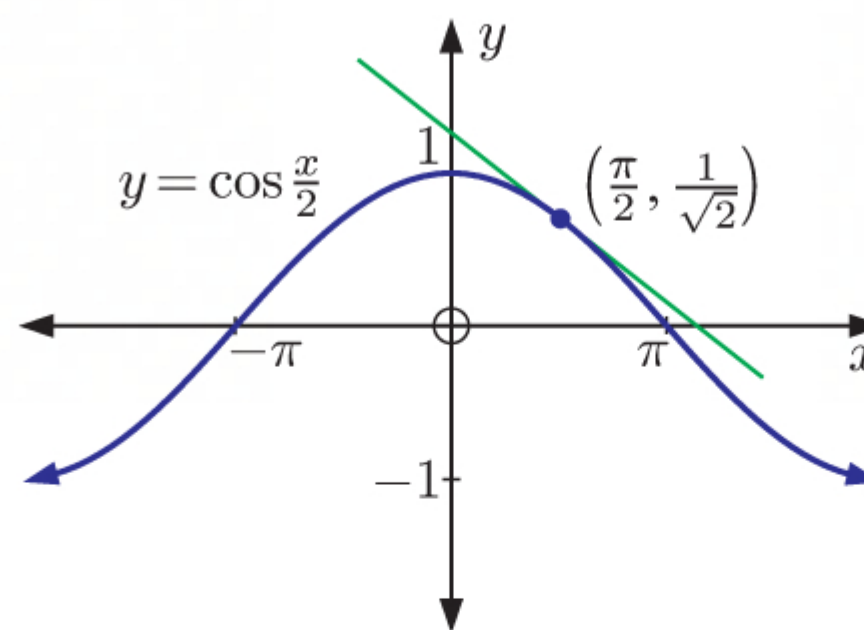
$$\begin{aligned} \text{When } x = \frac{\pi}{2}, \quad \frac{dy}{dx} &= -\frac{1}{2} \sin \frac{\pi}{4} \\ &= -\frac{1}{2\sqrt{2}} \end{aligned}$$

$$\text{So, the tangent has equation } y = -\frac{1}{2\sqrt{2}} \left(x - \frac{\pi}{2} \right) + \frac{1}{\sqrt{2}}$$

$$\therefore 2\sqrt{2}y = -\left(x - \frac{\pi}{2} \right) + 2$$

$$= -x + \frac{\pi}{2} + 2$$

$$\therefore x + 2\sqrt{2}y = \frac{\pi}{2} + 2$$



- 9 The curves $y = \sqrt{3x+1}$ and $y = \sqrt{5x-x^2}$ meet when

$$\sqrt{3x+1} = \sqrt{5x-x^2}$$

$$\text{Squaring both sides, } 3x+1 = 5x-x^2$$

$$\therefore x^2 - 2x + 1 = 0$$

$$\therefore (x-1)^2 = 0$$

$$\therefore x = 1$$

When $x = 1$, $y = \sqrt{3+1} = 2$, so the curves meet at $(1, 2)$.

$$\text{For } y = \sqrt{3x+1} = (3x+1)^{\frac{1}{2}}$$

$$\frac{dy}{dx} = \frac{1}{2}(3x+1)^{-\frac{1}{2}}(3) \quad \{\text{chain rule}\}$$

$$= \frac{3}{2\sqrt{3x+1}}$$

$$\therefore \text{ at } (1, 2), \quad \frac{dy}{dx} = \frac{3}{2\sqrt{3+1}} = \frac{3}{4}$$

$$\text{For } y = \sqrt{5x-x^2} = (5x-x^2)^{\frac{1}{2}}$$

$$\frac{dy}{dx} = \frac{1}{2}(5x-x^2)^{-\frac{1}{2}}(5-2x) \quad \{\text{chain rule}\}$$

$$= \frac{5-2x}{2\sqrt{5x-x^2}}$$

$$\therefore \text{ at } (1, 2), \quad \frac{dy}{dx} = \frac{5-2}{2\sqrt{5-1}} = \frac{3}{4}$$

\therefore the curves have a common tangent at their point of intersection.

The common tangent has equation $y = \frac{3}{4}(x-1) + 2$

$$\therefore y = \frac{3}{4}x + \frac{5}{4}$$

$$\begin{aligned}
 10 \quad y &= \frac{ax+b}{\sqrt{x}} \\
 &= a\sqrt{x} + \frac{b}{\sqrt{x}} = ax^{\frac{1}{2}} + bx^{-\frac{1}{2}} \\
 \therefore \frac{dy}{dx} &= \frac{a}{2}x^{-\frac{1}{2}} - \frac{b}{2}x^{-\frac{3}{2}} \\
 &= \frac{a}{2\sqrt{x}} - \frac{b}{2x\sqrt{x}}
 \end{aligned}$$

The equation of the tangent at $x = 1$ is $2x - y = 1$
 which is $y = 2x - 1$

so the gradient of the tangent is 2

$$\begin{aligned}
 \therefore \text{ at } x = 1, \quad \frac{dy}{dx} &= \frac{a}{2} - \frac{b}{2} = 2 \\
 \therefore a - b &= 4 \\
 \therefore a &= b + 4 \quad \dots (*)
 \end{aligned}$$

Also at $x = 1$, the tangent touches the curve

$$\begin{aligned}
 \therefore \frac{a(1)+b}{\sqrt{1}} &= 2(1) - 1 \\
 \therefore a + b &= 1 \\
 \therefore b + 4 + b &= 1 \quad \{\text{using } (*)\} \\
 \therefore 2b &= -3 \\
 \therefore b &= -\frac{3}{2} \quad \text{and} \quad a = -\frac{3}{2} + 4 = \frac{5}{2}
 \end{aligned}$$

$$\begin{aligned}
 11 \quad f(x) &= x^4 - 4x^3 - 8x^2 + 5 \\
 \therefore f'(x) &= 4x^3 - 12x^2 - 16x \\
 &= 4x(x^2 - 3x - 4) \\
 &= 4x(x+1)(x-4)
 \end{aligned}$$

which has sign diagram: $\begin{array}{ccccccc} & - & | & + & | & - & | & + \\ & & -1 & & 0 & & 4 & & x \end{array} \quad f'(x)$

a $f(x)$ is increasing for $-1 \leq x \leq 0$ and $x \geq 4$.

b $f(x)$ is decreasing for $x \leq -1$ and $0 \leq x \leq 4$.

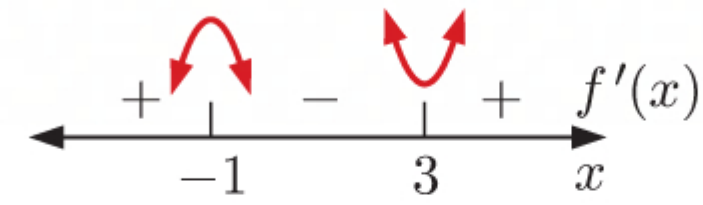
$$\begin{aligned}
 12 \quad f(x) &= x^3 - 3x^2 + ax + 50 \\
 \therefore f'(x) &= 3x^2 - 6x + a
 \end{aligned}$$

a $f(x)$ has a stationary point at $x = 3$

$$\begin{aligned}
 \therefore f'(3) &= 0 \\
 \therefore 3(3)^2 - 6(3) + a &= 0 \\
 \therefore 27 - 18 + a &= 0 \\
 \therefore a &= -9
 \end{aligned}$$

b Since $a = -9$, then $f(x) = x^3 - 3x^2 - 9x + 50$
 and $f'(x) = 3x^2 - 6x - 9$
 $= 3(x^2 - 2x - 3)$
 $= 3(x + 1)(x - 3)$

which has sign diagram:



When $x = -1$, $f(-1) = (-1)^3 - 3(-1)^2 - 9(-1) + 50 = 55$

When $x = 3$, $f(3) = 3^3 - 3(3)^2 - 9(3) + 50 = 23$

So, there is a local maximum at $(-1, 55)$ and a local minimum at $(3, 23)$.

13 $f(x) = x^3 - 4x^2 + 4x$

a $f(0) = 0$, so the y -intercept is 0.

When $f(x) = 0$, $x^3 - 4x^2 + 4x = 0$

$\therefore x(x^2 - 4x + 4) = 0$

$\therefore x(x - 2)^2 = 0$

$\therefore x = 0$ or 2

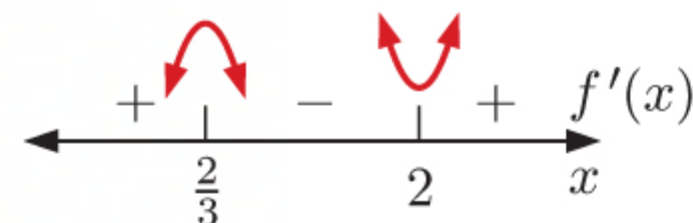
So, the x -intercepts are 0 and 2.

b $f'(x) = 3x^2 - 8x + 4$

$f'(x) = 0$ when $3x^2 - 8x + 4 = 0$

$\therefore (3x - 2)(x - 2) = 0$

$f'(x)$ has sign diagram:



When $x = \frac{2}{3}$, $f(\frac{2}{3}) = (\frac{2}{3})^3 - 4(\frac{2}{3})^2 + 4(\frac{2}{3}) = \frac{8}{27} - \frac{16}{9} + \frac{8}{3} = \frac{32}{27}$

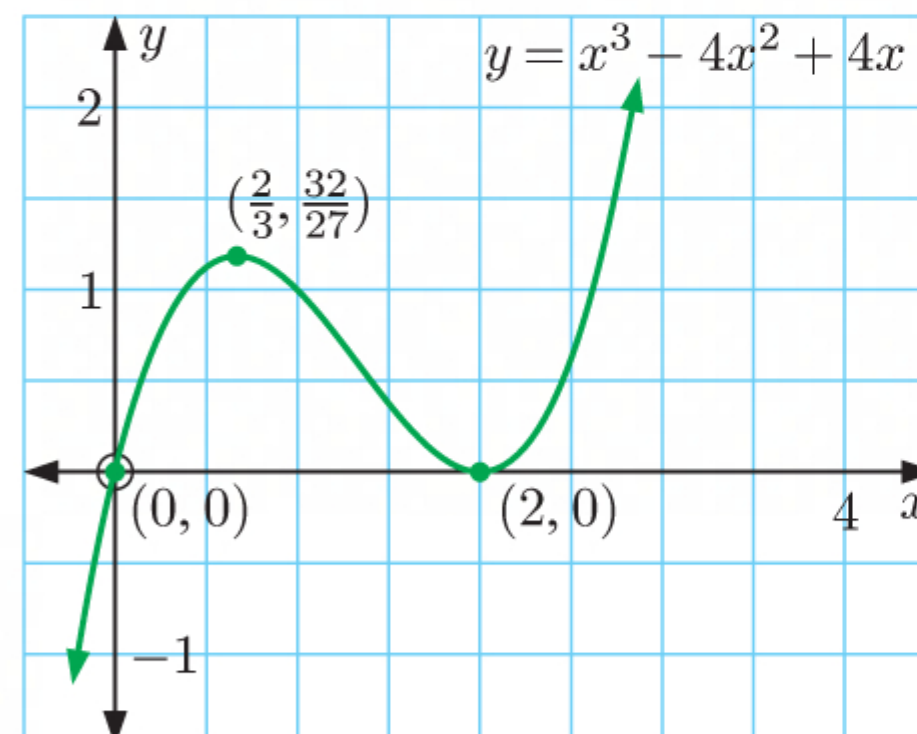
When $x = 2$, $f(2) = 2^3 - 4(2)^2 + 4(2) = 8 - 16 + 8 = 0$

So, there is a local maximum at $(\frac{2}{3}, \frac{32}{27})$, and a local minimum at $(2, 0)$.

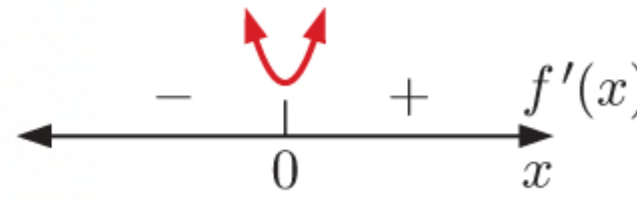
c From the sign diagram of $f'(x)$ in **b**, $f(x)$ is increasing for $x \leq \frac{2}{3}$ and $x \geq 2$, and $f(x)$ is decreasing for $\frac{2}{3} \leq x \leq 2$.

d As $x \rightarrow \infty$, $y \rightarrow \infty$,
 as $x \rightarrow -\infty$, $y \rightarrow -\infty$.

e



14 a $f(x) = e^x - x$
 $\therefore f'(x) = e^x - 1$ which has sign diagram:



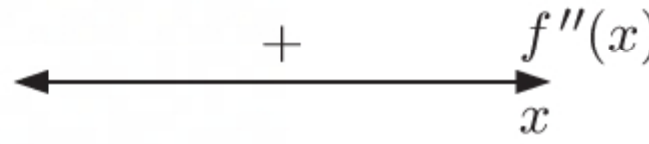
Now $f(0) = e^0 - 0 = 1$

$\therefore y = f(x)$ has a local minimum at $(0, 1)$.

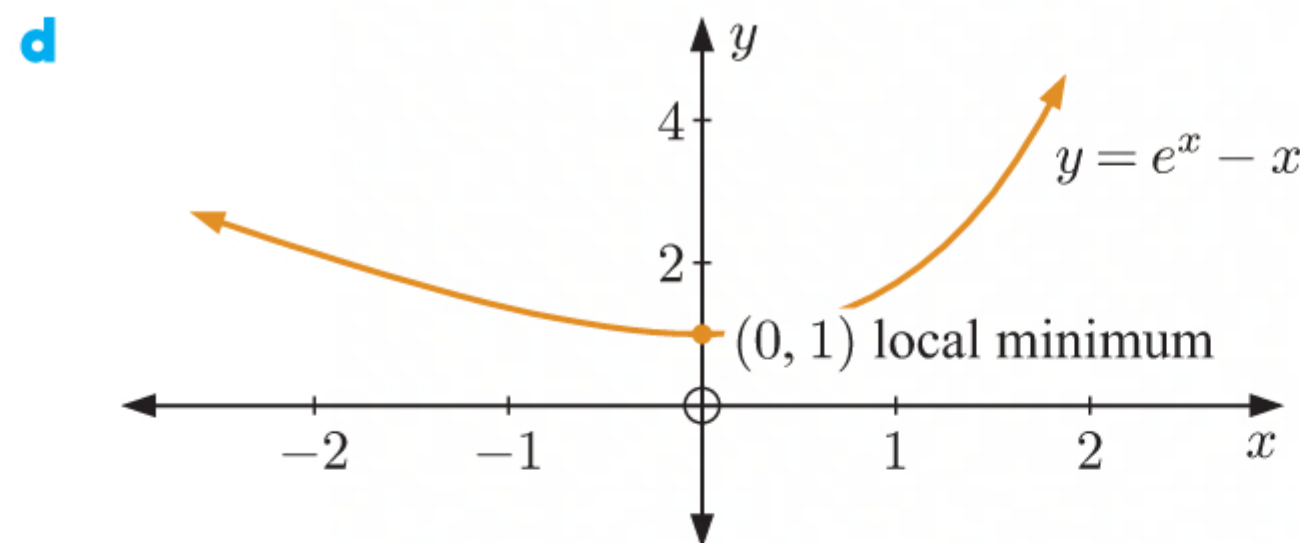
b As $x \rightarrow \infty$, $e^x \rightarrow \infty$ {at a much faster rate than x }

\therefore as $x \rightarrow \infty$, $f(x) \rightarrow \infty$

c $f''(x) = e^x$ which has sign diagram:



$\therefore f(x)$ is concave up for all $x \in \mathbb{R}$.



e $y = f(x)$ has a local minimum at $(0, 1)$

$$\therefore f(x) \geq 1$$

$$\therefore e^x - x \geq 1$$

$$\therefore e^x \geq x + 1 \quad \text{for all } x$$

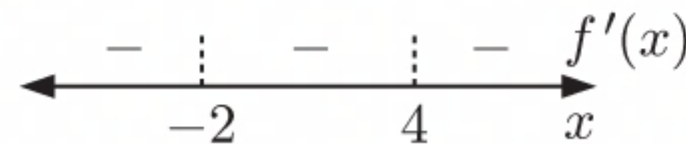
15 a $f(x) = \frac{x+1}{x^2-2x-8}$
 $\therefore f'(x) = \frac{(1)(x^2-2x-8) - (x+1)(2x-2)}{(x^2-2x-8)^2}$ {quotient rule}

$$= \frac{x^2 - 2x - 8 - (2x^2 - 2)}{(x^2 - 2x - 8)^2}$$

$$= \frac{x^2 - 2x - 8 - 2x^2 + 2}{(x^2 - 2x - 8)^2}$$

$$= \frac{-x^2 - 2x - 6}{(x^2 - 2x - 8)^2}$$

$$= -\frac{x^2 + 2x + 6}{(x^2 - 2x - 8)^2} \quad \text{which has sign diagram:}$$



b $f'(x) = -\frac{x^2 + 2x + 1 + 5}{(x^2 - 2x - 8)^2}$
 $= -\frac{(x+1)^2 + 5}{(x^2 - 2x - 8)^2} < 0$ for all $x \in \mathbb{R}$, $x \neq -2, 4$

$\therefore f(x)$ is decreasing for all $x \in \mathbb{R}$, $x \neq -2, 4$

$\therefore f(x)$ is never increasing.

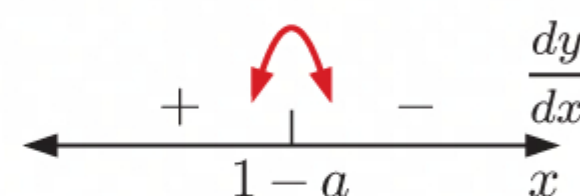
16 $y = \frac{x+a}{e^x}$

$\therefore \frac{dy}{dx} = \frac{(1)e^x - (x+a)e^x}{(e^x)^2}$ {quotient rule}

$$= \frac{e^x - xe^x - ae^x}{e^{2x}}$$

$$= \frac{e^x(1-x-a)}{e^{2x}}$$

$$= \frac{(1-a)-x}{e^x} \quad \text{which has sign diagram:}$$



$$\frac{dy}{dx} = 0 \text{ when } x = 1 - a$$

$$\begin{aligned} \text{When } x = 1 - a, \quad y &= \frac{(1 - a) + a}{e^{1-a}} \\ &= \frac{1}{e^{1-a}} \\ &= e^{a-1} \end{aligned}$$

\therefore the stationary point of $y = \frac{x+a}{e^x}$ where a is a constant, is a local maximum $(1 - a, e^{a-1})$.

17

$$\begin{aligned} f(x) &= \frac{\ln(ax)}{bx} \\ &= \frac{\ln a + \ln x}{bx} \\ \therefore f'(x) &= \frac{\left(\frac{1}{x}\right) \times bx - (\ln a + \ln x) \times b}{b^2x^2} \quad \{\text{quotient rule}\} \\ &= \frac{b - b(\ln a + \ln x)}{b^2x^2} \\ &= \frac{b(1 - \ln a - \ln x)}{b^2x^2} \\ &= \frac{1 - \ln a - \ln x}{bx^2} \end{aligned}$$

$$f'(x) = 0 \text{ when } \ln x = 1 - \ln a$$

$$\begin{aligned} \therefore \ln\left(\frac{e}{2}\right) &= 1 - \ln a \quad \left\{ \left(\frac{e}{2}, \frac{2}{3e}\right) \text{ is a stationary point} \right\} \\ \therefore \ln e - \ln 2 &= 1 - \ln a \\ \therefore 1 - \ln 2 &= 1 - \ln a \\ \therefore a &= 2 \end{aligned}$$

$$\text{Now } f\left(\frac{e}{2}\right) = \frac{2}{3e}$$

$$\therefore \frac{\ln(2 \times \frac{e}{2})}{b(\frac{e}{2})} = \frac{2}{3e}$$

$$\therefore \frac{\ln e}{\frac{be}{2}} = \frac{2}{3e}$$

$$\therefore 1 \times \frac{2}{be} = \frac{2}{3e}$$

$$\therefore b = 3$$

18

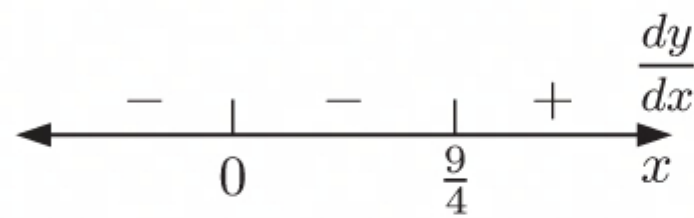
$$\begin{aligned} f(x) &= -\frac{1}{2}x^4 + x^3 + 6x^2 - 3x + 2 \\ \therefore f'(x) &= -2x^3 + 3x^2 + 12x - 3 \quad \text{which has sign diagram: } \begin{array}{ccccccc} & + & & - & & + & & - \\ & | & & | & & | & & | \\ \longleftarrow & -1.96 & & 0.238 & & 3.22 & & \longrightarrow x \end{array} f'(x) \\ \therefore f''(x) &= -6x^2 + 6x + 12 \\ &= -6(x^2 - x - 2) \\ &= -6(x+1)(x-2) \quad \text{which has sign diagram: } \begin{array}{ccccccc} & - & & + & & - \\ & | & & | & & | \\ \longleftarrow & -1 & & 2 & & & \longrightarrow x \end{array} f''(x) \end{aligned}$$

a $f(x)$ is increasing for $x \leq -1.96$ and $0.238 \leq x \leq 3.22$.

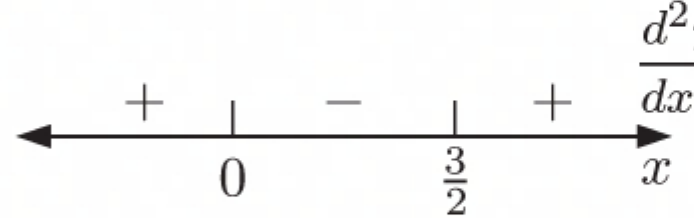
b $f(x)$ is decreasing for $-1.96 \leq x \leq 0.238$ and $x \geq 3.22$.

- c $f(x)$ is concave upwards for $-1 \leq x \leq 2$.
- d $f(x)$ is concave downwards for $x \leq -1$ and $x \geq 2$.

19 a $y = x^4 - 3x^3 + 9$

$$\begin{aligned} \therefore \frac{dy}{dx} &= 4x^3 - 9x^2 \\ &= x^2(4x - 9) \quad \text{which has sign diagram:} \end{aligned}$$


$$\therefore \frac{d^2y}{dx^2} = 12x^2 - 18x$$

$$= 6x(2x - 3) \quad \text{which has sign diagram:}$$


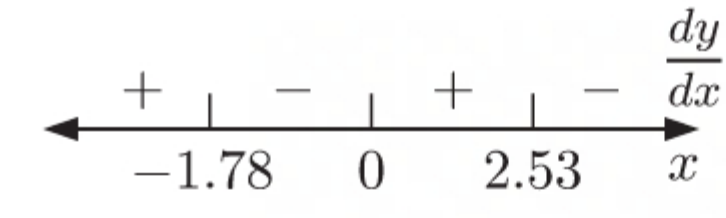
Since the sign of $\frac{d^2y}{dx^2}$ changes at $x = 0$ and $x = \frac{3}{2}$, both of these points are points of inflection.

When $x = 0$, $y = (0)^4 - 3(0)^3 + 9 = 9$
and $\frac{dy}{dx} = 0$

When $x = \frac{3}{2}$, $y = \left(\frac{3}{2}\right)^4 - 3\left(\frac{3}{2}\right)^3 + 9$
 $= \frac{81}{16} - \frac{81}{8} + 9$
 $= \frac{63}{16}$
and $\frac{dy}{dx} \neq 0$

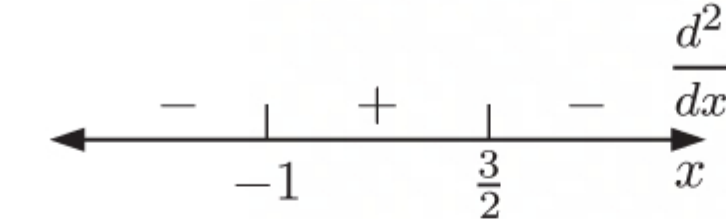
$\therefore (0, 9)$ is a stationary inflection point, and $\left(\frac{3}{2}, \frac{63}{16}\right)$ is a non-stationary inflection point.

b $y = -x^4 + x^3 + 9x^2 + 1$

$$\therefore \frac{dy}{dx} = -4x^3 + 3x^2 + 18x \quad \text{which has sign diagram:}$$


$$\therefore \frac{d^2y}{dx^2} = -12x^2 + 6x + 18$$

$$= -6(2x^2 - x - 3)$$

$$= -6(2x - 3)(x + 1) \quad \text{which has sign diagram:}$$


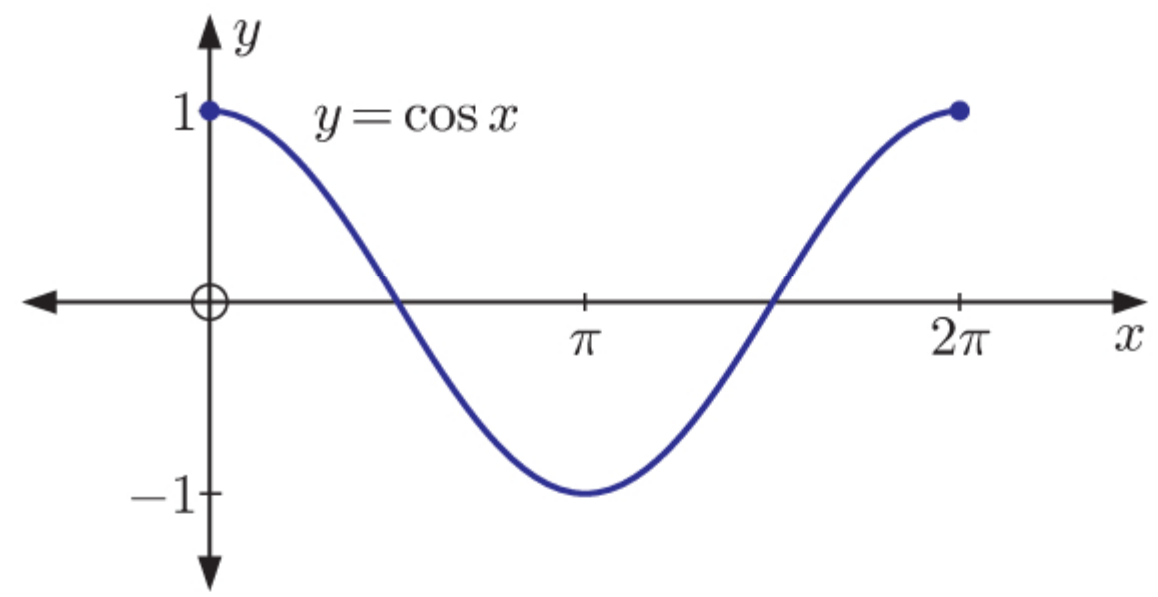
Since the sign of $\frac{d^2y}{dx^2}$ changes at $x = -1$ and $x = \frac{1}{2}$, both of these points are points of inflection.

When $x = -1$,
 $y = -(-1)^4 + (-1)^3 + 9(-1)^2 + 1$
 $= -1 - 1 + 9 + 1$
 $= 8$
and $\frac{dy}{dx} \neq 0$

When $x = \frac{3}{2}$,
 $y = -\left(\frac{3}{2}\right)^4 + \left(\frac{3}{2}\right)^3 + 9\left(\frac{3}{2}\right)^2 + 1$
 $= -\frac{81}{16} + \frac{27}{8} + \frac{81}{4} + 1$
 $= \frac{313}{16}$
and $\frac{dy}{dx} \neq 0$

$\therefore (-1, 8)$ and $\left(\frac{3}{2}, \frac{313}{16}\right)$ are non-stationary inflection points.

- 20 a** $f(x) = \sqrt{\cos x}$, $0 \leq x \leq 2\pi$ is defined
when $\cos x \geq 0$
 $\therefore 0 \leq x \leq \frac{\pi}{2}$ and $\frac{3\pi}{2} \leq x \leq 2\pi$



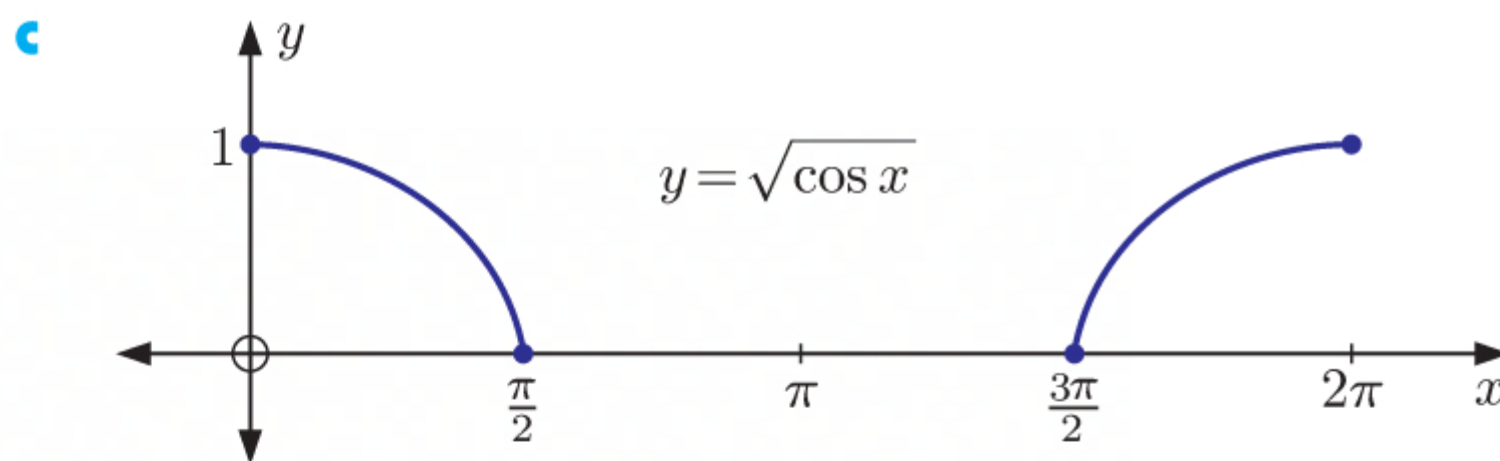
b $f(x) = \sqrt{\cos x} = (\cos x)^{\frac{1}{2}}$
 $\therefore f'(x) = \frac{1}{2}(\cos x)^{-\frac{1}{2}}(-\sin x)$
 $= -\frac{\sin x}{2\sqrt{\cos x}}$

From **a**, since $f(x)$ is only defined when $0 \leq x \leq \frac{\pi}{2}$ and $\frac{3\pi}{2} \leq x \leq 2\pi$, we only consider these values of x .

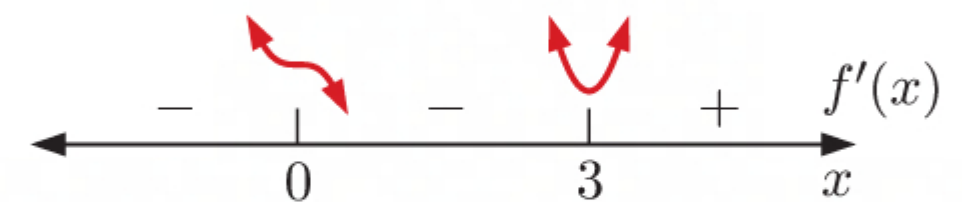
When $0 \leq x < \frac{\pi}{2}$, $f'(x) \leq 0$

When $\frac{3\pi}{2} < x \leq 2\pi$, $f'(x) \geq 0$

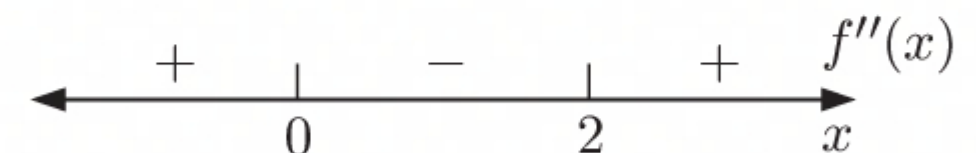
$\therefore f(x)$ is increasing for $\frac{3\pi}{2} \leq x \leq 2\pi$, and decreasing for $0 \leq x \leq \frac{\pi}{2}$.



- 21 a** $f(x) = x^4 - 4x^3 + 7$
 $\therefore f'(x) = 4x^3 - 12x^2$
 $= 4x^2(x - 3)$ which has sign diagram:



$\therefore f''(x) = 12x^2 - 24x$
 $= 12x(x - 2)$ which has sign diagram:



b $f(3) = 3^4 - 4(3)^3 + 7$
 $= 81 - 108 + 7$
 $= -20$

$\therefore (3, -20)$ is a local minimum.

- c** Since the signs of $f''(x)$ change about $x = 0$ and $x = 2$, these are points of inflection.

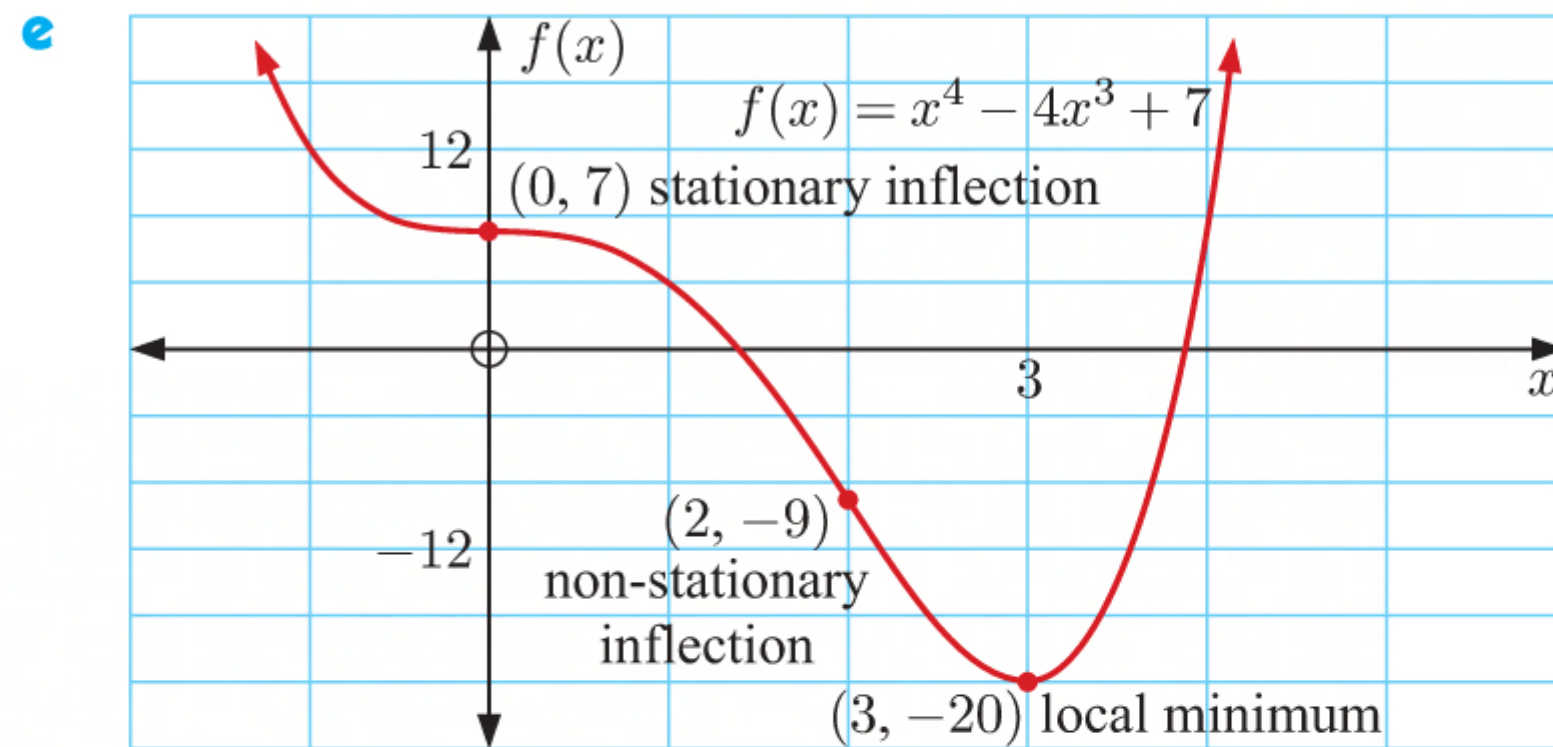
$f(0) = 7$ and $f'(0) = 0$

$\therefore (0, 7)$ is a stationary inflection.

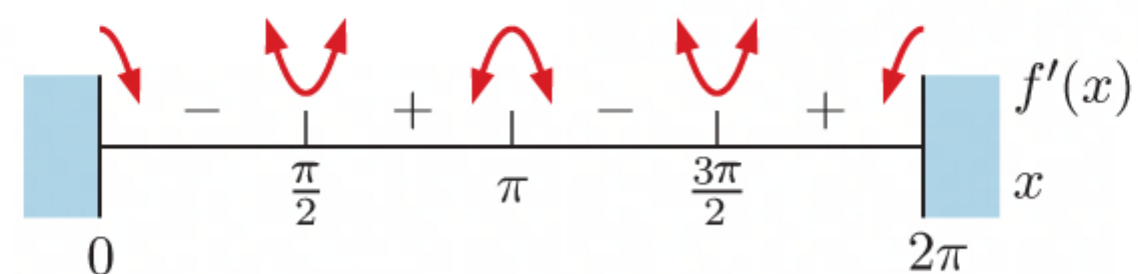
$f(2) = 2^4 - 4(2)^3 + 7$ and $f'(2) = 4(2)^3 - 12(2)^2$
 $= 16 - 32 + 7$ $= 32 - 48 \neq 0$
 $= -9$

$\therefore (2, -9)$ is a non-stationary inflection.

- d** **i** $f(x)$ is increasing for $x \geq 3$. **ii** $f(x)$ is decreasing for $x \leq 3$.
iii $f(x)$ is concave up for $x \leq 0$ and $x \geq 2$.
iv $f(x)$ is concave down for $0 \leq x \leq 2$.



- 22 a** $f(x) = \cos^2 x$, $0 \leq x \leq 2\pi$
 $\therefore f'(x) = 2 \cos x(-\sin x)$
 $= -2 \sin x \cos x$
 $= -\sin 2x$ which has sign diagram:



$$f'(x) = 0 \text{ when } -\sin 2x = 0$$

$$\therefore 2x = 0 + k\pi, \quad k \in \mathbb{Z}$$

$$\therefore x = 0 + \frac{k\pi}{2}$$

$$\therefore x = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, \text{ or } 2\pi$$

$$f(0) = \cos^2 0 = 1$$

$$f\left(\frac{\pi}{2}\right) = \cos^2\left(\frac{\pi}{2}\right) = 0$$

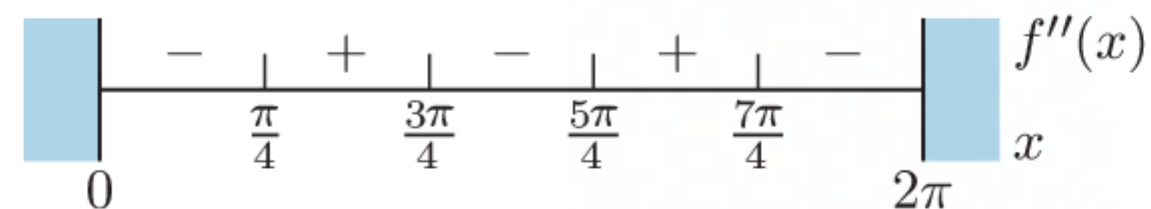
$$f(\pi) = \cos^2 \pi = 1$$

$$f\left(\frac{3\pi}{2}\right) = \cos^2\left(\frac{3\pi}{2}\right) = 0$$

$$f(2\pi) = \cos^2 2\pi = 1$$

\therefore there are local maxima at $(0, 1)$, $(\pi, 1)$, $(2\pi, 1)$, and local minima at $(\frac{\pi}{2}, 0)$, $(\frac{3\pi}{2}, 0)$.

- b** $f''(x) = -\cos 2x \times 2$
 $= -2 \cos 2x$ which has sign diagram:



$$f''(x) = 0 \text{ when } -2 \cos 2x = 0$$

$$\therefore 2x = \frac{\pi}{2} + k\pi, \quad k \in \mathbb{Z}$$

$$\therefore x = \frac{\pi}{4} + \frac{k\pi}{2}$$

$$\therefore x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \text{ or } \frac{7\pi}{4}$$

$$f\left(\frac{\pi}{4}\right) = \cos^2\left(\frac{\pi}{4}\right) = \left(\frac{1}{\sqrt{2}}\right)^2 = \frac{1}{2}$$

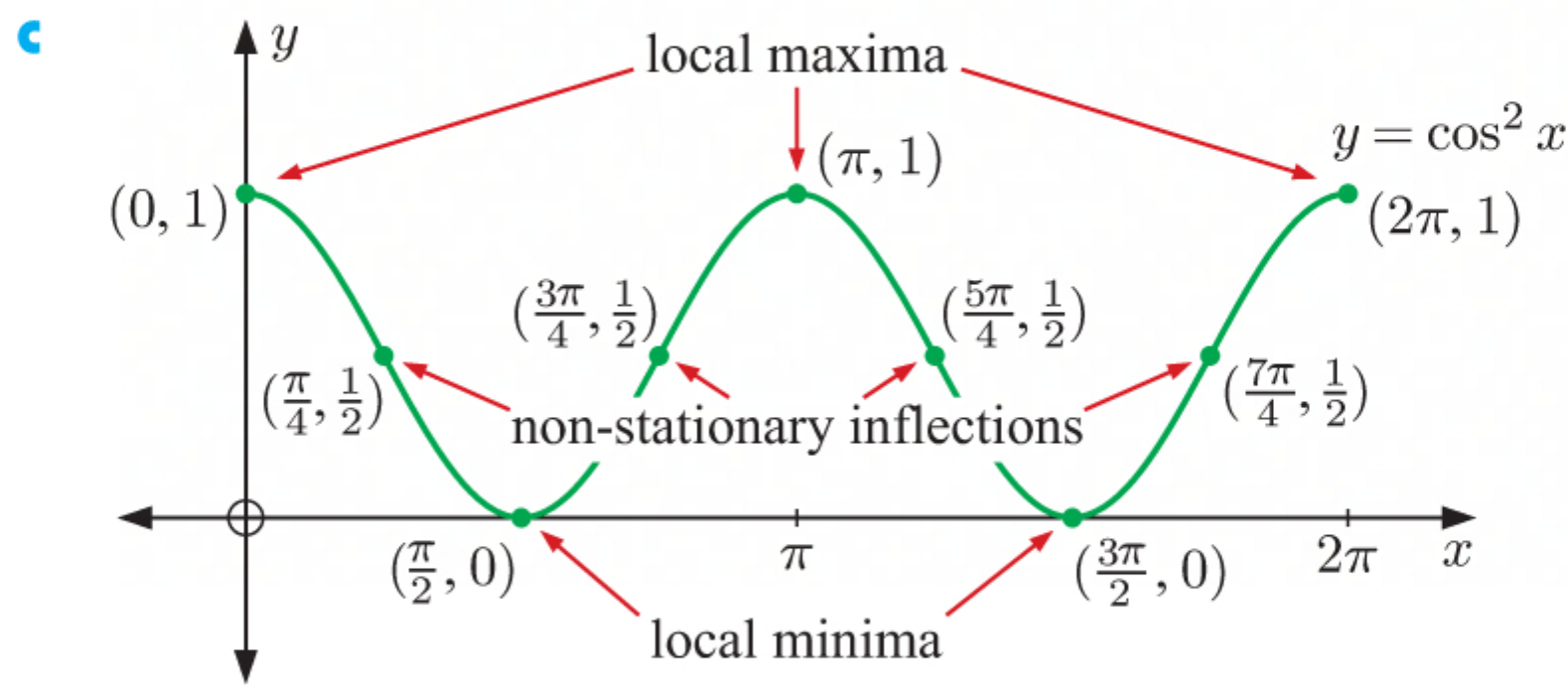
$$f\left(\frac{3\pi}{4}\right) = \cos^2\left(\frac{3\pi}{4}\right) = \left(-\frac{1}{\sqrt{2}}\right)^2 = \frac{1}{2}$$

$$f\left(\frac{5\pi}{4}\right) = \cos^2\left(\frac{5\pi}{4}\right) = \left(-\frac{1}{\sqrt{2}}\right)^2 = \frac{1}{2}$$

$$f\left(\frac{7\pi}{4}\right) = \cos^2\left(\frac{7\pi}{4}\right) = \left(\frac{1}{\sqrt{2}}\right)^2 = \frac{1}{2}$$

and $f'(x) \neq 0$ for $x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \text{ or } \frac{7\pi}{4}$

\therefore there are non-stationary points of inflection at $(\frac{\pi}{4}, \frac{1}{2})$, $(\frac{3\pi}{4}, \frac{1}{2})$, $(\frac{5\pi}{4}, \frac{1}{2})$, $(\frac{7\pi}{4}, \frac{1}{2})$.



23 a $f(x) = \frac{e^x}{x-1}$

Now $f(0) = \frac{e^0}{-1} = -1$ so the y -intercept is -1 .

b $f(x)$ is defined when $x - 1 \neq 0$
 $\therefore x \neq 1$

c $f'(x) = \frac{e^x(x-1) - e^x(1)}{(x-1)^2}$ {quotient rule}

$= \frac{e^x(x-2)}{(x-1)^2}$ which has sign diagram:

$\therefore f'(x) \leq 0$ for $x < 1$ and $1 < x \leq 2$ and $f'(x) \geq 0$ for $x \geq 2$

$\therefore f(x)$ is decreasing for all defined values of $x \leq 2$, and increasing for $x \geq 2$.

$f''(x) = \frac{[e^x(x-2) + e^x(1)](x-1)^2 - e^x(x-2)[2(x-1)^1(1)]}{(x-1)^4}$ {product and quotient rules}

$= \frac{[e^x(x-2+1)(x-1)^2] - 2e^x(x-2)(x-1)}{(x-1)^4}$

$= \frac{e^x(x-1)(x-1)^2 - 2e^x(x-2)(x-1)}{(x-1)^4}$

$= \frac{e^x(x-1)[(x-1)^2 - 2(x-2)]}{(x-1)^4}$

$= \frac{e^x(x-1)[x^2 - 2x + 1 - 2x + 4]}{(x-1)^4}$

$= \frac{e^x(x^2 - 4x + 5)}{(x-1)^3}$ where the quadratic term has $\Delta = (-4)^2 - 4(1)(5)$

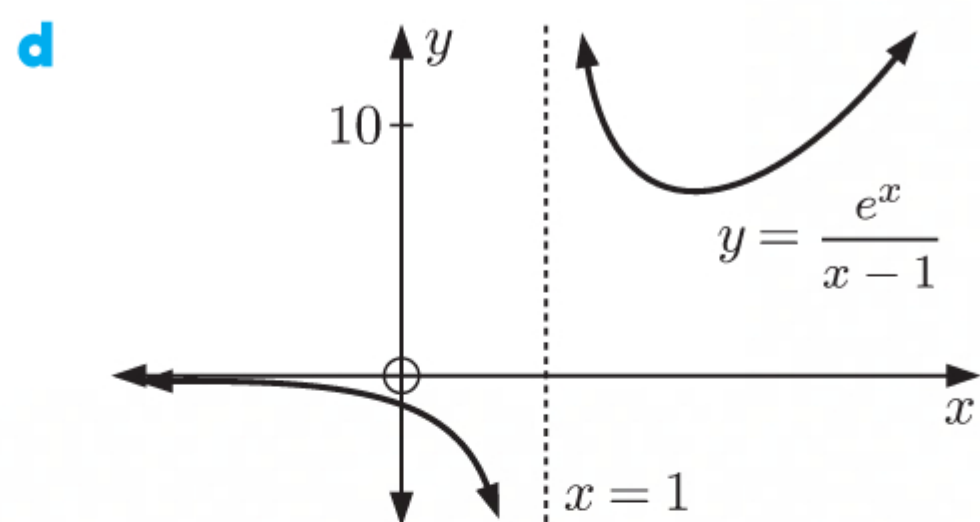
$= 16 - 20$

$= -4 < 0$

The sign diagram of $f''(x)$ is:

$\therefore f(x)$ is concave down for all $x < 1$

and concave up for all $x > 1$.



e $f(2) = \frac{e^2}{2-1} = e^2$

Using **c**, we have a local minimum at $(2, e^2)$

\therefore the tangent at $x = 2$ is horizontal
with equation $y = e^2$.

24 At $x = A$, $f'(x) = 0$ and $f''(x) = 0$

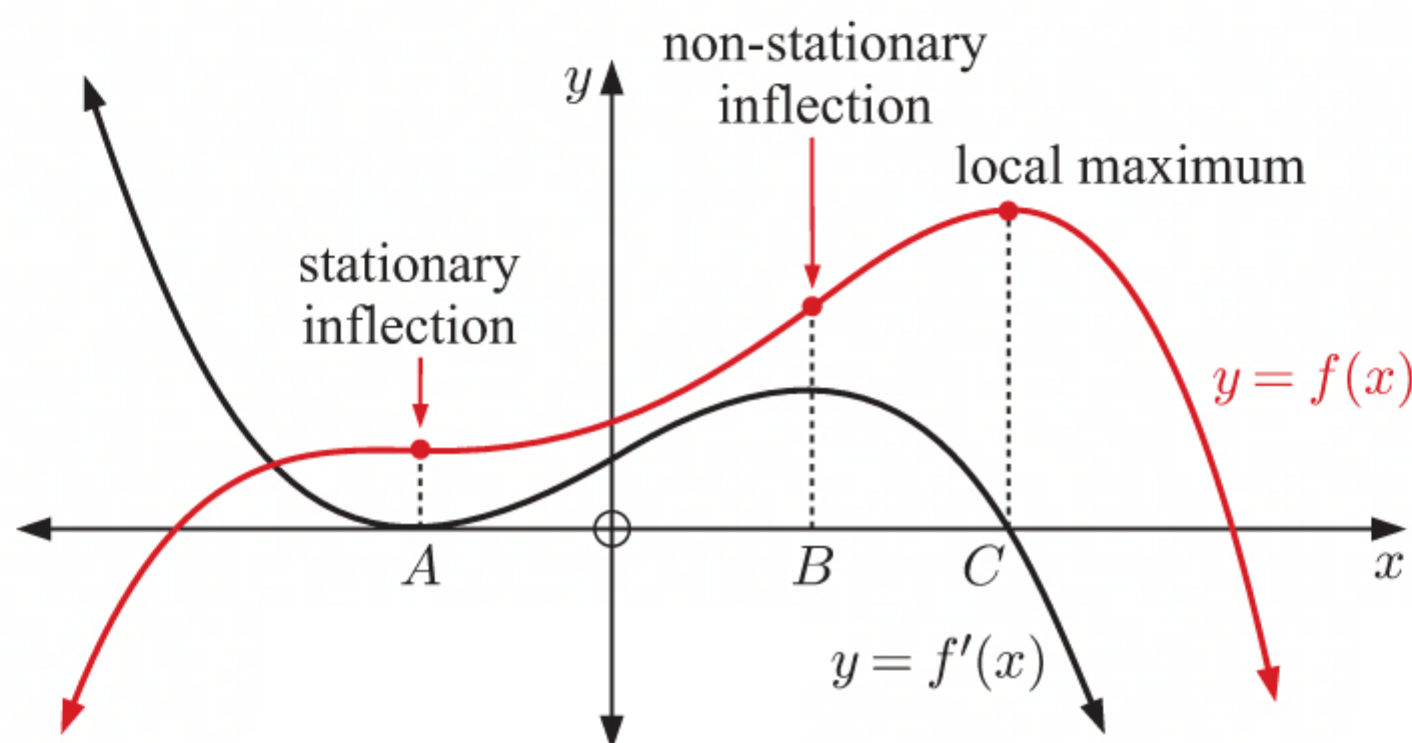
$\therefore f(x)$ has a stationary inflection point at $x = A$.

At $x = B$, $f''(x) = 0$ but $f'(x) \neq 0$

$\therefore f(x)$ has a non-stationary inflection point at $x = B$.

$f'(x)$ is above the x -axis for $x \leq C$, and below the x -axis for $x \geq C$

$\therefore f(x)$ is increasing for $x \leq C$ and decreasing for $x \geq C$, so $f(x)$ has a local maximum at $x = C$.



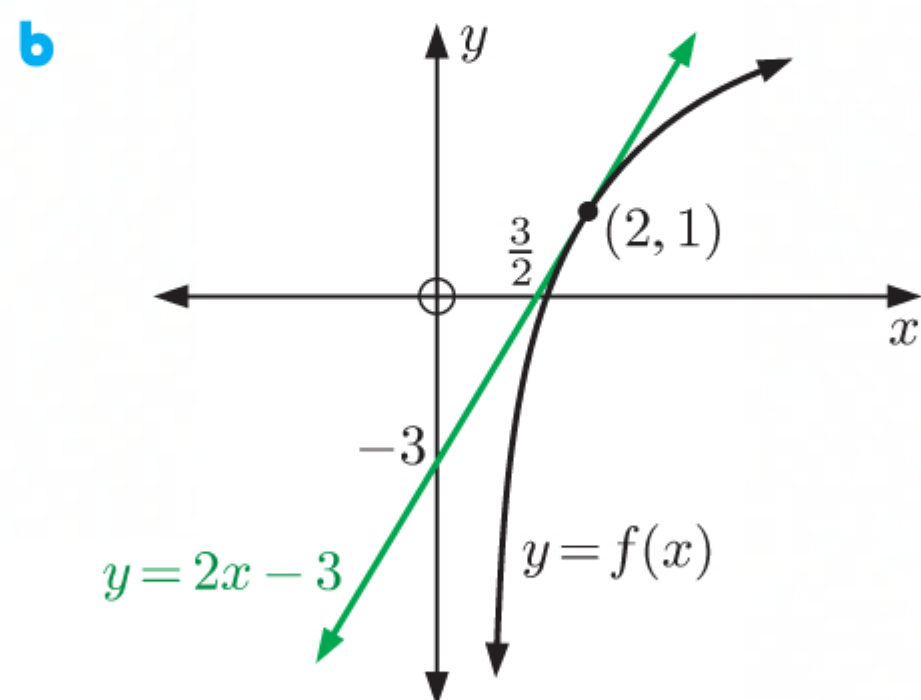
25 $f'(x) > 0$ and $f''(x) < 0$ for all x

$\therefore f(x)$ is increasing and concave downwards for all x .

a $f(2) = 1$ and $f'(2) = 2$

$\therefore (2, 1)$ lies on the curve and the tangent at this point has gradient 2

\therefore the tangent has equation $y = 2(x - 2) + 1$
 $= 2x - 4 + 1$
 $= 2x - 3$



c $f(x)$ is always increasing with $f'(x) > 0$ so it has *at most one* zero.

$f(x)$ is also concave downwards for all x , so it always lies below the tangent shown.

So, for $x < \frac{3}{2}$, the tangent and $f(x)$ lie below the x -axis.

$\therefore f(x)$ has exactly one zero.

d From the graph, the x -intercept of $y = f(x)$ lies in the interval $\frac{3}{2} < x < 2$.

Chapter 14

APPLICATIONS OF DIFFERENTIATION

EXERCISE 14A

1 a $P(t) = 2t^2 - 12t + 118$ thousand dollars

$$\begin{aligned} P(0) &= 2(0)^2 - 12(0) + 118 \\ &= 118 \end{aligned}$$

\therefore the current annual profit is \$118 000.

b $P = 2t^2 - 12t + 118$

$$\therefore \frac{dP}{dt} = 4t - 12 \text{ thousand dollars per year}$$

c When $t = 8$, $\frac{dP}{dt} = 4(8) - 12$
 $= 32 - 12$
 $= 20$

This means that in 8 years from now, profits will be increasing at a rate of \$20 000 per year.

2 a $V = 2(50 - t)^2 \text{ m}^3$

$$\begin{aligned} \text{When } t = 0, \quad V &= 2(50)^2 \\ &= 2 \times 2500 \\ &= 5000 \text{ m}^3 \end{aligned}$$

$$\begin{aligned} \text{When } t = 5, \quad V &= 2(50 - 5)^2 \\ &= 2(45)^2 \\ &= 2 \times 2025 \\ &= 4050 \text{ m}^3 \end{aligned}$$

\therefore the average rate at which the water evaporates in the first 5 days is

$$\frac{5000 - 4050}{5} = \frac{950}{5} = 190 \text{ m}^3 \text{ per day.}$$

b $V = 2(50 - t)^2$

$$\begin{aligned} \therefore \frac{dV}{dt} &= 2 \times 2(50 - t)(-1) \quad \{\text{chain rule}\} \\ &= -4(50 - t) \\ &= 4t - 200 \end{aligned}$$

$$\begin{aligned} \text{When } t = 5, \quad \frac{dV}{dt} &= 4(5) - 200 \\ &= 20 - 200 \\ &= -180 \end{aligned}$$

\therefore the instantaneous rate at which the water is evaporating at $t = 5$ days is 180 m³ per day.

3 a $Q(t) = 100 - 10\sqrt{t}$

i $Q(0) = 100 - 10\sqrt{0}$
 $= 100$

ii $Q(25) = 100 - 10\sqrt{25}$
 $= 50$

iii $Q(100) = 100 - 10\sqrt{100}$
 $= 0$

b $Q(t) = 100 - 10\sqrt{t} = 100 - 10t^{\frac{1}{2}}$ units, $t \geq 0$

$$\begin{aligned}\therefore Q'(t) &= -5t^{-\frac{1}{2}} \\ &= -\frac{5}{\sqrt{t}} \text{ units per year}\end{aligned}$$

i $Q'(25) = -\frac{5}{\sqrt{25}} = -1$

\therefore when the person is aged 25 years, the quantity of the chemical is decreasing by 1 unit per year.

ii $Q'(50) = -\frac{5}{\sqrt{50}}$
 $= -\frac{5}{5\sqrt{2}}$
 $= -\frac{1}{\sqrt{2}}$

\therefore when the person is aged 50 years, the quantity of the chemical is decreasing by $\frac{1}{\sqrt{2}}$ units per year.

c $Q'(t) = -\frac{5}{\sqrt{t}} < 0$ for all $t > 0$

\therefore the quantity of the chemical is decreasing for all $t > 0$.

4 a $H = 35 - \frac{172.5}{t+5}$ metres

When $t = 0$, $H = 35 - \frac{172.5}{5}$
 $= 0.5$

\therefore the tree was 0.5 m tall when it was planted.

b i When $t = 4$, $H = 35 - \frac{172.5}{9}$
 ≈ 15.8

\therefore after 4 years, the tree is about 15.8 m tall.

ii When $t = 8$, $H = 35 - \frac{172.5}{13}$
 ≈ 21.7

\therefore after 8 years, the tree is about 21.7 m tall.

iii When $t = 12$, $H = 35 - \frac{172.5}{17}$
 ≈ 24.9

\therefore after 12 years, the tree is about 24.9 m tall.

c $H = 35 - \frac{172.5}{t+5}$ m, $t \geq 0$

$$= 35 - 172.5(t+5)^{-1}$$

$$\begin{aligned}\therefore \frac{dH}{dt} &= 172.5(t+5)^{-2} \\ &= \frac{172.5}{(t+5)^2} \text{ m per year}\end{aligned}$$

When $t = 0$, $\frac{dH}{dt} = \frac{172.5}{(0+5)^2}$
 $= 6.9$

\therefore the tree is initially growing at a rate of 6.9 m per year.

When $t = 5$, $\frac{dH}{dt} = \frac{172.5}{10^2}$
 $= 1.725$

\therefore the tree is growing at a rate of 1.725 m per year after 5 years.

$$\text{When } t = 10, \quad \frac{dH}{dt} = \frac{172.5}{15^2} \\ \approx 0.767$$

\therefore the tree is growing at a rate of about 0.767 m per year after 10 years.

$$\text{d} \quad \frac{dH}{dt} = \frac{172.5}{(t+5)^2} > 0 \quad \text{for all } t \geq 0$$

This model predicts that the tree will continue to grow forever.

$$\text{5} \quad C(x) = 7800 + 6x + 12x^{0.7} \quad \text{dollars}$$

$$\text{a} \quad \text{The marginal cost function is } C'(x) = 6 + 12(0.7x^{-0.3}) \\ = 6 + 8.4x^{-0.3} \quad \text{dollars per pair}$$

$$\text{b} \quad C'(220) = 6 + 8.4(220)^{-0.3} \\ \approx \$7.67$$

This estimates the cost of making the 221st pair of jeans if 220 pairs are currently being made.

$$\text{c} \quad C(221) - C(220) = 7800 + 6(221) + 12(221)^{0.7} - (7800 + 6(220) + 12(220)^{0.7}) \\ \approx \$7.66$$

This is the actual cost of making the 221st pair of jeans.

The answer in **b** is a very good estimate.

$$\text{6} \quad \text{a} \quad C(v) = \frac{1}{5}v^2 + \frac{200\,000}{v} \quad \text{euros}$$

$$\text{i} \quad C(50) = \frac{1}{5}(50)^2 + \frac{200\,000}{50} \\ = 500 + 4000 \\ = 4500$$

\therefore if the average speed is 50 km h^{-1} , the total cost of the journey is 4500 euros.

$$\text{ii} \quad C(100) = \frac{1}{5}(100)^2 + \frac{200\,000}{100} \\ = 2000 + 2000 \\ = 4000$$

\therefore if the average speed is 100 km h^{-1} , the total cost of the journey is 4000 euros.

$$\text{b} \quad C(v) = \frac{1}{5}v^2 + \frac{200\,000}{v} \quad \text{euros, } v > 0 \\ = \frac{1}{5}v^2 + 200\,000v^{-1} \\ \therefore C'(v) = \frac{2}{5}v - 200\,000v^{-2} \\ = \frac{2}{5}v - \frac{200\,000}{v^2} \quad \text{euros per km h}^{-1}$$

$$\text{i} \quad C'(30) = \frac{2}{5}(30) - \frac{200\,000}{30^2} \\ = 12 - \frac{2000}{9} \\ \approx -210.22$$

\therefore if the average speed is 30 km h^{-1} , the rate of change in the cost of running the train is decreasing at about 210.22 euros per km h^{-1} .

$$\begin{aligned}
 \text{ii } C'(90) &= \frac{2}{5}(90) - \frac{200\,000}{90^2} \\
 &= 36 - \frac{2000}{81} \\
 &\approx 11.31
 \end{aligned}$$

\therefore if the average speed is 90 km h^{-1} , the rate of change in the cost of running the train is increasing at about $11.31 \text{ euros per km h}^{-1}$.

c $C(v)$ is a minimum when $C'(v) = 0$

$$\therefore \frac{2}{5}v - \frac{200\,000}{v^2} = 0$$

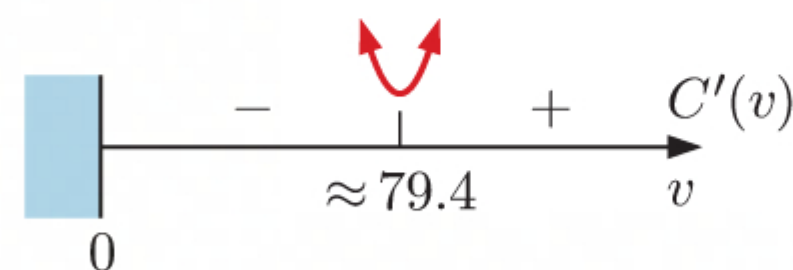
$$\therefore \frac{2}{5}v^3 - 200\,000 = 0$$

$$\therefore \frac{2}{5}v^3 = 200\,000$$

$$\therefore v^3 = 500\,000$$

$$\therefore v \approx 79.4 \text{ km h}^{-1}$$

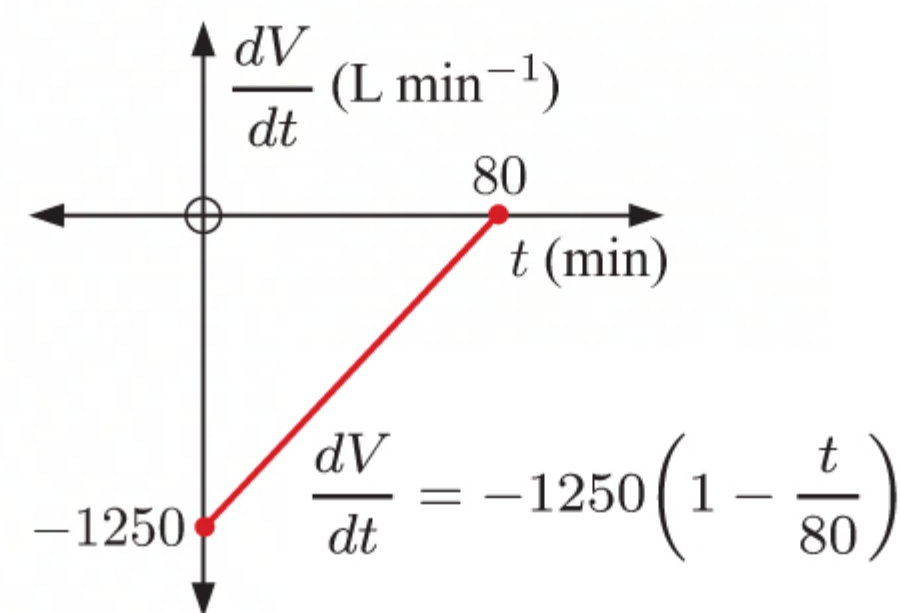
$C'(v)$ has sign diagram:



\therefore the cost of running the train is a minimum when the average speed of the train is about 79.4 km h^{-1} .

7 a $V = 50\,000 \left(1 - \frac{t}{80}\right)^2 \text{ L}, \quad 0 \leq t \leq 80$

$$\begin{aligned}
 \therefore \frac{dV}{dt} &= 100\,000 \left(1 - \frac{t}{80}\right) \left(-\frac{1}{80}\right) \quad \{\text{chain rule}\} \\
 &= -1250 \left(1 - \frac{t}{80}\right) \text{ L min}^{-1}
 \end{aligned}$$



b The outflow is fastest when $\frac{dV}{dt} = -1250 \left(1 - \frac{t}{80}\right)$ is smallest.

Looking at the graph in **a**, the minimum value of $\frac{dV}{dt}$ is -1250 L min^{-1} which occurs when $t = 0$.

So, the outflow is fastest at $t = 0$, when the tap was first opened.

c
$$\begin{aligned}
 \frac{dV}{dt} &= -1250 \left(1 - \frac{t}{80}\right) \\
 &= -1250 + \frac{125}{8}t
 \end{aligned}$$

$$\therefore \frac{d^2V}{dt^2} = \frac{125}{8} > 0$$

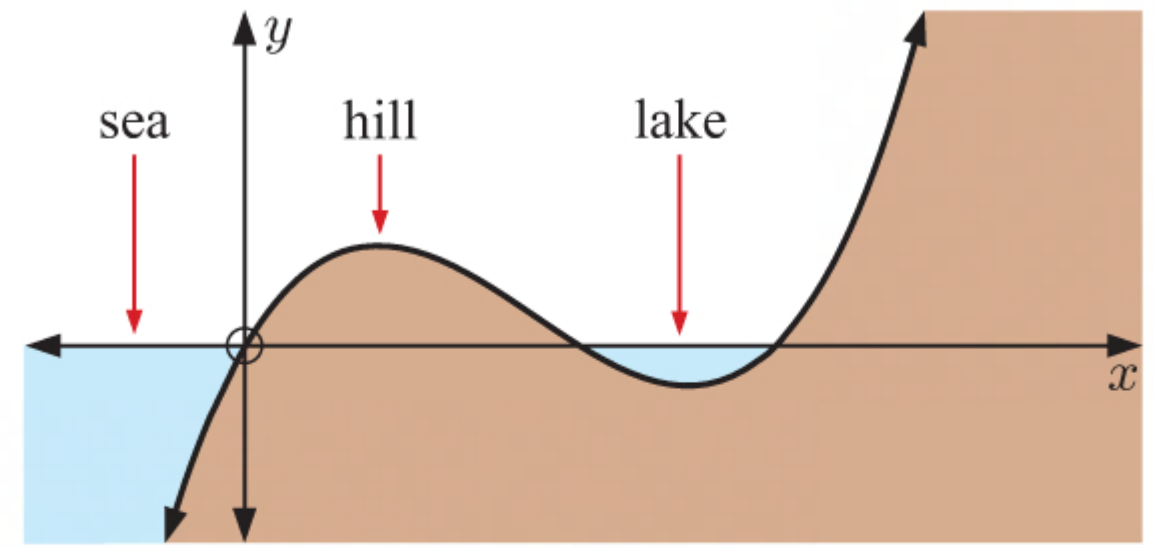
This shows that the rate of change of V is constantly increasing, so the outflow is increasing at a constant rate.

8 a $y = \frac{1}{10}x(x-2)(x-3)$

The edges of the lake correspond to values of x such that $y = 0$.

$$\begin{aligned}\therefore \frac{1}{10}x(x-2)(x-3) &= 0 \\ \therefore x &= 0, 2, \text{ or } 3\end{aligned}$$

From the graph, we can see that the near part of the lake is 2 km from the sea, and the furthest part is 3 km.



b

$$\begin{aligned}y &= \frac{1}{10}x(x-2)(x-3) \\ &= \frac{1}{10}x(x^2 - 5x + 6) \\ &= \frac{1}{10}x^3 - \frac{1}{2}x^2 + \frac{3}{5}x \\ \therefore \frac{dy}{dx} &= \frac{3}{10}x^2 - x + \frac{3}{5}\end{aligned}$$

$$\begin{aligned}\text{When } x = \frac{1}{2}, \quad \frac{dy}{dx} &= \frac{3}{10}\left(\frac{1}{2}\right)^2 - \frac{1}{2} + \frac{3}{5} \\ &= 0.175\end{aligned}$$

\therefore the height of the hill is increasing when $x = \frac{1}{2}$ km, as the gradient is positive.
So the land is sloping upwards at this point.

$$\begin{aligned}\text{When } x = 1\frac{1}{2} = \frac{3}{2}, \quad \frac{dy}{dx} &= \frac{3}{10}\left(\frac{3}{2}\right)^2 - \frac{3}{2} + \frac{3}{5} \\ &= -0.225\end{aligned}$$

\therefore the height of the hill is decreasing when $x = 1\frac{1}{2}$ km, as the gradient is negative.
So the land is sloping downwards at this point.

This means the top of the hill is between $x = \frac{1}{2}$ km and $x = 1\frac{1}{2}$ km.

c The deepest point of the lake occurs when the slope of the land is 0, which is when $\frac{dy}{dx} = 0$.

$$\begin{aligned}\therefore \frac{3}{10}x^2 - x + \frac{3}{5} &= 0 \\ \therefore x &\approx 0.785 \text{ or } 2.55 \quad \{\text{using technology}\}\end{aligned}$$

The deepest point of the lake is a turning point which lies between the edges of the lake at $x = 2$ and $x = 3$. So we only consider the value of x which lies between 2 and 3.

$$\begin{aligned}\text{When } x = 2.55, \quad y &\approx -0.0631 \text{ km} \\ &\approx -63.1 \text{ m}\end{aligned}$$

So, the deepest point of the lake is about 2.55 km from the sea, and about 63.1 m deep.

9 a $W = 20e^{-kt}$ grams, $t \geq 0$

$$\text{When } t = 50, \quad W = 10$$

$$\begin{aligned}\therefore 20e^{-50k} &= 10 \\ \therefore e^{-50k} &= \frac{1}{2} \\ \therefore -50k &= \ln\left(\frac{1}{2}\right) \\ \therefore k &= -\frac{1}{50} \ln\left(\frac{1}{2}\right) \\ \therefore k &= \frac{1}{50} \ln 2 \approx 0.0139\end{aligned}$$

b i When $t = 0$, $W = 20e^{-\frac{0}{50} \ln 2}$
 $= 20e^0$
 $= 20$

\therefore there are initially 20 grams of radioactive substance present.

ii When $t = 24$, $W = 20e^{-\frac{24}{50} \ln 2}$
 ≈ 14.3

\therefore after 24 hours, there are about 14.3 grams of radioactive substance present.

iii 1 week $= 7 \times 24 = 168$ hours

When $t = 168$, $W = 20e^{-\frac{168}{50} \ln 2}$
 ≈ 1.95

\therefore after 1 week, there are about 1.95 grams of radioactive substance present.

c When $W = 1$, $20e^{-\frac{t}{50} \ln 2} = 1$
 $\therefore e^{-\frac{t}{50} \ln 2} = \frac{1}{20}$
 $\therefore -\frac{t}{50} \ln 2 = \ln\left(\frac{1}{20}\right)$
 $\therefore t \ln 2 = 50 \ln 20$
 $\therefore t = \frac{50 \ln 20}{\ln 2}$
 ≈ 216

\therefore it will take about 216 hours or about 9 days for the weight of the radioactive substance to reach 1 gram.

d $W = 20e^{-\frac{t}{50} \ln 2}$
 $\therefore \frac{dW}{dt} = -\frac{1}{50} \ln 2 \times 20e^{-\frac{t}{50} \ln 2} \quad \{\text{chain rule}\}$

i When $t = 100$, $\frac{dW}{dt} = -\frac{1}{50} \ln 2 \times 20e^{-2 \ln 2}$
 ≈ -0.0693

\therefore after 100 hours, the rate of radioactive decay is about -0.0693 g h^{-1} .

ii When $t = 1000$, $\frac{dW}{dt} = -\frac{1}{50} \ln 2 \times 20e^{-20 \ln 2}$
 $\approx -2.64 \times 10^{-7}$

\therefore after 1000 hours, the rate of radioactive decay is about $-2.64 \times 10^{-7} \text{ g h}^{-1}$.

e $\frac{dW}{dt} = -\frac{1}{50} \ln 2 \times 20e^{-\frac{t}{50} \ln 2}$
 $= bW \quad \{\text{where } b = -\frac{1}{50} \ln 2 \text{ is constant}\}$

10 a $T = 5 + 95e^{-kt} \text{ }^{\circ}\text{C}, \quad t \geq 0$

When $t = 15$, $T = 20$

$$\therefore 5 + 95e^{-15k} = 20$$

$$\therefore 95e^{-15k} = 15$$

$$\therefore e^{-15k} = \frac{3}{19}$$

$$\therefore -15k = \ln\left(\frac{3}{19}\right)$$

$$\therefore k = \frac{1}{15} \ln\left(\frac{19}{3}\right) \approx 0.123$$

b When $t = 0$, $T = 5 + 95e^{-\frac{0}{15} \ln(\frac{19}{3})}$
 $= 5 + 95e^0$
 $= 100$

\therefore the temperature of the liquid when it was first placed in the refrigerator was 100°C .

c $T = 5 + 95e^{-\frac{t}{15} \ln(\frac{19}{3})}, \quad T - 5 = 95e^{-\frac{t}{15} \ln(\frac{19}{3})}$
 $\therefore \frac{dT}{dt} = -\frac{1}{15} \ln\left(\frac{19}{3}\right) \times 95e^{-\frac{t}{15} \ln(\frac{19}{3})}$
 $= -\frac{1}{15} \ln\left(\frac{19}{3}\right) \times (T - 5)$
 $= c(T - 5) \quad \{\text{where } c = -\frac{1}{15} \ln\left(\frac{19}{3}\right) = -k \text{ is constant}\}$

d $\frac{dT}{dt} = -\frac{1}{15} \ln\left(\frac{19}{3}\right) \times 95e^{-\frac{t}{15} \ln(\frac{19}{3})} \text{ }^{\circ}\text{C per minute} \quad \{\text{using c}\}$

i When $t = 0$, $\frac{dT}{dt} = -\frac{1}{15} \ln\left(\frac{19}{3}\right) \times 95e^{-\frac{0}{15} \ln(\frac{19}{3})}$
 $= -\frac{19}{3} \ln\left(\frac{19}{3}\right)$
 ≈ -11.7

\therefore the temperature is initially decreasing at about 11.7°C per minute.

ii When $t = 10$, $\frac{dT}{dt} = -\frac{1}{15} \ln\left(\frac{19}{3}\right) \times 95e^{-\frac{2}{3} \ln(\frac{19}{3})}$
 ≈ -3.42

\therefore after 10 minutes, the temperature is decreasing at about 3.42°C per minute.

iii When $t = 20$, $\frac{dT}{dt} = -\frac{1}{15} \ln\left(\frac{19}{3}\right) \times 95e^{-\frac{4}{3} \ln(\frac{19}{3})}$
 ≈ -0.998

\therefore after 20 minutes, the temperature is decreasing at about 0.998°C per minute.

11 a $H(t) = 20 \ln(3t + 2) + 30 \text{ cm}, \quad t \geq 0$

$$\therefore H(0) = 20 \ln 2 + 30$$

$$\approx 43.9$$

\therefore the shrub was about 43.9 cm high when it was planted.

b When $H(t) = 100$ cm,

$$20 \ln(3t + 2) + 30 = 100$$

$$\therefore 20 \ln(3t + 2) = 70$$

$$\therefore \ln(3t + 2) = \frac{7}{2}$$

$$\therefore 3t + 2 = e^{\frac{7}{2}}$$

$$\therefore 3t = e^{\frac{7}{2}} - 2$$

$$\therefore t = \frac{e^{\frac{7}{2}} - 2}{3}$$

$$\approx 10.4$$

\therefore it will take about 10.4 years for the shrub to reach a height of 1 m.

c $H(t) = 20 \ln(3t + 2) + 30$ cm, $t \geq 0$

$$\therefore H'(t) = 20 \times \frac{3}{3t + 2}$$

$$= \frac{60}{3t + 2}$$

i $H'(3) = \frac{60}{3(3) + 2}$

$$= \frac{60}{11}$$

$$\approx 5.45$$

\therefore 3 years after being planted, the shrub is growing at about 5.45 cm per year.

ii $H'(10) = \frac{60}{3(10) + 2}$

$$= \frac{60}{32}$$

$$= 1.875$$

\therefore 10 years after being planted, the shrub is growing at 1.875 cm per year.

12 a $A = s(1 - e^{-kt})$ litres

When $t = 0$, $A = s(1 - e^{-k(0)})$

$$= s(1 - e^0)$$

$$= 0$$

b i $s = 10$, and when $t = 3$, $A = 5$

$$\therefore 5 = 10(1 - e^{-3k})$$

$$\therefore 1 - e^{-3k} = \frac{1}{2}$$

$$\therefore e^{-3k} = \frac{1}{2}$$

$$\therefore -3k = \ln \frac{1}{2}$$

$$\therefore k = \frac{\ln 2}{3} \approx 0.231$$

$$\text{ii} \quad A = 10 \left(1 - e^{-\frac{\ln 2}{3}t} \right) \text{ L}$$

$$\therefore \frac{dA}{dt} = 10 \left(\frac{\ln 2}{3} e^{-\frac{\ln 2}{3}t} \right) \text{ L h}^{-1}$$

$$\text{When } t = 5, \quad \frac{dA}{dt} = 10 \left(\frac{\ln 2}{3} e^{-\frac{5 \ln 2}{3}} \right)$$

$$\approx 0.728$$

\therefore after 5 hours, the speed of the reaction is about 0.728 litres of alcohol produced per hour.

13 Triangle PQR has area $A = \frac{1}{2} \times 6 \times 7 \times \sin \theta$

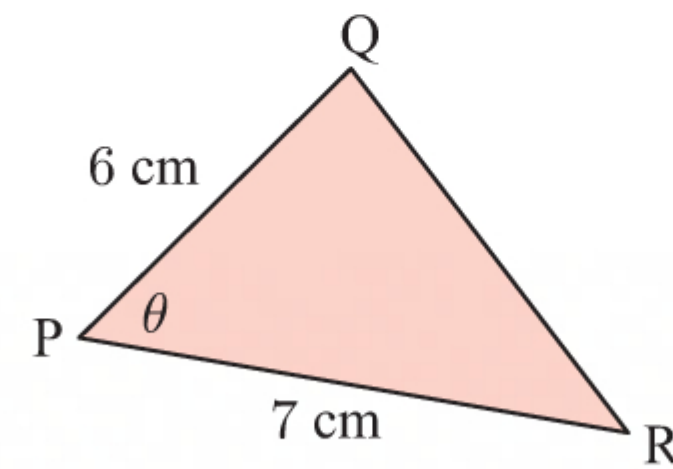
$$\therefore A = 21 \sin \theta \text{ cm}^2$$

$$\therefore \frac{dA}{d\theta} = 21 \cos \theta \text{ cm}^2 \text{ per radian}$$

$$\text{When } \theta = 45^\circ = \frac{\pi}{4}, \quad \frac{dA}{d\theta} = 21 \cos \frac{\pi}{4}$$

$$= 21 \times \frac{1}{\sqrt{2}}$$

$$= \frac{21}{\sqrt{2}} \text{ cm}^2 \text{ per radian}$$



\therefore the area of triangle PQR is changing at a rate of $\frac{21}{\sqrt{2}} \text{ cm}^2$ per radian at the time when $\theta = 45^\circ$.

14 a Using the cosine rule,

$$l^2 = 20^2 + 20^2 - 2 \times 20 \times 20 \times \cos \theta$$

$$\therefore l^2 = 400 + 400 - 800 \cos \theta$$

$$\therefore l^2 = 800 - 800 \cos \theta$$

$$\therefore l = \sqrt{800 - 800 \cos \theta} \text{ cm} \quad \{l > 0\}$$

b

$$l = (800 - 800 \cos \theta)^{\frac{1}{2}}$$

$$\therefore \frac{dl}{d\theta} = \frac{1}{2} (800 - 800 \cos \theta)^{-\frac{1}{2}} (800 \sin \theta)$$

$$= \frac{400 \sin \theta}{\sqrt{800 - 800 \cos \theta}} \text{ cm per radian}$$

$$\text{When } \theta = 120^\circ = \frac{2\pi}{3}, \quad \frac{dl}{d\theta} = \frac{400 \sin \frac{2\pi}{3}}{\sqrt{800 - 800 \cos \frac{2\pi}{3}}}$$

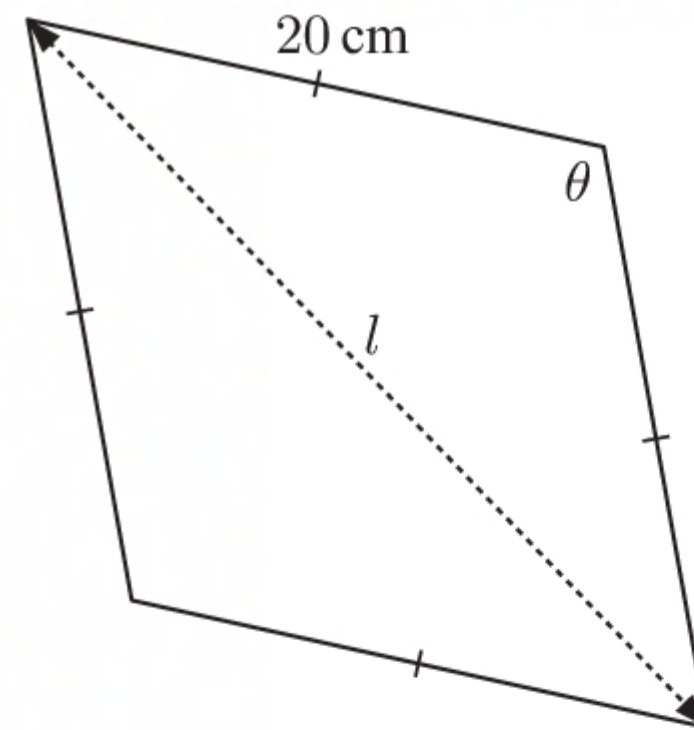
$$= \frac{400 \left(\frac{\sqrt{3}}{2} \right)}{\sqrt{800 - 800 \left(-\frac{1}{2} \right)}}$$

$$= \frac{200\sqrt{3}}{\sqrt{800 + 400}}$$

$$= \frac{200\sqrt{3}}{\sqrt{1200}}$$

$$= \frac{200\sqrt{3}}{20\sqrt{3}}$$

$$= 10 \text{ cm per radian}$$



$\therefore l$ is changing at a rate of 10 cm per radian at the time when $\theta = 120^\circ$.

15 a $V(t) = 340 \sin(100\pi t)$ volts

i $V(0) = 340 \sin(100\pi(0))$
 $= 340 \sin 0$
 $= 0$

\therefore there are initially 0 volts in the circuit.

ii $V(0.125) = 340 \sin(100\pi(0.125))$
 $= 340 \sin(12.5\pi)$
 $= 340$

\therefore there are 340 volts in the circuit after 0.125 seconds.

b $V(t) = 340 \sin(100\pi t)$ volts

$\therefore V'(t) = 34\,000\pi \cos(100\pi t)$ volts per second

i $V'(0.01) = 34\,000\pi \cos(100\pi(0.01))$
 $= 34\,000\pi \cos \pi$
 $= -34\,000\pi$

\therefore after 0.01 seconds, the voltage is changing at $-34\,000\pi$ volts per second.

ii $V(t)$ is a maximum when $V'(t) = 0$.

So the voltage is changing at 0 volts per second.

16 $B(t) = \frac{3000}{1 + 0.5e^{-1.73t}}$

a $B(0) = \frac{3000}{1 + 0.5e^{-1.73(0)}}$
 $= \frac{3000}{1 + 0.5e^0}$
 $= \frac{3000}{1.5}$
 $= 2000$

\therefore the initial bee population is 2000.

b $B(1) = \frac{3000}{1 + 0.5e^{-1.73(1)}}$
 ≈ 2756

The percentage increase after 1 month $\approx \frac{2756 - 2000}{2000} \times 100\%$
 $\approx 37.8\%$

\therefore the percentage increase in the population after 1 month is about 37.8%.

c As $t \rightarrow \infty$, $e^{-1.73t} \rightarrow 0$

$\therefore \frac{3000}{1 + 0.5e^{-1.73t}} \rightarrow \frac{3000}{1} = 3000$

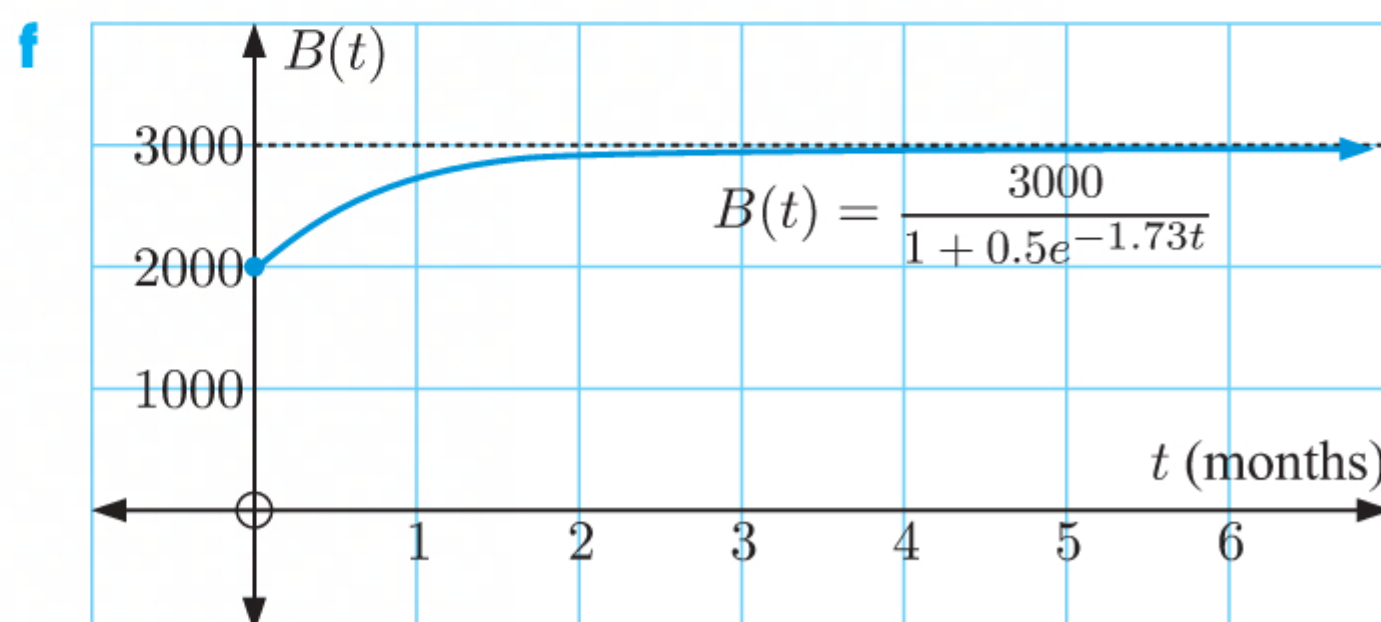
\therefore the population is limited to 3000 bees.

$$\begin{aligned}
 \text{d} \quad B(t) &= \frac{3000}{1 + 0.5e^{-1.73t}} = 3000(1 + 0.5e^{-1.73t})^{-1} \text{ bees} \\
 \therefore B'(t) &= -3000(1 + 0.5e^{-1.73t})^{-2}(0.5(-1.73)e^{-1.73t}) \quad \{\text{chain rule}\} \\
 &= \frac{2595}{e^{1.73t}(1 + 0.5e^{-1.73t})^2} \text{ bees per month} \\
 &> 0 \quad \text{for all } t
 \end{aligned}$$

\therefore the population is increasing over time.

$$\begin{aligned}
 \text{e} \quad B'(6) &= \frac{2595}{e^{1.73(6)}(1 + 0.5e^{-1.73(6)})^2} \\
 &\approx 0.0806
 \end{aligned}$$

\therefore after 6 months, the population is increasing at about 0.0806 bees per month.



EXERCISE 14B

$$1 \quad P(x) = -0.022x^2 + 11x - 720$$

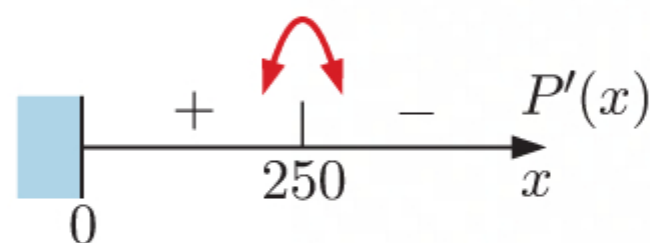
$$\therefore P'(x) = -0.044x + 11$$

$$\text{Now, } P'(x) = 0 \text{ when } -0.044x + 11 = 0$$

$$\therefore 11 = 0.044x$$

$$\begin{aligned}
 \therefore x &= \frac{11}{0.044} \\
 &= 250
 \end{aligned}$$

So, $P'(x)$ has sign diagram:



So, the profit is maximised when 250 items are made per day.

$$2 \quad \text{Production cost } C(x) = \frac{1}{4}x^2 + 8x + 20 \text{ pounds}$$

$$\text{Selling price } p(x) = 23 - \frac{1}{2}x \text{ pounds per blanket}$$

$$\text{Revenue } R(x) = xp(x) = 23x - \frac{1}{2}x^2 \text{ pounds}$$

$$\text{Profit } P(x) = \text{revenue} - \text{cost}$$

$$= (23x - \frac{1}{2}x^2) - (\frac{1}{4}x^2 + 8x + 20)$$

$$= -\frac{3}{4}x^2 + 15x - 20$$

$$\therefore P'(x) = -\frac{3}{2}x + 15$$

Now $P'(x) = 0$ when $-\frac{3}{2}x + 15 = 0$

$$\therefore x = \frac{15}{\frac{3}{2}} = 10$$

$P'(x)$ has sign diagram:

So, the profit is maximised when 10 blankets are produced per day.

3 a Let the remaining fence have length y m.

The total length of the fence is 60 m

$$\therefore 2x + y = 60$$

$$\therefore y = 60 - 2x$$

The area of the enclosure $A = \text{width} \times \text{length}$

$$= xy$$

$$= x(60 - 2x) \text{ m}^2$$

\therefore the area of the enclosure is given by $A(x) = x(60 - 2x) \text{ m}^2$.

b
$$A(x) = x(60 - 2x)$$

$$= 60x - 2x^2$$

$$\therefore A'(x) = 60 - 4x$$

So, $A'(x) = 0$ when $60 - 4x = 0$

$$\therefore 4x = 60$$

$$\therefore x = 15$$

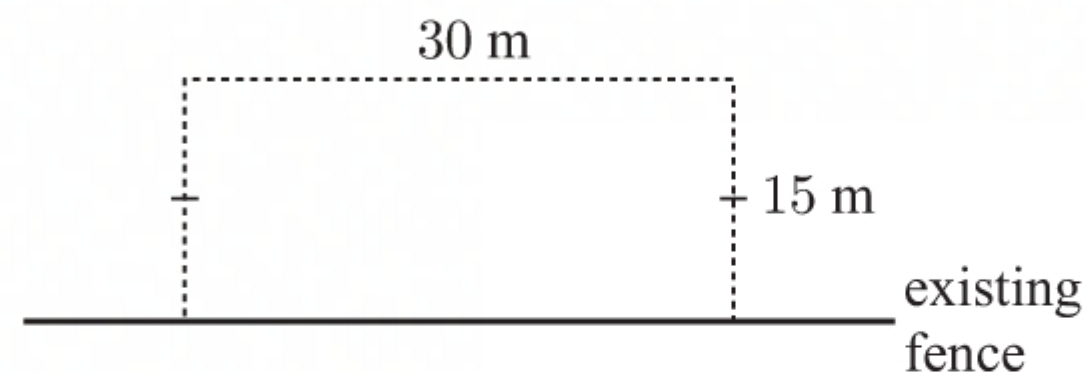
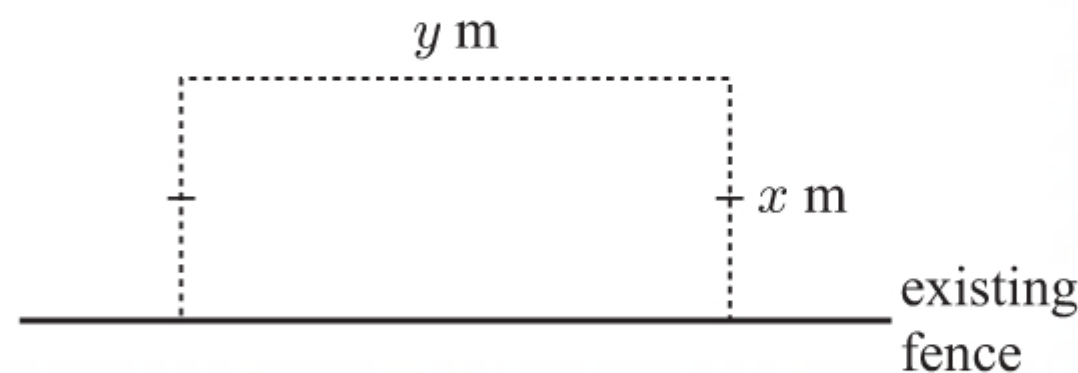
$A'(x)$ has sign diagram:

The area is maximised when $x = 15$

$$\text{and } y = 60 - 2(15)$$

$$= 30$$

The area of the enclosure is maximised by constructing a fence with width 15 m and length 30 m.



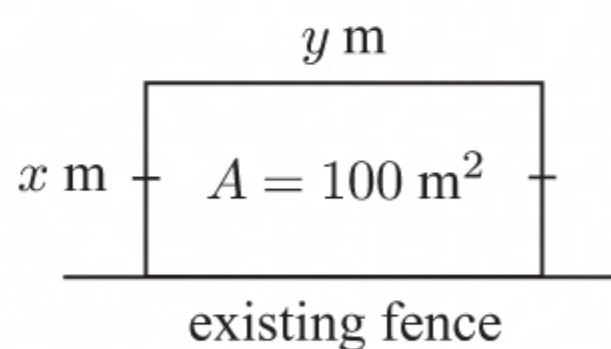
4 a Now $A = 100$

$$\therefore xy = 100$$

$$\therefore y = \frac{100}{x}$$

$$\text{So, } L = 2x + y$$

$$\therefore L = 2x + \frac{100}{x}$$



b $L = 2x + 100x^{-1}$

$$\therefore \frac{dL}{dx} = 2 - 100x^{-2} = 2 - \frac{100}{x^2}$$

which is 0 when $\frac{100}{x^2} = 2$

$$\therefore x^2 = 50$$

$$\therefore x = \sqrt{50} \quad \{x > 0\}$$

So, $\frac{dL}{dx}$ has sign diagram:

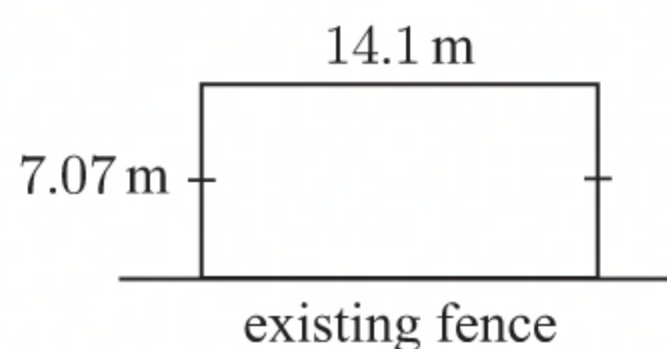
So, the length is a minimum when $x = \sqrt{50}$.

$$\text{When } x = \sqrt{50}, \quad L = 2\sqrt{50} + \frac{100}{\sqrt{50}} \\ \approx 28.3$$

So, the minimum value of L is about 28.3 m which occurs when $x = \sqrt{50} \approx 7.07$.

c $x = \sqrt{50} \approx 7.07$ and $y = \frac{100}{\sqrt{50}} \approx 14.1$

So, the optimal situation is:

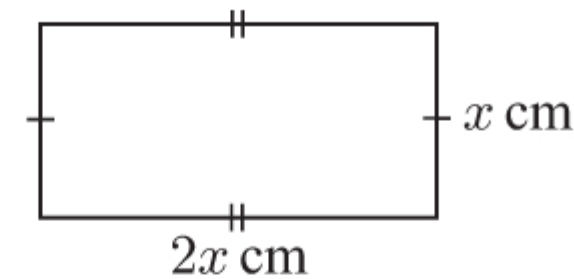


5 a The base has dimensions in the ratio 2 : 1.

\therefore if one side is x cm, then the other side must be $2x$ cm.

$$\begin{aligned} V &= \text{area of base} \times \text{height} \\ &= 2x \times x \times h \\ &= 2x^2h \end{aligned}$$

$$\begin{aligned} \text{but } V &= 200 \text{ cm}^3, \quad \therefore 2x^2h = 200 \\ &\therefore x^2h = 100 \end{aligned}$$



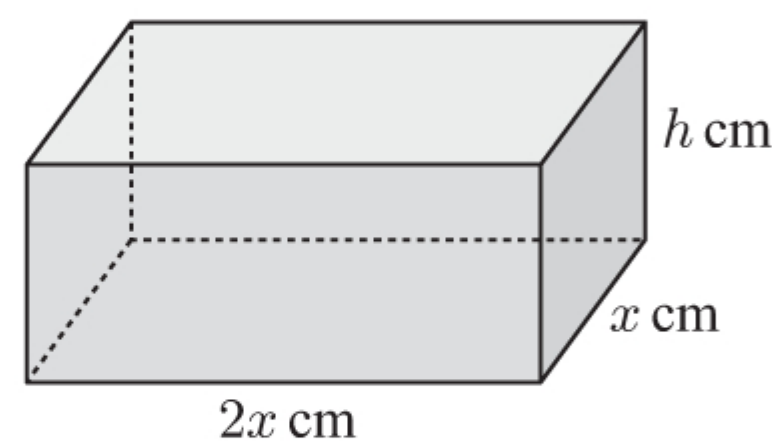
b $x^2h = 100 \quad \{\text{from a}\}$

$$\therefore h = \frac{100}{x^2}$$

Inner surface area

$$\begin{aligned} A(x) &= 2(2x \times x) + 2\left(x \times \frac{100}{x^2}\right) + 2\left(2x \times \frac{100}{x^2}\right) \\ &= 2\left(2x^2 + \frac{100}{x} + \frac{200}{x}\right) \\ &= 2\left(2x^2 + \frac{300}{x}\right) \end{aligned}$$

$$\therefore A(x) = 4x^2 + \frac{600}{x} \text{ cm}^2$$



$$\text{c} \quad A(x) = 4x^2 + 600x^{-1}$$

$$\therefore A'(x) = 8x - 600x^{-2}$$

$$= 8x - \frac{600}{x^2}$$

$$A'(x) = 0 \quad \text{when} \quad 8x - \frac{600}{x^2} = 0$$

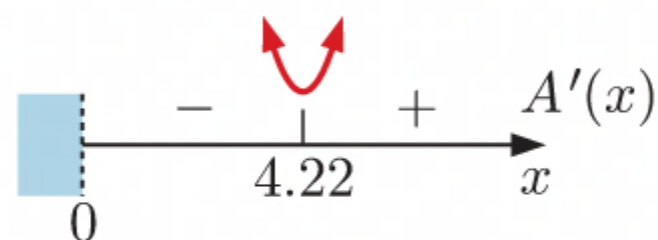
$$\therefore 8x = \frac{600}{x^2}$$

$$\therefore 8x^3 = 600$$

$$\therefore x^3 = 75$$

$$\therefore x = \sqrt[3]{75} \approx 4.22$$

$A'(x)$ has sign diagram:



So, the inner surface area of the box is a minimum when $x = \sqrt[3]{75} \approx 4.22$.

$$A(\sqrt[3]{75}) = 4(\sqrt[3]{75})^2 + \frac{600}{\sqrt[3]{75}} \approx 213$$

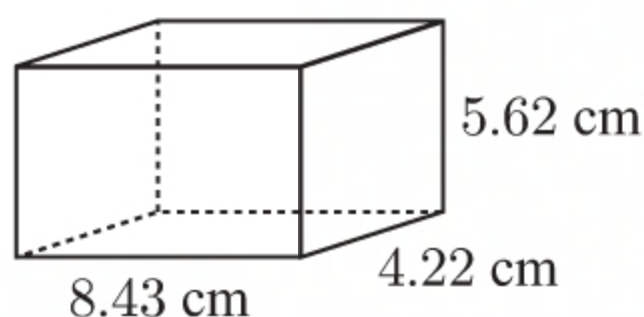
So, the minimum inner surface area is about 213 cm², when $x = \sqrt[3]{75} \approx 4.22$.

$$\text{d} \quad x = \sqrt[3]{75} \approx 4.22$$

$$\therefore 2x = 2\sqrt[3]{75} \approx 8.43$$

$$h = \frac{100}{x^2} = \frac{100}{(\sqrt[3]{75})^2} \approx 5.62$$

So, the optimal box shape is:



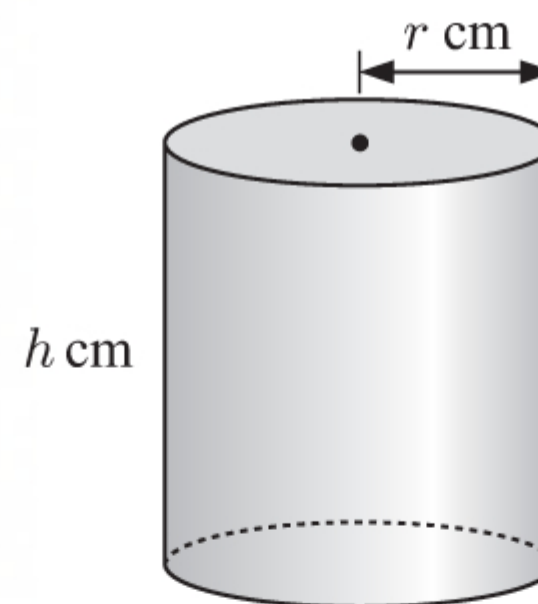
$$\text{6 a} \quad \text{volume of cylindrical can} = \pi r^2 h$$

Now, capacity is 1 litre which is equivalent to 1000 cm³.

So, the volume = 1000 cm³

$$\therefore \pi r^2 h = 1000$$

$$\therefore h = \frac{1000}{\pi r^2} \text{ cm}$$



$$\text{b} \quad \text{Total surface area of cylindrical can } A = 2\pi r^2 + 2\pi r h$$

$$\therefore A = 2\pi r^2 + 2\pi r \left(\frac{1000}{\pi r^2} \right) \quad \{\text{using a}\}$$

$$\therefore A = 2\pi r^2 + \frac{2000}{r} \text{ cm}^2$$

$$\text{c} \quad A = 2\pi r^2 + 2000r^{-1}$$

$$\begin{aligned}\therefore \frac{dA}{dr} &= 4\pi r - 2000r^{-2} \\ &= 4\pi r - \frac{2000}{r^2}\end{aligned}$$

$$\frac{dA}{dr} = 0 \quad \text{when} \quad 4\pi r - \frac{2000}{r^2} = 0$$

$$\therefore 4\pi r = \frac{2000}{r^2}$$

$$\therefore 4\pi r^3 = 2000$$

$$\therefore r^3 = \frac{500}{\pi}$$

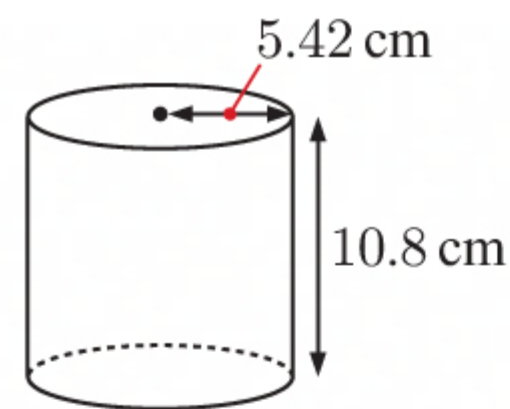
$$\therefore r = \sqrt[3]{\frac{500}{\pi}} \approx 5.42$$

$\frac{dA}{dr}$ has sign diagram:

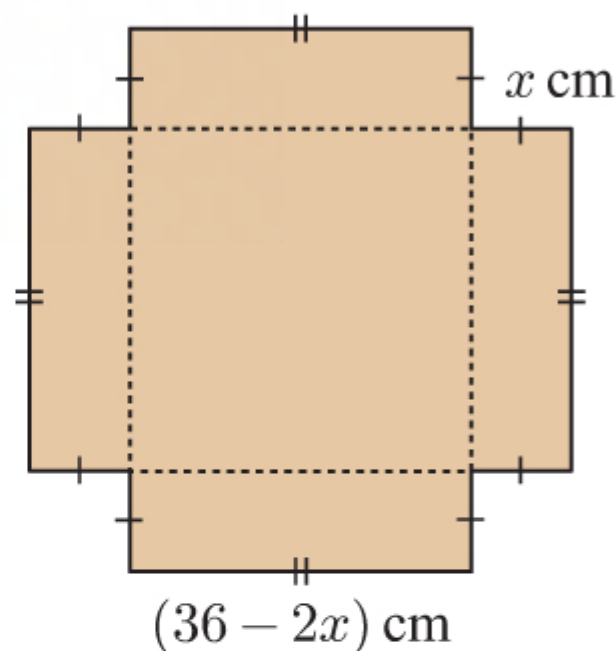
So, the total surface area is a minimum when $r = \sqrt[3]{\frac{500}{\pi}} \approx 5.42$.

When $r = \sqrt[3]{\frac{500}{\pi}}$, $h = \frac{1000}{\pi \left(\sqrt[3]{\frac{500}{\pi}} \right)^2} \approx 10.8$

So, the can should have dimensions:



7 a



The volume of the container is $V(x) = \text{area of base} \times \text{height}$

$$\therefore V(x) = (36 - 2x)^2 \times x$$

$$\therefore V(x) = x(36 - 2x)^2 \text{ cm}^3$$

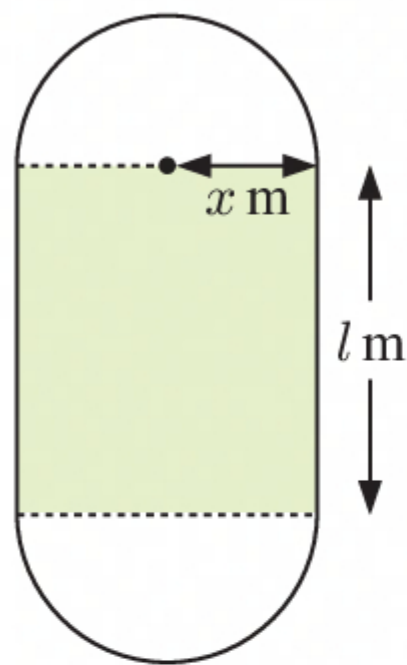
$$\begin{aligned}
 \text{b} \quad V(x) &= x(36 - 2x)^2 \\
 &= x(1296 - 144x + 4x^2) \\
 &= 1296x - 144x^2 + 4x^3 \\
 \therefore V'(x) &= 1296 - 288x + 12x^2 \\
 V'(x) = 0 \quad &\text{when} \quad 1296 - 288x + 12x^2 = 0 \\
 &\therefore 12(108 - 24x + x^2) = 0 \\
 &\therefore 12(x^2 - 24x + 108) = 0 \\
 &\therefore 12(x - 6)(x - 18) = 0
 \end{aligned}$$

$V'(x)$ has sign diagram:

The volume is a maximum when $x = 6$.

So, $6 \text{ cm} \times 6 \text{ cm}$ squares should be cut out to produce the container of greatest capacity.

8 a



$$\begin{aligned}
 \text{Perimeter} &= 2l + 2\pi x \\
 \therefore 400 &= 2l + 2\pi x \\
 \therefore 2l &= 400 - 2\pi x \\
 \therefore l &= 200 - \pi x \\
 x > 0 \quad &\text{and} \quad l > 0 \quad \text{for the track to exist} \\
 \therefore 200 - \pi x &> 0 \\
 \therefore \pi x &< 200 \\
 \therefore x &< \frac{200}{\pi} \\
 \text{So, } 0 < x &< \frac{200}{\pi} \approx 63.7
 \end{aligned}$$

$$\begin{aligned}
 \text{b} \quad \text{Area of rectangle } A &= 2x \times l \\
 &= 2x \times (200 - \pi x) \\
 \therefore A &= 400x - 2\pi x^2 \\
 \therefore \frac{dA}{dx} &= 400 - 4\pi x
 \end{aligned}$$

$$\begin{aligned}
 \frac{dA}{dx} = 0 \quad &\text{when} \quad 400 - 4\pi x = 0 \\
 \therefore 4\pi x &= 400 \\
 \therefore x &= \frac{100}{\pi}
 \end{aligned}$$

$\frac{dA}{dx}$ has sign diagram:

The area is a maximum when $x = \frac{100}{\pi} \approx 31.8$

$$\begin{aligned}
 \therefore l &= 200 - \pi \left(\frac{100}{\pi} \right) \\
 &= 100
 \end{aligned}$$

$$\begin{aligned}
 \text{When } x = \frac{100}{\pi}, \quad A &= 400\left(\frac{100}{\pi}\right) - 2\pi\left(\frac{100}{\pi}\right)^2 \\
 &= \frac{40\,000}{\pi} - \frac{20\,000}{\pi} \\
 &= \frac{20\,000}{\pi} \\
 &\approx 6370
 \end{aligned}$$

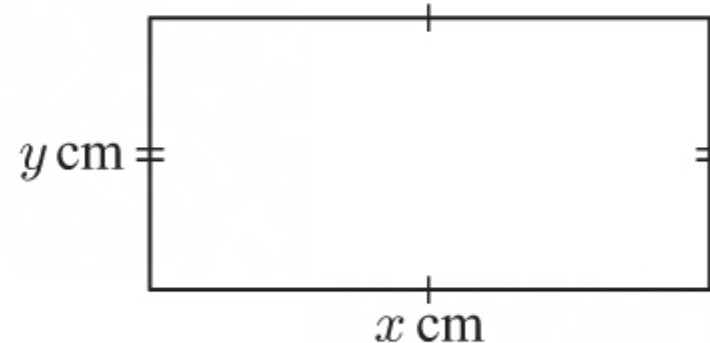
So, $l = 100$ and $x = \frac{100}{\pi} \approx 31.8$ give the maximum area $A = \frac{20\,000}{\pi} \approx 6370 \text{ m}^2$.

9 a Perimeter = 60 cm

$$\therefore 2x + 2y = 60$$

$$\therefore x + y = 30$$

$$\therefore y = 30 - x$$



b Area of rectangle $A = xy$

$$\therefore A(x) = x(30 - x) \text{ cm}^2 \quad \{\text{using a}\}$$

c $A(x) = 30x - x^2$

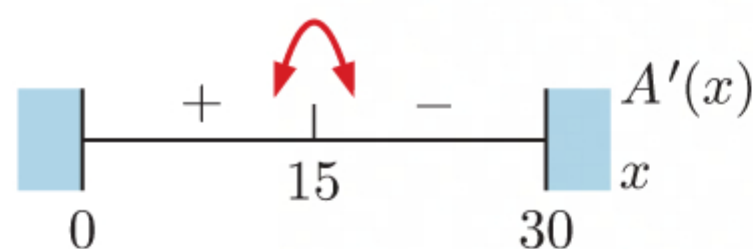
$$\therefore A'(x) = 30 - 2x$$

d $A'(x) = 0$ when $30 - 2x = 0$

$$\therefore 2x = 30$$

$$\therefore x = 15$$

$A'(x)$ has sign diagram:



The area is a maximum when $x = 15$

$$\begin{aligned}
 \therefore y &= 30 - 15 \quad \{\text{using a}\} \\
 &= 15
 \end{aligned}$$

\therefore the dimensions of the rectangle with maximum area are $15 \text{ cm} \times 15 \text{ cm}$.

10 a Let $CN = y \text{ cm}$

Now $x^2 + y^2 = 5^2$ {Pythagoras}

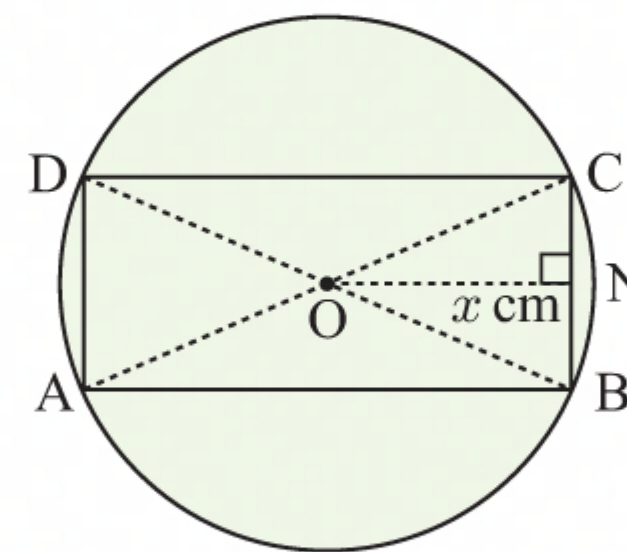
$$\therefore y = \sqrt{25 - x^2} \quad \{y > 0\}$$

Rectangle ABCD has area $A = \text{length} \times \text{width}$

$$= 2x \times 2y$$

$$= 4xy$$

$$= 4x\sqrt{25 - x^2} \text{ cm}^2$$



b $A = 4x\sqrt{25 - x^2} = 4x(25 - x^2)^{\frac{1}{2}}$

$$\therefore \frac{dA}{dx} = 4(25 - x^2)^{\frac{1}{2}} + 2x(25 - x^2)^{-\frac{1}{2}}(-2x) \quad \{\text{product rule and chain rule}\}$$

$$= 4\sqrt{25 - x^2} - \frac{4x^2}{\sqrt{25 - x^2}}$$

$$= \frac{4(25 - x^2) - 4x^2}{\sqrt{25 - x^2}}$$

$$= \frac{100 - 4x^2 - 4x^2}{\sqrt{25 - x^2}}$$

$$= \frac{100 - 8x^2}{\sqrt{25 - x^2}}$$

So, $\frac{dA}{dx} = 0$ when $100 - 8x^2 = 0$

$$\therefore 8x^2 = 100$$

$$\therefore x^2 = \frac{25}{2}$$

$$\therefore x = \frac{5}{\sqrt{2}} \quad \{\text{as } x > 0\}$$

$\frac{dA}{dx}$ has sign diagram:

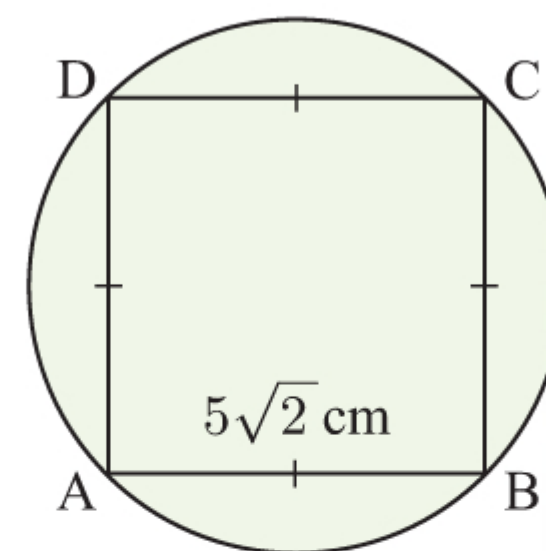
Area ABCD is maximised when $x = \frac{5}{\sqrt{2}}$ and $y = \sqrt{25 - \left(\frac{5}{\sqrt{2}}\right)^2}$

$$= \sqrt{25 - \frac{25}{2}}$$

$$= \sqrt{\frac{25}{2}}$$

$$= \frac{5}{\sqrt{2}}$$

Area ABCD is maximised when it is a square with side lengths $5\sqrt{2} \text{ cm} \times 5\sqrt{2} \text{ cm}$.



11 $C(x) = 4 \ln x + \left(\frac{30 - x}{10}\right)^2$ pounds, $x \geq 10$

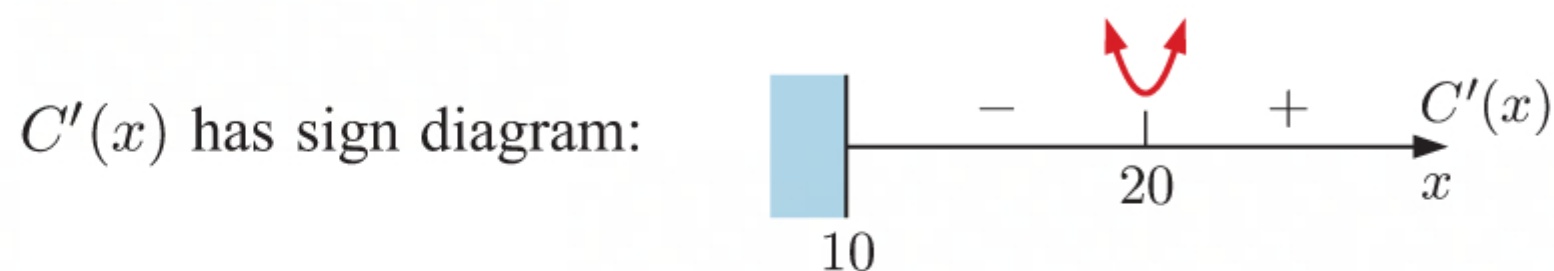
$$\therefore C'(x) = \frac{4}{x} + 2\left(\frac{30 - x}{10}\right)\left(-\frac{1}{10}\right)$$

$$= \frac{4}{x} - \left(\frac{30 - x}{50}\right)$$

$$= \frac{200 - x(30 - x)}{50x}$$

$$= \frac{x^2 - 30x + 200}{50x}$$

$$\begin{aligned}\text{So, } C'(x) = 0 \text{ when } x^2 - 30x + 200 &= 0 \\ \therefore (x - 10)(x - 20) &= 0 \\ \therefore x &= 10 \text{ or } 20\end{aligned}$$

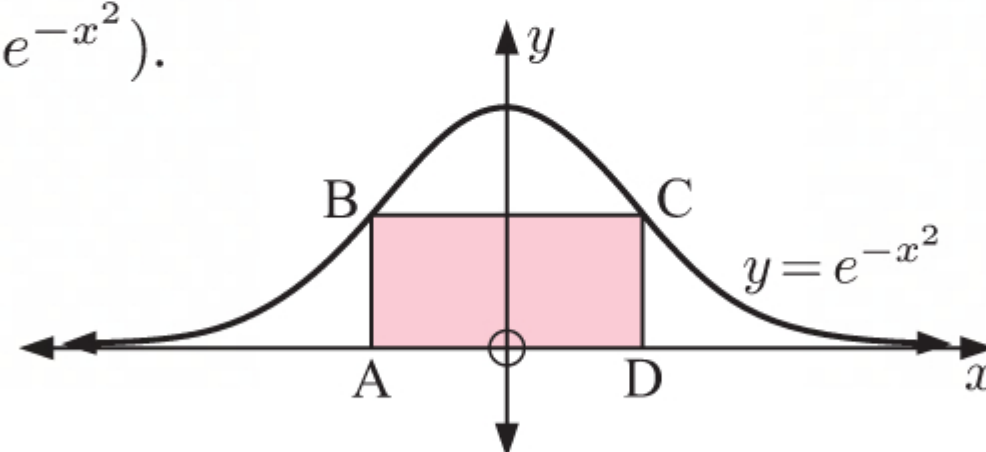


The cost per kettle is minimised when 20 kettles are manufactured per day.

12 Let D have x -coordinate x , so C has coordinates (x, e^{-x^2}) .

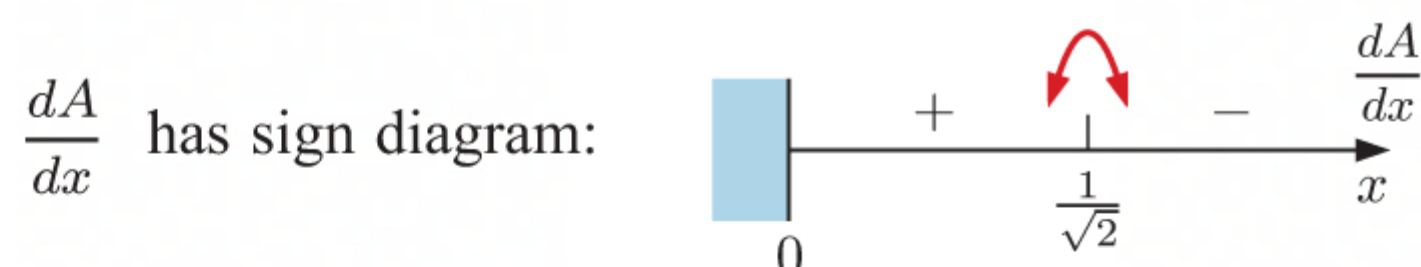
$$\therefore CD = e^{-x^2} \text{ and } AD = 2x$$

$$\begin{aligned}\therefore \text{rectangle ABCD has area } A &= \text{length} \times \text{width} \\ &= 2xe^{-x^2}\end{aligned}$$



$$\begin{aligned}A &= 2xe^{-x^2} \\ \therefore \frac{dA}{dx} &= 2e^{-x^2} + 2xe^{-x^2}(-2x) \quad \{\text{product rule and chain rule}\} \\ &= 2e^{-x^2} - 4x^2e^{-x^2} \\ &= 2e^{-x^2}(1 - 2x^2)\end{aligned}$$

$$\begin{aligned}\text{So, } \frac{dA}{dx} = 0 \text{ when } 1 - 2x^2 &= 0 \quad \{\text{as } e^{-x^2} > 0\} \\ \therefore 2x^2 &= 1 \\ \therefore x^2 &= \frac{1}{2} \\ \therefore x &= \frac{1}{\sqrt{2}} \quad \{\text{as } x > 0\}\end{aligned}$$



Area ABCD is maximised when $x = \frac{1}{\sqrt{2}}$.

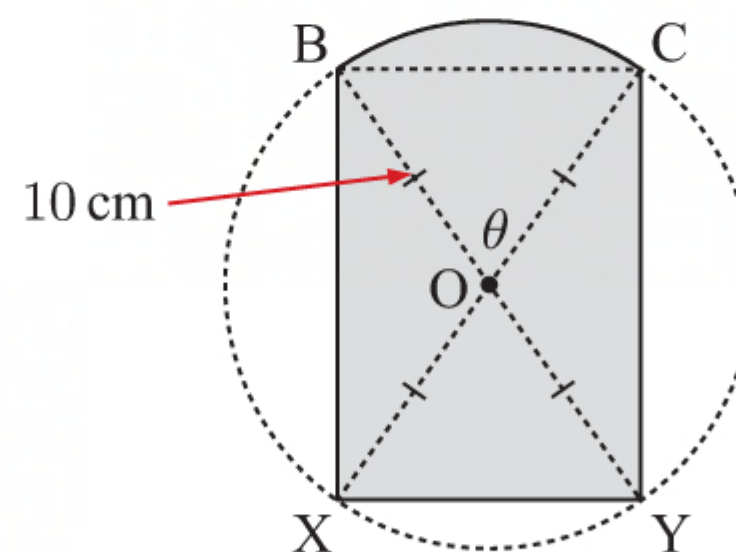
$$\begin{aligned}\text{When } x = \frac{1}{\sqrt{2}}, \quad y &= e^{-\left(\frac{1}{\sqrt{2}}\right)^2} \\ &= e^{-\frac{1}{2}}\end{aligned}$$

\therefore ABCD has maximum area when C is at $\left(\frac{1}{\sqrt{2}}, e^{-\frac{1}{2}}\right)$.

13 a Area of sector BOC $= \frac{1}{2}\theta r^2$
 $= \frac{1}{2}\theta(10)^2$
 $= 50\theta \text{ cm}^2$

Now $\widehat{BOX} = \frac{\pi}{2} - \theta$ {angles on a line}

$$\begin{aligned}\therefore \text{area of } \triangle BOX &= \frac{1}{2}ab \sin\left(\frac{\pi}{2} - \theta\right) \\ &= \frac{1}{2} \times 10 \times 10 \times \sin \theta \\ &\quad \{\sin(\frac{\pi}{2} - \theta) = \sin \theta\} \\ &= 50 \sin \theta \text{ cm}^2\end{aligned}$$



Similarly, $\widehat{COY} = \frac{\pi}{2} - \theta$ and area of $\triangle COY = 50 \sin \theta \text{ cm}^2$

Also, $\widehat{XOY} = \widehat{BOC} = \theta$ {vertically opposite angles}

$$\begin{aligned}\therefore \text{area of } \triangle XOY &= \frac{1}{2}ab \sin \theta \\ &= \frac{1}{2} \times 10 \times 10 \times \sin \theta \\ &= 50 \sin \theta \text{ cm}^2\end{aligned}$$

$$\begin{aligned}\text{Shaded area } A &= \text{area of sector } BOC + \text{area of } \triangle BOX \\ &\quad + \text{area of } \triangle COY + \text{area of } \triangle XOY \\ &= 50\theta + 50 \sin \theta + 50 \sin \theta + 50 \sin \theta \\ &= 50\theta + 150 \sin \theta \\ &= 50(\theta + 3 \sin \theta) \text{ cm}^2\end{aligned}$$

b $A = 50(\theta + 3 \sin \theta)$

$$\therefore \frac{dA}{d\theta} = 50(1 + 3 \cos \theta)$$

So, $\frac{dA}{d\theta} = 0$ when $1 + 3 \cos \theta = 0$

$$\therefore 3 \cos \theta = -1$$

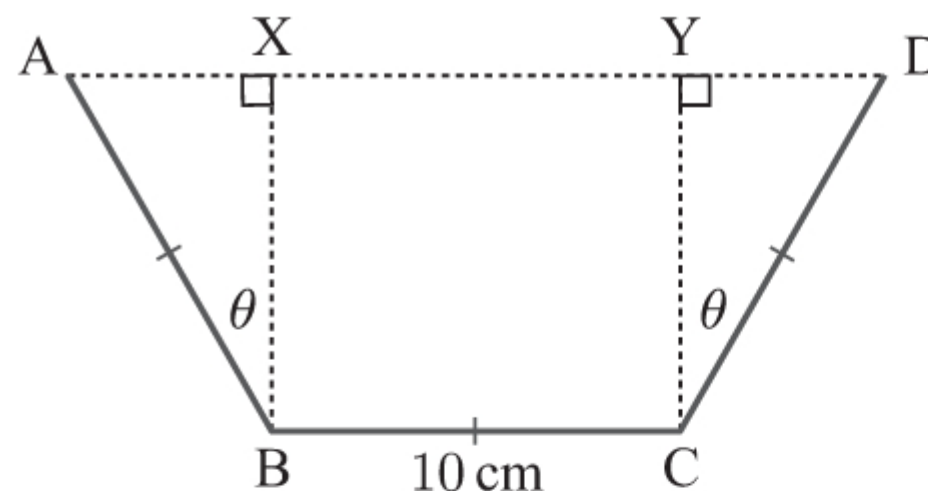
$$\therefore \cos \theta = -\frac{1}{3}$$

$$\therefore \theta \approx 1.91 \quad \{0 < \theta < \pi\}$$

$$\begin{aligned}\text{When } \theta \approx 1.91, \quad A &\approx 50(1.91 + 3 \sin 1.91) \\ &\approx 237\end{aligned}$$

The area A has a maximum of about 237 cm^2 when $\theta \approx 1.91$.

14 a In $\triangle ABX$, $\cos \theta = \frac{BX}{10}$
 $\therefore BX = CY = 10 \cos \theta \text{ cm}$
 $\sin \theta = \frac{AX}{10}$
 $\therefore AX = DY = 10 \sin \theta \text{ cm}$



Now, the cross-sectional area A

$$\begin{aligned}&= \text{area } BCYX + \text{area of } \triangle ABX + \text{area of } \triangle CDY \\ &= 10 \times 10 \cos \theta + \frac{1}{2} \times 10 \sin \theta \times 10 \cos \theta + \frac{1}{2} \times 10 \sin \theta \times 10 \cos \theta \\ &= 100 \cos \theta + 100 \sin \theta \cos \theta \\ &= 100 \cos \theta(1 + \sin \theta) \text{ cm}^2\end{aligned}$$

b $A = 100 \cos \theta(1 + \sin \theta)$

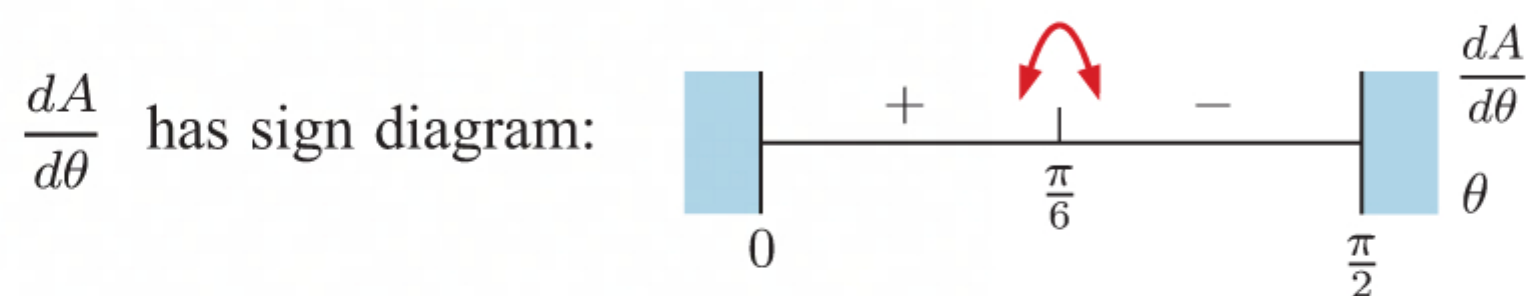
$$\begin{aligned}\therefore \frac{dA}{d\theta} &= 100(-\sin \theta(1 + \sin \theta) + \cos \theta \times \cos \theta) \quad \{\text{product rule}\} \\ &= 100(-\sin \theta - \sin^2 \theta + \cos^2 \theta) \\ &= 100(-\sin \theta - \sin^2 \theta + 1 - \sin^2 \theta) \\ &= -100(2 \sin^2 \theta + \sin \theta - 1) \\ &= -100(2 \sin \theta - 1)(\sin \theta + 1)\end{aligned}$$

$$\begin{aligned}\therefore \frac{dA}{d\theta} = 0 \quad \text{when} \quad 2 \sin \theta - 1 = 0 \quad \text{or} \quad \sin \theta + 1 = 0 \\ \therefore \sin \theta = \frac{1}{2} \quad \text{or} \quad \sin \theta = -1\end{aligned}$$

- c The gutter has maximum carrying capacity when the cross-sectional area A is maximised.

$$\frac{dA}{d\theta} = 0 \quad \text{when} \quad \sin \theta = \frac{1}{2} \quad \text{or} \quad -1$$

$$\therefore \theta = \frac{\pi}{6} \quad \{0 \leq \theta \leq \frac{\pi}{2}\}$$



The carrying capacity is maximised when $\theta = \frac{\pi}{6}$.

$$\begin{aligned} \text{When } \theta = \frac{\pi}{6}, \quad A &= 100 \cos \frac{\pi}{6} \left(1 + \sin \frac{\pi}{6}\right) \\ &= 100 \times \frac{\sqrt{3}}{2} \times \frac{3}{2} \\ &= 75\sqrt{3} \\ &\approx 130 \end{aligned}$$

\therefore the cross-sectional area for the maximum carrying capacity is about 130 cm^2 .

15 a $E(t) = 750te^{-1.5t}$ units, $t \geq 0$

$$\therefore E'(t) = 750e^{-1.5t} + 750te^{-1.5t}(-1.5) \quad \{\text{product rule}\}$$

$$= 750e^{-1.5t}(1 - 1.5t)$$

b $E'(t) = 0$ when $750e^{-1.5t}(1 - 1.5t) = 0$

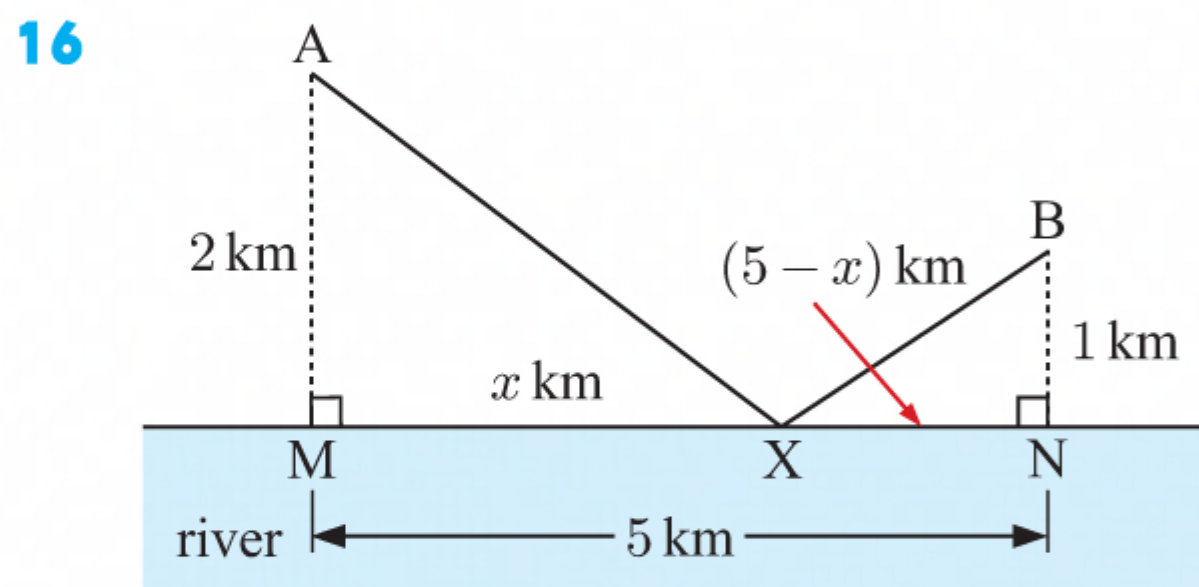
$$\therefore 1 - 1.5t = 0 \quad \{\text{as } e^{-1.5t} > 0\}$$

$$\therefore 1.5t = 1$$

$$\therefore t = \frac{2}{3} \text{ hours } (= 40 \text{ minutes})$$



The anaesthetic is most effective 40 minutes after the injection.



Let $MX = x$ km, so $XN = 5 - x$ km

$$\therefore AX = \sqrt{2^2 + x^2} \text{ km} \quad \text{and} \quad XB = \sqrt{1^2 + (5 - x)^2} \text{ km} \quad \{\text{Pythagoras}\}$$

Let the total length of pipeline required be P km.

Now $P = AX + XB$

$$= (4 + x^2)^{\frac{1}{2}} + (26 - 10x + x^2)^{\frac{1}{2}}$$

$$\therefore \frac{dP}{dx} = \frac{1}{2}(4 + x^2)^{-\frac{1}{2}}(2x) + \frac{1}{2}(26 - 10x + x^2)^{-\frac{1}{2}}(-10 + 2x) \quad \{\text{chain rule}\}$$

$$= \frac{x}{\sqrt{4 + x^2}} + \frac{x - 5}{\sqrt{x^2 - 10x + 26}}$$

$$\text{Now } \frac{dP}{dx} = 0 \text{ when } \frac{x}{\sqrt{4 + x^2}} + \frac{x - 5}{\sqrt{x^2 - 10x + 26}} = 0$$

$$\therefore \frac{x}{\sqrt{4 + x^2}} = \frac{5 - x}{\sqrt{x^2 - 10x + 26}}$$

$$\therefore \frac{x^2}{4 + x^2} = \frac{(5 - x)^2}{x^2 - 10x + 26}$$

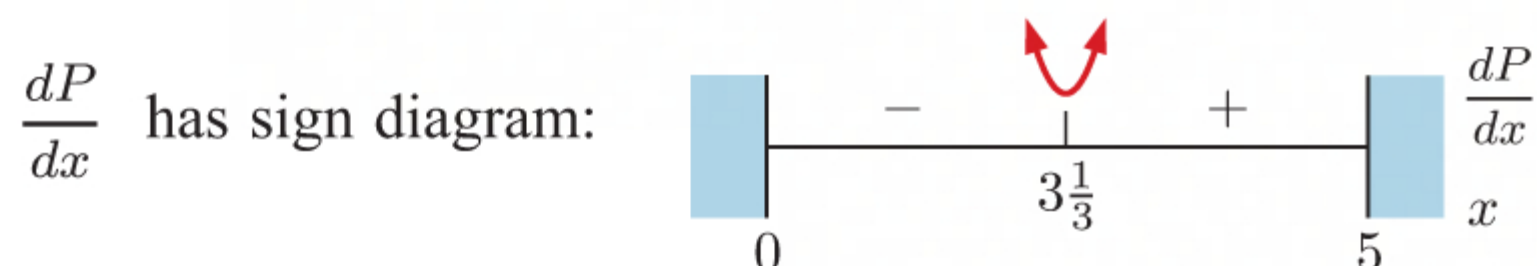
$$\therefore x^2(x^2 - 10x + 26) = (4 + x^2)(25 - 10x + x^2)$$

$$\therefore \cancel{x^4} - \cancel{10x^3} + 26x^2 = 100 - 40x + 4x^2 + 25x^2 - \cancel{10x^3} + \cancel{x^4}$$

$$\therefore -3x^2 + 40x - 100 = 0$$

$$\therefore -(3x - 10)(x - 10) = 0$$

$$\therefore x = \frac{10}{3} = 3\frac{1}{3} \quad \{\text{as } 0 \leq x \leq 5\}$$



The minimum length pipeline occurs when $x = 3\frac{1}{3}$ km.

\therefore X should be $3\frac{1}{3}$ km from M to minimise the total length of pipeline.

17

$$P(t) = \frac{50\,000}{1 + 1000e^{-0.5t}}, \quad 0 \leq t \leq 25$$

$$= 50\,000(1 + 1000e^{-0.5t})^{-1}$$

$$\therefore P'(t) = -50\,000(1 + 1000e^{-0.5t})^{-2}(-500e^{-0.5t})$$

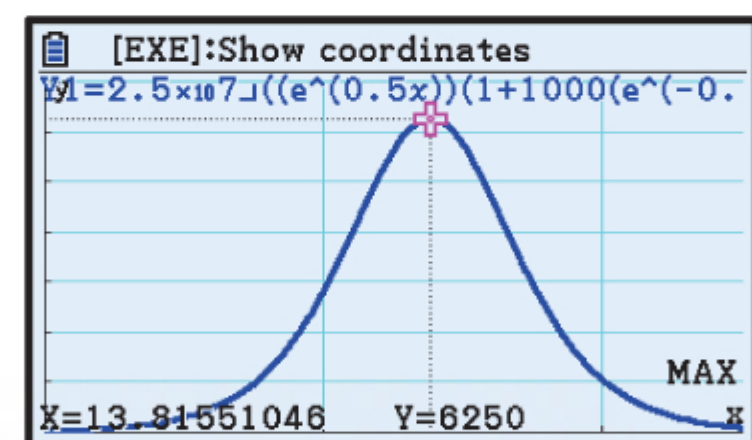
$$= \frac{2.5 \times 10^7}{e^{0.5t}(1 + 1000e^{-0.5t})^2}$$

The wasp population is growing the fastest when $\frac{dP}{dt}$ is a maximum.

We use technology to draw the graph of $P'(t)$ and find where it is maximised.

The maximum occurs when $t \approx 13.8$.

\therefore the wasp population is growing fastest after about 13.8 weeks.



- 18 a** Consider each boat's position t hours after 1:00 pm.

$$PA = 12t \quad \text{and} \quad QB = 8t$$

$$\therefore PB = 100 - 8t$$

Using the cosine rule in $\triangle PAB$,

$$\begin{aligned} [D(t)]^2 &= PA^2 + PB^2 - 2PA \times PB \cos 60^\circ \\ &= (12t)^2 + (100 - 8t)^2 - 2(12t)(100 - 8t)\left(\frac{1}{2}\right) \\ &= 144t^2 + 10\,000 - 1600t + 64t^2 - 12t(100 - 8t) \\ &= 144t^2 + 10\,000 - 1600t + 64t^2 - 1200t + 96t^2 \\ &= 304t^2 - 2800t + 10\,000 \end{aligned}$$

$$\therefore D(t) = \sqrt{304t^2 - 2800t + 10\,000} \quad \{D(t) > 0\}$$

b Now $\frac{d[D(t)]^2}{dt} = 608t - 2800$

$$\therefore \frac{d[D(t)]^2}{dt} = 0 \quad \text{when} \quad t = \frac{2800}{608} \approx 4.605$$

$\frac{d[D(t)]^2}{dt}$ has sign diagram:

$\therefore D(t)$ is a minimum when $t \approx 4.605$ hours after 1:00 pm

$$\begin{aligned} \text{When } t \approx 4.605, \quad [D(t)]^2 &\approx 304(4.605)^2 - 2800(4.605) + 10\,000 \\ &\approx 3550 \end{aligned}$$

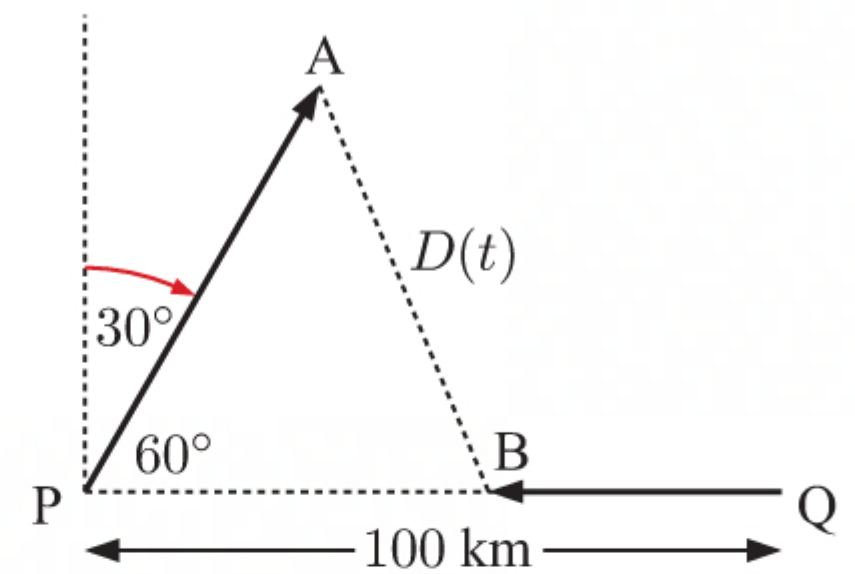
\therefore the minimum value of D^2 is about 3550.

- c** The ships are closest when $t \approx 4.605$ hours

$$0.605 \text{ hours} \approx 0.605 \times 60$$

$$\approx 36 \text{ minutes}$$

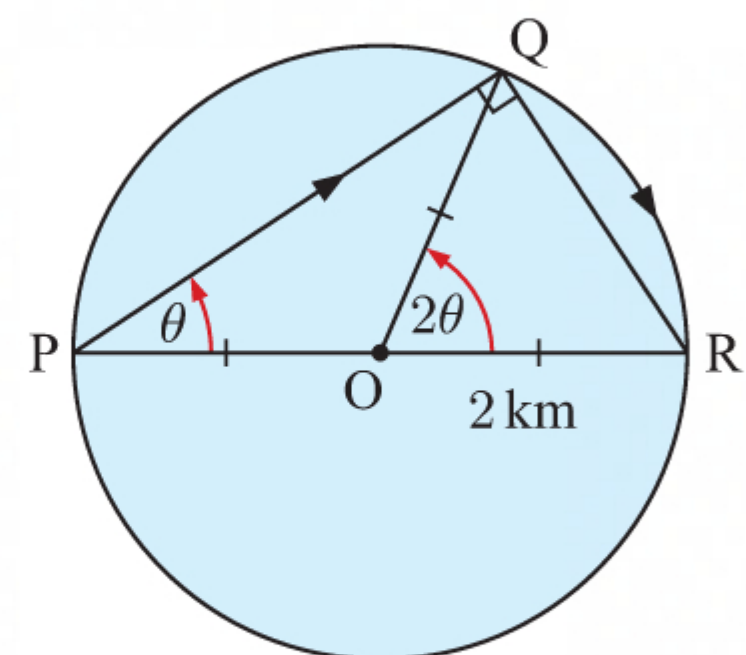
\therefore the ships are closest together 4 hours and 36 minutes after 1:00 pm, which is 5:36 pm.



- 19 a** $\widehat{PQR} = 90^\circ$ {angle in a semi-circle theorem}

$$\therefore \cos \theta = \frac{PQ}{4}$$

$$\therefore PQ = 4 \cos \theta \text{ km}$$



- b** Hieu can row at 3 km h^{-1} .

$$\begin{aligned} \therefore \text{time taken to row from P to Q} &= \frac{\text{distance}}{\text{speed}} \\ &= \frac{4 \cos \theta}{3} \\ &= \frac{4}{3} \cos \theta \text{ hours} \end{aligned}$$

Now $\widehat{QOR} = 2\theta$ {angle at the centre theorem}

$$\begin{aligned}\therefore \text{arc QR} &= 2\theta \times r \\ &= 2\theta \times 2 \\ &= 4\theta \text{ km}\end{aligned}$$

Hieu can walk at 6 km h^{-1} .

$$\begin{aligned}\therefore \text{time taken to walk along arc QR} &= \frac{\text{distance}}{\text{speed}} \\ &= \frac{4\theta}{6} \\ &= \frac{2\theta}{3} \text{ hours}\end{aligned}$$

\therefore the time taken for Hieu's journey $T = \frac{4}{3} \cos \theta + \frac{2\theta}{3}$ hours where $0 \leq \theta \leq \frac{\pi}{2}$.

c $T = \frac{4}{3} \cos \theta + \frac{2\theta}{3}, \quad 0 \leq \theta \leq \frac{\pi}{2}$

$$\therefore \frac{dT}{d\theta} = -\frac{4}{3} \sin \theta + \frac{2}{3}$$

$$\therefore \frac{dT}{d\theta} = 0 \text{ when } -\frac{4}{3} \sin \theta + \frac{2}{3} = 0$$

$$\therefore \frac{4}{3} \sin \theta = \frac{2}{3}$$

$$\therefore \sin \theta = \frac{1}{2}$$

$$\therefore \theta = \frac{\pi}{6} \quad \{0 \leq \theta \leq \frac{\pi}{2}\}$$

d $\frac{dT}{d\theta}$ has sign diagram:

e i From the sign diagram in **d**, the time T is a maximum when $\theta = \frac{\pi}{6}$. So the longest time taken involves Hieu rowing from P to Q at an angle of $\frac{\pi}{6}$ to the diameter of the lake, then walking from Q to R.

ii The maximum value of T occurs when $\theta = \frac{\pi}{6}$.

\therefore the minimum value of T must occur at one of the end points, that is, when $\theta = 0$ or $\theta = \frac{\pi}{2}$.

When $\theta = 0$, this means Hieu must row directly from P to R.

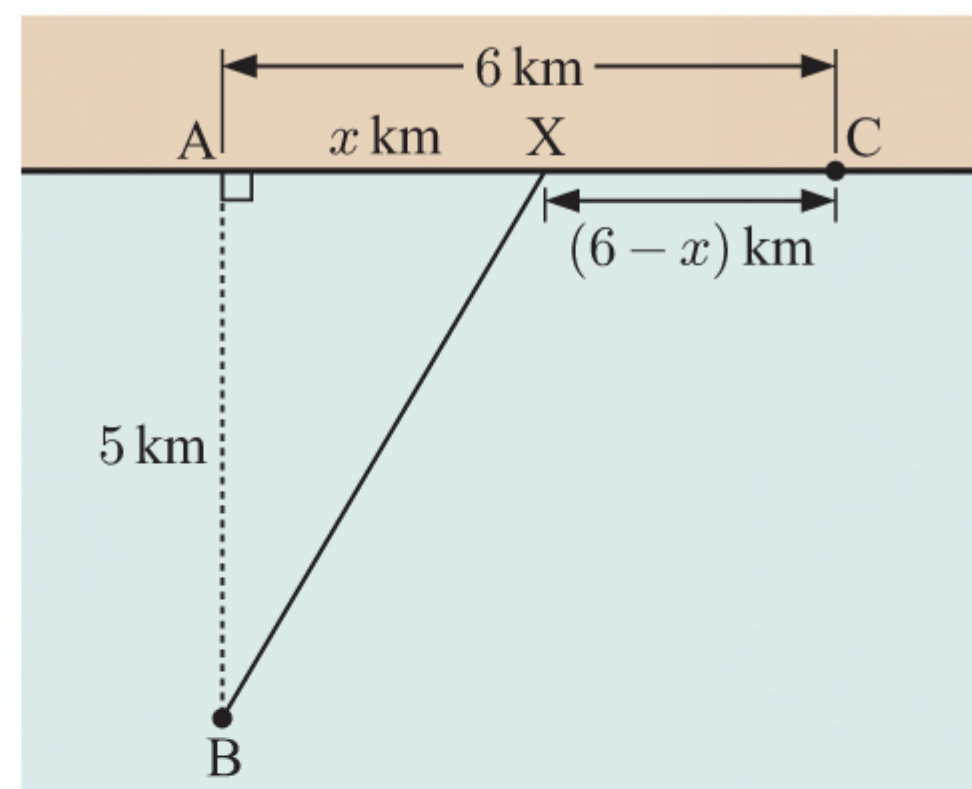
When $\theta = \frac{\pi}{2}$, this means Hieu must walk the whole way around the shore from P to R.

Firstly, when $\theta = 0$, $T = \frac{4}{3} \cos 0 + \frac{2(0)}{3} = \frac{4}{3} \approx 1.33$ hours.

Secondly, when $\theta = \frac{\pi}{2}$, $T = \frac{4}{3} \cos \frac{\pi}{2} + \frac{2(\frac{\pi}{2})}{3} = \frac{\pi}{3} \approx 1.05$ hours.

So, the shortest time taken involves Hieu walking the whole way around the shore from P to R.

- 20 a** AC has length 6 km and X lies between A and C.
 $\therefore 0 \leq x \leq 6$



- b** Now $XC = 6 - x$ and $BX = \sqrt{x^2 + 5^2}$ {Pythagoras}

$$\therefore \text{the time taken to row from B to X} = \frac{\text{distance}}{\text{speed}} = \frac{BX}{8} = \frac{\sqrt{x^2 + 5^2}}{8} \text{ hours}$$

$$\text{and the time taken to run from X to C} = \frac{\text{distance}}{\text{speed}} = \frac{XC}{17} = \frac{6 - x}{17} \text{ hours}$$

$$\therefore \text{the total time } T = \frac{\sqrt{x^2 + 25}}{8} + \frac{6 - x}{17} \text{ hours, } 0 \leq x \leq 6$$

c

$$T = \frac{\sqrt{x^2 + 25}}{8} + \frac{6 - x}{17}$$

$$= \frac{1}{8}(x^2 + 25)^{\frac{1}{2}} + \frac{6}{17} - \frac{x}{17}$$

$$\therefore \frac{dT}{dx} = \frac{1}{16}(x^2 + 25)^{-\frac{1}{2}}(2x) - \frac{1}{17} \quad \{\text{chain rule}\}$$

$$= \frac{x}{8\sqrt{x^2 + 25}} - \frac{1}{17}$$

$$\text{So, } \frac{dT}{dx} = 0 \text{ when } \frac{x}{8\sqrt{x^2 + 25}} = \frac{1}{17}$$

$$\therefore 17x = 8\sqrt{x^2 + 25}$$

$$\therefore 289x^2 = 64(x^2 + 25)$$

$$\therefore 289x^2 = 64x^2 + 1600$$

$$\therefore 225x^2 = 1600$$

$$\therefore x^2 = \frac{1600}{225}$$

$$\therefore x = \frac{40}{15} \quad \{x > 0\}$$

$$= \frac{8}{3} \approx 2.67$$

$\frac{dT}{dx}$ has sign diagram:

$\therefore x \approx 2.67$ is the distance in km from A to X which minimises the time taken for Peter to travel from B to C.

- 21 a** At time t , the mosquito is at $(3 - t^2, 2 + \sqrt{t}, 2 - \sqrt{t})$.
 \therefore the mosquito's distance from the origin

$$\begin{aligned} D &= \sqrt{(3 - t^2)^2 + (2 + \sqrt{t})^2 + (2 - \sqrt{t})^2} \\ &= \sqrt{9 - 6t^2 + t^4 + 4 + 4\sqrt{t} + t + 4 - 4\sqrt{t} + t} \\ &= \sqrt{t^4 - 6t^2 + 2t + 17} \end{aligned}$$

$$\therefore D^2 = t^4 - 6t^2 + 2t + 17$$

b $\frac{d}{dt}(D^2) = 4t^3 - 12t + 2$

Using technology,

$$\frac{d}{dt}(D^2) = 0 \text{ when } t \approx 1.6418$$

or $\approx 0.1683 \quad \{t \geq 0\}$

$\frac{d}{dt}(D^2)$ has sign diagram:

Now, D is minimised when D^2 is minimised, since $D > 0$.

$$D(1.6418) \approx 3.37$$

So, the closest the mosquito came to the source of the repellent was about 3.37 m.

ACTIVITY

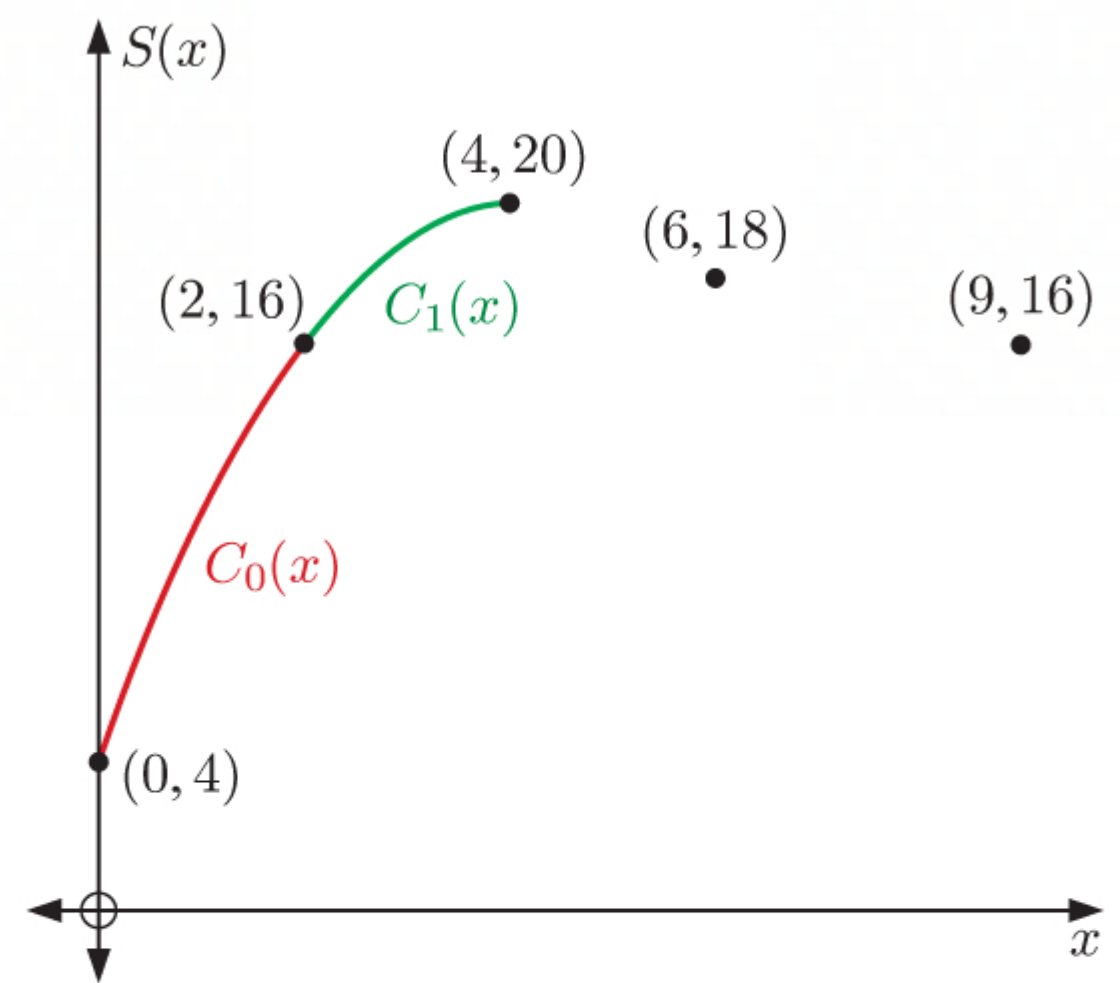
CUBIC SPLINES

- 2 a i** $C_i(x) = a_i(x - x_i)^3 + b_i(x - x_i)^2 + c_i(x - x_i) + d_i$
 $\therefore C_i'(x) = a_i \times 3(x - x_i)^2(1) + b_i \times 2(x - x_i)(1) + c_i(1) \quad \{\text{chain rule}\}$
 $\therefore C_i'(x) = 3a_i(x - x_i)^2 + 2b_i(x - x_i) + c_i$
- ii** $C_i''(x) = 3a_i \times 2(x - x_i)(1) + 2b_i(1) \quad \{\text{chain rule}\}$
 $\therefore C_i''(x) = 6a_i(x - x_i) + 2b_i$

b $C_i(x_i) = a_i(x_i - x_i)^3 + b_i(x_i - x_i)^2 + c_i(x_i - x_i) + d_i$
 $= d_i$
 $C_i'(x_i) = 3a_i(x_i - x_i)^2 + 2b_i(x_i - x_i) + c_i$
 $= c_i$
 $C_i''(x_i) = 6a_i(x_i - x_i) + 2b_i$
 $= 2b_i$

- 3**
- $C_i(x_i) = C_{i-1}(x_i)$ ensures that as we transition from the $(i - 1)$ th cubic to the i th cubic at x_i , the curve is continuous. This requirement gives us the data point at the left end of $C_i(x)$.
 - $C_i'(x_i) = C_{i-1}'(x_i)$ ensures that as we transition from the $(i - 1)$ th cubic to the i th cubic at x_i , the gradients are the same. This requirement gives us the gradient at the left end of $C_i(x)$.
 - $C_i''(x_i) = C_{i-1}''(x_i)$ ensures that as we transition from the $(i - 1)$ th cubic to the i th cubic at x_i , the cubics have the same shape. This requirement gives us the curvature at the left end of $C_i(x)$.

- 4 a i** $Q(x) = -x^2 + 8x + 4$
 $Q(0) = 4$ ✓
 $Q(2) = -(2)^2 + 8(2) + 4$
 $= -4 + 16 + 4$
 $= 16$ ✓
 $Q(4) = -(4)^2 + 8(4) + 4$
 $= -16 + 32 + 4$
 $= 20$ ✓
 $\therefore (0, 4), (2, 16), \text{ and } (4, 20)$ all lie
on $Q(x) = -x^2 + 8x + 4$.



- ii** $C_0''(x_0) = 2b_0$ {from **2 b**}

Now $Q(x) = -x^2 + 8x + 4$

$$\therefore Q'(x) = -2x + 8$$

$$\therefore Q''(x) = -2$$

$$\therefore Q''(x_0) = -2$$

If $C_0''(x_0) = Q''(x_0)$

then $2b_0 = -2$

$$\therefore b_0 = -1$$

- iii** $C_0'(x_0) = c_0$ {from **2 b**}

$$Q'(x) = -2x + 8$$

$$\therefore Q'(x_0) = -2x_0 + 8$$

$$= 8 \quad \{x_0 = 0\}$$

If $C_0'(x_0) = Q'(x_0)$

then $c_0 = 8$

- iv** $C_0(x_0) = d_0$ {from **2 b**}

If $C_0(x_0) = y_0$

then $d_0 = 4$ { $y_0 = 4$ }

v $C_0(x) = a_0(x - x_0)^3 + b_0(x - x_0)^2 + c_0(x - x_0) + d_0$
 $\therefore C_0(x_1) = a_0(x_1 - x_0)^3 + b_0(x_1 - x_0)^2 + c_0(x_1 - x_0) + d_0$
 $= a_0(2 - 0)^3 + b_0(2 - 0)^2 + c_0(2 - 0) + d_0 \quad \{x_1 = 2\}$
 $= 8a_0 - 4 + 16 + 4$
 $= 8a_0 + 16$

If $C_0(x_1) = y_1$

then $8a_0 + 16 = 16$ { $y_1 = 16$ }

$$\therefore 8a_0 = 0$$

$$\therefore a_0 = 0$$

So, $C_0(x) = -x^2 + 8x + 4, \quad 0 \leq x \leq 2$

b i $C_1''(x_1) = 2b_1$ {from **2 b**}

$$C_0(x) = -x^2 + 8x + 4$$

$$\therefore C_0'(x) = -2x + 8$$

$$\therefore C_0''(x) = -2$$

$$\therefore C_0''(x_1) = -2$$

If $C_1''(x_1) = C_0''(x_1)$

then $2b_1 = -2$

$$\therefore b_1 = -1$$

ii $C_1'(x_1) = c_1$ {from **2 b**}

$$C_0'(x) = -2x + 8$$

$$\therefore C_0'(x_1) = -2x_1 + 8$$

$$= -2(2) + 8 \quad \{x_1 = 2\}$$

$$= -4 + 8$$

$$= 4$$

If $C_1'(x_1) = C_0'(x_1)$

then $c_1 = 4$

iii $C_1(x_1) = d_1$ {from **2 b**}

If $C_1(x_1) = y_1$

then $d_1 = 16$ $\{y_1 = 16\}$

iv $C_1(x) = a_1(x - x_1)^3 + b_1(x - x_1)^2 + c_1(x - x_1) + d_1$

$$\therefore C_1(x_2) = a_1(x_2 - x_1)^3 + b_1(x_2 - x_1)^2 + c_1(x_2 - x_1) + d_1$$

$$= a_1(4 - 2)^3 - (4 - 2)^2 + 4(4 - 2) + 16 \quad \{x_2 = 4\}$$

$$= 8a_1 - 4 + 8 + 16$$

$$= 8a_1 + 20$$

If $C_1(x_2) = y_2$

then $8a_1 + 20 = 20$ $\{y_2 = 20\}$

$$\therefore 8a_1 = 0$$

$$\therefore a_1 = 0$$

So, $C_1(x) = -(x - 2)^2 + 4(x - 2) + 16, \quad 2 < x \leq 4$

c i From the spreadsheet, $a_2 = \frac{1}{4}, \quad b_2 = -1, \quad c_2 = 0, \quad d_2 = 20$

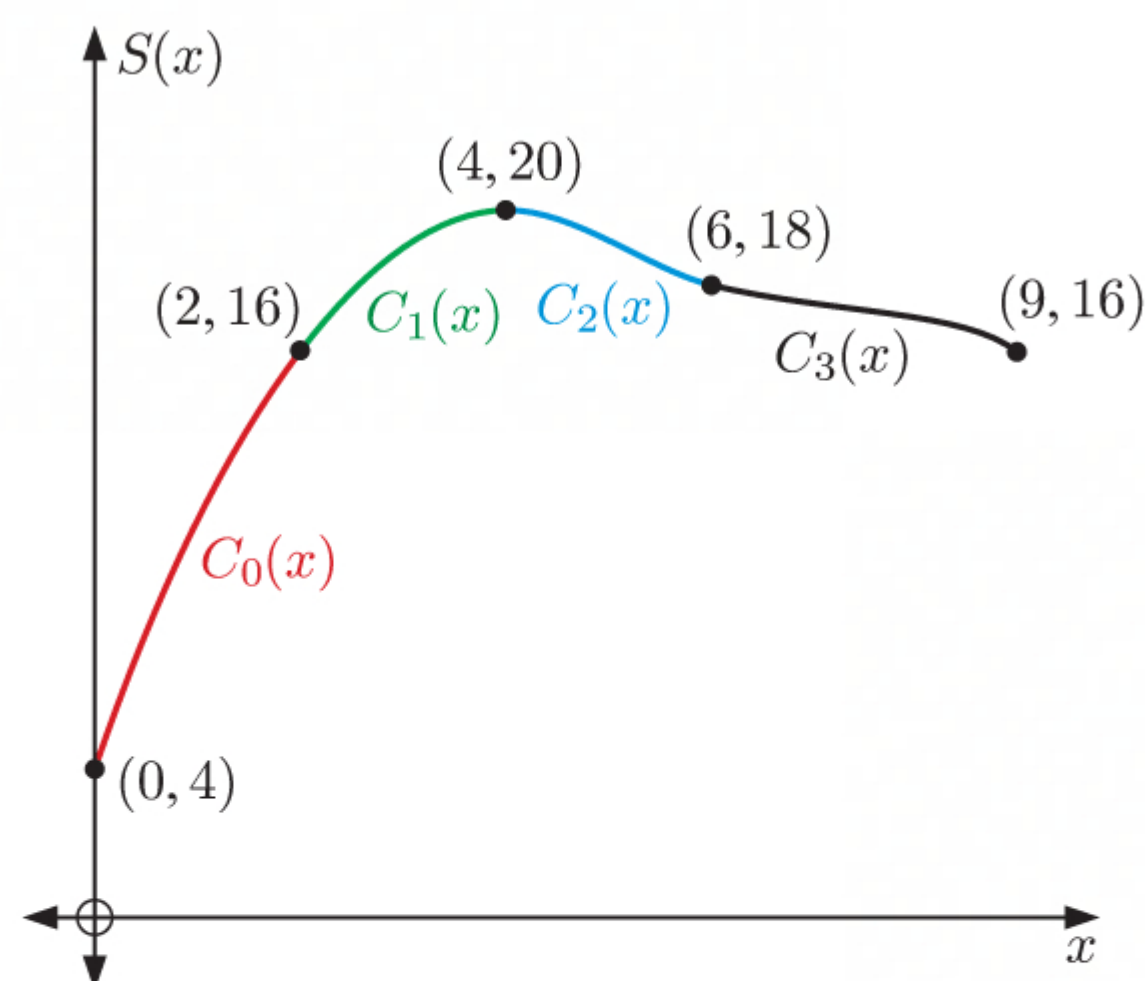
$$a_3 = -\frac{7}{54}, \quad b_3 = \frac{1}{2}, \quad c_3 = -1, \quad d_3 = 18$$

Now $x_2 = 4, \quad x_3 = 6$

$$\therefore C_2(x) = \frac{1}{4}(x - 4)^3 - (x - 4)^2 + 20,$$

$$C_3(x) = -\frac{7}{54}(x - 6)^3 + \frac{1}{2}(x - 6)^2 - (x - 6) + 18$$

- iii The cubic spline passes through each of the data points, and appears to be a good representation of the data.



- 5 a The data points $(0, 1)$, $(2, 7.39)$, $(4, 54.6)$, $(6, 403.4)$, $(8, 2981)$, and $(10, 22\,026)$ are of the form (x, e^x) where $x = 0, 2, 4, 6, 8, 10$.

x	1	3.5	4.25	5.25	7.5	9
e^x	2.718	33.135	70.105	190.566	1808.042	8103.084
$S(x)$	-0.907	38.971	63.881	168.500	1895.985	7724.231

- b For the x -values 3.5, 4.25, and 7.5, which are close to our original x -values, $S(x)$ approximates e^x reasonably well. For the x -values 1, 5.25, and 9, which are not as close to our original x -values, $S(x)$ is a less accurate approximation of e^x .
- c In a, $S(x)$ predicts that $e^1 = -0.907$, which we know is absurd since $e^x > 0$ for all x . If we add more data values, such as $(1, 2.718)$, to the spreadsheet, the accuracy of the approximation $S(x)$ is greatly improved.

REVIEW SET 14A

1 $H(t) = 60 + 40 \ln(2t + 1)$ cm, $t \geq 0$

a $H(0) = 60 + 40 \ln(2(0) + 1)$
 $= 60$

\therefore the tree was 60 cm tall when it was planted.

b i When $H = 150$,

$$60 + 40 \ln(2t + 1) = 150$$

$$\therefore 40 \ln(2t + 1) = 90$$

$$\therefore \ln(2t + 1) = \frac{9}{4}$$

$$\therefore 2t + 1 = e^{\frac{9}{4}}$$

$$\therefore 2t = e^{\frac{9}{4}} - 1$$

$$\therefore t = \frac{e^{\frac{9}{4}} - 1}{2} \approx 4.24$$

\therefore it will take about 4.24 years for the tree to reach a height of 150 cm.

ii When $H = 300$,

$$60 + 40 \ln(2t + 1) = 300$$

$$\therefore 40 \ln(2t + 1) = 240$$

$$\therefore \ln(2t + 1) = 6$$

$$\therefore 2t + 1 = e^6$$

$$\therefore 2t = e^6 - 1$$

$$\therefore t = \frac{e^6 - 1}{2} \approx 201$$

\therefore it will take about 201 years for the tree to reach a height of 300 cm.

c $H(t) = 60 + 40 \ln(2t + 1) \text{ cm}, \quad t \geq 0$

$$\begin{aligned} \therefore H'(t) &= 40 \left(\frac{2}{2t+1} \right) \\ &= \frac{80}{2t+1} \text{ cm per year} \end{aligned}$$

i $H'(2) = \frac{80}{2(2)+1} = 16$

\therefore after 2 years, the tree's height is increasing at a rate of 16 cm per year.

ii $H'(20) = \frac{80}{2(20)+1}$
 $= \frac{80}{41} \approx 1.95$

\therefore after 20 years, the tree's height is increasing at a rate of about 1.95 cm per year.

2 a $V = 20\,000e^{-0.4t} \text{ pounds}$

When $t = 0$, $V = 20\,000e^{-0.4(0)}$
 $= 20\,000$

\therefore the purchase price of the car is £20 000.

b $V = 20\,000e^{-0.4t} \text{ pounds}$

$$\begin{aligned} \therefore \frac{dV}{dt} &= 20\,000e^{-0.4t}(-0.4) \\ &= -8000e^{-0.4t} \text{ pounds per year} \end{aligned}$$

When $t = 10$, $\frac{dV}{dt} = -8000e^{-0.4(10)}$
 $= -8000e^{-4}$
 ≈ -146.53

\therefore after 10 years, the value of the car is decreasing at about £146.53 per year.

3 a $C(v) = \frac{v^2}{20} + \frac{50\,000}{v} \text{ dollars}$

$$\begin{aligned} C(64) &= \frac{64^2}{20} + \frac{50\,000}{64} \\ &= 204.8 + 781.25 \\ &= 986.05 \end{aligned}$$

\therefore the cost of running the train for 1 hour at 64 km h^{-1} is \$986.05

\therefore the cost of running the train for 5 hours at 64 km h^{-1} is \$4930.25.

$$\begin{aligned}
 \text{b} \quad C(v) &= \frac{v^2}{20} + \frac{50\,000}{v} \text{ euros, } v > 0 \\
 &= \frac{1}{20}v^2 + 50\,000v^{-1} \\
 \therefore C'(v) &= \frac{1}{10}v - 50\,000v^{-2} \\
 &= \frac{1}{10}v - \frac{50\,000}{v^2}
 \end{aligned}$$

$$\begin{aligned}
 \text{i} \quad C'(75) &= \frac{1}{10}(75) - \frac{50\,000}{75^2} \\
 &= \frac{75}{10} - \frac{80}{9} \\
 &= -\frac{25}{18} \\
 &\approx -1.39
 \end{aligned}$$

\therefore if the average speed is 75 km h^{-1} , the rate of change in the cost of running the train is decreasing at about $\$1.39$ per km h^{-1} .

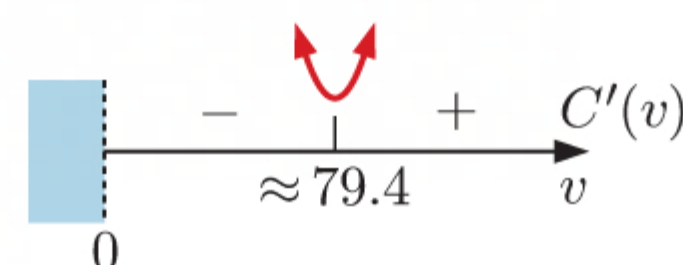
$$\begin{aligned}
 \text{ii} \quad C'(90) &= \frac{1}{10}(90) - \frac{50\,000}{90^2} \\
 &= 9 - \frac{500}{81} \\
 &= \frac{229}{81} \\
 &\approx 2.83
 \end{aligned}$$

\therefore if the average speed is 90 km h^{-1} , the rate of change in the cost of running the train is increasing at about $\$2.83$ per km h^{-1} .

c $C(v)$ is a minimum when $C'(v) = 0$

$$\begin{aligned}
 \therefore \frac{1}{10}v - \frac{50\,000}{v^2} &= 0 \\
 \therefore \frac{1}{10}v^3 - 50\,000 &= 0 \\
 \therefore \frac{1}{10}v^3 &= 50\,000 \\
 \therefore v^3 &= 500\,000 \\
 \therefore v &\approx 79.4 \text{ km h}^{-1}
 \end{aligned}$$

$C'(v)$ has sign diagram:



\therefore the cost of running the train is a minimum when the average speed of the train is about 79.4 km h^{-1} .

4 a Let $OD = x$, so C has coordinates $(x, 9 - x^2)$.

Area of rectangle ABCD = length \times width

$$\begin{aligned}
 \therefore A(x) &= 2x \times (9 - x^2) \\
 &= 18x - 2x^3
 \end{aligned}$$

$$\text{b} \quad A(x) = 18x - 2x^3$$

$$\therefore A'(x) = 18 - 6x^2$$

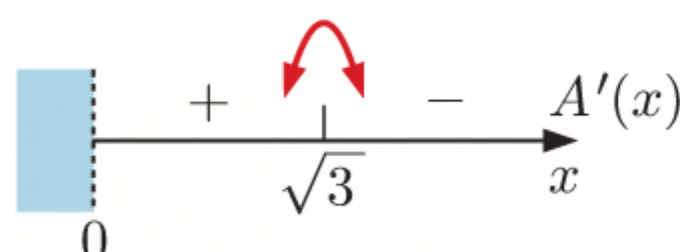
$$A'(x) = 0 \text{ when } 18 - 6x^2 = 0$$

$$\therefore 6x^2 = 18$$

$$\therefore x^2 = 3$$

$$\therefore x = \sqrt{3} \quad \{x > 0\}$$

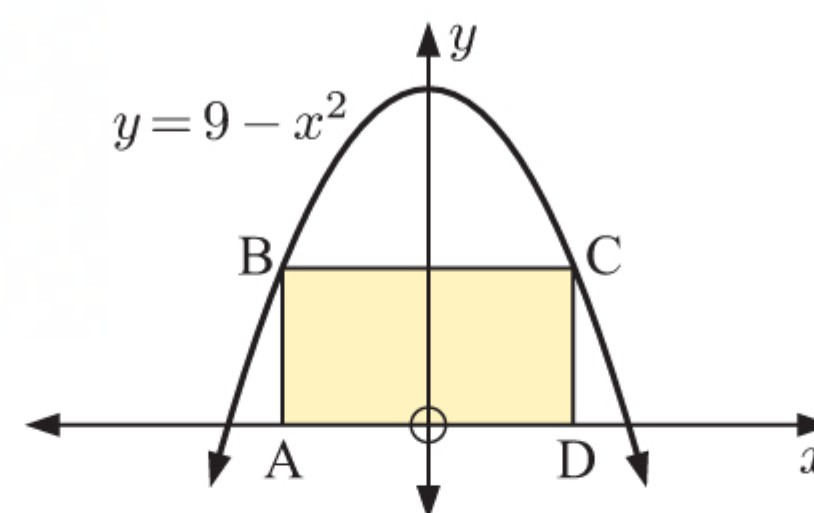
which has sign diagram:



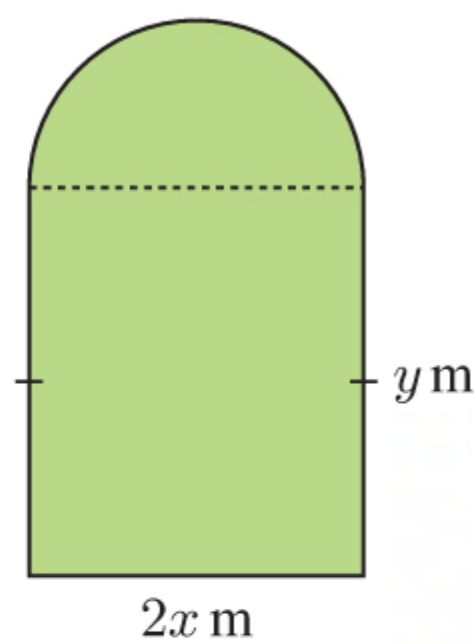
So, the area is a maximum when $x = \sqrt{3}$.

$$\text{When } x = \sqrt{3}, y = 9 - (\sqrt{3})^2 = 6$$

So, C has coordinates $(\sqrt{3}, 6)$.



5 a perimeter $= 2x + 2y + \pi x$
 $\therefore 200 = 2x + 2y + \pi x$
 $\therefore 2y = 200 - 2x - \pi x$
 $\therefore y = 100 - x - \frac{\pi}{2}x$



b area of lawn $A = \text{area of rectangle} + \text{area of semi-circle}$
 $= 2x \times y + \frac{1}{2}\pi x^2$
 $= 2x(100 - x - \frac{\pi}{2}x) + \frac{\pi}{2}x^2 \quad \{\text{using a}\}$
 $= 200x - 2x^2 - \pi x^2 + \frac{\pi}{2}x^2$
 $\therefore A = 200x - 2x^2 - \frac{\pi}{2}x^2 \text{ m}^2$

c $\frac{dA}{dx} = 200 - 4x - \pi x$

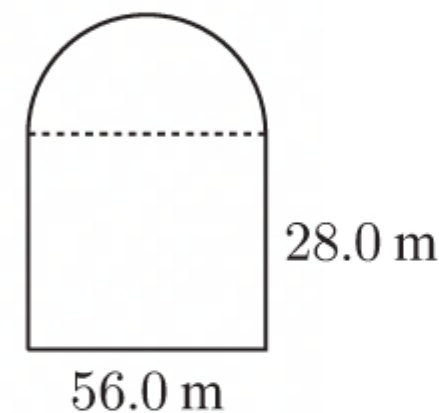
Now $\frac{dA}{dx} = 0$ when $200 - 4x - \pi x = 0$
 $\therefore 4x + \pi x = 200$
 $\therefore x(4 + \pi) = 200$
 $\therefore x = \frac{200}{4 + \pi}$
 $\therefore x \approx 28.0$



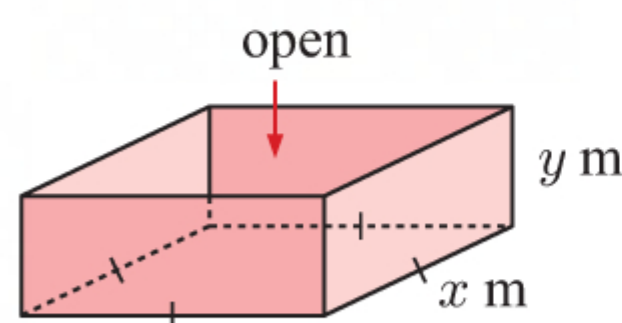
The area of the lawn is maximised when $x = \frac{200}{4 + \pi} \approx 28.0$

and $y = 100 - \frac{200}{4 + \pi} - \frac{\pi}{2} \left(\frac{200}{4 + \pi} \right)$
 ≈ 28.0

The dimensions of the lawn of maximum area are:



6 a capacity $= 1 \text{ kL} \equiv 1 \text{ m}^3$
volume of box $= \text{area of base} \times \text{height}$
 $\therefore 1 = x^2 y$
 $\therefore y = \frac{1}{x^2}, \quad x > 0$



b area of steel needed = area of base + area of 4 sides

$$\begin{aligned}
 &= x^2 + 4xy \\
 &= x^2 + 4x\left(\frac{1}{x^2}\right) \quad \{\text{from a}\} \\
 &= x^2 + \frac{4}{x}
 \end{aligned}$$

Steel costs £2 per m², so total cost of steel = $\left(x^2 + \frac{4}{x}\right) \times 2$

$$\therefore C(x) = 2x^2 + \frac{8}{x} \text{ pounds}$$

c $C(x) = 2x^2 + \frac{8}{x} = 2x^2 + 8x^{-1}$

$$\therefore C'(x) = 4x - 8x^{-2} = 4x - \frac{8}{x^2}$$

$$C'(x) = 0 \text{ when } 4x - \frac{8}{x^2} = 0$$

$$\therefore 4x = \frac{8}{x^2}$$

$$\therefore 4x^3 = 8$$

$$\therefore x^3 = 2$$

$$\therefore x = \sqrt[3]{2}$$

$C'(x)$ has sign diagram:

So, the cost is a minimum when $x = \sqrt[3]{2} \approx 1.26$

$$\text{When } x = \sqrt[3]{2}, \quad y = \frac{1}{(\sqrt[3]{2})^2} \approx 0.630$$

So, the box which would cost the least to make would have square base with sides about 1.26 m and height about 0.630 m.

7 a $D = 9.3 + 6.8 \cos(0.507t) \text{ m}$

$$\begin{aligned}
 \therefore \frac{dD}{dt} &= 6.8(-\sin(0.507t))(0.507) \\
 &= -3.4476 \sin(0.507t)
 \end{aligned}$$

This tells us the rate at which the depth of water is increasing or decreasing t hours after midnight.

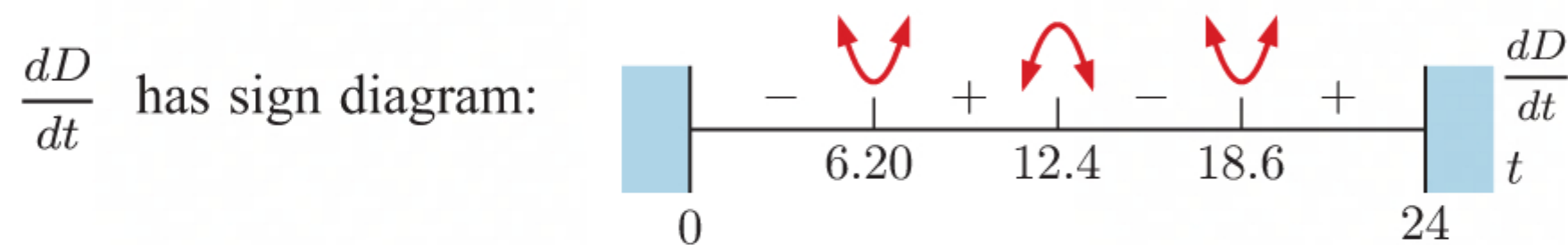
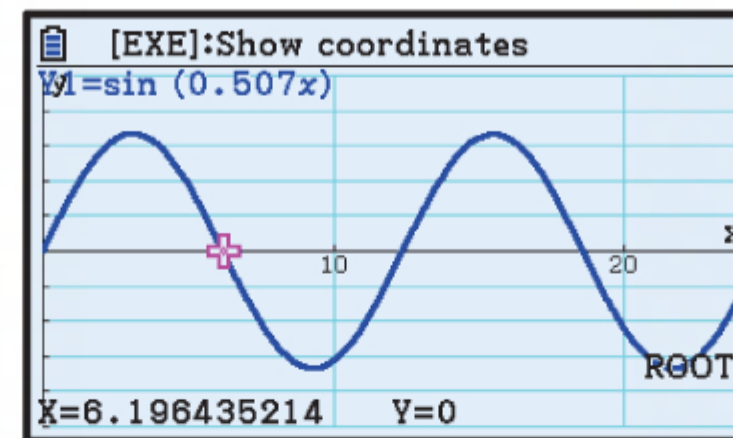
b When $t = 8$, $D = 9.3 + 6.8 \cos(0.507 \times 8)$
 ≈ 5.15

\therefore the depth of the water at 8 am is about 5.15 m.

c When $t = 8$, $\frac{dD}{dt} = -3.4476 \sin(0.507 \times 8)$
 $\approx 2.73 > 0$

\therefore the tide is rising at 8 am at a rate of about 2.73 m per hour.

$$\begin{aligned}
 \text{d } \frac{dD}{dt} = 0 \text{ when } -3.4476 \sin(0.507t) &= 0 \\
 \therefore \sin(0.507t) &= 0 \\
 \therefore t &= 0, \\
 &\text{or } \approx 6.20, 12.4, 18.6 \\
 &\text{\{on } 0 \leq t \leq 24\}}
 \end{aligned}$$



So, the tide is highest 0 hours and about 12.4 hours after midnight.

$$\begin{aligned}
 0.4 \text{ hours} &= 0.4 \times 60 \\
 &= 24 \text{ minutes}
 \end{aligned}$$

\therefore the tide will be highest at midnight and at about 12:24 pm.

$$\begin{aligned}
 \text{When } t = 0, \quad D &= 9.3 + 6.8 \cos 0 \\
 &= 16.1 \text{ m}
 \end{aligned}$$

\therefore the maximum depth is 16.1 m.

REVIEW SET 14B

$$\begin{aligned}
 \text{1 a } \quad C(x) &= 850 + 3.3x^{0.85} + 2.8x^{0.5} \text{ euros} \\
 \therefore C'(x) &= 3.3(0.85x^{-0.15}) + 2.8(0.5x^{-0.5}) \\
 &= 2.805x^{-0.15} + 1.4x^{-0.5} \text{ euros per item}
 \end{aligned}$$

$$\begin{aligned}
 \text{b } \quad C'(1000) &= 2.805(1000)^{-0.15} + 1.4(1000)^{-0.5} \\
 &\approx \text{€}1.04
 \end{aligned}$$

This estimates the cost of making the 1001st item each day.

$$\begin{aligned}
 \text{c } \quad C(1001) - C(1000) \\
 &= 850 + 3.3(1001)^{0.85} + 2.8(1001)^{0.5} - (850 + 3.3(1000)^{0.85} + 2.8(1000)^{0.5}) \\
 &\approx \text{€}1.04
 \end{aligned}$$

This is the actual cost of making the 1001st item each day. The answer in **b** is a very good estimate.

$$\begin{aligned}
 \text{2 } \quad P(t) &= 60\,000 \left(1 + 2e^{-\frac{t}{4}}\right)^{-1} \\
 &= \frac{60\,000}{1 + 2e^{-\frac{t}{4}}}, \quad t \geq 0
 \end{aligned}$$

$$\begin{aligned}
 \text{a } \quad P(0) &= \frac{60\,000}{1 + 2e^0} \\
 &= \frac{60\,000}{3} \\
 &= 20\,000
 \end{aligned}$$

\therefore the initial population is 20 000.

$$\begin{aligned} \text{b } P'(t) &= -60\,000 \left(1 + 2e^{-\frac{t}{4}}\right)^{-2} \times 2e^{-\frac{t}{4}} \left(-\frac{1}{4}\right) \quad \{\text{chain rule}\} \\ &= \frac{30\,000e^{-\frac{t}{4}}}{\left(1 + 2e^{-\frac{t}{4}}\right)^2} \end{aligned}$$

$$\text{c } \text{Since } e^{-\frac{t}{4}} > 0 \text{ for all } t \text{ and } \left(1 + 2e^{-\frac{t}{4}}\right)^2 > 0 \text{ for all } t, \quad P'(t) > 0 \text{ for all } t \geq 0.$$

This means $P(t)$ is increasing for all $t \geq 0$.

$$\text{d } P''(t) = \frac{30\,000e^{-\frac{t}{4}} \left(-\frac{1}{4}\right) \left(1 + 2e^{-\frac{t}{4}}\right)^2 - 30\,000e^{-\frac{t}{4}} \times 2 \left(1 + 2e^{-\frac{t}{4}}\right)^1 \times 2e^{-\frac{t}{4}} \left(-\frac{1}{4}\right)}{\left(1 + 2e^{-\frac{t}{4}}\right)^4} \quad \{\text{quotient rule}\}$$

$$= \frac{-7500e^{-\frac{t}{4}} \left(1 + 2e^{-\frac{t}{4}}\right)^2 + 30\,000e^{-\frac{t}{2}} \left(1 + 2e^{-\frac{t}{4}}\right)}{\left(1 + 2e^{-\frac{t}{4}}\right)^4}$$

$$= \frac{-7500e^{-\frac{t}{4}} \left(1 + 2e^{-\frac{t}{4}}\right) + 30\,000e^{-\frac{t}{2}}}{\left(1 + 2e^{-\frac{t}{4}}\right)^3}$$

$$= \frac{-7500e^{-\frac{t}{4}} - 15\,000e^{-\frac{t}{2}} + 30\,000e^{-\frac{t}{2}}}{\left(1 + 2e^{-\frac{t}{4}}\right)^3}$$

$$= \frac{15\,000e^{-\frac{t}{2}} - 7500e^{-\frac{t}{4}}}{\left(1 + 2e^{-\frac{t}{4}}\right)^3}$$

$$= \frac{7500e^{-\frac{t}{4}} \left(2e^{-\frac{t}{4}} - 1\right)}{\left(1 + 2e^{-\frac{t}{4}}\right)^3}$$

$$\text{e } \text{The growth is given by } P'(t) \text{ which is maximised when } P''(t) = 0.$$

$$P''(t) = 0 \quad \text{when} \quad 2e^{-\frac{t}{4}} - 1 = 0 \quad \{e^{-\frac{t}{4}} > 0\}$$

$$\therefore 2e^{-\frac{t}{4}} = 1$$

$$\therefore e^{-\frac{t}{4}} = \frac{1}{2}$$

$$\therefore -\frac{t}{4} = \ln\left(\frac{1}{2}\right)$$

$$\therefore t = -4 \ln(2^{-1})$$

$$\therefore t = 4 \ln 2$$

$$\text{When } t = 4 \ln 2, \quad e^{-\frac{t}{4}} = e^{-\ln 2}$$

$$= e^{\ln\left(\frac{1}{2}\right)}$$

$$= \frac{1}{2}$$

and so $P'(t) = \frac{30\,000(\frac{1}{2})}{\left(1 + 2(\frac{1}{2})\right)^2} = \frac{15\,000}{4} = 3750$

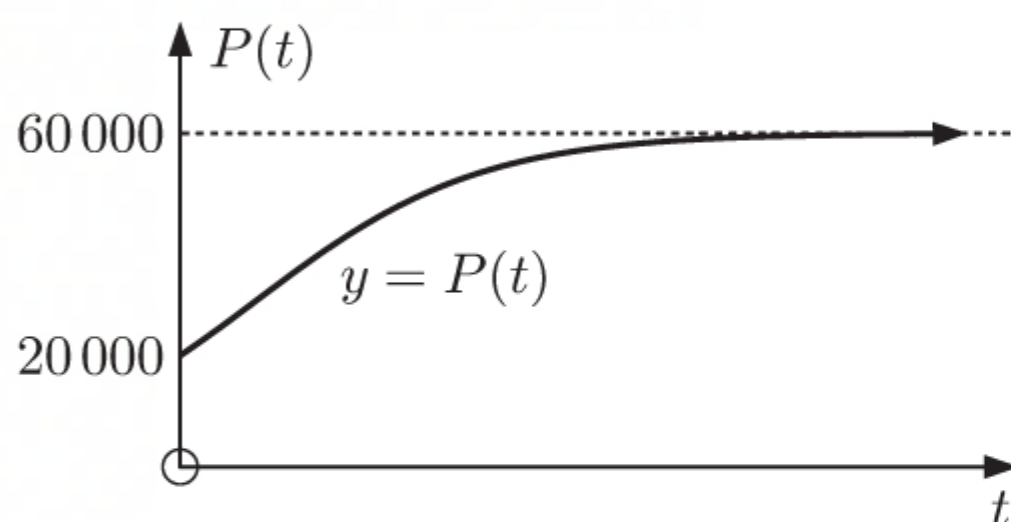
\therefore the maximum growth rate is 3750 per year when $t = 4 \ln 2$ years.

f As $t \rightarrow \infty$, $e^{-\frac{t}{4}} \rightarrow 0^+$

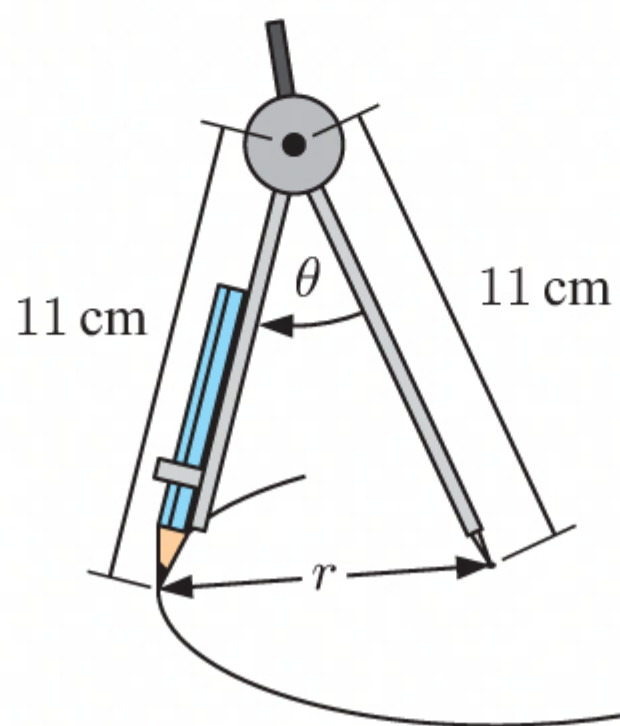
$$\therefore P(t) \rightarrow \frac{60\,000}{1 + 2(0)}$$

$$\therefore P(t) \rightarrow 60\,000^-$$

g



3 a



Using the cosine rule, $r^2 = 11^2 + 11^2 - 2 \times 11 \times 11 \times \cos \theta$
 $= 121 + 121 - 242 \cos \theta$
 $= 242 - 242 \cos \theta$
 $= 242(1 - \cos \theta)$

Now, the area of the circle $A = \pi r^2$
 $= \pi \times 242(1 - \cos \theta)$
 $= 242\pi(1 - \cos \theta) \text{ cm}^2$

b $A = 242\pi(1 - \cos \theta) \text{ cm}^2$
 $\therefore \frac{dA}{d\theta} = 242\pi(\sin \theta)$
 $= 242\pi \sin \theta \text{ cm}^2 \text{ per radian}$

When $\theta = \frac{\pi}{4}$, $\frac{dA}{d\theta} = 242\pi \sin \frac{\pi}{4}$
 $= 242\pi \times \frac{1}{\sqrt{2}}$
 $= 121\sqrt{2}\pi \approx 538 \text{ cm}^2 \text{ per radian}$

\therefore the area of the circle is changing at a rate of $121\sqrt{2}\pi \approx 538 \text{ cm}^2 \text{ per radian}$ when $\theta = \frac{\pi}{4}$.

4 Suppose the sheet is bent x cm from each end. To maximise the water carried we need to maximise the area of the cross-section.

$$A = x(24 - 2x), \quad 0 \leq x \leq 12$$

$$= 24x - 2x^2$$

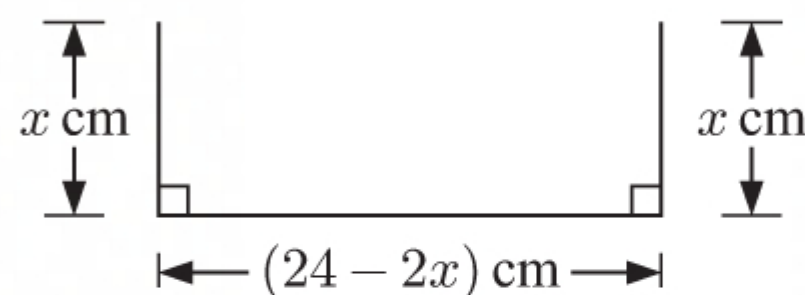
$$\therefore \frac{dA}{dx} = 24 - 4x$$

So, $\frac{dA}{dx} = 0$ when $24 - 4x = 0$
 $\therefore x = 6$

$\frac{dA}{dx}$ has sign diagram:

The maximum water is held when $x = 6$

\therefore the bends must be made 6 cm from each end.



5 Let $OA = x \quad \therefore AP = ae^{-x}$

\therefore rectangle OAPB has perimeter $P = 2x + 2ae^{-x}$

$$\therefore \frac{dP}{dx} = 2 - 2ae^{-x}$$

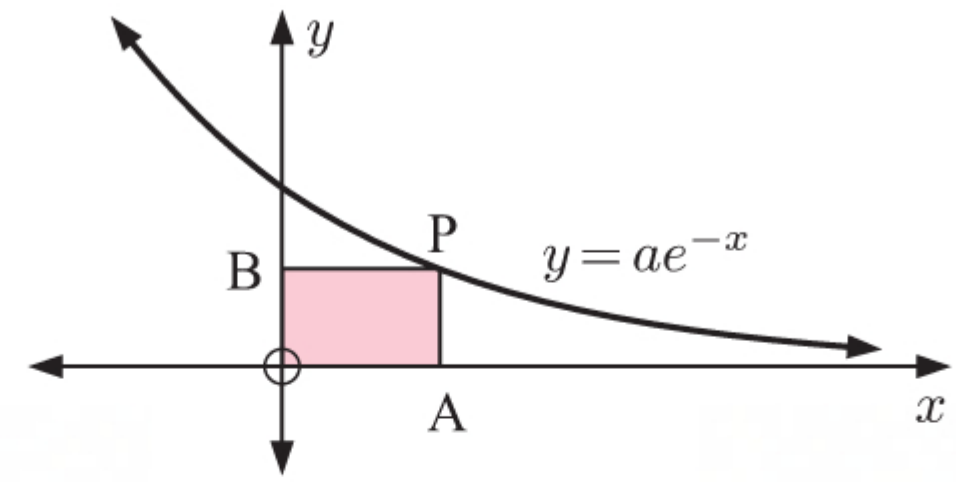
$$\frac{dP}{dx} = 0 \quad \text{when} \quad 2 - 2ae^{-x} = 0$$

$$\therefore 2ae^{-x} = 2$$

$$\therefore ae^{-x} = 1$$

$$\therefore a = e^x$$

$$\therefore x = \ln a$$



$\frac{dP}{dx}$ has sign diagram:

$$\begin{array}{c} \text{---} - \quad \quad \quad + \quad \text{---} \\ \quad \quad \quad \uparrow \\ \quad \quad \quad \ln a \end{array} \quad \frac{dP}{dx}$$

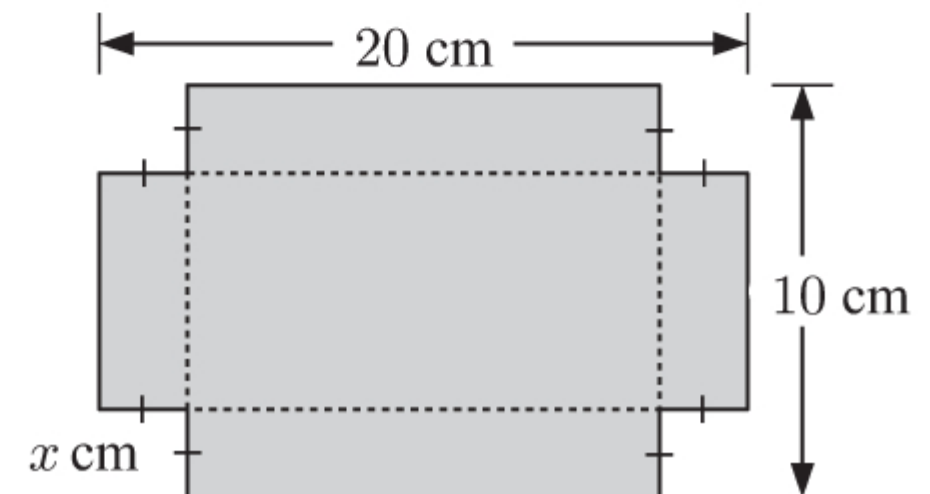
\therefore the rectangle OAPB has minimum perimeter when the x -coordinate of P is $\ln a$.

6 4 squares with sides x cm are cut from the corners.

\therefore the remaining sides have length $(20 - 2x)$ cm and $(10 - 2x)$ cm.

Now, volume $V = \text{length} \times \text{width} \times \text{depth}$

$$\begin{aligned} &= (20 - 2x)(10 - 2x)x \\ &= (200 - 40x - 20x + 4x^2)x \\ &= (200 - 60x + 4x^2)x \\ &= 200x - 60x^2 + 4x^3 \text{ cm}^3 \end{aligned}$$



Since the side lengths must be positive, $x > 0$ and $10 - 2x > 0$

$$\therefore 2x < 10$$

$$\therefore 0 < x < 5$$

$$V = 4x^3 - 60x^2 + 200x$$

$$\therefore \frac{dV}{dx} = 12x^2 - 120x + 200$$

So, $\frac{dV}{dx} = 0$ when $12x^2 - 120x + 200 = 0$

$$\therefore 3x^2 - 30x + 50 = 0$$

$$\therefore x \approx 2.11 \quad \{\text{using technology, } 0 < x < 5\}$$

$\frac{dV}{dx}$ has sign diagram:

$$\begin{array}{c} \text{---} + \quad \quad \quad - \quad \text{---} \\ \quad \quad \quad \uparrow \\ \quad \quad \quad \approx 2.11 \end{array} \quad \frac{dV}{dx}$$

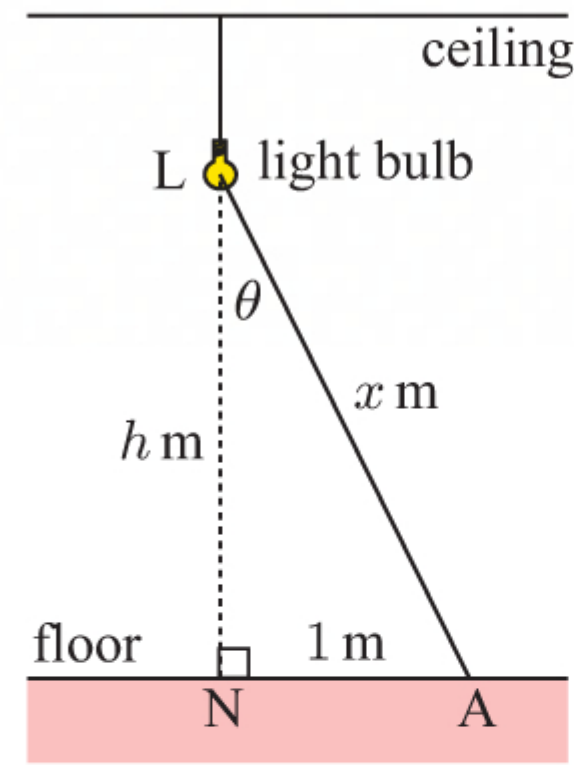
0 5

\therefore the capacity of the container is maximised when $x \approx 2.11$.

7 $I = \frac{\sqrt{8} \cos \theta}{x^2}$ units

a If $NA = 1$ metre, $\sin \theta = \frac{NA}{x} = \frac{1}{x}$
 $\therefore \frac{1}{x^2} = \sin^2 \theta$

\therefore at A, $I = \frac{\sqrt{8} \cos \theta}{x^2} = \sqrt{8} \cos \theta \sin^2 \theta$ units



b $\frac{dI}{d\theta} = \sqrt{8}(-\sin \theta) \sin^2 \theta + \sqrt{8} \cos \theta (2 \sin \theta \cos \theta)$ {product rule}
 $= \sqrt{8} \sin \theta [2 \cos^2 \theta - \sin^2 \theta]$
 $= \sqrt{8} \sin \theta [2(1 - \sin^2 \theta) - \sin^2 \theta]$ { $\sin^2 \theta + \cos^2 \theta = 1$ }
 $= \sqrt{8} \sin \theta [2 - 3 \sin^2 \theta]$

$\frac{dI}{d\theta} = 0$ when $\sin \theta = \sqrt{\frac{2}{3}}$, $0 < \theta < \frac{\pi}{2}$
 $\therefore \theta = \sin^{-1}\left(\sqrt{\frac{2}{3}}\right)$

$\frac{dI}{d\theta}$ has sign diagram:

	+	-	
0	$\sin^{-1}\left(\sqrt{\frac{2}{3}}\right)$	$\frac{\pi}{2}$	$\frac{dI}{d\theta}$
			θ

A red curved arrow points from the '+' region to the '-' region.

\therefore the maximum illumination at A is obtained when $\sin \theta = \sqrt{\frac{2}{3}}$

$\therefore x = \frac{1}{\sin \theta} = \sqrt{\frac{3}{2}}$

and $h = \sqrt{x^2 - NA^2}$ {Pythagoras}

$= \sqrt{\frac{3}{2} - 1^2} = \frac{1}{\sqrt{2}}$

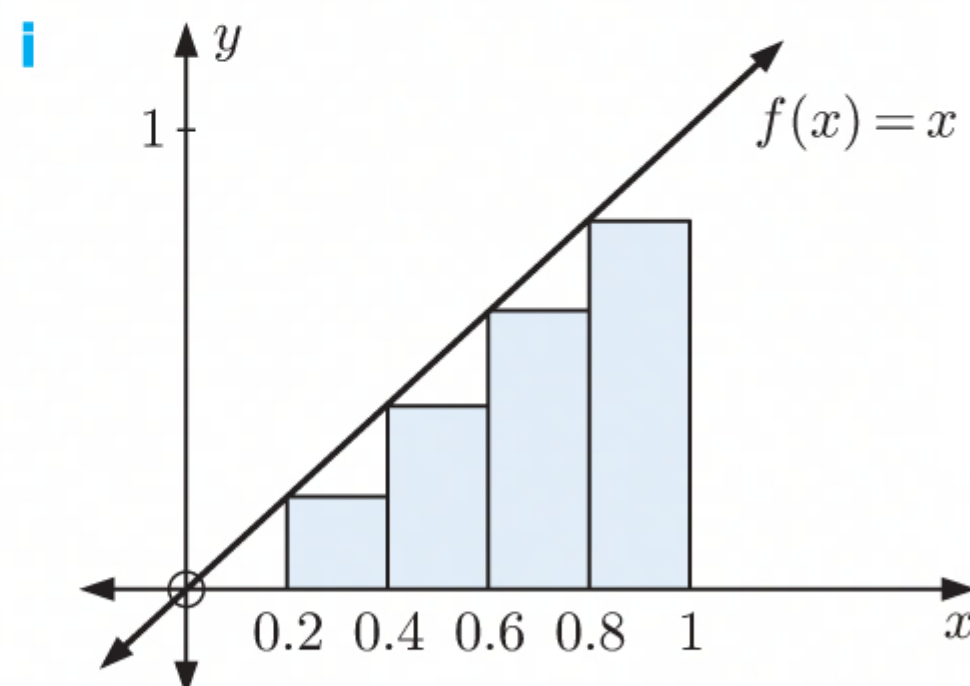
\therefore the bulb should be $\frac{1}{\sqrt{2}}$ m above the floor to provide the greatest illumination at A.

Chapter 15

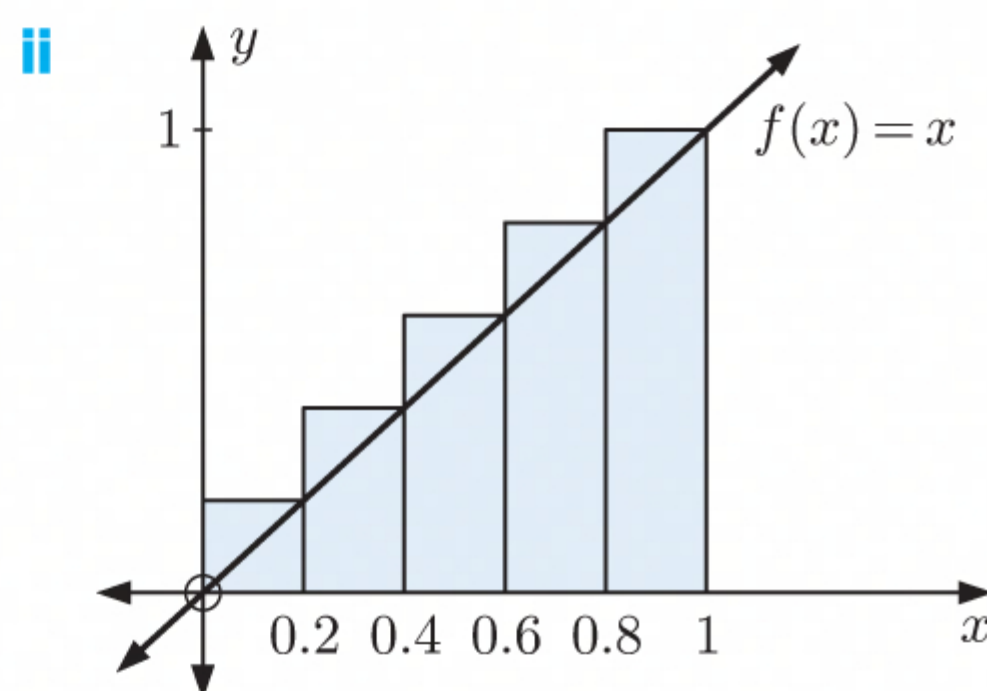
INTRODUCTION TO INTEGRATION

EXERCISE 15A

- 1 a The rectangles are $\frac{1}{5} = 0.2$ units wide.



$$\begin{aligned} A_L &= 0.2 \times f(0) + 0.2 \times f(0.2) + 0.2 \times f(0.4) \\ &\quad + 0.2 \times f(0.6) + 0.2 \times f(0.8) \\ &= (0.2 \times 0) + (0.2 \times 0.2) + (0.2 \times 0.4) \\ &\quad + (0.2 \times 0.6) + (0.2 \times 0.8) \\ &= 0.4 \text{ units}^2 \end{aligned}$$



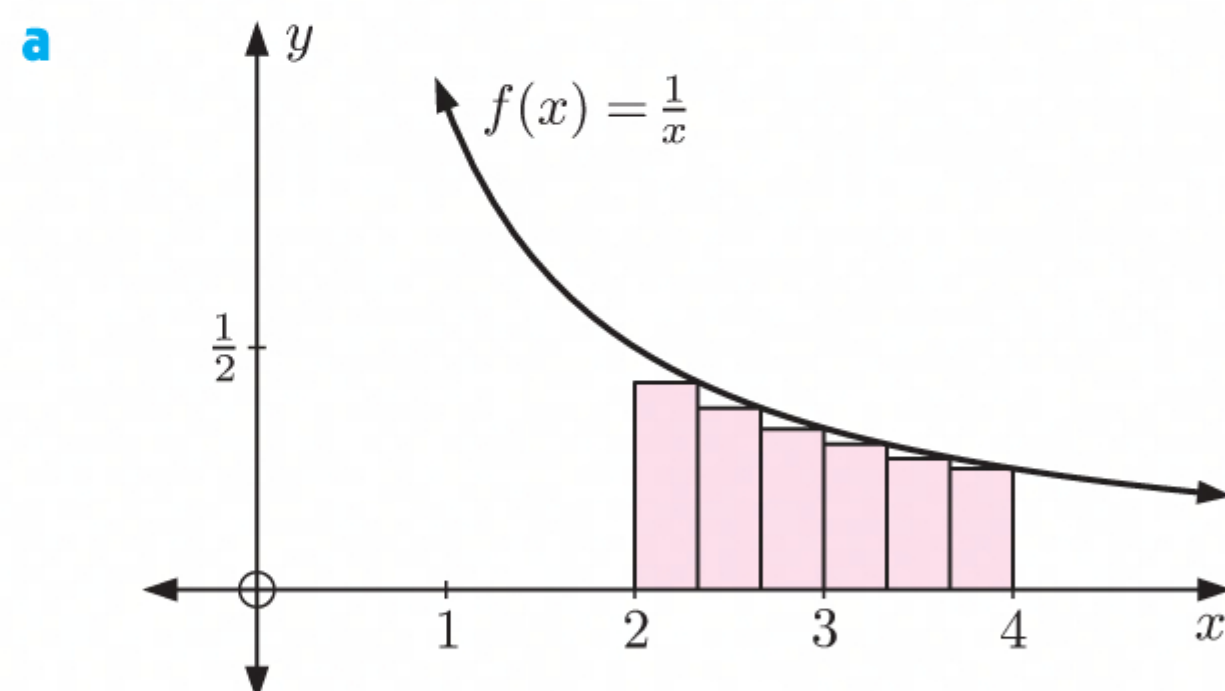
$$\begin{aligned} A_U &= 0.2 \times f(0.2) + 0.2 \times f(0.4) + 0.2 \times f(0.6) \\ &\quad + 0.2 \times f(0.8) + 0.2 \times f(1) \\ &= (0.2 \times 0.2) + (0.2 \times 0.4) + (0.2 \times 0.6) \\ &\quad + (0.2 \times 0.8) + (0.2 \times 1) \\ &= 0.6 \text{ units}^2 \end{aligned}$$

- b The area between $y = x$ and the x -axis from $x = 0$ to $x = 1$ is a triangle.

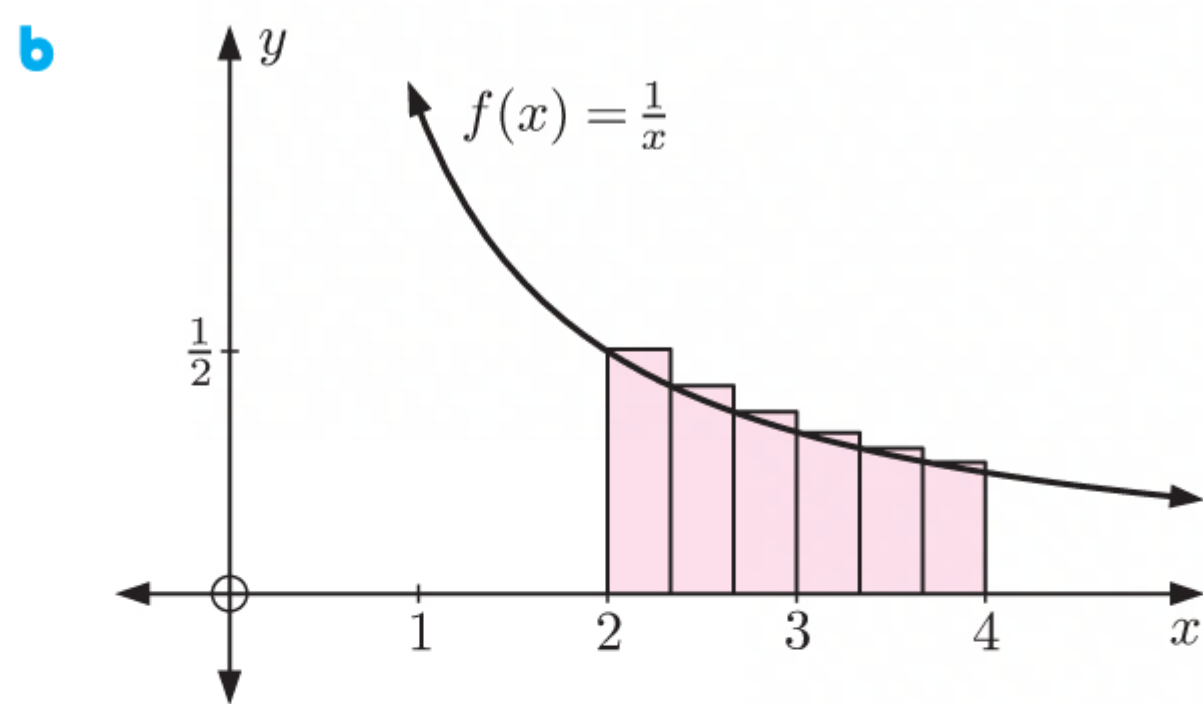
$$\begin{aligned} \therefore \text{area} &= \frac{1}{2} \times \text{base} \times \text{height} \\ &= \frac{1}{2} \times 1 \times 1 \\ &= 0.5 \text{ units}^2 \end{aligned}$$

$\therefore A_L < \text{area} < A_U$, and both A_L and A_U are within 0.1 units^2 , or 20%, of the actual area.

- 2 The rectangles are $\frac{2}{6} = \frac{1}{3}$ units wide.



$$\begin{aligned} A_L &= \frac{1}{3} \times f\left(\frac{7}{3}\right) + \frac{1}{3} \times f\left(\frac{8}{3}\right) + \frac{1}{3} \times f(3) + \frac{1}{3} \times f\left(\frac{10}{3}\right) + \frac{1}{3} \times f\left(\frac{11}{3}\right) + \frac{1}{3} \times f(4) \\ &= \left(\frac{1}{3} \times \frac{3}{7}\right) + \left(\frac{1}{3} \times \frac{3}{8}\right) + \left(\frac{1}{3} \times \frac{1}{3}\right) + \left(\frac{1}{3} \times \frac{3}{10}\right) + \left(\frac{1}{3} \times \frac{3}{11}\right) + \left(\frac{1}{3} \times \frac{1}{4}\right) \\ &\approx 0.653 \text{ units}^2 \end{aligned}$$



$$\begin{aligned}
 A_U &= \frac{1}{3} \times f(2) + \frac{1}{3} \times f\left(\frac{7}{3}\right) + \frac{1}{3} \times f\left(\frac{8}{3}\right) + \frac{1}{3} \times f(3) + \frac{1}{3} \times f\left(\frac{10}{3}\right) + \frac{1}{3} \times f\left(\frac{11}{3}\right) \\
 &= \left(\frac{1}{3} \times \frac{1}{2}\right) + \left(\frac{1}{3} \times \frac{3}{7}\right) + \left(\frac{1}{3} \times \frac{3}{8}\right) + \left(\frac{1}{3} \times \frac{1}{3}\right) + \left(\frac{1}{3} \times \frac{3}{10}\right) + \left(\frac{1}{3} \times \frac{3}{11}\right) \\
 &\approx 0.737 \text{ units}^2
 \end{aligned}$$

3 Using provided software,

n	A_L	A_U
10	2.1850	2.4850
25	2.2736	2.3936
50	2.3034	2.3634
100	2.3184	2.3484
500	2.3303	2.3363

A_L and A_U converge to $\frac{7}{3} = 2.\bar{3}$

4 a i

n	A_L	A_U
5	0.160 00	0.360 00
10	0.202 50	0.302 50
50	0.240 10	0.260 10
100	0.245 03	0.255 03
500	0.249 00	0.251 00
1000	0.249 50	0.250 50
10 000	0.249 95	0.250 05

ii

n	A_L	A_U
5	0.400 00	0.600 00
10	0.450 00	0.550 00
50	0.490 00	0.510 00
100	0.495 00	0.505 00
500	0.499 00	0.501 00
1000	0.499 50	0.500 50
10 000	0.499 95	0.500 05

iii

n	A_L	A_U
5	0.549 74	0.749 74
10	0.610 51	0.710 51
50	0.656 10	0.676 10
100	0.661 46	0.671 46
500	0.665 65	0.667 65
1000	0.666 16	0.667 16
10 000	0.666 62	0.666 72

iv

n	A_L	A_U
5	0.618 67	0.818 67
10	0.687 40	0.787 40
50	0.738 51	0.758 51
100	0.744 41	0.754 41
500	0.748 93	0.750 93
1000	0.749 47	0.750 47
10 000	0.749 95	0.750 05

b i A_L and A_U converge to $0.25 = \frac{1}{4} = \frac{1}{3+1}$

ii A_L and A_U converge to $0.5 = \frac{1}{2} = \frac{1}{1+1}$

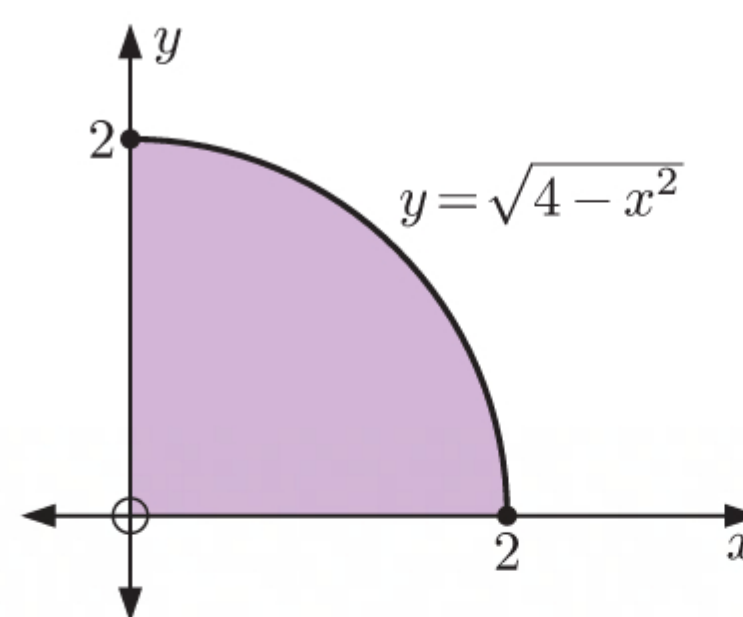
iii A_L and A_U converge to $0.\bar{6} = \frac{2}{3} = \frac{1}{\frac{1}{2}+1}$

iv A_L and A_U converge to $0.75 = \frac{3}{4} = \frac{1}{\frac{1}{3}+1}$

c From **b**, it appears that the area between the graph of $y = x^a$ and the x -axis for $0 \leq x \leq 1$ and any number $a > 0$ is $\frac{1}{a+1}$.

5 a

n	Rational bounds for π
10	$2.9045 < \pi < 3.3045$
50	$3.0983 < \pi < 3.1783$
100	$3.1204 < \pi < 3.1604$
200	$3.1312 < \pi < 3.1512$
1000	$3.1396 < \pi < 3.1436$
10 000	$3.1414 < \pi < 3.1418$



b $3\frac{10}{71} < \pi < 3\frac{1}{7}$ is approximately $3.1408 < \pi < 3.1429$

This is a better approximation than our estimates in **a** using $n = 10, 50, 100, 200$, or 1000 rectangles. Only $n = 10\,000$ gives us a better estimate than that of Archimedes.

INVESTIGATION 1**THE AREA UNDER $f(x) = x^2$**

1 a $f(x) = x^2$, $0 \leq x \leq 1$ is divided into n subintervals.

Each lower rectangle has width $\frac{1}{n}$.

Since $f(x)$ is increasing on $0 \leq x \leq 1$, the i th lower rectangle has height $f\left(\frac{i-1}{n}\right)$.

$$\begin{aligned} \therefore \text{the total area of lower rectangles } A_L &= \sum_{i=1}^n \left(\frac{1}{n} \times f\left(\frac{i-1}{n}\right) \right) \\ &= \frac{1}{n} \sum_{i=1}^n f\left(\frac{i-1}{n}\right) \end{aligned}$$

b

$$\begin{aligned} \frac{1}{n} \sum_{i=1}^n f\left(\frac{i-1}{n}\right) &= \frac{1}{n} \sum_{i=1}^n \left(\frac{i-1}{n}\right)^2 \\ &= \frac{1}{n} \sum_{i=1}^n \frac{i^2 - 2i + 1}{n^2} \\ &= \frac{1}{n^3} \sum_{i=1}^n (i^2 - 2i + 1) \\ &= \frac{1}{n^3} \left(\sum_{i=1}^n i^2 - 2 \sum_{i=1}^n i + \sum_{i=1}^n 1 \right) \\ &= \frac{1}{n^3} \left(\frac{n(n+1)(2n+1)}{6} - 2 \times \frac{n(n+1)}{2} + n \right) \\ &\quad \text{\{using provided formulae\}} \\ &= \frac{1}{n^3} \left(\frac{n(n+1)(2n+1)}{6} - n(n+1) + n \right) \\ &= \frac{1}{n^3} \left(\frac{2n^3 + 3n^2 + n}{6} - n^2 - n + n \right) \\ &= \frac{1}{n^3} \left(\frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n - n^2 \right) \\ &= \frac{1}{n^3} \left(\frac{1}{3}n^3 - \frac{1}{2}n^2 + \frac{1}{6}n \right) \end{aligned}$$

$$\therefore A_L = \frac{1}{n} \sum_{i=1}^n f\left(\frac{i-1}{n}\right) = \frac{1}{3} - \frac{1}{2n} + \frac{1}{6n^2}$$

c As $n \rightarrow \infty$, $\frac{1}{2n} \rightarrow 0$ and $\frac{1}{6n^2} \rightarrow 0$
 \therefore as $n \rightarrow \infty$, $A_L \rightarrow \frac{1}{3}$.

2 a Each upper rectangle has width $\frac{1}{n}$.

Since $f(x)$ is increasing on $0 \leq x \leq 1$, the i th upper rectangle has height $f\left(\frac{i}{n}\right)$.

$$\begin{aligned} \therefore \text{the total area of upper rectangles } A_U &= \sum_{i=1}^n \left(\frac{1}{n} \times f\left(\frac{i}{n}\right) \right) \\ &= \frac{1}{n} \sum_{i=1}^n f\left(\frac{i}{n}\right) \end{aligned}$$

b

$$\begin{aligned} \frac{1}{n} \sum_{i=1}^n f\left(\frac{i}{n}\right) &= \frac{1}{n} \sum_{i=1}^n \left(\frac{i}{n}\right)^2 \\ &= \frac{1}{n} \sum_{i=1}^n \frac{i^2}{n^2} \\ &= \frac{1}{n^3} \sum_{i=1}^n i^2 \\ &= \frac{1}{n^3} \left(\frac{n(n+1)(2n+1)}{6} \right) \quad \left\{ \text{using } \sum_{i=1}^n i^2 = \frac{n(n+1)(2n+1)}{6} \right\} \\ &= \frac{1}{n^3} \left(\frac{2n^3 + 3n^2 + n}{6} \right) \\ &= \frac{1}{n^3} \left(\frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n \right) \end{aligned}$$

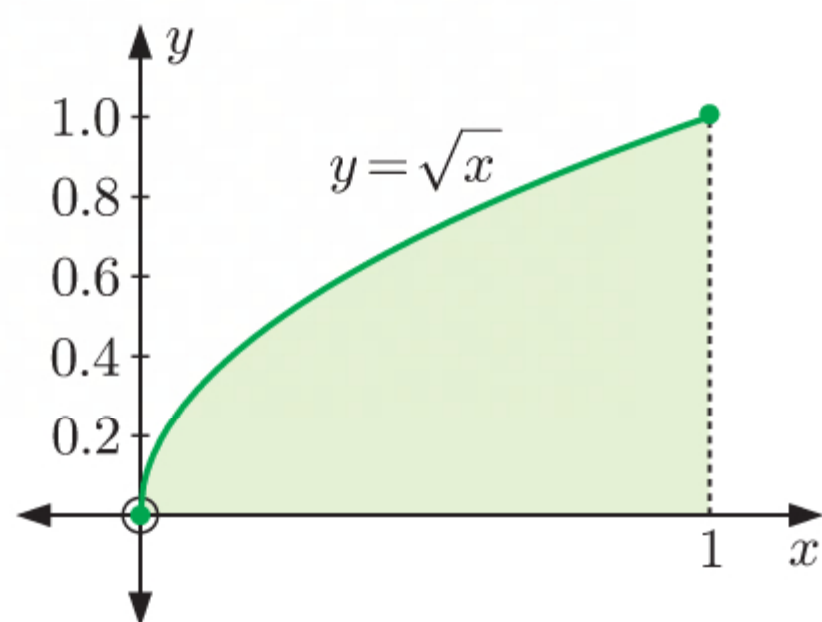
$$\therefore A_U = \frac{1}{n} \sum_{i=1}^n f\left(\frac{i}{n}\right) = \frac{1}{3} + \frac{1}{2n} + \frac{1}{6n^2}$$

c As $n \rightarrow \infty$, $\frac{1}{2n} \rightarrow 0$ and $\frac{1}{6n^2} \rightarrow 0$
 \therefore as $n \rightarrow \infty$, $A_U \rightarrow \frac{1}{3}$.

3 We know that $A_L < A < A_U$, and as $n \rightarrow \infty$, A_L and A_U both converge to the value $\frac{1}{3}$.
 $\therefore A = \frac{1}{3}$ units².

EXERCISE 15B

1 a



b

n	A_L	A_U
5	0.5497	0.7497
10	0.6105	0.7105
50	0.6561	0.6761
100	0.6615	0.6715
500	0.6656	0.6676

c $\int_0^1 \sqrt{x} \, dx \approx 0.67$

- 2 a** The rectangles will have width $\frac{2-0}{n} = \frac{2}{n}$.

Let $x_i = \frac{2i}{n}$ for $i = 0, \dots, n$.

Since $y = \sqrt{1+x^3}$ is increasing on $0 \leq x \leq 2$, the i th lower rectangle has height $\sqrt{1+x_{i-1}^3}$ and the i th upper rectangle has height $\sqrt{1+x_i^3}$.

The lower rectangle sum will be

$$\begin{aligned} A_L &= \frac{2}{n} \times \sqrt{1+x_0^3} + \frac{2}{n} \times \sqrt{1+x_1^3} + \dots + \frac{2}{n} \times \sqrt{1+x_{n-1}^3} \\ &= \frac{2}{n} \sum_{i=0}^{n-1} \sqrt{1+x_i^3} \end{aligned}$$

and the upper rectangle sum will be

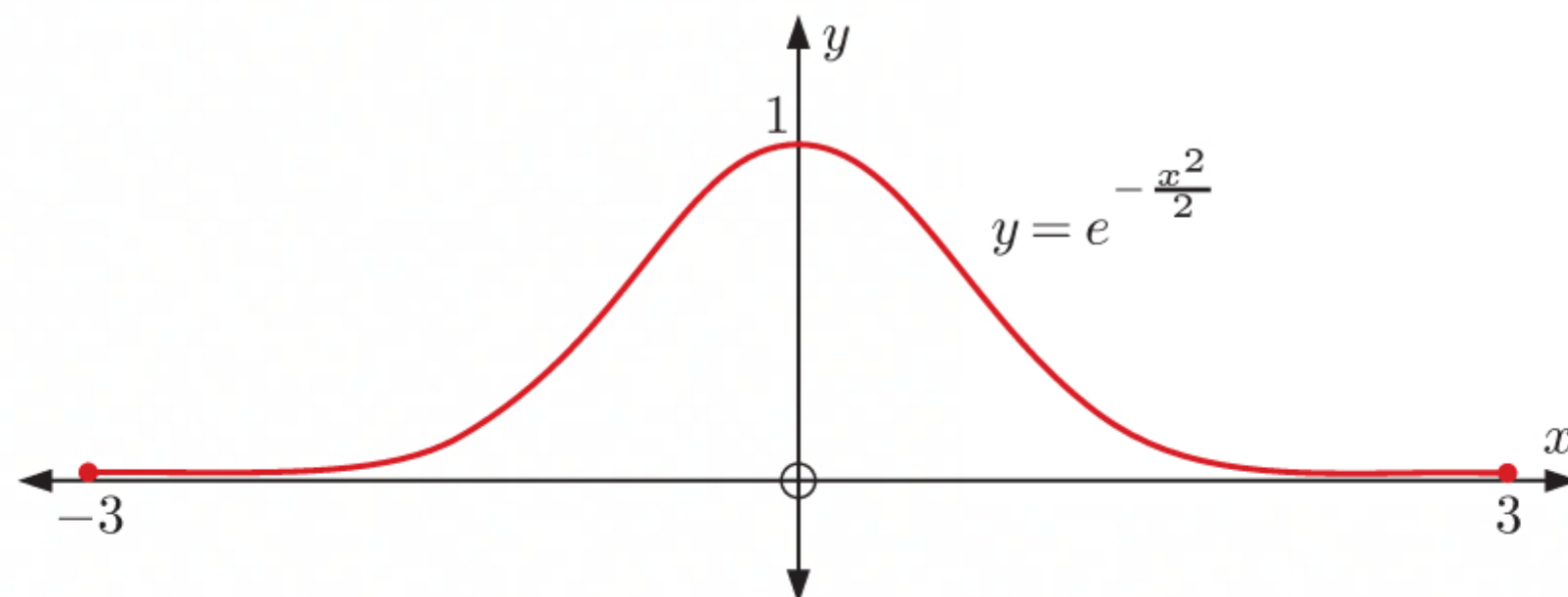
$$\begin{aligned} A_U &= \frac{2}{n} \sqrt{1+x_1^3} + \frac{2}{n} \sqrt{1+x_2^3} + \dots + \frac{2}{n} \sqrt{1+x_n^3} \\ &= \frac{2}{n} \sum_{i=1}^n \sqrt{1+x_i^3} \end{aligned}$$

b

n	A_L	A_U
50	3.2016	3.2816
100	3.2214	3.2614
500	3.2373	3.2453

c $\int_0^2 \sqrt{1+x^3} dx \approx 3.24$

- 3 a**



- b** Using provided software with the following settings:

From: 0

To: 3

Method: Upper/lower rectangles

Partitions: 2250

we find that $A_L \approx 1.2493$ and $A_U \approx 1.2506$

- c** Since $y = e^{-\frac{x^2}{2}}$ is symmetrical about the y -axis, the lower and upper rectangle sums for $-3 \leq x \leq 0$ with $n = 2250$ is equivalent to the lower and upper rectangle sums for $0 \leq x \leq 3$ with $n = 2250$.

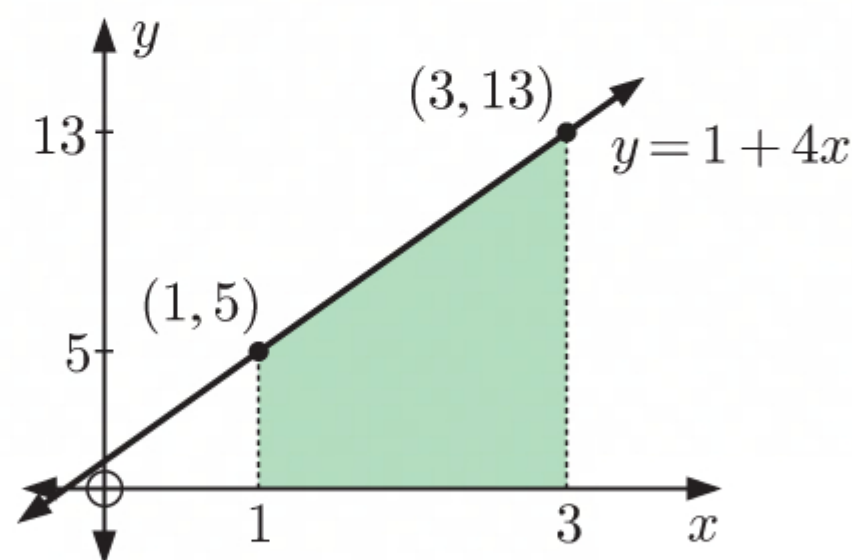
$\therefore A_L \approx 1.2493$ and $A_U \approx 1.2506$

d Using lower rectangles, $\int_{-3}^3 e^{-\frac{x^2}{2}} dx \approx 2 \times 1.2493 \approx 2.4986$

Using upper rectangles, $\int_{-3}^3 e^{-\frac{x^2}{2}} dx \approx 2 \times 1.2506 \approx 2.5012$

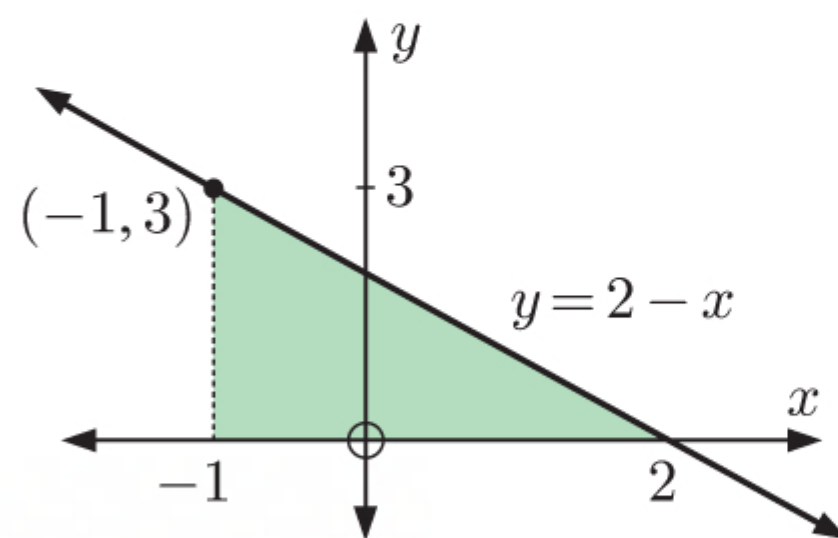
So, $\int_{-3}^3 e^{-\frac{x^2}{2}} dx \approx \frac{2.4986 + 2.5012}{2} \approx 2.4999$, $\sqrt{2\pi} \approx 2.5066$

4 a



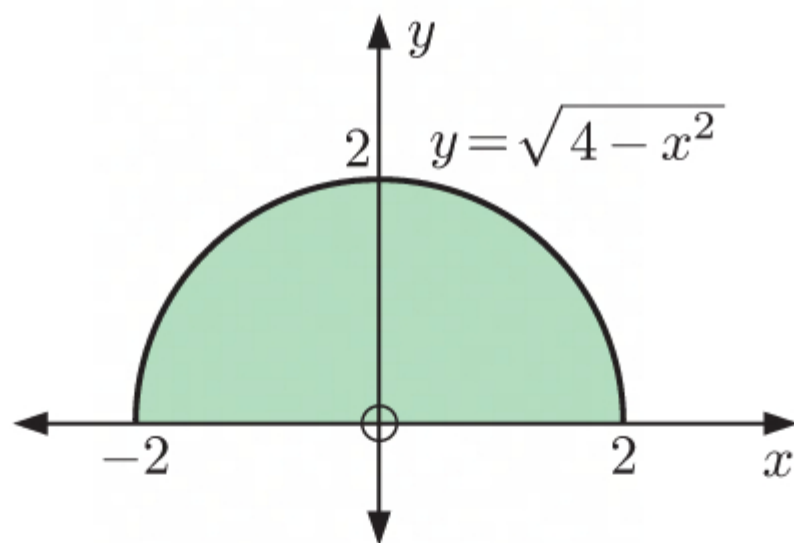
$$\begin{aligned} \int_1^3 (1 + 4x) dx &= \text{shaded area} \\ &= \left(\frac{5 + 13}{2} \right) \times 2 \\ &= 18 \end{aligned}$$

b



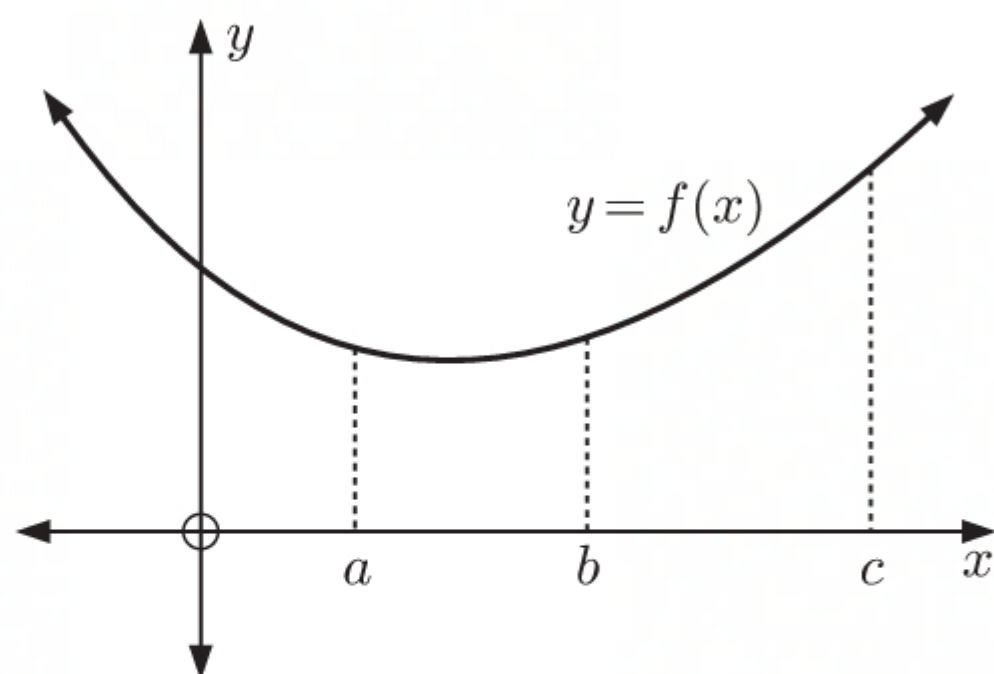
$$\begin{aligned} \int_{-1}^2 (2 - x) dx &= \text{shaded area} \\ &= \frac{1}{2}(3 \times 3) \\ &= 4.5 \end{aligned}$$

c



$$\begin{aligned} \int_{-2}^2 \sqrt{4 - x^2} dx &= \text{shaded area} \\ &= \frac{1}{2}(\pi \times 2^2) \\ &= 2\pi \end{aligned}$$

5 a



i From the diagram, we can see that there is no area under the curve for $a \leq x \leq a$.

$$\therefore \int_a^a f(x) dx = 0 \text{ for any positive function } f(x).$$

ii From the diagram, we can see that the area under the curve for $a \leq x \leq c$ is made up of the area under the curve for $a \leq x \leq b$ and $b \leq x \leq c$.

$$\begin{aligned} \therefore \int_a^b f(x) dx + \int_b^c f(x) dx &= \int_a^c f(x) dx \text{ for any positive function } f(x), \\ &\text{provided that } a \leq b \leq c. \end{aligned}$$

b i $\int_5^5 f(x) dx = 0 \quad \left\{ \int_a^a f(x) dx = 0 \right\}$

$$\begin{aligned}
 \text{ii} \quad & \int_2^9 f(x) \, dx \\
 &= \int_2^5 f(x) \, dx + \int_5^9 f(x) \, dx \quad \left\{ \int_a^b f(x) \, dx + \int_b^c f(x) \, dx = \int_a^c f(x) \, dx \right\} \\
 &= 10 + 12 \\
 &= 22
 \end{aligned}$$

EXERCISE 15C

$$\text{1 a i} \quad \frac{d}{dx}(x^2) = 2x$$

$$\therefore \frac{d}{dx}\left(\frac{1}{2}x^2\right) = x$$

\therefore the antiderivative of x is $\frac{1}{2}x^2$ or $\frac{x^2}{2}$.

$$\text{iii} \quad \frac{d}{dx}(x^6) = 6x^5$$

$$\therefore \frac{d}{dx}\left(\frac{1}{6}x^6\right) = x^5$$

\therefore the antiderivative of x^5 is $\frac{1}{6}x^6$ or $\frac{x^6}{6}$.

$$\text{v} \quad \frac{d}{dx}(x^{-3}) = -3x^{-4}$$

$$\therefore \frac{d}{dx}\left(-\frac{1}{3}x^{-3}\right) = x^{-4}$$

\therefore the antiderivative of x^{-4} is $-\frac{1}{3}x^{-3} = -\frac{1}{3x^3}$.

$$\text{vii} \quad \frac{d}{dx}(x^{\frac{2}{3}}) = \frac{2}{3}x^{-\frac{1}{3}}$$

$$\therefore \frac{d}{dx}\left(\frac{3}{2}x^{\frac{2}{3}}\right) = x^{-\frac{1}{3}}$$

\therefore the antiderivative of $x^{-\frac{1}{3}}$ is $\frac{3}{2}x^{\frac{2}{3}}$.

$$\text{ii} \quad \frac{d}{dx}(x^3) = 3x^2$$

$$\therefore \frac{d}{dx}\left(\frac{1}{3}x^3\right) = x^2$$

\therefore the antiderivative of x^2 is $\frac{1}{3}x^3$ or $\frac{x^3}{3}$.

$$\text{iv} \quad \frac{d}{dx}(x^{-1}) = -x^{-2}$$

$$\therefore \frac{d}{dx}(-x^{-1}) = x^{-2}$$

\therefore the antiderivative of x^{-2} is $-x^{-1}$ or $-\frac{1}{x}$.

$$\text{vi} \quad \frac{d}{dx}(x^{\frac{4}{3}}) = \frac{4}{3}x^{\frac{1}{3}}$$

$$\therefore \frac{d}{dx}\left(\frac{3}{4}x^{\frac{4}{3}}\right) = x^{\frac{1}{3}}$$

\therefore the antiderivative of $x^{\frac{1}{3}}$ is $\frac{3}{4}x^{\frac{4}{3}}$.

$$\text{viii} \quad \frac{d}{dx}(x^{\frac{5}{3}}) = \frac{5}{3}x^{\frac{2}{3}}$$

$$\therefore \frac{d}{dx}\left(\frac{3}{5}x^{\frac{5}{3}}\right) = x^{\frac{2}{3}}$$

\therefore the antiderivative of $x^{\frac{2}{3}}$ is $\frac{3}{5}x^{\frac{5}{3}}$.

b The antiderivative of x^n is $\frac{x^{n+1}}{n+1}$, for $n \neq -1$.

$$\text{2 a i} \quad \frac{d}{dx}(e^{3x}) = 3e^{3x}$$

$$\therefore \frac{d}{dx}\left(\frac{1}{3}e^{3x}\right) = e^{3x}$$

\therefore the antiderivative of e^{3x} is $\frac{1}{3}e^{3x}$.

$$\text{ii} \quad \frac{d}{dx}(e^{5x}) = 5e^{5x}$$

$$\therefore \frac{d}{dx}\left(\frac{1}{5}e^{5x}\right) = e^{5x}$$

\therefore the antiderivative of e^{5x} is $\frac{1}{5}e^{5x}$.

$$\text{iii} \quad \frac{d}{dx}(e^{\frac{1}{2}x}) = \frac{1}{2}e^{\frac{1}{2}x}$$

$$\therefore \frac{d}{dx}(2e^{\frac{1}{2}x}) = e^{\frac{1}{2}x}$$

\therefore the antiderivative of $e^{\frac{1}{2}x}$ is $2e^{\frac{1}{2}x}$.

$$\text{v} \quad \frac{d}{dx}(e^{\pi x}) = \pi e^{\pi x}$$

$$\therefore \frac{d}{dx}\left(\frac{1}{\pi}e^{\pi x}\right) = e^{\pi x}$$

\therefore the antiderivative of $e^{\pi x}$ is $\frac{1}{\pi}e^{\pi x}$.

$$\text{iv} \quad \frac{d}{dx}(e^{0.01x}) = 0.01e^{0.01x}$$

$$\therefore \frac{d}{dx}(100e^{0.01x}) = e^{0.01x}$$

\therefore the antiderivative of $e^{0.01x}$ is $100e^{0.01x}$.

$$\text{vi} \quad \frac{d}{dx}(e^{\frac{x}{3}}) = \frac{1}{3}e^{\frac{x}{3}}$$

$$\therefore \frac{d}{dx}(3e^{\frac{x}{3}}) = e^{\frac{x}{3}}$$

\therefore the antiderivative of $e^{\frac{x}{3}}$ is $3e^{\frac{x}{3}}$.

b The antiderivative of e^{kx} is $\frac{1}{k}e^{kx}$, where $k \neq 0$ is a constant.

$$\text{3 a} \quad \frac{d}{dx}(x^3 + x^2) = 3x^2 + 2x$$

$$\therefore \frac{d}{dx}(2x^3 + 2x^2) = 6x^2 + 4x$$

\therefore the antiderivative of $6x^2 + 4x$ is $2x^3 + 2x^2$.

$$\text{b} \quad \frac{d}{dx}(x\sqrt{x}) = \frac{d}{dx}(x^{\frac{3}{2}}) = \frac{3}{2}x^{\frac{1}{2}} = \frac{3}{2}\sqrt{x}$$

$$\therefore \frac{d}{dx}\left(\frac{2}{3}x\sqrt{x}\right) = \sqrt{x}$$

\therefore the antiderivative of \sqrt{x} is $\frac{2}{3}x\sqrt{x}$.

$$\text{c} \quad \frac{d}{dx}\left(\frac{1}{\sqrt{x}}\right) = \frac{d}{dx}(x^{-\frac{1}{2}}) = -\frac{1}{2}x^{-\frac{3}{2}} = -\frac{1}{2x\sqrt{x}}$$

$$\therefore \frac{d}{dx}\left(-\frac{2}{\sqrt{x}}\right) = \frac{1}{x\sqrt{x}}$$

\therefore the antiderivative of $\frac{1}{x\sqrt{x}}$ is $-\frac{2}{\sqrt{x}}$.

INVESTIGATION 2

THE AREA FUNCTION

$$\text{1} \quad F(t) = 5t$$

$$\therefore F'(t) = 5 = f(t)$$

$\therefore F(t)$ is the antiderivative of $f(t)$.

$$\text{2 a} \quad A(x) = \left(\frac{x+a}{2}\right)(x-a)$$

$$= \frac{x^2 - a^2}{2}$$

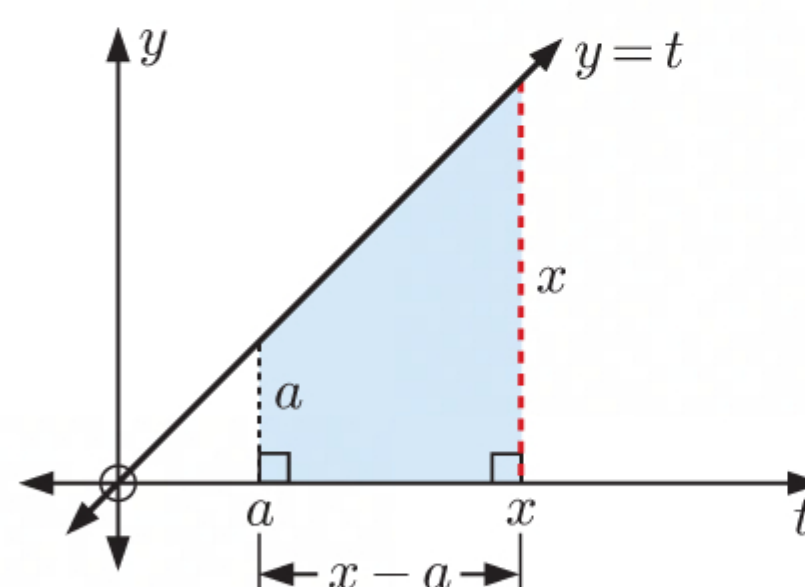
$$= \frac{x^2}{2} - \frac{a^2}{2}$$

$$= F(x) - F(a) \quad \text{where } F(t) = \frac{t^2}{2}$$

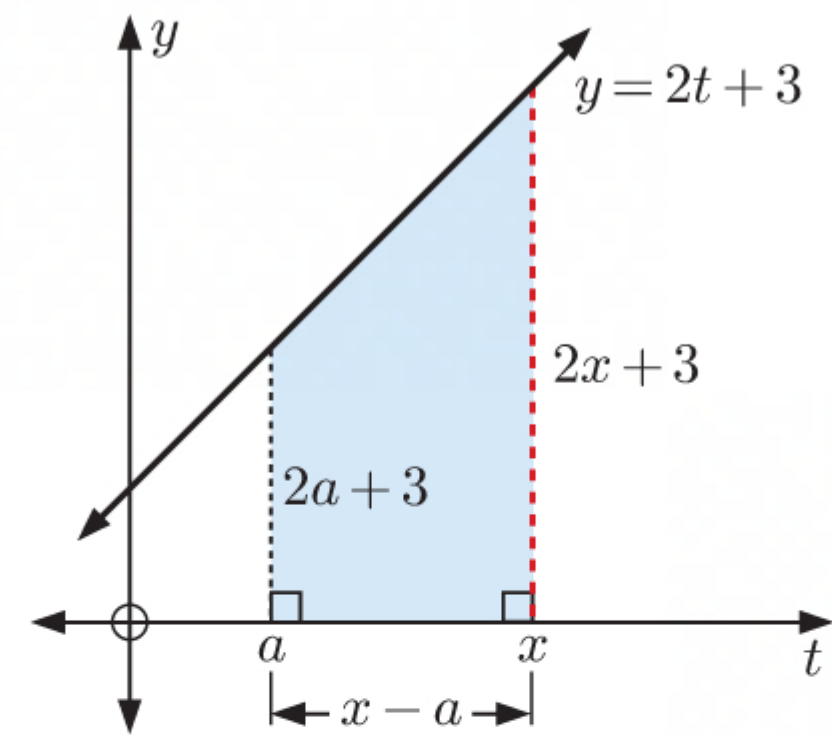
$$\text{b} \quad F(t) = \frac{t^2}{2}$$

$$\therefore F'(t) = t = f(t)$$

$\therefore F(t)$ is the antiderivative of $f(t)$.



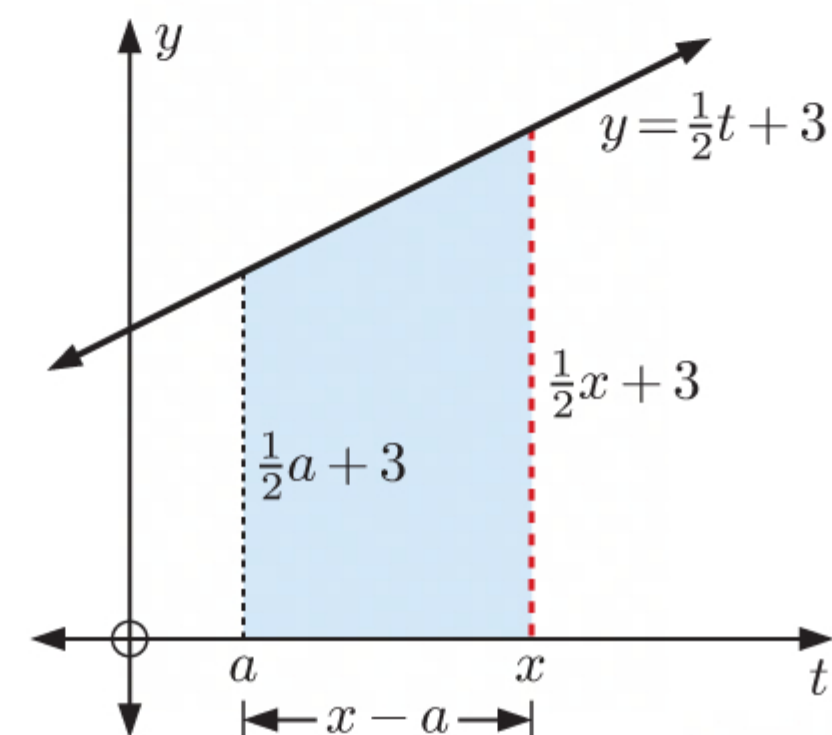
$$\begin{aligned}
 \text{3 a } A(x) &= \left(\frac{2x + 3 + 2a + 3}{2} \right) (x - a) \\
 &= \left(\frac{2(x + a) + 6}{2} \right) (x - a) \\
 &= (x + a + 3)(x - a) \\
 &= (x + a)(x - a) + 3(x - a) \\
 &= x^2 - a^2 + 3x - 3a \\
 &= x^2 + 3x - (a^2 + 3a) \\
 &= F(x) - F(a) \quad \text{where } F(t) = t^2 + 3t
 \end{aligned}$$



$$\begin{aligned}
 \text{b } F(t) &= t^2 + 3t \\
 \therefore F'(t) &= 2t + 3 = f(t) \\
 \therefore F(t) &\text{ is the antiderivative of } f(t).
 \end{aligned}$$

4 a Consider $f(t) = \frac{1}{2}t + 3$.
The corresponding area function is

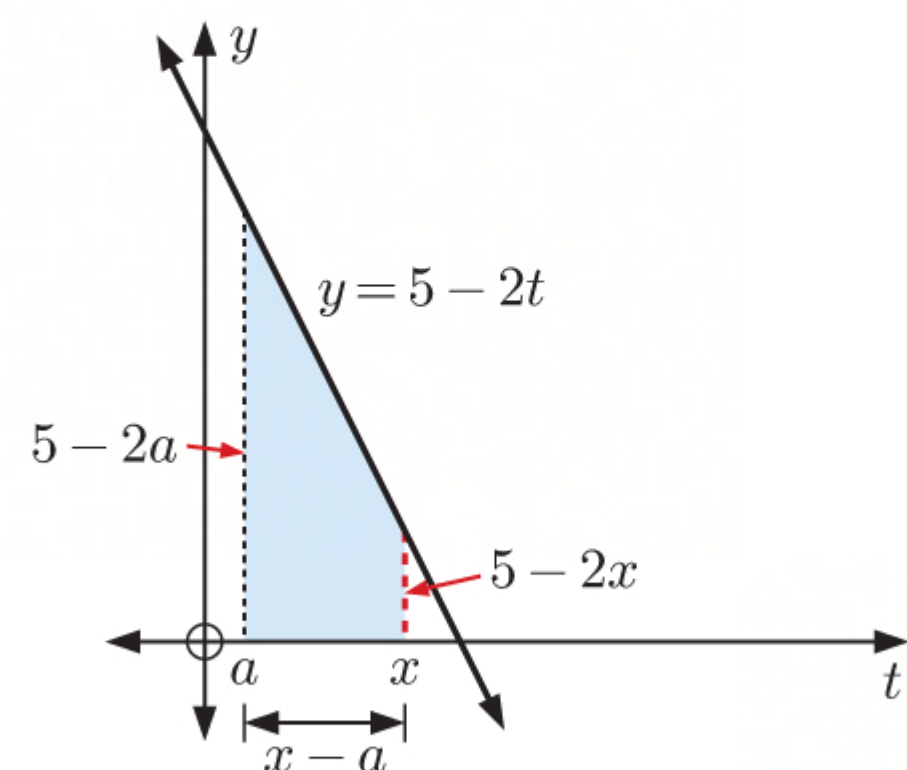
$$\begin{aligned}
 A(x) &= \int_a^x \left(\frac{1}{2}t + 3 \right) dt \\
 &= \text{shaded area} \\
 &= \left(\frac{\frac{1}{2}x + 3 + \frac{1}{2}a + 3}{2} \right) (x - a) \\
 &= \left(\frac{1}{4}x + \frac{3}{2} + \frac{1}{4}a + \frac{3}{2} \right) (x - a) \\
 &= \frac{1}{4}x^2 - \cancel{\frac{1}{4}ax} + \frac{3}{2}x - \frac{3}{2}a + \cancel{\frac{1}{4}ax} - \frac{1}{4}a^2 + \frac{3}{2}x - \frac{3}{2}a \\
 &= \frac{1}{4}x^2 + 3x - \frac{1}{4}a^2 - 3a \\
 &= \frac{1}{4}x^2 + 3x - \left(\frac{1}{4}a^2 + 3a \right) \\
 &= F(x) - F(a) \quad \text{where } F(t) = \frac{1}{4}t^2 + 3t
 \end{aligned}$$



$$\begin{aligned}
 \text{Now } F(t) &= \frac{1}{4}t^2 + 3t \\
 \therefore F'(t) &= \frac{1}{2}t + 3 = f(t) \\
 \therefore F(t) &\text{ is the antiderivative of } f(t).
 \end{aligned}$$

b Consider $f(t) = 5 - 2t$.
The corresponding area function is

$$\begin{aligned}
 A(x) &= \int_a^x (5 - 2t) dt \\
 &= \text{shaded area} \\
 &= \left(\frac{5 - 2x + 5 - 2a}{2} \right) (x - a) \\
 &= \left(\frac{5}{2} - x + \frac{5}{2} - a \right) (x - a) \\
 &= \frac{5}{2}x - \frac{5}{2}a - x^2 + \cancel{ax} + \frac{5}{2}x - \frac{5}{2}a - \cancel{ax} + a^2 \\
 &= 5x - x^2 - 5a + a^2 \\
 &= 5x - x^2 - (5a - a^2) \\
 &= F(x) - F(a) \quad \text{where } F(t) = 5t - t^2
 \end{aligned}$$



Now $F(t) = 5t - t^2$

$$\therefore F'(t) = 5 - 2t = f(t)$$

$\therefore F(t)$ is the antiderivative of $f(t)$.

5 $f(t) = 3t^2 + 4t + 5$

We predict that $F(t)$ is the antiderivative of $f(t)$.

$$\frac{d}{dt}(t^3 + 2t^2 + 5t) = 3t^2 + 4t + 5 = f(t)$$

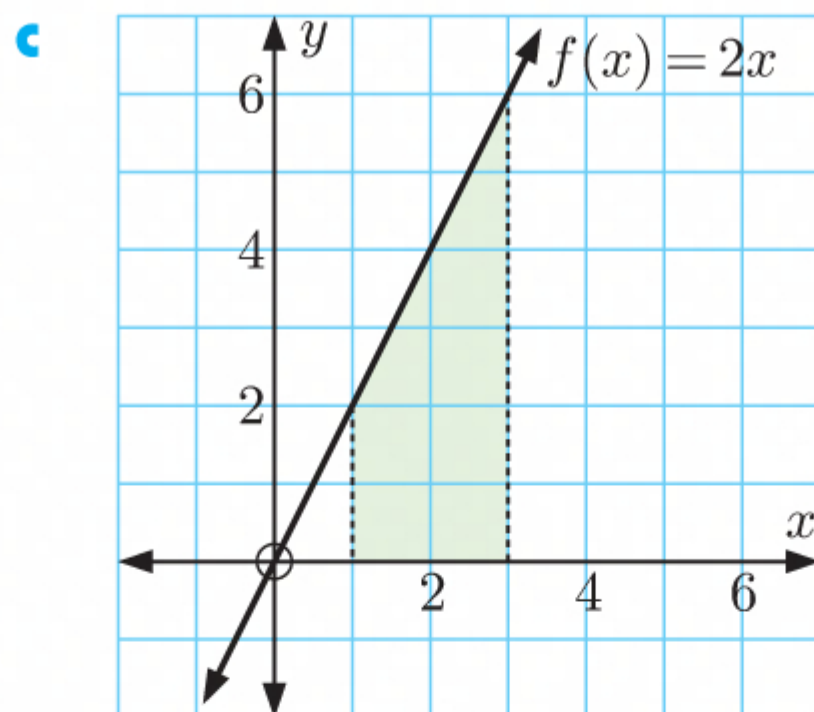
$$\therefore F(t) = t^3 + 2t^2 + 5t$$

EXERCISE 15D

1 a $\frac{d}{dx}(x^2) = 2x$

\therefore the antiderivative of $f(x) = 2x$ is $F(x) = x^2$.

b
$$\begin{aligned}\int_1^3 2x \, dx &= F(3) - F(1) \\ &= 3^2 - 1^2 \\ &= 8 \text{ units}^2\end{aligned}$$



$$\begin{aligned}\int_1^3 2x \, dx &= \text{shaded area} \\ &= \left(\frac{2+6}{2}\right) \times 2 \\ &= 8 \text{ units}^2\end{aligned}$$

2 a
$$\begin{aligned}\frac{d}{dx}(x\sqrt{x}) &= \frac{d}{dx}(x^{\frac{3}{2}}) \\ &= \frac{3}{2}x^{\frac{1}{2}} \\ &= \frac{3}{2}\sqrt{x}\end{aligned}$$

$$\begin{aligned}\therefore \frac{d}{dx}\left(\frac{2}{3}x^{\frac{3}{2}}\right) &= x^{\frac{1}{2}} \\ &= \sqrt{x}\end{aligned}$$

\therefore the antiderivative of $f(x) = \sqrt{x}$

is $F(x) = \frac{2}{3}x^{\frac{3}{2}}$.

b
$$\begin{aligned}\int_0^1 \sqrt{x} \, dx &= F(1) - F(0) \\ &= \frac{2}{3} - 0 \\ &= \frac{2}{3} \text{ units}^2\end{aligned}$$

c $\frac{2}{3} \approx 0.67$ to 2 significant figures
 \therefore the answer is the same as
Exercise 15B question 1.

3 a $\frac{d}{dx}(x^4) = 4x^3$

$\therefore \frac{d}{dx}\left(\frac{1}{4}x^4\right) = x^3$

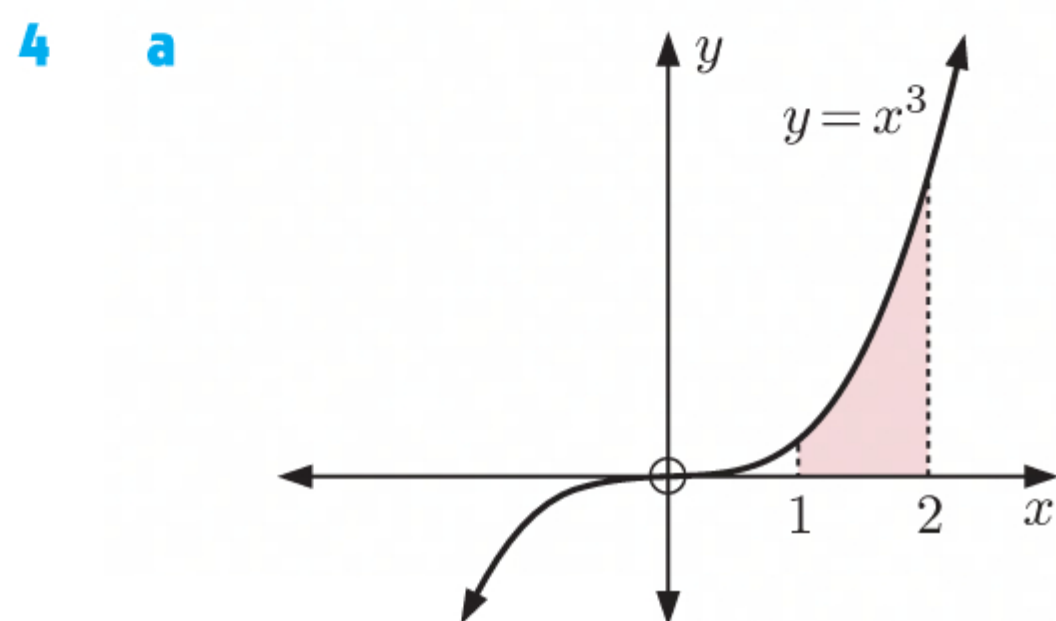
\therefore the antiderivative of $f(x) = x^3$ is $F(x) = \frac{1}{4}x^4$.

i $\int_0^2 x^3 dx$
 $= F(2) - F(0)$
 $= 4 - 0$
 $= 4 \text{ units}^2$

ii $\int_2^3 x^3 dx$
 $= F(3) - F(2)$
 $= \frac{81}{4} - 4$
 $= 16\frac{1}{4} \text{ units}^2$

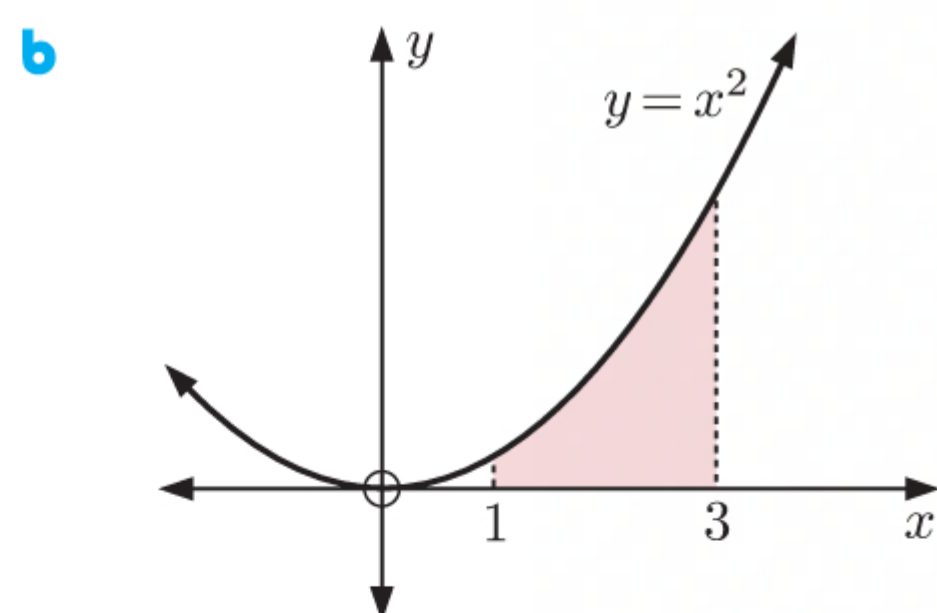
iii $\int_0^3 x^3 dx$
 $= F(3) - F(0)$
 $= \frac{81}{4} - 0$
 $= 20\frac{1}{4} \text{ units}^2$

b $\int_0^3 x^3 dx = \int_0^2 x^3 dx + \int_2^3 x^3 dx$



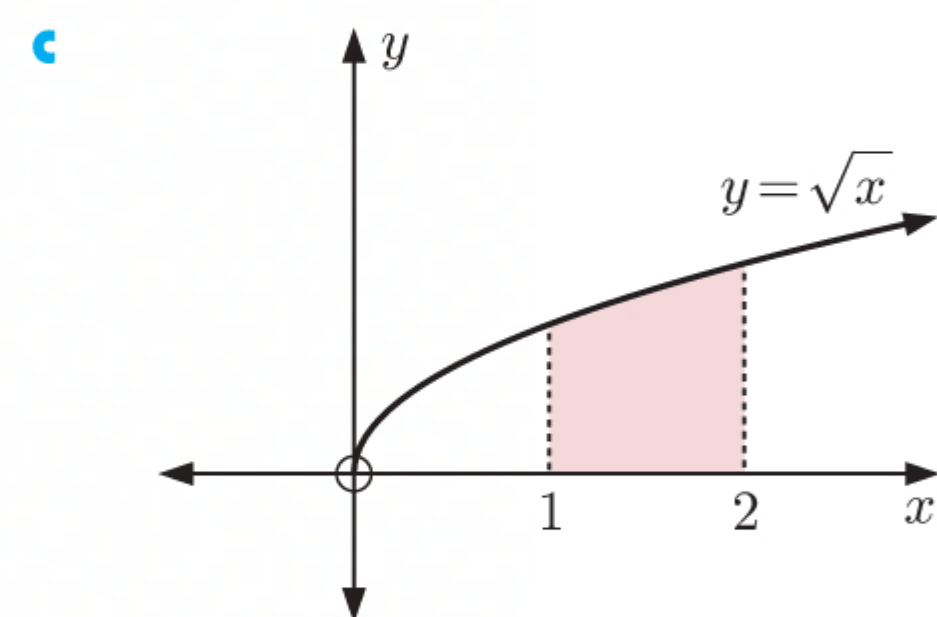
$f(x) = x^3$ has antiderivative $F(x) = \frac{x^4}{4}$

\therefore shaded area $= \int_1^2 x^3 dx$
 $= F(2) - F(1)$
 $= 4 - \frac{1}{4}$
 $= 3\frac{3}{4} \text{ units}^2$



$f(x) = x^2$ has antiderivative $F(x) = \frac{x^3}{3}$

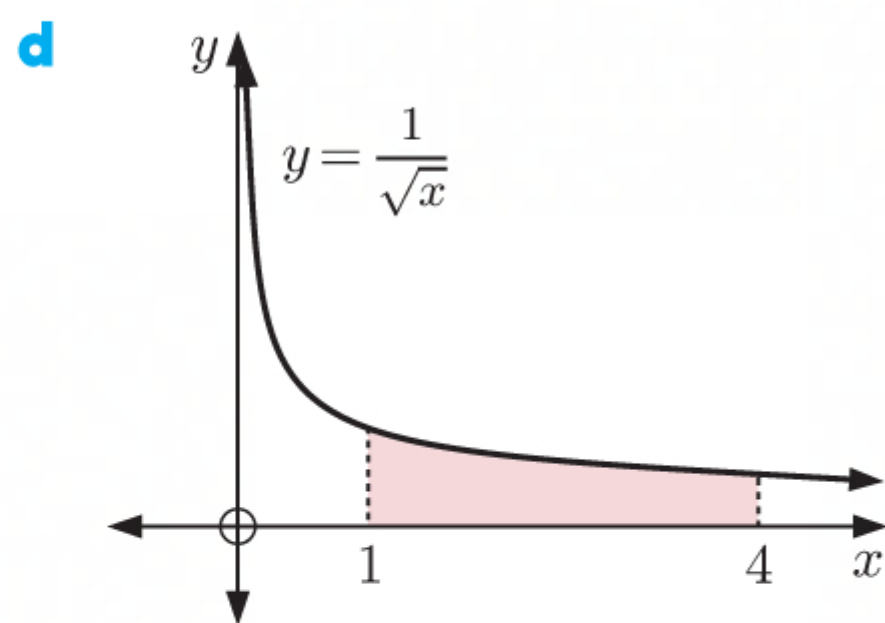
\therefore shaded area $= \int_1^3 x^2 dx$
 $= F(3) - F(1)$
 $= 9 - \frac{1}{3}$
 $= 8\frac{2}{3} \text{ units}^2$



$f(x) = \sqrt{x} = x^{\frac{1}{2}}$ has antiderivative

$F(x) = \frac{x^{\frac{3}{2}}}{\frac{3}{2}} = \frac{2}{3}x\sqrt{x}$

\therefore shaded area $= \int_1^2 \sqrt{x} dx$
 $= F(2) - F(1)$
 $= \frac{2}{3}(2\sqrt{2}) - \frac{2}{3}(1\sqrt{1})$
 $= \frac{4\sqrt{2}}{3} - \frac{2}{3}$
 $= \frac{4\sqrt{2} - 2}{3} \text{ units}^2$



$$f(x) = \frac{1}{\sqrt{x}} = x^{-\frac{1}{2}} \text{ has antiderivative}$$

$$F(x) = \frac{x^{\frac{1}{2}}}{\frac{1}{2}} = 2\sqrt{x}$$

$$\begin{aligned} \therefore \text{shaded area} &= \int_1^4 \frac{1}{\sqrt{x}} dx \\ &= F(4) - F(1) \\ &= 2\sqrt{4} - 2\sqrt{1} \\ &= 2 \text{ units}^2 \end{aligned}$$

5 Let $F(x)$ be the antiderivative of $f(x)$ and $G(x)$ be the antiderivative of $g(x)$.

a $\int_a^a f(x) dx = F(a) - F(a) = 0$

$\int_a^a f(x) dx = \text{area of the region under the curve } y = f(x) \text{ between } x = a \text{ and } x = a.$

This region has 0 width, so its area = 0.

c
$$\begin{aligned} \int_b^a f(x) dx &= F(a) - F(b) \\ &= -[F(b) - F(a)] \\ &= -\int_a^b f(x) dx \end{aligned}$$

e $\frac{d}{dx} F(x) = f(x) \quad \text{and} \quad \frac{d}{dx} G(x) = g(x)$

$$\therefore \frac{d}{dx} [F(x) + G(x)] = f(x) + g(x)$$

$\therefore F(x) + G(x)$ is the antiderivative of $f(x) + g(x)$.

$$\begin{aligned} \text{So, } \int_a^b [f(x) + g(x)] dx &= [F(b) + G(b)] - [F(a) + G(a)] \\ &= [F(b) - F(a)] + [G(b) - G(a)] \\ &= \int_a^b f(x) dx + \int_a^b g(x) dx \end{aligned}$$

b The antiderivative of $f(x) = k$ is $F(x) = kx$.

$$\begin{aligned} \therefore \int_a^b k dx &= F(b) - F(a) \\ &= kb - ka \\ &= k(b - a) \end{aligned}$$

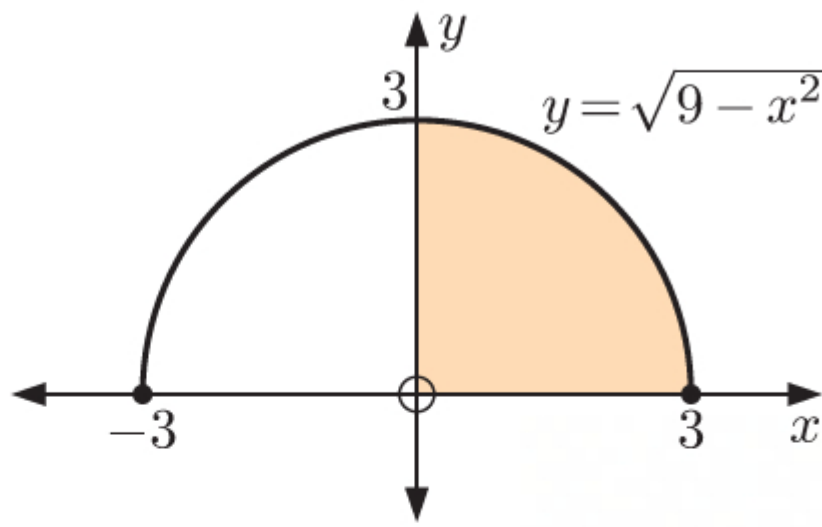
d $\frac{d}{dx} F(x) = f(x)$

$$\therefore \frac{d}{dx} (k F(x)) = k f(x)$$

$\therefore k F(x)$ is the antiderivative of $k f(x)$.

$$\begin{aligned} \text{So, } \int_a^b k f(x) dx &= k F(b) - k F(a) \\ &= k[F(b) - F(a)] \\ &= k \int_a^b f(x) dx \end{aligned}$$

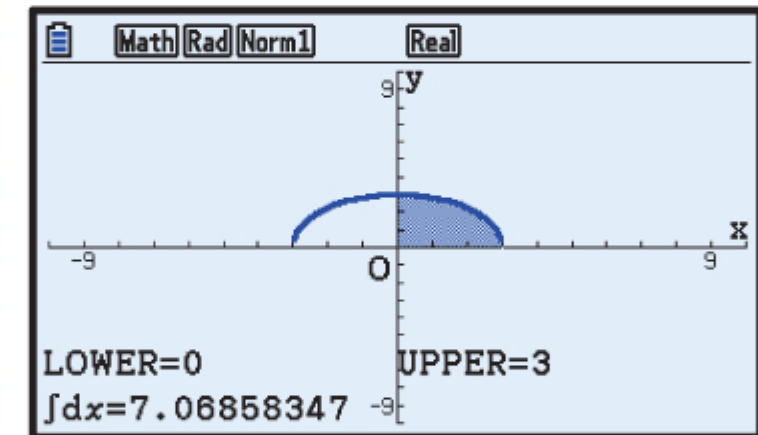
6



Using technology, $\text{area} = \int_0^3 \sqrt{9 - x^2} \, dx \approx 7.07 \text{ units}^2$

Check: The area is a quarter circle with radius 3 units.

$$\begin{aligned} \therefore \text{area} &= \frac{1}{4} \pi r^2 \\ &= \frac{1}{4} \times \pi \times 3^2 \\ &= \frac{9\pi}{4} \\ &\approx 7.07 \text{ units}^2 \quad \checkmark \end{aligned}$$

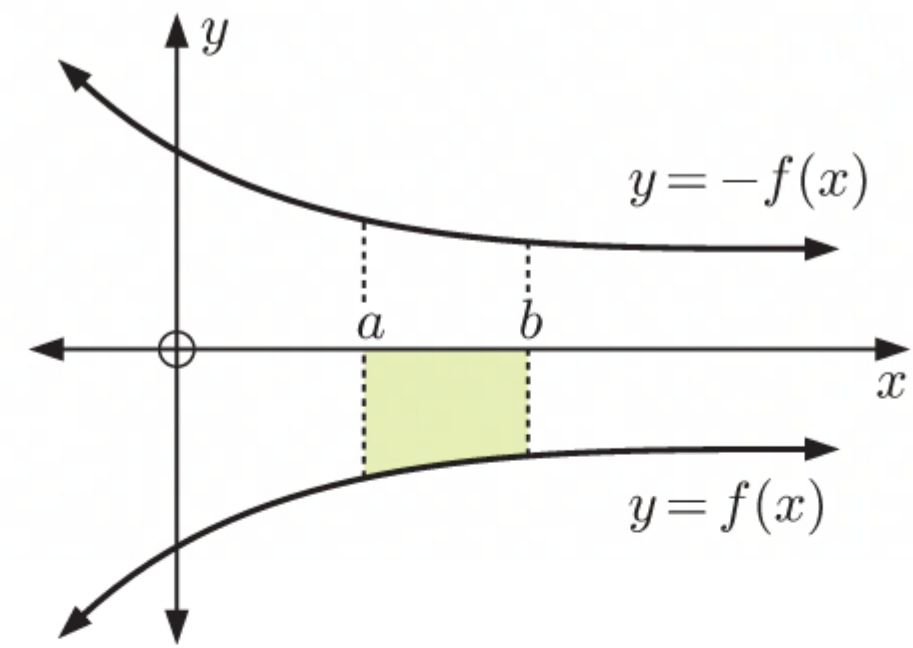


7 a If $\frac{d}{dx} F(x) = f(x)$ then $\frac{d}{dx} (-F(x)) = -f(x)$

$$\begin{aligned} \therefore \int_a^b (-f(x)) \, dx &= -F(b) - (-F(a)) \\ &= -(F(b) - F(a)) \\ &= -\int_a^b f(x) \, dx \end{aligned}$$

b Since $y = -f(x)$ is a reflection of $y = f(x)$ in the x -axis, then

$$\begin{aligned} &\text{shaded area} \\ &= \text{area between the } x\text{-axis and } y = -f(x) \\ &\text{from } x = a \text{ to } x = b \\ &= \int_a^b (-f(x)) \, dx \\ &= -\int_a^b f(x) \, dx \quad \{\text{using a}\} \end{aligned}$$

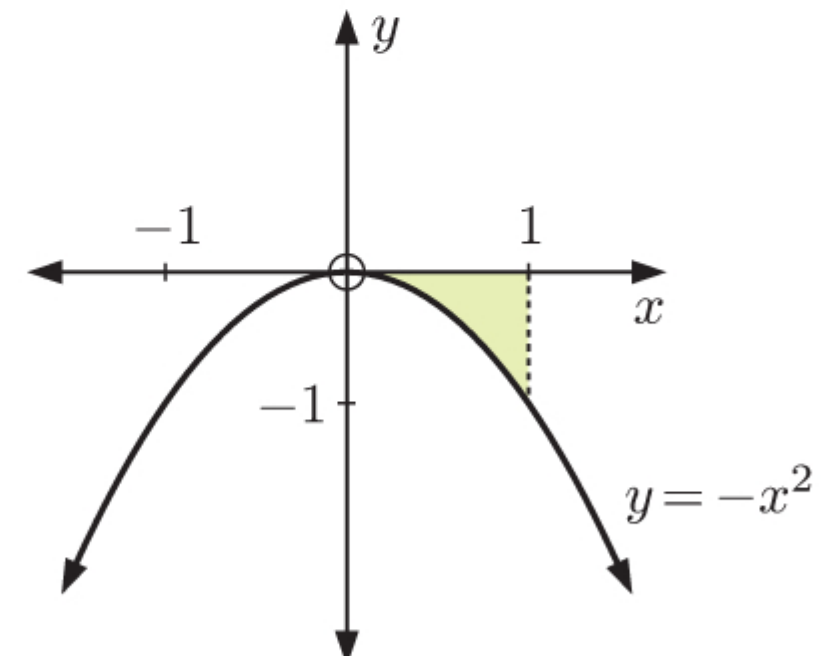


c i $\int_0^1 (-x^2) \, dx = -\int_0^1 x^2 \, dx$

Now $f(x) = x^2$ has antiderivative $F(x) = \frac{1}{3}x^3$

$$\begin{aligned} \therefore \int_0^1 (-x^2) \, dx &= -(F(1) - F(0)) \\ &= -\left(\frac{1}{3} - 0\right) \\ &= -\frac{1}{3} \end{aligned}$$

The shaded region has area $\frac{1}{3} \text{ units}^2$.



$$\text{ii} \quad \int_0^1 (x^2 - x) dx = - \int_0^1 (x - x^2) dx$$

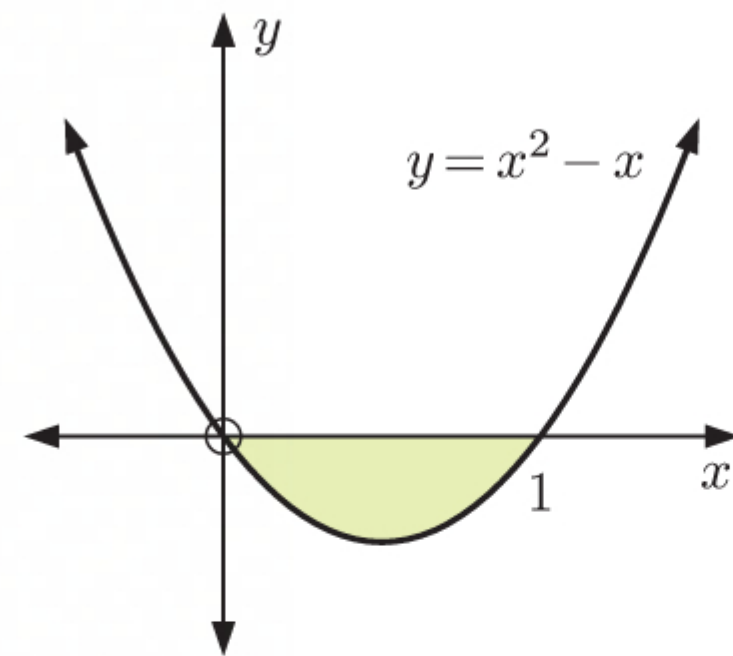
$$\{x^2 - x \leq 0 \text{ for all } 0 \leq x \leq 1\}$$

Now $f(x) = x - x^2$ has antiderivative

$$F(x) = \frac{1}{2}x^2 - \frac{1}{3}x^3$$

$$\begin{aligned} \therefore \int_0^1 (x^2 - x) dx &= -(F(1) - F(0)) \\ &= -\left(\frac{1}{2} - \frac{1}{3} - (0 - 0)\right) \\ &= -\frac{1}{6} \end{aligned}$$

The shaded region has area $\frac{1}{6}$ units².



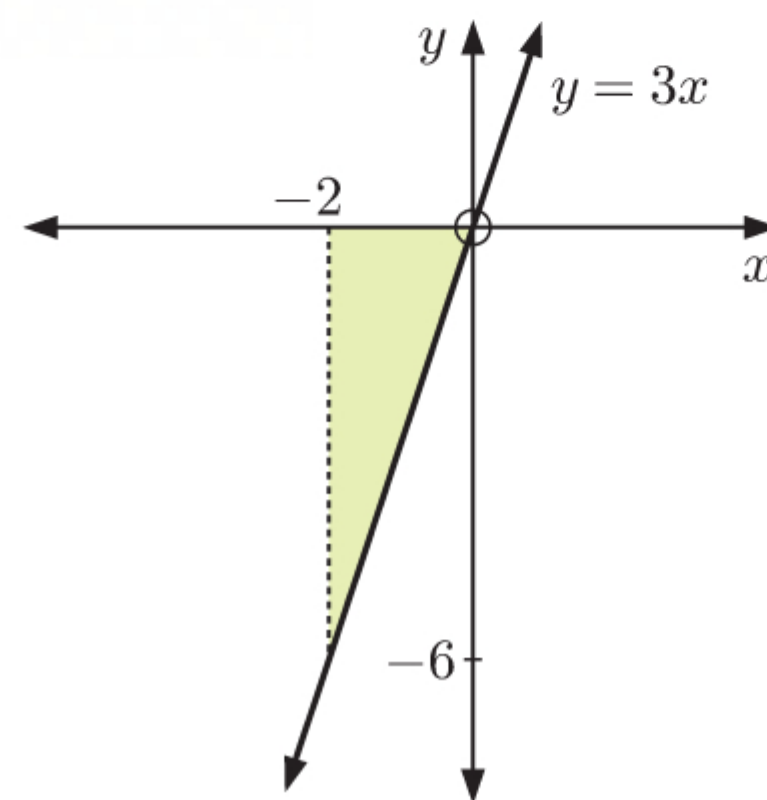
$$\text{iii} \quad \int_{-2}^0 3x dx = - \int_{-2}^0 -3x dx$$

Now $f(x) = -3x$ has antiderivative

$$F(x) = -\frac{3}{2}x^2$$

$$\begin{aligned} \therefore \int_{-2}^0 3x dx &= -(F(0) - F(-2)) \\ &= -(0 - (-6)) \\ &= -6 \end{aligned}$$

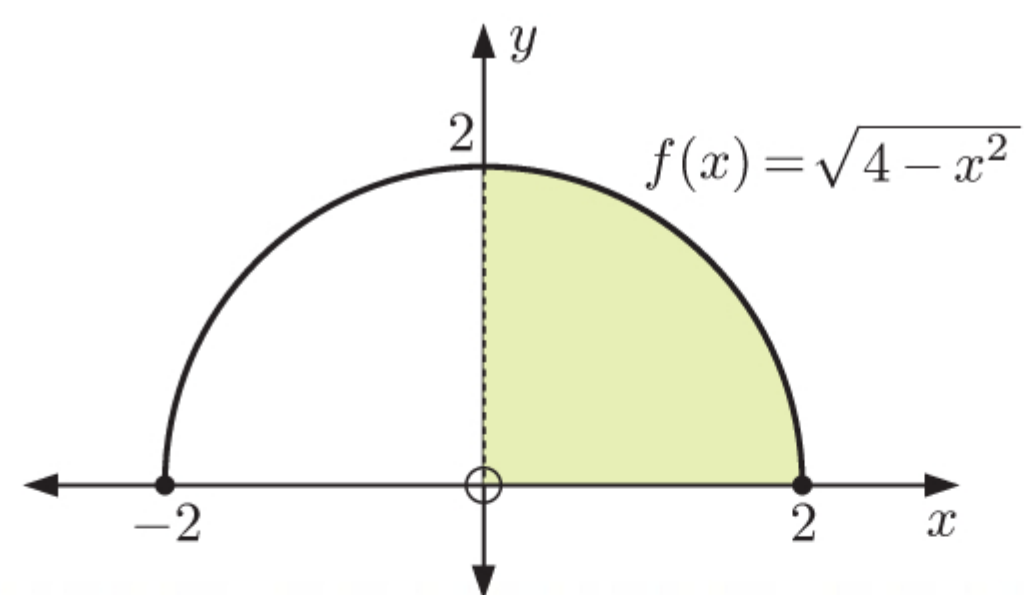
The shaded region has area 6 units².



$$\text{d} \quad \int_0^2 \left(-\sqrt{4-x^2}\right) dx = - \int_0^2 \sqrt{4-x^2} dx$$

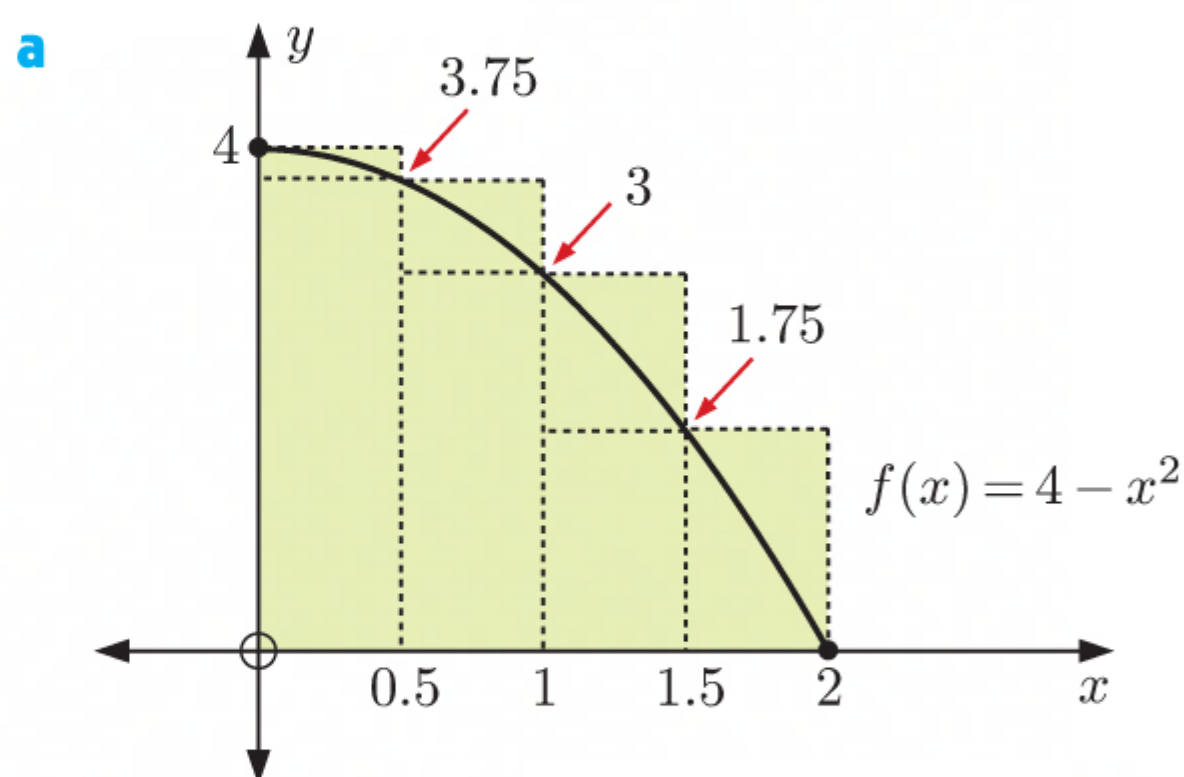
Now $f(x) = \sqrt{4-x^2}$ is the top half of a circle with radius 2 units and centre (0, 0).

$$\begin{aligned} \therefore \int_0^2 \left(-\sqrt{4-x^2}\right) dx &= - \int_0^2 \sqrt{4-x^2} dx \\ &= -(\text{shaded area}) \\ &= -\frac{1}{4} \times \pi \times 2^2 \\ &= -\pi \end{aligned}$$



REVIEW SET 15A

- 1 The rectangles are $\frac{2}{4} = \frac{1}{2}$ units wide.



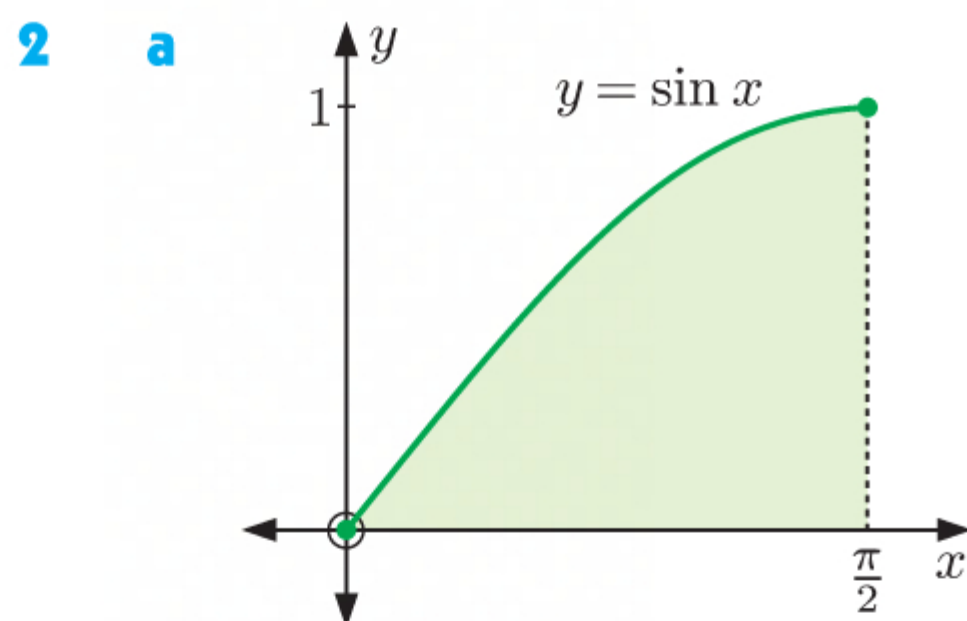
$$\begin{aligned} A_L &= \frac{1}{2} \left[f\left(\frac{1}{2}\right) + f(1) + f\left(\frac{3}{2}\right) + f(2) \right] \\ &= \frac{1}{2} \left(\frac{15}{4} + 3 + \frac{7}{4} + 0 \right) \\ &= \frac{17}{4} \end{aligned}$$

$$\begin{aligned} A_U &= \frac{1}{2} \left[f(0) + f\left(\frac{1}{2}\right) + f(1) + f\left(\frac{3}{2}\right) \right] \\ &= \frac{1}{2} \left(4 + \frac{15}{4} + 3 + \frac{7}{4} \right) \\ &= \frac{25}{4} \end{aligned}$$

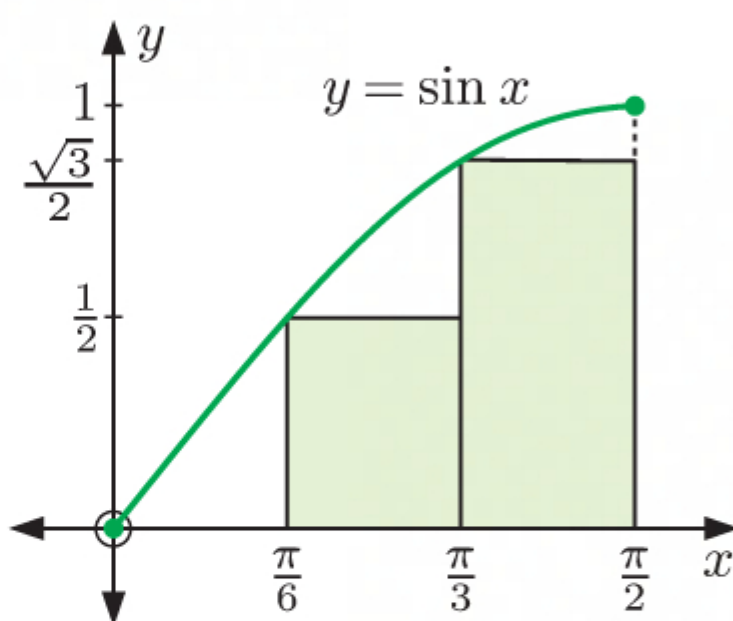
$$\therefore \frac{17}{4} < \int_0^2 (4 - x^2) dx < \frac{25}{4}$$

$$\therefore A = \frac{17}{4}, \quad B = \frac{25}{4}$$

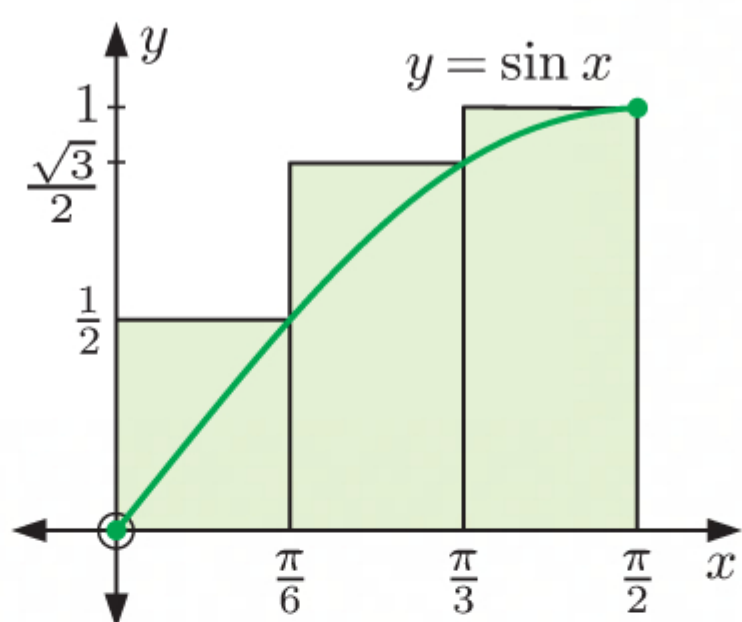
b $\int_0^2 (4 - x^2) dx \approx \frac{A+B}{2} \approx \frac{42}{8} \approx \frac{21}{4}$



- b The rectangles are $\frac{\pi/2}{3} = \frac{\pi}{6}$ units wide.



$$\begin{aligned} A_L &= \frac{\pi}{6} (\sin 0 + \sin \frac{\pi}{6} + \sin \frac{\pi}{3}) \\ &= \frac{\pi}{6} \left(0 + \frac{1}{2} + \frac{\sqrt{3}}{2} \right) \\ &= \frac{\pi(1 + \sqrt{3})}{12} \end{aligned}$$



$$\begin{aligned}
 A_U &= \frac{\pi}{6} \left(\sin \frac{\pi}{6} + \sin \frac{\pi}{3} + \sin \frac{\pi}{2} \right) \\
 &= \frac{\pi}{6} \left(\frac{1}{2} + \frac{\sqrt{3}}{2} + 1 \right) \\
 &= \frac{\pi}{6} \left(\frac{1 + \sqrt{3} + 2}{2} \right) \\
 &= \frac{\pi}{6} \left(\frac{3 + \sqrt{3}}{2} \right) \\
 &= \frac{\pi(3 + \sqrt{3})}{12}
 \end{aligned}$$

$$\frac{\pi(1 + \sqrt{3})}{12} < \int_0^{\frac{\pi}{2}} \sin x \, dx < \frac{\pi(3 + \sqrt{3})}{12}$$

$$\text{or } 0.715 < \int_0^{\frac{\pi}{2}} \sin x \, dx < 1.24$$

3 a

$$\frac{d}{dx}(x^5) = 5x^4$$

$$\therefore \frac{d}{dx}\left(\frac{1}{5}x^5\right) = x^4$$

\therefore the antiderivative of x^4 is $\frac{1}{5}x^5$ or $\frac{x^5}{5}$.

c

$$\frac{d}{dx}(e^{-\frac{1}{2}x}) = -\frac{1}{2}e^{-\frac{1}{2}x}$$

$$\therefore \frac{d}{dx}(-2e^{-\frac{1}{2}x}) = e^{-\frac{1}{2}x}$$

\therefore the antiderivative of $e^{-\frac{1}{2}x}$ is $-2e^{-\frac{1}{2}x}$.

b

$$\frac{1}{2x^2} = \frac{1}{2}x^{-2}$$

$$\text{Now, } \frac{d}{dx}(x^{-1}) = -x^{-2}$$

$$\therefore \frac{d}{dx}\left(-\frac{1}{2}x^{-1}\right) = \frac{1}{2}x^{-2}$$

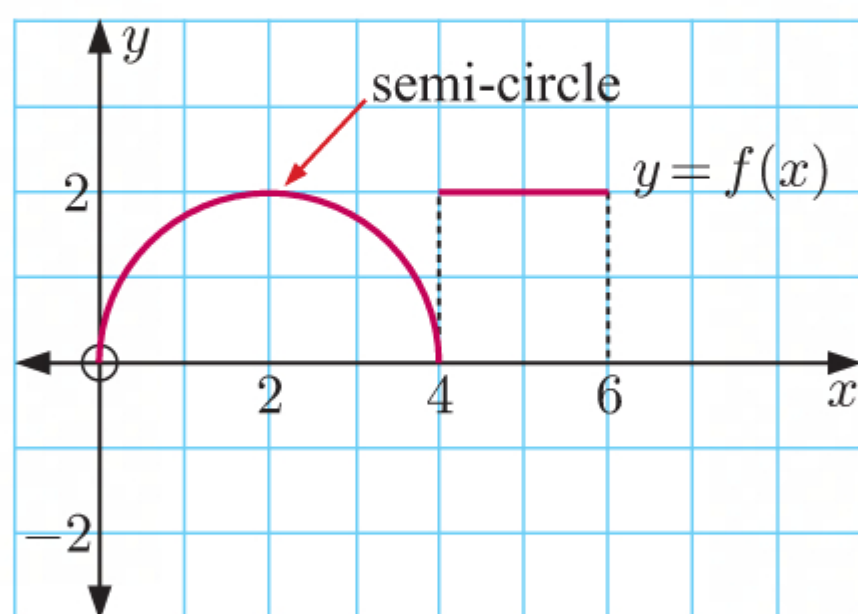
\therefore the antiderivative of $\frac{1}{2x^2}$ is $-\frac{1}{2}x^{-1} = -\frac{1}{2x}$.

d

$$\frac{d}{dx}(\sin x) = \cos x$$

\therefore the antiderivative of $\cos x$ is $\sin x$.

4



a

$$\begin{aligned}
 &\int_0^4 f(x) \, dx \\
 &= \text{area of semi-circle} \\
 &\quad \text{with radius 2} \\
 &= \frac{1}{2} \times \pi(2)^2 \\
 &= 2\pi
 \end{aligned}$$

b

$$\begin{aligned}
 &\int_4^6 f(x) \, dx \\
 &= \text{area of square} \\
 &= 2^2 \\
 &= 4
 \end{aligned}$$

5 a $\frac{d}{dx}(x^3) = 3x^2$

$\therefore \frac{d}{dx}\left(\frac{1}{3}x^3\right) = x^2$

\therefore the antiderivative of $f(x) = x^2$ is $F(x) = \frac{1}{3}x^3$.

i $\int_0^1 x^2 dx$
 $= F(1) - F(0)$
 $= \frac{1}{3} - 0$
 $= \frac{1}{3} \text{ units}^2$

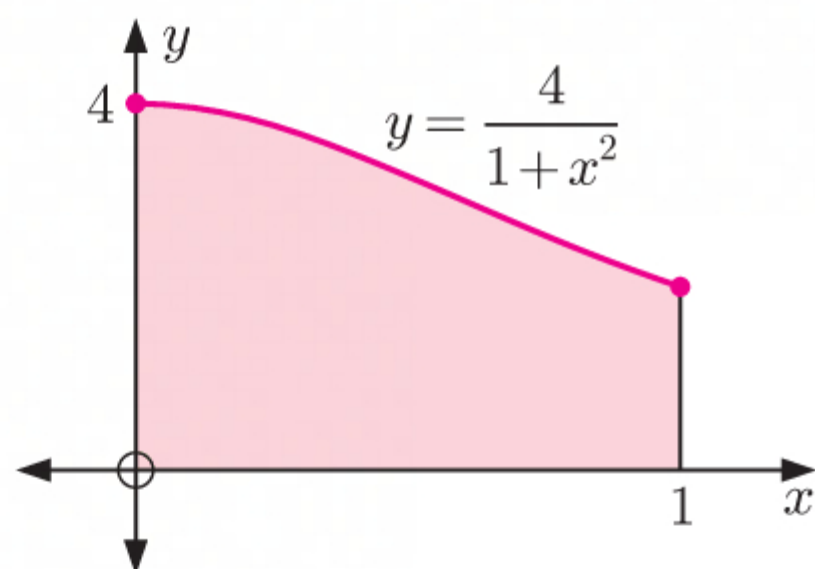
ii $\int_1^2 x^2 dx$
 $= F(2) - F(1)$
 $= \frac{8}{3} - \frac{1}{3}$
 $= \frac{7}{3} = 2\frac{1}{3} \text{ units}^2$

iii $\int_0^2 x^2 dx$
 $= F(2) - F(0)$
 $= \frac{8}{3} - 0$
 $= \frac{8}{3} = 2\frac{2}{3} \text{ units}^2$

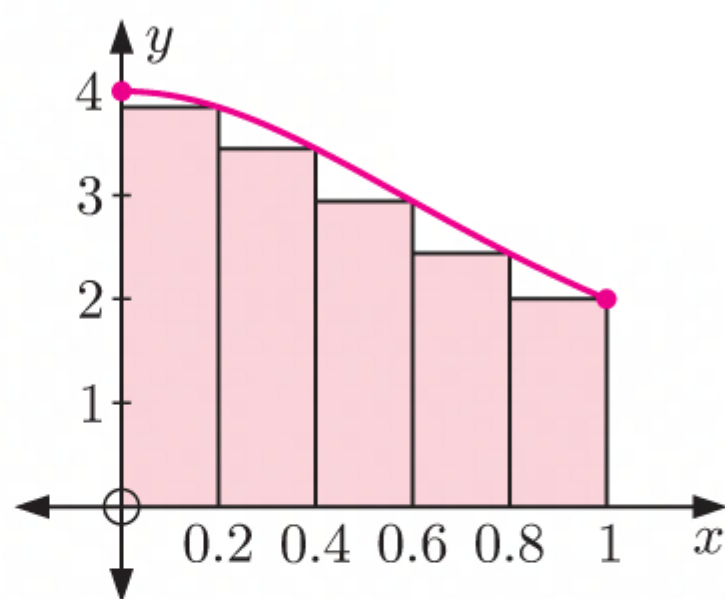
b $\int_0^2 x^2 dx = \int_0^1 x^2 dx + \int_1^2 x^2 dx$

REVIEW SET 15B

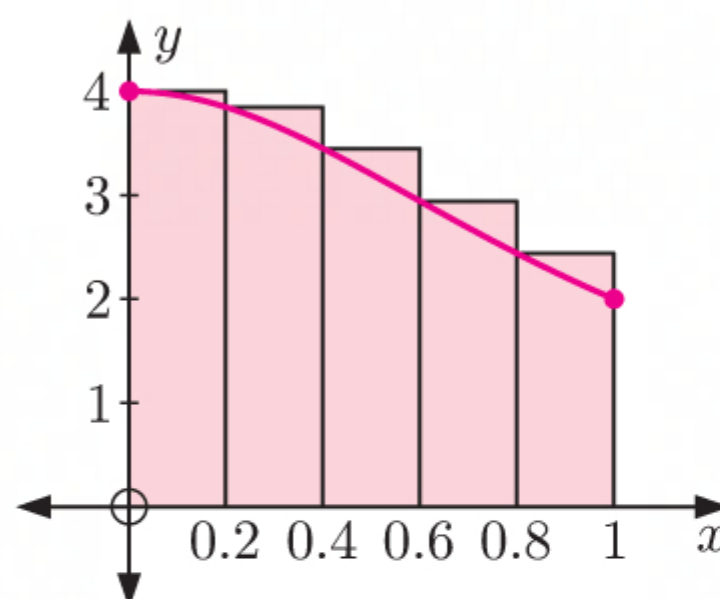
1 a



lower rectangles



upper rectangles

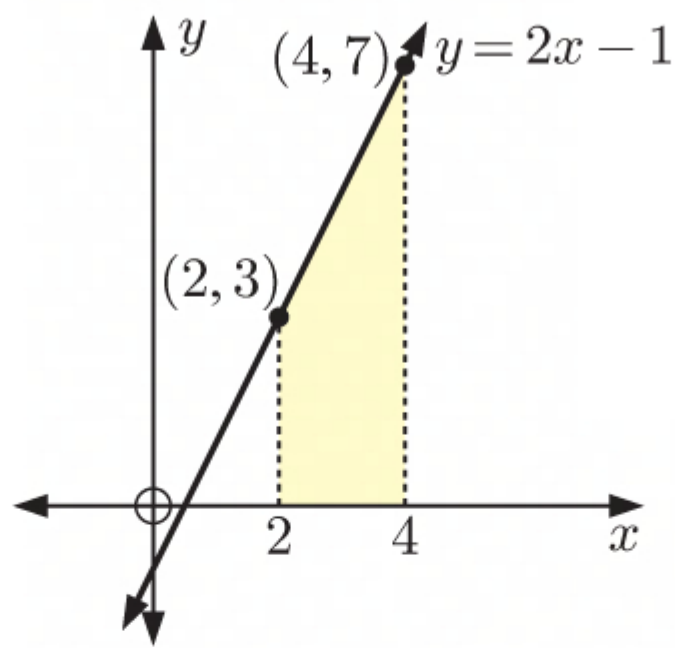


b

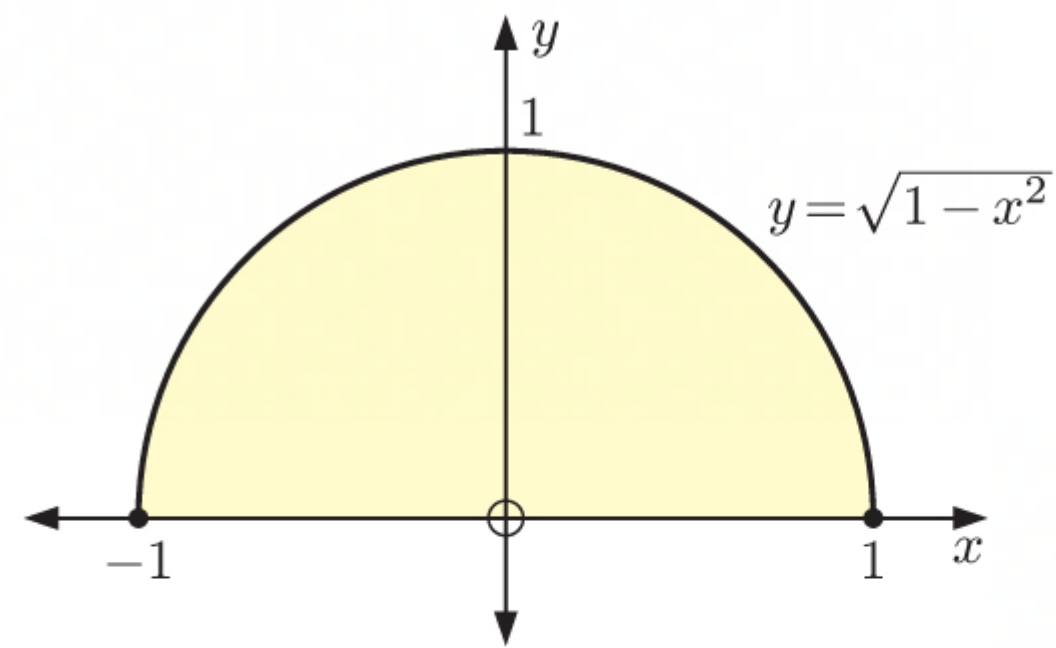
n	A_L	A_U
5	2.9349	3.3349
50	3.1215	3.1615
100	3.1316	3.1516
500	3.1396	3.1436

c Using $n = 500$ rectangles,

$$\begin{aligned} \int_0^1 \frac{4}{1+x^2} dx &\approx \frac{A_L + A_U}{2} \\ &\approx \frac{3.1396 + 3.1436}{2} \\ &\approx 3.1416 \approx \pi \end{aligned}$$

2 a

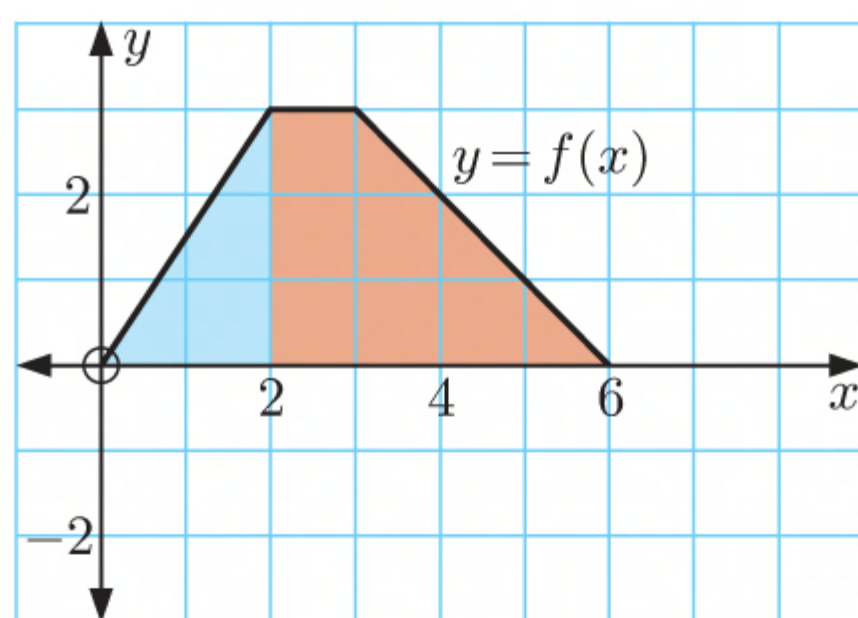
$$\begin{aligned}\int_2^4 (2x - 1) dx &= \text{shaded area} \\ &= \left(\frac{3+7}{2}\right) \times 2 \\ &= 10\end{aligned}$$

b

$$\begin{aligned}\int_{-1}^1 \sqrt{1-x^2} dx &= \text{shaded area} \\ &= \frac{1}{2}\pi r^2 \\ &= \frac{1}{2}(\pi \times 1^2) \\ &= \frac{\pi}{2}\end{aligned}$$

3 a $\frac{d}{dx}(x^3 - 2x) = 3x^2 - 2$
 \therefore the antiderivative of $3x^2 - 2$ is $x^3 - 2x$.

b $\frac{d}{dx}(x^{\frac{4}{3}}) = \frac{4}{3}x^{\frac{1}{3}} = \frac{4}{3}\sqrt[3]{x}$
 $\therefore \frac{d}{dx}(\frac{3}{4}x^{\frac{4}{3}}) = \sqrt[3]{x}$
 \therefore the antiderivative of $\sqrt[3]{x}$ is $\frac{3}{4}x^{\frac{4}{3}}$.

4

a $\int_0^2 f(x) dx$
 $=$ area of blue triangle
 $= \frac{1}{2} \times 2 \times 3$
 $= 3$

b $\int_2^6 f(x) dx$
 $=$ area of red trapezium
 $= \left(\frac{1+4}{2}\right) \times 3$
 $= \frac{15}{2}$

5 a $\frac{d}{dx}(x^2) = 2x$
 $\therefore \frac{d}{dx}(2x^2) = 4x$
 \therefore the antiderivative of $f(x) = 4x$ is $F(x) = 2x^2$.

$$\begin{aligned}\int_0^3 4x dx &= F(3) - F(0) \\ &= 2(3)^2 - 2(0)^2 \\ &= 18 \text{ units}^2\end{aligned}$$

b $\frac{d}{dx}(x^{\frac{3}{2}}) = \frac{3}{2}x^{\frac{1}{2}} = \frac{3}{2}\sqrt{x}$
 $\therefore \frac{d}{dx}(\frac{2}{3}x^{\frac{3}{2}}) = \sqrt{x}$
 \therefore the antiderivative of $f(x) = \sqrt{x}$ is $F(x) = \frac{2}{3}x^{\frac{3}{2}}$.

$$\begin{aligned}\int_0^9 \sqrt{x} dx &= F(9) - F(0) \\ &= \frac{2}{3}(9)^{\frac{3}{2}} - \frac{2}{3}(0)^{\frac{3}{2}} \\ &= \frac{2}{3} \times 9 \times 3 \\ &= 18 \text{ units}^2\end{aligned}$$

Chapter 16

TECHNIQUES FOR INTEGRATION

EXERCISE 16A

1 a $\frac{d}{dx}(x^7) = 7x^6$

$$\therefore \int 7x^6 dx = x^7 + c$$

$$\therefore 7 \int x^6 dx = x^7 + c$$

$$\therefore \int x^6 dx = \frac{1}{7}x^7 + c$$

c $\frac{d}{dx}(x^{-\frac{1}{2}}) = -\frac{1}{2}x^{-\frac{3}{2}}$

$$\therefore \int -\frac{1}{2}x^{-\frac{3}{2}} dx = x^{-\frac{1}{2}} + c$$

$$\therefore -\frac{1}{2} \int x^{-\frac{3}{2}} dx = x^{-\frac{1}{2}} + c$$

$$\therefore \int x^{-\frac{3}{2}} dx = -2x^{-\frac{1}{2}} + c$$

b $\frac{d}{dx}(x^{\frac{3}{2}}) = \frac{3}{2}x^{\frac{1}{2}} = \frac{3}{2}\sqrt{x}$

$$\therefore \int \frac{3}{2}\sqrt{x} dx = x^{\frac{3}{2}} + c$$

$$\therefore \frac{3}{2} \int \sqrt{x} dx = x^{\frac{3}{2}} + c$$

$$\therefore \int \sqrt{x} dx = \frac{2}{3}x^{\frac{3}{2}} + c$$

d $\frac{d}{dx}(x^{n+1}) = (n+1)x^n, \quad n \neq -1$

$$\therefore \int (n+1)x^n dx = x^{n+1} + c, \quad n \neq -1$$

$$\therefore (n+1) \int x^n dx = x^{n+1} + c, \quad n \neq -1$$

$$\therefore \int x^n dx = \frac{x^{n+1}}{n+1} + c, \quad n \neq -1$$

2 a $\frac{d}{dx}(e^{4x}) = 4e^{4x}$

$$\therefore \int 4e^{4x} dx = e^{4x} + c$$

$$\therefore 4 \int e^{4x} dx = e^{4x} + c$$

$$\therefore \int e^{4x} dx = \frac{1}{4}e^{4x} + c$$

c $\frac{d}{dx}(e^{kx}) = ke^{kx}, \quad k \neq 0$

$$\therefore \int ke^{kx} dx = e^{kx} + c, \quad k \neq 0$$

$$\therefore k \int e^{kx} dx = e^{kx} + c, \quad k \neq 0$$

$$\therefore \int e^{kx} dx = \frac{1}{k}e^{kx} + c, \quad k \neq 0$$

b $\frac{d}{dx}(e^{-\frac{x}{2}}) = -\frac{1}{2}e^{-\frac{x}{2}}$

$$\therefore \int -\frac{1}{2}e^{-\frac{x}{2}} dx = e^{-\frac{x}{2}} + c$$

$$\therefore -\frac{1}{2} \int e^{-\frac{x}{2}} dx = e^{-\frac{x}{2}} + c$$

$$\therefore \int e^{-\frac{x}{2}} dx = -2e^{-\frac{x}{2}} + c$$

$$\mathbf{3} \quad \mathbf{a} \quad \frac{d}{dx}(\sin x) = \cos x$$

$$\therefore \int \cos x \, dx = \sin x + c$$

$$\mathbf{b} \quad \frac{d}{dx}(\cos x) = -\sin x$$

$$\therefore \int (-\sin x) \, dx = \cos x + c$$

$$\therefore -\int \sin x \, dx = \cos x + c$$

$$\therefore \int \sin x \, dx = -\cos x + c$$

$$\mathbf{4} \quad \frac{d}{dx}(x^3 + x^2) = 3x^2 + 2x$$

$$\therefore \int (3x^2 + 2x) \, dx = x^3 + x^2 + c$$

$$\mathbf{5} \quad \frac{d}{dx}(3x^4 - 2x^2) = 12x^3 - 4x$$

$$\therefore \int (12x^3 - 4x) \, dx = 3x^4 - 2x^2 + c$$

$$\therefore 4 \int (3x^3 - x) \, dx = 3x^4 - 2x^2 + c$$

$$\therefore \int (3x^3 - x) \, dx = \frac{3}{4}x^4 - \frac{1}{2}x^2 + c$$

$$\mathbf{6} \quad \mathbf{a} \quad \frac{d}{dx}[F(x) + G(x)] = F'(x) + G'(x) \\ = f(x) + g(x)$$

$$\mathbf{b} \quad \text{Using } \mathbf{a}, \quad \int [f(x) + g(x)] \, dx = F(x) + G(x) + c \\ = \int f(x) \, dx + \int g(x) \, dx$$

$$\mathbf{7} \quad \mathbf{a} \quad \frac{d}{dx}(\sin 3x) = \cos 3x \times 3 \\ = 3 \cos 3x$$

$$\therefore \int 3 \cos 3x \, dx = \sin 3x + c$$

$$\therefore 3 \int \cos 3x \, dx = \sin 3x + c$$

$$\therefore \int \cos 3x \, dx = \frac{1}{3} \sin 3x + c$$

$$\mathbf{b} \quad \frac{d}{dx}\left(\cos\left(\frac{\pi}{3} - x\right)\right) = -\sin\left(\frac{\pi}{3} - x\right) \times (-1) \\ = \sin\left(\frac{\pi}{3} - x\right)$$

$$\therefore \int \sin\left(\frac{\pi}{3} - x\right) \, dx = \cos\left(\frac{\pi}{3} - x\right) + c$$

$$\mathbf{c} \quad \frac{d}{dx}(e^{3x+1}) = 3e^{3x+1}$$

$$\therefore \int 3e^{3x+1} \, dx = e^{3x+1} + c$$

$$\therefore 3 \int e^{3x+1} \, dx = e^{3x+1} + c$$

$$\therefore \int e^{3x+1} \, dx = \frac{1}{3}e^{3x+1} + c$$

$$\mathbf{d} \quad \frac{d}{dx}(\sqrt{5x-1}) = \frac{d}{dx}((5x-1)^{\frac{1}{2}}) \\ = \frac{1}{2}(5x-1)^{-\frac{1}{2}} \times 5 \\ = \frac{5}{2\sqrt{5x-1}}$$

$$\therefore \int \frac{5}{2\sqrt{5x-1}} \, dx = \sqrt{5x-1} + c$$

$$\therefore \frac{5}{2} \int \frac{1}{\sqrt{5x-1}} \, dx = \sqrt{5x-1} + c$$

$$\therefore \int \frac{1}{\sqrt{5x-1}} \, dx = \frac{2}{5}\sqrt{5x-1} + c$$

$$\begin{aligned} \text{e} \quad \frac{d}{dx} ((2x+1)^4) &= 4(2x+1)^3 \times 2 \\ &= 8(2x+1)^3 \end{aligned}$$

$$\therefore \int 8(2x+1)^3 dx = (2x+1)^4 + c$$

$$\therefore 8 \int (2x+1)^3 dx = (2x+1)^4 + c$$

$$\therefore \int (2x+1)^3 dx = \frac{1}{8}(2x+1)^4 + c$$

$$8 \quad \text{a} \quad \text{For } x > 0, \quad \frac{d}{dx} (\ln x) = \frac{1}{x} \qquad \text{b} \quad \text{For } x < 0, \quad \frac{d}{dx} (\ln(-x)) = \frac{-1}{-x} = \frac{1}{x}$$

$$\text{c} \quad \int \frac{1}{x} dx = \begin{cases} \ln x + c & \text{if } x > 0 \\ \ln(-x) + c & \text{if } x < 0 \end{cases}$$

$$\therefore \int \frac{1}{x} dx = \ln |x| + c, \quad x \neq 0$$

EXERCISE 16B

$$\begin{aligned} 1 \quad \text{a} \quad \int (x^2 + 3x - 2) dx \\ = \frac{1}{3}x^3 + \frac{3}{2}x^2 - 2x + c \end{aligned}$$

$$\begin{aligned} \text{c} \quad \int (-x^3 + 4x^2 - 3) dx \\ = -\frac{1}{4}x^4 + \frac{4}{3}x^3 - 3x + c \end{aligned}$$

$$\begin{aligned} \text{e} \quad \int (x^4 - x^2 - x + 2) dx \\ = \frac{1}{5}x^5 - \frac{1}{3}x^3 - \frac{1}{2}x^2 + 2x + c \end{aligned}$$

$$\begin{aligned} \text{g} \quad \int \left(2\sqrt{x} - \frac{3}{\sqrt{x}} \right) dx \\ = \int (2x^{\frac{1}{2}} - 3x^{-\frac{1}{2}}) dx \\ = \frac{2x^{\frac{3}{2}}}{\frac{3}{2}} - \frac{3x^{\frac{1}{2}}}{\frac{1}{2}} + c \\ = \frac{4}{3}x^{\frac{3}{2}} - 6x^{\frac{1}{2}} + c \end{aligned}$$

$$\begin{aligned} \text{b} \quad \int (2x^2 - 3x + 1) dx \\ = \frac{2}{3}x^3 - \frac{3}{2}x^2 + x + c \end{aligned}$$

$$\begin{aligned} \text{d} \quad \int \left(\frac{1}{2}x + x^2 + x^3 \right) dx \\ = \frac{1}{4}x^2 + \frac{1}{3}x^3 + \frac{1}{4}x^4 + c \end{aligned}$$

$$\begin{aligned} \text{f} \quad \int \left(4x^2 + \frac{1}{x} \right) dx \\ = \frac{4}{3}x^3 + \ln |x| + c \end{aligned}$$

$$\begin{aligned} \text{h} \quad \int \left(\frac{1}{3x} - \frac{2}{x^2} \right) dx \\ = \int \left(\frac{1}{3}x^{-1} - 2x^{-2} \right) dx \\ = \frac{1}{3} \ln |x| - \frac{2x^{-1}}{(-1)} + c \\ = \frac{1}{3} \ln |x| + \frac{2}{x} + c \end{aligned}$$

$$\begin{aligned}
 \text{i} \quad \int (x\sqrt{x} - 9) dx &= \int (x^{\frac{3}{2}} - 9) dx \\
 &= \frac{x^{\frac{5}{2}}}{\frac{5}{2}} - 9x + c \\
 &= \frac{2}{5}x^{\frac{5}{2}} - 9x + c
 \end{aligned}$$

$$\begin{aligned}
 \text{j} \quad \int (3x^{-\frac{3}{2}} + x^{\frac{1}{4}}) dx &= \frac{3x^{-\frac{1}{2}}}{(-\frac{1}{2})} + \frac{x^{\frac{5}{4}}}{\frac{5}{4}} + c \\
 &= -6x^{-\frac{1}{2}} + \frac{4}{5}x^{\frac{5}{4}} + c
 \end{aligned}$$

$$\begin{aligned}
 \text{2 a} \quad \int (2e^x - 3x) dx &= 2e^x - \frac{3}{2}x^2 + c
 \end{aligned}$$

$$\begin{aligned}
 \text{b} \quad \int \left(\frac{4}{x} + x^2 - e^x \right) dx &= 4 \ln |x| + \frac{1}{3}x^3 - e^x + c
 \end{aligned}$$

$$\begin{aligned}
 \text{c} \quad \int (5e^x + \frac{1}{2}x^2) dx &= 5e^x + \frac{1}{6}x^3 + c
 \end{aligned}$$

$$\begin{aligned}
 \text{3 a} \quad \int (3 \sin x - 2) dx &= -3 \cos x - 2x + c
 \end{aligned}$$

$$\begin{aligned}
 \text{b} \quad \int (4x - 2 \cos x) dx &= 2x^2 - 2 \sin x + c
 \end{aligned}$$

$$\begin{aligned}
 \text{c} \quad \int (\sin x - 2 \cos x + e^x) dx &= -\cos x - 2 \sin x + e^x + c
 \end{aligned}$$

$$\begin{aligned}
 \text{d} \quad \int (x^2\sqrt{x} - 10 \sin x) dx &= \int (x^{\frac{5}{2}} - 10 \sin x) dx \\
 &= \frac{x^{\frac{7}{2}}}{\frac{7}{2}} + 10 \cos x + c \\
 &= \frac{2}{7}x^{\frac{7}{2}} + 10 \cos x + c
 \end{aligned}$$

$$\begin{aligned}
 \text{e} \quad \int \left(\frac{x(x-1)}{3} + \cos x \right) dx &= \int \left(\frac{1}{3}x^2 - \frac{1}{3}x + \cos x \right) dx \\
 &= \frac{1}{9}x^3 - \frac{1}{6}x^2 + \sin x + c
 \end{aligned}$$

$$\begin{aligned}
 \text{f} \quad \int (-\sin x + 2\sqrt{x}) dx &= \int (-\sin x + 2x^{\frac{1}{2}}) dx \\
 &= \cos x + \frac{2x^{\frac{3}{2}}}{\frac{3}{2}} + c \\
 &= \cos x + \frac{4}{3}x^{\frac{3}{2}} + c
 \end{aligned}$$

$$\begin{aligned}
 \text{4 a} \quad \frac{dy}{dx} &= 6 \\
 \therefore y &= \int 6 dx \\
 &= 6x + c
 \end{aligned}$$

$$\begin{aligned}
 \text{b} \quad \frac{dy}{dx} &= 4x^2 \\
 \therefore y &= \int 4x^2 dx \\
 &= \frac{4}{3}x^3 + c
 \end{aligned}$$

$$\begin{aligned}
 \text{c} \quad \frac{dy}{dx} &= \frac{1}{x^2} = x^{-2} \\
 \therefore y &= \int x^{-2} dx \\
 &= \frac{x^{-1}}{(-1)} + c \\
 &= -\frac{1}{x} + c
 \end{aligned}$$

$$\begin{aligned}
 \text{d} \quad \frac{dy}{dx} &= \frac{2}{\sqrt[3]{x}} = 2x^{-\frac{1}{3}} \\
 \therefore y &= \int 2x^{-\frac{1}{3}} dx \\
 &= \frac{2x^{\frac{2}{3}}}{\frac{2}{3}} + c \\
 &= 3x^{\frac{2}{3}} + c
 \end{aligned}$$

$$\begin{aligned}
 \text{e} \quad \frac{dy}{dx} &= 2x^3 - 4 \\
 \therefore y &= \int (2x^3 - 4) dx \\
 &= \frac{1}{2}x^4 - 4x + c
 \end{aligned}$$

$$\begin{aligned}
 \text{f} \quad \frac{dy}{dx} &= 4x^3 + 3x^2 \\
 \therefore y &= \int (4x^3 + 3x^2) dx \\
 &= x^4 + x^3 + c
 \end{aligned}$$

$$\begin{aligned}
 \text{g} \quad \frac{dy}{dx} &= 2 - \frac{1}{x} \\
 \therefore y &= \int \left(2 - \frac{1}{x}\right) dx \\
 &= 2x - \ln|x| + c
 \end{aligned}$$

$$\begin{aligned}
 \text{h} \quad \frac{dy}{dx} &= \sin x + 2 \cos x \\
 \therefore y &= \int (\sin x + 2 \cos x) dx \\
 &= -\cos x + 2 \sin x + c
 \end{aligned}$$

$$\begin{aligned}
 \text{i} \quad \frac{dy}{dx} &= 2e^x - 5 + x \\
 \therefore y &= \int (2e^x - 5 + x) dx \\
 &= 2e^x - 5x + \frac{1}{2}x^2 + c
 \end{aligned}$$

$$\begin{aligned}
 \text{5 a} \quad &\int \left(\frac{1}{2}x^3 - x^4 + x^{\frac{1}{3}}\right) dx \\
 &= \frac{1}{2} \left(\frac{x^4}{4}\right) - \frac{x^5}{5} + \frac{x^{\frac{4}{3}}}{\frac{4}{3}} + c \\
 &= \frac{1}{8}x^4 - \frac{1}{5}x^5 + \frac{3}{4}x^{\frac{4}{3}} + c
 \end{aligned}$$

$$\begin{aligned}
 \text{b} \quad &\int \left(\frac{4}{x^2} + x^2 - \frac{1}{4}x^3\right) dx \\
 &= \int \left(4x^{-2} + x^2 - \frac{1}{4}x^3\right) dx \\
 &= \frac{4x^{-1}}{(-1)} + \frac{1}{3}x^3 - \frac{1}{4} \left(\frac{x^4}{4}\right) + c \\
 &= -\frac{4}{x} + \frac{1}{3}x^3 - \frac{1}{16}x^4 + c
 \end{aligned}$$

$$\begin{aligned}
 \text{c} \quad &\int (5x^4 + \frac{1}{3}x^3 - \sqrt{x}) dx \\
 &= \int (5x^4 + \frac{1}{3}x^3 - x^{\frac{1}{2}}) dx \\
 &= x^5 + \frac{1}{3} \left(\frac{x^4}{4}\right) - \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + c \\
 &= x^5 + \frac{1}{12}x^4 - \frac{2}{3}x^{\frac{3}{2}} + c
 \end{aligned}$$

$$\begin{aligned}
 \text{6 a} \quad & \int (2x + 1)^2 dx \\
 &= \int (4x^2 + 4x + 1) dx \\
 &= \frac{4x^3}{3} + \frac{4x^2}{2} + x + c \\
 &= \frac{4}{3}x^3 + 2x^2 + x + c
 \end{aligned}$$

$$\begin{aligned}
 \text{c} \quad & \int \left(\frac{1 - 4x}{x\sqrt{x}} \right) dx \\
 &= \int \left(\frac{1}{x\sqrt{x}} - \frac{4}{\sqrt{x}} \right) dx \\
 &= \int (x^{-\frac{3}{2}} - 4x^{-\frac{1}{2}}) dx \\
 &= \frac{x^{-\frac{1}{2}}}{(-\frac{1}{2})} - \frac{4x^{\frac{1}{2}}}{\frac{1}{2}} + c \\
 &= -2x^{-\frac{1}{2}} - 8x^{\frac{1}{2}} + c \\
 &= -\frac{2}{\sqrt{x}} - 8\sqrt{x} + c
 \end{aligned}$$

$$\begin{aligned}
 \text{e} \quad & \int \left(\sqrt{x} - \frac{1}{\sqrt{x}} \right)^2 dx \\
 &= \int \left(x - 2 + \frac{1}{x} \right) dx \\
 &= \frac{1}{2}x^2 - 2x + \ln|x| + c
 \end{aligned}$$

$$\begin{aligned}
 \text{g} \quad & \int \left(\frac{2}{x} + 1 \right)^2 dx \\
 &= \int \left(\frac{4}{x^2} + \frac{4}{x} + 1 \right) dx \\
 &= \int (4x^{-2} + 4x^{-1} + 1) dx \\
 &= \frac{4x^{-1}}{(-1)} + 4\ln|x| + x + c \\
 &= -\frac{4}{x} + 4\ln|x| + x + c
 \end{aligned}$$

$$\begin{aligned}
 \text{b} \quad & \int \left(x + \frac{1}{x} \right)^2 dx \\
 &= \int \left(x^2 + 2 + \frac{1}{x^2} \right) dx \\
 &= \int (x^2 + 2 + x^{-2}) dx \\
 &= \frac{x^3}{3} + 2x + \frac{x^{-1}}{(-1)} + c \\
 &= \frac{1}{3}x^3 + 2x - \frac{1}{x} + c
 \end{aligned}$$

$$\begin{aligned}
 \text{d} \quad & \int \frac{2x - 1}{\sqrt{x}} dx \\
 &= \int \left(2\sqrt{x} - \frac{1}{\sqrt{x}} \right) dx \\
 &= \int (2x^{\frac{1}{2}} - x^{-\frac{1}{2}}) dx \\
 &= \frac{2x^{\frac{3}{2}}}{\frac{3}{2}} - \frac{x^{\frac{1}{2}}}{\frac{1}{2}} + c \\
 &= \frac{4}{3}x^{\frac{3}{2}} - 2x^{\frac{1}{2}} + c \\
 &= \frac{4}{3}x\sqrt{x} - 2\sqrt{x} + c
 \end{aligned}$$

$$\begin{aligned}
 \text{f} \quad & \int \left(\frac{1 - x^2}{x} \right) dx \\
 &= \int \left(\frac{1}{x} - x \right) dx \\
 &= \ln|x| - \frac{1}{2}x^2 + c
 \end{aligned}$$

$$\begin{aligned}
 \text{h} \quad & \int (x + 1)^3 dx \\
 &= \int (x^3 + 3x^2 + 3x + 1) dx \\
 & \quad \text{\{binomial theorem\}} \\
 &= \frac{x^4}{4} + \frac{3x^3}{3} + \frac{3x^2}{2} + x + c \\
 &= \frac{1}{4}x^4 + x^3 + \frac{3}{2}x^2 + x + c
 \end{aligned}$$

$$\begin{aligned}
 \text{i} \quad & \int (x-1)^4 dx \\
 &= \int (x^4 - 4x^3 + 6x^2 - 4x + 1) dx \\
 &\quad \{\text{binomial theorem}\} \\
 &= \frac{x^5}{5} - \frac{4x^4}{4} + \frac{6x^3}{3} - \frac{4x^2}{2} + x + c \\
 &= \frac{1}{5}x^5 - x^4 + 2x^3 - 2x^2 + x + c
 \end{aligned}$$

$$\begin{aligned}
 \text{j} \quad & \int \frac{x^2 - 4}{x\sqrt{x}} dx \\
 &= \int \frac{x^2 - 4}{x^{\frac{3}{2}}} dx \\
 &= \int (x^{\frac{1}{2}} - 4x^{-\frac{3}{2}}) dx \\
 &= \frac{x^{\frac{3}{2}}}{\frac{3}{2}} - \frac{4x^{-\frac{1}{2}}}{(-\frac{1}{2})} + c \\
 &= \frac{2}{3}x^{\frac{3}{2}} + 8x^{-\frac{1}{2}} + c \\
 &= \frac{2}{3}x\sqrt{x} + \frac{8}{\sqrt{x}} + c
 \end{aligned}$$

$$\begin{aligned}
 \text{k} \quad & \int \frac{x^2 - 4x + 10}{x} dx \\
 &= \int \left(x - 4 + \frac{10}{x}\right) dx \\
 &= \frac{1}{2}x^2 - 4x + 10 \ln|x| + c
 \end{aligned}$$

$$\begin{aligned}
 \text{l} \quad & \int \frac{3x^3 - 2x^2 + 5}{x^2} dx \\
 &= \int \left(3x - 2 + \frac{5}{x^2}\right) dx \\
 &= \int (3x - 2 + 5x^{-2}) dx \\
 &= \frac{3x^2}{2} - 2x + \frac{5x^{-1}}{(-1)} + c \\
 &= \frac{3}{2}x^2 - 2x - \frac{5}{x} + c
 \end{aligned}$$

$$\begin{aligned}
 \text{7 a} \quad & f'(x) = (1 - 2x)^2 \\
 \therefore f(x) &= \int (1 - 2x)^2 dx \\
 &= \int (1 - 4x + 4x^2) dx \\
 &= x - \frac{4x^2}{2} + \frac{4x^3}{3} + c \\
 &= x - 2x^2 + \frac{4}{3}x^3 + c
 \end{aligned}$$

$$\begin{aligned}
 \text{b} \quad & f'(x) = \sqrt{x} - \frac{2}{\sqrt{x}} \\
 &= x^{\frac{1}{2}} - 2x^{-\frac{1}{2}} \\
 \therefore f(x) &= \int (x^{\frac{1}{2}} - 2x^{-\frac{1}{2}}) dx \\
 &= \frac{x^{\frac{3}{2}}}{\frac{3}{2}} - \frac{2x^{\frac{1}{2}}}{\frac{1}{2}} + c \\
 &= \frac{2}{3}x^{\frac{3}{2}} - 4x^{\frac{1}{2}} + c \\
 &= \frac{2}{3}x\sqrt{x} - 4\sqrt{x} + c
 \end{aligned}$$

$$\begin{aligned}
 \text{c} \quad & f'(x) = \frac{x^2 - 5}{x^2} \\
 &= 1 - 5x^{-2} \\
 \therefore f(x) &= \int (1 - 5x^{-2}) dx \\
 &= x - \frac{5x^{-1}}{(-1)} + c \\
 &= x + \frac{5}{x} + c
 \end{aligned}$$

$$\begin{aligned}
 8 \quad a \quad & \int (\sqrt{x} + \tfrac{1}{2} \cos x) dx \\
 &= \int (x^{\frac{1}{2}} + \tfrac{1}{2} \cos x) dx \\
 &= \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + \tfrac{1}{2} \sin x + c \\
 &= \tfrac{2}{3} x^{\frac{3}{2}} + \tfrac{1}{2} \sin x + c
 \end{aligned}$$

$$\begin{aligned}
 9 \quad a \quad & \int \left(x^2 - \frac{1}{x}\right)^2 dx \\
 &= \int \left(x^4 - 2x + \frac{1}{x^2}\right) dx \\
 &= \int (x^4 - 2x + x^{-2}) dx \\
 &= \frac{x^5}{5} - \frac{2x^2}{2} + \frac{x^{-1}}{(-1)} + c \\
 &= \tfrac{1}{5} x^5 - x^2 - \frac{1}{x} + c
 \end{aligned}$$

$$\begin{aligned}
 c \quad & \int \sqrt{x}(3x-1)^2 dx \\
 &= \int x^{\frac{1}{2}} \times (9x^2 - 6x + 1) dx \\
 &= \int (9x^{\frac{5}{2}} - 6x^{\frac{3}{2}} + x^{\frac{1}{2}}) dx \\
 &= \frac{9x^{\frac{7}{2}}}{\frac{7}{2}} - \frac{6x^{\frac{5}{2}}}{\frac{5}{2}} + \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + c \\
 &= \frac{18}{7} x^{\frac{7}{2}} - \frac{12}{5} x^{\frac{5}{2}} + \frac{2}{3} x^{\frac{3}{2}} + c
 \end{aligned}$$

$$\begin{aligned}
 b \quad & \int (2e^x - 4 \sin x) dx \\
 &= 2e^x + 4 \cos x + c
 \end{aligned}$$

$$\begin{aligned}
 c \quad & \int (3 \cos x - \sin x) dx \\
 &= 3 \sin x + \cos x + c
 \end{aligned}$$

$$\begin{aligned}
 b \quad & \int \frac{x^2 - 4x + 2}{\sqrt{x}} dx \\
 &= \int \frac{x^2 - 4x + 2}{x^{\frac{1}{2}}} dx \\
 &= \int (x^{\frac{3}{2}} - 4x^{\frac{1}{2}} + 2x^{-\frac{1}{2}}) dx \\
 &= \frac{x^{\frac{5}{2}}}{\frac{5}{2}} - \frac{4x^{\frac{3}{2}}}{\frac{3}{2}} + \frac{2x^{\frac{1}{2}}}{\frac{1}{2}} + c \\
 &= \tfrac{2}{5} x^{\frac{5}{2}} - \tfrac{8}{3} x^{\frac{3}{2}} + 4x^{\frac{1}{2}} + c
 \end{aligned}$$

EXERCISE 16C

$$\begin{aligned}
 1 \quad a \quad & f'(x) = 2x - 1 \\
 \therefore f(x) &= \int (2x - 1) dx \\
 &= \frac{2x^2}{2} - x + c \\
 &= x^2 - x + c
 \end{aligned}$$

$$\begin{aligned}
 \text{But } f(0) &= 3, \text{ so } 0 - 0 + c = 3 \\
 \therefore c &= 3
 \end{aligned}$$

$$\therefore f(x) = x^2 - x + 3$$

$$\begin{aligned}
 b \quad & f'(x) = 3x^2 + 2x \\
 \therefore f(x) &= \int (3x^2 + 2x) dx \\
 &= \frac{3x^3}{3} + \frac{2x^2}{2} + c \\
 &= x^3 + x^2 + c
 \end{aligned}$$

$$\begin{aligned}
 \text{But } f(2) &= 5, \text{ so } 8 + 4 + c = 5 \\
 \therefore c &= -7
 \end{aligned}$$

$$\therefore f(x) = x^3 + x^2 - 7$$

$$\begin{aligned}
 \text{c} \quad f'(x) &= 2 + \frac{1}{\sqrt{x}} = 2 + x^{-\frac{1}{2}} \\
 \therefore f(x) &= \int (2 + x^{-\frac{1}{2}}) dx \\
 &= 2x + \frac{x^{\frac{1}{2}}}{\frac{1}{2}} + c \\
 &= 2x + 2x^{\frac{1}{2}} + c \\
 \text{But } f(1) &= 1, \text{ so } 2 + 2 + c = 1 \\
 &\therefore c = -3 \\
 \therefore f(x) &= 2x + 2\sqrt{x} - 3
 \end{aligned}$$

$$\begin{aligned}
 \text{e} \quad f'(x) &= \sqrt{x} - 2 = x^{\frac{1}{2}} - 2 \\
 \therefore f(x) &= \int (x^{\frac{1}{2}} - 2) dx \\
 &= \frac{x^{\frac{3}{2}}}{\frac{3}{2}} - 2x + c \\
 &= \frac{2}{3}x^{\frac{3}{2}} - 2x + c \\
 \text{But } f(4) &= 0, \text{ so } \frac{2}{3}(4)^{\frac{3}{2}} - 2(4) + c = 0 \\
 &\therefore \frac{16}{3} - 8 + c = 0 \\
 &\therefore c = \frac{8}{3} \\
 \therefore f(x) &= \frac{2}{3}x^{\frac{3}{2}} - 2x + \frac{8}{3}
 \end{aligned}$$

$$\begin{aligned}
 \text{2} \quad \frac{dy}{dx} &= x - 2x^2 \\
 \therefore y &= \int (x - 2x^2) dx \\
 &= \frac{1}{2}x^2 - \frac{2}{3}x^3 + c \\
 \text{But the curve passes through } (2, 4), \\
 \text{so when } x &= 2, y = 4. \\
 \therefore 4 &= \frac{1}{2}(2)^2 - \frac{2}{3}(2)^3 + c \\
 \therefore 4 &= 2 - \frac{16}{3} + c \\
 \therefore c &= \frac{22}{3} \\
 \therefore y &= \frac{1}{2}x^2 - \frac{2}{3}x^3 + \frac{22}{3}
 \end{aligned}$$

$$\begin{aligned}
 \text{d} \quad f'(x) &= x - \frac{2}{\sqrt{x}} = x - 2x^{-\frac{1}{2}} \\
 \therefore f(x) &= \int (x - 2x^{-\frac{1}{2}}) dx \\
 &= \frac{x^2}{2} - \frac{2x^{\frac{1}{2}}}{\frac{1}{2}} + c \\
 &= \frac{1}{2}x^2 - 4\sqrt{x} + c \\
 \text{But } f(1) &= 2, \text{ so } \frac{1}{2} - 4 + c = 2 \\
 &\therefore c = \frac{11}{2} \\
 \therefore f(x) &= \frac{1}{2}x^2 - 4\sqrt{x} + \frac{11}{2}
 \end{aligned}$$

$$\begin{aligned}
 \text{f} \quad f'(x) &= \frac{1}{x} \\
 \therefore f(x) &= \int \frac{1}{x} dx \\
 &= \ln|x| + c \\
 \text{But } f(e) &= 2, \text{ so } \ln e + c = 2 \\
 &\therefore c = 1 \\
 \therefore f(x) &= \ln|x| + 1
 \end{aligned}$$

$$\begin{aligned}
 \text{3} \quad \frac{dy}{dx} &= 1 - e^x \\
 \therefore y &= \int (1 - e^x) dx \\
 &= x - e^x + c \\
 \text{But the curve passes through } (3, e^3), \\
 \text{so when } x &= 3, y = e^3. \\
 \therefore e^3 &= 3 - e^3 + c \\
 \therefore c &= 2e^3 - 3 \\
 \therefore y &= x - e^x + 2e^3 - 3
 \end{aligned}$$

4 a $f'(x) = x^2 - 4 \cos x$

$$\begin{aligned}\therefore f(x) &= \int (x^2 - 4 \cos x) dx \\ &= \frac{1}{3}x^3 - 4 \sin x + c\end{aligned}$$

But $f(0) = 3$, so $c = 3$

$$\therefore f(x) = \frac{1}{3}x^3 - 4 \sin x + 3$$

b $f'(x) = 2 \cos x - 3 \sin x$

$$\begin{aligned}\therefore f(x) &= \int (2 \cos x - 3 \sin x) dx \\ &= 2 \sin x + 3 \cos x + c\end{aligned}$$

But $f(\frac{\pi}{4}) = \frac{1}{\sqrt{2}}$,

$$\text{so } 2 \sin \frac{\pi}{4} + 3 \cos \frac{\pi}{4} + c = \frac{1}{\sqrt{2}}$$

$$\therefore \frac{2}{\sqrt{2}} + \frac{3}{\sqrt{2}} + c = \frac{1}{\sqrt{2}}$$

$$\therefore c = -\frac{4}{\sqrt{2}}$$

$$\therefore c = -2\sqrt{2}$$

$$\therefore f(x) = 2 \sin x + 3 \cos x - 2\sqrt{2}$$

c $f'(x) = \sqrt{x} - 2 \sin x = x^{\frac{1}{2}} - 2 \sin x$

$$\begin{aligned}\therefore f(x) &= \int (x^{\frac{1}{2}} - 2 \sin x) dx \\ &= \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + 2 \cos x + c \\ &= \frac{2}{3}x^{\frac{3}{2}} + 2 \cos x + c\end{aligned}$$

But $f(0) = -2$,

$$\text{so } \frac{2}{3}(0)^{\frac{3}{2}} + 2 \cos 0 + c = -2$$

$$\therefore 2 + c = -2$$

$$\therefore c = -4$$

$$\therefore f(x) = \frac{2}{3}x^{\frac{3}{2}} + 2 \cos x - 4$$

d $f'(x) = e^x + 3 \cos x$

$$\begin{aligned}\therefore f(x) &= \int (e^x + 3 \cos x) dx \\ &= e^x + 3 \sin x + c\end{aligned}$$

But $f(\pi) = 0$, so $e^\pi + 3 \sin \pi + c = 0$

$$\therefore e^\pi + c = 0$$

$$\therefore c = -e^\pi$$

$$\therefore f(x) = e^x + 3 \sin x - e^\pi$$

5 $f'(x) = ax + 1$

$$\begin{aligned}\therefore f(x) &= \int (ax + 1) dx \\ &= \frac{ax^2}{2} + x + c\end{aligned}$$

Now $f(0) = 3$, so $c = 3$

$$\therefore f(x) = \frac{1}{2}ax^2 + x + 3$$

and $f(3) = -3$

$$\therefore \frac{1}{2}a(3)^2 + 3 + 3 = -3$$

$$\therefore \frac{9}{2}a + 6 = -3$$

$$\therefore \frac{9}{2}a = -9$$

$$\therefore a = -2$$

$$\begin{aligned}\therefore f(x) &= \frac{1}{2}(-2)x^2 + x + 3 \\ &= -x^2 + x + 3\end{aligned}$$

6 $f'(x) = ax^2 + bx$

$$\begin{aligned}\therefore f(x) &= \int (ax^2 + bx) dx \\ &= \frac{ax^3}{3} + \frac{bx^2}{2} + c\end{aligned}$$

Now $f(0) = 1$, so $c = 1$

$$\therefore f(x) = \frac{1}{3}ax^3 + \frac{1}{2}bx^2 + 1,$$

$$f(-1) = -2 \quad \text{and}$$

$$f(1) = 4$$

$$\therefore \frac{1}{3}a(-1)^3 + \frac{1}{2}b(-1)^2 + 1 = -2$$

$$\therefore \frac{1}{3}a + \frac{1}{2}b + 1 = 4$$

$$\therefore -\frac{1}{3}a + \frac{1}{2}b + 1 = -2$$

$$\therefore \frac{1}{3}a + \frac{1}{2}b = 3 \quad \dots (2)$$

$$\therefore -\frac{1}{3}a + \frac{1}{2}b = -3 \quad \dots (1)$$

Adding (1) and (2) together gives: $b = 0$

Substituting $b = 0$ into (1) gives: $-\frac{1}{3}a = -3$

$$\therefore a = 9$$

$$\begin{aligned}\therefore f(x) &= \frac{1}{3}(9)x^3 + \frac{1}{2}(0)x^2 + 1 \\ &= 3x^3 + 1\end{aligned}$$

7 a $f''(x) = 2x + 1$

$$\begin{aligned}\therefore f'(x) &= \int (2x + 1) dx \\ &= x^2 + x + c\end{aligned}$$

But $f'(1) = 3$, so $1 + 1 + c = 3$

$$\therefore c = 1$$

$$\therefore f'(x) = x^2 + x + 1$$

$$\begin{aligned}\therefore f(x) &= \int (x^2 + x + 1) dx \\ &= \frac{1}{3}x^3 + \frac{1}{2}x^2 + x + d\end{aligned}$$

But $f(2) = 7$,

so $\frac{1}{3}(2)^3 + \frac{1}{2}(2)^2 + 2 + d = 7$

$$\therefore \frac{8}{3} + 2 + 2 + d = 7$$

$$\therefore d = \frac{1}{3}$$

$$\therefore f(x) = \frac{1}{3}x^3 + \frac{1}{2}x^2 + x + \frac{1}{3}$$

b $f''(x) = 15\sqrt{x} + \frac{3}{\sqrt{x}} = 15x^{\frac{1}{2}} + 3x^{-\frac{1}{2}}$

$$\therefore f'(x) = \int (15x^{\frac{1}{2}} + 3x^{-\frac{1}{2}}) dx$$

$$= \frac{15x^{\frac{3}{2}}}{\frac{3}{2}} + \frac{3x^{\frac{1}{2}}}{\frac{1}{2}} + c$$

$$= 10x^{\frac{3}{2}} + 6x^{\frac{1}{2}} + c$$

But $f'(1) = 12$, so $10 + 6 + c = 12$

$$\therefore c = -4$$

$$\therefore f'(x) = 10x^{\frac{3}{2}} + 6x^{\frac{1}{2}} - 4$$

$$\therefore f(x) = \int (10x^{\frac{3}{2}} + 6x^{\frac{1}{2}} - 4) dx$$

$$= \frac{10x^{\frac{5}{2}}}{\frac{5}{2}} + \frac{6x^{\frac{3}{2}}}{\frac{3}{2}} - 4x + d$$

$$= 4x^{\frac{5}{2}} + 4x^{\frac{3}{2}} - 4x + d$$

But $f(0) = 5$, so $d = 5$

$$\therefore f(x) = 4x^{\frac{5}{2}} + 4x^{\frac{3}{2}} - 4x + 5$$

$$\text{c} \quad f''(x) = \cos x$$

$$\begin{aligned} \therefore f'(x) &= \int \cos x \, dx \\ &= \sin x + c \end{aligned}$$

$$\text{But } f'\left(\frac{\pi}{2}\right) = 0, \text{ so } \sin \frac{\pi}{2} + c = 0$$

$$\therefore 1 + c = 0$$

$$\therefore c = -1$$

$$\therefore f'(x) = \sin x - 1$$

$$\begin{aligned} \therefore f(x) &= \int (\sin x - 1) \, dx \\ &= -\cos x - x + d \end{aligned}$$

$$\text{But } f(0) = 3, \text{ so } -\cos 0 + d = 3$$

$$\therefore -1 + d = 3$$

$$\therefore d = 4$$

$$\therefore f(x) = -\cos x - x + 4$$

$$\text{d} \quad f''(x) = 2x$$

$$\begin{aligned} \therefore f'(x) &= \int 2x \, dx \\ &= x^2 + c \end{aligned}$$

$$\begin{aligned} \therefore f(x) &= \int (x^2 + c) \, dx \\ &= \frac{1}{3}x^3 + cx + d \end{aligned}$$

But $(1, 0)$ and $(0, 5)$ lie on the curve $y = f(x)$

$$\therefore f(1) = 0$$

$$\therefore \frac{1}{3} + c + d = 0$$

$$\therefore c + d = -\frac{1}{3} \quad \dots (*)$$

$$\text{and } f(0) = 5$$

$$\therefore d = 5$$

Substituting $d = 5$ into $(*)$ gives:

$$c + 5 = -\frac{1}{3}$$

$$\therefore c = -\frac{16}{3}$$

$$\therefore f(x) = \frac{1}{3}x^3 - \frac{16}{3}x + 5$$

$$\text{8} \quad f''(x) = 3e^{-x}$$

$$\begin{aligned} \therefore f'(x) &= \int 3e^{-x} \, dx \\ &= -3e^{-x} + c \end{aligned}$$

$$\begin{aligned} \therefore f(x) &= \int (-3e^{-x} + c) \, dx \\ &= 3e^{-x} + cx + d \end{aligned}$$

$$\text{But } f(1) = \frac{3}{e}$$

and

$$f(3) = \frac{3}{e^3} - 2$$

$$\therefore 3e^{-1} + c + d = \frac{3}{e}$$

$$\therefore 3e^{-3} + 3c + d = \frac{3}{e^3} - 2$$

$$\therefore \frac{3}{e} + c + d = \frac{3}{e}$$

$$\therefore \frac{3}{e^3} + 3c + d = \frac{3}{e^3} - 2$$

$$\therefore c + d = 0$$

$$\therefore 3c + d = -2 \quad \dots (2)$$

$$\therefore d = -c \quad \dots (1)$$

Substituting (1) into (2) gives: $3c + (-c) = -2$

$$\therefore 2c = -2$$

$$\therefore c = -1$$

Substituting $c = -1$ into (1) gives $d = -(-1)$
 $= 1$

$$\therefore f(x) = 3e^{-x} - x + 1$$

EXERCISE 16D

$$\begin{aligned}
 \mathbf{1} \quad \mathbf{a} \quad & \int (2x + 5)^3 dx \\
 &= \frac{1}{2} \times \frac{(2x + 5)^4}{4} + c \\
 &= \frac{1}{8}(2x + 5)^4 + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad & \int \frac{1}{(3 - 2x)^2} dx \\
 &= \int (3 - 2x)^{-2} dx \\
 &= \frac{1}{-2} \times \frac{(3 - 2x)^{-1}}{-1} + c \\
 &= \frac{1}{2(3 - 2x)} + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{c} \quad & \int \frac{4}{(2x - 1)^4} dx \\
 &= 4 \int (2x - 1)^{-4} dx \\
 &= 4 \times \frac{1}{2} \times \frac{(2x - 1)^{-3}}{-3} + c \\
 &= \frac{-2}{3(2x - 1)^3} + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{d} \quad & \int (4x - 3)^7 dx \\
 &= \frac{1}{4} \times \frac{(4x - 3)^8}{8} + c \\
 &= \frac{1}{32}(4x - 3)^8 + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{e} \quad & \int \sqrt{3x - 4} dx \\
 &= \int (3x - 4)^{\frac{1}{2}} dx \\
 &= \frac{1}{3} \times \frac{(3x - 4)^{\frac{3}{2}}}{\frac{3}{2}} + c \\
 &= \frac{2}{9}(3x - 4)^{\frac{3}{2}} + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{f} \quad & \int \frac{10}{\sqrt{1 - 5x}} dx \\
 &= 10 \int (1 - 5x)^{-\frac{1}{2}} dx \\
 &= 10 \times \frac{1}{-5} \times \frac{(1 - 5x)^{\frac{1}{2}}}{\frac{1}{2}} + c \\
 &= -4\sqrt{1 - 5x} + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{g} \quad & \int 3(1 - x)^4 dx \\
 &= 3 \int (1 - x)^4 dx \\
 &= 3 \times \frac{1}{-1} \times \frac{(1 - x)^5}{5} + c \\
 &= -\frac{3}{5}(1 - x)^5 + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{h} \quad & \int \frac{4}{\sqrt{3 - 4x}} dx \\
 &= 4 \int (3 - 4x)^{-\frac{1}{2}} dx \\
 &= 4 \times \frac{1}{-4} \times \frac{(3 - 4x)^{\frac{1}{2}}}{\frac{1}{2}} + c \\
 &= -2\sqrt{3 - 4x} + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{i} \quad & \int \frac{5}{(3x - 2)^3} dx \\
 &= 5 \int (3x - 2)^{-3} dx \\
 &= 5 \times \frac{1}{3} \times \frac{(3x - 2)^{-2}}{-2} + c \\
 &= -\frac{5}{6(3x - 2)^2} + c
 \end{aligned}$$

$$2 \quad \frac{dy}{dx} = \sqrt{2x-7} = (2x-7)^{\frac{1}{2}}$$

$$\therefore y = \int (2x-7)^{\frac{1}{2}} dx$$

$$= \frac{1}{2} \times \frac{(2x-7)^{\frac{3}{2}}}{\frac{3}{2}} + c$$

$$\therefore y = f(x) = \frac{1}{3}(2x-7)^{\frac{3}{2}} + c$$

But $f(8) = 11$, so $\frac{1}{3}(2(8)-7)^{\frac{3}{2}} + c = 11$

$$\therefore \frac{1}{3}(9)^{\frac{3}{2}} + c = 11$$

$$\therefore \frac{1}{3}(27) + c = 11$$

$$\therefore 9 + c = 11$$

$$\therefore c = 2$$

$$\therefore y = \frac{1}{3}(2x-7)^{\frac{3}{2}} + 2$$

$$3 \quad f'(x) = \frac{4}{\sqrt{1-x}} = 4(1-x)^{-\frac{1}{2}}$$

$$\therefore f(x) = \int 4(1-x)^{-\frac{1}{2}} dx$$

$$= 4 \int (1-x)^{-\frac{1}{2}} dx$$

$$= 4 \times \frac{1}{-\frac{1}{2}} \times \frac{(1-x)^{\frac{1}{2}}}{\frac{1}{2}} + c$$

$$= -8\sqrt{1-x} + c$$

But $y = f(x)$ passes through $(-3, -11)$, so $-8\sqrt{1-(-3)} + c = -11$

$$\therefore -8\sqrt{4} + c = -11$$

$$\therefore -16 + c = -11$$

$$\therefore c = 5$$

$$\therefore f(x) = 5 - 8\sqrt{1-x}$$

$$\begin{aligned} \text{Now } f(-8) &= 5 - 8\sqrt{1-(-8)} \\ &= 5 - 8(3) \\ &= -19 \end{aligned}$$

So, the point on the graph of $y = f(x)$ with x -coordinate -8 is $(-8, -19)$.

$$4 \quad \mathbf{a} \quad \int 3(2x-1)^2 dx$$

$$= 3 \int (2x-1)^2 dx$$

$$= 3 \times \frac{1}{2} \times \frac{(2x-1)^3}{3} + c$$

$$= \frac{1}{2}(2x-1)^3 + c$$

$$\mathbf{b} \quad \int (4x-5)^2 dx$$

$$= \frac{1}{4} \times \frac{(4x-5)^3}{3} + c$$

$$= \frac{1}{12}(4x-5)^3 + c$$

$$\begin{aligned}
 \text{c} \quad & \int (1-3x)^3 dx \\
 &= \frac{1}{-3} \times \frac{(1-3x)^4}{4} + c \\
 &= -\frac{1}{12}(1-3x)^4 + c
 \end{aligned}$$

$$\begin{aligned}
 \text{e} \quad & \int 4\sqrt{5-x} dx \\
 &= 4 \int (5-x)^{\frac{1}{2}} dx \\
 &= 4 \times \frac{1}{-1} \times \frac{(5-x)^{\frac{3}{2}}}{\frac{3}{2}} + c \\
 &= -\frac{8}{3}(5-x)^{\frac{3}{2}} + c
 \end{aligned}$$

$$\begin{aligned}
 5 \quad & \frac{dy}{dx} = x - \frac{5}{(1-x)^2} \\
 &= x - 5(1-x)^{-2} \\
 \therefore y &= \int (x - 5(1-x)^{-2}) dx \\
 &= \frac{1}{2}x^2 - 5 \times \frac{1}{-1} \times \frac{(1-x)^{-1}}{-1} + c \\
 &= \frac{1}{2}x^2 - \frac{5}{1-x} + c
 \end{aligned}$$

But when $x = 2$, $y = 0$

$$\begin{aligned}
 \therefore 0 &= \frac{1}{2}(2)^2 - \frac{5}{1-2} + c \\
 \therefore 0 &= \frac{1}{2} \times 4 - \frac{5}{-1} + c \\
 \therefore 0 &= 2 + 5 + c \\
 \therefore c &= -7
 \end{aligned}$$

$$\therefore y = \frac{1}{2}x^2 - \frac{5}{1-x} - 7$$

$$\begin{aligned}
 6 \quad \text{a} \quad & \int \sin 3x dx \\
 &= -\frac{1}{3} \cos 3x + c
 \end{aligned}$$

$$\begin{aligned}
 \text{c} \quad & \int 3 \cos \frac{x}{2} dx \\
 &= 6 \sin \frac{x}{2} + c
 \end{aligned}$$

$$\begin{aligned}
 \text{e} \quad & \int 2 \sin\left(2x + \frac{\pi}{6}\right) dx \\
 &= 2 \times \left(-\frac{1}{2}\right) \cos\left(2x + \frac{\pi}{6}\right) + c \\
 &= -\cos\left(2x + \frac{\pi}{6}\right) + c
 \end{aligned}$$

$$\begin{aligned}
 \text{d} \quad & \int (2-5x)^2 dx \\
 &= \frac{1}{-5} \times \frac{(2-5x)^3}{3} + c \\
 &= -\frac{1}{15}(2-5x)^3 + c
 \end{aligned}$$

$$\begin{aligned}
 \text{f} \quad & \int (7x+1)^4 dx \\
 &= \frac{1}{7} \times \frac{(7x+1)^5}{5} + c \\
 &= \frac{1}{35}(7x+1)^5 + c
 \end{aligned}$$

$$\begin{aligned}
 \text{b} \quad & \int (2 \cos(-4x) + 1) dx \\
 &= 2 \times \left(\frac{1}{-4}\right) \sin(-4x) + x + c \\
 &= -\frac{1}{2} \sin(-4x) + x + c
 \end{aligned}$$

$$\begin{aligned}
 \text{d} \quad & \int (3 \sin 2x - e^{-x}) dx \\
 &= -\frac{3}{2} \cos 2x + e^{-x} + c
 \end{aligned}$$

$$\begin{aligned}
 \text{f} \quad & \int -3 \cos\left(\frac{\pi}{4} - x\right) dx \\
 &= -3 \times (-1) \sin\left(\frac{\pi}{4} - x\right) + c \\
 &= 3 \sin\left(\frac{\pi}{4} - x\right) + c
 \end{aligned}$$

$$\begin{aligned} \mathbf{g} \quad & \int (\cos 2x + \sin 2x) dx \\ &= \frac{1}{2} \sin 2x - \frac{1}{2} \cos 2x + c \end{aligned}$$

$$\begin{aligned} \mathbf{h} \quad & \int (2 \sin 3x + 5 \cos 4x) dx \\ &= -\frac{2}{3} \cos 3x + \frac{5}{4} \sin 4x + c \end{aligned}$$

$$\begin{aligned} \mathbf{i} \quad & \int \left(\frac{1}{2} \cos 8x - 3 \sin x \right) dx = \frac{1}{2} \times \frac{1}{8} \sin 8x + 3 \cos x + c \\ &= \frac{1}{16} \sin 8x + 3 \cos x + c \end{aligned}$$

$$\begin{aligned} \mathbf{7} \quad \mathbf{a} \quad & \int (2e^x + 5e^{2x}) dx \\ &= 2e^x + 5\left(\frac{1}{2}\right)e^{2x} + c \\ &= 2e^x + \frac{5}{2}e^{2x} + c \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad & \int (3e^{5x-2}) dx \\ &= 3\left(\frac{1}{5}\right)e^{5x-2} + c \\ &= \frac{3}{5}e^{5x-2} + c \end{aligned}$$

$$\begin{aligned} \mathbf{c} \quad & \int (e^{7-3x}) dx \\ &= \left(\frac{1}{-3}\right)e^{7-3x} + c \\ &= -\frac{1}{3}e^{7-3x} + c \end{aligned}$$

$$\begin{aligned} \mathbf{d} \quad & \int (e^x + e^{-x})^2 dx \\ &= \int (e^{2x} + 2 + e^{-2x}) dx \\ &= \frac{1}{2}e^{2x} + 2x + \left(\frac{1}{-2}\right)e^{-2x} + c \\ &= \frac{1}{2}e^{2x} + 2x - \frac{1}{2}e^{-2x} + c \end{aligned}$$

$$\begin{aligned} \mathbf{e} \quad & \int (e^{-x} + 2)^2 dx \\ &= \int (e^{-2x} + 4e^{-x} + 4) dx \\ &= \left(\frac{1}{-2}\right)e^{-2x} + 4\left(\frac{1}{-1}\right)e^{-x} + 4x + c \\ &= -\frac{1}{2}e^{-2x} - 4e^{-x} + 4x + c \end{aligned}$$

$$\begin{aligned} \mathbf{f} \quad & \int \frac{(e^{2x} - 5)^2}{e^x} dx \\ &= \int \frac{e^{4x} - 10e^{2x} + 25}{e^x} dx \\ &= \int (e^{3x} - 10e^x + 25e^{-x}) dx \\ &= \frac{1}{3}e^{3x} - 10e^x - 25e^{-x} + c \end{aligned}$$

$$\mathbf{8} \quad \frac{dy}{dx} = (1 - e^x)^2 = 1 - 2e^x + e^{2x}$$

$$\begin{aligned} \therefore y &= \int (1 - 2e^x + e^{2x}) dx \\ &= x - 2e^x + \frac{1}{2}e^{2x} + c \end{aligned}$$

When $x = 0$, $y = 4$

$$\begin{aligned} \therefore 0 - 2 + \frac{1}{2} + c &= 4 \\ \therefore c &= \frac{11}{2} \end{aligned}$$

$$\therefore y = x - 2e^x + \frac{1}{2}e^{2x} + \frac{11}{2}$$

$$9 \quad f'(x) = p \sin \frac{x}{2}$$

$$\begin{aligned} \therefore f(x) &= \int p \sin \frac{x}{2} dx \\ &= p \int \sin \frac{x}{2} dx \\ &= p \times 2 \left(-\cos \frac{x}{2} \right) + c \\ &= -2p \cos \frac{x}{2} + c \end{aligned}$$

But $f(0) = 1$, so $-2p + c = 1$ (1)

and $f(2\pi) = 0$, so $2p + c = 0$ (2)

(2) - (1) gives: $2p + c - (-2p + c) = 0 - 1$

$$\therefore 4p = -1$$

$$\begin{aligned} \therefore p &= -\frac{1}{4} \quad \text{and} \quad \therefore c = -2p \quad \{\text{using (2)}\} \\ &= -2\left(-\frac{1}{4}\right) \\ &= \frac{1}{2} \end{aligned}$$

$$\therefore f(x) = \frac{1}{2} \cos \frac{x}{2} + \frac{1}{2}$$

$$10 \quad g''(x) = -\sin 2x$$

$$\begin{aligned} \therefore g'(x) &= \int -\sin 2x dx \\ &= \frac{1}{2} \cos 2x + c \end{aligned}$$

Now $g'(\pi) = \frac{1}{2} \cos 2\pi + c$

$$= \frac{1}{2} + c$$

and $g'(-\pi) = \frac{1}{2} \cos(-2\pi) + c$

$$= \frac{1}{2} + c$$

$$= g'(\pi)$$

\therefore the gradients of the tangents to $y = g(x)$
at $x = \pi$ and $x = -\pi$ are equal.

$$11 \quad f'(x) = 2e^{-2x}$$

$$\begin{aligned} \therefore f(x) &= 2\left(\frac{1}{-2}\right)e^{-2x} + c \\ &= -e^{-2x} + c \end{aligned}$$

But $f(0) = 3$, so $-e^0 + c = 3$

$$\therefore -1 + c = 3$$

$$\therefore c = 4$$

$$\therefore f(x) = -e^{-2x} + 4$$

$$12 \quad \frac{dy}{dx} = \sqrt{x} + \frac{1}{2}e^{-4x} = x^{\frac{1}{2}} + \frac{1}{2}e^{-4x}$$

$$\begin{aligned} \therefore y &= \int \left(x^{\frac{1}{2}} + \frac{1}{2}e^{-4x} \right) dx \\ &= \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + \frac{1}{2}\left(\frac{1}{-4}\right)e^{-4x} + c \\ &= \frac{2}{3}x^{\frac{3}{2}} - \frac{1}{8}e^{-4x} + c \end{aligned}$$

But $y = 0$ when $x = 1$ so $\frac{2}{3} - \frac{1}{8}e^{-4} + c = 0$

$$\therefore c = \frac{1}{8}e^{-4} - \frac{2}{3}$$

$$\therefore y = \frac{2}{3}x^{\frac{3}{2}} - \frac{1}{8}e^{-4x} + \frac{1}{8}e^{-4} - \frac{2}{3}$$

$$\begin{aligned}
13 \quad (\sin x + \cos x)^2 &= \sin^2 x + 2 \sin x \cos x + \cos^2 x \\
&= \sin^2 x + \cos^2 x + \sin 2x && \{\sin 2x = 2 \sin x \cos x\} \\
&= 1 + \sin 2x && \{\sin^2 x + \cos^2 x = 1\} \\
\therefore \int (\sin x + \cos x)^2 dx &= \int (1 + \sin 2x) dx \\
&= x + \frac{1}{2}(-\cos 2x) + c \\
&= x - \frac{1}{2} \cos 2x + c
\end{aligned}$$

$$\begin{aligned}
14 \quad \mathbf{a} \quad \cos 2x &= 1 - 2 \sin^2 x && \cos 2x = 2 \cos^2 x - 1 \\
\therefore 2 \sin^2 x &= 1 - \cos 2x && \therefore 2 \cos^2 x = 1 + \cos 2x \\
\therefore \sin^2 x &= \frac{1}{2} - \frac{1}{2} \cos 2x && \therefore \cos^2 x = \frac{1}{2} + \frac{1}{2} \cos 2x
\end{aligned}$$

$$\begin{aligned}
\mathbf{b} \quad \mathbf{i} \quad \int \sin^2 x dx &= \int \left(\frac{1}{2} - \frac{1}{2} \cos 2x \right) dx && \{\text{from } \mathbf{a}\} \\
&= \frac{1}{2}x - \frac{1}{2} \left(\frac{1}{2} \sin 2x \right) + c \\
&= \frac{1}{2}x - \frac{1}{4} \sin 2x + c
\end{aligned}$$

$$\begin{aligned}
\mathbf{ii} \quad \int \cos^2 x dx &= \int \left(\frac{1}{2} + \frac{1}{2} \cos 2x \right) dx && \{\text{from } \mathbf{a}\} \\
&= \frac{1}{2}x + \frac{1}{2} \left(\frac{1}{2} \sin 2x \right) + c \\
&= \frac{1}{2}x + \frac{1}{4} \sin 2x + c
\end{aligned}$$

$$\begin{aligned}
15 \quad \mathbf{a} \quad \int (\cos^2 x + 2) dx \\
&= \int \left(\frac{1}{2} + \frac{1}{2} \cos 2x + 2 \right) dx \\
&= \int \left(\frac{5}{2} + \frac{1}{2} \cos 2x \right) dx \\
&= \frac{5}{2}x + \frac{1}{4} \sin 2x + c
\end{aligned}$$

$$\begin{aligned}
\mathbf{b} \quad \int (\sin^2 x + 4x) dx \\
&= \int \left(\frac{1}{2} - \frac{1}{2} \cos 2x + 4x \right) dx \\
&= \frac{1}{2}x - \frac{1}{4} \sin 2x + 2x^2 + c
\end{aligned}$$

$$\begin{aligned}
\mathbf{c} \quad \int (1 + \cos^2 2x) dx \\
&= \int \left(1 + \frac{1}{2} + \frac{1}{2} \cos(2(2x)) \right) dx \\
&= \int \left(\frac{3}{2} + \frac{1}{2} \cos 4x \right) dx \\
&= \frac{3}{2}x + \frac{1}{8} \sin 4x + c
\end{aligned}$$

$$\begin{aligned}
\mathbf{d} \quad \int (3 - \sin^2 3x) dx \\
&= \int \left(3 - \left(\frac{1}{2} - \frac{1}{2} \cos(2(3x)) \right) \right) dx \\
&= \int \left(\frac{5}{2} + \frac{1}{2} \cos 6x \right) dx \\
&= \frac{5}{2}x + \frac{1}{12} \sin 6x + c
\end{aligned}$$

$$\begin{aligned}
 \text{e} \quad & \int \frac{1}{2} \cos^2 4x \, dx \\
 &= \int \frac{1}{2} \left(\frac{1}{2} + \frac{1}{2} \cos(2(4x)) \right) dx \\
 &= \int \left(\frac{1}{4} + \frac{1}{4} \cos 8x \right) dx \\
 &= \frac{1}{4}x + \frac{1}{32} \sin 8x + c
 \end{aligned}$$

$$\begin{aligned}
 \text{g} \quad & \int \sin x(2 \sin x - 1) \, dx \\
 &= \int (2 \sin^2 x - \sin x) \, dx \\
 &= \int \left(2 \left(\frac{1}{2} - \frac{1}{2} \cos 2x \right) - \sin x \right) dx \\
 &= \int (1 - \cos 2x - \sin x) \, dx \\
 &= x - \frac{1}{2} \sin 2x + \cos x + c
 \end{aligned}$$

$$16 \quad \text{a} \quad \int \frac{6}{x+4} \, dx = 6 \ln |x+4| + c$$

$$\begin{aligned}
 \text{c} \quad & \int \frac{3}{1-x} \, dx \\
 &= 3 \int \frac{1}{1-x} \, dx \\
 &= 3 \left(\frac{1}{-1} \right) \ln |1-x| + c \\
 &= -3 \ln |1-x| + c
 \end{aligned}$$

$$\begin{aligned}
 \text{e} \quad & \int \left(1 - 2x + \frac{4}{x-3} \right) dx \\
 &= x - x^2 + 4 \ln |x-3| + c
 \end{aligned}$$

$$\begin{aligned}
 \text{g} \quad & \int \left(e^{-x} - \frac{4}{2x+1} \right) dx \\
 &= \left(\frac{1}{-1} \right) e^{-x} - 4 \left(\frac{1}{2} \right) \ln |2x+1| + c \\
 &= -e^{-x} - 2 \ln |2x+1| + c
 \end{aligned}$$

$$\begin{aligned}
 \text{i} \quad & \int \left(\frac{5}{x-6} - \frac{2}{3x-1} \right) dx \\
 &= 5 \ln |x-6| - 2 \left(\frac{1}{3} \right) \ln |3x-1| + c \\
 &= 5 \ln |x-6| - \frac{2}{3} \ln |3x-1| + c
 \end{aligned}$$

$$\begin{aligned}
 \text{f} \quad & \int (1 + \cos x)^2 \, dx \\
 &= \int (1 + 2 \cos x + \cos^2 x) \, dx \\
 &= \int \left(1 + 2 \cos x + \frac{1}{2} + \frac{1}{2} \cos 2x \right) dx \\
 &= \int \left(\frac{3}{2} + 2 \cos x + \frac{1}{2} \cos 2x \right) dx \\
 &= \frac{3}{2}x + 2 \sin x + \frac{1}{4} \sin 2x + c
 \end{aligned}$$

$$\begin{aligned}
 \text{h} \quad & \int (1 - 3 \sin x)^2 \, dx \\
 &= \int (1 - 6 \sin x + 9 \sin^2 x) \, dx \\
 &= \int \left(1 - 6 \sin x + 9 \left(\frac{1}{2} - \frac{1}{2} \cos 2x \right) \right) dx \\
 &= \int \left(1 - 6 \sin x + \frac{9}{2} - \frac{9}{2} \cos 2x \right) dx \\
 &= \int \left(\frac{11}{2} - 6 \sin x - \frac{9}{2} \cos 2x \right) dx \\
 &= \frac{11}{2}x + 6 \cos x - \frac{9}{4} \sin 2x + c
 \end{aligned}$$

$$\begin{aligned}
 \text{b} \quad & \int \frac{1}{2x-1} \, dx \\
 &= \frac{1}{2} \ln |2x-1| + c
 \end{aligned}$$

$$\begin{aligned}
 \text{d} \quad & \int \frac{5}{1-3x} \, dx \\
 &= 5 \int \frac{1}{1-3x} \, dx \\
 &= 5 \left(\frac{1}{-3} \right) \ln |1-3x| + c \\
 &= -\frac{5}{3} \ln |1-3x| + c
 \end{aligned}$$

$$\begin{aligned}
 \text{f} \quad & \int \left(4 + \frac{1}{5x-2} \right) dx \\
 &= 4x + \frac{1}{5} \ln |5x-2| + c
 \end{aligned}$$

$$\begin{aligned}
 \text{h} \quad & \int \left(\frac{1}{x+2} + \frac{2}{x-3} \right) dx \\
 &= \ln |x+2| + 2 \ln |x-3| + c
 \end{aligned}$$

17 Differentiating Tracy's answer: $\frac{d}{dx} \left(\frac{1}{4} \ln 4x + c \right) = \frac{1}{4} \left(\frac{4}{4x} \right) + 0, \quad x > 0$

$$= \frac{1}{4x}, \quad x > 0$$

Differentiating Nadine's answer: $\frac{d}{dx} \left(\frac{1}{4} \ln x + c \right) = \frac{1}{4} \left(\frac{1}{x} \right) + 0, \quad x > 0$

$$= \frac{1}{4x}, \quad x > 0$$

Both answers give the correct derivative and both are correct. This result occurs because $\ln 4x = \ln 4 + \ln x$. So, their answers differ by a constant which is accounted for by c .

18
$$\frac{3x-1}{x+2} = \frac{3(x+2)-6-1}{x+2}$$

$$= \frac{3(x+2)-7}{x+2}$$

$$= 3 - \frac{7}{x+2}$$

$$\therefore \int \frac{3x-1}{x+2} dx = \int \left(3 - \frac{7}{x+2} \right) dx$$

$$= 3x - 7 \ln |x+2| + c$$

19
$$f'(x) = 2x - \frac{2}{1-x}$$

$$\therefore f(x) = \int \left(2x - \frac{2}{1-x} \right) dx$$

$$= x^2 - 2 \left(\frac{1}{-1} \right) \ln |1-x| + c$$

$$= x^2 + 2 \ln |1-x| + c$$

But $f(-1) = 3$

$$\therefore (-1)^2 + 2 \ln |1 - (-1)| + c = 3$$

$$\therefore 1 + 2 \ln 2 + c = 3$$

$$\therefore c = 2 - 2 \ln 2$$

$$\therefore f(x) = x^2 + 2 \ln |1-x| + 2 - 2 \ln 2$$

EXERCISE 16E

1
$$\frac{d}{dx} ((x^2 - x)^3) = 3(x^2 - x)^2(2x - 1) \quad \{\text{chain rule}\}$$

$$\therefore \int 3(x^2 - x)^2(2x - 1) dx = (x^2 - x)^3 + c$$

$$\therefore 3 \int (x^2 - x)^2(2x - 1) dx = (x^2 - x)^3 + c$$

$$\therefore \int (2x - 1)(x^2 - x)^2 dx = \frac{1}{3}(x^2 - x)^3 + c$$

$$\begin{aligned} \mathbf{2} \quad \frac{d}{dx}(\sin(x^2)) &= \cos(x^2) \times 2x \\ &= 2x \cos(x^2) \end{aligned}$$

$$\therefore \int 2x \cos(x^2) dx = \sin(x^2) + c$$

$$\therefore 2 \int x \cos(x^2) dx = \sin(x^2) + c$$

$$\therefore \int x \cos(x^2) dx = \frac{1}{2} \sin(x^2) + c$$

$$\mathbf{3} \quad \frac{d}{dx}(\ln(5 - 3x + x^2)) = \frac{-3 + 2x}{5 - 3x + x^2}$$

$$\therefore \int \frac{-3 + 2x}{5 - 3x + x^2} dx = \ln |5 - 3x + x^2| + c$$

$$\therefore 2 \int \frac{-3 + 2x}{5 - 3x + x^2} dx = 2 \ln |5 - 3x + x^2| + c$$

$$\therefore \int \frac{4x - 6}{5 - 3x + x^2} dx = 2 \ln |5 - 3x + x^2| + c$$

$$\mathbf{4} \quad \mathbf{a} \quad u = x^3 + 1, \quad \frac{du}{dx} = 3x^2$$

$$\begin{aligned} \therefore \int 3x^2(x^3 + 1)^4 dx &= \int u^4 \frac{du}{dx} dx \\ &= \int u^4 du \\ &= \frac{1}{5} u^5 + c \\ &= \frac{1}{5} (x^3 + 1)^5 + c \end{aligned}$$

$$\mathbf{b} \quad u = x^3 + 1, \quad \frac{du}{dx} = 3x^2$$

$$\begin{aligned} \therefore \int x^2 e^{x^3+1} dx &= \frac{1}{3} \int (3x^2) e^{x^3+1} dx \\ &= \frac{1}{3} \int e^u \frac{du}{dx} dx \\ &= \frac{1}{3} \int e^u du \\ &= \frac{1}{3} e^u + c \\ &= \frac{1}{3} e^{x^3+1} + c \end{aligned}$$

$$\mathbf{c} \quad u = \sin x, \quad \frac{du}{dx} = \cos x$$

$$\begin{aligned} \therefore \int \sin^4 x \cos x dx &= \int u^4 \frac{du}{dx} dx \\ &= \int u^4 du \\ &= \frac{u^5}{5} + c \\ &= \frac{1}{5} \sin^5 x + c \end{aligned}$$

$$\mathbf{d} \quad u = x^2 - 3, \quad \frac{du}{dx} = 2x$$

$$\begin{aligned} \therefore \int 2x \cos(x^2 - 3) dx &= \int \cos u \frac{du}{dx} dx \\ &= \int \cos u du \\ &= \sin u + c \\ &= \sin(x^2 - 3) + c \end{aligned}$$

$$\begin{aligned}
 \mathbf{5} \quad \mathbf{a} \quad \int 4x^3(2+x^4)^3 dx &= \int u^3 \frac{du}{dx} dx \quad \{u = 2 + x^4, \quad \frac{du}{dx} = 4x^3\} \\
 &= \int u^3 du \\
 &= \frac{u^4}{4} + c \\
 &= \frac{1}{4}(2+x^4)^4 + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad \int \frac{2x}{\sqrt{x^2+3}} dx &= \int 2x(x^2+3)^{-\frac{1}{2}} dx \\
 &= \int u^{-\frac{1}{2}} \frac{du}{dx} dx \\
 &\quad \{u = x^2 + 3, \quad \frac{du}{dx} = 2x\} \\
 &= \int u^{-\frac{1}{2}} du \\
 &= \frac{u^{\frac{1}{2}}}{\frac{1}{2}} + c \\
 &= 2\sqrt{u} + c \\
 &= 2\sqrt{x^2+3} + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{d} \quad \int (x^3+2x+1)^4(3x^2+2) dx &= \int u^4 \frac{du}{dx} dx \\
 &\quad \{u = x^3 + 2x + 1, \quad \frac{du}{dx} = 3x^2 + 2\} \\
 &= \int u^4 du \\
 &= \frac{u^5}{5} + c \\
 &= \frac{1}{5}(x^3+2x+1)^5 + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{c} \quad \int \frac{6x^2}{(2x^3-1)^4} dx &= \int 6x^2(2x^3-1)^{-4} dx \\
 &= \int u^{-4} \frac{du}{dx} dx \\
 &\quad \{u = 2x^3 - 1, \quad \frac{du}{dx} = 6x^2\} \\
 &= \int u^{-4} du \\
 &= \frac{u^{-3}}{-3} + c \\
 &= -\frac{1}{3u^3} + c \\
 &= -\frac{1}{3(2x^3-1)^3} + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{e} \quad \int \frac{x}{(1-x^2)^5} dx &= -\frac{1}{2} \int (1-x^2)^{-5} \times (-2x) dx \\
 &= -\frac{1}{2} \int u^{-5} \frac{du}{dx} dx \\
 &\quad \{u = 1 - x^2, \quad \frac{du}{dx} = -2x\} \\
 &= -\frac{1}{2} \int u^{-5} du \\
 &= -\frac{1}{2} \frac{u^{-4}}{-4} + c \\
 &= \frac{1}{8(1-x^2)^4} + c
 \end{aligned}$$

$$\begin{aligned}
 \text{f} \quad \int \frac{x+2}{(x^2+4x-3)^2} dx &= \frac{1}{2} \int (x^2+4x-3)^{-2} (2x+4) dx \\
 &= \frac{1}{2} \int u^{-2} \frac{du}{dx} dx \quad \{u = x^2 + 4x - 3, \quad \frac{du}{dx} = 2x + 4\} \\
 &= \frac{1}{2} \int u^{-2} du \\
 &= \frac{1}{2} \frac{u^{-1}}{-1} + c \\
 &= -\frac{1}{2(x^2+4x-3)} + c
 \end{aligned}$$

$$\begin{aligned}
 \text{6 a} \quad \int -2e^{1-2x} dx \\
 &= \int e^u \frac{du}{dx} dx \\
 &\quad \{u = 1 - 2x, \quad \frac{du}{dx} = -2\} \\
 &= \int e^u du \\
 &= e^u + c \\
 &= e^{1-2x} + c
 \end{aligned}$$

$$\begin{aligned}
 \text{b} \quad \int 2xe^{x^2} dx \\
 &= \int e^u \frac{du}{dx} dx \quad \{u = x^2, \quad \frac{du}{dx} = 2x\} \\
 &= \int e^u du \\
 &= e^u + c \\
 &= e^{x^2} + c
 \end{aligned}$$

$$\begin{aligned}
 \text{c} \quad \int \frac{e^{\sqrt{x}}}{\sqrt{x}} dx &= 2 \int \frac{e^{\sqrt{x}}}{2\sqrt{x}} dx \\
 &= 2 \int e^u \frac{du}{dx} dx \quad \{u = \sqrt{x}, \quad \frac{du}{dx} = \frac{1}{2\sqrt{x}}\} \\
 &= 2 \int e^u du \\
 &= 2e^u + c \\
 &= 2e^{\sqrt{x}} + c
 \end{aligned}$$

$$\begin{aligned}
 \text{7 a} \quad \int \frac{2x}{x^2+1} dx \\
 &= \int \frac{1}{x^2+1} (2x) dx \\
 &= \int \frac{1}{u} \frac{du}{dx} dx \\
 &\quad \{u = x^2 + 1, \quad \frac{du}{dx} = 2x\} \\
 &= \int \frac{1}{u} du \\
 &= \ln |u| + c \\
 &= \ln |x^2 + 1| + c
 \end{aligned}$$

$$\begin{aligned}
 \text{b} \quad \int \frac{x}{2-x^2} dx \\
 &= -\frac{1}{2} \int \frac{1}{2-x^2} (-2x) dx \\
 &= -\frac{1}{2} \int \frac{1}{u} \frac{du}{dx} dx \\
 &\quad \{u = 2 - x^2, \quad \frac{du}{dx} = -2x\} \\
 &= -\frac{1}{2} \int \frac{1}{u} du \\
 &= -\frac{1}{2} \ln |u| + c \\
 &= -\frac{1}{2} \ln |2 - x^2| + c
 \end{aligned}$$

$$\begin{aligned}
& \int \frac{2x-3}{x^2-3x} dx \\
&= \int \frac{1}{x^2-3x} (2x-3) dx \\
&= \int \frac{1}{u} \frac{du}{dx} dx \quad \{u = x^2 - 3x, \quad \frac{du}{dx} = 2x - 3\} \\
&= \int \frac{1}{u} du \\
&= \ln |u| + c \\
&= \ln |x^2 - 3x| + c
\end{aligned}$$

$$\begin{aligned}
& \int x^2(3-x^3)^2 dx \\
&= -\frac{1}{3} \int (-3x^2)(3-x^3)^2 dx \\
&= -\frac{1}{3} \int u^2 \frac{du}{dx} dx \\
&\quad \{u = 3 - x^3, \quad \frac{du}{dx} = -3x^2\} \\
&= -\frac{1}{3} \int u^2 du \\
&= -\frac{1}{3} \times \frac{u^3}{3} + c \\
&= -\frac{1}{9}(3-x^3)^3 + c
\end{aligned}$$

$$\begin{aligned}
& \int x\sqrt{1-x^2} dx \\
&= -\frac{1}{2} \int (-2x)(1-x^2)^{\frac{1}{2}} dx \\
&= -\frac{1}{2} \int u^{\frac{1}{2}} \frac{du}{dx} dx \\
&\quad \{u = 1 - x^2, \quad \frac{du}{dx} = -2x\} \\
&= -\frac{1}{2} \int u^{\frac{1}{2}} du \\
&= -\frac{1}{2} \times \frac{u^{\frac{3}{2}}}{\frac{3}{2}} + c \\
&= -\frac{1}{3} u^{\frac{3}{2}} + c \\
&= -\frac{1}{3}(1-x^2)^{\frac{3}{2}} + c
\end{aligned}$$

$$\begin{aligned}
& \int x e^{1-x^2} dx \\
&= -\frac{1}{2} \int (-2x) e^{1-x^2} dx \\
&= -\frac{1}{2} \int e^u \frac{du}{dx} dx \\
&\quad \{u = 1 - x^2, \quad \frac{du}{dx} = -2x\} \\
&= -\frac{1}{2} \int e^u du \\
&= -\frac{1}{2} e^u + c \\
&= -\frac{1}{2} e^{1-x^2} + c
\end{aligned}$$

$$\begin{aligned}
& \int \frac{(\ln x)^3}{x} dx \\
&= \int u^3 \frac{du}{dx} dx \\
&\quad \{u = \ln x, \quad \frac{du}{dx} = \frac{1}{x}\} \\
&= \int u^3 du \\
&= \frac{u^4}{4} + c \\
&= \frac{1}{4}(\ln x)^4 + c
\end{aligned}$$

$$\begin{aligned}
 \text{e} \quad & \int (2x - 1)e^{x-x^2} dx \\
 &= - \int (1 - 2x)e^{x-x^2} dx \\
 &= - \int e^u \frac{du}{dx} dx \\
 &\quad \{u = x - x^2, \quad \frac{du}{dx} = 1 - 2x\} \\
 &= - \int e^u du \\
 &= -e^u + c \\
 &= -e^{x-x^2} + c
 \end{aligned}$$

$$\begin{aligned}
 \text{f} \quad & \int \frac{1-x^2}{x^3-3x} dx \\
 &= -\frac{1}{3} \int \frac{1}{x^3-3x} (3x^2-3) dx \\
 &= -\frac{1}{3} \int \frac{1}{u} \frac{du}{dx} dx \\
 &\quad \{u = x^3 - 3x, \quad \frac{du}{dx} = 3x^2 - 3\} \\
 &= -\frac{1}{3} \int \frac{1}{u} du \\
 &= -\frac{1}{3} \ln |u| + c \\
 &= -\frac{1}{3} \ln |x^3 - 3x| + c
 \end{aligned}$$

$$\begin{aligned}
 \text{9 a} \quad & \int \sin^7 x \cos x dx \\
 &= \int u^7 \frac{du}{dx} dx \\
 &\quad \{u = \sin x, \quad \frac{du}{dx} = \cos x\} \\
 &= \int u^7 du \\
 &= \frac{u^8}{8} + c \\
 &= \frac{1}{8} \sin^8 x + c
 \end{aligned}$$

$$\begin{aligned}
 \text{b} \quad & \int \cos^5 x \sin x dx \\
 &= - \int \cos^5 x (-\sin x) dx \\
 &= - \int u^5 \frac{du}{dx} dx \\
 &\quad \{u = \cos x, \quad \frac{du}{dx} = -\sin x\} \\
 &= - \int u^5 du \\
 &= -\frac{1}{6} u^6 + c \\
 &= -\frac{1}{6} \cos^6 x + c
 \end{aligned}$$

$$\begin{aligned}
 \text{c} \quad & \int \frac{\sin x}{\sqrt{\cos x}} dx \\
 &= - \int \frac{-\sin x}{\sqrt{\cos x}} dx \\
 &= - \int u^{-\frac{1}{2}} \frac{du}{dx} dx \\
 &\quad \{u = \cos x, \quad \frac{du}{dx} = -\sin x\} \\
 &= - \int u^{-\frac{1}{2}} du \\
 &= -\frac{u^{\frac{1}{2}}}{\frac{1}{2}} + c \\
 &= -2(\cos x)^{\frac{1}{2}} + c \\
 &= -2\sqrt{\cos x} + c
 \end{aligned}$$

$$\begin{aligned}
 \text{d} \quad & \int \tan x dx \\
 &= \int \frac{\sin x}{\cos x} dx \\
 &= - \int \frac{-\sin x}{\cos x} dx \\
 &= - \int \frac{1}{u} \frac{du}{dx} dx \\
 &\quad \{u = \cos x, \quad \frac{du}{dx} = -\sin x\} \\
 &= - \int \frac{1}{u} du \\
 &= -\ln |u| + c \\
 &= -\ln |\cos x| + c
 \end{aligned}$$

$$\begin{aligned}
 \text{e} \quad & \int \sqrt{\sin x} \cos x \, dx \\
 &= \int u^{\frac{1}{2}} \frac{du}{dx} \, dx \\
 &\quad \{u = \sin x, \quad \frac{du}{dx} = \cos x\} \\
 &= \int u^{\frac{1}{2}} \, du \\
 &= \frac{2}{3} u^{\frac{3}{2}} + c \\
 &= \frac{2}{3} (\sin x)^{\frac{3}{2}} + c
 \end{aligned}$$

$$\begin{aligned}
 \text{f} \quad & \int \frac{\cos x}{(2 + \sin x)^2} \, dx \\
 &= \int u^{-2} \frac{du}{dx} \, dx \\
 &\quad \{u = 2 + \sin x, \quad \frac{du}{dx} = \cos x\} \\
 &= \int u^{-2} \, du \\
 &= -u^{-1} + c \\
 &= -(2 + \sin x)^{-1} + c \\
 &= -\frac{1}{2 + \sin x} + c
 \end{aligned}$$

$$\begin{aligned}
 \text{g} \quad & \int \frac{\sin x}{1 - \cos x} \, dx \\
 &= \int \frac{1}{u} \frac{du}{dx} \, dx \\
 &\quad \{u = 1 - \cos x, \quad \frac{du}{dx} = \sin x\} \\
 &= \int \frac{1}{u} \, du \\
 &= \ln |u| + c \\
 &= \ln |1 - \cos x| + c
 \end{aligned}$$

$$\begin{aligned}
 \text{h} \quad & \int \frac{\cos 2x}{\sin 2x - 3} \, dx \\
 &= \frac{1}{2} \int \frac{2 \cos 2x}{\sin 2x - 3} \, dx \\
 &= \frac{1}{2} \int \frac{1}{u} \frac{du}{dx} \, dx \\
 &\quad \{u = \sin 2x - 3, \quad \frac{du}{dx} = 2 \cos 2x\} \\
 &= \frac{1}{2} \int \frac{1}{u} \, du \\
 &= \frac{1}{2} \ln |u| + c \\
 &= \frac{1}{2} \ln |\sin 2x - 3| + c
 \end{aligned}$$

$$\begin{aligned}
 \text{i} \quad & \int x \sin(x^2) \, dx = \frac{1}{2} \int (2x) \sin(x^2) \, dx \\
 &= \frac{1}{2} \int \sin u \frac{du}{dx} \, dx \quad \{u = x^2, \quad \frac{du}{dx} = 2x\} \\
 &= \frac{1}{2} \int \sin u \, du \\
 &= \frac{1}{2} (-\cos u) + c \\
 &= -\frac{1}{2} \cos(x^2) + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{10} \quad \mathbf{a} \quad & \int \cos^3 x \, dx \\
 &= \int \cos^2 x \cos x \, dx \\
 &= \int (1 - \sin^2 x) \cos x \, dx \\
 &= \int (1 - u^2) \frac{du}{dx} \, dx \\
 &\quad \{u = \sin x, \quad \frac{du}{dx} = \cos x\} \\
 &= \int (1 - u^2) \, du \\
 &= u - \frac{u^3}{3} + c \\
 &= \sin x - \frac{\sin^3 x}{3} + c \\
 &= \sin x - \frac{1}{3} \sin^3 x + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad & \int \sin^3 2x \cos 2x \, dx \\
 &= \frac{1}{2} \int \sin^3 2x (2 \cos 2x) \, dx \\
 &= \frac{1}{2} \int u^3 \frac{du}{dx} \, dx \\
 &\quad \{u = \sin 2x, \quad \frac{du}{dx} = 2 \cos 2x\} \\
 &= \frac{1}{2} \int u^3 \, du \\
 &= \frac{1}{2} \times \frac{u^4}{4} + c \\
 &= \frac{1}{8} \sin^4 2x + c
 \end{aligned}$$

REVIEW SET 16A

$$\begin{aligned}
 \mathbf{1} \quad & \frac{d}{dx}(x^4 - x^2) = 4x^3 - 2x \\
 \therefore & \int (4x^3 - 2x) \, dx = x^4 - x^2 + c \\
 \therefore & 2 \int (2x^3 - x) \, dx = x^4 - x^2 + c \\
 \therefore & \int (2x^3 - x) \, dx = \frac{1}{2}x^4 - \frac{1}{2}x^2 + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{2} \quad & \frac{d}{dx} \left(\sin\left(\frac{\pi}{3} - 2x\right) \right) = \cos\left(\frac{\pi}{3} - 2x\right) \times (-2) \\
 &= -2 \cos\left(\frac{\pi}{3} - 2x\right) \\
 \therefore & \int -2 \cos\left(\frac{\pi}{3} - 2x\right) \, dx = \sin\left(\frac{\pi}{3} - 2x\right) + c \\
 \therefore & -2 \int \cos\left(\frac{\pi}{3} - 2x\right) \, dx = \sin\left(\frac{\pi}{3} - 2x\right) + c \\
 \therefore & \int \cos\left(\frac{\pi}{3} - 2x\right) \, dx = -\frac{1}{2} \sin\left(\frac{\pi}{3} - 2x\right) + c
 \end{aligned}$$

$$\begin{aligned}
 \text{3 a } \int \left(\sqrt{x} - \frac{2}{x^2} \right) dx \\
 &= \int (x^{\frac{1}{2}} - 2x^{-2}) dx \\
 &= \frac{x^{\frac{3}{2}}}{\frac{3}{2}} - 2 \times \frac{x^{-1}}{-1} + c \\
 &= \frac{2}{3}x^{\frac{3}{2}} + 2x^{-1} + c \\
 &= \frac{2}{3}x\sqrt{x} + \frac{2}{x} + c
 \end{aligned}$$

$$\begin{aligned}
 \text{b } \int \left(2x - \frac{3}{\sqrt[3]{x}} \right) dx \\
 &= \int (2x - 3x^{-\frac{1}{3}}) dx \\
 &= x^2 - \frac{3x^{\frac{2}{3}}}{\frac{2}{3}} + c \\
 &= x^2 - \frac{9}{2}x^{\frac{2}{3}} + c
 \end{aligned}$$

$$\begin{aligned}
 \text{c } \int \frac{6x+5}{\sqrt{x}} dx \\
 &= \int (6x^{\frac{1}{2}} + 5x^{-\frac{1}{2}}) dx \\
 &= 6 \times \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + 5 \times \frac{x^{\frac{1}{2}}}{\frac{1}{2}} + c \\
 &= 4x^{\frac{3}{2}} + 10x^{\frac{1}{2}} + c \\
 &= 4x\sqrt{x} + 10\sqrt{x} + c
 \end{aligned}$$

$$\begin{aligned}
 \text{4 a } \int \frac{4}{\sqrt{x}} dx \\
 &= 4 \int x^{-\frac{1}{2}} dx \\
 &= 4 \times \frac{x^{\frac{1}{2}}}{\frac{1}{2}} + c \\
 &= 8\sqrt{x} + c
 \end{aligned}$$

$$\begin{aligned}
 \text{b } \int \left(\frac{1}{3}x^3 + 2x \right) dx \\
 &= \frac{1}{3} \times \frac{x^4}{4} + x^2 + c \\
 &= \frac{1}{12}x^4 + x^2 + c
 \end{aligned}$$

$$\begin{aligned}
 \text{c } \int \frac{1-2x}{x^3} dx \\
 &= \int \left(\frac{1}{x^3} - \frac{2}{x^2} \right) dx \\
 &= \int (x^{-3} - 2x^{-2}) dx \\
 &= \frac{x^{-2}}{-2} - \frac{2x^{-1}}{-1} + c \\
 &= -\frac{1}{2}x^{-2} + 2x^{-1} + c \\
 &= -\frac{1}{2x^2} + \frac{2}{x} + c
 \end{aligned}$$

$$\begin{aligned}
 \text{5 a } \int (-3x^4 + 6x^2) dx \\
 &= -\frac{3x^5}{5} + \frac{6x^3}{3} + c \\
 &= -\frac{3}{5}x^5 + 2x^3 + c
 \end{aligned}$$

$$\begin{aligned}
 \text{b } \int \frac{3x^3 - x^2 - 1}{x^2} dx \\
 &= \int \left(3x - 1 - \frac{1}{x^2} \right) dx \\
 &= \int (3x - 1 - x^{-2}) dx \\
 &= \frac{3x^2}{2} - x - \frac{x^{-1}}{-1} + c \\
 &= \frac{3}{2}x^2 - x + \frac{1}{x} + c
 \end{aligned}$$

$$\begin{aligned}
 \text{c } \int (2x - \sqrt{x})^2 dx \\
 &= \int (4x^2 - 4x\sqrt{x} + x) dx \\
 &= \int (4x^2 - 4x^{\frac{3}{2}} + x) dx \\
 &= \frac{4x^3}{3} - \frac{4x^{\frac{5}{2}}}{\frac{5}{2}} + \frac{x^2}{2} + c \\
 &= \frac{4}{3}x^3 - \frac{8}{5}x^{\frac{5}{2}} + \frac{1}{2}x^2 + c
 \end{aligned}$$

$$\begin{aligned}
 \text{d } \int \left(4e^x - \frac{3}{x} \right) dx \\
 &= 4e^x - 3 \ln |x| + c
 \end{aligned}$$

$$\begin{aligned}
 \text{e } \int \sin(4x - 5) dx \\
 &= -\frac{1}{4} \cos(4x - 5) + c
 \end{aligned}$$

$$\begin{aligned}
 \text{f } \int e^{4-3x} dx \\
 &= \left(\frac{1}{-3} \right) e^{4-3x} + c \\
 &= -\frac{1}{3}e^{4-3x} + c
 \end{aligned}$$

$$\begin{aligned}
 \text{6 a } \frac{dy}{dx} &= 3e^{-x} - 2 \sin\left(\frac{\pi}{2} - x\right) \\
 \therefore y &= \int (3e^{-x} - 2 \sin(\frac{\pi}{2} - x)) dx \\
 &= 3\left(\frac{1}{-1}\right)e^{-x} - 2\left(\frac{1}{-1}\right)(-\cos(\frac{\pi}{2} - x)) + c \\
 &= -3e^{-x} - 2 \cos\left(\frac{\pi}{2} - x\right) + c
 \end{aligned}$$

$$\begin{aligned}
 \text{b } \frac{dy}{dx} &= \cos 4x - \frac{1}{2}x^2 \\
 \therefore y &= \int (\cos 4x - \frac{1}{2}x^2) dx \\
 &= \frac{1}{4} \sin 4x - \frac{1}{2} \times \frac{x^3}{3} + c \\
 &= \frac{1}{4} \sin 4x - \frac{1}{6}x^3 + c
 \end{aligned}$$

7 $f'(x) = 3x^2 - 4x + 1$

$$\begin{aligned}\therefore f(x) &= \int (3x^2 - 4x + 1) dx \\ &= \frac{3x^3}{3} - \frac{4x^2}{2} + x + c \\ &= x^3 - 2x^2 + x + c\end{aligned}$$

But $f(0) = 2$, so $c = 2$

$$\therefore f(x) = x^3 - 2x^2 + x + 2$$

8 $f'(x) = ax + 3$

$$\begin{aligned}\therefore f(x) &= \int (ax + 3) dx \\ &= \frac{ax^2}{2} + 3x + c\end{aligned}$$

But the y -intercept is 2, so $f(0) = 2$

$$\therefore c = 2$$

$$\therefore f(x) = \frac{ax^2}{2} + 3x + 2$$

and the x -intercept is 2, so $f(2) = 0$

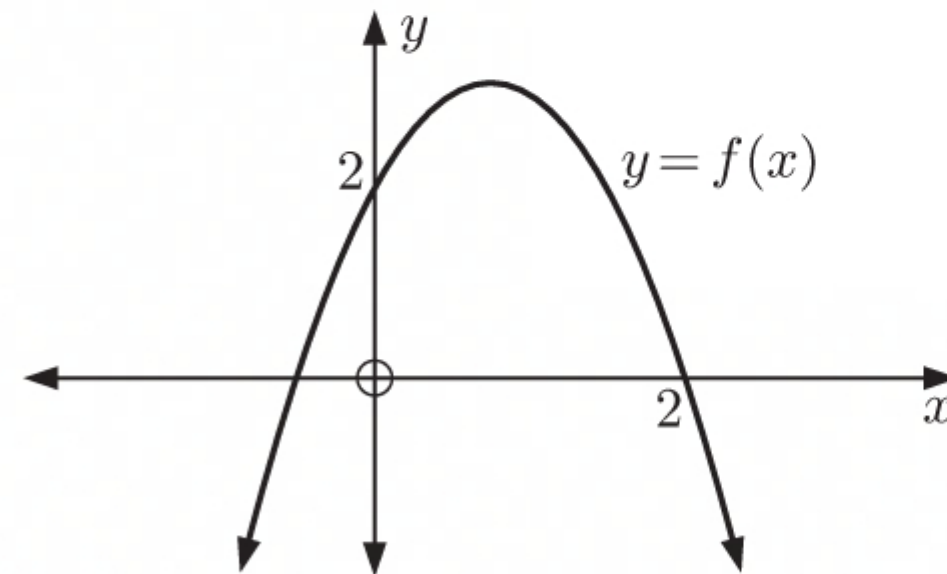
$$\therefore \frac{a(2)^2}{2} + 3(2) + 2 = 0$$

$$\therefore 2a + 6 + 2 = 0$$

$$\therefore 2a = -8$$

$$\therefore a = -4$$

$$\begin{aligned}\therefore \text{the equation of the curve is } y = f(x) &= \frac{(-4)x^2}{2} + 3x + 2 \\ &= -2x^2 + 3x + 2\end{aligned}$$



9 $f'(x) = 3e^{2x}$

$$\begin{aligned}\therefore f(x) &= \int 3e^{2x} dx \\ &= \frac{3}{2}e^{2x} + c\end{aligned}$$

But $f(0) = 2$, so $\frac{3}{2} + c = 2$

$$\therefore c = \frac{1}{2}$$

$$\therefore f(x) = \frac{3}{2}e^{2x} + \frac{1}{2}$$

10 a
$$\begin{aligned}\int \frac{x^2 - 7}{x} dx \\ &= \int \left(x - \frac{7}{x}\right) dx \\ &= \frac{1}{2}x^2 - 7 \ln |x| + c\end{aligned}$$

b
$$\begin{aligned}\int \left(e^{2x-3} - \frac{2}{3x-1}\right) dx \\ &= \frac{1}{2}e^{2x-3} - \frac{2}{3} \ln |3x-1| + c\end{aligned}$$

$$\begin{aligned}
 & \int ((4-3x)^3 + \sin(-2x)) dx \\
 &= \left(\frac{1}{-3}\right) \frac{(4-3x)^4}{4} + \left(\frac{1}{-2}\right)(-\cos(-2x)) + c \\
 &= -\frac{1}{12}(4-3x)^4 + \frac{1}{2}\cos(-2x) + c
 \end{aligned}$$

$$\begin{aligned}
 11 \quad & f'(x) = a \cos 3x \\
 \therefore & f(x) = \int a \cos 3x dx \\
 &= \frac{a}{3} \sin 3x + c
 \end{aligned}$$

But $f(0) = -1$, so $c = -1$

$$\therefore f(x) = \frac{a}{3} \sin 3x - 1$$

$$\text{and } f\left(\frac{\pi}{4}\right) = 1$$

$$\therefore \frac{a}{3} \sin \frac{3\pi}{4} - 1 = 1$$

$$\therefore \frac{a}{3} \times \frac{1}{\sqrt{2}} = 2$$

$$\therefore \frac{a}{3\sqrt{2}} = 2$$

$$\therefore a = 6\sqrt{2}$$

$$\begin{aligned}
 \therefore f(x) &= \frac{6\sqrt{2}}{3} \sin 3x - 1 \\
 &= 2\sqrt{2} \sin 3x - 1
 \end{aligned}$$

$$\begin{aligned}
 12 \quad & \int (1 - \sin x)^2 \\
 &= \int (1 - 2\sin x + \sin^2 x) dx \\
 &= \int \left(1 - 2\sin x + \frac{1}{2} - \frac{1}{2}\cos 2x\right) dx \quad \{\sin^2 x = \frac{1}{2} - \frac{1}{2}\cos 2x\} \\
 &= \int \left(\frac{3}{2} - 2\sin x - \frac{1}{2}\cos 2x\right) dx \\
 &= \frac{3}{2}x + 2\cos x - \frac{1}{4}\sin 2x + c
 \end{aligned}$$

$$\begin{aligned}
 13 \quad & \sqrt{x^2 - 4} = (x^2 - 4)^{\frac{1}{2}} \\
 \therefore \frac{d}{dx} \left(\sqrt{x^2 - 4} \right) &= \frac{1}{2}(x^2 - 4)^{-\frac{1}{2}} \times 2x \\
 &= \frac{x}{\sqrt{x^2 - 4}}
 \end{aligned}$$

$$\therefore \int \frac{x}{\sqrt{x^2 - 4}} dx = \sqrt{x^2 - 4} + c$$

$$\mathbf{14} \quad u = x^2 + \frac{\pi}{3}, \quad \frac{du}{dx} = 2x$$

$$\begin{aligned} \therefore \int x \sin\left(x^2 + \frac{\pi}{3}\right) dx &= \frac{1}{2} \int 2x \sin\left(x^2 + \frac{\pi}{3}\right) dx \\ &= \frac{1}{2} \int \sin u \frac{du}{dx} dx \\ &= \frac{1}{2} \int \sin u \, du \\ &= \frac{1}{2}(-\cos u) + c \\ &= -\frac{1}{2} \cos\left(x^2 + \frac{\pi}{3}\right) + c \end{aligned}$$

$$\begin{aligned} \mathbf{15} \quad \mathbf{a} \quad & \int \frac{x+2}{x^2+4x} dx \\ &= \frac{1}{2} \int \frac{2x+4}{x^2+4x} dx \\ &= \frac{1}{2} \int \frac{1}{u} \frac{du}{dx} dx \\ & \quad \{u = x^2 + 4x, \quad \frac{du}{dx} = 2x + 4\} \\ &= \frac{1}{2} \int \frac{1}{u} du \\ &= \frac{1}{2} \ln |u| + c \\ &= \frac{1}{2} \ln |x^2 + 4x| + c \end{aligned}$$

$$\begin{aligned} \mathbf{c} \quad & \int \sin^9 x \cos x \, dx \\ &= \int u^9 \frac{du}{dx} dx \\ & \quad \{u = \sin x, \quad \frac{du}{dx} = \cos x\} \\ &= \int u^9 du \\ &= \frac{1}{10} u^{10} + c \\ &= \frac{1}{10} \sin^{10} x + c \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad & \int 2xe^{x^2-1} dx \\ &= \int e^u \frac{du}{dx} dx \\ & \quad \{u = x^2 - 1, \quad \frac{du}{dx} = 2x\} \\ &= \int e^u du \\ &= e^u + c \\ &= e^{x^2-1} + c \\ \\ \mathbf{d} \quad & \int \tan 2x \, dx \\ &= \int \frac{\sin 2x}{\cos 2x} dx \\ &= -\frac{1}{2} \int \frac{-2 \sin 2x}{\cos 2x} dx \\ &= -\frac{1}{2} \int \frac{1}{u} \frac{du}{dx} dx \\ & \quad \{u = \cos 2x, \quad \frac{du}{dx} = -2 \sin 2x\} \\ &= -\frac{1}{2} \int \frac{1}{u} du \\ &= -\frac{1}{2} \ln |u| + c \\ &= -\frac{1}{2} \ln |\cos 2x| + c \end{aligned}$$

REVIEW SET 16B

$$\begin{aligned}
 1 \quad \frac{d}{dx}(6e^{-2x}) &= 6e^{-2x} \times (-2) \\
 &= -12e^{-2x} \\
 \therefore \int -12e^{-2x} dx &= 6e^{-2x} + c \\
 \therefore -12 \int e^{-2x} dx &= 6e^{-2x} + c \\
 \therefore \int e^{-2x} dx &= -\frac{1}{2}e^{-2x} + c
 \end{aligned}$$

$$\begin{aligned}
 2 \quad \frac{d}{dx}(\ln(2x+1)) &= \frac{2}{2x+1} \\
 \therefore \int \frac{2}{2x+1} dx &= \ln|2x+1| + c \\
 \therefore 2 \int \frac{1}{2x+1} dx &= \ln|2x+1| + c \\
 \therefore \int \frac{1}{2x+1} dx &= \frac{1}{2} \ln|2x+1| + c
 \end{aligned}$$

$$\begin{aligned}
 3 \quad a \quad \int \frac{x^2-2}{x^2} dx \\
 = \int \left(1 - \frac{2}{x^2}\right) dx \\
 = \int (1 - 2x^{-2}) dx \\
 = x - \frac{2x^{-1}}{-1} + c \\
 = x + \frac{2}{x} + c
 \end{aligned}$$

$$\begin{aligned}
 b \quad \int (3x-4)^2 dx \\
 = \int (9x^2 - 24x + 16) dx \\
 = \frac{9x^3}{3} - \frac{24x^2}{2} + 16x + c \\
 = 3x^3 - 12x^2 + 16x + c
 \end{aligned}$$

$$\begin{aligned}
 c \quad \int (4-2x^2) dx \\
 = 4x - \frac{2}{3}x^3 + c
 \end{aligned}$$

$$\begin{aligned}
 4 \quad a \quad \int (x^{\frac{1}{3}} + 3) dx \\
 = \frac{x^{\frac{4}{3}}}{\frac{4}{3}} + 3x + c \\
 = \frac{3}{4}x^{\frac{4}{3}} + 3x + c
 \end{aligned}$$

$$\begin{aligned}
 b \quad \int (3x^2 - 2) dx \\
 = \frac{3x^3}{3} - 2x + c \\
 = x^3 - 2x + c
 \end{aligned}$$

$$\begin{aligned}
 c \quad \int (3+2x)^2 dx \\
 = \int (9 + 12x + 4x^2) dx \\
 = 9x + \frac{12x^2}{2} + \frac{4x^3}{3} + c \\
 = 9x + 6x^2 + \frac{4}{3}x^3 + c
 \end{aligned}$$

$$\begin{aligned}
 5 \quad f'(x) &= x^2 - 3x + 2 \\
 \therefore f(x) &= \int (x^2 - 3x + 2) dx \\
 &= \frac{x^3}{3} - \frac{3x^2}{2} + 2x + c \\
 &= \frac{1}{3}x^3 - \frac{3}{2}x^2 + 2x + c
 \end{aligned}$$

$$\text{But } f(1) = 3, \text{ so } \frac{1}{3} - \frac{3}{2} + 2 + c = 3$$

$$\therefore c = \frac{13}{6} = 2\frac{1}{6}$$

$$\therefore f(x) = \frac{1}{3}x^3 - \frac{3}{2}x^2 + 2x + 2\frac{1}{6}$$

$$6 \quad a \quad \frac{dy}{dx} = (x^2 - 1)^2 = x^4 - 2x^2 + 1$$

$$\begin{aligned} \therefore y &= \int (x^4 - 2x^2 + 1) dx \\ &= \frac{1}{5}x^5 - \frac{2}{3}x^3 + x + c \end{aligned}$$

$$b \quad \frac{dy}{dx} = 400 - 20x^{-\frac{1}{2}}$$

$$\begin{aligned} \therefore y &= \int (400 - 20x^{-\frac{1}{2}}) dx \\ &= 400x - \frac{20x^{\frac{1}{2}}}{\frac{1}{2}} + c \\ &= 400x - 40x^{\frac{1}{2}} + c \end{aligned}$$

$$7 \quad a \quad \int (2x^3 - 5x + 7) dx$$

$$\begin{aligned} &= \frac{2x^4}{4} - \frac{5x^2}{2} + 7x + c \\ &= \frac{1}{2}x^4 - \frac{5}{2}x^2 + 7x + c \end{aligned}$$

$$b \quad \int \left(3x - \frac{1}{x}\right) dx$$

$$= \frac{3}{2}x^2 - \ln|x| + c$$

$$c \quad \int (1 - x^2)^3 dx = \int (1^3 + 3(1)^2(-x^2) + 3(1)(-x^2)^2 + (-x^2)^3) dx \quad \{\text{binomial theorem}\}$$

$$= \int (1 - 3x^2 + 3x^4 - x^6) dx$$

$$= x - \frac{3x^3}{3} + \frac{3x^5}{5} - \frac{x^7}{7} + c$$

$$= x - x^3 + \frac{3}{5}x^5 - \frac{1}{7}x^7 + c$$

$$d \quad \int (2e^{-x} + 3) dx$$

$$= 2\left(\frac{1}{-1}\right)e^{-x} + 3x + c$$

$$= -2e^{-x} + 3x + c$$

$$e \quad \int 4 \cos 2x dx$$

$$= 4 \times \frac{1}{2} \sin 2x + c$$

$$= 2 \sin 2x + c$$

$$f \quad \int (3 + e^{2x-1})^2 dx = \int (9 + 6e^{2x-1} + (e^{2x-1})^2) dx$$

$$= \int (9 + 6e^{2x-1} + e^{4x-2}) dx$$

$$= 9x + 6\left(\frac{1}{2}\right)e^{2x-1} + \frac{1}{4}e^{4x-2} + c$$

$$= 9x + 3e^{2x-1} + \frac{1}{4}e^{4x-2} + c$$

$$8 \quad f'(x) = \frac{2}{x} - 1$$

$$\therefore f(x) = \int \left(\frac{2}{x} - 1\right) dx$$

$$= 2 \ln|x| - x + c$$

But $f(2) = e$, so $2 \ln 2 - 2 + c = e$

$$\therefore c = e + 2 - 2 \ln 2$$

$$\therefore f(x) = 2 \ln|x| - x + e + 2 - 2 \ln 2$$

$$9 \quad \frac{dy}{dx} = ax^2 + b\sqrt{x-1} = ax^2 + b(x-1)^{\frac{1}{2}}$$

$$\therefore y = \int (ax^2 + b(x-1)^{\frac{1}{2}}) dx$$

$$= \frac{ax^3}{3} + \frac{b(x-1)^{\frac{3}{2}}}{\frac{3}{2}} + c$$

$$= \frac{1}{3}ax^3 + \frac{2}{3}b(x-1)^{\frac{3}{2}} + c$$

But the curve passes through (1, 4), (2, 4), and (5, 1)

$$\therefore \frac{1}{3}a(1)^3 + \frac{2}{3}b(1-1)^{\frac{3}{2}} + c = 4$$

$$\therefore \frac{1}{3}a + c = 4 \quad \dots (1)$$

$$\text{and } \frac{1}{3}a(2)^3 + \frac{2}{3}b(2-1)^{\frac{3}{2}} + c = 4$$

$$\therefore \frac{8}{3}a + \frac{2}{3}b + c = 4 \quad \dots (2)$$

$$\text{and } \frac{1}{3}a(5)^3 + \frac{2}{3}b(5-1)^{\frac{3}{2}} + c = 1$$

$$\therefore \frac{125}{3}a + \frac{2}{3}b(4)^{\frac{3}{2}} + c = 1$$

$$\therefore \frac{125}{3}a + \frac{16}{3}b + c = 1 \quad \dots (3)$$

We solve (1), (2), and (3) simultaneously using technology.

$$\therefore a = -\frac{9}{68}, \quad b = \frac{63}{136}, \quad c = \frac{275}{68}$$

$$\therefore y = \frac{1}{3}\left(-\frac{9}{68}\right)x^3 + \frac{2}{3}\left(\frac{63}{136}\right)(x-1)^{\frac{3}{2}} + \frac{275}{68}$$

$$\therefore y = -\frac{3}{68}x^3 + \frac{21}{68}(x-1)^{\frac{3}{2}} + \frac{275}{68}$$

$$10 \quad f'(x) = \frac{3}{\sqrt{4-3x}} = 3(4-3x)^{-\frac{1}{2}}$$

$$\therefore f(x) = \int 3(4-3x)^{-\frac{1}{2}} dx$$

$$= 3\left(\frac{1}{-\frac{1}{2}}\right) \frac{(4-3x)^{\frac{1}{2}}}{\frac{1}{2}} + c$$

$$= -2\sqrt{4-3x} + c$$

$$\text{But } f(-4) = 0, \text{ so } -2\sqrt{4-3(-4)} + c = 0$$

$$\therefore -2\sqrt{16} + c = 0$$

$$\therefore -8 + c = 0$$

$$\therefore c = 8$$

$$\therefore f(x) = -2\sqrt{4-3x} + 8$$

$$11 \quad (\sin x - \cos x)^2 = \sin^2 x - 2\sin x \cos x + \cos^2 x \\ = 1 - \sin 2x \quad \{ \sin^2 x + \cos^2 x = 1, \quad \sin 2x = 2\sin x \cos x \}$$

$$\therefore \int (\sin x - \cos x)^2 dx = \int (1 - \sin 2x) dx \\ = x + \frac{1}{2} \cos 2x + c$$

$$\begin{aligned} \mathbf{12} \quad \mathbf{a} \quad \int \frac{1}{3-2x} dx &= \frac{1}{-2} \ln |3-2x| + c \\ &= -\frac{1}{2} \ln |3-2x| + c \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad \int \frac{4}{5x+1} dx &= 4 \int \frac{1}{5x+1} dx \\ &= \frac{4}{5} \ln |5x+1| + c \end{aligned}$$

$$\mathbf{13} \quad \frac{d}{dx} ((3x^2 + x)^3) = 3(3x^2 + x)^2(6x + 1)$$

$$\therefore \int 3(3x^2 + x)^2(6x + 1) dx = (3x^2 + x)^3 + c$$

$$\therefore 3 \int (3x^2 + x)^2(6x + 1) dx = (3x^2 + x)^3 + c$$

$$\therefore \int (3x^2 + x)^2(6x + 1) dx = \frac{1}{3}(3x^2 + x)^3 + c$$

$$\begin{aligned} \mathbf{14} \quad \mathbf{a} \quad \int \frac{2x}{\sqrt{x^2 - 5}} dx \\ &= \int 2x(x^2 - 5)^{-\frac{1}{2}} dx \\ &= \int u^{-\frac{1}{2}} \frac{du}{dx} dx \\ &\quad \{u = x^2 - 5, \quad \frac{du}{dx} = 2x\} \\ &= \int u^{-\frac{1}{2}} du \\ &= \frac{u^{\frac{1}{2}}}{\frac{1}{2}} + c \\ &= 2\sqrt{u} + c \\ &= 2\sqrt{x^2 - 5} + c \end{aligned}$$

$$\begin{aligned} \mathbf{c} \quad \int 4xe^{-x^2} dx \\ &= -2 \int -2xe^{-x^2} dx \\ &= -2 \int e^u \frac{du}{dx} dx \\ &\quad \{u = -x^2, \quad \frac{du}{dx} = -2x\} \\ &= -2 \int e^u du \\ &= -2e^u + c \\ &= -2e^{-x^2} + c \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad \int \frac{\sin x}{\cos^4 x} dx \\ &= - \int \frac{-\sin x}{\cos^4 x} dx \\ &= - \int u^{-4} \frac{du}{dx} dx \\ &\quad \{u = \cos x, \quad \frac{du}{dx} = -\sin x\} \\ &= - \int u^{-4} du \\ &= -\frac{u^{-3}}{-3} + c \\ &= \frac{1}{3\cos^3 x} + c \end{aligned}$$

$$\begin{aligned} \mathbf{d} \quad \int \sin^3 x dx \\ &= \int \sin^2 x \sin x dx \\ &= \int (1 - \cos^2 x) \sin x dx \\ &= - \int (1 - \cos^2 x)(-\sin x) dx \\ &= - \int (1 - u^2) \frac{du}{dx} dx \\ &\quad \{u = \cos x, \quad \frac{du}{dx} = -\sin x\} \\ &= - \int (1 - u^2) du \\ &= -(u - \frac{1}{3}u^3 + c) \\ &= -u + \frac{1}{3}u^3 + c \\ &= -\cos x + \frac{1}{3}\cos^3 x + c \end{aligned}$$

$$\begin{aligned}
 \mathbf{15} \quad \mathbf{a} \quad & \int \frac{x}{x^2 - 9} dx \\
 &= \frac{1}{2} \int \frac{2x}{x^2 - 9} dx \\
 &= \frac{1}{2} \int \frac{1}{u} \frac{du}{dx} dx \\
 &\quad \{u = x^2 - 9, \quad \frac{du}{dx} = 2x\} \\
 &= \frac{1}{2} \int \frac{1}{u} du \\
 &= \frac{1}{2} \ln |u| + c \\
 &= \frac{1}{2} \ln |x^2 - 9| + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad & \int \frac{x}{x^2 - 9} dx \\
 & \text{If } x = 3 \sin t, \text{ then } \frac{dx}{dt} = 3 \cos t \\
 & \therefore \int \frac{x}{x^2 - 9} dx \\
 &= \int \frac{3 \sin t}{(3 \sin t)^2 - 9} \frac{dx}{dt} dt \\
 &= \int \frac{3 \sin t}{9 \sin^2 t - 9} (3 \cos t) dt \\
 &= \int \frac{9 \sin t \cos t}{-9(1 - \sin^2 t)} dt \\
 &= \int \frac{\sin t \cos t}{-\cos^2 t} dt \\
 &= \int -\frac{\sin t}{\cos t} dt \\
 &= \int \frac{1}{u} \frac{du}{dt} dt \quad \{u = \cos t, \quad \frac{du}{dt} = -\sin t\} \\
 &= \int \frac{1}{u} du \\
 &= \ln |u| + c \\
 &= \ln |\cos t| + c
 \end{aligned}$$

Now, $x = 3 \sin t$

$$\therefore \frac{x}{3} = \sin t$$

$$\therefore t = \sin^{-1}\left(\frac{x}{3}\right)$$

$$\therefore \int \frac{x}{x^2 - 9} dx = \ln \left| \cos\left(\sin^{-1}\left(\frac{x}{3}\right)\right) \right| + c$$

Chapter 17

DEFINITE INTEGRALS

EXERCISE 17A

$$\begin{aligned}
 \mathbf{1} \quad \mathbf{a} \quad \int_1^4 \sqrt{x} \, dx &= \int_1^4 x^{\frac{1}{2}} \, dx & \int_1^4 (-\sqrt{x}) \, dx &= \int_1^4 (-x^{\frac{1}{2}}) \, dx \\
 &= \left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}} \right]_1^4 & &= \left[-\frac{x^{\frac{3}{2}}}{\frac{3}{2}} \right]_1^4 \\
 &= \left[\frac{2}{3} x^{\frac{3}{2}} \right]_1^4 & &= \left[-\frac{2}{3} x^{\frac{3}{2}} \right]_1^4 \\
 &= \frac{2}{3}(8) - \frac{2}{3}(1) & &= -\frac{2}{3}(8) - \left(-\frac{2}{3}(1)\right) \\
 &= \frac{14}{3} & &= -\frac{14}{3}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad \int_0^1 x^7 \, dx &= \left[\frac{x^8}{8} \right]_0^1 & \int_0^1 (-x^7) \, dx &= \left[-\frac{x^8}{8} \right]_0^1 \\
 &= \frac{1}{8} - 0 & &= -\frac{1}{8} - 0 \\
 &= \frac{1}{8} & &= -\frac{1}{8}
 \end{aligned}$$

Property: $\int_a^b [-f(x)] \, dx = -\int_a^b f(x) \, dx$

$$\begin{aligned}
 \mathbf{2} \quad \mathbf{a} \quad \int_0^1 x^2 \, dx &= \left[\frac{x^3}{3} \right]_0^1 & \mathbf{b} \quad \int_1^2 x^2 \, dx &= \left[\frac{x^3}{3} \right]_1^2 \\
 &= \frac{1}{3} - 0 & &= \frac{8}{3} - \frac{1}{3} \\
 &= \frac{1}{3} & &= \frac{7}{3} \\
 \mathbf{c} \quad \int_0^2 x^2 \, dx &= \left[\frac{x^3}{3} \right]_0^2 & \mathbf{d} \quad \int_0^1 3x^2 \, dx &= \left[x^3 \right]_0^1 \\
 &= \frac{8}{3} - 0 & &= 1 - 0 \\
 &= \frac{8}{3} & &= 1
 \end{aligned}$$

Properties: $\int_a^b f(x) \, dx + \int_b^c f(x) \, dx = \int_a^c f(x) \, dx$

$\int_a^b c f(x) \, dx = c \int_a^b f(x) \, dx$, where c is a constant

$$\begin{aligned}
 \text{3 a } \int_0^2 (x^3 - 4x) \, dx &= \left[\frac{x^4}{4} - 2x^2 \right]_0^2 \\
 &= \left(\frac{16}{4} - 2(4) \right) - (0 - 0) \\
 &= -4
 \end{aligned}$$

$$\begin{aligned}
 \text{c } \int_0^3 (x^3 - 4x) \, dx &= \left[\frac{x^4}{4} - 2x^2 \right]_0^3 \\
 &= \left(\frac{81}{4} - 2(9) \right) - (0 - 0) \\
 &= \frac{9}{4} \\
 &= 2\frac{1}{4}
 \end{aligned}$$

$$\text{Property: } \int_a^b f(x) \, dx + \int_b^c f(x) \, dx = \int_a^c f(x) \, dx$$

$$\begin{aligned}
 \text{4 a } \int_0^1 x^2 \, dx &= \left[\frac{x^3}{3} \right]_0^1 \\
 &= \frac{1}{3}(1) - 0 \\
 &= \frac{1}{3}
 \end{aligned}$$

$$\begin{aligned}
 \text{c } \int_0^1 (x^2 + \sqrt{x}) \, dx &= \int_0^1 (x^2 + x^{\frac{1}{2}}) \, dx \\
 &= \left[\frac{x^3}{3} + \frac{x^{\frac{3}{2}}}{\frac{3}{2}} \right]_0^1 \\
 &= \left[\frac{x^3}{3} + \frac{2}{3}x^{\frac{3}{2}} \right]_0^1 \\
 &= \left(\frac{1}{3} + \frac{2}{3}(1) \right) - (0 + 0) \\
 &= 1
 \end{aligned}$$

$$\text{Property: } \int_a^b f(x) \, dx + \int_a^b g(x) \, dx = \int_a^b [f(x) + g(x)] \, dx$$

$$\begin{aligned}
 \text{5 a } \int_0^1 x^3 \, dx &= \left[\frac{x^4}{4} \right]_0^1 \\
 &= \frac{1}{4} - 0 \\
 &= \frac{1}{4}
 \end{aligned}$$

$$\begin{aligned}
 \text{b } \int_2^3 (x^3 - 4x) \, dx &= \left[\frac{x^4}{4} - 2x^2 \right]_2^3 \\
 &= \left(\frac{81}{4} - 2(9) \right) - \left(\frac{16}{4} - 2(4) \right) \\
 &= \frac{25}{4} \\
 &= 6\frac{1}{4}
 \end{aligned}$$

$$\begin{aligned}
 \text{b } \int_0^1 \sqrt{x} \, dx &= \int_0^1 x^{\frac{1}{2}} \, dx \\
 &= \left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}} \right]_0^1 \\
 &= \left[\frac{2}{3}x^{\frac{3}{2}} \right]_0^1 \\
 &= \frac{2}{3}(1) - 0 \\
 &= \frac{2}{3}
 \end{aligned}$$

$$\begin{aligned}
 \text{b } \int_0^2 (x^2 - x) \, dx &= \left[\frac{x^3}{3} - \frac{x^2}{2} \right]_0^2 \\
 &= \left(\frac{8}{3} - 2 \right) - (0 - 0) \\
 &= \frac{2}{3}
 \end{aligned}$$

$$\begin{aligned}
 \text{c} \quad & \int_0^2 (3x^2 - x + 6) dx \\
 &= \left[\frac{3x^3}{3} - \frac{x^2}{2} + 6x \right]_0^2 \\
 &= \left[x^3 - \frac{x^2}{2} + 6x \right]_0^2 \\
 &= (8 - 2 + 12) - 0 \\
 &= 18
 \end{aligned}$$

$$\begin{aligned}
 \text{e} \quad & \int_1^4 (x + 2\sqrt{x}) dx \\
 &= \int_1^4 (x + 2x^{\frac{1}{2}}) dx \\
 &= \left[\frac{x^2}{2} + \frac{2x^{\frac{3}{2}}}{\frac{3}{2}} \right]_1^4 \\
 &= \left[\frac{x^2}{2} + \frac{4}{3}x^{\frac{3}{2}} \right]_1^4 \\
 &= \left(\frac{16}{2} + \frac{4}{3}(8) \right) - \left(\frac{1}{2} + \frac{4}{3} \right) \\
 &= 16\frac{5}{6}
 \end{aligned}$$

$$\begin{aligned}
 \text{g} \quad & \int_1^3 \frac{1}{x^2} dx = \int_1^3 x^{-2} dx \\
 &= \left[\frac{x^{-1}}{-1} \right]_1^3 \\
 &= \left[-\frac{1}{x} \right]_1^3 \\
 &= -\frac{1}{3} - (-1) \\
 &= \frac{2}{3}
 \end{aligned}$$

$$\begin{aligned}
 \text{i} \quad & \int_1^4 \left(x^2 + \frac{1}{x} \right) dx = \left[\frac{x^3}{3} + \ln|x| \right]_1^4 \\
 &= \left(\frac{64}{3} + \ln 4 \right) - \left(\frac{1}{3} + \ln 1 \right) \\
 &= 21 + \ln 4
 \end{aligned}$$

$$\begin{aligned}
 \text{d} \quad & \int_1^4 \left(x - \frac{3}{\sqrt{x}} \right) dx \\
 &= \int_1^4 (x - 3x^{-\frac{1}{2}}) dx \\
 &= \left[\frac{x^2}{2} - \frac{3x^{\frac{1}{2}}}{\frac{1}{2}} \right]_1^4 \\
 &= \left[\frac{x^2}{2} - 6\sqrt{x} \right]_1^4 \\
 &= \left(\frac{16}{2} - 12 \right) - \left(\frac{1}{2} - 6 \right) \\
 &= 1\frac{1}{2}
 \end{aligned}$$

$$\begin{aligned}
 \text{f} \quad & \int_4^9 \frac{x-3}{\sqrt{x}} dx \\
 &= \int_4^9 (x^{\frac{1}{2}} - 3x^{-\frac{1}{2}}) dx \\
 &= \left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}} - \frac{3x^{\frac{1}{2}}}{\frac{1}{2}} \right]_4^9 \\
 &= \left[\frac{2}{3}x^{\frac{3}{2}} - 6\sqrt{x} \right]_4^9 \\
 &= \left(\frac{2}{3}(27) - 6(3) \right) - \left(\frac{2}{3}(8) - 6(2) \right) \\
 &= (18 - 18) - \left(\frac{16}{3} - 12 \right) \\
 &= 6\frac{2}{3}
 \end{aligned}$$

$$\begin{aligned}
 \text{h} \quad & \int_1^2 (x+3)^2 dx \\
 &= \int_1^2 (x^2 + 6x + 9) dx \\
 &= \left[\frac{x^3}{3} + \frac{6x^2}{2} + 9x \right]_1^2 \\
 &= \left[\frac{x^3}{3} + 3x^2 + 9x \right]_1^2 \\
 &= \left(\frac{8}{3} + 12 + 18 \right) - \left(\frac{1}{3} + 3 + 9 \right) \\
 &= 20\frac{1}{3}
 \end{aligned}$$

$$\begin{aligned}
6 \quad & \int_m^{2m} (2x - 1) dx = 4 \\
& \therefore [x^2 - x]_m^{2m} = 4 \\
& \therefore (4m^2 - 2m) - (m^2 - m) = 4 \\
& \therefore 3m^2 - m - 4 = 0 \\
& \therefore (3m - 4)(m + 1) = 0 \\
& \therefore m = -1 \text{ or } \frac{4}{3}
\end{aligned}$$

$$\begin{aligned}
7 \quad a \quad & \int_0^1 (3x + 1)^4 dx \\
& = \left[\frac{1}{3} \frac{(3x + 1)^5}{5} \right]_0^1 \\
& = \left[\frac{(3x + 1)^5}{15} \right]_0^1 \\
& = \frac{4^5}{15} - \frac{1^5}{15} \\
& = \frac{1024}{15} - \frac{1}{15} \\
& = \frac{1023}{15} = 68\frac{1}{5}
\end{aligned}$$

$$\begin{aligned}
c \quad & \int_{-3}^0 \sqrt{1-x} dx = \int_{-3}^0 (1-x)^{\frac{1}{2}} dx \\
& = \left[\left(\frac{1}{-1} \right) \frac{(1-x)^{\frac{3}{2}}}{\frac{3}{2}} \right]_{-3}^0 \\
& = \left[-\frac{2}{3} (1-x)^{\frac{3}{2}} \right]_{-3}^0 \\
& = \left(-\frac{2}{3} (1)^{\frac{3}{2}} \right) - \left(-\frac{2}{3} (4)^{\frac{3}{2}} \right) \\
& = -\frac{2}{3} - \left(-\frac{16}{3} \right) \\
& = \frac{14}{3} = 4\frac{2}{3}
\end{aligned}$$

$$\begin{aligned}
8 \quad a \quad & \int_0^1 e^x dx = [e^x]_0^1 \\
& = e^1 - e^0 \\
& = e - 1
\end{aligned}$$

$$\begin{aligned}
c \quad & \int_0^2 e^{3x} dx = \left[\frac{1}{3} e^{3x} \right]_0^2 \\
& = \frac{1}{3} e^6 - \frac{1}{3} e^0 \\
& = \frac{1}{3} e^6 - \frac{1}{3} \\
& = \frac{1}{3} (e^6 - 1)
\end{aligned}$$

$$\begin{aligned}
b \quad & \int_2^6 \frac{1}{\sqrt{2x-3}} dx \\
& = \int_2^6 (2x-3)^{-\frac{1}{2}} dx \\
& = \left[\frac{1}{2} \frac{(2x-3)^{\frac{1}{2}}}{\frac{1}{2}} \right]_2^6 \\
& = [\sqrt{2x-3}]_2^6 \\
& = \sqrt{9} - \sqrt{1} \\
& = 2
\end{aligned}$$

$$\begin{aligned}
b \quad & \int_0^3 (2e^x - 3) dx = [2e^x - 3x]_0^3 \\
& = (2e^3 - 9) - (2 - 0) \\
& = 2e^3 - 11
\end{aligned}$$

$$\begin{aligned}
d \quad & \int_0^1 e^{1-x} dx = \left[\left(\frac{1}{-1} \right) e^{1-x} \right]_0^1 \\
& = [-e^{1-x}]_0^1 \\
& = -e^0 - (-e^1) \\
& = e - 1
\end{aligned}$$

$$\begin{aligned}
 \text{e} \quad \int_0^{\ln 4} e^x(e^x - 2) dx &= \int_0^{\ln 4} (e^{2x} - 2e^x) dx \\
 &= \left[\frac{1}{2}e^{2x} - 2e^x \right]_0^{\ln 4} \\
 &= \left(\frac{1}{2}e^{2\ln 4} - 2e^{\ln 4} \right) - \left(\frac{1}{2}e^0 - 2e^0 \right) \\
 &= \left(\frac{1}{2}e^{\ln 16} - 2e^{\ln 4} \right) - \left(\frac{1}{2} - 2 \right) \\
 &= \left(\frac{1}{2}(16) - 2(4) \right) - \left(-\frac{3}{2} \right) \\
 &= \frac{3}{2}
 \end{aligned}$$

$$\begin{aligned}
 \text{f} \quad \int_1^2 (e^{-x} + 1)^2 dx &= \int_1^2 (e^{-2x} + 2e^{-x} + 1) dx \\
 &= \left[\left(-\frac{1}{2} \right) e^{-2x} + 2 \left(-\frac{1}{1} \right) e^{-x} + x \right]_1^2 \\
 &= \left[-\frac{e^{-2x}}{2} - 2e^{-x} + x \right]_1^2 \\
 &= \left(-\frac{e^{-4}}{2} - 2e^{-2} + 2 \right) - \left(-\frac{e^{-2}}{2} - 2e^{-1} + 1 \right) \\
 &= -\frac{1}{2e^4} - \frac{3}{2e^2} + \frac{2}{e} + 1
 \end{aligned}$$

$$\begin{aligned}
 \text{9 a} \quad \int_0^\pi \sin x dx &= [-\cos x]_0^\pi \\
 &= (-\cos \pi) - (-\cos 0) \\
 &= 1 - (-1) \\
 &= 2
 \end{aligned}$$

$$\begin{aligned}
 \text{b} \quad \int_0^{\frac{\pi}{6}} \cos x dx &= [\sin x]_0^{\frac{\pi}{6}} \\
 &= \sin \frac{\pi}{6} - \sin 0 \\
 &= \frac{1}{2} - 0 \\
 &= \frac{1}{2}
 \end{aligned}$$

$$\begin{aligned}
 \text{c} \quad \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \sin x dx &= [-\cos x]_{\frac{\pi}{3}}^{\frac{\pi}{2}} \\
 &= (-\cos \frac{\pi}{2}) - (-\cos \frac{\pi}{3}) \\
 &= 0 - (-\frac{1}{2}) \\
 &= \frac{1}{2}
 \end{aligned}$$

$$\begin{aligned}
 \text{d} \quad \int_0^{\frac{\pi}{6}} \sin 3x dx &= \left[-\frac{1}{3} \cos 3x \right]_0^{\frac{\pi}{6}} \\
 &= \left(-\frac{1}{3} \cos \frac{\pi}{2} \right) - \left(-\frac{1}{3} \cos 0 \right) \\
 &= 0 - \left(-\frac{1}{3} \right) \\
 &= \frac{1}{3}
 \end{aligned}$$

$$\begin{aligned}
 \text{e} \quad \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \cos \left(x - \frac{\pi}{3} \right) dx &= \left[\sin \left(x - \frac{\pi}{3} \right) \right]_{\frac{\pi}{6}}^{\frac{\pi}{2}} \\
 &= \sin \left(\frac{\pi}{2} - \frac{\pi}{3} \right) - \sin \left(\frac{\pi}{6} - \frac{\pi}{3} \right) \\
 &= \sin \frac{\pi}{6} - \sin \left(-\frac{\pi}{6} \right) \\
 &= \frac{1}{2} - \left(-\frac{1}{2} \right) \\
 &= 1
 \end{aligned}$$

$$\begin{aligned}
 \text{f} \quad \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin \left(2x - \frac{\pi}{4} \right) dx &= \left[-\frac{1}{2} \cos \left(2x - \frac{\pi}{4} \right) \right]_{\frac{\pi}{4}}^{\frac{\pi}{2}} \\
 &= \left(-\frac{1}{2} \cos \frac{3\pi}{4} \right) - \left(-\frac{1}{2} \cos \frac{\pi}{4} \right) \\
 &= \frac{1}{2\sqrt{2}} + \frac{1}{2\sqrt{2}} \\
 &= \frac{1}{\sqrt{2}}
 \end{aligned}$$

$$\mathbf{10} \quad \cos^2 x = \frac{1}{2} + \frac{1}{2} \cos 2x$$

$$\begin{aligned} \therefore \int_0^{\frac{\pi}{4}} \cos^2 x \, dx &= \int_0^{\frac{\pi}{4}} \left(\frac{1}{2} + \frac{1}{2} \cos 2x \right) dx \\ &= \left[\frac{1}{2}x + \frac{1}{4} \sin 2x \right]_0^{\frac{\pi}{4}} \\ &= \left(\frac{\pi}{8} + \frac{1}{4}(1) \right) - \left(0 + \frac{1}{4}(0) \right) \\ &= \frac{\pi}{8} + \frac{1}{4} \end{aligned}$$

$$\mathbf{11} \quad \sin^2 x = \frac{1}{2} - \frac{1}{2} \cos 2x$$

$$\begin{aligned} \therefore \int_0^{\frac{\pi}{2}} \sin^2 x \, dx &= \int_0^{\frac{\pi}{2}} \left(\frac{1}{2} - \frac{1}{2} \cos 2x \right) dx \\ &= \left[\frac{1}{2}x - \frac{1}{4} \sin 2x \right]_0^{\frac{\pi}{2}} \\ &= \left(\frac{\pi}{4} - \frac{1}{4}(0) \right) - \left(0 - \frac{1}{4}(0) \right) \\ &= \frac{\pi}{4} \end{aligned}$$

$$\begin{aligned} \mathbf{12} \quad \int_0^{\frac{\pi}{6}} (\sin 3x - \cos x) \, dx &= \left[-\frac{1}{3} \cos 3x - \sin x \right]_0^{\frac{\pi}{6}} \\ &= \left(-\frac{1}{3}(0) - \frac{1}{2} \right) - \left(-\frac{1}{3}(1) - 0 \right) \\ &= -\frac{1}{6} \end{aligned}$$

$$\begin{aligned} \mathbf{13} \quad \int_3^{12} \frac{1}{x} \, dx &= [\ln |x|]_3^{12} \\ &= \ln 12 - \ln 3 \\ &= \ln \left(\frac{12}{3} \right) \quad \left\{ \ln a - \ln b = \ln \left(\frac{a}{b} \right) \right\} \\ &= \ln 4 \end{aligned}$$

$$\begin{aligned} \mathbf{14} \quad \mathbf{a} \quad \int_{-6}^{-2} \frac{1}{x} \, dx &= [\ln |x|]_{-6}^{-2} \\ &= \ln 2 - \ln 6 \\ &= \ln \left(\frac{2}{6} \right) \quad \left\{ \ln a - \ln b = \ln \left(\frac{a}{b} \right) \right\} \\ &= \ln \left(\frac{1}{3} \right) \\ &= \ln(3^{-1}) \\ &= -\ln 3 \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad \int_{-1}^5 \frac{1}{x+4} \, dx &= [\ln |x+4|]_{-1}^5 \\ &= \ln 9 - \ln 3 \\ &= \ln \left(\frac{9}{3} \right) \quad \left\{ \ln a - \ln b = \ln \left(\frac{a}{b} \right) \right\} \\ &= \ln 3 \end{aligned}$$

$$\begin{aligned}
\text{c} \quad & \int_1^8 \frac{2}{3x+4} dx \\
&= \left[2\left(\frac{1}{3}\right) \ln |3x+4| \right]_1^8 \\
&= \left[\frac{2}{3} \ln |3x+4| \right]_1^8 \\
&= \frac{2}{3} \ln 28 - \frac{2}{3} \ln 7 \\
&= \frac{2}{3} (\ln 28 - \ln 7) \\
&= \frac{2}{3} \ln \left(\frac{28}{7} \right) \quad \left\{ \ln a - \ln b = \ln \left(\frac{a}{b} \right) \right\} \\
&= \frac{2}{3} \ln 4 \\
&= \frac{4}{3} \ln 2
\end{aligned}$$

$$\begin{aligned}
\text{d} \quad & \int_{-4}^0 \frac{4}{5-2x} dx \\
&= \left[4\left(\frac{1}{-2}\right) \ln |5-2x| \right]_{-4}^0 \\
&= \left[-2 \ln |5-2x| \right]_{-4}^0 \\
&= -2 \ln 5 - (-2 \ln 13) \\
&= -2 \ln 5 + 2 \ln 13 \\
&= 2(\ln 13 - \ln 5) \\
&= 2 \ln \left(\frac{13}{5} \right) \quad \left\{ \ln a - \ln b = \ln \left(\frac{a}{b} \right) \right\}
\end{aligned}$$

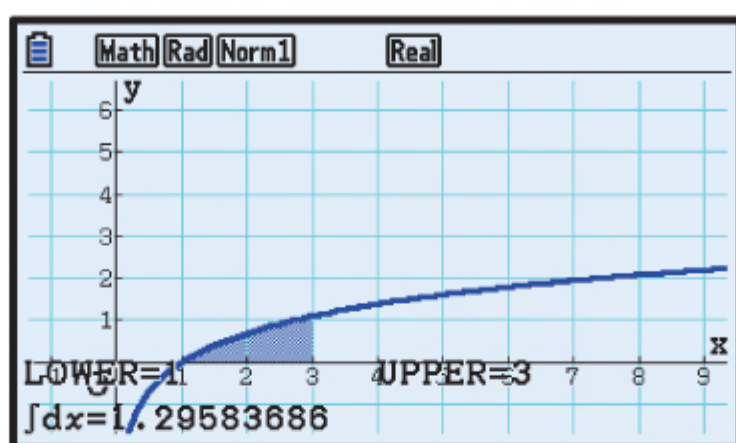
$$\begin{aligned}
15 \quad & \int_m^{-2} \frac{1}{4-x} dx = \ln \left(\frac{3}{2} \right) \\
& \therefore \left[-\ln |4-x| \right]_m^{-2} = \ln \left(\frac{3}{2} \right) \\
& \therefore -\ln |4-(-2)| + \ln |4-m| = \ln \left(\frac{3}{2} \right) \\
& \therefore \ln |4-m| - \ln 6 = \ln \left(\frac{3}{2} \right) \\
& \therefore \ln \left| \frac{4-m}{6} \right| = \ln \left(\frac{3}{2} \right) \\
& \therefore \left| \frac{4-m}{6} \right| = \frac{3}{2} \\
& \therefore \frac{4-m}{6} = \pm \frac{3}{2} \\
& \therefore 4-m = \pm 9 \\
& \therefore m = 4 \pm 9 \\
& \therefore m = -5 \text{ or } 13
\end{aligned}$$

However, the solution $m = 13$ is invalid, since the vertical asymptote $x = 4$ lies between -2 and 13 .

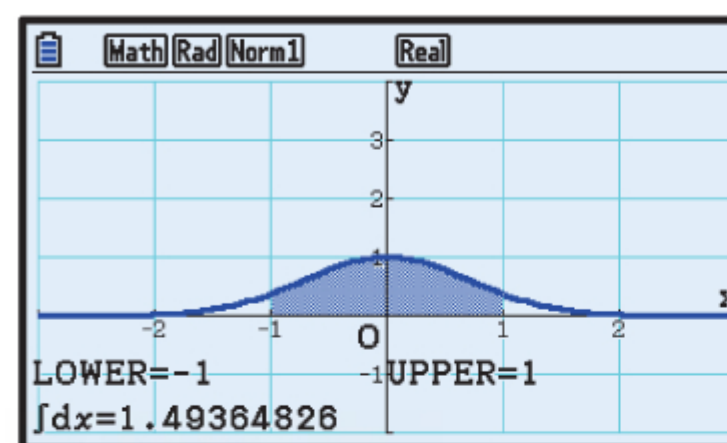
$\therefore m = -5$ is the only valid answer.

$$\begin{aligned}
16 \quad & \frac{4x+1}{x-1} = \frac{4x-4+1+4}{x-1} \\
&= \frac{4(x-1)+5}{x-1} \\
&= \frac{4\cancel{(x-1)}}{\cancel{x-1}} + \frac{5}{x-1} \\
&= 4 + \frac{5}{x-1} \quad \text{as required}
\end{aligned}$$

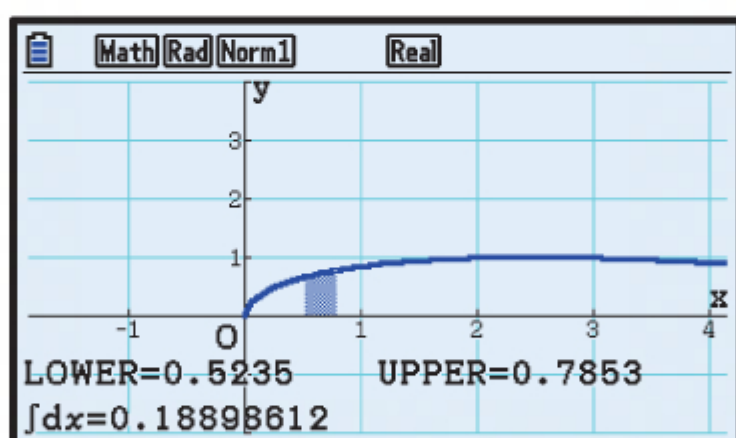
$$\begin{aligned}
& \therefore \int_3^5 \frac{4x+1}{x-1} dx \\
&= \int_3^5 \left(4 + \frac{5}{x-1} \right) dx \\
&= \left[4x + 5 \ln |x-1| \right]_3^5 \\
&= (4(5) + 5 \ln |5-1|) - (4(3) + 5 \ln |3-1|) \\
&= 20 + 5 \ln 4 - 12 - 5 \ln 2 \\
&= 20 + 5 \ln(2^2) - 12 - 5 \ln 2 \\
&= 8 + 10 \ln 2 - 5 \ln 2 \\
&= 8 + 5 \ln 2
\end{aligned}$$

17 a

$$\therefore \int_1^3 \ln x \, dx \approx 1.30$$

b

$$\therefore \int_{-1}^1 e^{-x^2} \, dx \approx 1.49$$

c

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \sin(\sqrt{x}) \, dx \approx 0.189$$

$$\therefore \int_{\frac{\pi}{4}}^{\frac{\pi}{6}} \sin(\sqrt{x}) \, dx \approx -0.189$$

18 Suppose $f(x)$ and $g(x)$ are continuous functions with antiderivatives $F(x)$ and $G(x)$ respectively.

$$\begin{aligned} \mathbf{a} \quad \int_a^b f(x) \, dx &= F(b) - F(a) \\ &= -(F(a) - F(b)) \\ &= - \int_b^a f(x) \, dx \end{aligned}$$

b $\frac{d}{dx}(kF(x)) = k f(x)$, so $kF(x)$ is the antiderivative of $k f(x)$.

$$\begin{aligned} \int_a^b k f(x) \, dx &= k F(b) - k F(a) \quad \{\text{since } k \text{ is a constant}\} \\ &= k(F(b) - F(a)) \\ &= k \int_a^b f(x) \, dx \end{aligned}$$

$$\begin{aligned} \mathbf{c} \quad \int_a^b f(x) \, dx + \int_b^c f(x) \, dx &= \cancel{F(b)} - F(a) + F(c) - \cancel{F(b)} \\ &= F(c) - F(a) \\ &= \int_a^c f(x) \, dx \end{aligned}$$

d $\frac{d}{dx}(F(x) + G(x)) = f(x) + g(x)$, so $F(x) + G(x)$ is the antiderivative of $f(x) + g(x)$.

$$\begin{aligned}\int_a^b [f(x) + g(x)] dx &= F(b) + G(b) - (F(a) + G(a)) \\ &= F(b) + G(b) - F(a) - G(a) \\ &= F(b) - F(a) + G(b) - G(a) \\ &= \int_a^b f(x) dx + \int_a^b g(x) dx\end{aligned}$$

19 a $\int_2^4 f(x) dx + \int_4^7 f(x) dx = \int_2^7 f(x) dx$

b $\int_4^5 f(x) dx - \int_6^5 f(x) dx = \int_4^5 f(x) dx + \int_5^6 f(x) dx$
 $= \int_4^6 f(x) dx$

c $\int_1^3 g(x) dx + \int_3^8 g(x) dx + \int_8^9 g(x) dx = \int_1^9 g(x) dx$

20 a $\int_1^3 f(x) dx + \int_3^6 f(x) dx = \int_1^6 f(x) dx$
 $\therefore \int_3^6 f(x) dx = \int_1^6 f(x) dx - \int_1^3 f(x) dx$
 $= -3 - 2$
 $= -5$

b $\int_0^2 f(x) dx + \int_2^4 f(x) dx + \int_4^6 f(x) dx = \int_0^6 f(x) dx$
 $\therefore \int_2^4 f(x) dx = \int_0^6 f(x) dx - \int_4^6 f(x) dx - \int_0^2 f(x) dx$
 $= 7 - (-2) - 5$
 $= 4$

21 a $\int_1^{-1} f(x) dx = - \int_{-1}^1 f(x) dx$
 $= -(-4)$
 $= 4$

b $\int_{-1}^1 (2 + f(x)) dx = \int_{-1}^1 2 dx + \int_{-1}^1 f(x) dx$
 $= [2x]_{-1}^1 + (-4)$
 $= (2 - (-2)) - 4$
 $= 0$

$$\begin{array}{ll}
 \text{c} \quad \int_{-1}^1 2f(x) \, dx = 2 \int_{-1}^1 f(x) \, dx & \text{d} \quad \int_{-1}^1 k f(x) \, dx = 7 \\
 = 2(-4) & \therefore k \int_{-1}^1 f(x) \, dx = 7 \\
 = -8 & \therefore k(-4) = 7 \\
 & \therefore k = -\frac{7}{4}
 \end{array}$$

22 Since $\frac{d}{dx}(g(x)) = g'(x)$, $g(x)$ is the antiderivative of $g'(x)$.

$$\begin{aligned}
 \int_2^3 (g'(x) - 1) \, dx &= \int_2^3 g'(x) \, dx + \int_2^3 -1 \, dx \\
 &= [g(x)]_2^3 + [-x]_2^3 \\
 &= (g(3) - g(2)) + (-3 - (-2)) \\
 &= 5 - 4 - 1 \\
 &= 0
 \end{aligned}$$

$$\begin{aligned}
 \text{23 a} \quad \int 2x(x^2 - 1)^3 \, dx &= \int u^3 \frac{du}{dx} \, dx \quad \{u = x^2 - 1, \frac{du}{dx} = 2x\} \\
 &= \int u^3 \, du \\
 &= \frac{u^4}{4} + c \\
 &= \frac{1}{4}(x^2 - 1)^4 + c
 \end{aligned}$$

$$\begin{aligned}
 \text{b} \quad \int_1^2 2x(x^2 - 1)^3 \, dx &= \left[\frac{1}{4}(x^2 - 1)^4 \right]_1^2 \\
 &= \frac{1}{4}(3)^4 - \frac{1}{4}(0)^4 \\
 &= \frac{81}{4}
 \end{aligned}$$

$$\begin{aligned}
 \text{24 a} \quad \int x e^{-2x^2} \, dx &= -\frac{1}{4} \int -4x e^{-2x^2} \, dx \\
 &= -\frac{1}{4} \int e^u \frac{du}{dx} \, dx \quad \{u = -2x^2, \frac{du}{dx} = -4x\} \\
 &= -\frac{1}{4} \int e^u \, du \\
 &= -\frac{1}{4} e^u + c \\
 &= -\frac{1}{4} e^{-2x^2} + c
 \end{aligned}$$

$$\begin{aligned}
 \text{b} \quad \int_1^2 x e^{-2x^2} dx &= \left[-\frac{1}{4} e^{-2x^2} \right]_1^2 \\
 &= -\frac{1}{4} e^{-8} - \left(-\frac{1}{4} e^{-2} \right) \\
 &= -\frac{1}{4e^8} + \frac{1}{4e^2} \times \frac{e^6}{e^6} \\
 &= -\frac{1}{4e^8} + \frac{e^6}{4e^8} \\
 &= \frac{e^6 - 1}{4e^8}
 \end{aligned}$$

$$\begin{aligned}
 \text{25 a} \quad \int \frac{x}{2-x^2} dx &= -\frac{1}{2} \int \frac{-2x}{2-x^2} dx \\
 &= -\frac{1}{2} \int \frac{1}{u} \frac{du}{dx} dx \quad \{u = 2 - x^2, \quad \frac{du}{dx} = -2x\} \\
 &= -\frac{1}{2} \int \frac{1}{u} du \\
 &= -\frac{1}{2} \ln |u| + c \\
 &= -\frac{1}{2} \ln |2 - x^2| + c
 \end{aligned}$$

$$\begin{aligned}
 \text{b} \quad \int_2^3 \frac{x}{2-x^2} dx &= \left[-\frac{1}{2} \ln |2 - x^2| \right]_2^3 \\
 &= -\frac{1}{2} \ln 7 - \left(-\frac{1}{2} \ln 2 \right) \\
 &= -\frac{1}{2} \ln 7 + \frac{1}{2} \ln 2 \\
 &= \frac{1}{2} (\ln 2 - \ln 7) \\
 &= \frac{1}{2} \ln \left(\frac{2}{7} \right) \quad \left\{ \ln a - \ln b = \ln \left(\frac{a}{b} \right) \right\}
 \end{aligned}$$

$$\begin{aligned}
 \text{26 a} \quad \int \frac{x}{(x^2+2)^2} dx &= \frac{1}{2} \int \frac{2x}{(x^2+2)^2} dx \\
 &= \frac{1}{2} \int \frac{1}{u^2} \frac{du}{dx} dx \quad \{u = x^2 + 2, \quad \frac{du}{dx} = 2x\} \\
 &= \frac{1}{2} \int \frac{1}{u^2} du \\
 &= \frac{1}{2} \int u^{-2} du \\
 &= \frac{1}{2} \frac{u^{-1}}{-1} + c \\
 &= -\frac{1}{2u} + c \\
 &= -\frac{1}{2(x^2+2)} + c
 \end{aligned}$$

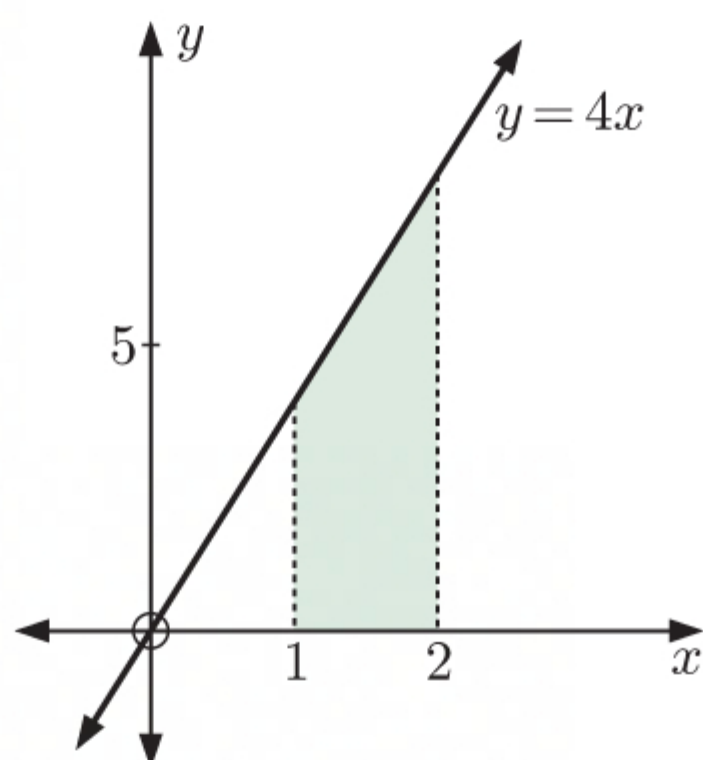
$$\begin{aligned}
 \text{b} \quad \int_1^2 \frac{x}{(x^2+2)^2} dx &= \left[-\frac{1}{2(x^2+2)} \right]_1^2 \\
 &= -\frac{1}{12} - \left(-\frac{1}{6} \right) \\
 &= \frac{1}{12}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{27} \quad \mathbf{a} \quad \int \sin^2 x \cos x \, dx &= \int u^2 \frac{du}{dx} \, dx \quad \{u = \sin x, \quad \frac{du}{dx} = \cos x\} \\
 &= \int u^2 \, du \\
 &= \frac{1}{3} u^3 + c \\
 &= \frac{1}{3} \sin^3 x + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad \int_0^{\frac{\pi}{6}} \sin^2 x \cos x \, dx &= \left[\frac{1}{3} \sin^3 x \right]_0^{\frac{\pi}{6}} \\
 &= \frac{1}{3} \left(\frac{1}{2} \right)^3 - \frac{1}{3} (0)^3 \\
 &= \frac{1}{24}
 \end{aligned}$$

EXERCISE 17B

1 a



When $x = 1$, $y = 4(1) = 4$

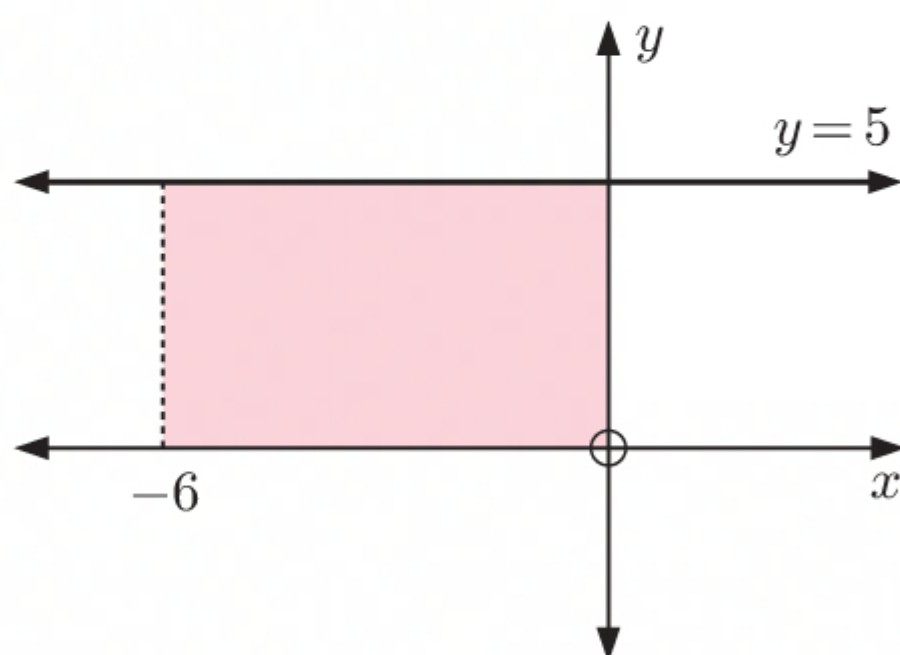
When $x = 2$, $y = 4(2) = 8$

Area = area of trapezium

$$\begin{aligned}
 &= \left(\frac{4 + 8}{2} \right) \times 1 \\
 &= 6 \text{ units}^2
 \end{aligned}$$

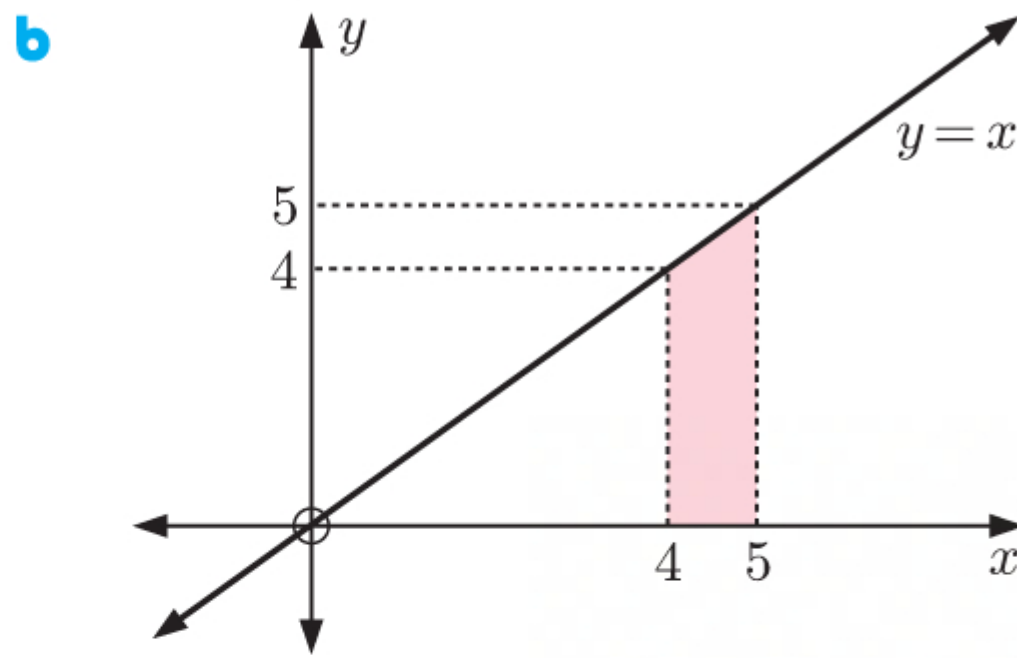
$$\begin{aligned}
 \mathbf{b} \quad \text{Area} &= \int_1^2 4x \, dx \\
 &= [2x^2]_1^2 \\
 &= 2(2)^2 - 2(1)^2 \\
 &= 6 \text{ units}^2
 \end{aligned}$$

2 a



$$\begin{aligned}
 \mathbf{i} \quad \text{Area} &= 6 \times 5 \\
 &= 30 \text{ units}^2
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{ii} \quad \text{Area} &= \int_{-6}^0 5 \, dx \\
 &= [5x]_{-6}^0 \\
 &= 5(0) - 5(-6) \\
 &= 30 \text{ units}^2
 \end{aligned}$$



i Area = area of trapezium

$$= \left(\frac{4+5}{2} \right) \times 1$$

$$= \frac{9}{2}$$

$$= 4\frac{1}{2} \text{ units}^2$$

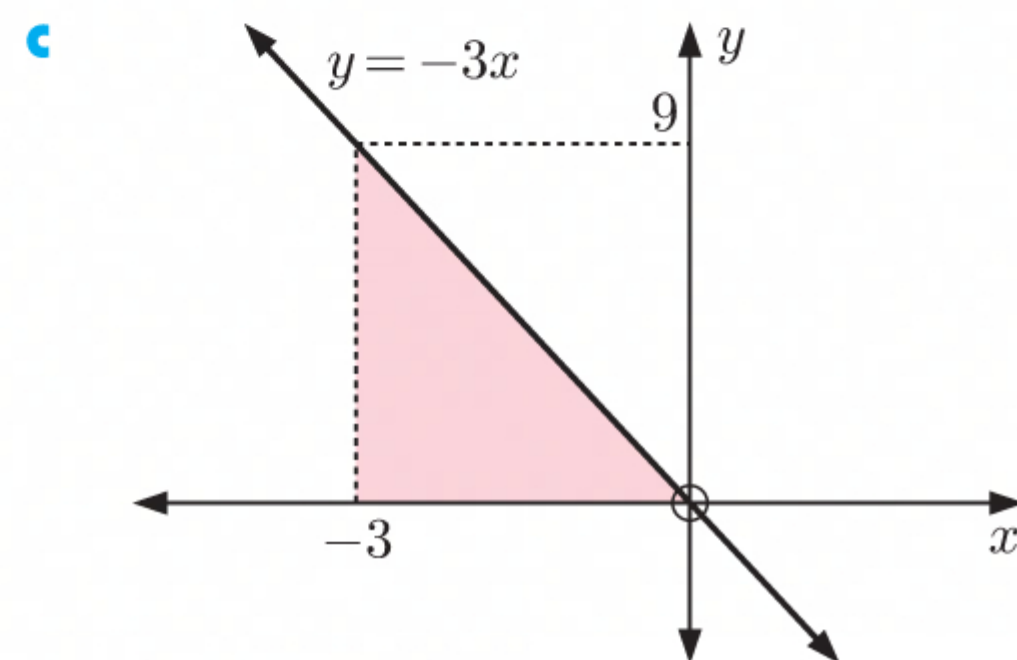
ii Area = $\int_4^5 x \, dx$

$$= \left[\frac{1}{2}x^2 \right]_4^5$$

$$= \frac{1}{2}(5)^2 - \frac{1}{2}(4)^2$$

$$= \frac{25}{2} - \frac{16}{2}$$

$$= 4\frac{1}{2} \text{ units}^2$$



i Area = $\frac{1}{2} \times 3 \times 9$

$$= 13\frac{1}{2} \text{ units}^2$$

ii Area = $\int_{-3}^0 -3x \, dx$

$$= \left[-\frac{3}{2}x^2 \right]_{-3}^0$$

$$= 0 - \left(-\frac{3}{2}(9) \right)$$

$$= 13\frac{1}{2} \text{ units}^2$$

3 a Area of blue shaded region

$$= \int_0^2 2x^2 \, dx$$

$$= \left[\frac{2}{3}x^3 \right]_0^2$$

$$= \frac{16}{3} - 0$$

$$= 5\frac{1}{3} \text{ units}^2$$

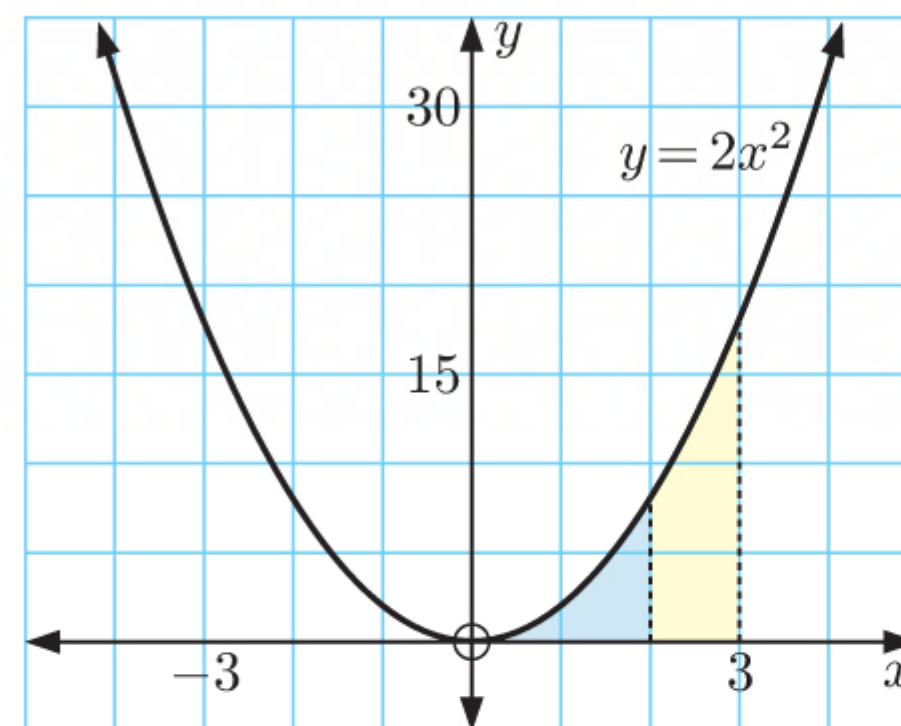
b Area of yellow shaded region

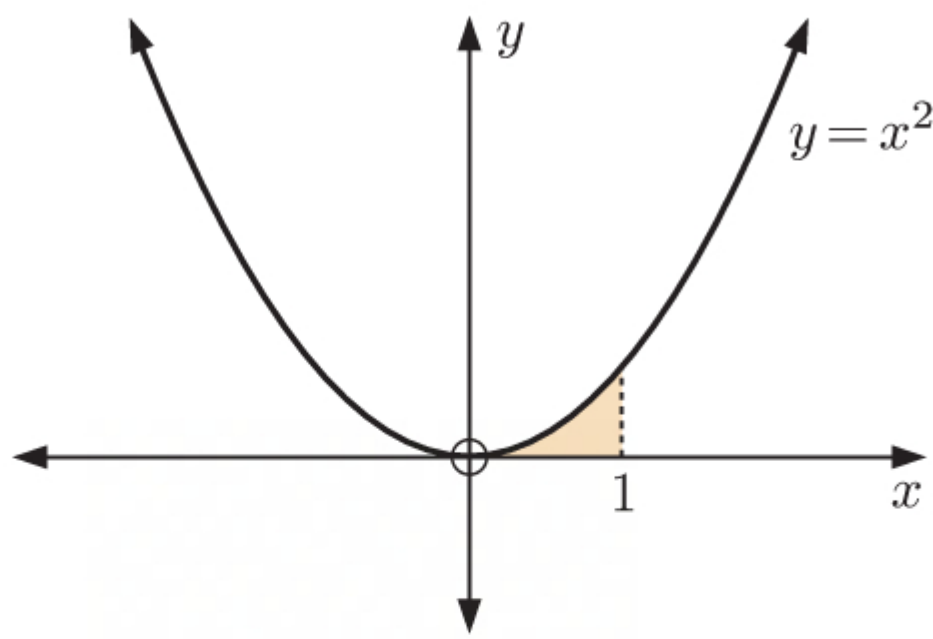
$$= \int_2^3 2x^2 \, dx$$

$$= \left[\frac{2}{3}x^3 \right]_2^3$$

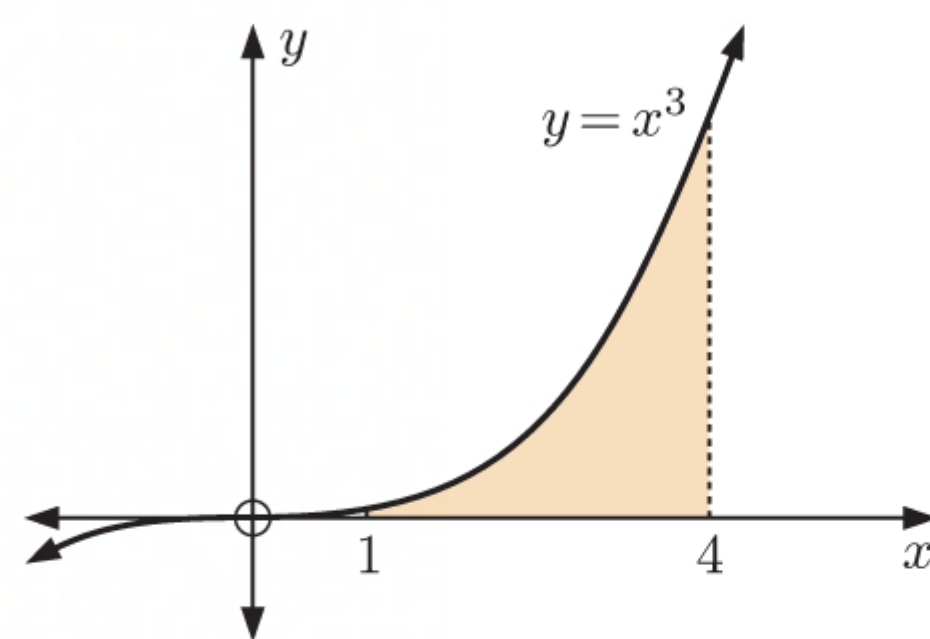
$$= \frac{54}{3} - \frac{16}{3}$$

$$= 12\frac{2}{3} \text{ units}^2$$

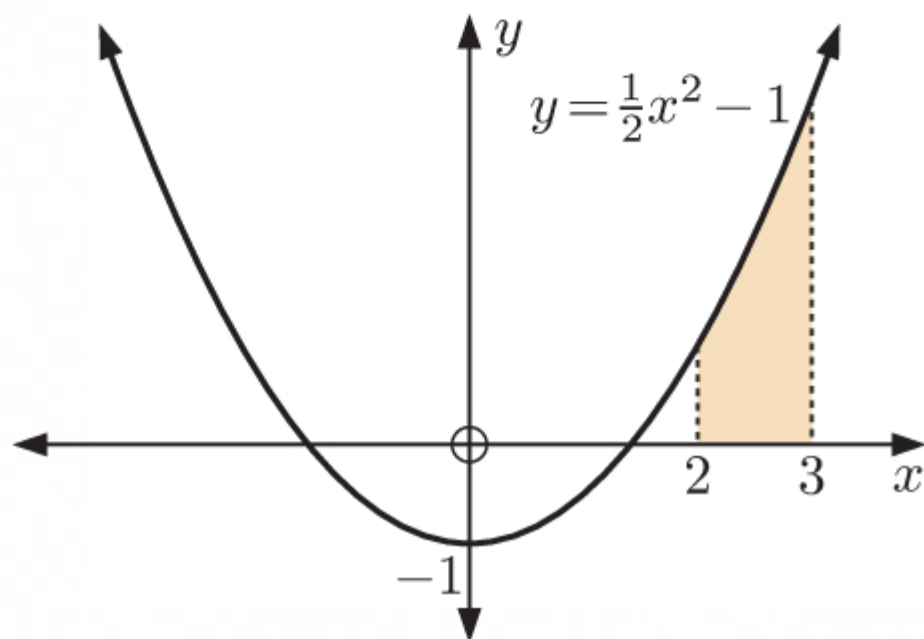


4 a

$$\begin{aligned}
 \text{Area} &= \int_0^1 x^2 \, dx \\
 &= \left[\frac{1}{3}x^3 \right]_0^1 \\
 &= \frac{1}{3} - 0 \\
 &= \frac{1}{3} \text{ units}^2
 \end{aligned}$$

b

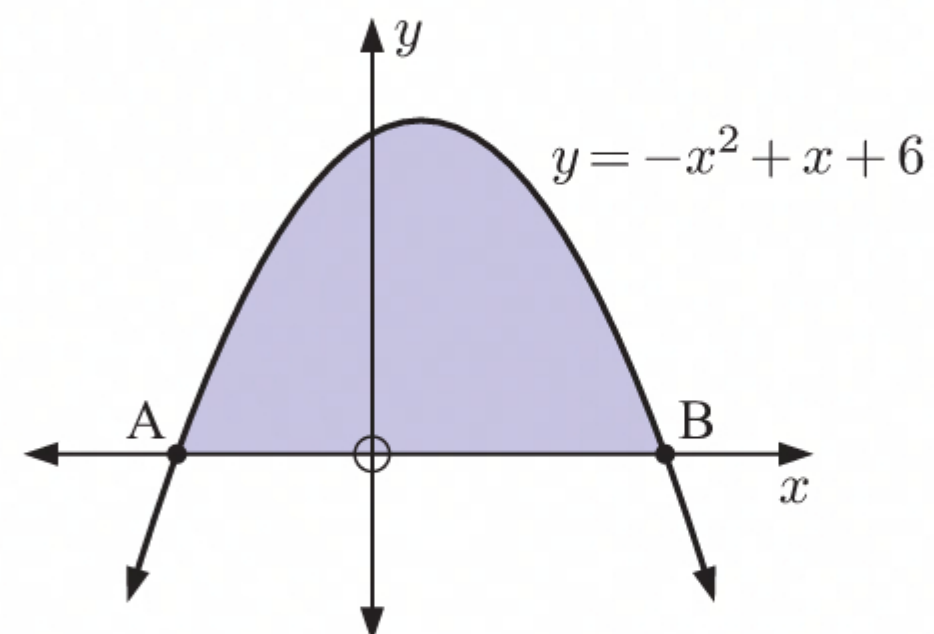
$$\begin{aligned}
 \text{Area} &= \int_1^4 x^3 \, dx \\
 &= \left[\frac{1}{4}x^4 \right]_1^4 \\
 &= 64 - \frac{1}{4} \\
 &= 63\frac{3}{4} \text{ units}^2
 \end{aligned}$$

c

$$\begin{aligned}
 \text{Area} &= \int_2^3 \left(\frac{1}{2}x^2 - 1 \right) dx \\
 &= \left[\frac{1}{6}x^3 - x \right]_2^3 \\
 &= \left(\frac{27}{6} - 3 \right) - \left(\frac{8}{6} - 2 \right) \\
 &= 2\frac{1}{6} \text{ units}^2
 \end{aligned}$$

5 a A and B are the x -intercepts of $y = -x^2 + x + 6$.

$$\begin{aligned}
 \text{When } y = 0, \quad -x^2 + x + 6 &= 0 \\
 \therefore x^2 - x - 6 &= 0 \\
 \therefore (x + 2)(x - 3) &= 0 \\
 \therefore x &= -2 \text{ or } 3
 \end{aligned}$$

 \therefore A is $(-2, 0)$ and B is $(3, 0)$.

$$\begin{aligned}
 \text{b Area} &= \int_{-2}^3 (-x^2 + x + 6) \, dx \\
 &= \left[-\frac{1}{3}x^3 + \frac{1}{2}x^2 + 6x \right]_{-2}^3 \\
 &= \left(-9 + \frac{9}{2} + 18 \right) - \left(\frac{8}{3} + 2 - 12 \right) \\
 &= 13\frac{1}{2} - \left(-7\frac{1}{3} \right) \\
 &= 20\frac{5}{6} \text{ units}^2
 \end{aligned}$$

6 a $y = -x^2 + 7x - 10$

When $y = 0$, $-x^2 + 7x - 10 = 0$

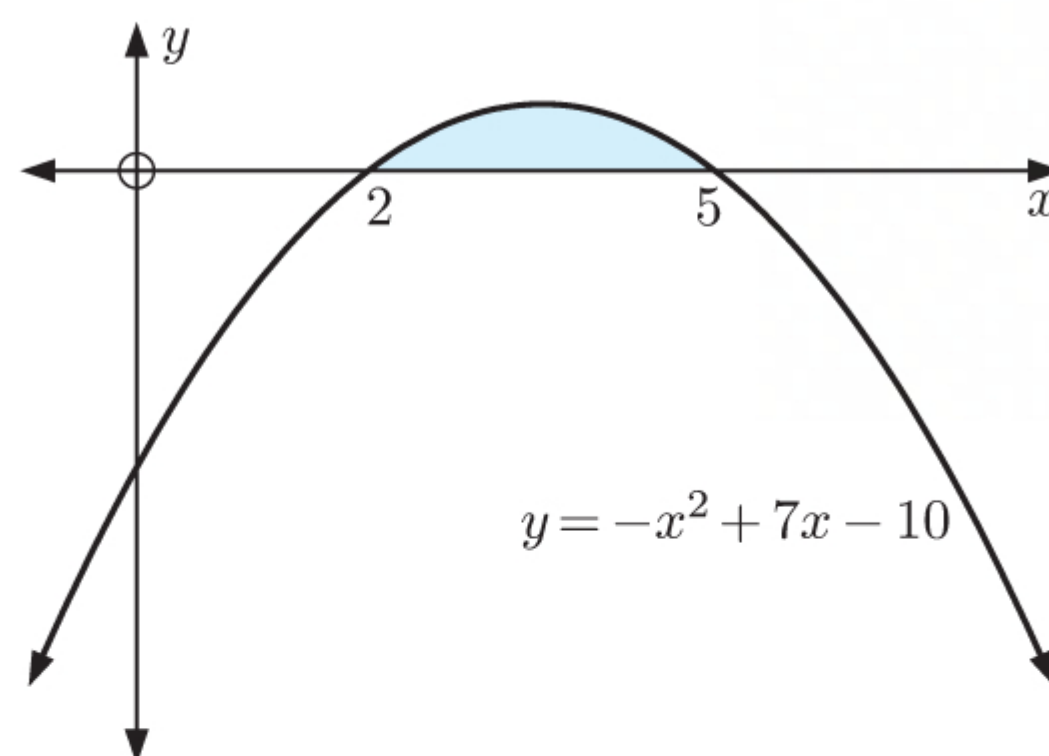
$$\therefore x^2 - 7x + 10 = 0$$

$$\therefore (x - 2)(x - 5) = 0$$

$$\therefore x = 2 \text{ or } 5$$

\therefore the x -intercepts are 2 and 5.

$$\begin{aligned} \therefore \text{enclosed area} &= \int_2^5 (-x^2 + 7x - 10) dx \\ &= \left[-\frac{1}{3}x^3 + \frac{7}{2}x^2 - 10x \right]_2^5 \\ &= \left(-\frac{125}{3} + \frac{175}{2} - 50 \right) - \left(-\frac{8}{3} + 14 - 20 \right) \\ &= 4\frac{1}{2} \text{ units}^2 \end{aligned}$$



b $y = -2x^2 + 2x + 4$

When $y = 0$, $-2x^2 + 2x + 4 = 0$

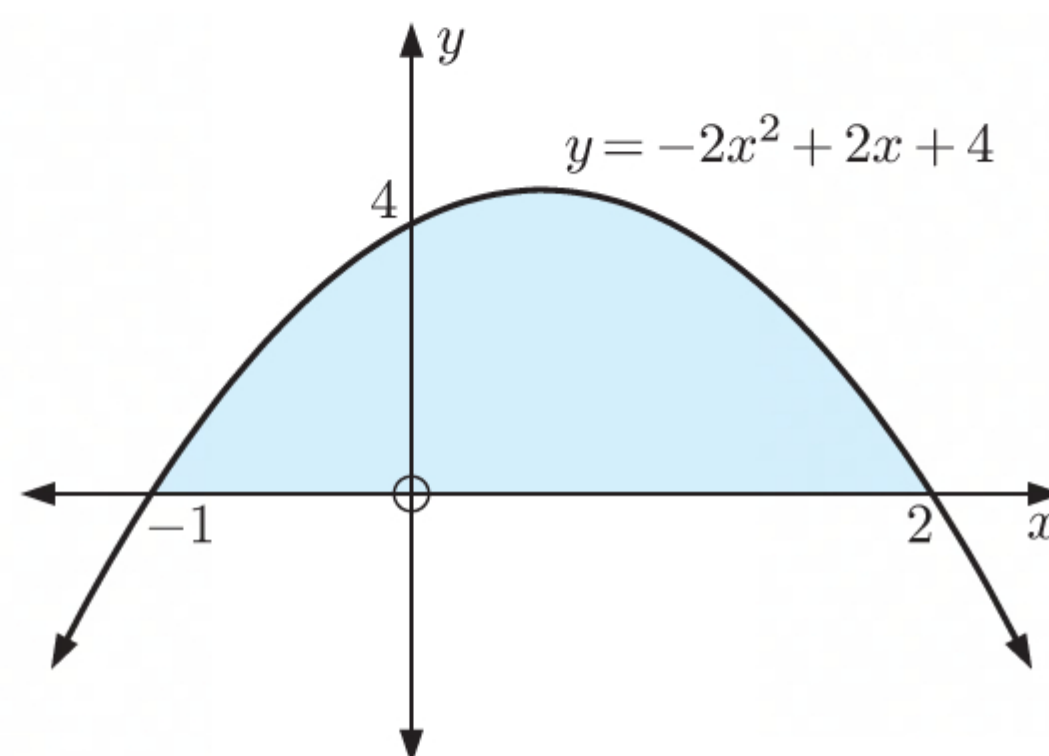
$$\therefore -2(x^2 - x - 2) = 0$$

$$\therefore -2(x + 1)(x - 2) = 0$$

$$\therefore x = -1 \text{ or } 2$$

\therefore the x -intercepts are -1 and 2 .

$$\begin{aligned} \therefore \text{enclosed area} &= \int_{-1}^2 (-2x^2 + 2x + 4) dx \\ &= \left[-\frac{2}{3}x^3 + x^2 + 4x \right]_{-1}^2 \\ &= \left(-\frac{16}{3} + 4 + 8 \right) - \left(\frac{2}{3} + 1 - 4 \right) \\ &= 9 \text{ units}^2 \end{aligned}$$



c $y = 3 - x^2$

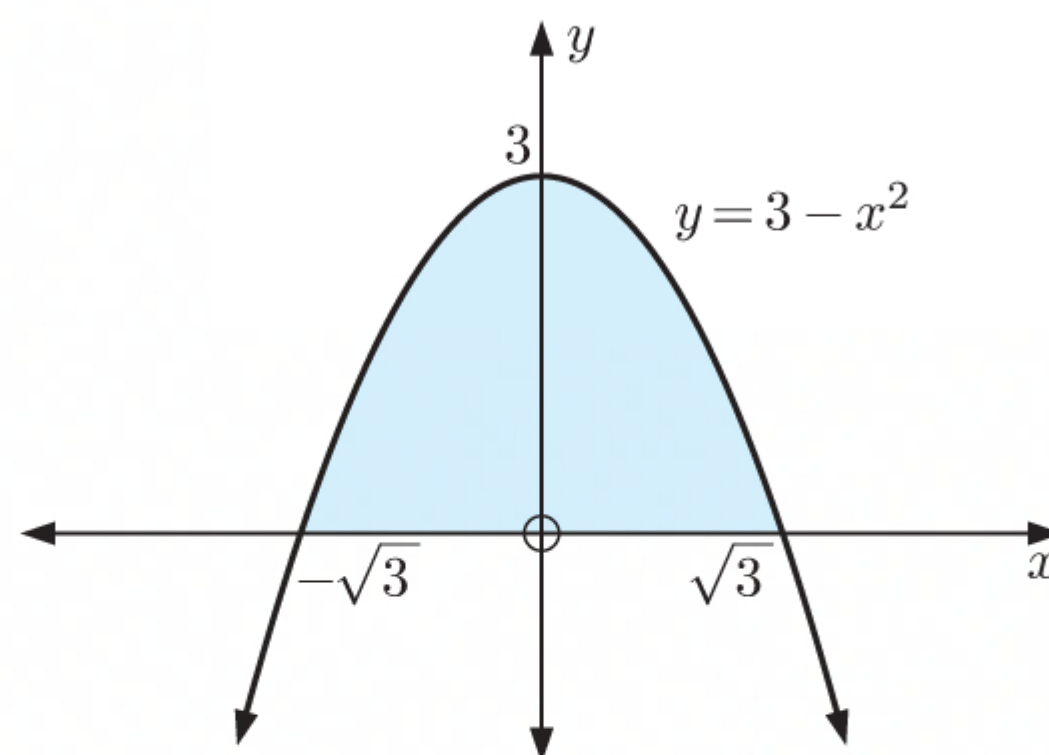
When $y = 0$, $3 - x^2 = 0$

$$\therefore x^2 = 3$$

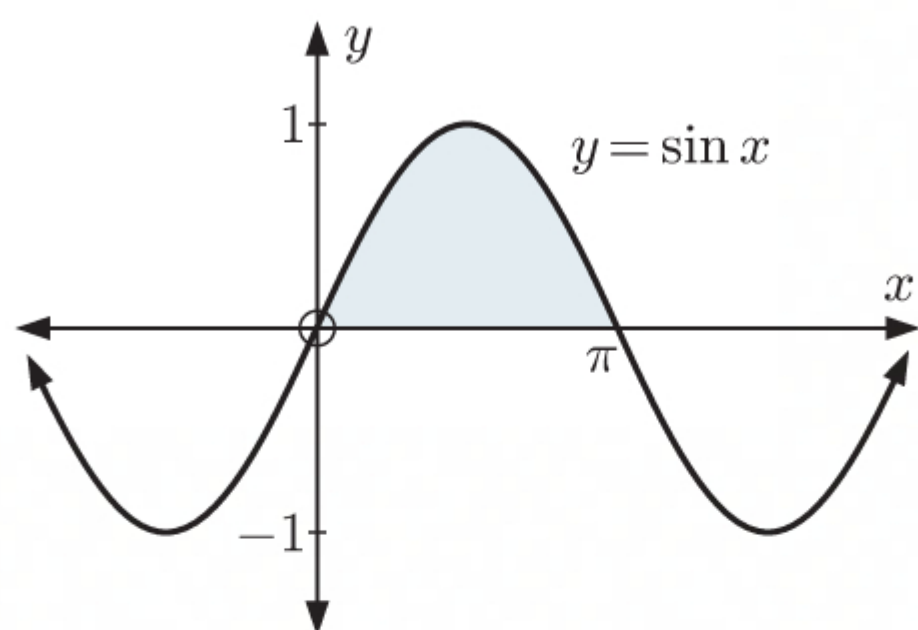
$$\therefore x = \pm\sqrt{3}$$

\therefore the x -intercepts are $\sqrt{3}$ and $-\sqrt{3}$.

$$\begin{aligned} \therefore \text{enclosed area} &= \int_{-\sqrt{3}}^{\sqrt{3}} (3 - x^2) dx \\ &= \left[3x - \frac{1}{3}x^3 \right]_{-\sqrt{3}}^{\sqrt{3}} \\ &= \left(3\sqrt{3} - \sqrt{3} \right) - \left(-3\sqrt{3} + \sqrt{3} \right) \\ &= 4\sqrt{3} \text{ units}^2 \end{aligned}$$

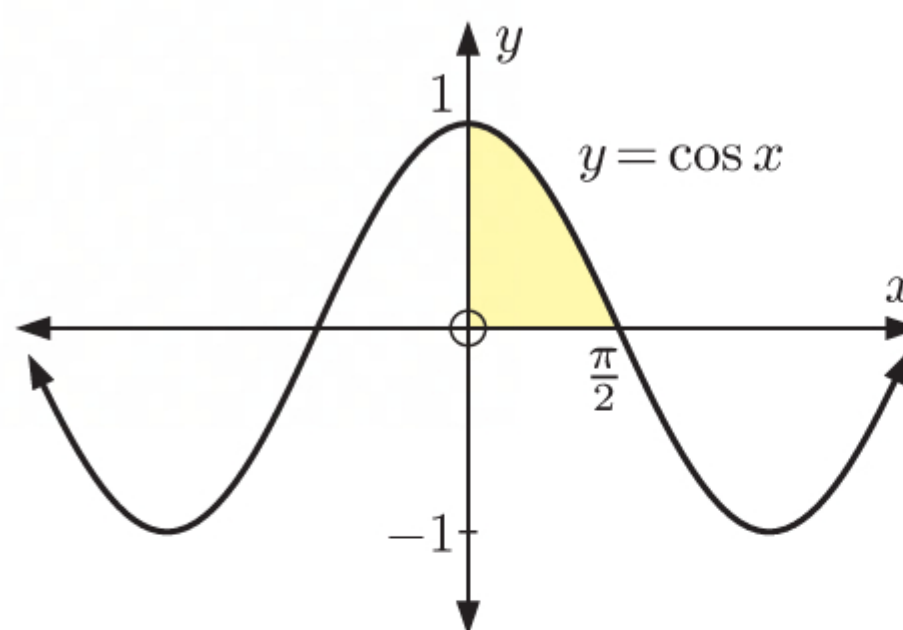


7



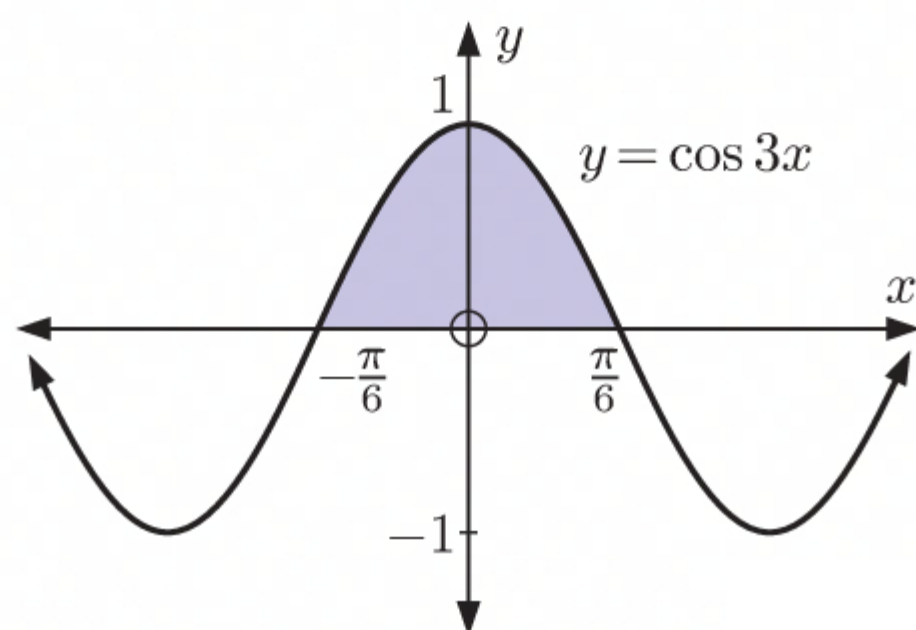
$$\begin{aligned}
 \text{Area} &= \int_0^{\pi} \sin x \, dx \\
 &= [-\cos x]_0^{\pi} \\
 &= [-\cos \pi + \cos 0] \\
 &= -(-1) + 1 \\
 &= 2 \text{ units}^2
 \end{aligned}$$

8



$$\begin{aligned}
 \text{Area} &= \int_0^{\frac{\pi}{2}} \cos x \, dx \\
 &= [\sin x]_0^{\frac{\pi}{2}} \\
 &= \sin \frac{\pi}{2} - \sin 0 \\
 &= 1 \text{ unit}^2
 \end{aligned}$$

9



The period of $y = \cos 3x$ is $\frac{2\pi}{3}$, so the x -intercepts under one arch are $-\frac{\pi}{6}$ and $\frac{\pi}{6}$.

$$\begin{aligned}
 \text{The required area} &= \int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \cos 3x \, dx \\
 &= \left[\frac{1}{3} \sin 3x \right]_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \\
 &= \frac{1}{3} \left(\sin \frac{\pi}{2} - \sin \left(-\frac{\pi}{2} \right) \right) \\
 &= \frac{1}{3} (1 - (-1)) \\
 &= \frac{2}{3} \text{ units}^2
 \end{aligned}$$

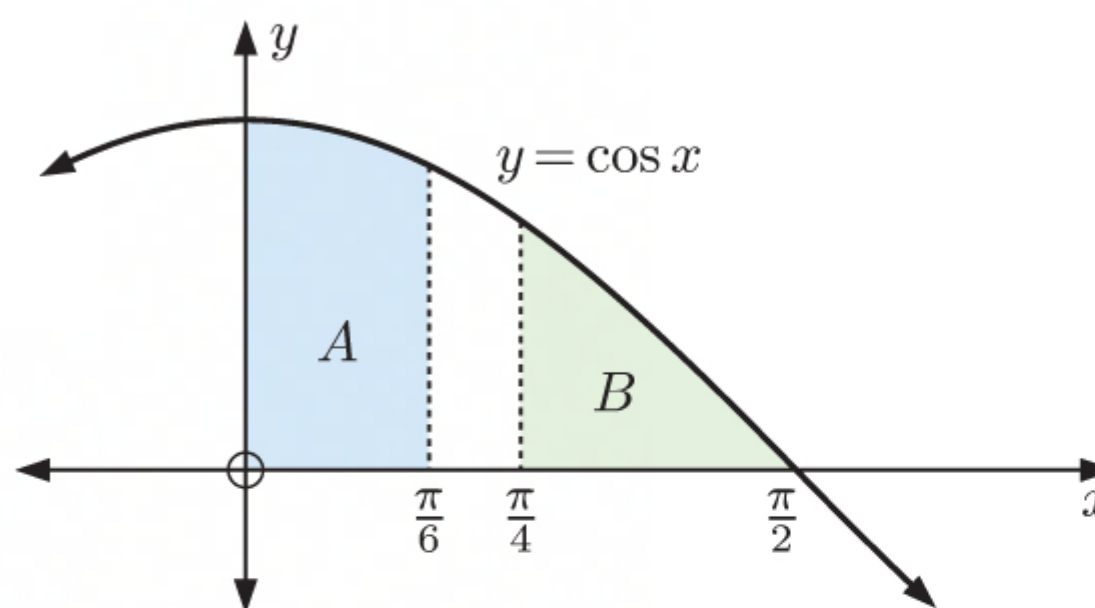
10 a Region A appears to be larger.

$$\begin{aligned}
 \text{b Area of region A} &= \int_0^{\frac{\pi}{6}} \cos x \, dx \\
 &= [\sin x]_0^{\frac{\pi}{6}} \\
 &= \frac{1}{2} - 0 \\
 &= \frac{1}{2} \text{ units}^2
 \end{aligned}$$

$y = \cos x = 0$ when $x = \frac{\pi}{2}$
 \therefore the x -intercept is $\frac{\pi}{2}$.

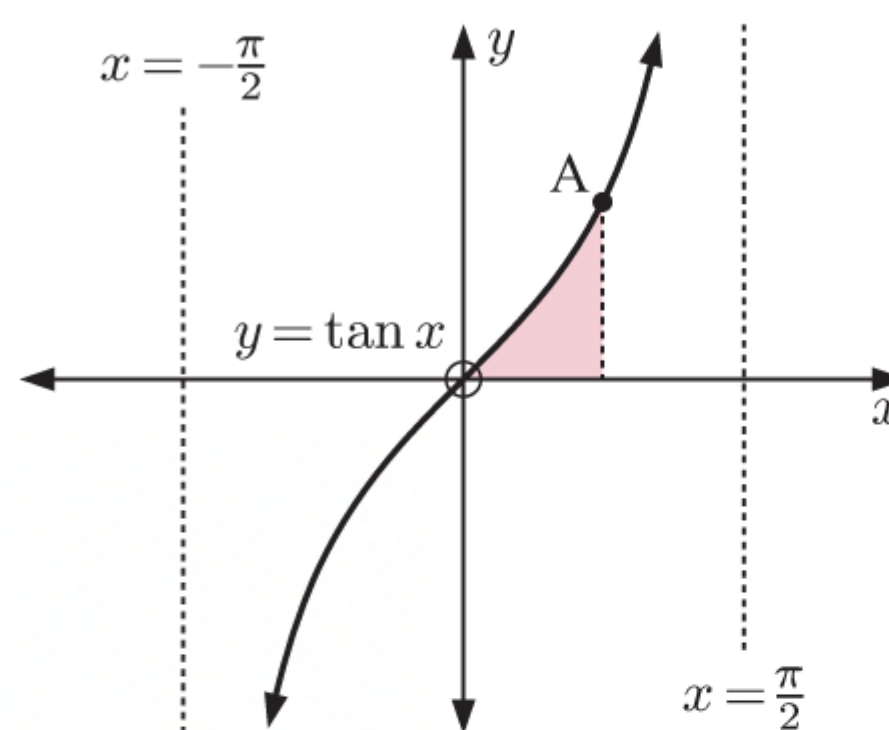
$$\begin{aligned}
 \text{Area of region B} &= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cos x \, dx \\
 &= [\sin x]_{\frac{\pi}{4}}^{\frac{\pi}{2}} \\
 &= \left(1 - \frac{1}{\sqrt{2}} \right) \text{ units}^2 \\
 &\approx 0.293 \text{ units}^2
 \end{aligned}$$

\therefore region A is larger than region B.



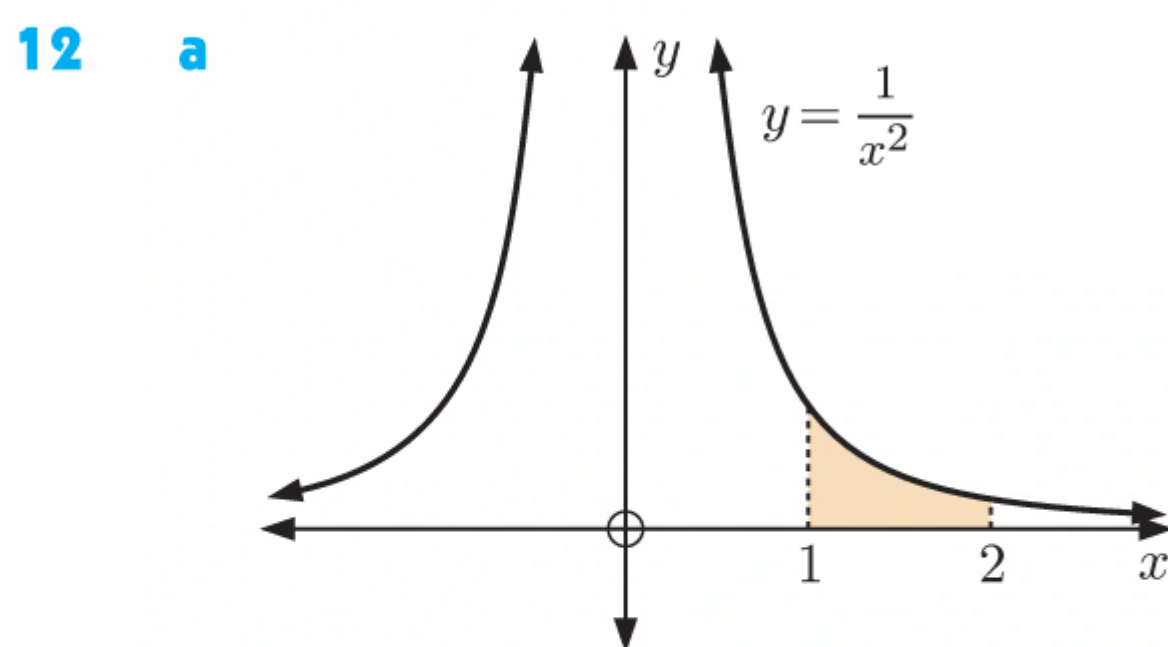
- 11 a** $y = \tan x$, $-\frac{\pi}{2} < x < \frac{\pi}{2}$
 $y = 1$ when $x = \frac{\pi}{4}$
 \therefore A has x -coordinate $\frac{\pi}{4}$.

b
$$\begin{aligned} \int \tan x \, dx &= \int \frac{\sin x}{\cos x} \, dx \\ &= - \int \frac{-\sin x}{\cos x} \, dx \\ &= - \int \frac{1}{u} \frac{du}{dx} \, dx \\ &\quad \{u = \cos x, \quad \frac{du}{dx} = -\sin x\} \\ &= - \int \frac{1}{u} \, du \\ &= -\ln |u| + c \\ &= -\ln |\cos x| + c \end{aligned}$$

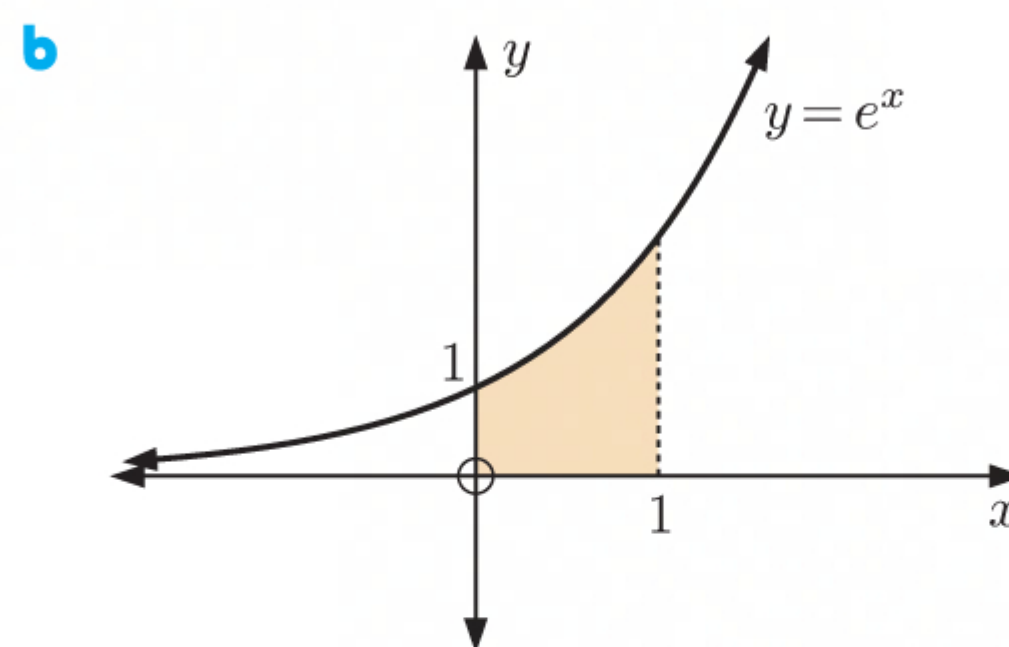


c Shaded area $= \int_0^{\frac{\pi}{4}} \tan x \, dx$

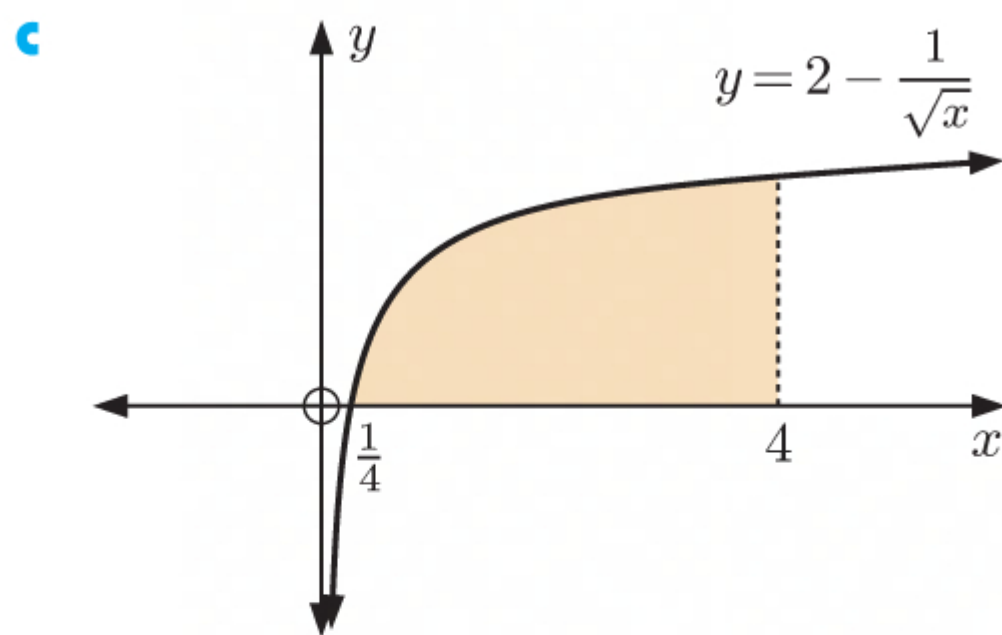
$$\begin{aligned} &= \left[-\ln |\cos x| \right]_0^{\frac{\pi}{4}} \\ &= -\ln \left(\frac{1}{\sqrt{2}} \right) - (-\ln 1) \\ &= -\ln(2^{-\frac{1}{2}}) \\ &= \ln(2^{\frac{1}{2}}) \\ &= \ln \sqrt{2} \text{ units}^2 \end{aligned}$$



$$\begin{aligned} \text{Area} &= \int_1^2 \frac{1}{x^2} \, dx \\ &= \int_1^2 x^{-2} \, dx \\ &= \left[-\frac{1}{x} \right]_1^2 \\ &= -\frac{1}{2} - (-1) \\ &= \frac{1}{2} \text{ units}^2 \end{aligned}$$



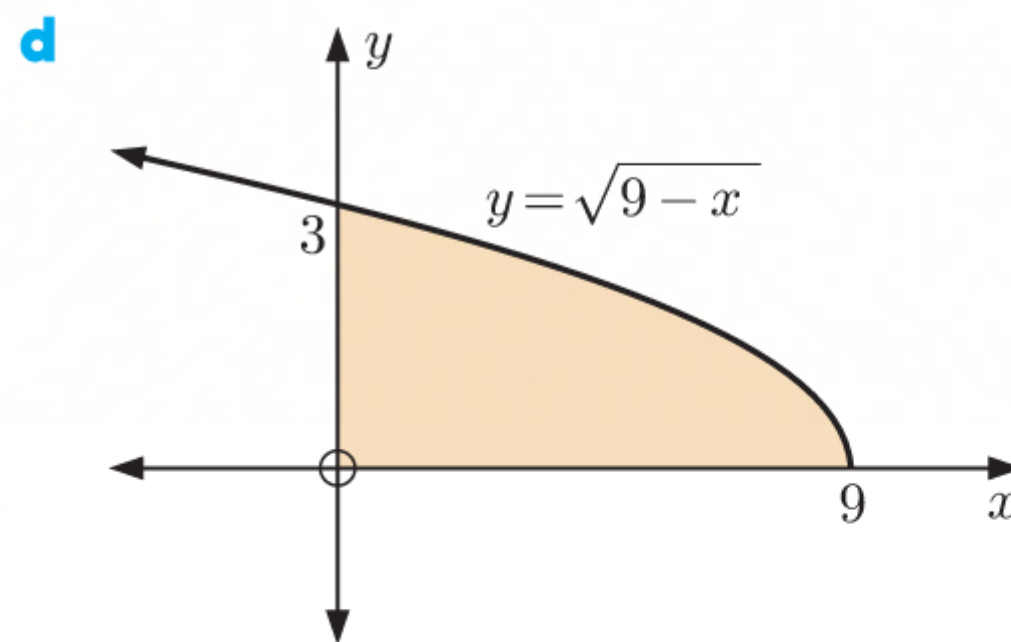
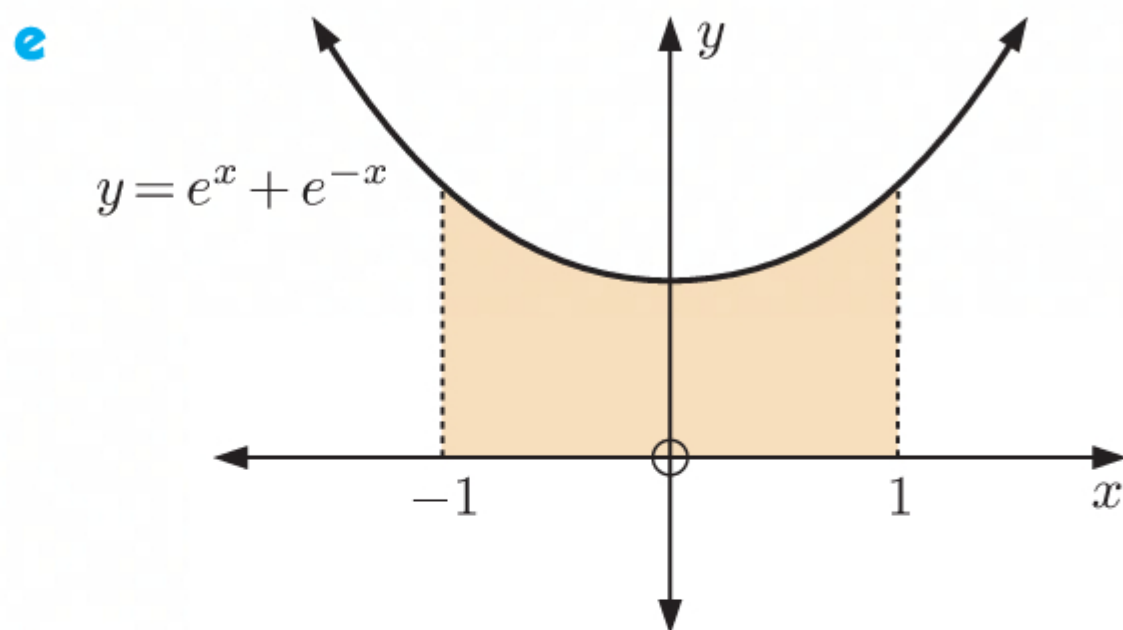
$$\begin{aligned} \text{Area} &= \int_0^1 e^x \, dx \\ &= [e^x]_0^1 \\ &= e^1 - e^0 \\ &= (e - 1) \text{ units}^2 \end{aligned}$$



$$\begin{aligned}\text{When } y = 0, \quad 2 - \frac{1}{\sqrt{x}} &= 0 \\ \therefore 2 &= \frac{1}{\sqrt{x}} \\ \therefore \sqrt{x} &= \frac{1}{2} \\ \therefore x &= \frac{1}{4} \quad \{x > 0\}\end{aligned}$$

\therefore the x -intercept is $\frac{1}{4}$.

$$\begin{aligned}\therefore \text{area} &= \int_{\frac{1}{4}}^4 \left(2 - \frac{1}{\sqrt{x}}\right) dx \\ &= \int_{\frac{1}{4}}^4 \left(2 - x^{-\frac{1}{2}}\right) dx \\ &= \left[2x - 2x^{\frac{1}{2}}\right]_{\frac{1}{4}}^4 \\ &= (8 - 4) - \left(\frac{1}{2} - 1\right) \\ &= 4\frac{1}{2} \text{ units}^2\end{aligned}$$



$$\begin{aligned}\text{When } y = 0, \quad \sqrt{9 - x} &= 0 \\ \therefore 9 - x &= 0 \\ \therefore x &= 9\end{aligned}$$

\therefore the x -intercept is 9.

$$\begin{aligned}\therefore \text{area} &= \int_0^9 \sqrt{9 - x} \, dx \\ &= \int_0^9 (9 - x)^{\frac{1}{2}} \, dx \\ &= \left[-\frac{2}{3}(9 - x)^{\frac{3}{2}}\right]_0^9 \\ &= 0 - \left(-\frac{2}{3}(27)\right) \\ &= 18 \text{ units}^2\end{aligned}$$

$$\begin{aligned}\text{Area} &= \int_{-1}^1 (e^x + e^{-x}) \, dx \\ &= [e^x - e^{-x}]_{-1}^1 \\ &= \left(e - \frac{1}{e}\right) - \left(\frac{1}{e} - e\right) \\ &= \left(2e - \frac{2}{e}\right) \text{ units}^2\end{aligned}$$

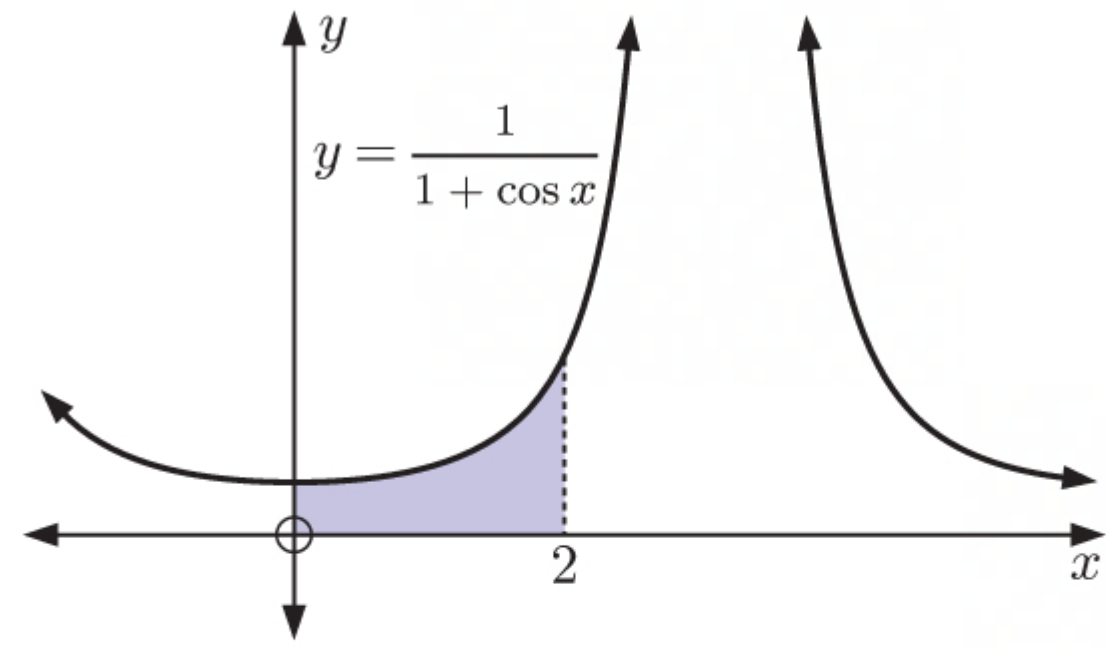
13 a

$$\begin{aligned}\frac{d}{dx} \left(\frac{\sin x}{1 + \cos x} \right) &= \frac{\cos x(1 + \cos x) - \sin x(-\sin x)}{(1 + \cos x)^2} \\ &= \frac{\cos x + \cos^2 x + \sin^2 x}{(1 + \cos x)^2} \\ &= \frac{1 + \cos x}{(1 + \cos x)^2} \\ &= \frac{1}{1 + \cos x}\end{aligned}$$

$$\mathbf{b} \quad \frac{d}{dx} \left(\frac{\sin x}{1 + \cos x} \right) = \frac{1}{1 + \cos x}$$

$$\therefore \int \frac{1}{1 + \cos x} dx = \frac{\sin x}{1 + \cos x} + c$$

$$\begin{aligned} \therefore \text{shaded area} &= \int_0^2 \frac{1}{1 + \cos x} dx \\ &= \left[\frac{\sin x}{1 + \cos x} \right]_0^2 \\ &= \frac{\sin 2}{1 + \cos 2} - \frac{\sin 0}{1 + \cos 0} \text{ units}^2 \\ &= \frac{\sin 2}{1 + \cos 2} \text{ units}^2 \quad (\approx 1.56 \text{ units}^2) \end{aligned}$$



$$\mathbf{14} \quad \mathbf{a} \quad \text{Area} = \int_0^b \sqrt{x} dx$$

$$\therefore 1 = \int_0^b x^{\frac{1}{2}} dx$$

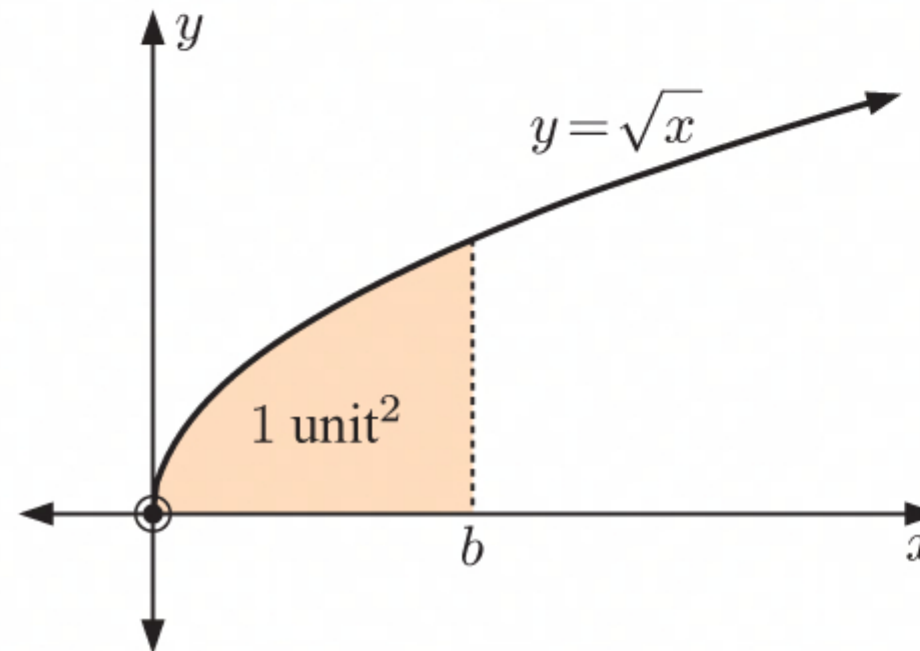
$$\therefore 1 = \left[\frac{2}{3} x^{\frac{3}{2}} \right]_0^b$$

$$\therefore 1 = \frac{2}{3} b^{\frac{3}{2}} - 0$$

$$\therefore \frac{3}{2} = b^{\frac{3}{2}}$$

$$\therefore b = \left(\frac{3}{2} \right)^{\frac{2}{3}}$$

$$\therefore b \approx 1.3104$$



$$\mathbf{b} \quad \text{Area} = \int_{-a}^a (x^2 + 2) dx$$

$$\begin{aligned} \therefore 6a &= \left[\frac{1}{3} x^3 + 2x \right]_{-a}^a \\ &= \left(\frac{1}{3} a^3 + 2a \right) - \left(-\frac{1}{3} a^3 - 2a \right) \\ &= \frac{2}{3} a^3 + 4a \end{aligned}$$

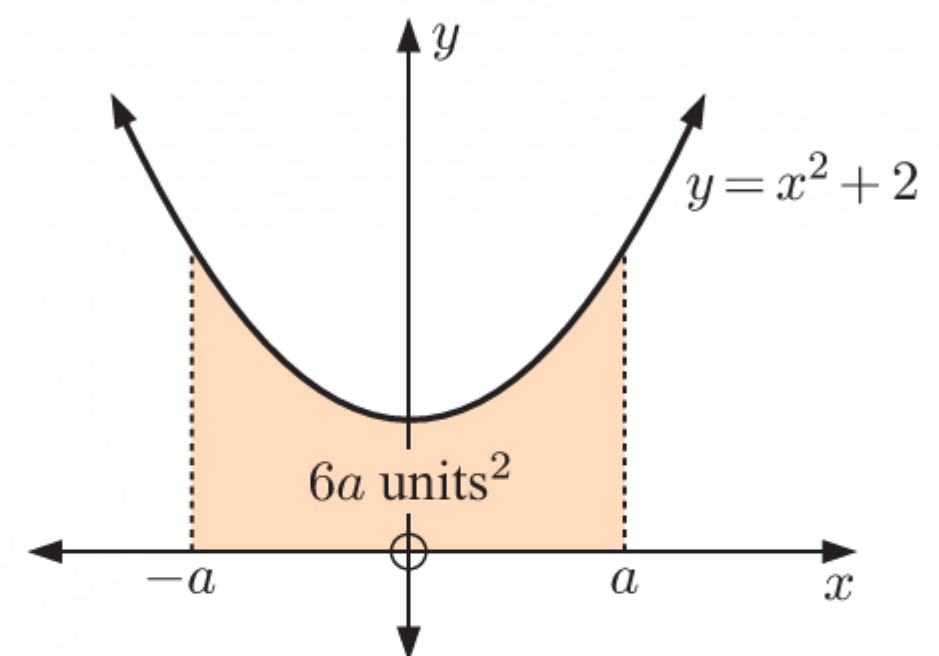
$$\therefore \frac{2}{3} a^3 - 2a = 0$$

$$\therefore a^3 - 3a = 0$$

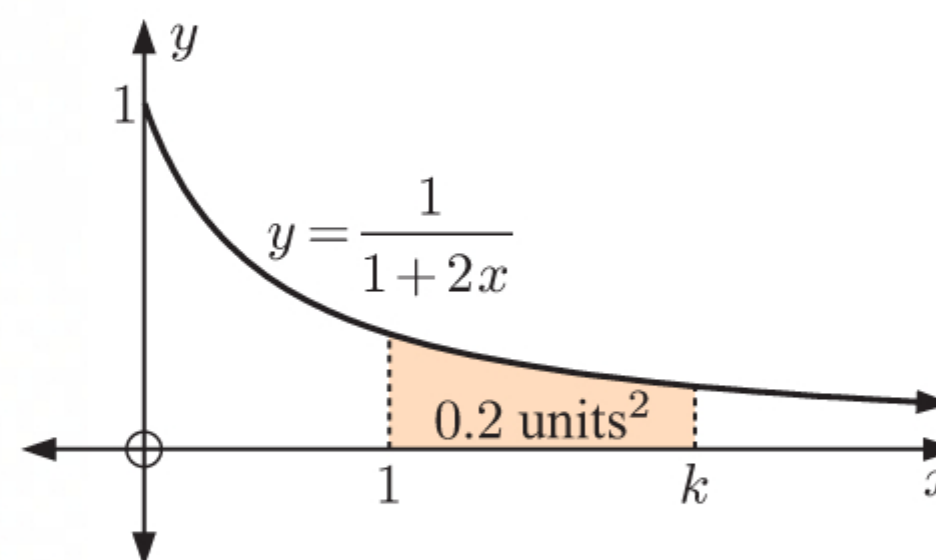
$$\therefore a(a^2 - 3) = 0$$

$$\therefore a = 0, \pm \sqrt{3}$$

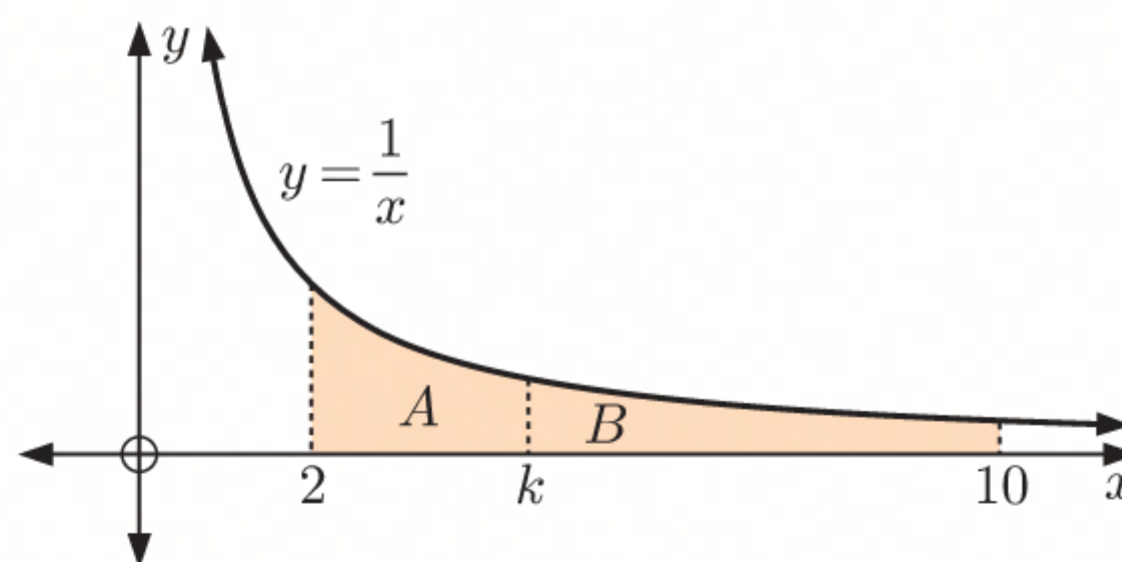
$$\text{but } a > 0, \therefore a = \sqrt{3}$$



$$\begin{aligned}
 \text{Area} &= \int_1^k \frac{1}{1+2x} dx \\
 \therefore 0.2 &= \left[\frac{1}{2} \ln |1+2x| \right]_1^k \\
 &= \frac{1}{2} \ln(1+2k) - \frac{1}{2} \ln 3 \quad \{k > 1\} \\
 &= \frac{1}{2} [\ln(1+2k) - \ln 3] \\
 \therefore 0.4 &= \ln \left(\frac{1+2k}{3} \right) \\
 \therefore \frac{1+2k}{3} &= e^{0.4} \\
 \therefore 1+2k &= 3e^{0.4} \\
 \therefore 2k &= 3e^{0.4} - 1 \\
 \therefore k &= \frac{3e^{0.4} - 1}{2} \approx 1.7377
 \end{aligned}$$



$$\begin{aligned}
 \text{Area of } A &= \text{Area of } B \\
 \therefore \int_2^k \frac{1}{x} dx &= \int_k^{10} \frac{1}{x} dx \\
 \therefore [\ln |x|]_2^k &= [\ln |x|]_k^{10} \\
 \therefore \ln k - \ln 2 &= \ln 10 - \ln k \quad \{2 < k < 10\} \\
 \therefore 2 \ln k &= \ln 10 + \ln 2 \\
 \therefore \ln k^2 &= \ln 20 \\
 \therefore k^2 &= 20 \\
 \therefore k &= 2\sqrt{5} \quad \{2 < k < 10\}
 \end{aligned}$$

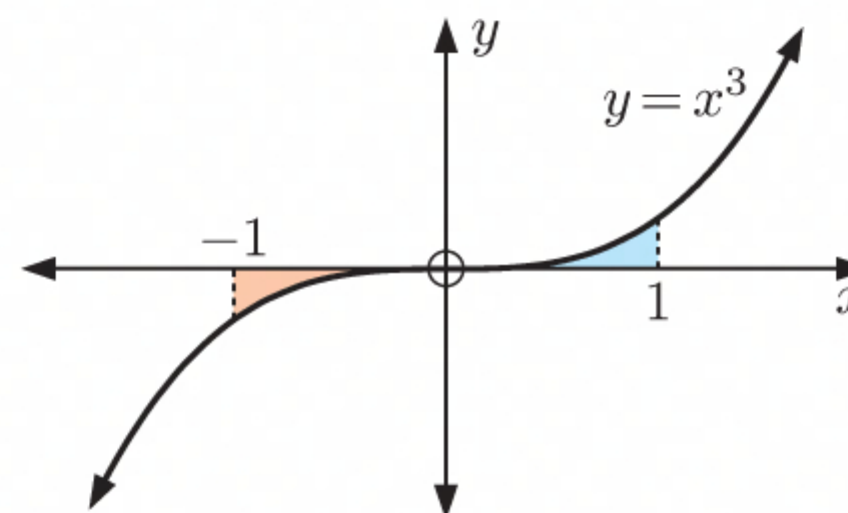


INVESTIGATION

$\int_a^b f(x) dx$ AND AREAS

$$\begin{aligned}
 \mathbf{1} \quad \mathbf{a} \quad \int_0^1 x^3 dx &= \left[\frac{1}{4} x^4 \right]_0^1 & \text{and} & \quad \int_{-1}^1 x^3 dx = \left[\frac{1}{4} x^4 \right]_{-1}^1 \\
 &= \frac{1}{4} - 0 & & \quad = \frac{1}{4} - \frac{1}{4}(-1)^4 \\
 &= \frac{1}{4} & & \quad = \frac{1}{4} - \frac{1}{4} \\
 & & & \quad = 0
 \end{aligned}$$

- b** Since the curve lies on or above the x -axis for $0 \leq x \leq 1$, the first integral in **a** is the area bounded by $y = x^3$, the x -axis, and the vertical lines $x = 0$ and $x = 1$.
The second integral in **a** does *not* give an area as the curve lies on or below the x -axis for $-1 \leq x \leq 0$.



$$\begin{aligned}
 \text{c} \quad \int_{-1}^0 x^3 dx &= \left[\frac{1}{4} x^4 \right]_{-1}^0 \\
 &= 0 - \frac{1}{4} (-1)^4 \\
 &= -\frac{1}{4}
 \end{aligned}$$

The answer is negative since the curve lies on or below the x -axis for $-1 \leq x \leq 0$.

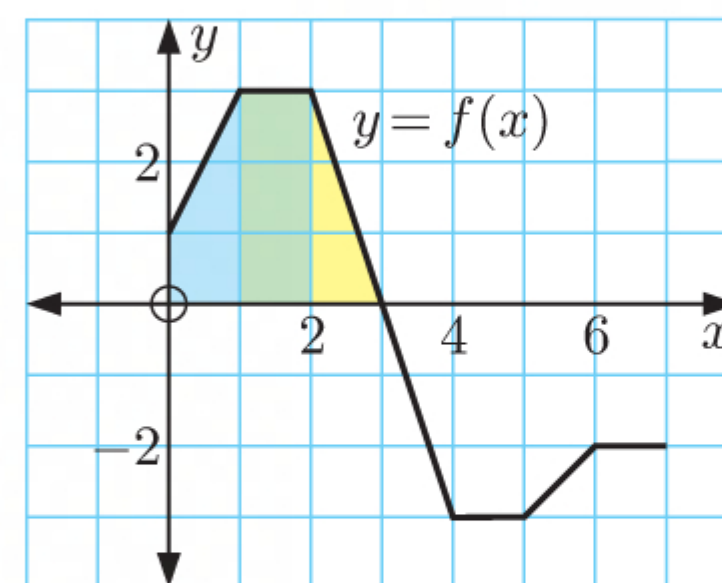
$$\begin{aligned}
 \text{d} \quad \int_{-1}^0 x^3 dx + \int_0^1 x^3 dx &= -\frac{1}{4} + \frac{1}{4} \quad \{\text{using a and c}\} \\
 &= 0 \\
 &= \int_{-1}^1 x^3 dx
 \end{aligned}$$

$$\begin{aligned}
 \text{e} \quad \int_0^{-1} x^3 dx &= - \int_{-1}^0 x^3 dx \\
 &= - \left(-\frac{1}{4} \right) \quad \{\text{from c}\} \\
 &= \frac{1}{4}
 \end{aligned}$$

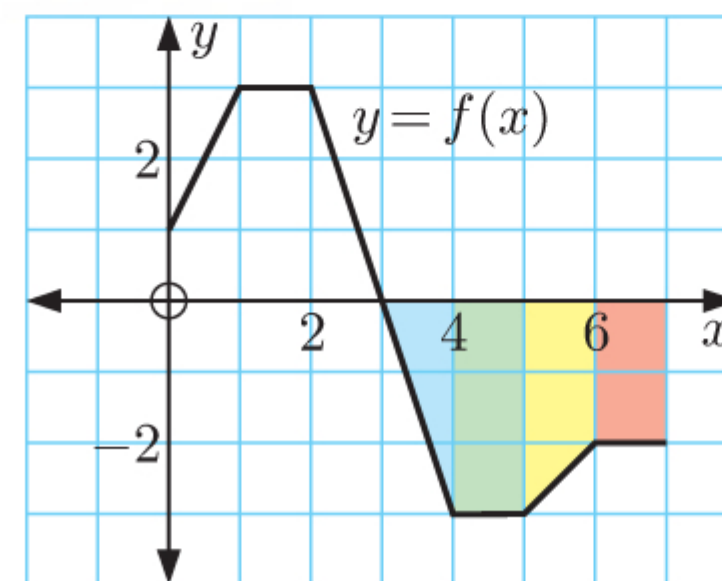
The area between the curve and the x -axis between $x = -1$ and $x = 0$ is $\frac{1}{4}$ units².

$$2 \quad \text{Area} = - \int_a^b f(x) dx$$

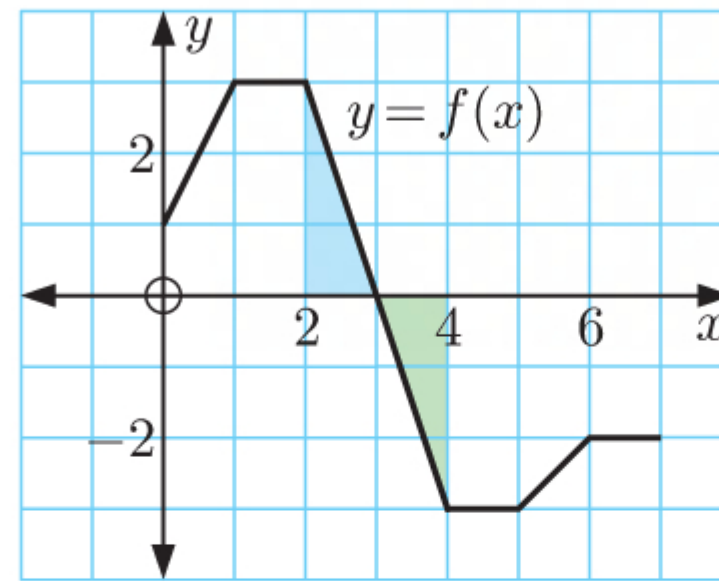
$$\begin{aligned}
 3 \quad \text{a} \quad \int_0^3 f(x) dx &= \text{area of blue trapezium} \\
 &\quad + \text{area of green rectangle} \\
 &\quad + \text{area of yellow triangle} \\
 &= \left(\frac{1+3}{2} \right) \times 1 + (1 \times 3) + \left(\frac{1}{2} \times 1 \times 3 \right) \\
 &= 2 + 3 + \frac{3}{2} \\
 &= \frac{13}{2}
 \end{aligned}$$



$$\begin{aligned}
 \text{b} \quad \int_3^7 f(x) dx &= -(\text{area of blue triangle} \\
 &\quad + \text{area of green rectangle} \\
 &\quad + \text{area of yellow trapezium} \\
 &\quad + \text{area of red rectangle}) \\
 &= - \left[\left(\frac{1}{2} \times 1 \times 3 \right) + (1 \times 3) + \left(\frac{2+3}{2} \right) \times 1 + (1 \times 2) \right] \\
 &= - \left(\frac{3}{2} + 3 + \frac{5}{2} + 2 \right) \\
 &= -9
 \end{aligned}$$



$$\begin{aligned}
 \text{c } \int_2^4 f(x) \, dx &= \text{area of blue triangle} \\
 &\quad - \text{area of green triangle} \\
 &= \left(\frac{1}{2} \times 1 \times 3\right) - \left(\frac{1}{2} \times 1 \times 3\right) \\
 &= 0
 \end{aligned}$$



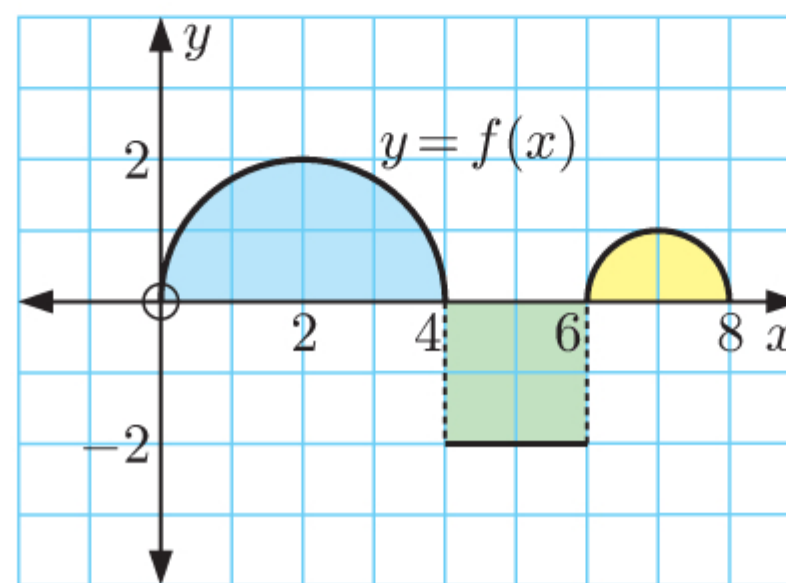
$$\begin{aligned}
 \text{d } \int_0^7 f(x) \, dx &= \int_0^3 f(x) \, dx + \int_3^7 f(x) \, dx \\
 &= \frac{13}{2} + (-9) \quad \{\text{using a and b}\} \\
 &= -\frac{5}{2}
 \end{aligned}$$

$$\begin{aligned}
 \text{4 a } \int_0^4 f(x) \, dx &= \text{area of blue semi-circle} \\
 &= \frac{1}{2} \times \pi \times 2^2 \\
 &= 2\pi
 \end{aligned}$$

$$\begin{aligned}
 \text{b } \int_4^6 f(x) \, dx &= -\text{area of green square} \\
 &= -(2 \times 2) \\
 &= -4
 \end{aligned}$$

$$\begin{aligned}
 \text{c } \int_6^8 f(x) \, dx &= \text{area of yellow semi-circle} \\
 &= \frac{1}{2} \times \pi \times 1^2 \\
 &= \frac{\pi}{2}
 \end{aligned}$$

$$\begin{aligned}
 \text{d } \int_0^8 f(x) \, dx &= \int_0^4 f(x) \, dx + \int_4^6 f(x) \, dx + \int_6^8 f(x) \, dx \\
 &= 2\pi + (-4) + \frac{\pi}{2} \quad \{\text{using a, b, and c}\} \\
 &= \frac{5\pi}{2} - 4
 \end{aligned}$$

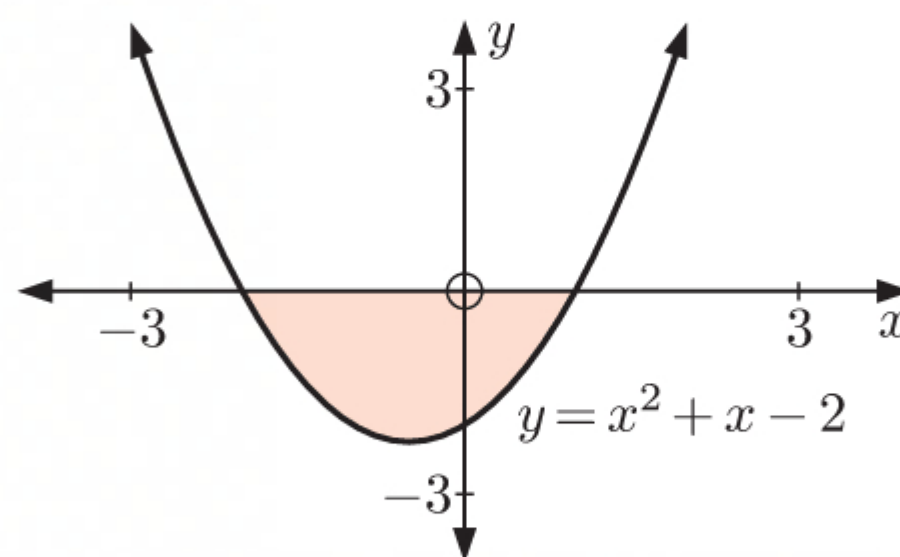


EXERCISE 17C

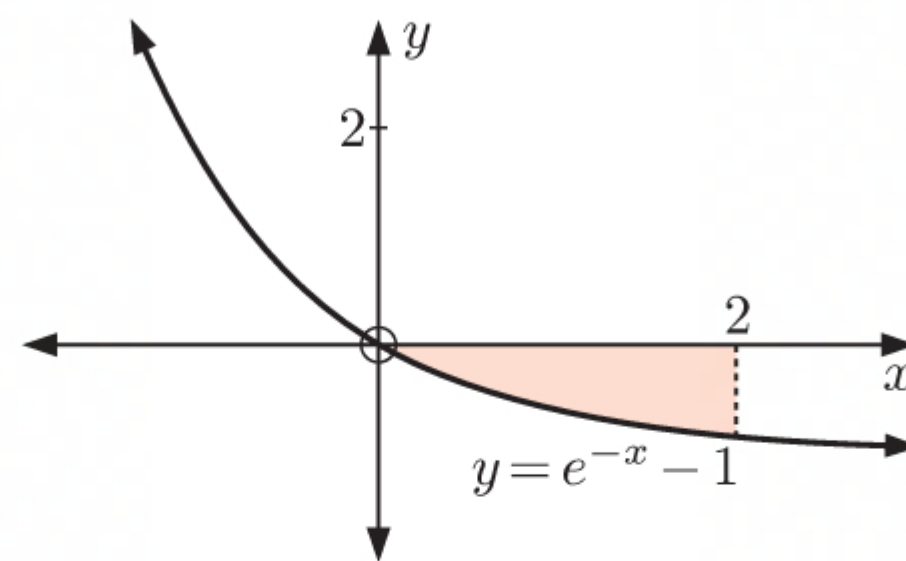
$$\begin{aligned}
 \text{1 a } \text{The curve cuts the } x\text{-axis when } y &= 0 \\
 \therefore x^2 + x - 2 &= 0 \\
 \therefore (x + 2)(x - 1) &= 0 \\
 \therefore x &= -2 \text{ or } 1
 \end{aligned}$$

\therefore the x -intercepts are -2 and 1 .

$$\begin{aligned}
 \text{Area} &= -\int_{-2}^1 (x^2 + x - 2) \, dx \\
 &= -\left[\frac{1}{3}x^3 + \frac{1}{2}x^2 - 2x\right]_{-2}^1 \\
 &= -\left[\left(\frac{1}{3} + \frac{1}{2} - 2\right) - \left(-\frac{8}{3} + 2 + 4\right)\right] \\
 &= -\left[-\frac{7}{6} - \frac{10}{3}\right] \\
 &= 4\frac{1}{2} \text{ units}^2
 \end{aligned}$$

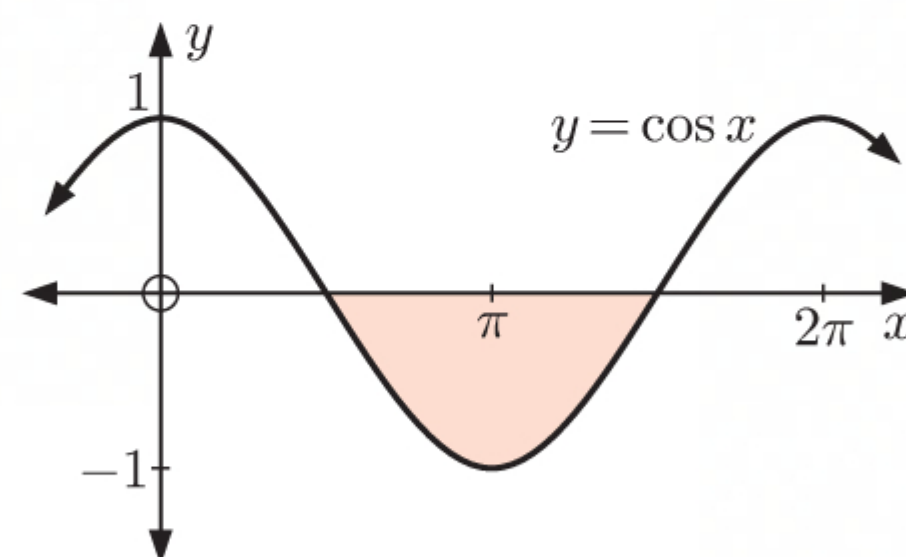


$$\begin{aligned}
 \text{b Area} &= - \int_0^2 (e^{-x} - 1) dx \\
 &= - [-e^{-x} - x]_0^2 \\
 &= - [(-e^{-2} - 2) - (-e^0 - 0)] \\
 &= -(-e^{-2} - 1) \\
 &= (1 + e^{-2}) \text{ units}^2
 \end{aligned}$$



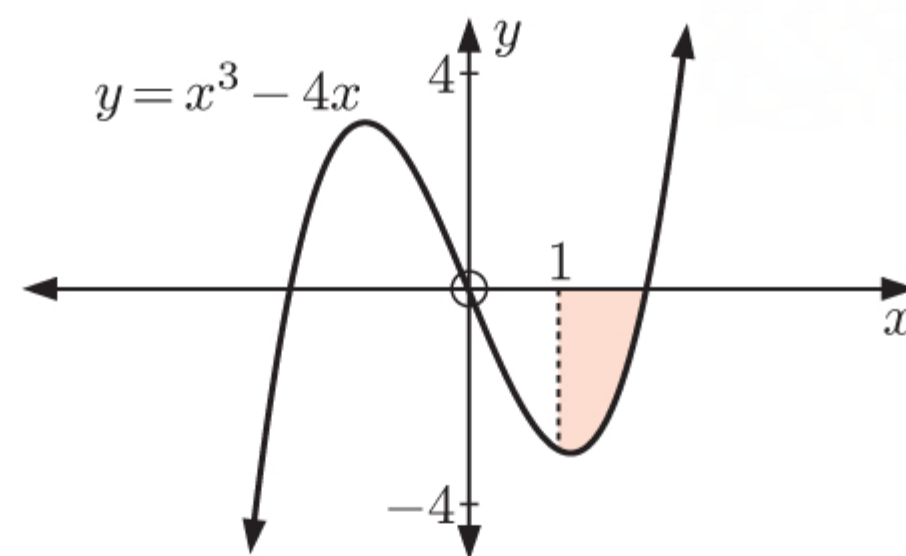
$$\begin{aligned}
 \text{c The curve cuts the } x\text{-axis when } y &= 0 \\
 \therefore \cos x &= 0 \\
 \therefore x &= \frac{\pi}{2} \text{ or } \frac{3\pi}{2}
 \end{aligned}$$

$$\begin{aligned}
 \text{Area} &= - \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \cos x dx \\
 &= - [\sin x]_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \\
 &= -[(-1) - 1] \\
 &= 2 \text{ units}^2
 \end{aligned}$$



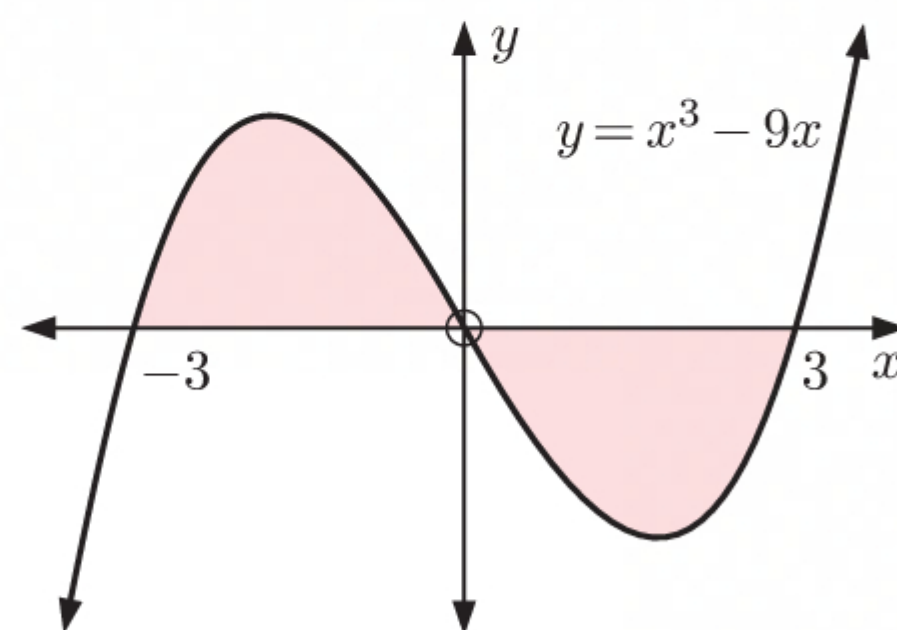
$$\begin{aligned}
 \text{d The curve cuts the } x\text{-axis when } y &= 0 \\
 \therefore x^3 - 4x &= 0 \\
 \therefore x(x^2 - 4) &= 0 \\
 \therefore x(x + 2)(x - 2) &= 0 \\
 \therefore \text{the } x\text{-intercepts are } -2, 0, \text{ and } 2.
 \end{aligned}$$

$$\begin{aligned}
 \text{Area} &= - \int_1^2 (x^3 - 4x) dx \\
 &= - \left[\frac{1}{4}x^4 - 2x^2 \right]_1^2 \\
 &= - \left[(4 - 8) - \left(\frac{1}{4} - 2 \right) \right] \\
 &= - \left[-4 - \left(-\frac{7}{4} \right) \right] \\
 &= 2\frac{1}{4} \text{ units}^2
 \end{aligned}$$



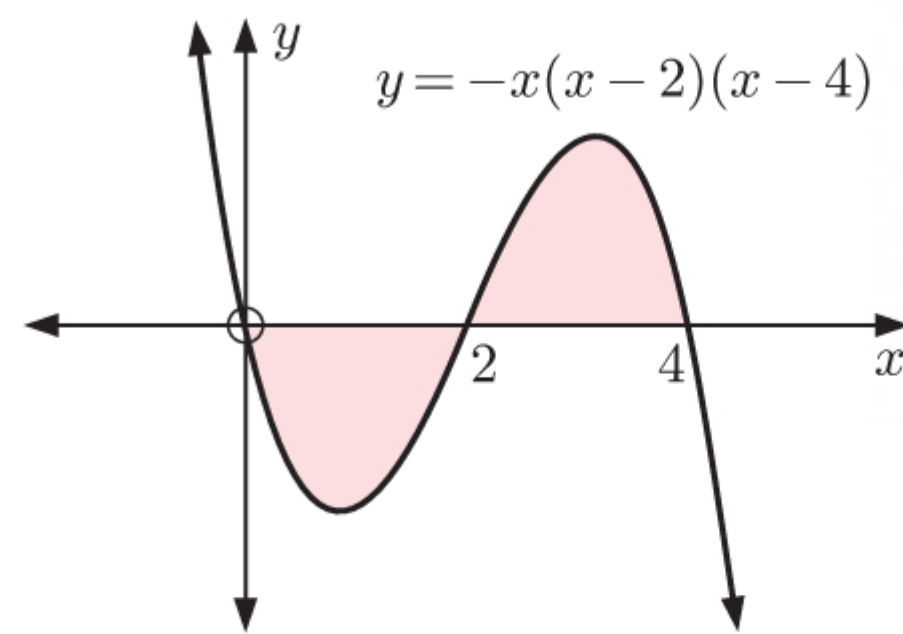
$$\begin{aligned}
 \text{2 a } f(x) &= x^3 - 9x \\
 &= x(x^2 - 9) \\
 &= x(x + 3)(x - 3) \\
 \therefore y = f(x) &\text{ cuts the } x\text{-axis at } -3, 0, \text{ and } 3.
 \end{aligned}$$

$$\begin{aligned}
 \text{Total area} &= \int_{-3}^0 (x^3 - 9x) dx - \int_0^3 (x^3 - 9x) dx \\
 &= \left[\frac{1}{4}x^4 - \frac{9}{2}x^2 \right]_{-3}^0 - \left[\frac{1}{4}x^4 - \frac{9}{2}x^2 \right]_0^3 \\
 &= \left(0 - \left(\frac{81}{4} - \frac{81}{2} \right) \right) - \left(\left(\frac{81}{4} - \frac{81}{2} \right) - 0 \right) \\
 &= 40\frac{1}{2} \text{ units}^2
 \end{aligned}$$



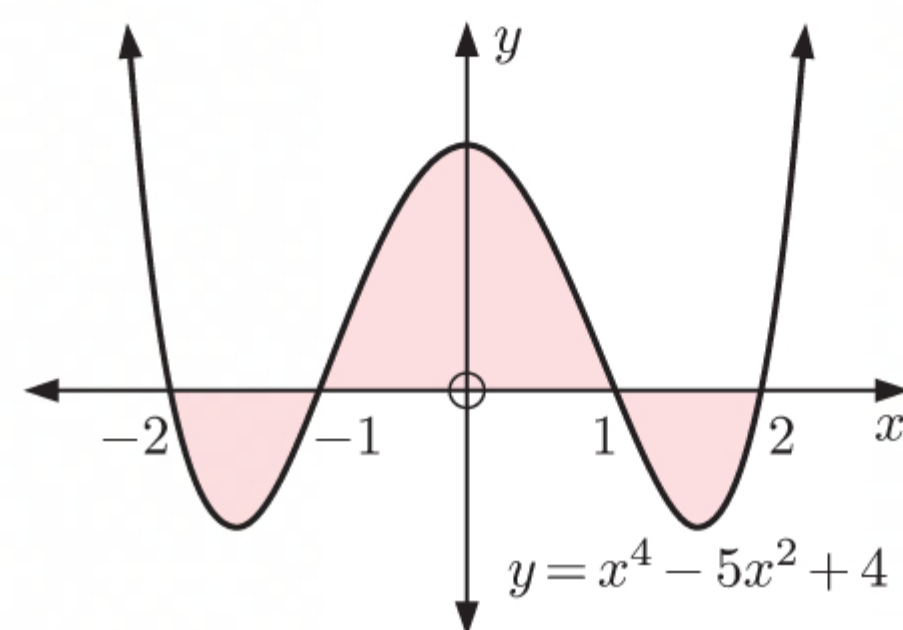
- b** $f(x) = -x(x-2)(x-4)$
 $\therefore y = f(x)$ cuts the x -axis at 0, 2, and 4.

$$\begin{aligned} f(x) &= -x(x-2)(x-4) \\ &= -x(x^2 - 6x + 8) \\ &= -x^3 + 6x^2 - 8x \end{aligned}$$



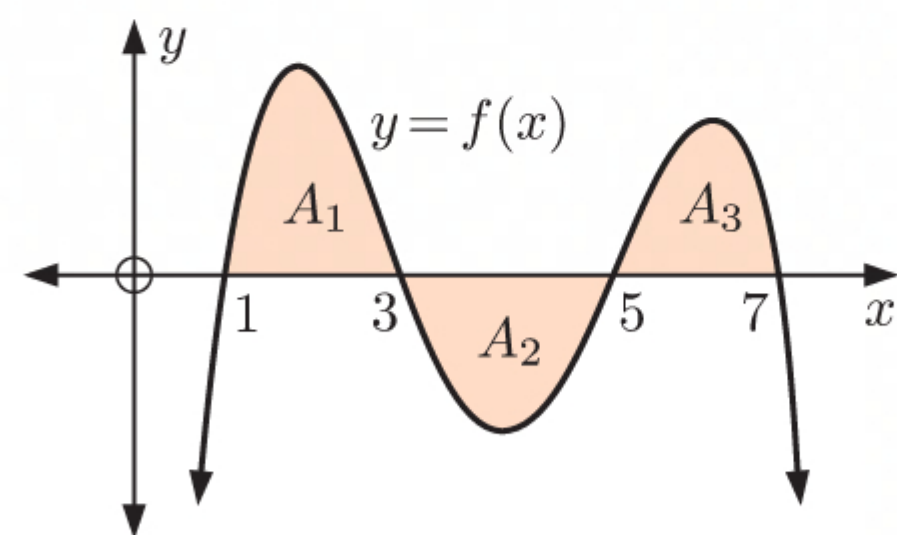
$$\begin{aligned} \text{Total area} &= -\int_0^2 (-x^3 + 6x^2 - 8x) dx + \int_2^4 (-x^3 + 6x^2 - 8x) dx \\ &= -\left[-\frac{1}{4}x^4 + 2x^3 - 4x^2\right]_0^2 + \left[-\frac{1}{4}x^4 + 2x^3 - 4x^2\right]_2^4 \\ &= -((-4 + 16 - 16) - 0) + ((-64 + 128 - 64) - (-4 + 16 - 16)) \\ &= 8 \text{ units}^2 \end{aligned}$$

- c** $f(x) = x^4 - 5x^2 + 4$
 $= (x^2 - 1)(x^2 - 4)$
 $= (x+1)(x-1)(x+2)(x-2)$
 $\therefore y = f(x)$ cuts the x -axis at -2, -1, 1, and 2.

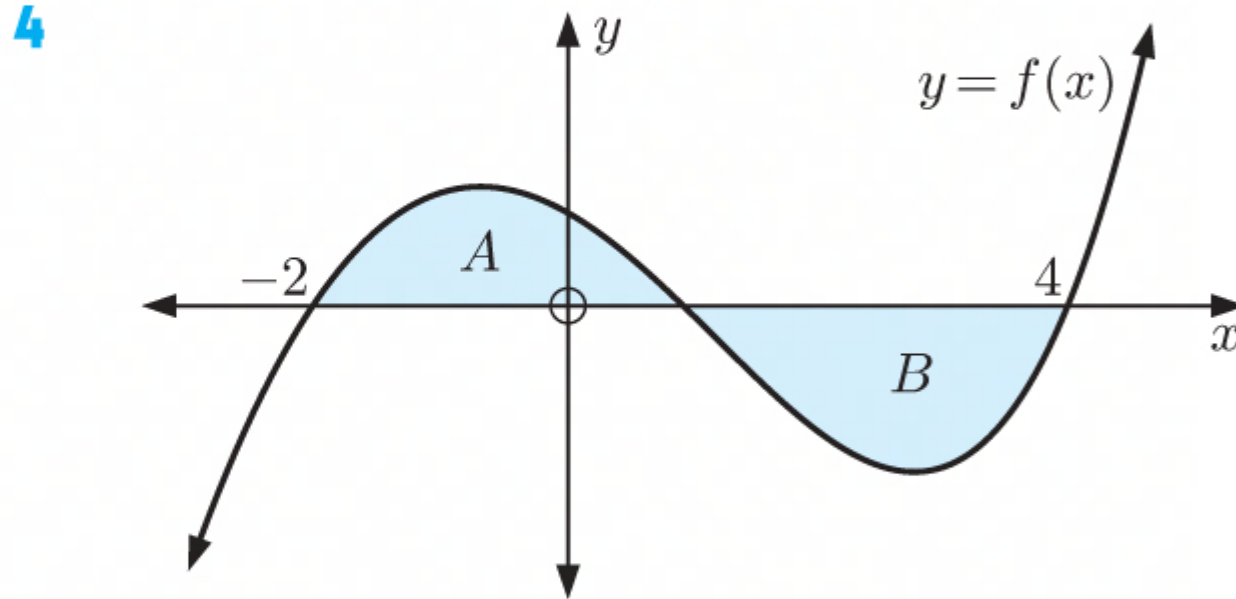


$$\begin{aligned} \text{Total area} &= -\int_{-2}^{-1} (x^4 - 5x^2 + 4) dx + \int_{-1}^1 (x^4 - 5x^2 + 4) dx - \int_1^2 (x^4 - 5x^2 + 4) dx \\ &= -\left[\frac{1}{5}x^5 - \frac{5}{3}x^3 + 4x\right]_{-2}^{-1} + \left[\frac{1}{5}x^5 - \frac{5}{3}x^3 + 4x\right]_{-1}^1 - \left[\frac{1}{5}x^5 - \frac{5}{3}x^3 + 4x\right]_1^2 \\ &= -\left[\left(-\frac{1}{5} + \frac{5}{3} - 4\right) - \left(-\frac{32}{5} + \frac{40}{3} - 8\right)\right] + \left[\left(\frac{1}{5} - \frac{5}{3} + 4\right) - \left(-\frac{1}{5} + \frac{5}{3} - 4\right)\right] \\ &\quad - \left[\left(\frac{32}{5} - \frac{40}{3} + 8\right) - \left(\frac{1}{5} - \frac{5}{3} + 4\right)\right] \\ &= -\left[-\frac{38}{15} + \frac{16}{15}\right] + \left[\frac{38}{15} + \frac{38}{15}\right] - \left[\frac{16}{15} - \frac{38}{15}\right] \\ &= \frac{22}{15} + \frac{76}{15} + \frac{22}{15} \\ &= 8 \text{ units}^2 \end{aligned}$$

- 3 a** $\int_1^7 f(x) dx$ only gives us the correct area provided that $f(x)$ is positive on the interval $1 \leq x \leq 7$. But $f(x)$ is not positive for $3 \leq x \leq 5$, so
- $$\int_1^7 f(x) dx = A_1 - A_2 + A_3 \quad \text{which is not the shaded area.}$$

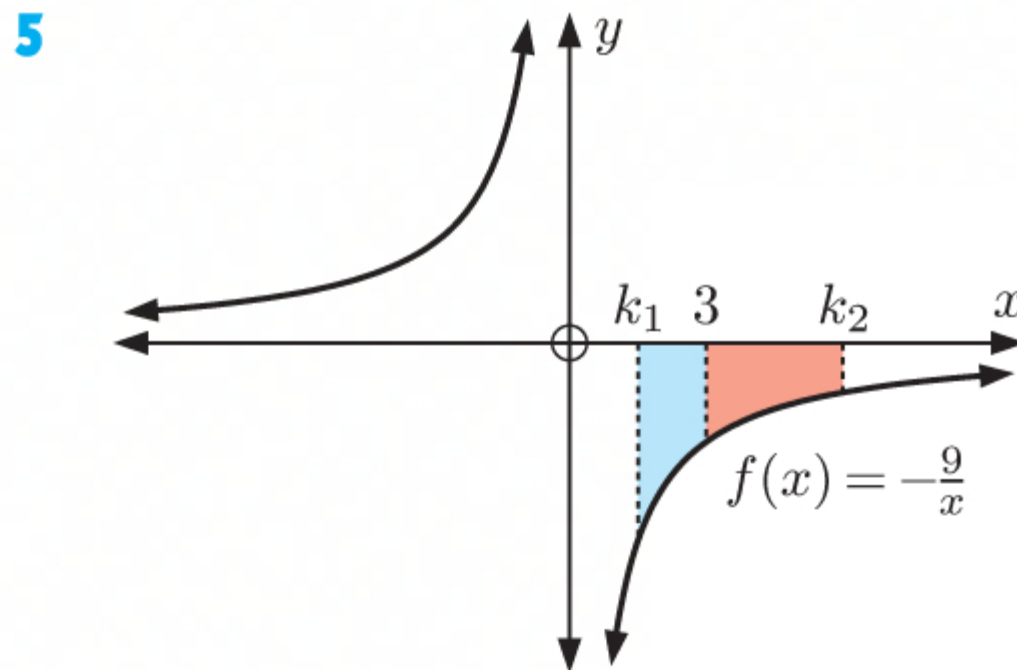


$$\begin{aligned}
 \text{b Total shaded area} &= \int_1^3 f(x) \, dx + \int_3^5 (-f(x)) \, dx + \int_5^7 f(x) \, dx \\
 &= \int_1^3 f(x) \, dx - \int_3^5 f(x) \, dx + \int_5^7 f(x) \, dx
 \end{aligned}$$



$$\begin{aligned}
 \int_{-2}^4 f(x) \, dx &= \text{area of region A} + (-\text{area of region B}) \quad \{\text{since region B is below the } x\text{-axis}\} \\
 &= \text{area of region A} - \text{area of region B} \\
 &= -6
 \end{aligned}$$

\therefore area of region B > area of region A



Blue area = red area = $9 \ln 2$ units²

$$\therefore - \int_{k_1}^3 -\frac{9}{x} \, dx = 9 \ln 2$$

$$\therefore \int_{k_1}^3 \frac{9}{x} \, dx = 9 \ln 2$$

$$\therefore [9 \ln |x|]_{k_1}^3 = 9 \ln 2$$

$$\therefore 9 \ln 3 - 9 \ln k_1 = 9 \ln 2 \quad \{k_1 > 0\}$$

$$\therefore \ln 3 - \ln k_1 = \ln 2$$

$$\therefore \ln k_1 = \ln 3 - \ln 2$$

$$\therefore \ln k_1 = \ln \left(\frac{3}{2} \right)$$

$$\therefore k_1 = \frac{3}{2}$$

$$\text{or} \quad - \int_3^{k_2} -\frac{9}{x} \, dx = 9 \ln 2$$

$$\therefore \int_3^{k_2} \frac{9}{x} \, dx = 9 \ln 2$$

$$\therefore [9 \ln |x|]_3^{k_2} = 9 \ln 2$$

$$\therefore 9 \ln k_2 - 9 \ln 3 = 9 \ln 2 \quad \{k_2 > 0\}$$

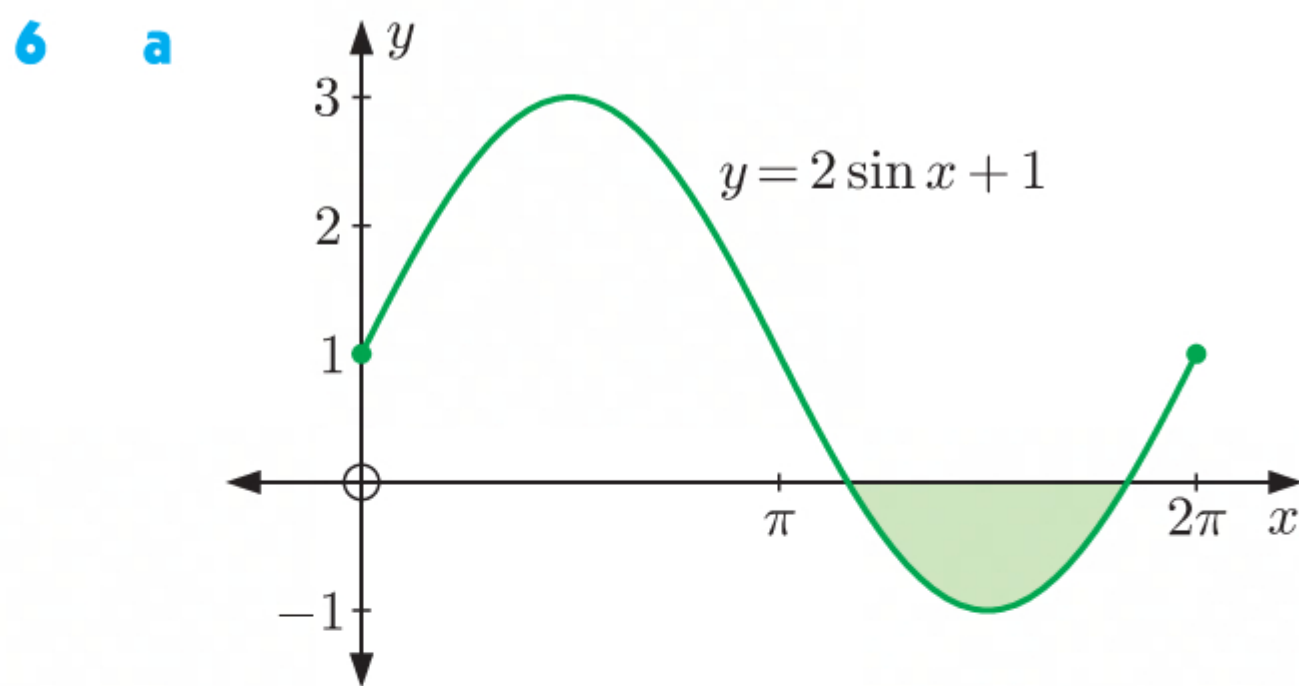
$$\therefore \ln k_2 - \ln 3 = \ln 2$$

$$\therefore \ln k_2 = \ln 2 + \ln 3$$

$$\therefore \ln k_2 = \ln 6$$

$$\therefore k_2 = 6$$

So, $k = \frac{3}{2}$ or 6



b The curve cuts the x -axis when $y = 0$

$$\therefore 2 \sin x + 1 = 0$$

$$\therefore 2 \sin x = -1$$

$$\therefore \sin x = -\frac{1}{2}$$

$$\therefore x = -\frac{\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}, \dots$$

\therefore the x -intercepts are $\frac{7\pi}{6}$ and $\frac{11\pi}{6}$.

$$\begin{aligned} \text{Area} &= - \int_{\frac{7\pi}{6}}^{\frac{11\pi}{6}} (2 \sin x + 1) dx \\ &= - \left[-2 \cos x + x \right]_{\frac{7\pi}{6}}^{\frac{11\pi}{6}} \\ &= - \left[\left(-2 \cos \frac{11\pi}{6} + \frac{11\pi}{6} \right) - \left(-2 \cos \frac{7\pi}{6} + \frac{7\pi}{6} \right) \right] \\ &= - \left[\left(-\sqrt{3} + \frac{11\pi}{6} \right) - \left(\sqrt{3} + \frac{7\pi}{6} \right) \right] \\ &= \left(2\sqrt{3} - \frac{2\pi}{3} \right) \text{ units}^2 \\ &\approx 1.37 \text{ units}^2 \end{aligned}$$

7 a Cross-sectional area of gutter

$$\begin{aligned} &= - \int_{-5}^5 (0.02x^4 - 0.4x^2 - 2.5) dx \\ &= - \left[0.004x^5 - \frac{0.4}{3}x^3 - 2.5x \right]_{-5}^5 \\ &= - \left[-16\frac{2}{3} - 16\frac{2}{3} \right] \\ &= 33\frac{1}{3} \text{ cm}^2 \end{aligned}$$

b Volume of gutter

$$= \text{area of cross-section} \times \text{length of gutter}$$

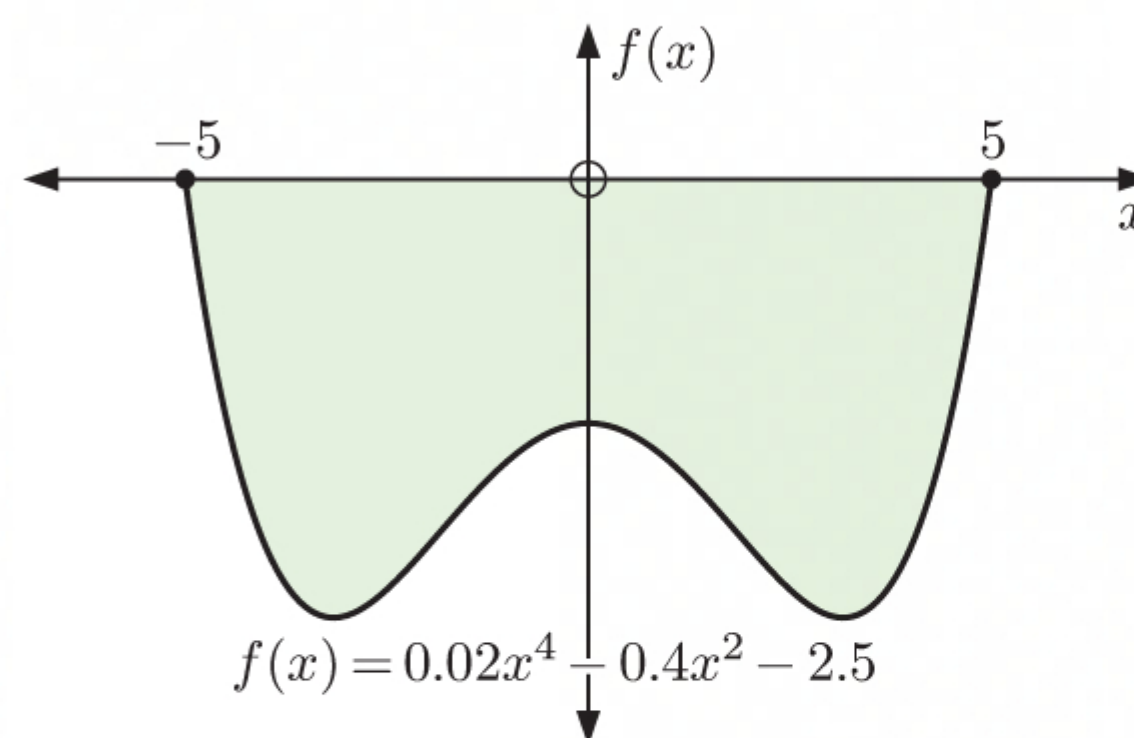
$$= 33\frac{1}{3} \text{ cm}^2 \times 20 \text{ m}$$

$$= 33\frac{1}{3} \text{ cm}^2 \times 2000 \text{ cm}$$

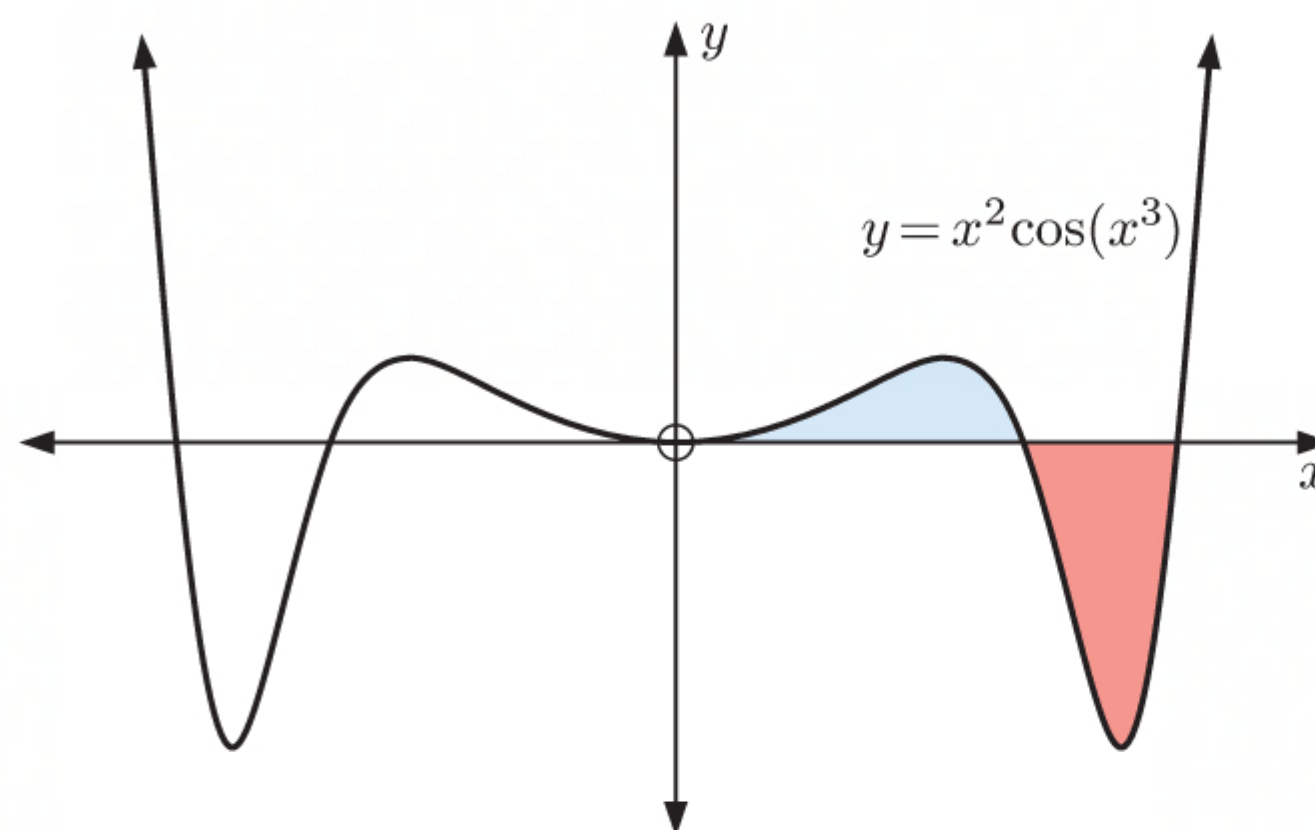
$$= 66\,666\frac{2}{3} \text{ cm}^3$$

\therefore the gutter can hold $66\,666\frac{2}{3} \text{ mL}$ $\{1 \text{ cm}^3 \equiv 1 \text{ mL}\}$

$\approx 66.7 \text{ L}$ of water in total.



$$\begin{aligned}
 8 \quad a \quad & \int x^2 \cos(x^3) dx \\
 &= \frac{1}{3} \int 3x^2 \cos(x^3) dx \\
 &= \frac{1}{3} \int \cos u \frac{du}{dx} dx \\
 &\quad \{u = x^3, \quad \frac{du}{dx} = 3x^2\} \\
 &= \frac{1}{3} \int \cos u du \\
 &= \frac{1}{3} \sin u + c \\
 &= \frac{1}{3} \sin(x^3) + c
 \end{aligned}$$



b $y = x^2 \cos(x^3)$ has x -intercepts when $y = 0$

$$\therefore x^2 \cos(x^3) = 0$$

$$\therefore x^2 = 0 \text{ or } \cos(x^3) = 0$$

$\therefore x = 0$, and the two smallest positive x -intercepts occur when $x^3 = \frac{\pi}{2}$ or $\frac{3\pi}{2}$

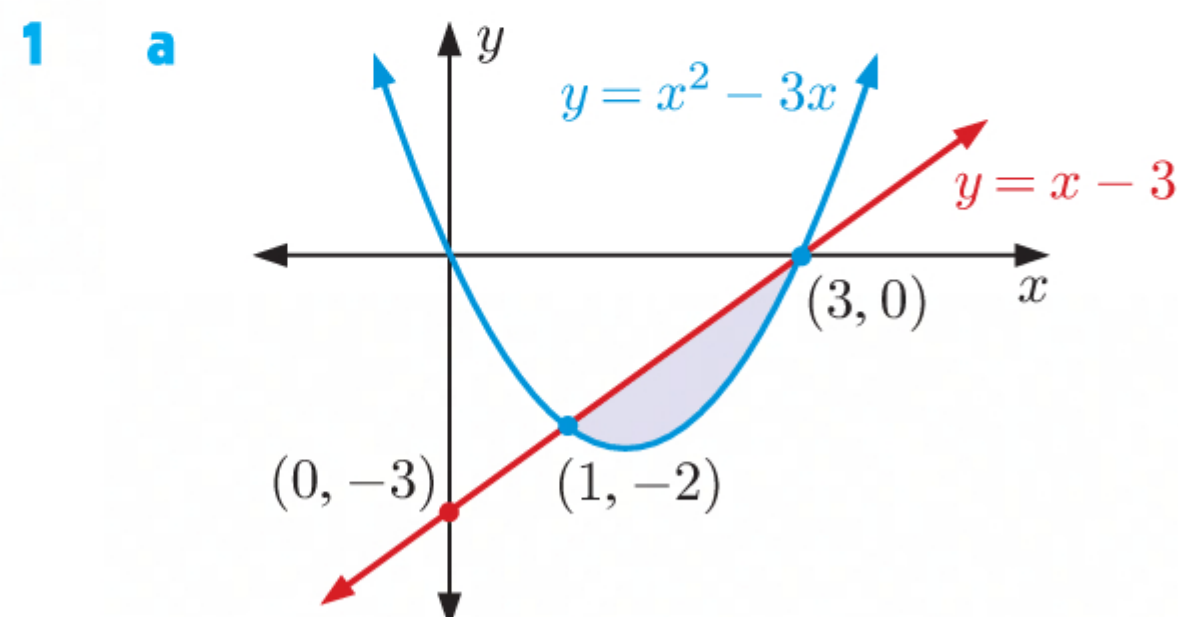
$$\therefore x = \sqrt[3]{\frac{\pi}{2}} \text{ or } \sqrt[3]{\frac{3\pi}{2}}$$

$$\begin{aligned}
 \therefore \text{blue area} &= \int_0^{\sqrt[3]{\frac{\pi}{2}}} x^2 \cos(x^3) dx \\
 &= \left[\frac{1}{3} \sin(x^3) \right]_0^{\sqrt[3]{\frac{\pi}{2}}} \\
 &= \left(\frac{1}{3} \sin \frac{\pi}{2} \right) - \left(\frac{1}{3} \sin 0 \right) \\
 &= \frac{1}{3} \text{ units}^2
 \end{aligned}$$

$$\begin{aligned}
 \therefore \text{red area} &= - \int_{\sqrt[3]{\frac{\pi}{2}}}^{\sqrt[3]{\frac{3\pi}{2}}} x^2 \cos(x^3) dx \\
 &= - \left[\frac{1}{3} \sin(x^3) \right]_{\sqrt[3]{\frac{\pi}{2}}}^{\sqrt[3]{\frac{3\pi}{2}}} \\
 &= - \left[\left(\frac{1}{3} \sin \frac{3\pi}{2} \right) - \left(\frac{1}{3} \sin \frac{\pi}{2} \right) \right] \\
 &= - \left[\left(-\frac{1}{3} \right) - \frac{1}{3} \right] \\
 &= \frac{2}{3} \text{ units}^2
 \end{aligned}$$

\therefore the red shaded region is twice as large as the blue shaded region.

EXERCISE 17D



b $y = x - 3$ meets $y = x^2 - 3x$
 where $x^2 - 3x = x - 3$
 $\therefore x^2 - 4x + 3 = 0$
 $\therefore (x - 1)(x - 3) = 0$
 $\therefore x = 1 \text{ or } 3$

When $x = 1$, $y = 1 - 3 = -2$

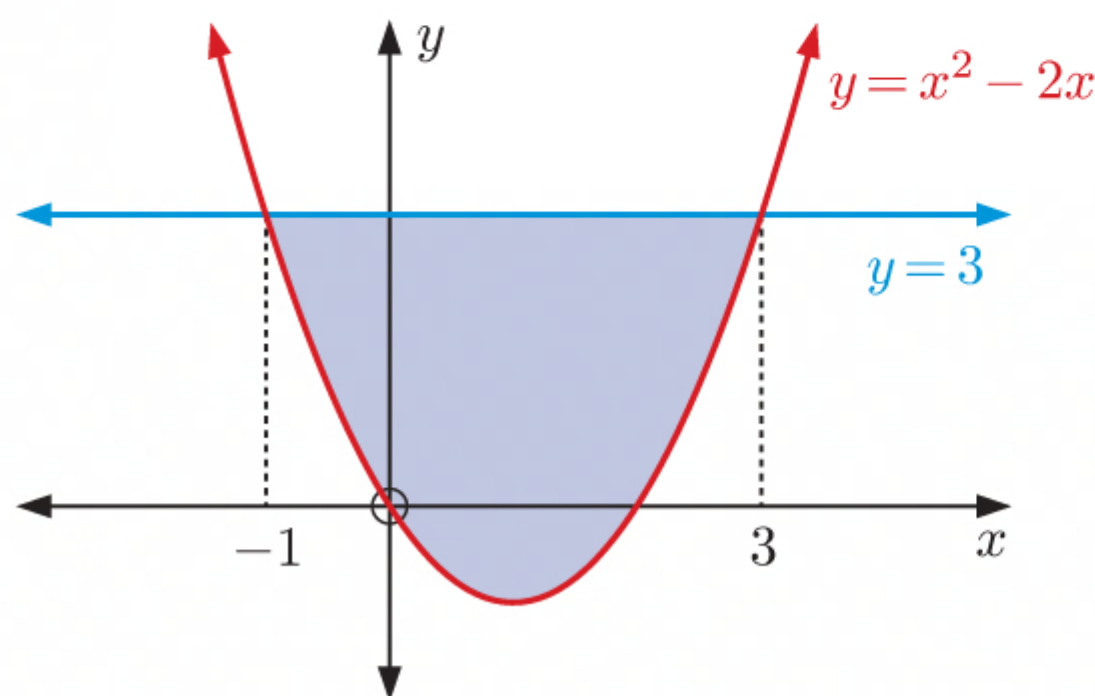
When $x = 3$, $y = 3 - 3 = 0$

\therefore the graphs meet at the points $(1, -2)$ and $(3, 0)$.

$$\begin{aligned}
 \text{c Area} &= \int_1^3 [(x-3) - (x^2-3x)] dx \\
 &= \int_1^3 (-x^2 + 4x - 3) dx \\
 &= \left[-\frac{1}{3}x^3 + 2x^2 - 3x\right]_1^3 \\
 &= (-9 + 18 - 9) - \left(-\frac{1}{3} + 2 - 3\right) \\
 &= 0 - \left(-1\frac{1}{3}\right) \\
 &= 1\frac{1}{3} \text{ units}^2
 \end{aligned}$$

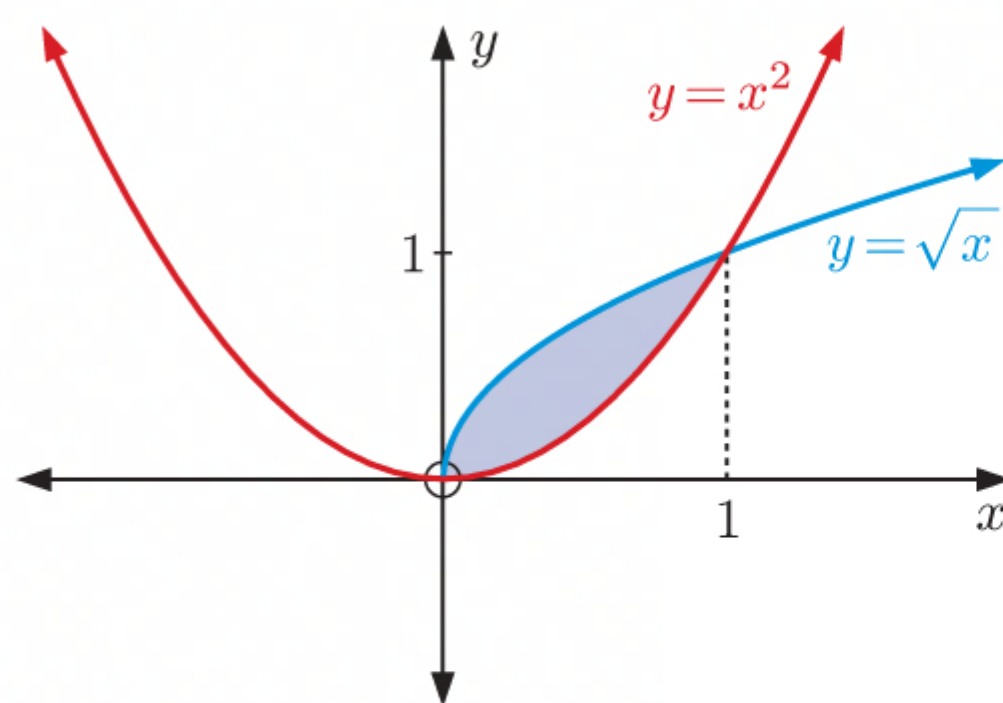
$$\begin{aligned}
 \text{2 a } y = x^2 - 2x \text{ meets } y = 3 \\
 \text{where } x^2 - 2x = 3 \\
 \therefore x^2 - 2x - 3 = 0 \\
 \therefore (x+1)(x-3) = 0 \\
 \therefore x = -1 \text{ or } 3
 \end{aligned}$$

$$\begin{aligned}
 \text{Area} &= \int_{-1}^3 [3 - (x^2 - 2x)] dx \\
 &= \int_{-1}^3 (-x^2 + 2x + 3) dx \\
 &= \left[-\frac{1}{3}x^3 + x^2 + 3x\right]_{-1}^3 \\
 &= (-9 + 9 + 9) - \left(\frac{1}{3} + 1 - 3\right) \\
 &= 9 + 1\frac{2}{3} \\
 &= 10\frac{2}{3} \text{ units}^2
 \end{aligned}$$



$$\begin{aligned}
 \text{b } y = \sqrt{x} \text{ meets } y = x^2 \\
 \text{where } \sqrt{x} = x^2 \\
 \therefore x = x^4 \\
 \therefore x^4 - x = 0 \\
 \therefore x(x^3 - 1) = 0 \\
 \therefore x = 0 \text{ or } 1
 \end{aligned}$$

$$\begin{aligned}
 \text{Area} &= \int_0^1 [\sqrt{x} - x^2] dx \\
 &= \int_0^1 (x^{\frac{1}{2}} - x^2) dx \\
 &= \left[\frac{2}{3}x^{\frac{3}{2}} - \frac{1}{3}x^3\right]_0^1 \\
 &= \left(\frac{2}{3} - \frac{1}{3}\right) - 0 \\
 &= \frac{1}{3} \text{ units}^2
 \end{aligned}$$



3 $y = 2e^x$ meets $y = e^{2x}$

where $2e^x = e^{2x}$

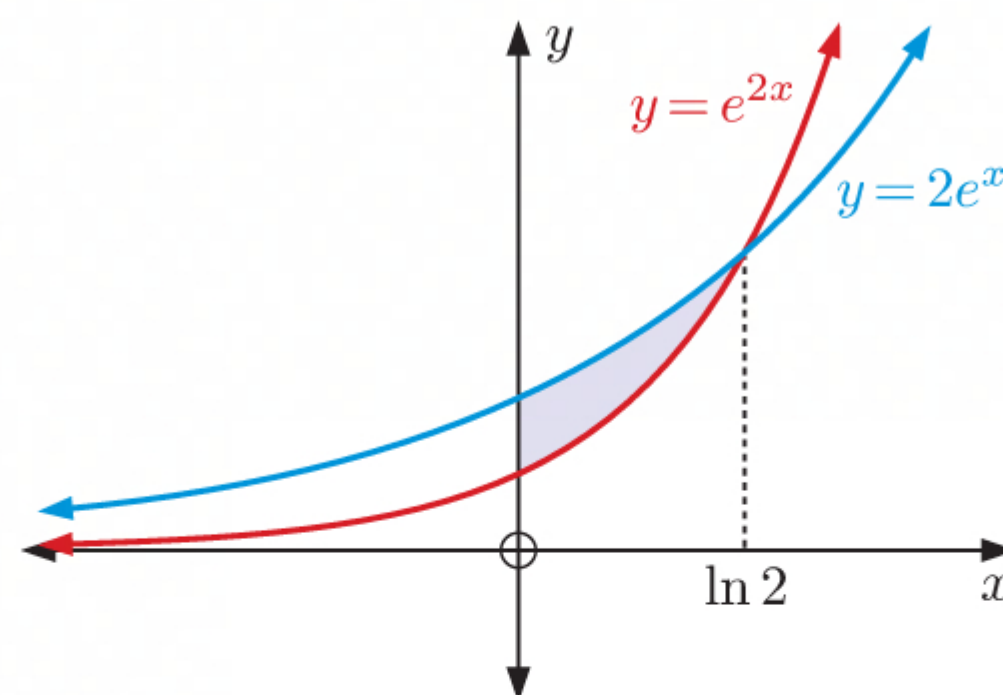
$$\therefore e^{2x} - 2e^x = 0$$

$$\therefore e^x(e^x - 2) = 0$$

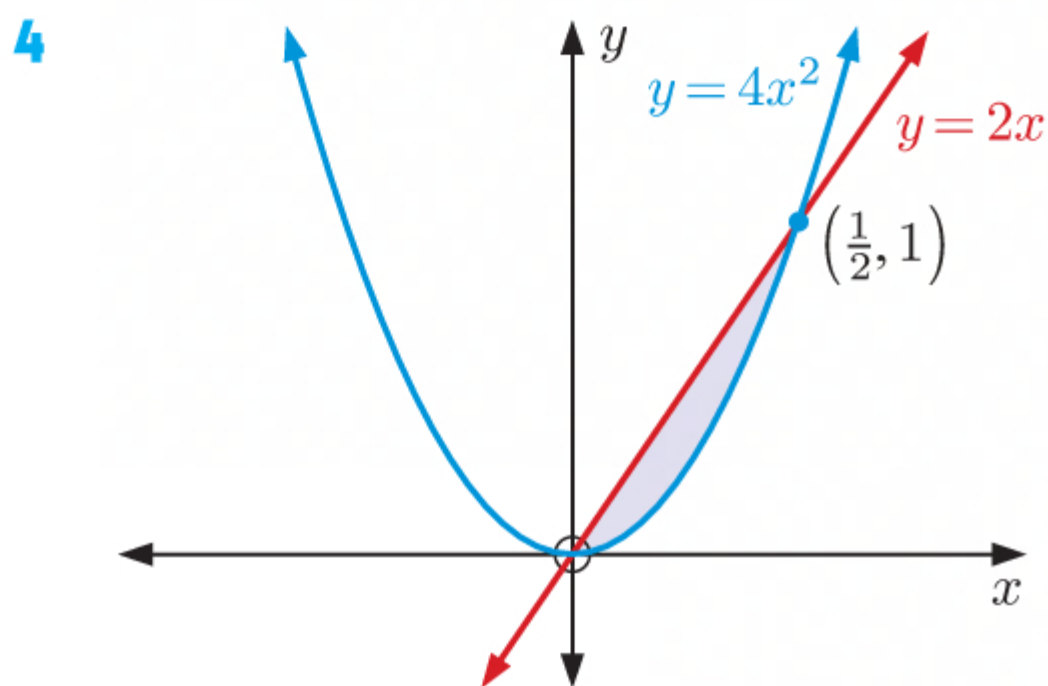
$$\therefore e^x = 0 \text{ or } 2$$

$$\therefore x = \ln 2$$

{since $e^x > 0$ for all $x \in \mathbb{R}$ }



$$\begin{aligned} \text{Area} &= \int_0^{\ln 2} [2e^x - e^{2x}] dx \\ &= \left[2e^x - \frac{1}{2}e^{2x} \right]_0^{\ln 2} \\ &= \left[2(2) - \frac{1}{2}(e^{\ln(2^2)}) \right] - \left[2(1) - \frac{1}{2}(1) \right] \\ &= (4 - 2) - 1\frac{1}{2} \\ &= \frac{1}{2} \text{ units}^2 \end{aligned}$$



$$\begin{aligned} y = 2x \text{ meets } y &= 4x^2 \\ \text{where } 4x^2 &= 2x \\ \therefore 4x^2 - 2x &= 0 \\ \therefore 2x(2x - 1) &= 0 \\ \therefore x &= 0 \text{ or } \frac{1}{2} \end{aligned}$$

$$\begin{aligned} \text{Area} &= \int_0^{\frac{1}{2}} [2x - 4x^2] dx \\ &= \left[x^2 - \frac{4}{3}x^3 \right]_0^{\frac{1}{2}} \\ &= \left(\frac{1}{4} - \frac{4}{3} \left(\frac{1}{8} \right) \right) - 0 \\ &= \frac{1}{12} \text{ units}^2 \end{aligned}$$

5 a $y = \frac{5}{2x+1}$ and $y = 3 - x$ meet where

$$\frac{5}{2x+1} = 3 - x$$

$$\therefore 5 = (3 - x)(2x + 1)$$

$$\therefore 5 = 6x + 3 - 2x^2 - x$$

$$\therefore 2x^2 - 5x + 2 = 0$$

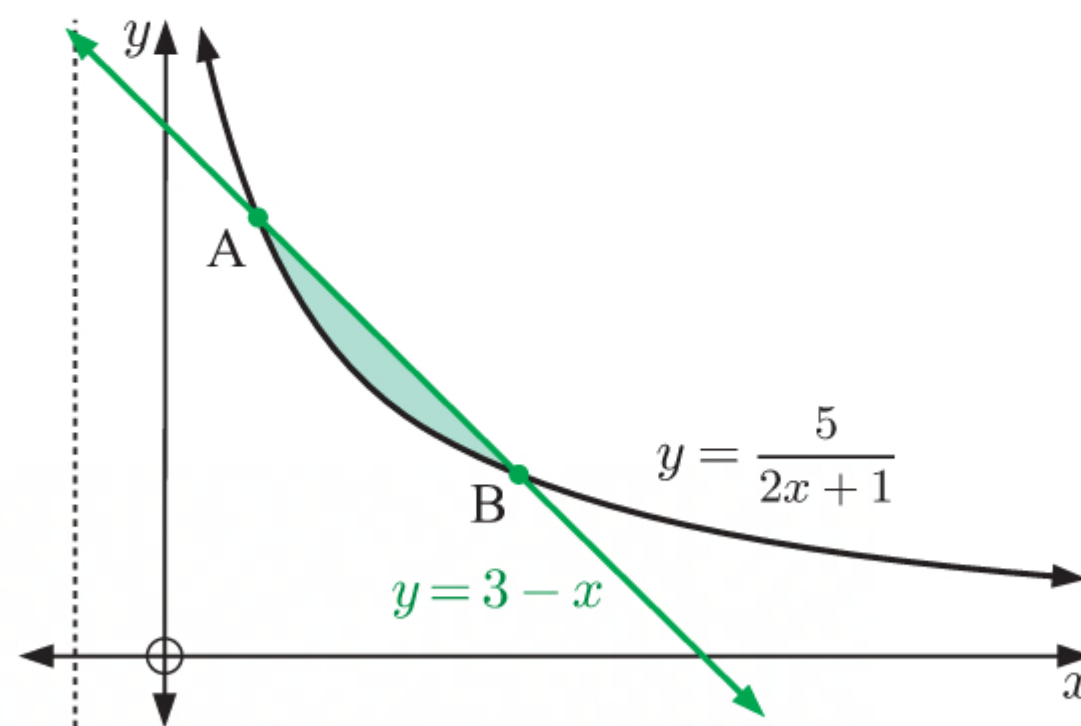
$$\therefore (2x - 1)(x - 2) = 0$$

$$\therefore x = \frac{1}{2} \text{ or } 2$$

When $x = \frac{1}{2}$, $y = 3 - \frac{1}{2} = \frac{5}{2}$

When $x = 2$, $y = 3 - 2 = 1$

\therefore A has coordinates $(\frac{1}{2}, \frac{5}{2})$ and B has coordinates $(2, 1)$.



$$\begin{aligned}
 \text{b Area} &= \int_{\frac{1}{2}}^2 \left((3-x) - \frac{5}{2x+1} \right) dx \\
 &= \left[3x - \frac{1}{2}x^2 - \frac{5}{2} \ln |2x+1| \right]_{\frac{1}{2}}^2 \\
 &= \left(6 - 2 - \frac{5}{2} \ln 5 \right) - \left(\frac{3}{2} - \frac{1}{8} - \frac{5}{2} \ln 2 \right) \\
 &= 4 - \frac{5}{2} \ln 5 - \frac{11}{8} + \frac{5}{2} \ln 2 \\
 &= \frac{21}{8} - \frac{5}{2} (\ln 5 - \ln 2) \\
 &= \left(\frac{21}{8} - \frac{5}{2} \ln \left(\frac{5}{2} \right) \right) \text{ units}^2
 \end{aligned}$$

$$\begin{aligned}
 \text{6 } y = x^2 \text{ meets } y = k \text{ where } x^2 = k \\
 \therefore x = \pm \sqrt{k}
 \end{aligned}$$

$$\text{Now, the area} = \int_0^{\sqrt{k}} (k - x^2) dx$$

$$\therefore \int_0^{\sqrt{k}} (k - x^2) dx = 2.4$$

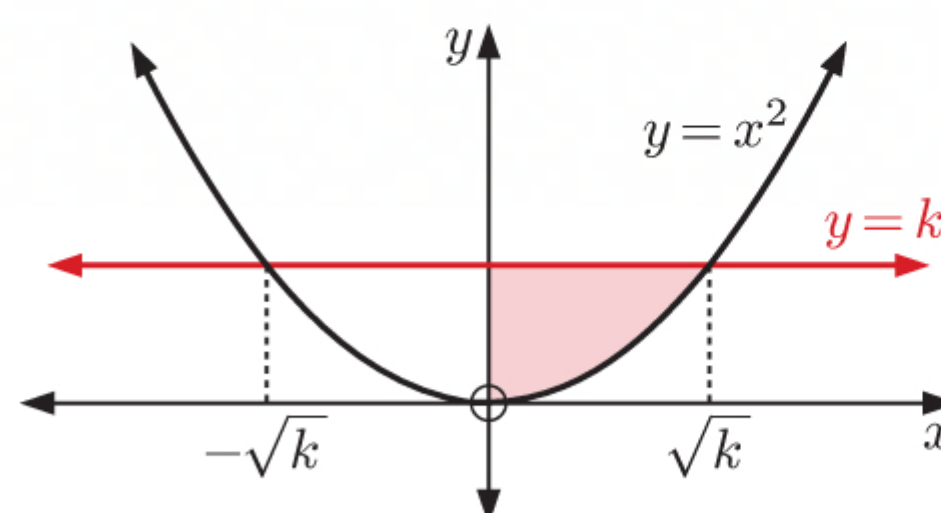
$$\therefore \left[kx - \frac{x^3}{3} \right]_0^{\sqrt{k}} = 2.4$$

$$\therefore \left(k\sqrt{k} - \frac{k\sqrt{k}}{3} \right) - 0 = 2.4$$

$$\therefore \frac{2k\sqrt{k}}{3} = 2.4$$

$$\therefore k^{\frac{3}{2}} = 3.6$$

$$\begin{aligned}
 \therefore k &= (3.6)^{\frac{2}{3}} \\
 &\approx 2.3489
 \end{aligned}$$



$$\text{7 } y = \sin x \text{ meets } y = \cos x$$

$$\text{where } \sin x = \cos x$$

$$\therefore \tan x = 1$$

$$\therefore x = \frac{\pi}{4}$$

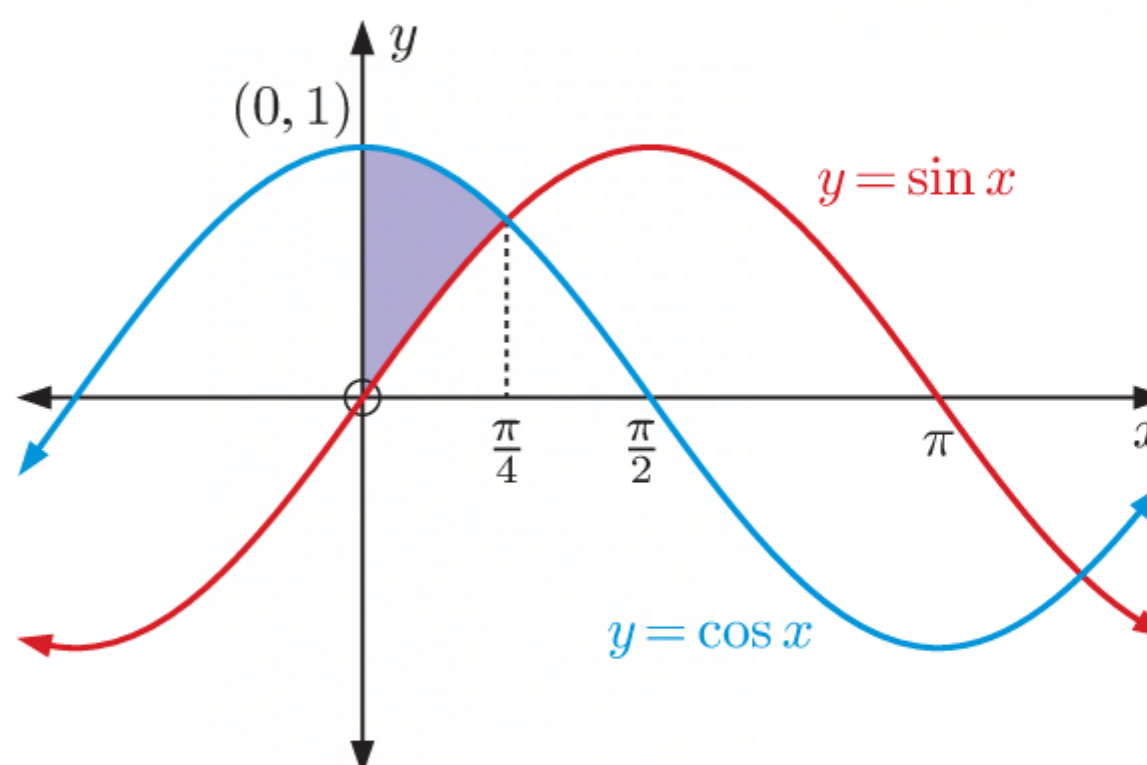
$$\text{Area} = \int_0^{\frac{\pi}{4}} [\cos x - \sin x] dx$$

$$= \left[\sin x + \cos x \right]_0^{\frac{\pi}{4}}$$

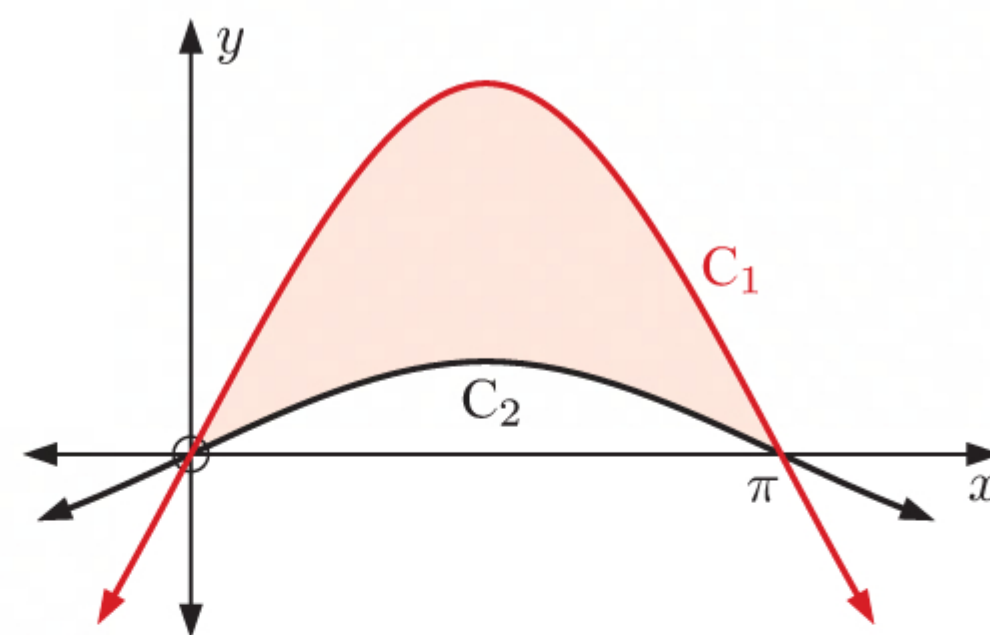
$$= \left(\sin \frac{\pi}{4} + \cos \frac{\pi}{4} \right) - (\sin 0 + \cos 0)$$

$$= \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} \right) - (0 + 1)$$

$$= (\sqrt{2} - 1) \text{ units}^2$$

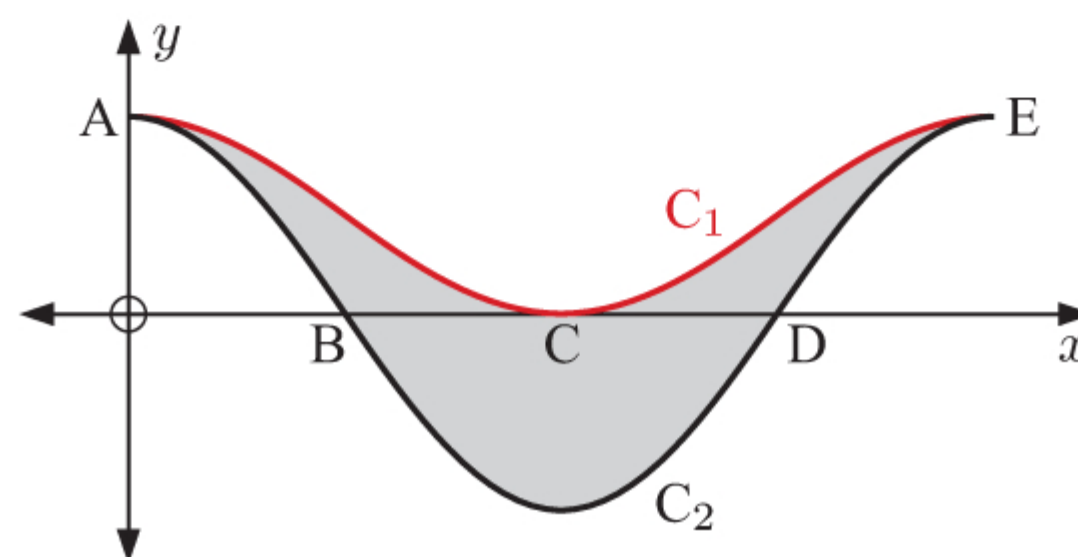


- 8 a** $y = 4 \sin x$ has amplitude 4, which is larger than the amplitude of $y = \sin x$ which has amplitude 1.
 $\therefore y = 4 \sin x$ is the curve C_1 and
 $y = \sin x$ is the curve C_2 .



b Area = $\int_0^{\pi} [4 \sin x - \sin x] dx$
 $= \int_0^{\pi} 3 \sin x dx$
 $= [-3 \cos x]_0^{\pi}$
 $= (-3 \cos \pi) - (-3 \cos 0)$
 $= 3 - (-3)$
 $= 6 \text{ units}^2$

- 9 a** $\cos^2 x \geq 0$ for all x , so $y = \cos^2 x$ must lie above the x -axis.
 $\therefore y = \cos^2 x$ is the curve C_1 and
 $y = \cos 2x$ is the curve C_2 .



b $y = \cos 2x$ meets $y = \cos^2 x$ where $\cos 2x = \cos^2 x$
 $\therefore 2 \cos^2 x - 1 = \cos^2 x$
 $\therefore \cos^2 x = 1$
 $\therefore \cos x = \pm 1$
 $\therefore x = 0, \pi$

So, points A, B, C, D, and E have x -coordinates between 0 and π .

Now, $y = \cos 2x$ cuts the x -axis where $\cos 2x = 0$
 $\therefore 2x = \frac{\pi}{2}, \frac{3\pi}{2}$
 $\therefore x = \frac{\pi}{4}, \frac{3\pi}{4}$

and $y = \cos^2 x$ cuts the x -axis where $\cos^2 x = 0$
 $\therefore \cos x = 0$
 $\therefore x = \frac{\pi}{2}$

Point A lies where the curves first meet, so has x -coordinate 0.

When $x = 0$, $y = \cos 0 = 1$, so A is $(0, 1)$.

Point B is the first x -intercept of $y = \cos 2x$, so B is $(\frac{\pi}{4}, 0)$.

Point C is the x -intercept of $y = \cos^2 x$, so C is $(\frac{\pi}{2}, 0)$.

Point D is the second x -intercept of $y = \cos 2x$, so D is $(\frac{3\pi}{4}, 0)$.

Point E lies where the curves meet for the second time, so has x -coordinate π .

When $x = \pi$, $y = \cos 2\pi = 1$, so E is $(\pi, 1)$.

$$\begin{aligned}
 \text{c Area} &= \int_0^{\pi} (\cos^2 x - \cos 2x) dx \\
 &= \int_0^{\pi} \left(\frac{1}{2} + \frac{1}{2} \cos 2x - \cos 2x \right) dx \\
 &= \int_0^{\pi} \left(\frac{1}{2} - \frac{1}{2} \cos 2x \right) dx \\
 &= \left[\frac{x}{2} - \frac{1}{4} \sin 2x \right]_0^{\pi} \\
 &= \left(\frac{\pi}{2} - 0 \right) - (0 - 0) \\
 &= \frac{\pi}{2} \text{ units}^2
 \end{aligned}$$

10 a $y = e^x - 1$ has no vertical asymptotes.

As $x \rightarrow \infty$, $e^x - 1 \rightarrow \infty$

As $x \rightarrow -\infty$, $e^x \rightarrow 0^+$
so $e^x - 1 \rightarrow -1^+$

$\therefore y = -1$ is a horizontal asymptote.

$y = 0$ when $e^x - 1 = 0$

$$\therefore e^x = 1$$

$$\therefore x = 0$$

\therefore the x -intercept is 0.

This is also the y -intercept.

$y = 2 - 2e^{-x}$ has no vertical asymptotes.

As $x \rightarrow \infty$, $e^{-x} \rightarrow 0^+$

so $2 - 2e^{-x} \rightarrow 2^-$

$\therefore y = 2$ is a horizontal asymptote.

As $x \rightarrow -\infty$, $2 - 2e^{-x} \rightarrow -\infty$

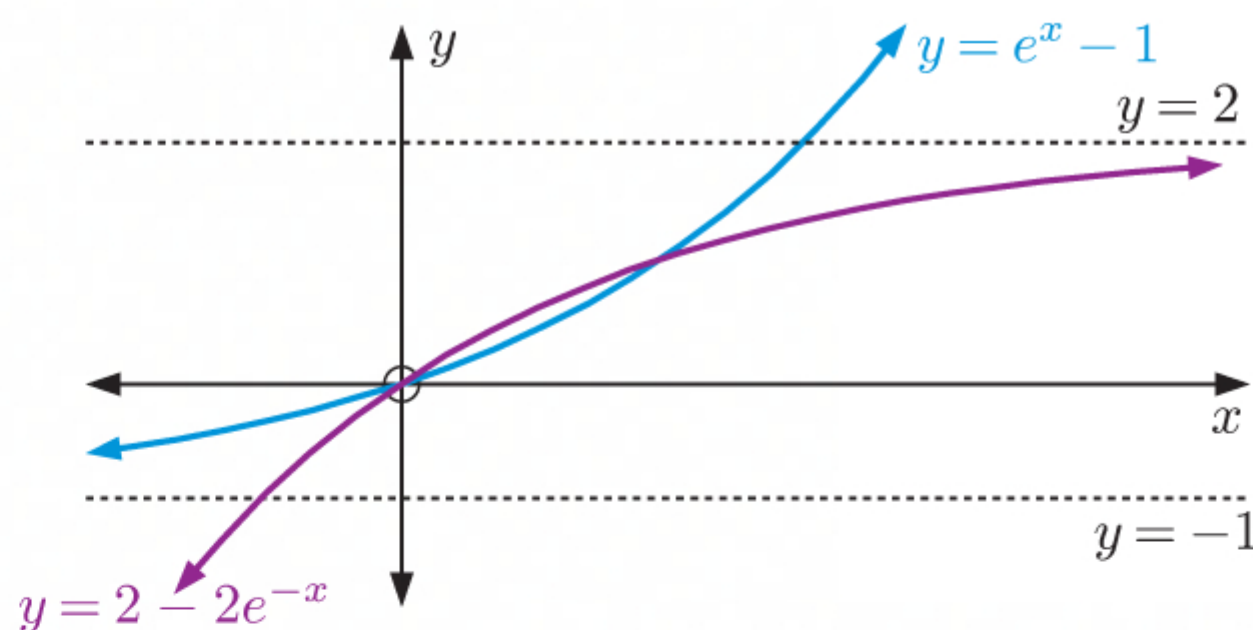
$y = 0$ when $2 - 2e^{-x} = 0$

$$\therefore e^{-x} = 1$$

$$\therefore x = 0$$

\therefore the x -intercept is 0.

This is also the y -intercept.



b $y = e^x - 1$ meets $y = 2 - 2e^{-x}$ where $e^x - 1 = 2 - 2e^{-x}$

$$\therefore e^{2x} - e^x = 2e^x - 2 \quad \{ \times e^x \}$$

$$\therefore e^{2x} - 3e^x + 2 = 0$$

$$\therefore (e^x - 1)(e^x - 2) = 0$$

$$\therefore e^x = 1 \text{ or } 2$$

$$\therefore x = 0 \text{ or } \ln 2$$

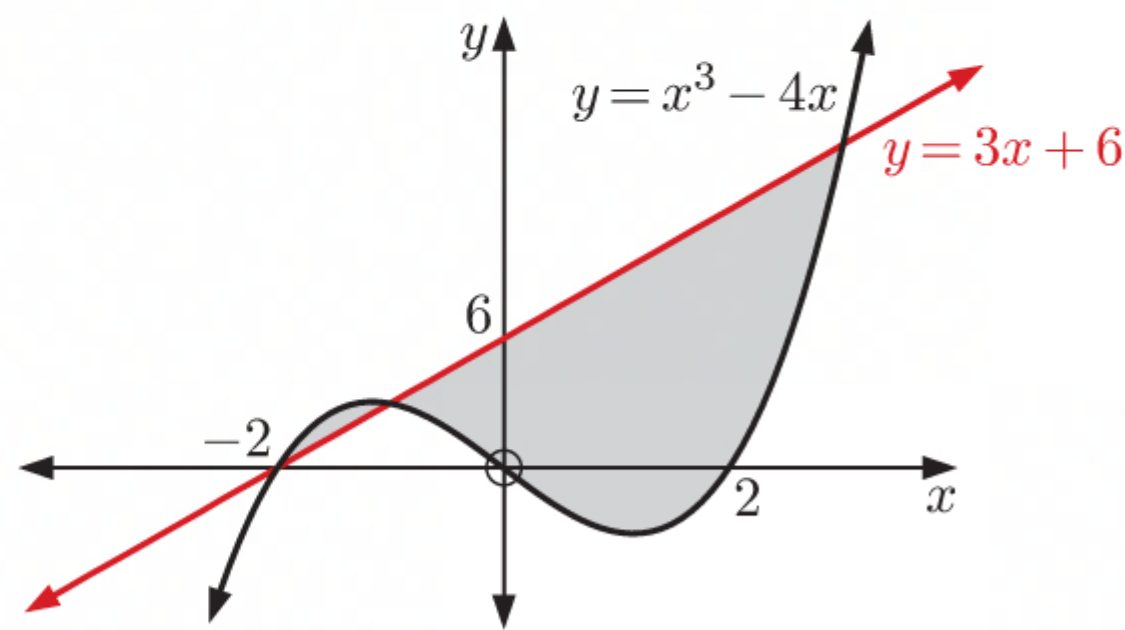
When $x = 0$, $y = e^0 - 1 = 0$

When $x = \ln 2$, $y = e^{\ln 2} - 1 = 1$

\therefore the graphs meet at $(0, 0)$ and $(\ln 2, 1)$.

$$\begin{aligned}
 \text{c Area} &= \int_0^{\ln 2} [(2 - 2e^{-x}) - (e^x - 1)] dx \\
 &= \int_0^{\ln 2} (3 - e^x - 2e^{-x}) dx \\
 &= [3x - e^x + 2e^{-x}]_0^{\ln 2} \\
 &= (3 \ln 2 - 2 + 1) - (0 - 1 + 2) \\
 &= (3 \ln 2 - 2) \text{ units}^2
 \end{aligned}$$

11 a



The graphs meet where $x^3 - 4x = 3x + 6$

$$\therefore x^3 - 7x - 6 = 0$$

$$\therefore (x + 2)(x^2 - 2x - 3) = 0 \quad \{\text{diagram shows intersection at } -2\}$$

$$\therefore (x + 2)(x + 1)(x - 3) = 0$$

$$\therefore x = -2, -1, \text{ or } 3$$

$$\text{Total area} = \int_{-2}^{-1} [(x^3 - 4x) - (3x + 6)] dx + \int_{-1}^3 [(3x + 6) - (x^3 - 4x)] dx$$

$$= \int_{-2}^{-1} (x^3 - 7x - 6) dx + \int_{-1}^3 (-x^3 + 7x + 6) dx$$

$$\text{b Total area} = \int_{-2}^{-1} (x^3 - 7x - 6) dx + \int_{-1}^3 (-x^3 + 7x + 6) dx \quad \{\text{from a}\}$$

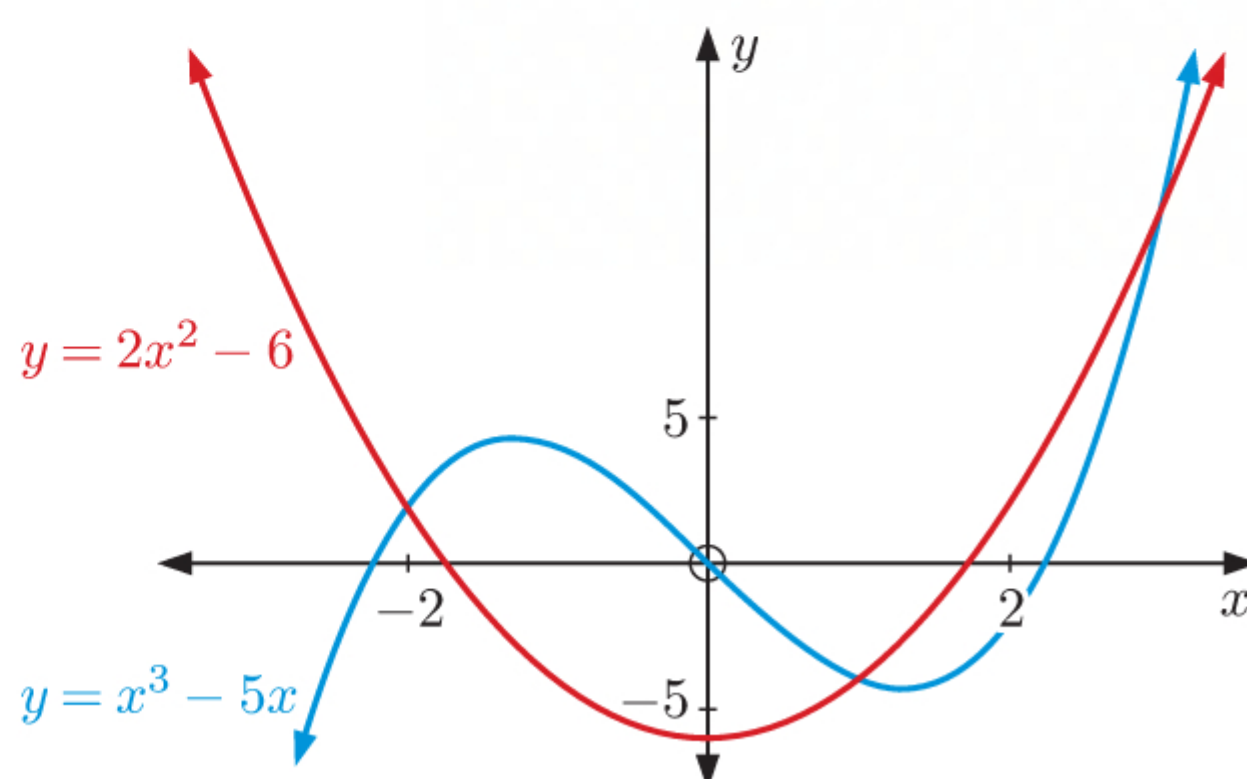
$$= \left[\frac{1}{4}x^4 - \frac{7}{2}x^2 - 6x \right]_{-2}^{-1} + \left[-\frac{1}{4}x^4 + \frac{7}{2}x^2 + 6x \right]_{-1}^3$$

$$= \left[\left(\frac{1}{4} - \frac{7}{2} - 6 \right) - (4 - 14 + 12) \right] + \left[\left(-\frac{81}{4} + \frac{63}{2} + 18 \right) - \left(-\frac{1}{4} + \frac{7}{2} - 6 \right) \right]$$

$$= \left(\frac{11}{4} - 2 \right) + \left(\frac{117}{4} + \frac{11}{4} \right)$$

$$= 32\frac{3}{4} \text{ units}^2$$

12 a



b $y = x^3 - 5x$ meets $y = 2x^2 - 6$ where $x^3 - 5x = 2x^2 - 6$

$$\therefore x^3 - 2x^2 - 5x + 6 = 0$$

$$\therefore (x+2)(x-1)(x-3) = 0$$

$$\therefore x = -2, 1, \text{ or } 3$$

\therefore the intersection points have x -coordinates $-2, 1$, and 3 respectively.

c Area = $\int_{-2}^1 [(x^3 - 5x) - (2x^2 - 6)] dx + \int_1^3 [(2x^2 - 6) - (x^3 - 5x)] dx$

$$= \int_{-2}^1 (x^3 - 2x^2 - 5x + 6) dx + \int_1^3 (-x^3 + 2x^2 + 5x - 6) dx$$

$$= \left[\frac{1}{4}x^4 - \frac{2}{3}x^3 - \frac{5}{2}x^2 + 6x \right]_{-2}^1 + \left[-\frac{1}{4}x^4 + \frac{2}{3}x^3 + \frac{5}{2}x^2 - 6x \right]_1^3$$

$$= \left[\left(\frac{1}{4} - \frac{2}{3} - \frac{5}{2} + 6 \right) - \left(4 + \frac{16}{3} - 10 - 12 \right) \right]$$

$$+ \left[\left(-\frac{81}{4} + 18 + \frac{45}{2} - 18 \right) - \left(-\frac{1}{4} + \frac{2}{3} + \frac{5}{2} - 6 \right) \right]$$

$$= \left(\frac{37}{12} + \frac{38}{3} \right) + \left(\frac{9}{4} + \frac{37}{12} \right)$$

$$= 21\frac{1}{12} \text{ units}^2$$

13 a $y = -x^3 + 3x^2 + 6x - 8$ meets $y = 5x - 5$

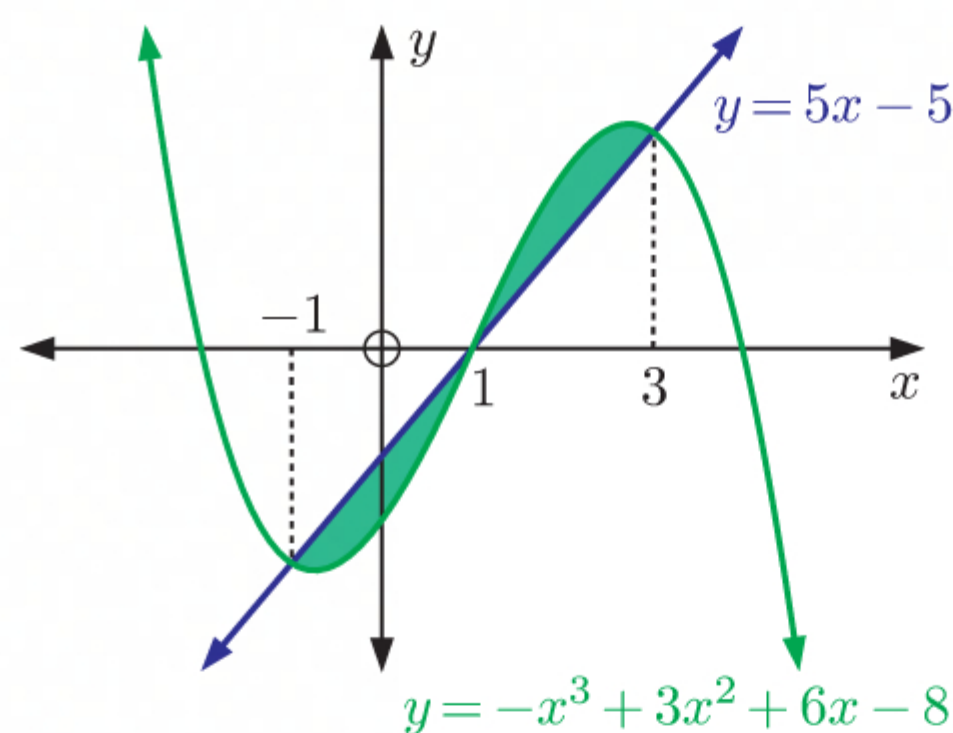
where $-x^3 + 3x^2 + 6x - 8 = 5x - 5$

$$\therefore x^3 - 3x^2 - x + 3 = 0$$

$$\therefore (x-1)(x^2 - 2x - 3) = 0$$

$$\therefore (x-1)(x-3)(x+1) = 0$$

$$\therefore x = -1, 1, \text{ or } 3$$



Total area = $\int_{-1}^1 [(5x - 5) - (-x^3 + 3x^2 + 6x - 8)] dx$

$$+ \int_1^3 [(-x^3 + 3x^2 + 6x - 8) - (5x - 5)] dx$$

$$= \int_{-1}^1 (x^3 - 3x^2 - x + 3) dx + \int_1^3 (-x^3 + 3x^2 + x - 3) dx$$

$$= \left[\frac{1}{4}x^4 - x^3 - \frac{1}{2}x^2 + 3x \right]_{-1}^1 + \left[-\frac{1}{4}x^4 + x^3 + \frac{1}{2}x^2 - 3x \right]_1^3$$

$$= \left[\left(\frac{1}{4} - 1 - \frac{1}{2} + 3 \right) - \left(\frac{1}{4} + 1 - \frac{1}{2} - 3 \right) \right]$$

$$+ \left[\left(-\frac{81}{4} + 27 + \frac{9}{2} - 9 \right) - \left(-\frac{1}{4} + 1 + \frac{1}{2} - 3 \right) \right]$$

$$= \left(\frac{7}{4} + \frac{9}{4} \right) + \left(\frac{9}{4} + \frac{7}{4} \right)$$

$$= 8 \text{ units}^2$$

b $y = 2x^3 - 3x^2 + 18$ meets $y = x^3 + 10x - 6$

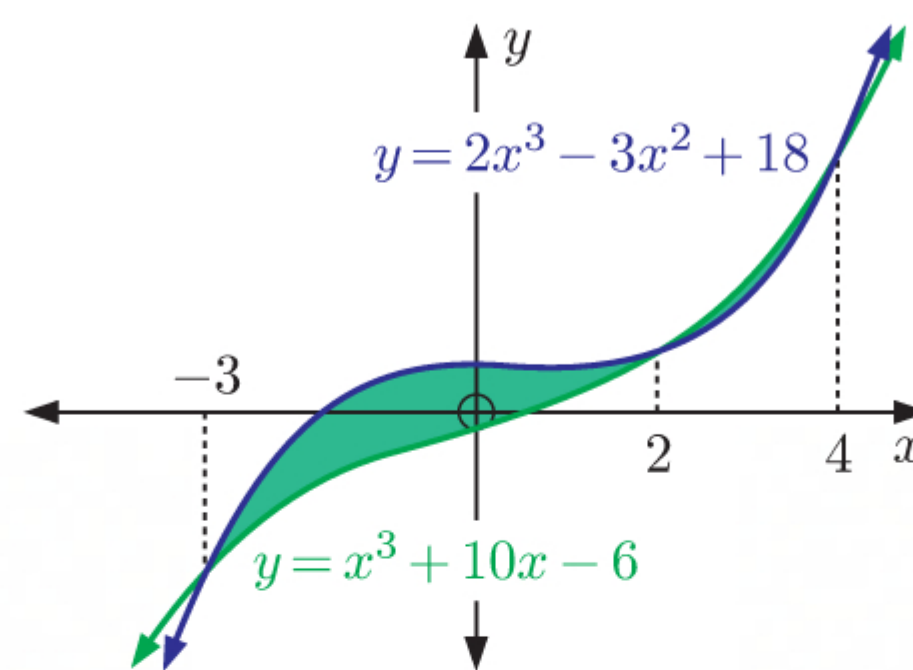
where $2x^3 - 3x^2 + 18 = x^3 + 10x - 6$

$$\therefore x^3 - 3x^2 - 10x + 24 = 0$$

$$\therefore (x-2)(x^2 - x - 12) = 0$$

$$\therefore (x-2)(x-4)(x+3) = 0$$

$$\therefore x = -3, 2, \text{ or } 4$$

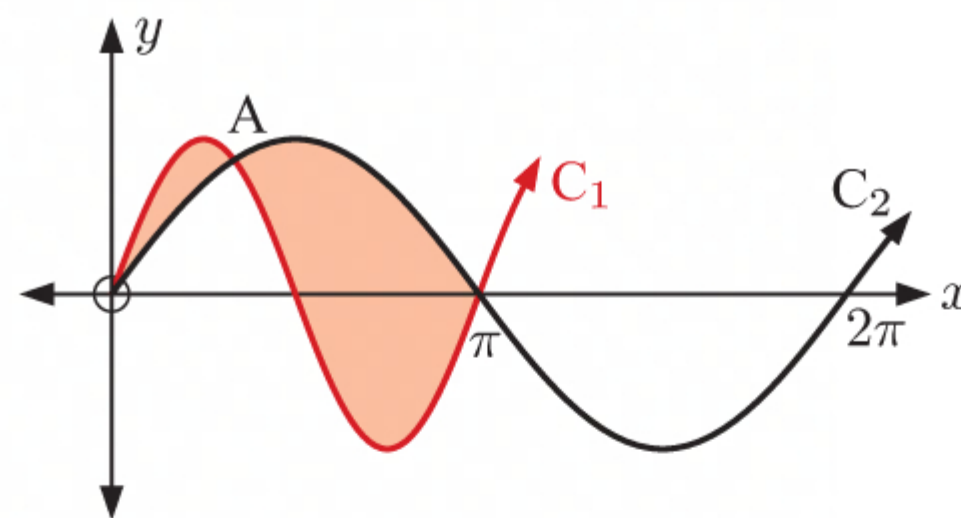


$$\begin{aligned} \text{Total area} &= \int_{-3}^2 [(2x^3 - 3x^2 + 18) - (x^3 + 10x - 6)] dx \\ &\quad + \int_2^4 [(x^3 + 10x - 6) - (2x^3 - 3x^2 + 18)] dx \\ &= \int_{-3}^2 (x^3 - 3x^2 - 10x + 24) dx + \int_2^4 (-x^3 + 3x^2 + 10x - 24) dx \\ &= \left[\frac{1}{4}x^4 - x^3 - 5x^2 + 24x \right]_{-3}^2 + \left[-\frac{1}{4}x^4 + x^3 + 5x^2 - 24x \right]_2^4 \\ &= [(4 - 8 - 20 + 48) - (\frac{81}{4} + 27 - 45 - 72)] \\ &\quad + [(-64 + 64 + 80 - 96) - (-4 + 8 + 20 - 48)] \\ &= (24 + \frac{279}{4}) + (-16 + 24) \\ &= 101\frac{3}{4} \text{ units}^2 \end{aligned}$$

14 a C_1 has period π , and C_2 has period 2π .

$\therefore y = \sin 2x$ is the curve C_1 and

$y = \sin x$ is the curve C_2 .



b The curves meet where $\sin 2x = \sin x$

$$\therefore 2 \sin x \cos x - \sin x = 0$$

$$\therefore \sin x(2 \cos x - 1) = 0$$

$$\therefore \sin x = 0 \text{ or } \cos x = \frac{1}{2}$$

$$\therefore x = 0 + k\pi \text{ or } x = \begin{cases} \frac{\pi}{3} \\ \frac{5\pi}{3} \end{cases} + 2k\pi, \quad k \in \mathbb{Z}$$

\therefore the x -coordinate of A $= \frac{\pi}{3}$ {smallest positive solution}

and when $x = \frac{\pi}{3}$, $y = \sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}$

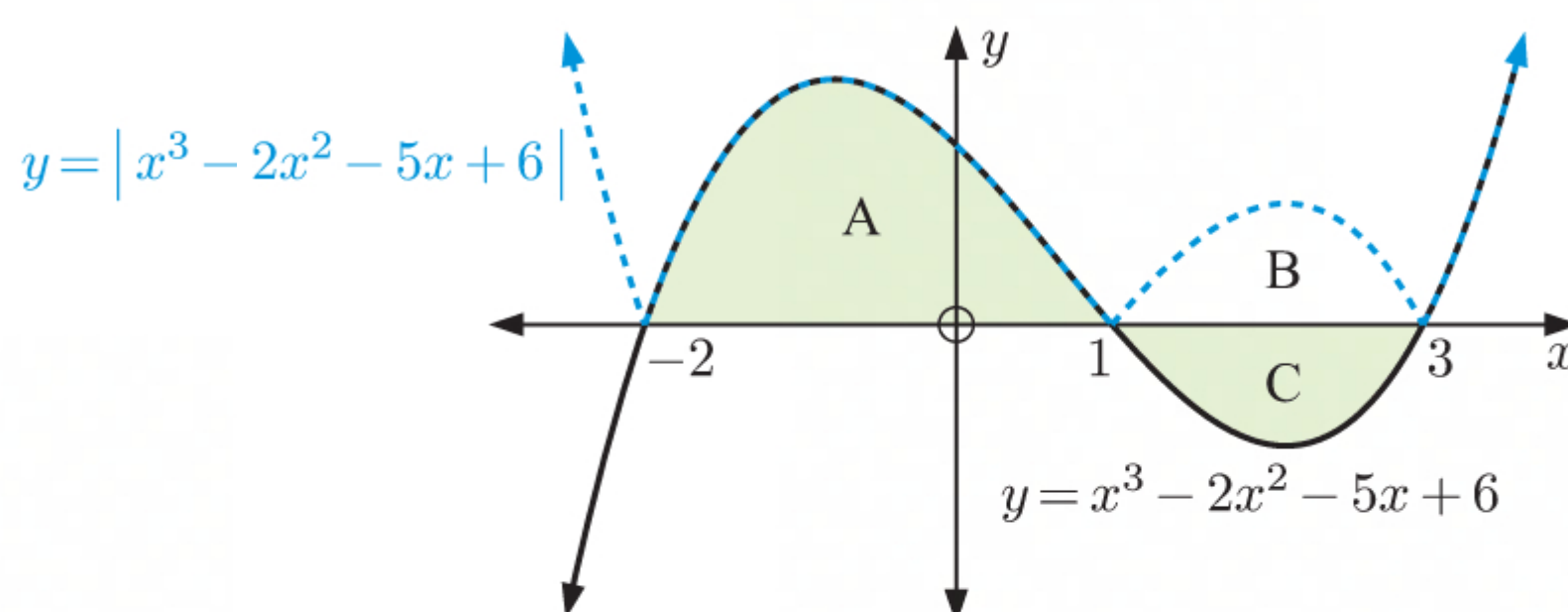
\therefore A is at $(\frac{\pi}{3}, \frac{\sqrt{3}}{2})$

$$\begin{aligned}
 \text{Area} &= \int_0^{\frac{\pi}{3}} (\sin 2x - \sin x) dx + \int_{\frac{\pi}{3}}^{\pi} (\sin x - \sin 2x) dx \\
 &= \left[-\frac{1}{2} \cos 2x + \cos x \right]_0^{\frac{\pi}{3}} + \left[-\cos x + \frac{1}{2} \cos 2x \right]_{\frac{\pi}{3}}^{\pi} \\
 &= \left[\left(-\frac{1}{2} \cos \frac{2\pi}{3} + \cos \frac{\pi}{3} \right) - \left(-\frac{1}{2} \cos 0 + \cos 0 \right) \right] \\
 &\quad + \left[\left(-\cos \pi + \frac{1}{2} \cos 2\pi \right) - \left(-\cos \frac{\pi}{3} + \frac{1}{2} \cos \frac{2\pi}{3} \right) \right] \\
 &= \left[\left(\frac{1}{4} + \frac{1}{2} \right) - \left(-\frac{1}{2} + 1 \right) \right] + \left[\left(1 + \frac{1}{2} \right) - \left(-\frac{1}{2} - \frac{1}{4} \right) \right] \\
 &= 2\frac{1}{2} \text{ units}^2
 \end{aligned}$$

ACTIVITY 1

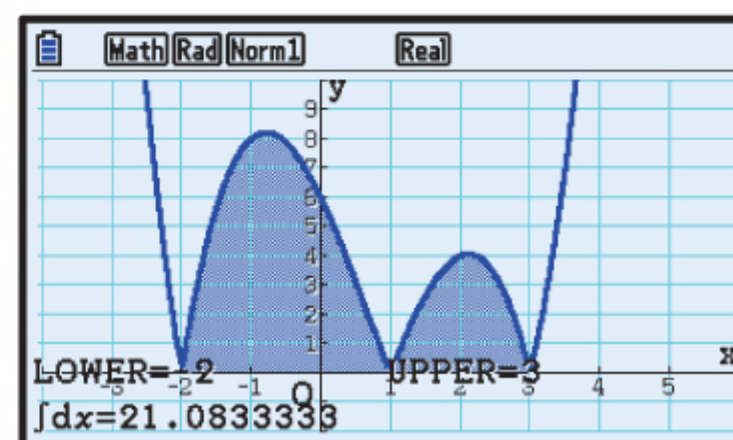
CALCULATING AREAS USING TECHNOLOGY

1 a



$$\begin{aligned}
 \text{b Enclosed area} &= A + C \\
 &= A + B \quad \{\text{area B} = \text{area C} \text{ since it is a reflection in the } x\text{-axis}\} \\
 &= \int_{-2}^3 |x^3 - 2x^2 - 5x + 6| dx
 \end{aligned}$$

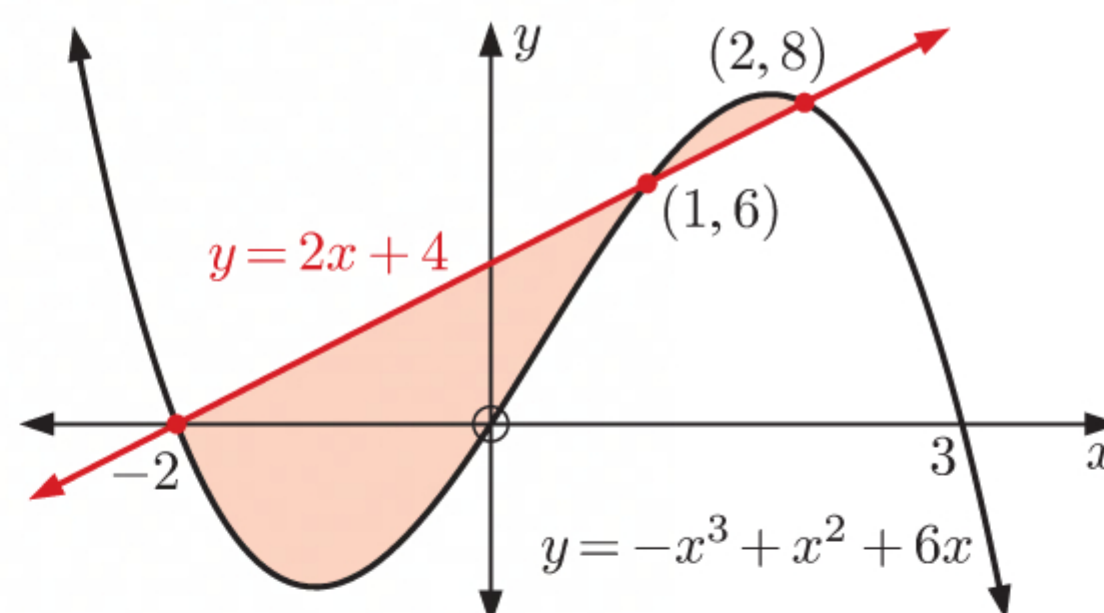
$$\text{c } \int_{-2}^3 |x^3 - 2x^2 - 5x + 6| dx \approx 21.1$$



\therefore the area enclosed between $y = x^3 - 2x^2 - 5x + 6$ and the x -axis is about 21.1 units².

2 a The graphs meet where

$$\begin{aligned}
 -x^3 + x^2 + 6x &= 2x + 4 \\
 \therefore -x^3 + x^2 + 4x - 4 &= 0 \\
 \therefore x^3 - x^2 - 4x + 4 &= 0 \\
 \therefore (x + 2)(x - 2)(x - 1) &= 0 \\
 \therefore x &= -2, 1, \text{ or } 2
 \end{aligned}$$



So, total area

$$\begin{aligned}
 &= \int_{-2}^1 [(2x+4) - (-x^3+x^2+6x)] dx + \int_1^2 [(-x^3+x^2+6x) - (2x+4)] dx \\
 &= \int_{-2}^1 |(-x^3+x^2+6x) - (2x+4)| dx + \int_1^2 |(-x^3+x^2+6x) - (2x+4)| dx \\
 &= \int_{-2}^2 |(-x^3+x^2+6x) - (2x+4)| dx \\
 &= \int_{-2}^2 |-x^3+x^2+4x-4| dx
 \end{aligned}$$

b $\int_{-2}^2 |-x^3+x^2+4x-4| dx = \frac{71}{6}$

\therefore the area enclosed between $y = -x^3 + x^2 + 6x$ and $y = 2x + 4$ is $\frac{71}{6}$ units².

3 Consider the interval $a \leq x \leq b$.

For intervals $c \leq x \leq d$ within $a \leq x \leq b$ where $y = f(x)$ lies *above* $y = g(x)$, the contribution to the total enclosed area is

$$\int_c^d [f(x) - g(x)] dx = \int_c^d |f(x) - g(x)| dx.$$

For intervals $c \leq x \leq d$ within $a \leq x \leq b$ where $y = f(x)$ lies *below* $y = g(x)$, the contribution to the total enclosed area is

$$\int_c^d [g(x) - f(x)] dx = \int_c^d |f(x) - g(x)| dx.$$

In each case, the integrand is $|f(x) - g(x)|$.

$$\therefore \text{total enclosed area} = \int_a^b |f(x) - g(x)| dx$$

EXERCISE 17E

1 a 8:20 am is 20 minutes after 8 am.

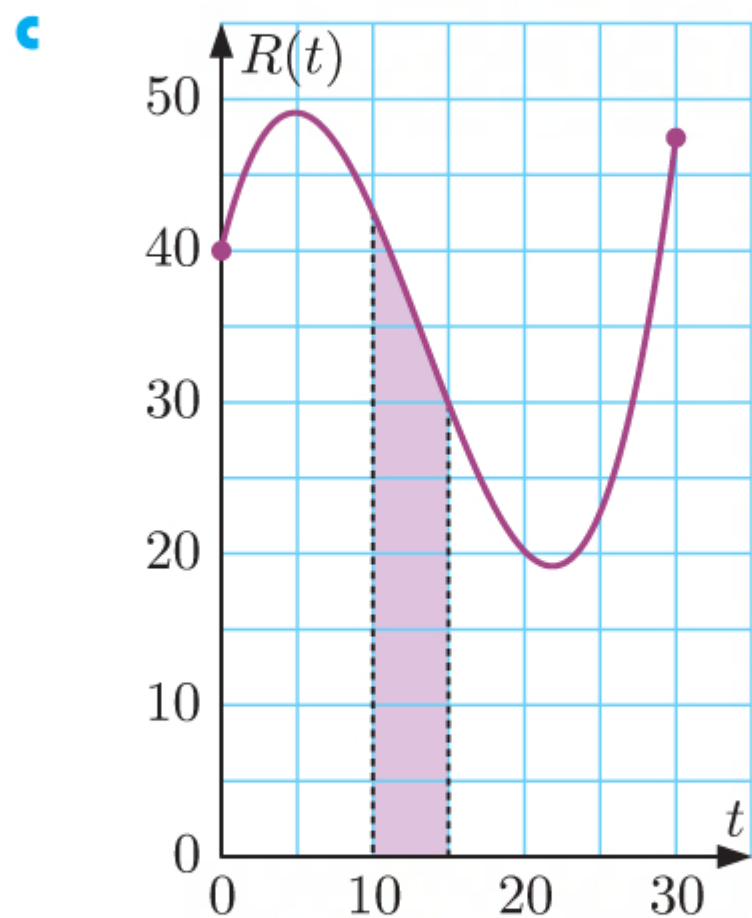
$$R(t) = \frac{t^3}{80} - \frac{t^2}{2} + 4t + 40$$

$$\begin{aligned}
 \therefore R(20) &= \frac{20^3}{80} - \frac{20^2}{2} + 4(20) + 40 \\
 &= 20
 \end{aligned}$$

So, the rate of traffic flow at 8:20 am was 20 cars per minute.

- b** The traffic flow was greatest when $R(t)$ was a maximum. Looking at the graph, the maximum value of $R(t)$ occurred when $t \approx 5$.

\therefore the traffic flow was greatest at about 5 minutes after 8 am, that is, about 8:05 am.



$\int_{10}^{15} R(t) dt$ represents the total number of cars going past the pedestrian crossing from 8:10 am to 8:15 am.

- d** 8 am is 0 minutes after 8 am and 8:30 am is 30 minutes after 8 am.

Total number of cars which passed the crossing between 8 am and 8:30 am

$$\begin{aligned}
 &= \int_0^{30} \left(\frac{t^3}{80} - \frac{t^2}{2} + 4t + 40 \right) dt \\
 &= \left[\frac{1}{320}t^4 - \frac{1}{6}t^3 + 2t^2 + 40t \right]_0^{30} \\
 &= 1031.25 \\
 &\approx 1031 \text{ cars}
 \end{aligned}$$

2 a $R_1(t) = 5 - 5e^{-0.2t}$, $R_2(t) = 6 - 6e^{-0.1t}$

i $R_1(2) = 5 - 5e^{-0.2(2)}$
 ≈ 1.65 litres per minute

ii $R_2(2) = 6 - 6e^{-0.1(2)}$
 ≈ 1.09 litres per minute

- b** The rate of water leaking into the kayak is greater than the rate of water being bailed from the kayak after 2 minutes. So, the amount of water in the kayak is increasing after 2 minutes.

c i $\int_0^3 R_1(t) dt = \int_0^3 (5 - 5e^{-0.2t}) dt$
 $= [5t + 25e^{-0.2t}]_0^3$
 $= (15 + 25e^{-0.6}) - (0 + 25)$
 ≈ 3.72

About 3.72 litres of water have leaked into the kayak in the first 3 minutes.

$$\begin{aligned}
 \text{ii} \quad \int_2^5 R_2(t) dt &= \int_2^5 (6 - 6e^{-0.1t}) dt \\
 &= [6t + 60e^{-0.1t}]_2^5 \\
 &= (30 + 60e^{-0.5}) - (12 + 60e^{-0.2}) \\
 &\approx 5.27
 \end{aligned}$$

About 5.27 litres of water have been bailed out of the kayak from $t = 2$ minutes to $t = 5$ minutes.

$$\begin{aligned}
 \text{iii} \quad \int_0^8 [R_1(t) - R_2(t)] dt &= \int_0^8 (5 - 5e^{-0.2t} - (6 - 6e^{-0.1t})) dt \\
 &= \int_0^8 (-1 - 5e^{-0.2t} + 6e^{-0.1t}) dt \\
 &= [-t + 25e^{-0.2t} - 60e^{-0.1t}]_0^8 \\
 &= (-8 + 25e^{-1.6} - 60e^{-0.8}) - (0 + 25 - 60) \\
 &\approx 5.09
 \end{aligned}$$

There are about 5.09 litres of water in the kayak 8 minutes after striking the rock.

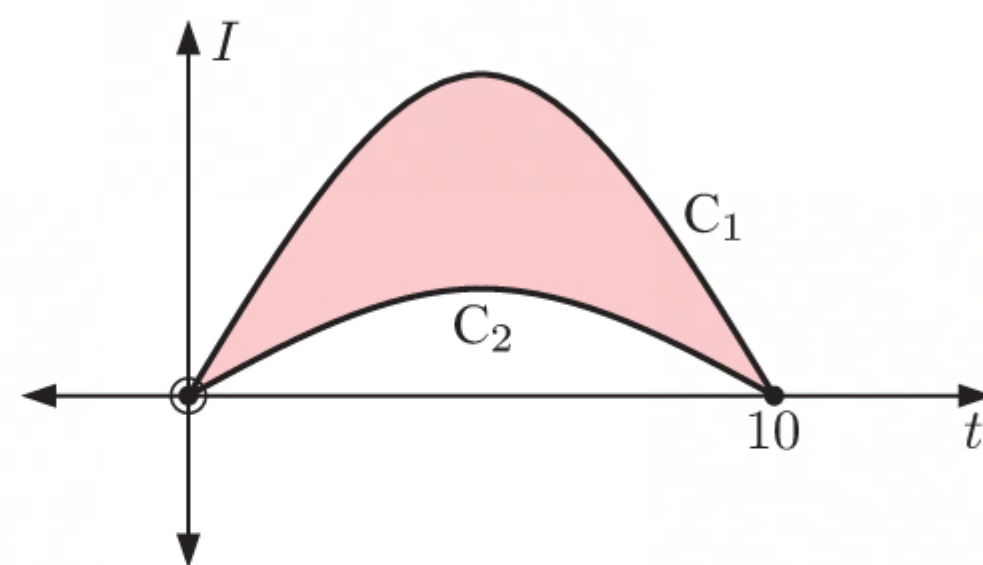
$$\begin{aligned}
 \text{d} \quad \int_0^{10} [R_1(t) - R_2(t)] dt &= \int_0^{10} (5 - 5e^{-0.2t} - (6 - 6e^{-0.1t})) dt \\
 &= \int_0^{10} (-1 - 5e^{-0.2t} + 6e^{-0.1t}) dt \\
 &= [-t + 25e^{-0.2t} - 60e^{-0.1t}]_0^{10} \\
 &= (-10 + 25e^{-2} - 60e^{-1}) - (0 + 25 - 60) \\
 &\approx 6.31
 \end{aligned}$$

There are about 6.31 litres of water in the kayak 10 minutes after striking the rock.

- 3 a $y = 3 \sin \frac{\pi t}{10}$ has amplitude 3, which is larger than the amplitude of $y = \sin \frac{\pi t}{10}$ which has amplitude 1.

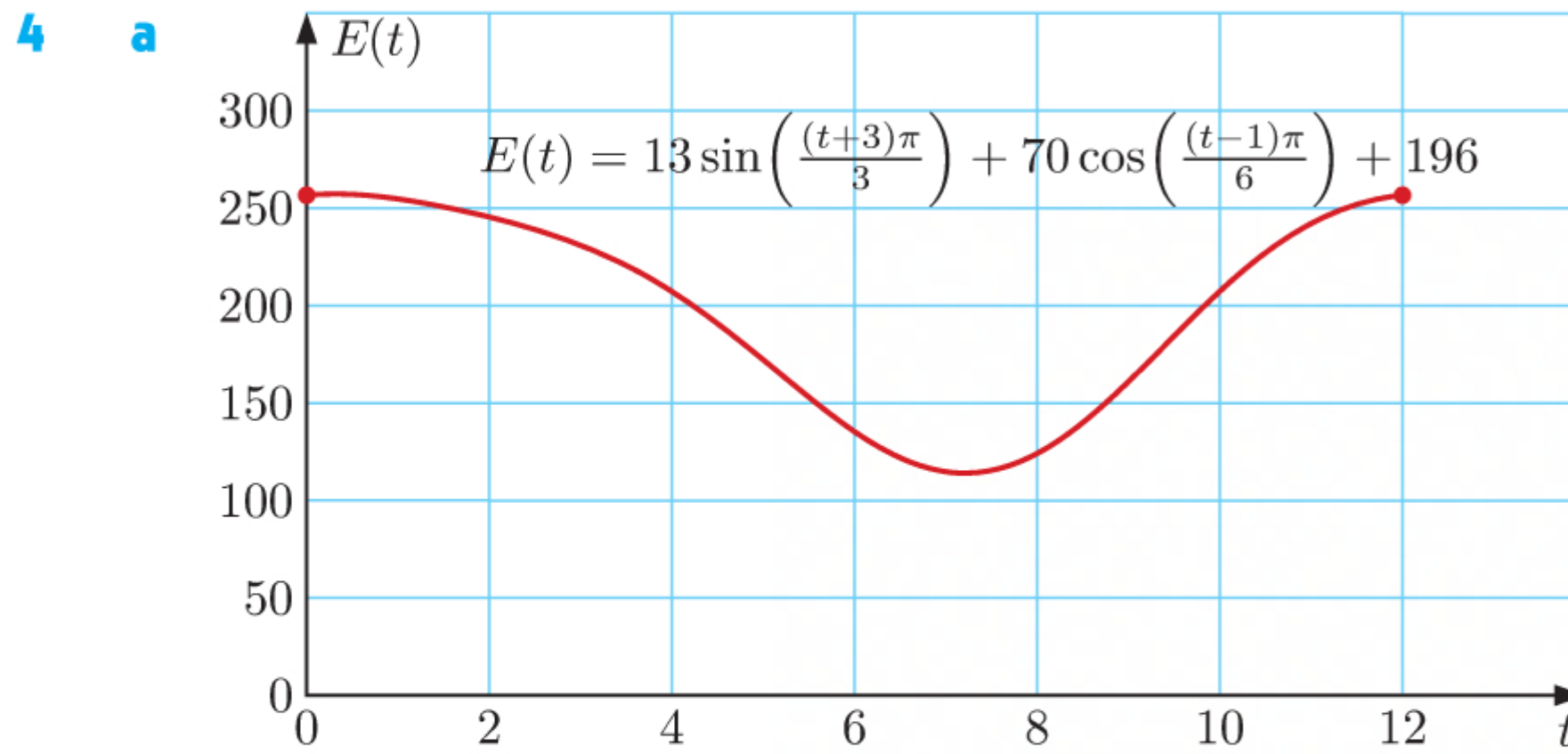
$\therefore C_1$ is $y = 3 \sin \frac{\pi t}{10}$ and

C_2 is $y = \sin \frac{\pi t}{10}$



$$\begin{aligned}
 \text{b} \quad \text{Area} &= \int_0^{10} \left(3 \sin \frac{\pi t}{10} - \sin \frac{\pi t}{10} \right) dt \\
 &= \int_0^{10} 2 \sin \frac{\pi t}{10} dt \\
 &= \left[-\frac{20}{\pi} \cos \frac{\pi t}{10} \right]_0^{10} \\
 &= -\frac{20}{\pi} \cos \pi + \frac{20}{\pi} \cos 0 \\
 &= \frac{20}{\pi} + \frac{20}{\pi} \\
 &= \frac{40}{\pi} \text{ units}
 \end{aligned}$$

- c The area in b represents the total amount of energy that enters the greenhouse in the first 10 hours.



b
$$E(t) = 13 \sin\left(\frac{(t+3)\pi}{3}\right) + 70 \cos\left(\frac{(t-1)\pi}{6}\right) + 196 \text{ TWh per month}$$

$$\begin{aligned} \therefore \int E(t) dt &= \int \left(13 \sin\left(\frac{(t+3)\pi}{3}\right) + 70 \cos\left(\frac{(t-1)\pi}{6}\right) + 196 \right) dt \\ &= 13 \left(-\cos\left(\frac{(t+3)\pi}{3}\right) \right) \left(\frac{3}{\pi} \right) + 70 \sin\left(\frac{(t-1)\pi}{6}\right) \left(\frac{6}{\pi} \right) + 196t + c \\ &= -\frac{39}{\pi} \cos\left(\frac{(t+3)\pi}{3}\right) + \frac{420}{\pi} \sin\left(\frac{(t-1)\pi}{6}\right) + 196t + c \end{aligned}$$

i
$$\int_3^4 E(t) dt = \left(-\frac{39}{\pi} \cos \frac{7\pi}{3} + \frac{420}{\pi} \sin \frac{\pi}{2} + 784 \right) - \left(-\frac{39}{\pi} \cos 2\pi + \frac{420}{\pi} \sin \frac{\pi}{3} + 588 \right) \\ \approx 220.12 \text{ TWh}$$

The power consumption of the United Kingdom in April is about 220.12 TWh.

ii
$$\int_5^8 E(t) dt = \left(-\frac{39}{\pi} \cos \frac{11\pi}{3} + \frac{420}{\pi} \sin \frac{7\pi}{6} + 1568 \right) - \left(-\frac{39}{\pi} \cos \frac{8\pi}{3} + \frac{420}{\pi} \sin \frac{2\pi}{3} + 980 \right) \\ \approx 392.96 \text{ TWh}$$

The power consumption of the United Kingdom for June 1st to September 1st is about 392.96 TWh.

iii
$$\int_0^{12} E(t) dt = \left(-\frac{39}{\pi} \cos 5\pi + \frac{420}{\pi} \sin \frac{11\pi}{6} + 2352 \right) - \left(-\frac{39}{\pi} \cos \pi + \frac{420}{\pi} \sin \left(-\frac{\pi}{6} \right) \right) \\ = 2352 \text{ TWh}$$

The yearly power consumption of the United Kingdom is 2352 TWh.

ACTIVITY 2

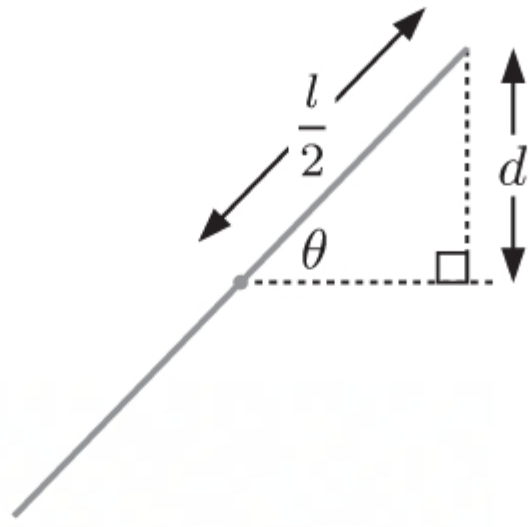
BUFFON'S NEEDLE PROBLEM

CASE 1: THE SHORT NEEDLE

1 a $0 \leq \theta \leq \pi$ b $0 \leq D \leq \frac{w}{2}$

- 2 Assuming that the needle toss is “random”, that is, it is tossed vertically rather than being cast on a particular orientation, it is reasonable to assume that θ and D will take values in their ranges with equal probability.

3



$$\sin \theta = \frac{d}{\left(\frac{l}{2}\right)}$$

$$\therefore d = \frac{l}{2} \sin \theta$$

The needle will lie on a line if $D \leq d$

$$\therefore D \leq \frac{l}{2} \sin \theta$$

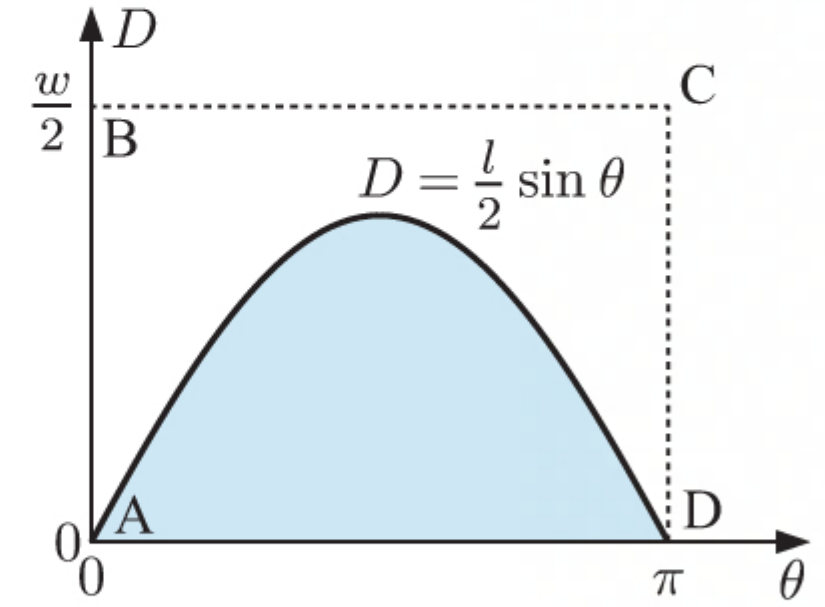
4 θ is on the horizontal axis. The length of the rectangle, AD, covers the range of values for θ found in 1 a.

D is on the vertical axis. The height of the rectangle, AB, covers the range of values for D found in 1 b.

So, any toss of the needle can be described by a unique point (θ, D) which lies in the rectangle. Under the assumption in 2, each possible outcome (θ, D) is equally likely.

Assuming $l \leq w$, the shaded area describes the set of points for which $D \leq \frac{l}{2} \sin \theta$. From 3, this is the set of points for which the needle will lie on a line.

$$\therefore P(\text{needle lies on a line}) = \frac{\text{shaded area}}{\text{area of rectangle ABCD}}$$



$$\begin{aligned} \text{5 Shaded area} &= \int_0^\pi \frac{l}{2} \sin \theta \, d\theta \\ &= \frac{l}{2} \int_0^\pi \sin \theta \, d\theta \\ &= \frac{l}{2} [-\cos \theta]_0^\pi \\ &= \frac{l}{2} (-\cos \pi + \cos 0) \\ &= l \end{aligned}$$

$$\begin{aligned} \text{6 Using 4 and 5, } P(\text{needle lies on a line}) &= \frac{l}{\pi \left(\frac{w}{2}\right)} \\ &= \frac{2l}{w\pi} \end{aligned}$$

CASE 2: THE LONG NEEDLE

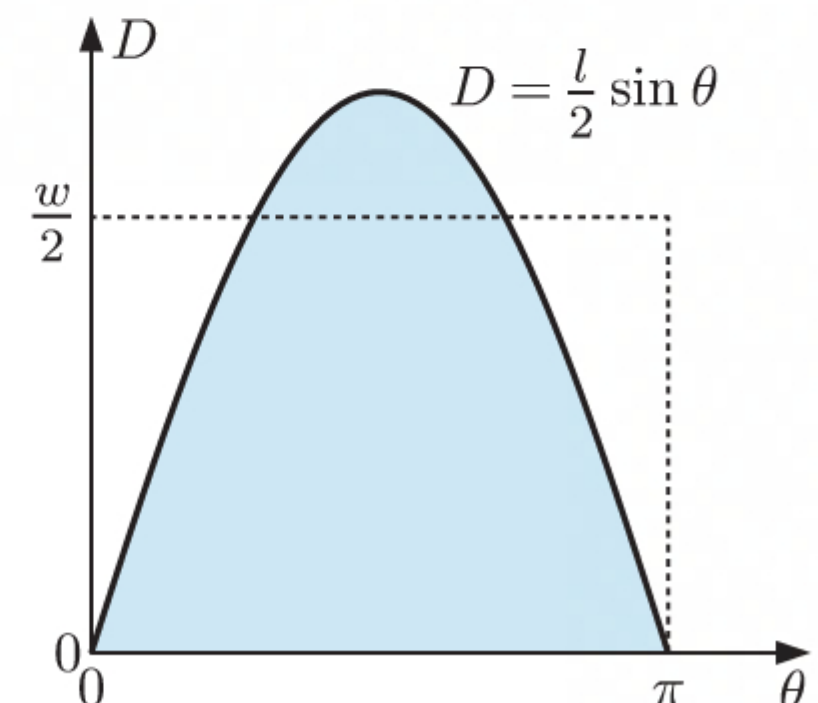
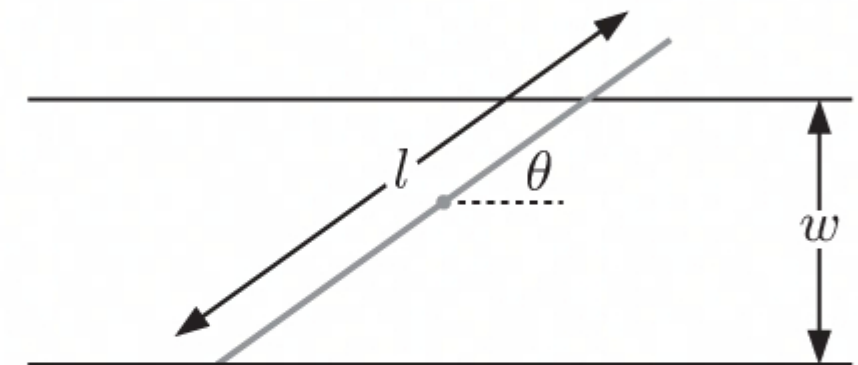
1 For a needle with length $l > w$, the curve $D = \frac{l}{2} \sin \theta$ leaves the rectangle.

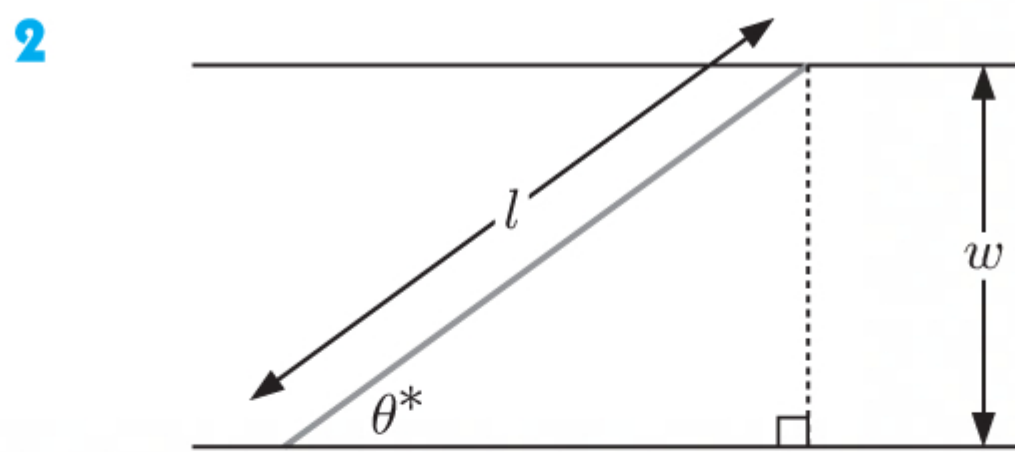
Physically, this means that there is a range of angles $\theta^* < \theta < \pi - \theta^*$ for which the needle is *certain* to lie on a line.

Since any toss of the needle is still described by the set of points in the rectangle, the part of the shaded area *outside* the rectangle is impossible.

So, for $l > w$,

$$P(\text{needle lies on a line}) = \frac{\text{shaded area within rectangle}}{\text{area of rectangle}}.$$

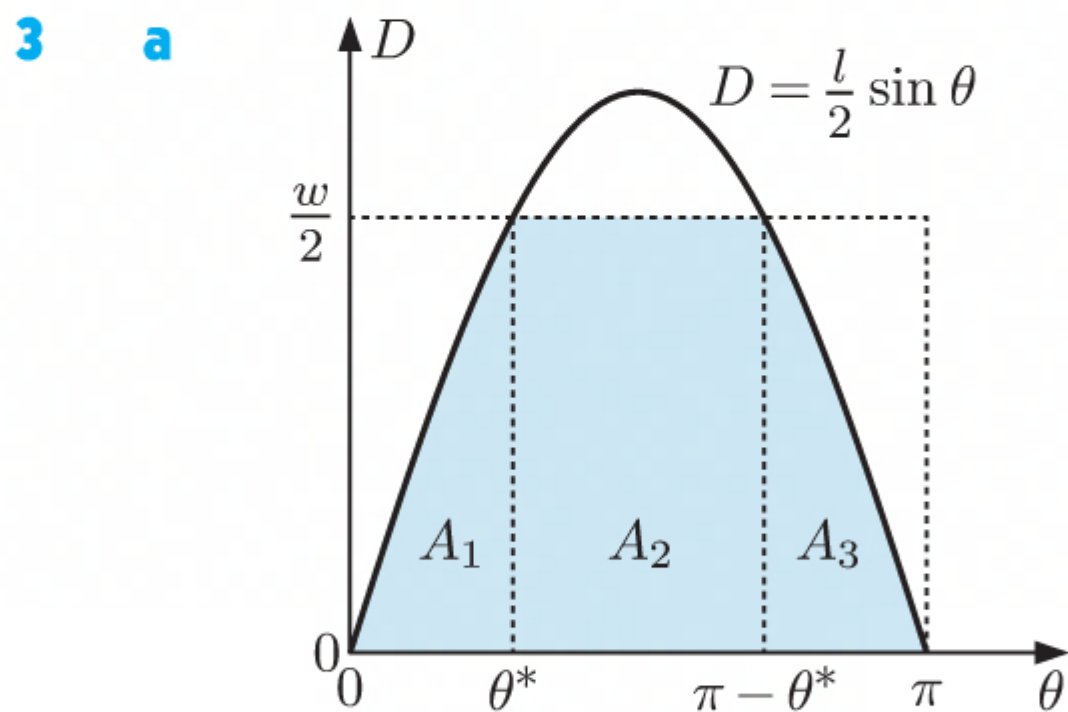




θ^* is the critical value of θ for which the needle *exactly* fits between two lines.

$$\sin \theta^* = \frac{w}{l}$$

$$\therefore \theta^* = \sin^{-1}\left(\frac{w}{l}\right)$$



$$A_1 = \int_0^{\theta^*} \frac{l}{2} \sin \theta \, d\theta$$

$$\begin{aligned} A_2 &= (\pi - \theta^* - \theta^*) \frac{w}{2} \\ &= (\pi - 2\theta^*) \frac{w}{2} \end{aligned}$$

$$A_3 = A_1 \quad \{\text{by symmetry}\}$$

$$\therefore \text{P(needle lies on a line)}$$

$$= \frac{A_1 + A_2 + A_3}{\pi\left(\frac{w}{2}\right)}$$

$$= \frac{(\pi - 2\theta^*)\frac{w}{2} + 2 \int_0^{\theta^*} \left(\frac{l}{2} \sin \theta\right) d\theta}{\frac{w\pi}{2}}$$

b Continuing from **a**, P(needle lies on a line)

$$= \frac{\pi - 2\theta^*}{\pi} + \frac{4}{w\pi} \frac{l}{2} \int_0^{\theta^*} \sin \theta \, d\theta$$

$$= 1 - \frac{2}{\pi} \theta^* + \frac{2l}{w\pi} [-\cos \theta]_0^{\theta^*}$$

$$= 1 - \frac{2}{\pi} \theta^* + \frac{2l}{w\pi} (-\cos \theta^* + \cos 0)$$

$$= 1 - \frac{2}{\pi} \theta^* + \frac{2l}{w\pi} (1 - \cos \theta^*)$$

$$= 1 - \frac{2}{\pi} \theta^* + \frac{2l}{w\pi} \left(1 - \sqrt{1 - \sin^2 \theta^*}\right) \quad \{\text{since } 0 < \theta^* \leq \frac{\pi}{2}\}$$

$$= 1 - \frac{2}{\pi} \sin^{-1}\left(\frac{w}{l}\right) + \frac{2l}{w\pi} \left(1 - \sqrt{1 - \left(\sin\left(\sin^{-1}\left(\frac{w}{l}\right)\right)\right)^2}\right) \quad \{\text{using } \mathbf{2}\}$$

$$= 1 - \frac{2}{\pi} \sin^{-1}\left(\frac{w}{l}\right) + \frac{2l}{w\pi} \left(1 - \sqrt{1 - \frac{w^2}{l^2}}\right)$$

$$= 1 - \frac{2}{\pi} \sin^{-1}\left(\frac{w}{l}\right) + \frac{2l}{w\pi} \left(1 - \frac{\sqrt{l^2 - w^2}}{l}\right)$$

4 For the boundary case $l = w$, the formula in **3 b** gives

$$\text{P(needle lies on a line)} = 1 - \frac{2}{\pi} \sin^{-1}(1) + \frac{2l}{w\pi} \left(1 - \frac{\sqrt{0}}{l}\right)$$

$$= 1 - \frac{2}{\pi} \times \frac{\pi}{2} + \frac{2l}{w\pi}$$

$$= \frac{2l}{w\pi}$$

which agrees with the formula for the short needle.

REVIEW SET 17A

$$\begin{aligned}
 1 \quad a \quad \int_{-2}^0 (1 - 3x) dx &= \left[x - \frac{3}{2}x^2 \right]_{-2}^0 \\
 &= 0 - \left(-2 - \frac{3}{2}(4) \right) \\
 &= 0 - (-8) \\
 &= 8
 \end{aligned}$$

$$\begin{aligned}
 c \quad \int_1^2 (x^2 + 1)^2 dx &= \int_1^2 (x^4 + 2x^2 + 1) dx \\
 &= \left[\frac{1}{5}x^5 + \frac{2}{3}x^3 + x \right]_1^2 \\
 &= \left(\frac{32}{5} + \frac{16}{3} + 2 \right) - \left(\frac{1}{5} + \frac{2}{3} + 1 \right) \\
 &= 11\frac{13}{15}
 \end{aligned}$$

$$\begin{aligned}
 2 \quad a \quad \int_0^b (x - b)^2 dx &= 9 \\
 \therefore \int_0^b (x^2 - 2bx + b^2) dx &= 9 \\
 \therefore \left[\frac{1}{3}x^3 - bx^2 + b^2x \right]_0^b &= 9 \\
 \therefore \left(\frac{1}{3}b^3 - \cancel{b^3} + \cancel{b^3} \right) - 0 &= 9 \\
 \therefore \frac{1}{3}b^3 &= 9 \\
 \therefore b^3 &= 27 \\
 \therefore b &= 3
 \end{aligned}$$

$$\begin{aligned}
 b \quad \int_0^b \left(x^2 + \frac{1}{2}x \right) dx &= 3 \\
 \therefore \left[\frac{1}{3}x^3 + \frac{1}{4}x^2 \right]_0^b &= 3 \\
 \therefore \left(\frac{1}{3}b^3 + \frac{1}{4}b^2 \right) - 0 &= 3 \\
 \therefore \frac{1}{3}b^3 + \frac{1}{4}b^2 &= 3 \\
 \therefore \frac{1}{3}b^3 + \frac{1}{4}b^2 - 3 &= 0 \\
 \therefore b &\approx 1.86 \\
 &\text{\{using technology\}}
 \end{aligned}$$

$$\begin{aligned}
 b \quad \int_0^{\frac{1}{2}} (x - \sqrt{x}) dx &= \int_0^{\frac{1}{2}} \left(x - x^{\frac{1}{2}} \right) dx \\
 &= \left[\frac{1}{2}x^2 - \frac{x^{\frac{3}{2}}}{\frac{3}{2}} \right]_0^{\frac{1}{2}} \\
 &= \left[\frac{1}{2}x^2 - \frac{2}{3}x^{\frac{3}{2}} \right]_0^{\frac{1}{2}} \\
 &= \frac{1}{2} \left(\frac{1}{4} \right) - \frac{2}{3} \left(\frac{1}{2\sqrt{2}} \right) - 0 \\
 &= \frac{1}{8} - \frac{1}{3\sqrt{2}}
 \end{aligned}$$

Math Rad Norm1 d/c Real
 $aX^3 + bX^2 + cX + d = 0$

a	b	c	d
0.3333	0.25	0	-3

 SOLVE DELETE CLEAR EDIT

Math Rad Norm1 d/c Real
 $aX^3 + bX^2 + cX + d = 0$
 X1 1.8577
 1.857753178
 REPEAT

$$\begin{aligned}
 \text{3 a } \int_{-5}^{-1} \sqrt{1-3x} \, dx &= \int_{-5}^{-1} (1-3x)^{\frac{1}{2}} \, dx \\
 &= \left[\left(\frac{1}{-3} \right) \frac{(1-3x)^{\frac{3}{2}}}{\frac{3}{2}} \right]_{-5}^{-1} \\
 &= \left[-\frac{2}{9} (1-3x)^{\frac{3}{2}} \right]_{-5}^{-1} \\
 &= -\frac{2}{9} (4)^{\frac{3}{2}} - \left(-\frac{2}{9} (16)^{\frac{3}{2}} \right) \\
 &= -\frac{16}{9} + \frac{128}{9} \\
 &= 12\frac{4}{9}
 \end{aligned}$$

$$\begin{aligned}
 \text{b } \int_0^{\frac{\pi}{2}} \cos \frac{x}{2} \, dx &= \left[2 \sin \frac{x}{2} \right]_0^{\frac{\pi}{2}} \\
 &= 2 \sin \frac{\pi}{4} - 2 \sin 0 \\
 &= \sqrt{2}
 \end{aligned}$$

$$\begin{aligned}
 \text{c } \int_2^6 \frac{2}{x} \, dx &= \left[2 \ln |x| \right]_2^6 \\
 &= 2 \ln 6 - 2 \ln 2 \\
 &= 2(\ln 6 - \ln 2) \\
 &= 2 \ln \left(\frac{6}{2} \right) \\
 &= 2 \ln 3
 \end{aligned}$$

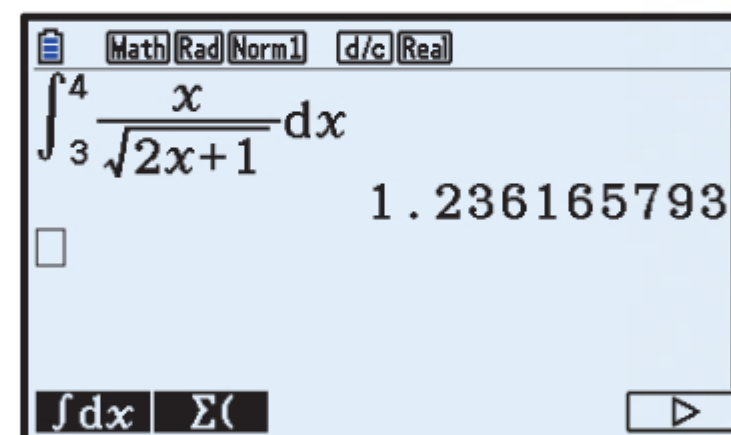
$$\begin{aligned}
 \text{4 } \int_0^a e^{1-2x} \, dx &= \frac{e}{4} \\
 \therefore \left[\left(\frac{1}{-2} \right) e^{1-2x} \right]_0^a &= \frac{e}{4} \\
 \therefore \left[-\frac{1}{2} e^{1-2x} \right]_0^a &= \frac{e}{4} \\
 \therefore -\frac{1}{2} e^{1-2a} - \left(-\frac{1}{2} e^1 \right) &= \frac{e}{4} \\
 \therefore -\frac{1}{2} e^{1-2a} + \frac{1}{2} e &= \frac{e}{4} \\
 \therefore -2e^{1-2a} + 2e &= e \\
 \therefore 2e^{1-2a} &= e \\
 \therefore e^{1-2a} &= \frac{e}{2} \\
 \therefore e^{-2a} &= \frac{1}{2} \\
 \therefore -2a &= \ln \left(\frac{1}{2} \right) \\
 \therefore -2a &= -\ln 2 \\
 \therefore a &= \frac{1}{2} \ln 2
 \end{aligned}$$

$$\begin{aligned}
 \text{5 } \sin^2 x &= \frac{1}{2} - \frac{1}{2} \cos 2x \\
 \therefore \sin^2 \left(\frac{x}{2} \right) &= \frac{1}{2} - \frac{1}{2} \cos x \\
 \therefore \int_0^{\frac{\pi}{6}} \sin^2 \left(\frac{x}{2} \right) \, dx &= \int_0^{\frac{\pi}{6}} \left(\frac{1}{2} - \frac{1}{2} \cos x \right) \, dx \\
 &= \left[\frac{1}{2} x - \frac{1}{2} \sin x \right]_0^{\frac{\pi}{6}} \\
 &= \left(\frac{\pi}{12} - \frac{1}{4} \right) - 0 \\
 &= \frac{\pi}{12} - \frac{1}{4}
 \end{aligned}$$

$$\begin{aligned} 6 \quad \frac{d}{dx} (e^{-2x} \sin x) &= -2e^{-2x} \sin x + e^{-2x} (\cos x) \\ &= e^{-2x} (\cos x - 2 \sin x) \end{aligned}$$

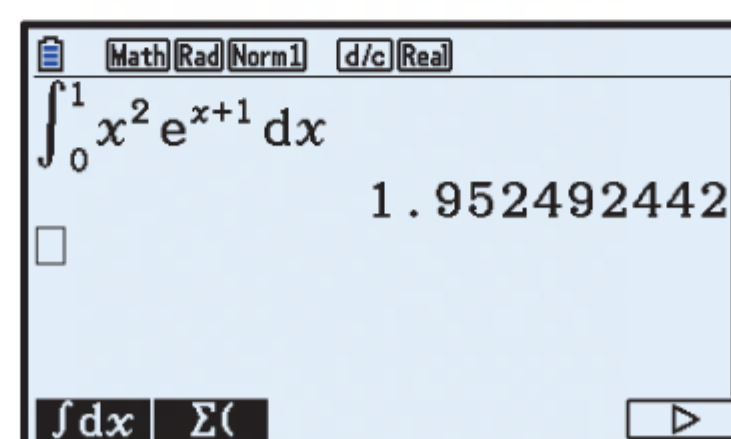
$$\begin{aligned} \therefore \int_0^{\frac{\pi}{2}} [e^{-2x} (\cos x - 2 \sin x)] &= \left[e^{-2x} \sin x \right]_0^{\frac{\pi}{2}} \\ &= (e^{-\pi} \sin \frac{\pi}{2}) - 0 \\ &= e^{-\pi} \end{aligned}$$

$$7 \quad \text{a} \quad \text{Using technology, } \int_3^4 \frac{x}{\sqrt{2x+1}} dx \approx 1.236\,17$$



Calculator screen showing the integral of $\frac{x}{\sqrt{2x+1}}$ from 3 to 4, resulting in 1.236165793.

$$\text{b} \quad \text{Using technology, } \int_0^1 x^2 e^{x+1} dx \approx 1.952\,49$$



Calculator screen showing the integral of $x^2 e^{x+1}$ from 0 to 1, resulting in 1.952492442.

$$\begin{aligned} 8 \quad \text{a} \quad \int 2x(x^2 + 1)^3 dx &= \int u^3 \frac{du}{dx} dx \quad \{u = x^2 + 1, \frac{du}{dx} = 2x\} \\ &= \int u^3 du \\ &= \frac{1}{4} u^4 + c \\ &= \frac{1}{4} (x^2 + 1)^4 + c \end{aligned}$$

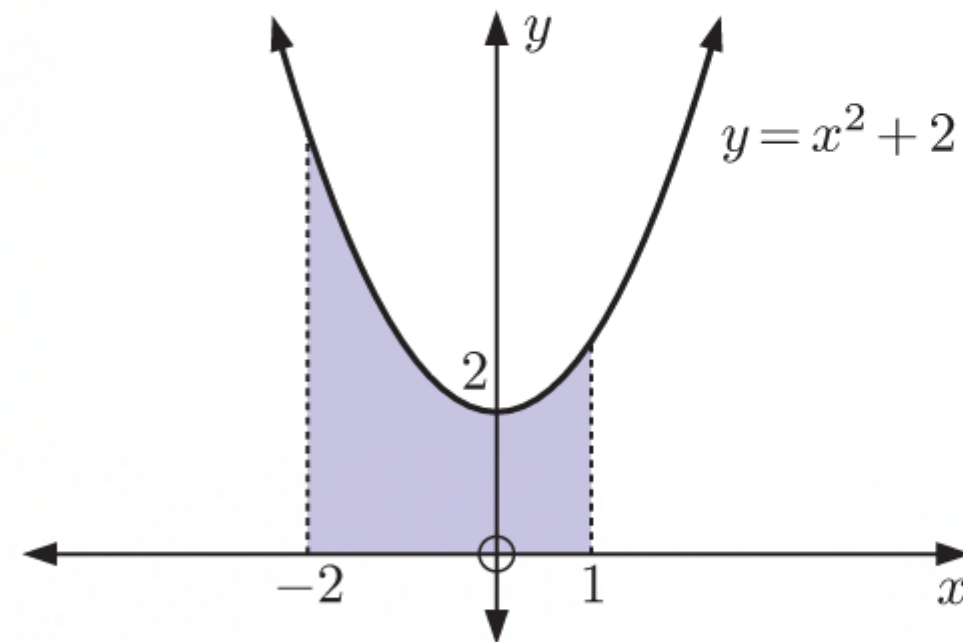
$$\begin{aligned} \text{b} \quad \text{i} \quad \int_0^1 2x(x^2 + 1)^3 dx &= \left[\frac{1}{4} (x^2 + 1)^4 \right]_0^1 \\ &= \frac{1}{4} (16) - \frac{1}{4} (1) \\ &= \frac{15}{4} \end{aligned}$$

$$\begin{aligned} \text{ii} \quad \int_{-1}^2 -x(1+x)^3 dx \\ &= -\frac{1}{2} \int_{-1}^2 2x(x^2 + 1)^3 dx \\ &= -\frac{1}{2} \left[\frac{1}{4} (x^2 + 1)^4 \right]_{-1}^2 \\ &= -\frac{1}{2} \left[\frac{1}{4} (5)^4 - \frac{1}{4} (2)^4 \right] \\ &= -\frac{1}{2} \left(\frac{625}{4} - 4 \right) \\ &= -\frac{609}{8} \end{aligned}$$

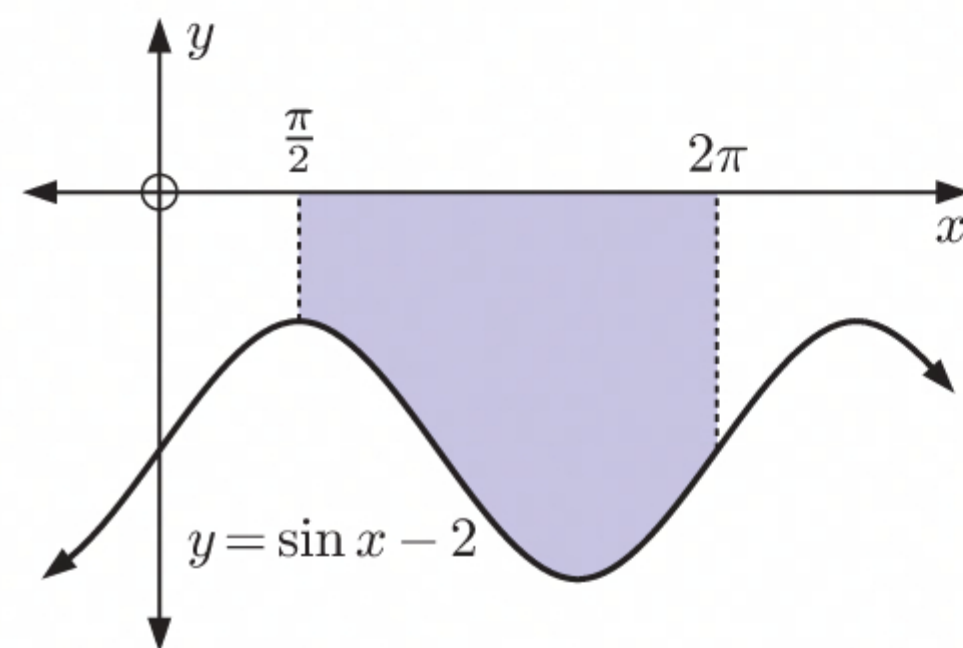
$$\begin{aligned} 9 \quad \text{a} \quad \int x^2 e^{1-x^3} dx &= -\frac{1}{3} \int -3x^2 e^{1-x^3} dx \\ &= -\frac{1}{3} \int e^u \frac{du}{dx} dx \quad \{u = 1 - x^3, \frac{du}{dx} = -3x^2\} \\ &= -\frac{1}{3} \int e^u du \\ &= -\frac{1}{3} e^u + c \\ &= -\frac{1}{3} e^{1-x^3} + c \end{aligned}$$

$$\begin{aligned}
 \text{b } \int_0^1 x^2 e^{1-x^3} dx &= \left[-\frac{1}{3} e^{1-x^3} \right]_0^1 \\
 &= -\frac{1}{3} e^0 - \left(-\frac{1}{3} e^1 \right) \\
 &= -\frac{1}{3} + \frac{1}{3} e \\
 &= \frac{e-1}{3}
 \end{aligned}$$

$$\begin{aligned}
 \text{10 a Area} &= \int_{-2}^1 (x^2 + 2) dx \\
 &= \left[\frac{1}{3} x^3 + 2x \right]_{-2}^1 \\
 &= \left(\frac{1}{3} + 2 \right) - \left(-\frac{8}{3} - 4 \right) \\
 &= 9 \text{ units}^2
 \end{aligned}$$



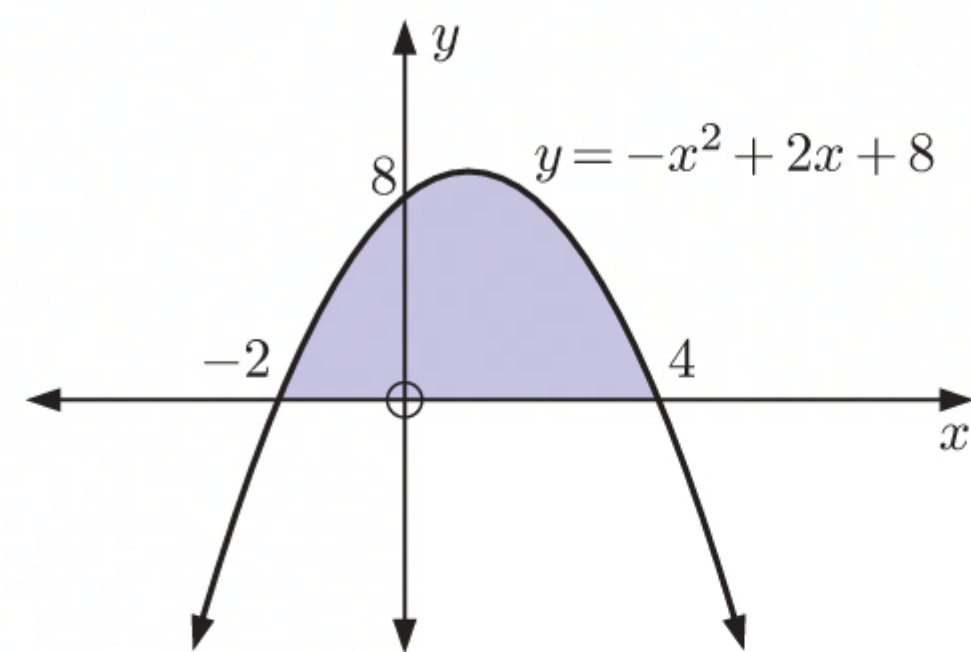
$$\begin{aligned}
 \text{b Area} &= - \int_{\frac{\pi}{2}}^{2\pi} (\sin x - 2) dx \\
 &= - \left[-\cos x - 2x \right]_{\frac{\pi}{2}}^{2\pi} \\
 &= -[(-1 - 4\pi) - (0 - \pi)] \\
 &= (3\pi + 1) \text{ units}^2
 \end{aligned}$$



$$\begin{aligned}
 \text{c The curve cuts the } x\text{-axis when } y &= 0 \\
 \therefore -x^2 + 2x + 8 &= 0 \\
 \therefore x^2 - 2x - 8 &= 0 \\
 \therefore (x+2)(x-4) &= 0 \\
 \therefore x &= -2 \text{ or } 4
 \end{aligned}$$

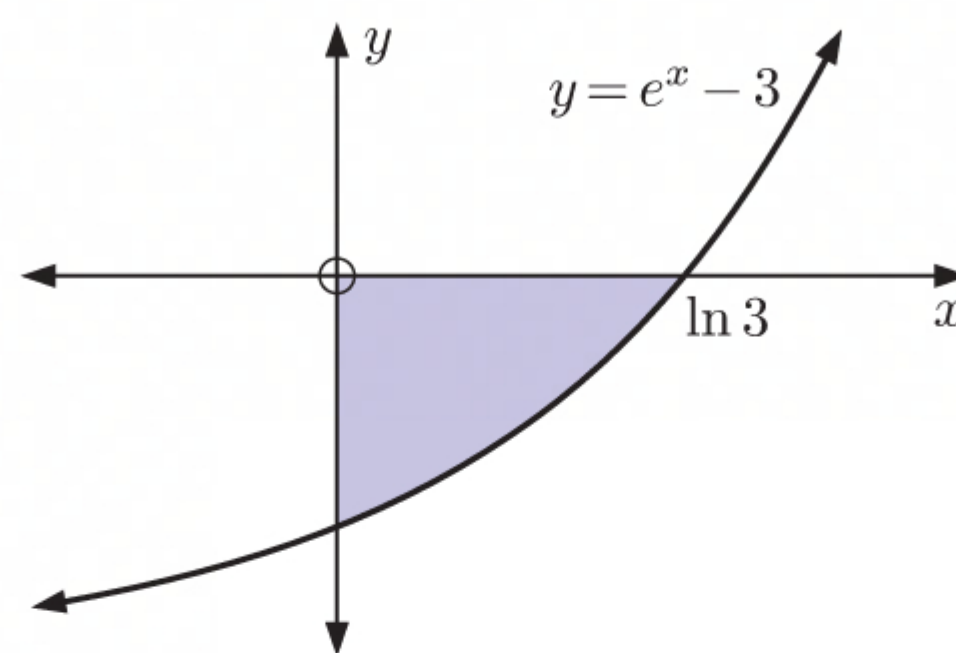
\therefore the x -intercepts are -2 and 4 .

$$\begin{aligned}
 \text{Area} &= \int_{-2}^4 (-x^2 + 2x + 8) dx \\
 &= \left[-\frac{1}{3} x^3 + x^2 + 8x \right]_{-2}^4 \\
 &= \left(-\frac{64}{3} + 16 + 32 \right) - \left(\frac{8}{3} + 4 - 16 \right) \\
 &= \frac{80}{3} - \left(-\frac{28}{3} \right) \\
 &= 36 \text{ units}^2
 \end{aligned}$$

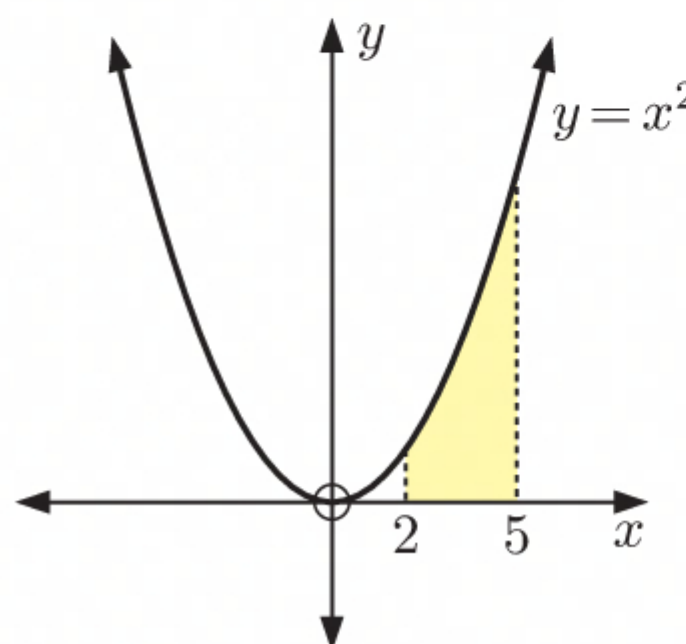


- d** The curve cuts the x -axis when $y = 0$
 $\therefore e^x - 3 = 0$
 $\therefore e^x = 3$
 $\therefore x = \ln 3$

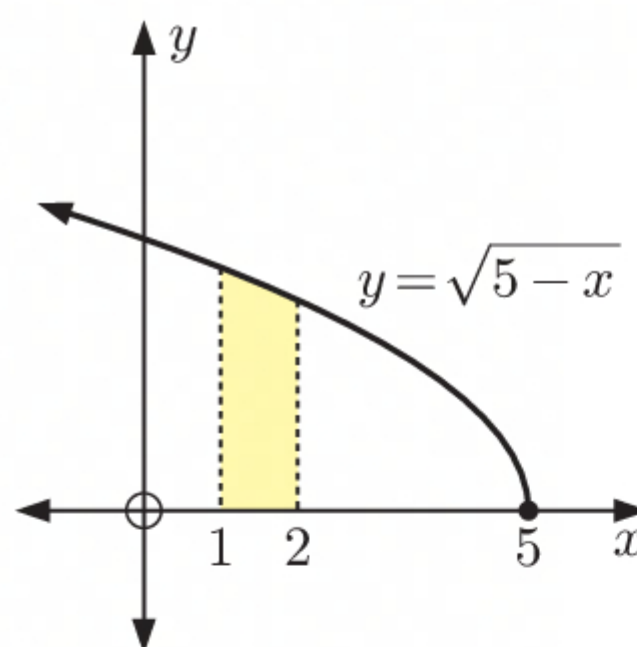
$$\begin{aligned}\text{Area} &= - \int_0^{\ln 3} (e^x - 3) dx \\ &= - [e^x - 3x]_0^{\ln 3} \\ &= - [(3 - 3 \ln 3) - 1] \\ &= (3 \ln 3 - 2) \text{ units}^2\end{aligned}$$



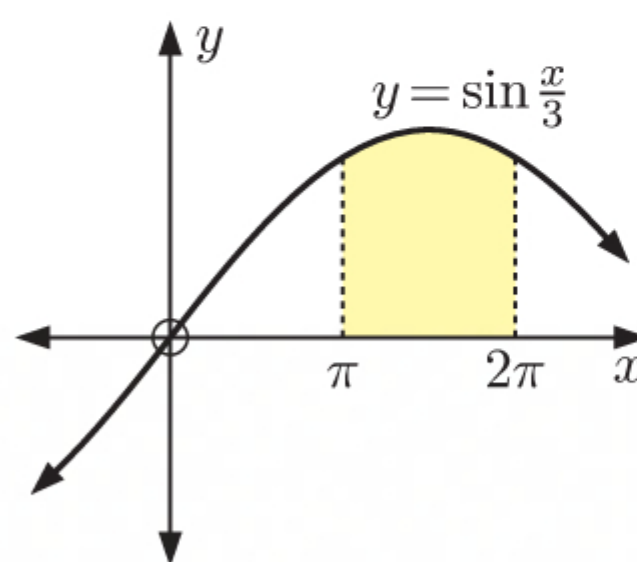
11 a
$$\begin{aligned}\text{Area} &= \int_2^5 x^2 dx \\ &= \left[\frac{1}{3} x^3 \right]_2^5 \\ &= \frac{125}{3} - \frac{8}{3} \\ &= 39 \text{ units}^2\end{aligned}$$

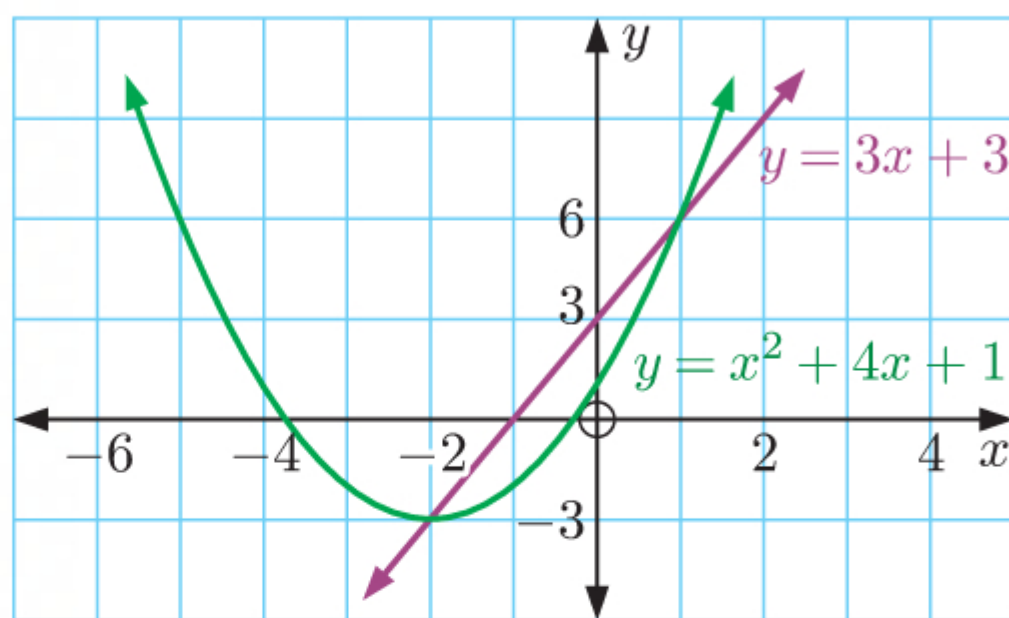


b
$$\begin{aligned}\text{Area} &= \int_1^2 \sqrt{5-x} dx \\ &= \int_1^2 (5-x)^{\frac{1}{2}} dx \\ &= \left[\left(\frac{1}{-1} \right) \frac{(5-x)^{\frac{3}{2}}}{\frac{3}{2}} \right]_1^2 \\ &= \left[-\frac{2}{3} (5-x)^{\frac{3}{2}} \right]_1^2 \\ &= -\frac{2}{3} (3)^{\frac{3}{2}} - \left(-\frac{2}{3} (4)^{\frac{3}{2}} \right) \\ &= \left(\frac{16}{3} - 2\sqrt{3} \right) \text{ units}^2\end{aligned}$$



c
$$\begin{aligned}\text{Area} &= \int_{\pi}^{2\pi} \sin \frac{x}{3} dx \\ &= \left[-3 \cos \frac{x}{3} \right]_{\pi}^{2\pi} \\ &= -3 \cos \frac{2\pi}{3} - \left(-3 \cos \frac{\pi}{3} \right) \\ &= \frac{3}{2} - \left(-\frac{3}{2} \right) \\ &= 3 \text{ units}^2\end{aligned}$$



12 a

b $y = x^2 + 4x + 1$ meets $y = 3x + 3$

where $x^2 + 4x + 1 = 3x + 3$

$$\therefore x^2 + x - 2 = 0$$

$$\therefore (x + 2)(x - 1) = 0$$

$$\therefore x = -2 \text{ or } 1$$

When $x = -2$, $y = 3(-2) + 3$

$$= -3$$

When $x = 1$, $y = 3(1) + 3$

$$= 6$$

\therefore the graphs meet at the points

$(-2, -3)$ and $(1, 6)$.

c Area $= \int_{-2}^1 [(3x + 3) - (x^2 + 4x + 1)] dx$

$$= \int_{-2}^1 (-x^2 - x + 2) dx$$

$$= \left[-\frac{1}{3}x^3 - \frac{1}{2}x^2 + 2x \right]_{-2}^1$$

$$= \left(-\frac{1}{3} - \frac{1}{2} + 2 \right) - \left(\frac{8}{3} - 2 - 4 \right)$$

$$= \frac{7}{6} - \left(-\frac{10}{3} \right)$$

$$= 4\frac{1}{2} \text{ units}^2$$

13 $\int_{-1}^3 f(x) dx$ only gives us the correct area provided that $f(x)$ is positive on the interval $-1 \leq x \leq 3$.

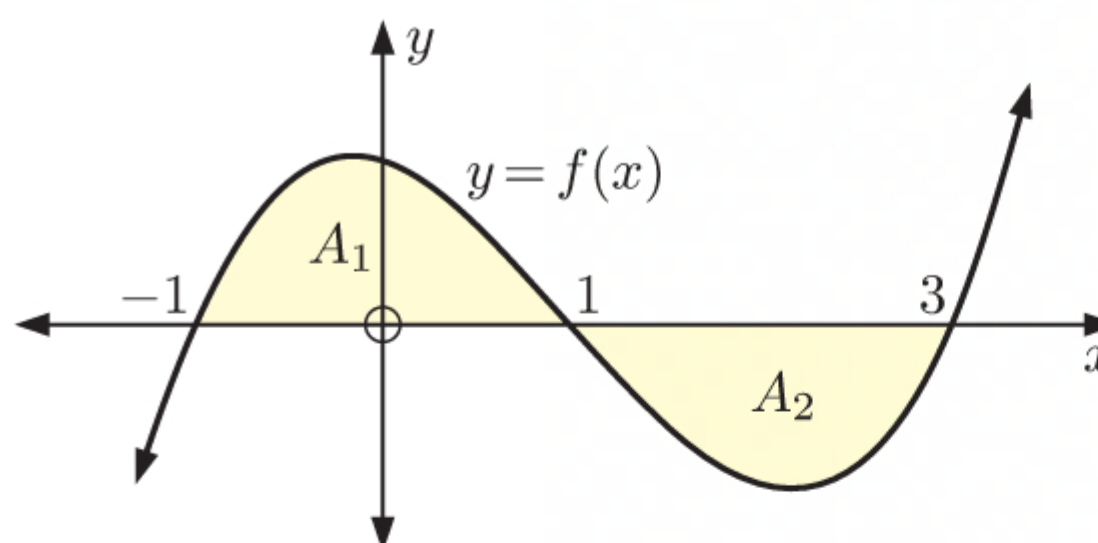
But $f(x)$ is not positive for $1 \leq x \leq 3$, so

$$\int_{-1}^3 f(x) dx = A_1 - A_2 \text{ which is not the shaded area.}$$

shaded area.

The area of the shaded region

$$= \int_{-1}^1 f(x) dx - \int_1^3 f(x) dx$$



14 $y = x^2$ meets $y = k$ where $x^2 = k$

$$\therefore x = \pm\sqrt{k}$$

Now, the area = $5\frac{1}{3}$

$$\therefore \int_{-\sqrt{k}}^{\sqrt{k}} (k - x^2) dx = 5\frac{1}{3}$$

$$\therefore \left[kx - \frac{x^3}{3} \right]_{-\sqrt{k}}^{\sqrt{k}} = 5\frac{1}{3}$$

$$\therefore \left(k\sqrt{k} - \frac{k\sqrt{k}}{3} \right) - \left(-k\sqrt{k} - \left(-\frac{k\sqrt{k}}{3} \right) \right) = \frac{16}{3}$$

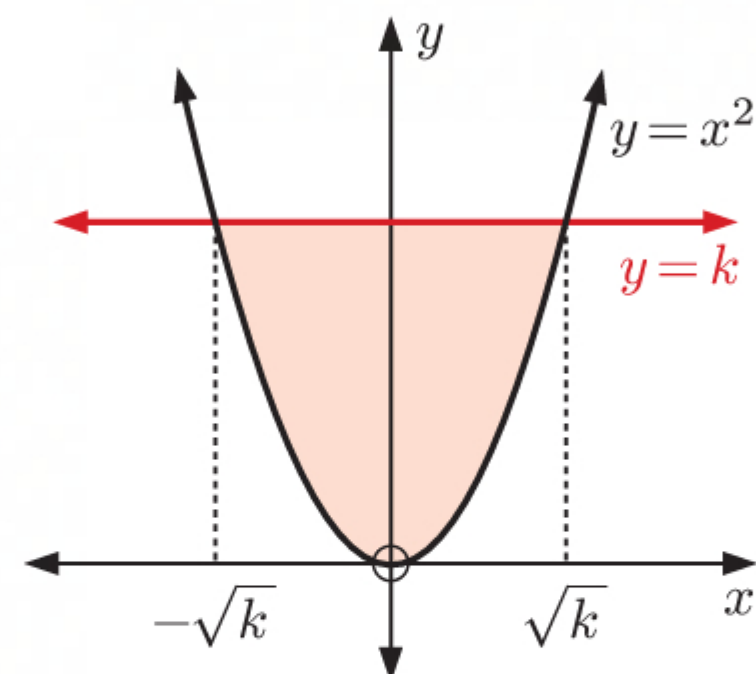
$$\therefore 2k\sqrt{k} - \frac{2k\sqrt{k}}{3} = \frac{16}{3}$$

$$\therefore \frac{4}{3}k^{\frac{3}{2}} = \frac{16}{3}$$

$$\therefore k^{\frac{3}{2}} = 4$$

$$\therefore k^3 = 16$$

$$\therefore k = \sqrt[3]{16}$$



15 a Area from $x = 0$ to $x = a$ is $\int_0^a e^x dx = 2$

$$\therefore [e^x]_0^a = 2$$

$$\therefore e^a - 1 = 2$$

$$\therefore e^a = 3$$

$$\therefore a = \ln 3$$

b Area from $x = a$ to $x = b$ is $\int_{\ln 3}^b e^x dx = 2$

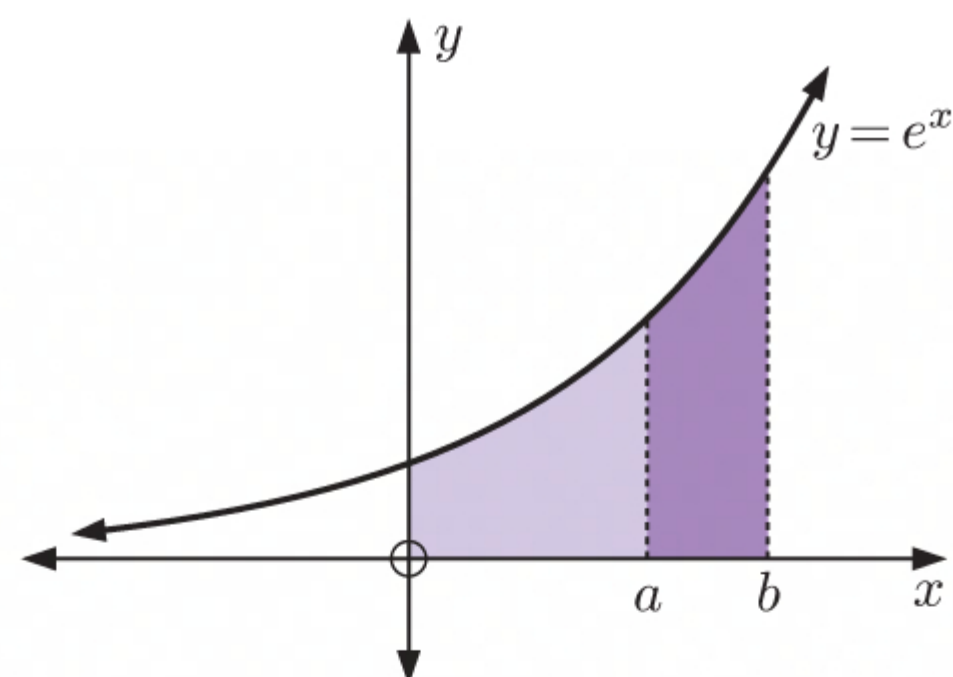
$$\therefore [e^x]_{\ln 3}^b = 2$$

$$\therefore e^b - e^{\ln 3} = 2$$

$$\therefore e^b - 3 = 2$$

$$\therefore e^b = 5$$

$$\therefore b = \ln 5$$

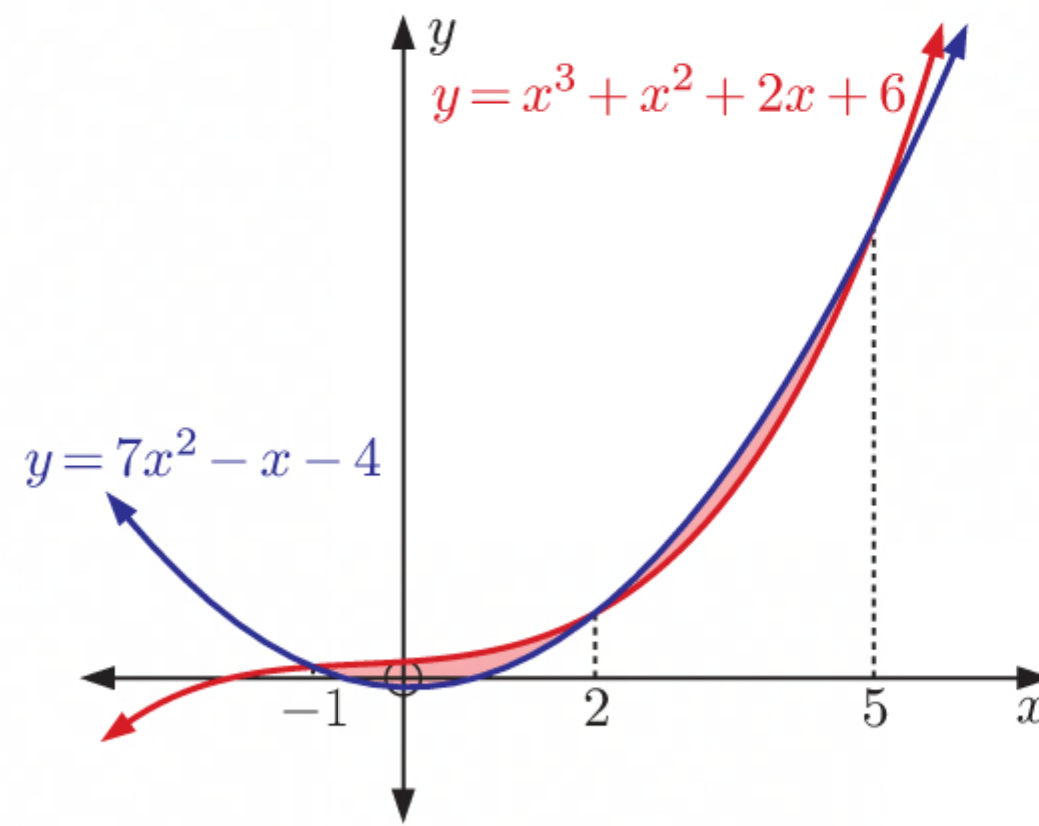
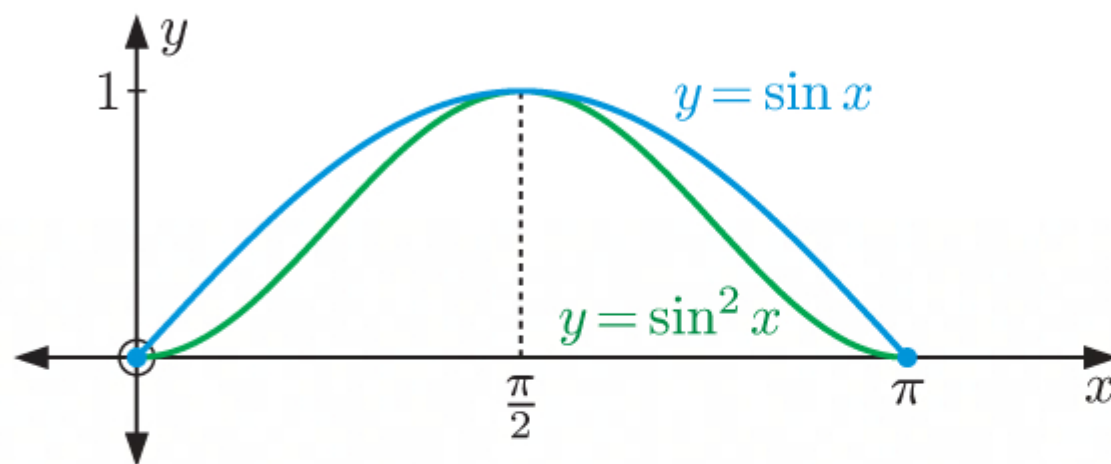


16 The curves meet where

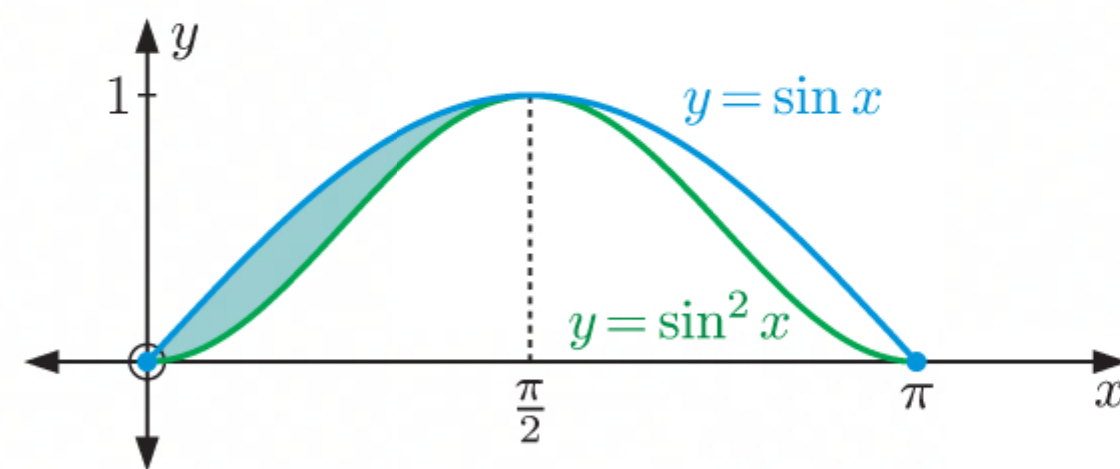
$$\begin{aligned}
 x^3 + x^2 + 2x + 6 &= 7x^2 - x - 4 \\
 \therefore x^3 - 6x^2 + 3x + 10 &= 0 \\
 \therefore (x+1)(x^2 - 7x + 10) &= 0 \\
 \therefore (x+1)(x-2)(x-5) &= 0 \\
 \therefore x &= -1, 2, \text{ or } 5
 \end{aligned}$$

 \therefore enclosed area

$$\begin{aligned}
 &= \int_{-1}^2 ((x^3 + x^2 + 2x + 6) - (7x^2 - x - 4)) dx \\
 &\quad + \int_2^5 ((7x^2 - x - 4) - (x^3 + x^2 + 2x + 6)) dx \\
 &= \int_{-1}^2 (x^3 - 6x^2 + 3x + 10) dx + \int_2^5 (-x^3 + 6x^2 - 3x - 10) dx \\
 &= \left[\frac{1}{4}x^4 - 2x^3 + \frac{3}{2}x^2 + 10x \right]_{-1}^2 + \left[-\frac{1}{4}x^4 + 2x^3 - \frac{3}{2}x^2 - 10x \right]_2^5 \\
 &= \left[(4 - 16 + 6 + 20) - \left(\frac{1}{4} + 2 + \frac{3}{2} - 10 \right) \right] \\
 &\quad + \left[\left(-\frac{625}{4} + 250 - \frac{75}{2} - 50 \right) - (-4 + 16 - 6 - 20) \right] \\
 &= 40\frac{1}{2} \text{ units}^2
 \end{aligned}$$

**17 a**

$$\begin{aligned}
 \text{b Area} &= \int_0^{\pi/2} (\sin x - \sin^2 x) dx \\
 &= \int_0^{\pi/2} \left(\sin x - \left(\frac{1}{2} - \frac{1}{2} \cos 2x \right) \right) dx \\
 &= \int_0^{\pi/2} \left(\sin x + \frac{1}{2} \cos 2x - \frac{1}{2} \right) dx \\
 &= \left[-\cos x + \frac{1}{4} \sin 2x - \frac{1}{2}x \right]_0^{\pi/2} \\
 &= \left(0 + \frac{1}{4}(0) - \frac{\pi}{4} \right) - (-1 + 0 - 0) \\
 &= \left(1 - \frac{\pi}{4} \right) \text{ units}^2
 \end{aligned}$$



18 $R_1(t) = 6.4, \quad R_2(t) = 2.5 - 1.25e^{-0.2t}$

a i
$$\begin{aligned} \int_0^{\frac{1}{2}} R_2(t) dt &= \int_0^{\frac{1}{2}} (2.5 - 1.25e^{-0.2t}) dt \\ &= \left[2.5t - 1.25\left(\frac{1}{-0.2}\right) e^{-0.2t} \right]_0^{\frac{1}{2}} \\ &= \left[2.5t + 6.25e^{-0.2t} \right]_0^{\frac{1}{2}} \\ &= (2.5(\frac{1}{2}) + 6.25e^{-0.2(\frac{1}{2})}) - (0 + 6.25) \\ &\approx 0.655 \end{aligned}$$

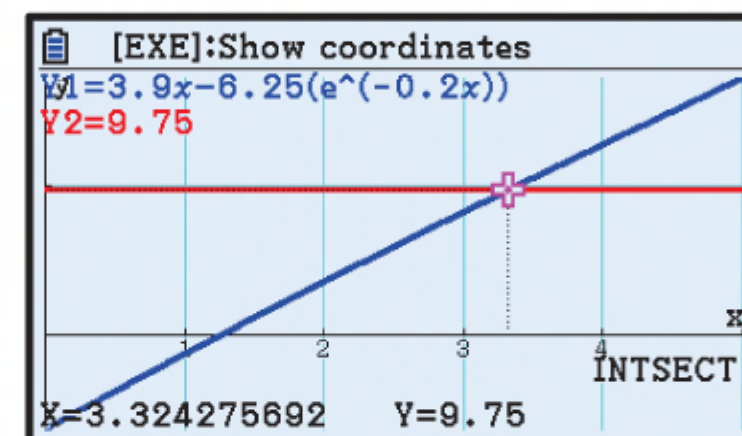
About 655 millilitres of water leak from the watering can in the first 30 seconds.

ii
$$\begin{aligned} \int_0^1 [R_1(t) - R_2(t)] dt &= \int_0^1 [6.4 - (2.5 - 1.25e^{-0.2t})] dt \\ &= \int_0^1 (3.9 + 1.25e^{-0.2t}) dt \\ &= \left[3.9t + 1.25\left(\frac{1}{-0.2}\right) e^{-0.2t} \right]_0^1 \\ &= \left[3.9t - 6.25e^{-0.2t} \right]_0^1 \\ &= (3.9 - 6.25e^{-0.2}) - (0 - 6.25) \\ &\approx 5.03 \end{aligned}$$

There are about 5.03 litres of water in the watering can after 1 minute.

b Suppose it takes x seconds for the watering can to be full.

We find x such that
$$\begin{aligned} \int_0^x [R_1(t) - R_2(t)] dt &= 16 \\ \therefore [3.9t - 6.25e^{-0.2t}]_0^x &= 16 \quad \{\text{using a ii}\} \\ \therefore (3.9x - 6.25e^{-0.2x}) - (0 - 6.25) &= 16 \\ \therefore 3.9x - 6.25e^{-0.2x} + 6.25 &= 16 \\ \therefore 3.9x - 6.25e^{-0.2x} &= 9.75 \\ \therefore x &\approx 3.324 \\ \{\text{using technology}\} \end{aligned}$$

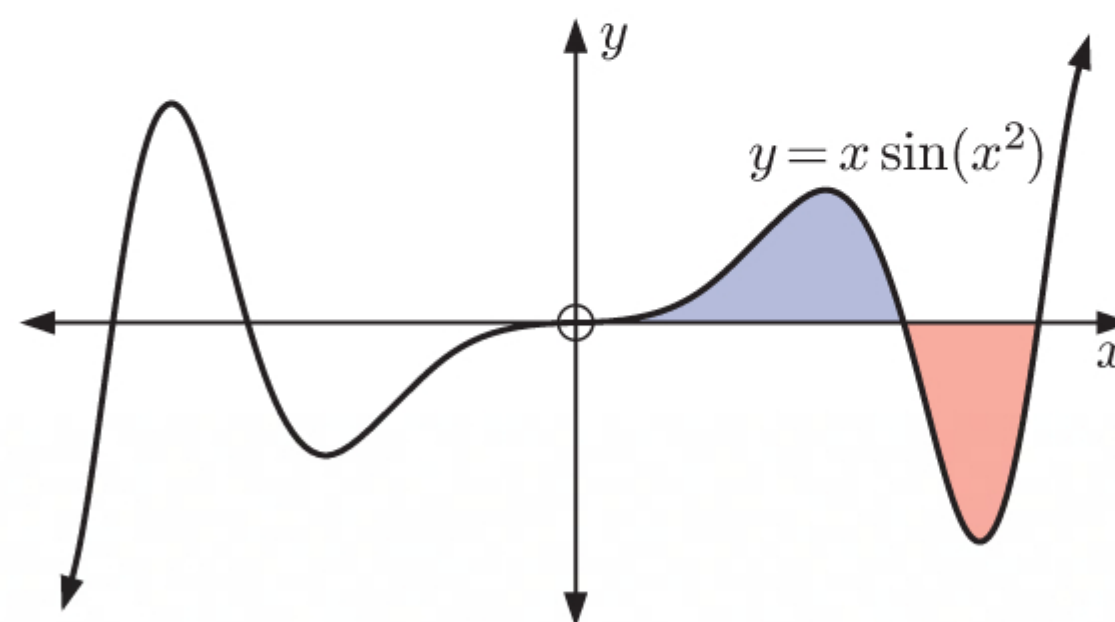


$$\begin{aligned} 3.324 \text{ minutes} &\approx 3.324 \times 60 \text{ seconds} \\ &\approx 199 \text{ seconds} \end{aligned}$$

\therefore it will take about 199 seconds for the watering can to be full.

$$\begin{aligned}
 19 \quad a \quad \int x \sin(x^2) dx &= \frac{1}{2} \int 2x \sin(x^2) dx \\
 &= \frac{1}{2} \int \sin u \frac{du}{dx} dx \quad \{u = x^2, \frac{du}{dx} = 2x\} \\
 &= \frac{1}{2} \int \sin u du \\
 &= \frac{1}{2}(-\cos u) + c \\
 &= -\frac{1}{2} \cos(x^2) + c
 \end{aligned}$$

- b** $y = x \sin(x^2)$ cuts the x -axis when $x = 0$
 or $\sin(x^2) = 0$
 $\therefore x = 0$, and the smallest positive
 x -intercepts occur when
 $x^2 = \pi$ or 2π
 $\therefore x = \sqrt{\pi}$ or $\sqrt{2\pi} \quad \{x > 0\}$



$$\begin{aligned}
 \text{Purple area} &= \int_0^{\sqrt{\pi}} x \sin(x^2) dx \\
 &= \left[-\frac{1}{2} \cos(x^2) \right]_0^{\sqrt{\pi}} \\
 &= -\frac{1}{2} \cos \pi - \left(-\frac{1}{2} \cos 0 \right) \\
 &= 1 \text{ unit}^2
 \end{aligned}$$

$$\begin{aligned}
 \text{Red area} &= - \int_{\sqrt{\pi}}^{\sqrt{2\pi}} x \sin(x^2) dx \\
 &= - \left[-\frac{1}{2} \cos(x^2) \right]_{\sqrt{\pi}}^{\sqrt{2\pi}} \\
 &= \frac{1}{2} \cos 2\pi - \frac{1}{2} \cos \pi \\
 &= 1 \text{ unit}^2
 \end{aligned}$$

\therefore the shaded regions have equal area.

REVIEW SET 17B

$$\begin{aligned}
 1 \quad a \quad \int_2^3 \frac{1}{\sqrt{3x}} dx &= \int_2^3 (3x)^{-\frac{1}{2}} dx \\
 &= \left[\left(\frac{1}{3} \right) \frac{(3x)^{\frac{1}{2}}}{\frac{1}{2}} \right]_2^3 \\
 &= \left[\frac{2}{3} \sqrt{3x} \right]_2^3 \\
 &= \frac{2}{3} \sqrt{9} - \frac{2}{3} \sqrt{6} \\
 &= 2 - \frac{2}{3} \sqrt{2} \sqrt{3} \\
 &= 2 - \frac{2\sqrt{2}}{\sqrt{3}}
 \end{aligned}$$

$$\begin{aligned}
 b \quad \int_1^4 \left(x - \frac{1}{2}x^2 \right) dx &= \left[\frac{1}{2}x^2 - \frac{1}{6}x^3 \right]_1^4 \\
 &= \left(\frac{1}{2}(16) - \frac{1}{6}(64) \right) - \left(\frac{1}{2} - \frac{1}{6} \right) \\
 &= -\frac{8}{3} - \frac{1}{3} \\
 &= -3
 \end{aligned}$$

$$\begin{aligned}
 c \quad \int_0^1 \left(x^2 + \frac{1}{3} \right)^2 dx &= \int_0^1 \left(x^4 + \frac{2}{3}x^2 + \frac{1}{9} \right) dx \\
 &= \left[\frac{1}{5}x^5 + \frac{2}{9}x^3 + \frac{1}{9}x \right]_0^1 \\
 &= \left(\frac{1}{5} + \frac{2}{9} + \frac{1}{9} \right) - 0 \\
 &= \frac{8}{15}
 \end{aligned}$$

$$\begin{aligned}
 2 \quad & \int_0^a (x^2 - \tfrac{1}{2}x) dx = \tfrac{9}{16} \\
 & \therefore \left[\tfrac{1}{3}x^3 - \tfrac{1}{4}x^2 \right]_0^a = \tfrac{9}{16} \\
 & \therefore \tfrac{1}{3}a^3 - \tfrac{1}{4}a^2 = \tfrac{9}{16} \\
 & \therefore 16a^3 - 12a^2 = 27 \\
 & \therefore 16a^3 - 12a^2 - 27 = 0 \\
 & \therefore a = \tfrac{3}{2} \\
 & \quad \text{\{using technology\}}
 \end{aligned}$$

Math Rad Norm1 d/c Real
 $aX^3 + bX^2 + cX + d = 0$

a	b	c	d
16	-12	0	-27

 SOLVE DELET CLEAR EDIT

Math Rad Norm1 d/c Real
 $aX^3 + bX^2 + cX + d = 0$
 X1 1.5
 REPEAT

$$\begin{aligned}
 3 \quad a \quad & \int_2^3 \frac{1}{\sqrt{3x-4}} dx = \int_2^3 (3x-4)^{-\frac{1}{2}} dx \\
 & = \left[\left(\frac{1}{3} \right) \frac{(3x-4)^{\frac{1}{2}}}{\frac{1}{2}} \right]_2^3 \\
 & = \left[\frac{2}{3} \sqrt{3x-4} \right]_2^3 \\
 & = \frac{2}{3} \sqrt{5} - \frac{2}{3} \sqrt{2} \\
 & = \frac{2}{3} (\sqrt{5} - \sqrt{2})
 \end{aligned}$$

$$\begin{aligned}
 b \quad & \int_{2e}^{3e} \frac{4}{x+e} dx = \left[4 \ln |x+e| \right]_{2e}^{3e} \\
 & = 4 \ln 4e - 4 \ln 3e \\
 & = 4(\ln 4e - \ln 3e) \\
 & = 4 \ln \left(\frac{4e}{3e} \right) \\
 & = 4 \ln \left(\frac{4}{3} \right)
 \end{aligned}$$

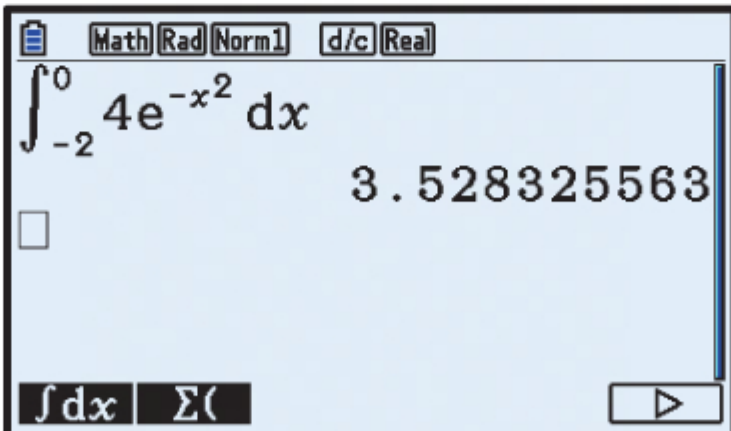
$$\begin{aligned}
 c \quad & \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (2 \sin x + 1) dx = \left[-2 \cos x + x \right]_{\frac{\pi}{4}}^{\frac{\pi}{2}} \\
 & = \left(-2 \cos \frac{\pi}{2} + \frac{\pi}{2} \right) - \left(-2 \cos \frac{\pi}{4} + \frac{\pi}{4} \right) \\
 & = \frac{\pi}{2} - \left(-\sqrt{2} + \frac{\pi}{4} \right) \\
 & = \frac{\pi}{2} + \sqrt{2} - \frac{\pi}{4} \\
 & = \frac{\pi}{4} + \sqrt{2}
 \end{aligned}$$

$$\begin{aligned}
 4 \quad & \int_0^b \cos x dx = \frac{1}{\sqrt{2}}, \quad 0 < b < \pi \\
 & \therefore [\sin x]_0^b = \frac{1}{\sqrt{2}} \\
 & \therefore \sin b - 0 = \frac{1}{\sqrt{2}} \\
 & \therefore \sin b = \frac{1}{\sqrt{2}} \\
 & \therefore b = \frac{\pi}{4} \text{ or } \frac{3\pi}{4}
 \end{aligned}$$

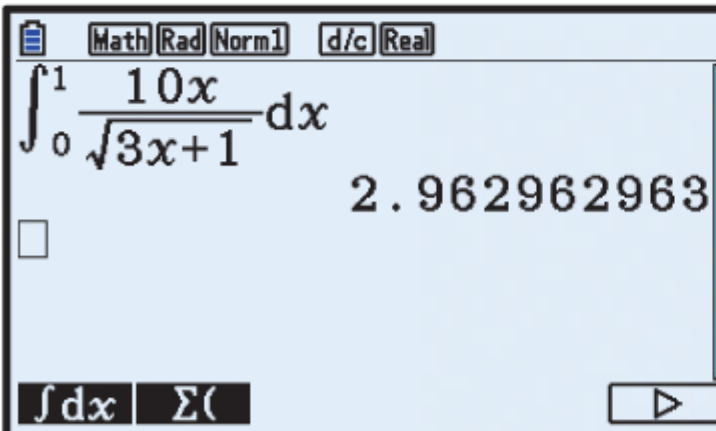
$$5 \quad \cos^2 x = \frac{1}{2} + \frac{1}{2} \cos 2x$$

$$\therefore \cos^2\left(\frac{x}{2}\right) = \frac{1}{2} + \frac{1}{2} \cos x$$

$$\begin{aligned} \therefore \int_0^{\frac{\pi}{3}} \cos^2\left(\frac{x}{2}\right) dx &= \int_0^{\frac{\pi}{3}} \left(\frac{1}{2} + \frac{1}{2} \cos x\right) dx \\ &= \left[\frac{1}{2}x + \frac{1}{2} \sin x\right]_0^{\frac{\pi}{3}} \\ &= \left(\frac{1}{2}\left(\frac{\pi}{3}\right) + \frac{1}{2} \sin \frac{\pi}{3}\right) - \left(\frac{1}{2}(0) + \frac{1}{2} \sin 0\right) \\ &= \left(\frac{\pi}{6} + \frac{1}{2} \times \frac{\sqrt{3}}{2}\right) - 0 \\ &= \frac{\pi}{6} + \frac{\sqrt{3}}{4} \end{aligned}$$

6 a 

$$\therefore \int_{-2}^0 4e^{-x^2} dx \approx 3.528$$

b 

$$\therefore \int_0^1 \frac{10x}{\sqrt{3x+1}} dx \approx 2.963$$

$$7 \quad \int_1^4 f(x) dx = 3$$

a
$$\begin{aligned} &\int_1^4 (f(x) + 1) dx \\ &= \int_1^4 f(x) dx + \int_1^4 1 dx \\ &= 3 + [x]_1^4 \\ &= 3 + (4 - 1) \\ &= 3 + 3 \\ &= 6 \end{aligned}$$

c
$$\begin{aligned} &\int_4^1 k f(x) dx = 5 \\ \therefore -k \int_1^4 f(x) dx &= 5 \\ \therefore -3k &= 5 \\ \therefore k &= -\frac{5}{3} \end{aligned}$$

b
$$\begin{aligned} &\int_1^2 f(x) dx - \int_4^2 f(x) dx \\ &= \int_1^2 f(x) dx + \int_2^4 f(x) dx \\ &= \int_1^4 f(x) dx \\ &= 3 \end{aligned}$$

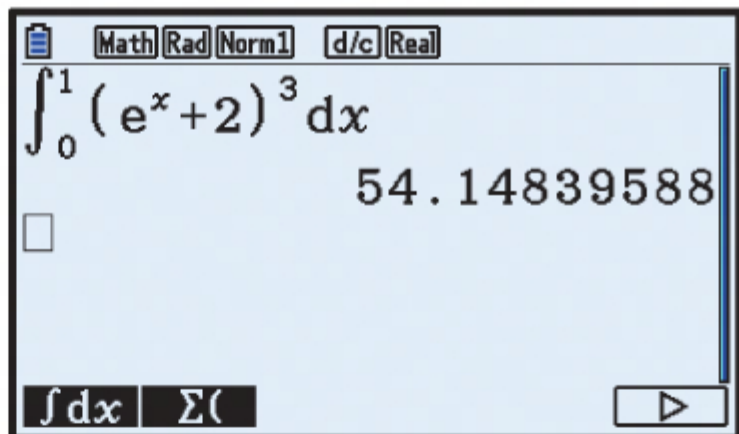
8 a
$$\begin{aligned} (\sin \theta - \cos \theta)^2 &= \sin^2 \theta - 2 \sin \theta \cos \theta + \cos^2 \theta \\ &= 1 - \sin 2\theta \quad \{\sin^2 \theta + \cos^2 \theta = 1, \quad \sin 2\theta = 2 \sin \theta \cos \theta\} \end{aligned}$$

$$\begin{aligned}
 \text{b} \quad \int_0^{\frac{\pi}{4}} (\sin \theta - \cos \theta)^2 d\theta &= \int_0^{\frac{\pi}{4}} (1 - \sin 2\theta) d\theta \\
 &= \left[\theta + \frac{1}{2} \cos 2\theta \right]_0^{\frac{\pi}{4}} \\
 &= \left(\frac{\pi}{4} + \frac{1}{2} \cos \frac{\pi}{2} \right) - \left(0 + \frac{1}{2} \cos 0 \right) \\
 &= \frac{\pi}{4} - \frac{1}{2}
 \end{aligned}$$

$$\begin{aligned}
 \text{9 a} \quad (e^x + 2)^3 &= (e^x)^3 + 3(e^x)^2(2) + 3(e^x)(2)^2 + 2^3 \\
 &= e^{3x} + 6e^{2x} + 12e^x + 8
 \end{aligned}$$

$$\begin{aligned}
 \text{b} \quad \int_0^1 (e^x + 2)^3 dx &= \int_0^1 (e^{3x} + 6e^{2x} + 12e^x + 8) dx \\
 &= \left[\frac{1}{3}e^{3x} + 3e^{2x} + 12e^x + 8x \right]_0^1 \\
 &= \left(\frac{1}{3}e^3 + 3e^2 + 12e^1 + 8 \right) - \left(\frac{1}{3}e^0 + 3e^0 + 12e^0 + 0 \right) \\
 &= \frac{1}{3}e^3 + 3e^2 + 12e + 8 - \left(\frac{1}{3} + 3 + 12 \right) \\
 &= \frac{1}{3}e^3 + 3e^2 + 12e + 8 - 15\frac{1}{3} \\
 &= \frac{1}{3}e^3 + 3e^2 + 12e - 7\frac{1}{3} \\
 &\approx 54.1
 \end{aligned}$$

c

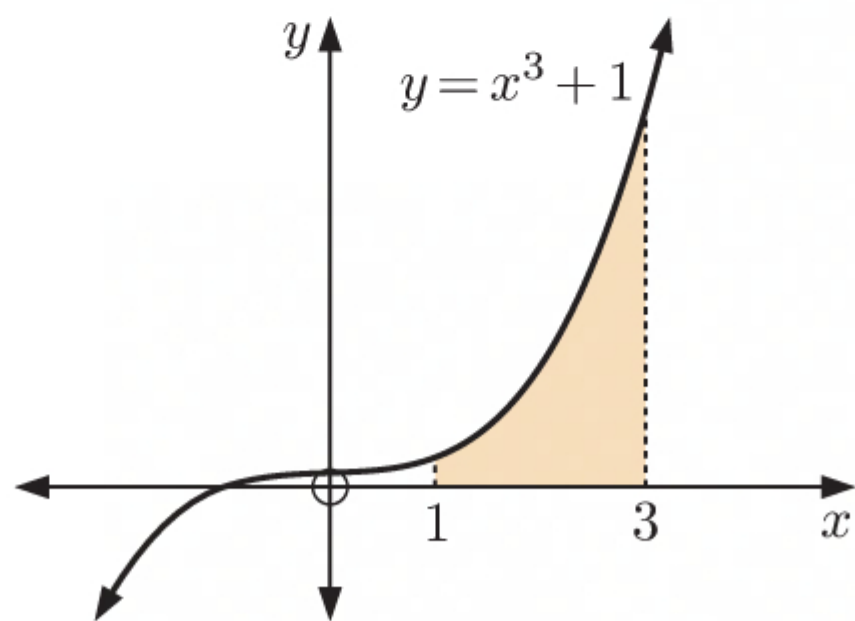


$$\int_0^1 (e^x + 2)^3 dx \approx 54.1$$

$$\begin{aligned}
 \text{10 a} \quad \int \frac{\cos x}{(1 + \sin x)^3} dx &= \int \frac{1}{u^3} \frac{du}{dx} dx \quad \{u = 1 + \sin x, \quad \frac{du}{dx} = \cos x\} \\
 &= \int \frac{1}{u^3} du \\
 &= \int u^{-3} du \\
 &= \frac{u^{-2}}{-2} + c \\
 &= -\frac{1}{2} \times \frac{1}{u^2} + c \\
 &= -\frac{1}{2(1 + \sin x)^2} + c
 \end{aligned}$$

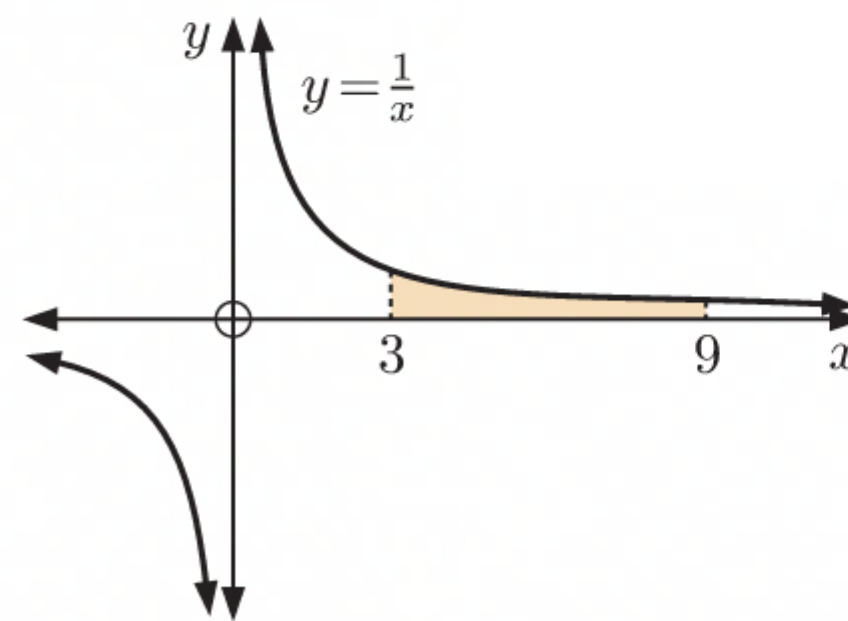
$$\begin{aligned}
 \text{b} \quad \int_0^{\frac{\pi}{6}} \frac{\cos x}{(1 + \sin x)^3} dx &= \left[-\frac{1}{2(1 + \sin x)^2} \right]_0^{\frac{\pi}{6}} \\
 &= -\frac{1}{2(1 + \frac{1}{2})^2} - \left(-\frac{1}{2(1)^2} \right) \\
 &= -\frac{1}{\frac{9}{2}} + \frac{1}{2} \\
 &= -\frac{2}{9} + \frac{1}{2} \\
 &= \frac{5}{18}
 \end{aligned}$$

11 a



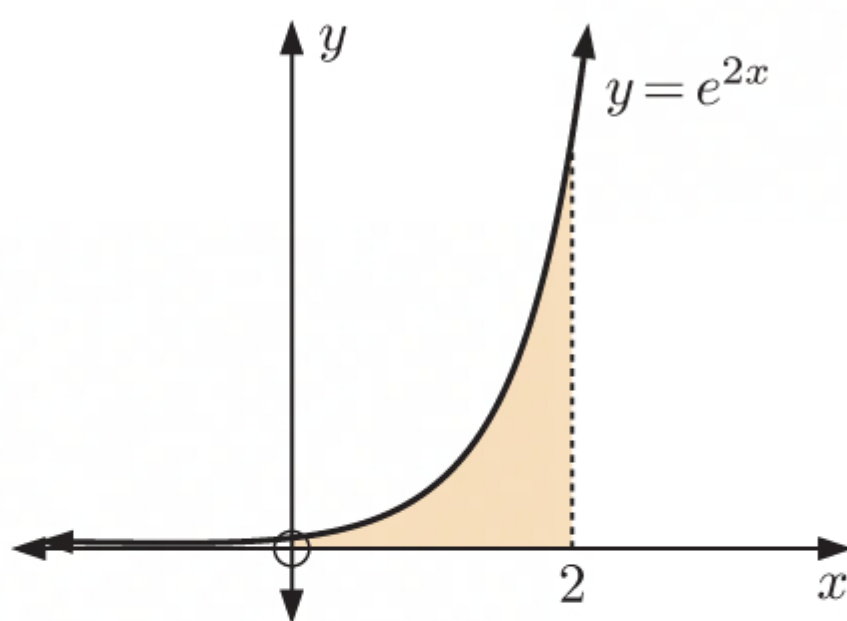
$$\begin{aligned}
 \text{Area} &= \int_1^3 (x^3 + 1) dx \\
 &= \left[\frac{1}{4}x^4 + x \right]_1^3 \\
 &= \left(\frac{81}{4} + 3 \right) - \left(\frac{1}{4} + 1 \right) \\
 &= 22 \text{ units}^2
 \end{aligned}$$

b



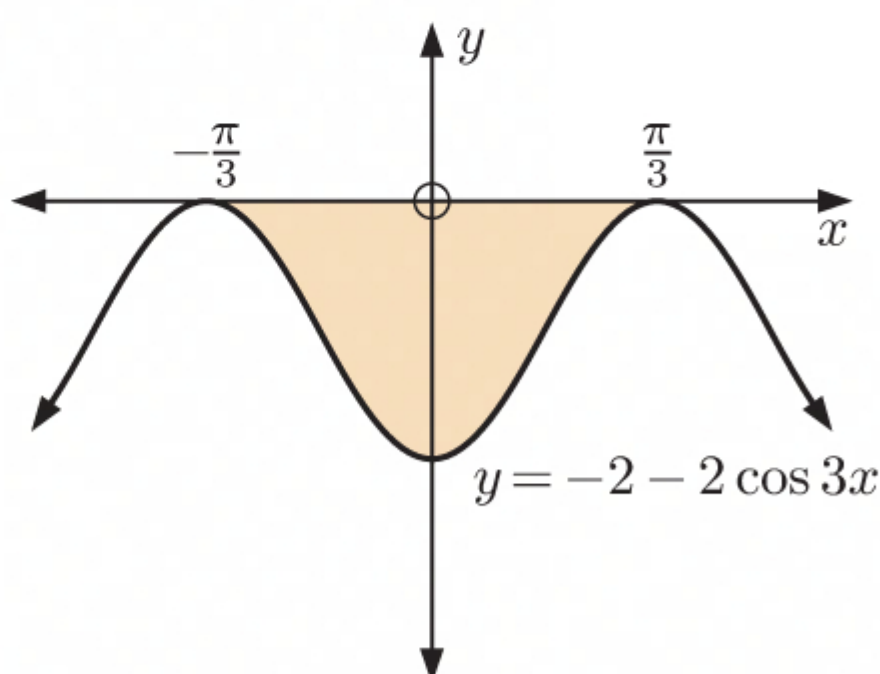
$$\begin{aligned}
 \text{Area} &= \int_3^9 \frac{1}{x} dx \\
 &= [\ln |x|]_3^9 \\
 &= \ln 9 - \ln 3 \\
 &= \ln\left(\frac{9}{3}\right) \\
 &= \ln 3 \text{ units}^2
 \end{aligned}$$

c

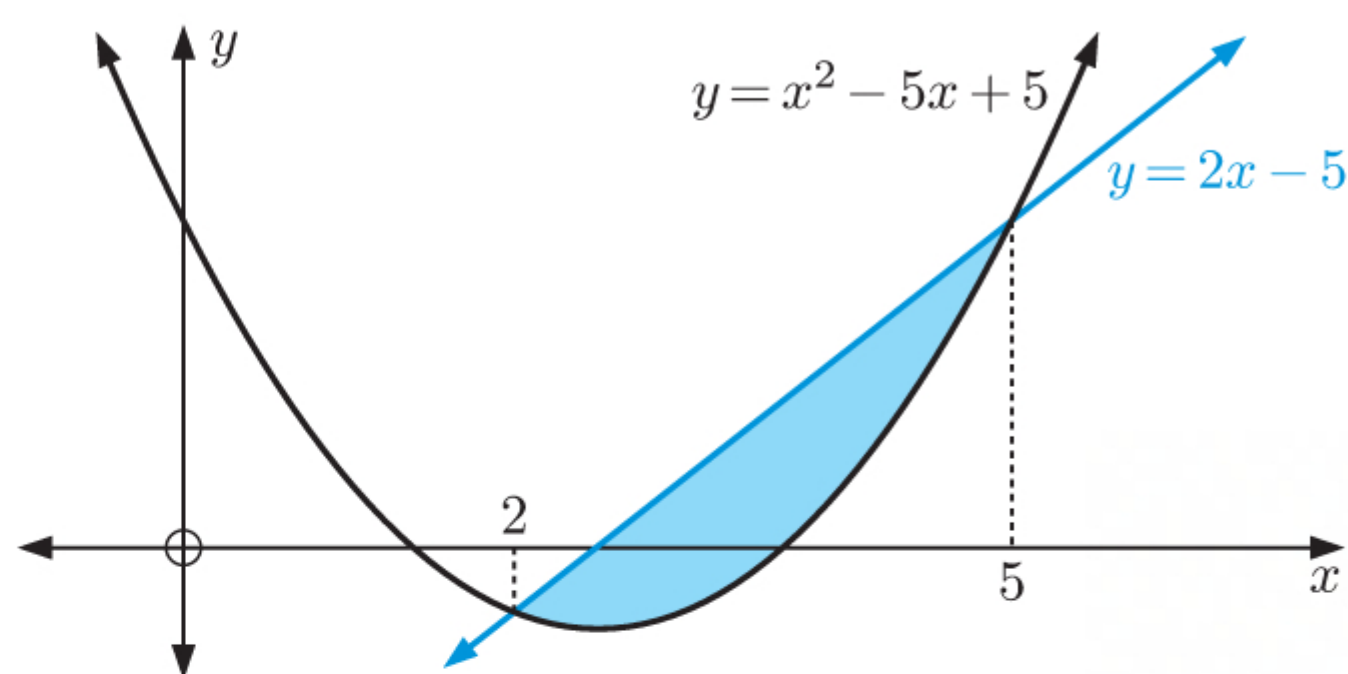


$$\begin{aligned}
 \text{Area} &= \int_0^2 e^{2x} dx \\
 &= \left[\frac{1}{2}e^{2x} \right]_0^2 \\
 &= \frac{1}{2}e^4 - \frac{1}{2} \\
 &= \frac{e^4 - 1}{2} \text{ units}^2
 \end{aligned}$$

d



$$\begin{aligned}
 \text{Area} &= - \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} (-2 - 2 \cos 3x) dx \\
 &= - \left[-2x - \frac{2}{3} \sin 3x \right]_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \\
 &= - \left[\left(-\frac{2\pi}{3} - \frac{2}{3} \sin \pi \right) - \left(\frac{2\pi}{3} - \frac{2}{3} \sin(-\pi) \right) \right] \\
 &= - \left(-\frac{2\pi}{3} - \frac{2\pi}{3} \right) \\
 &= \frac{4\pi}{3} \text{ units}^2
 \end{aligned}$$

12

$$y = x^2 - 5x + 5 \text{ meets } y = 2x - 5$$

$$\text{where } x^2 - 5x + 5 = 2x - 5$$

$$\therefore x^2 - 7x + 10 = 0$$

$$\therefore (x - 2)(x - 5) = 0$$

$$\therefore x = 2 \text{ or } 5$$

$$\begin{aligned} \text{Area} &= \int_2^5 [(2x - 5) - (x^2 - 5x + 5)] dx \\ &= \int_2^5 (-x^2 + 7x - 10) dx \\ &= \left[-\frac{1}{3}x^3 + \frac{7}{2}x^2 - 10x \right]_2^5 \\ &= \left(-\frac{125}{3} + \frac{175}{2} - 50 \right) - \left(-\frac{8}{3} + 14 - 20 \right) \\ &= -\frac{25}{6} - \left(-\frac{26}{3} \right) \\ &= 4\frac{1}{2} \text{ units}^2 \end{aligned}$$

13 The curve cuts the x -axis when $y = 0$

$$\therefore 4e^x - 1 = 0$$

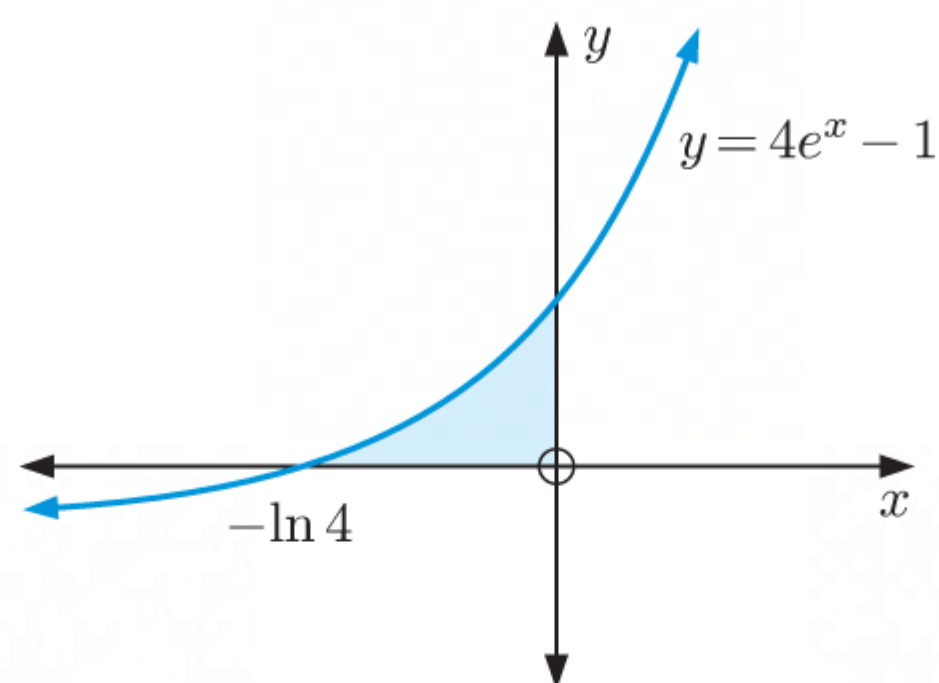
$$\therefore 4e^x = 1$$

$$\therefore e^x = \frac{1}{4}$$

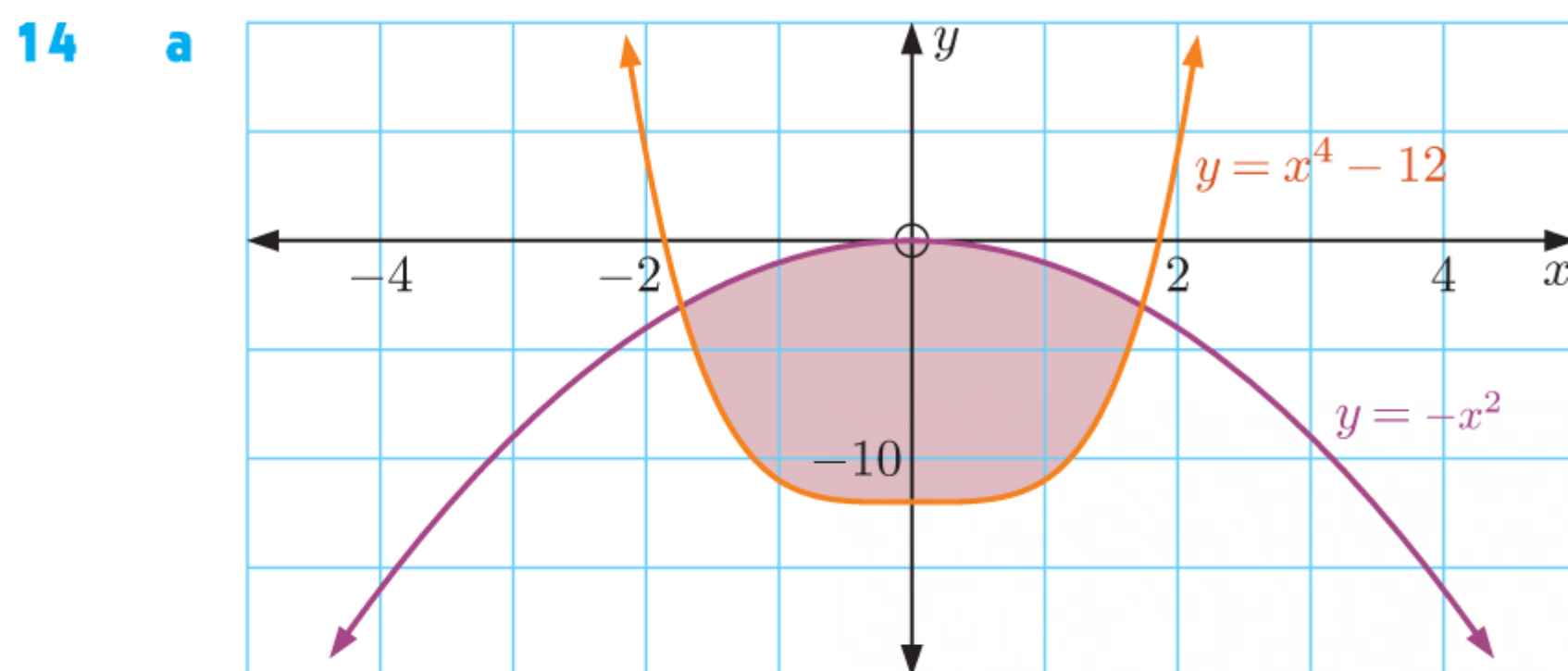
$$\therefore x = \ln\left(\frac{1}{4}\right)$$

$$= \ln(4^{-1})$$

$$= -\ln 4$$



$$\begin{aligned} \text{Area} &= \int_{-\ln 4}^0 (4e^x - 1) dx \\ &= [4e^x - x]_{-\ln 4}^0 \\ &= 4 - (4e^{-\ln 4} + \ln 4) \\ &= 4 - (1 + \ln 4) \\ &= (3 - \ln 4) \text{ units}^2 \end{aligned}$$



- b** The graphs meet where $x^4 - 12 = -x^2$
 $\therefore x^4 + x^2 - 12 = 0$
 $\therefore (x^2 + 4)(x^2 - 3) = 0$
 $\therefore x^2 - 3 = 0 \quad \{x^2 + 4 > 0 \text{ for all } x\}$
 $\therefore x^2 = 3$
 $\therefore x = \pm\sqrt{3}$

$$\begin{array}{ll} \text{When } x = \sqrt{3}, & y = -(\sqrt{3})^2 \\ & = -3 \end{array} \qquad \begin{array}{ll} \text{When } x = -\sqrt{3}, & y = -(-\sqrt{3})^2 \\ & = -3 \end{array}$$

\therefore the graphs meet at $(-\sqrt{3}, -3)$ and $(\sqrt{3}, -3)$.

c Area enclosed $= \int_{-\sqrt{3}}^{\sqrt{3}} (-x^2 - (x^4 - 12)) dx$
 $= \int_{-\sqrt{3}}^{\sqrt{3}} (-x^4 - x^2 + 12) dx$
 $= \left[-\frac{1}{5}x^5 - \frac{1}{3}x^3 + 12x \right]_{-\sqrt{3}}^{\sqrt{3}}$
 $= \left(-\frac{1}{5}(\sqrt{3})^5 - \frac{1}{3}(\sqrt{3})^3 + 12\sqrt{3} \right) - \left(-\frac{1}{5}(-\sqrt{3})^5 - \frac{1}{3}(-\sqrt{3})^3 - 12\sqrt{3} \right)$
 $= -\frac{9}{5}\sqrt{3} - \sqrt{3} + 12\sqrt{3} - \left(\frac{9}{5}\sqrt{3} + \sqrt{3} - 12\sqrt{3} \right)$
 $= -\frac{9}{5}\sqrt{3} - \sqrt{3} + 12\sqrt{3} - \frac{9}{5}\sqrt{3} - \sqrt{3} + 12\sqrt{3}$
 $= \frac{92\sqrt{3}}{5} \text{ units}^2$

- 15** B has coordinates $(2, 2^2 + k)$, or $(2, 4 + k)$.
 \therefore the horizontal line from A to B is $y = 4 + k$.

Now, upper area U = lower area L

$$\therefore \int_0^2 [(4 + k) - (x^2 + k)] dx = \int_0^2 (x^2 + k) dx$$

$$\therefore \int_0^2 (-x^2 + 4) dx = \int_0^2 (x^2 + k) dx$$

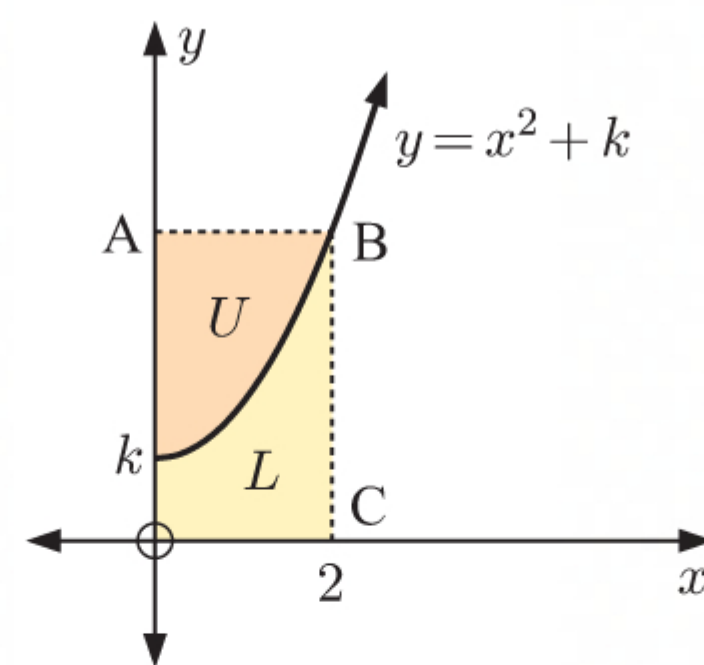
$$\therefore \left[-\frac{1}{3}x^3 + 4x\right]_0^2 = \left[\frac{1}{3}x^3 + kx\right]_0^2$$

$$\therefore \left(-\frac{8}{3} + 8\right) - 0 = \left(\frac{8}{3} + 2k\right) - 0$$

$$\therefore \frac{16}{3} = \frac{8}{3} + 2k$$

$$\therefore \frac{8}{3} = 2k$$

$$\therefore k = \frac{4}{3}$$



16 Area = $\int_0^m \sin x dx = \frac{1}{2}$

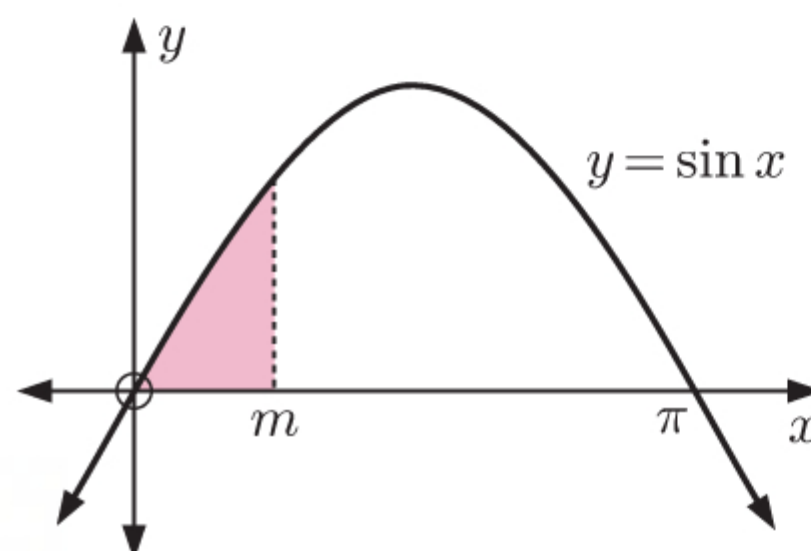
$$\therefore [-\cos x]_0^m = \frac{1}{2}$$

$$\therefore -\cos m - (-1) = \frac{1}{2}$$

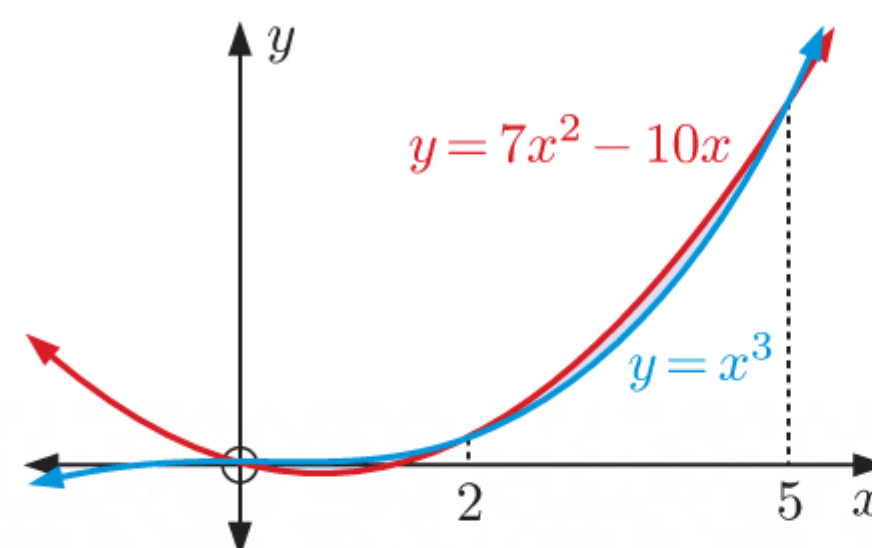
$$\therefore -\cos m = -\frac{1}{2}$$

$$\therefore \cos m = \frac{1}{2}$$

$$\therefore m = \frac{\pi}{3}$$



- 17** $y = x^3$ meets $y = 7x^2 - 10x$
 where $x^3 = 7x^2 - 10x$
 $\therefore x^3 - 7x^2 + 10x = 0$
 $\therefore x(x^2 - 7x + 10) = 0$
 $\therefore x(x - 2)(x - 5) = 0$
 $\therefore x = 0, 2, \text{ or } 5$



$$\begin{aligned} \text{Area} &= \int_0^2 [x^3 - (7x^2 - 10x)] dx + \int_2^5 [7x^2 - 10x - x^3] dx \\ &= \left[\frac{1}{4}x^4 - \frac{7}{3}x^3 + 5x^2\right]_0^2 + \left[\frac{7}{3}x^3 - 5x^2 - \frac{1}{4}x^4\right]_2^5 \\ &= \left(\left(4 - \frac{56}{3} + 20\right) - 0\right) + \left(\left(\frac{875}{3} - 125 - \frac{625}{4}\right) - \left(\frac{56}{3} - 20 - 4\right)\right) \\ &= \frac{16}{3} + \left(\frac{125}{12} - \left(-\frac{16}{3}\right)\right) \\ &= 21\frac{1}{12} \text{ units}^2 \end{aligned}$$

18 a Area = $\int_0^2 ax(x-2) dx = 4$

$$\therefore \int_0^2 (ax^2 - 2ax) dx = 4$$

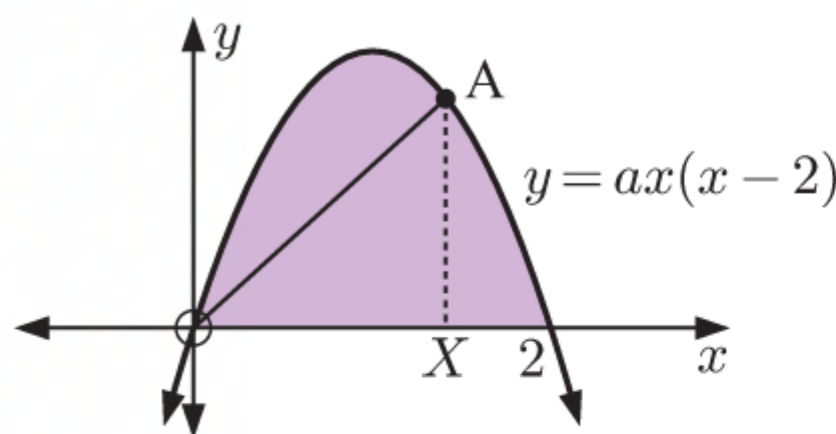
$$\therefore \left[\frac{1}{3}ax^3 - ax^2 \right]_0^2 = 4$$

$$\therefore \left(\frac{8}{3}a - 4a \right) - 0 = 4$$

$$\therefore \frac{8}{3}a - 4a = 4$$

$$\therefore -\frac{4}{3}a = 4$$

$$\therefore a = -3$$



b Let A have x -coordinate X , then A has y -coordinate $-3X(X-2)$.

If [OA] divides the shaded region into equal parts, then each region has area 2 units².

We consider the area of the region bounded by [OA], $y = -3x(x-2)$, and the x -axis.

This is the area between [OA] and the x -axis from O to X , and the area between $y = -3x(x-2)$ and the x -axis from X to 2.

$$\text{So, } \frac{1}{2}(X)(-3X(X-2)) + \int_X^2 -3x(x-2) dx = 2$$

$$\therefore -\frac{3}{2}X^2(X-2) + \int_X^2 (-3x^2 + 6x) dx = 2$$

$$\therefore -\frac{3}{2}X^3 + 3X^2 + [-x^3 + 3x^2]_X^2 = 2$$

$$\therefore -\frac{3}{2}X^3 + 3X^2 + ((-8 + 12) - (-X^3 + 3X^2)) = 2$$

$$\therefore -\frac{3}{2}X^3 + 3X^2 + (4 + X^3 - 3X^2) = 2$$

$$\therefore -\frac{3}{2}X^3 + \cancel{3X^2} + 4 + X^3 - \cancel{3X^2} = 2$$

$$\therefore -\frac{1}{2}X^3 = -2$$

$$\therefore X^3 = 4$$

$$\therefore X = \sqrt[3]{4}$$

\therefore A has x -coordinate $\sqrt[3]{4}$.

19 $E(t) = 2 \sin\left(\frac{t-5}{5}\right) + \frac{1}{2} \sin\left(\frac{t-5}{4}\right)$ kW

$$\begin{aligned} \therefore \int E(t) dt &= \int \left(2 \sin\left(\frac{t-5}{5}\right) + \frac{1}{2} \sin\left(\frac{t-5}{4}\right) \right) dt \\ &= 2 \left(-\cos\left(\frac{t-5}{5}\right) (5) \right) + \frac{1}{2} \left(-\cos\left(\frac{t-5}{4}\right) (4) \right) + c \\ &= -10 \cos\left(\frac{t-5}{5}\right) - 2 \cos\left(\frac{t-5}{4}\right) + c \end{aligned}$$

a $\int_5^{12} E(t) dt = \left[-10 \cos\left(\frac{t-5}{5}\right) - 2 \cos\left(\frac{t-5}{4}\right) \right]_5^{12}$

$$= \left(-10 \cos \frac{7}{5} - 2 \cos \frac{7}{4} \right) - (-10 \cos 0 - 2 \cos 0)$$

$$\approx 10.66$$

The solar energy transferred into Callum's solar panels from 5 am to 12 pm is about 10.7 kWh.

$$\begin{aligned}
 \text{b} \quad \int_{12}^{20} E(t) \, dt &= \left[-10 \cos\left(\frac{t-5}{5}\right) - 2 \cos\left(\frac{t-5}{4}\right) \right]_{12}^{20} \\
 &= \left(-10 \cos 3 - 2 \cos \frac{15}{4} \right) - \left(-10 \cos \frac{7}{5} - 2 \cos \frac{7}{4} \right) \\
 &\approx 12.88
 \end{aligned}$$

The solar energy transferred into Callum's solar panels from 12 pm to 8 pm is about 12.9 kWh.

$$\begin{aligned}
 \text{c} \quad \int_5^{20} E(t) \, dt &= \int_5^{12} E(t) \, dt + \int_{12}^{20} E(t) \, dt \\
 &\approx 10.66 + 12.88 \quad \{\text{using a and b}\} \\
 &\approx 23.54
 \end{aligned}$$

The solar energy transferred into Callum's solar panels from 5 am to 8 pm is about 23.5 kWh.

Chapter 18

KINEMATICS

EXERCISE 18A

1 $s(t) = 5 - t$ cm, $0 \leq t \leq 10$ s

a $s(0) = 5 - 0 = 5$ cm

\therefore the initial displacement of the object is 5 cm to the right of the origin.

b i $s(3) = 5 - 3 = 2$ cm

At time $t = 3$ seconds, the object is 2 cm to the right of the origin.

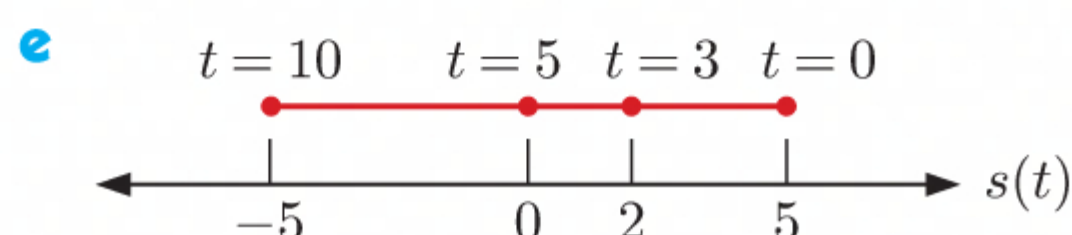
ii $s(10) = 5 - 10 = -5$ cm

At time $t = 10$ seconds, the object is 5 cm to the left of the origin.

c $s(t) = 5 - t = 0$ when $t = 5$

\therefore the object reaches the origin at time $t = 5$ seconds.

d No, the displacement function $s(t)$ is linear, so it has no turning points.

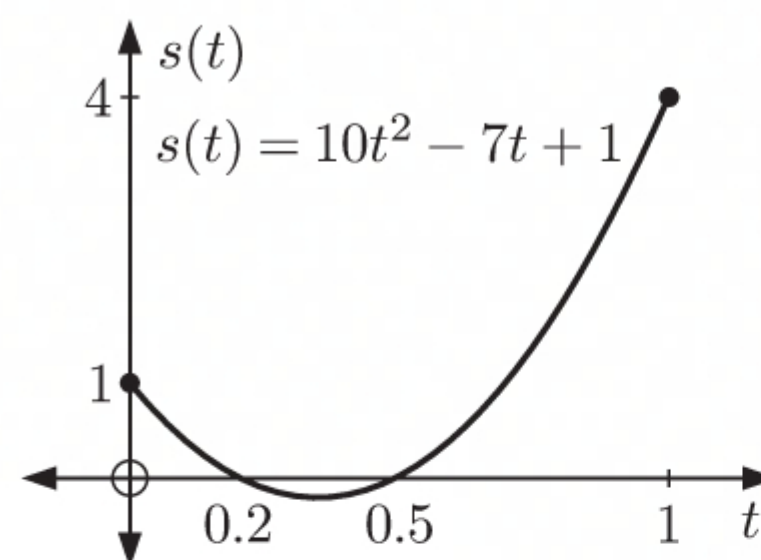


2 $s(t) = 10t^2 - 7t + 1$ m, $0 \leq t \leq 1$ s

a $s(0) = 1$ m

\therefore the initial displacement of the object is 1 m to the right of the origin.

b $s(t) = 10t^2 - 7t + 1$
 $= (5t - 1)(2t - 1)$



c The object changes direction at the turning point of $s(t)$.

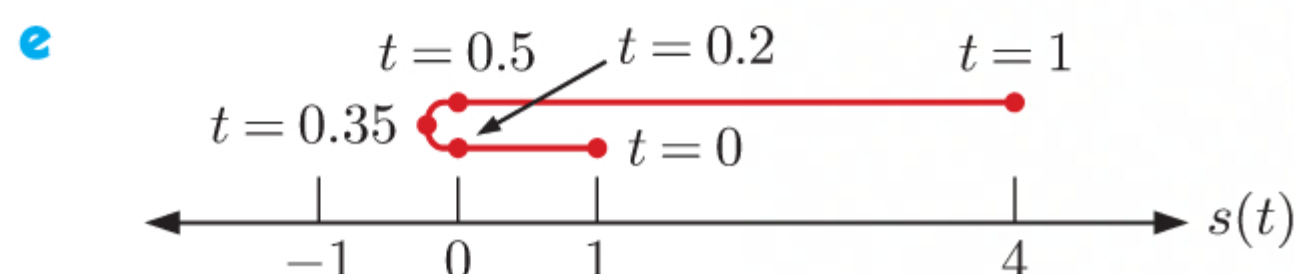
$$\begin{aligned} \text{This occurs when } t &= \frac{-(-7)}{2(10)} \\ &= \frac{7}{20} = 0.35 \text{ s} \end{aligned}$$

$$\begin{aligned} s(0.35) &= 10(0.35)^2 - 7(0.35) + 1 \\ &= -0.225 \end{aligned}$$

\therefore the object changes direction after 0.35 seconds, when it is 0.225 m to the left of the origin.

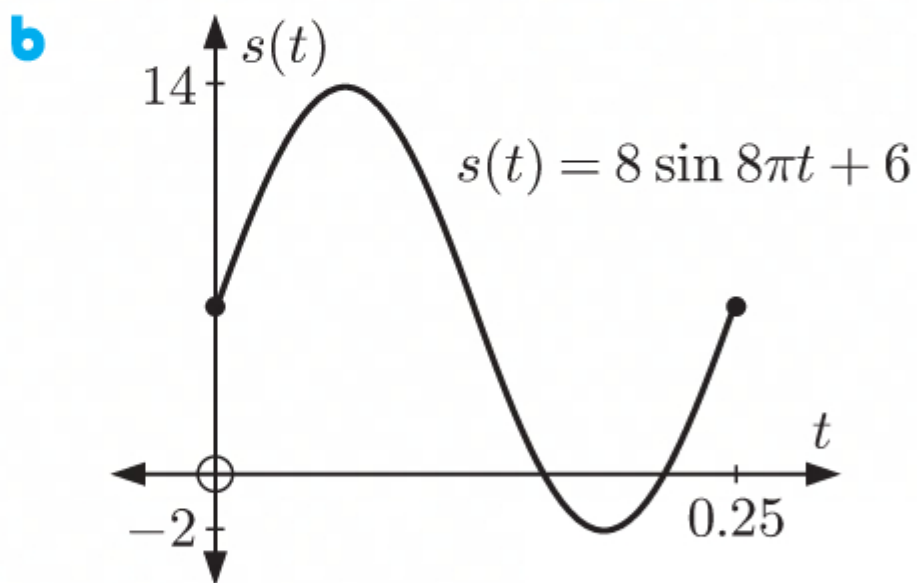
d The object is to the right of the origin when $s(t) > 0$.

This occurs for $0 \leq t < 0.2$ s and $0.5 \text{ s} < t \leq 1$ s.



3 $s(t) = 8 \sin 8\pi t + 6 \text{ cm}, 0 \leq t \leq 0.25 \text{ s}$

a $s(t) = 6$
 $\therefore 8 \sin 8\pi t + 6 = 6$
 $\therefore 8 \sin 8\pi t = 0$
 $\therefore \sin 8\pi t = 0$
 $\therefore 8\pi t = 0, \pi, 2\pi, \dots$ {since $t \geq 0$ }
 $\therefore t = 0 \text{ s}, 0.125 \text{ s}, \text{ or } 0.25 \text{ s}$ $\{0 \leq t \leq 0.25 \text{ s}\}$



c The mass changes direction at the turning points of $s(t)$.

This occurs when $\sin 8\pi t = 1$ or $\sin 8\pi t = -1, 0 \leq t \leq 0.25$

$$\begin{aligned} \therefore 8\pi t &= \frac{\pi}{2} & \therefore \sin 8\pi t &= \frac{3\pi}{2} \\ \therefore t &= \frac{1}{16} & \therefore t &= \frac{3}{16} \\ &= 0.0625 & &= 0.1875 \end{aligned}$$

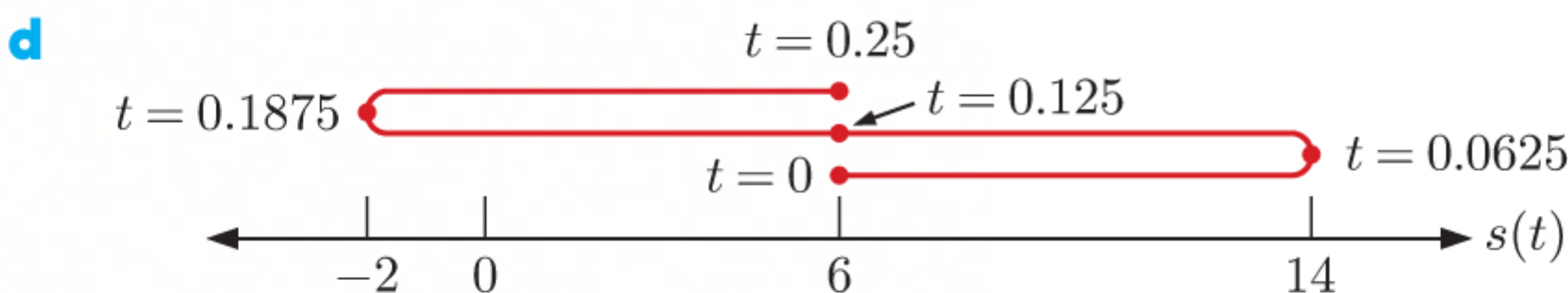
When $t = 0.0625$, $\sin 8\pi t = 1$

$$\therefore 8 \sin 8\pi t + 6 = 8(1) + 6 = 14$$

When $t = 0.1875$, $\sin 8\pi t = -1$

$$\therefore 8 \sin 8\pi t + 6 = 8(-1) + 6 = -2$$

\therefore the mass changes direction 14 cm to the right of the origin, at $t = 0.0625$ seconds, and 2 cm to the left of the origin, at $t = 0.1875$ seconds.

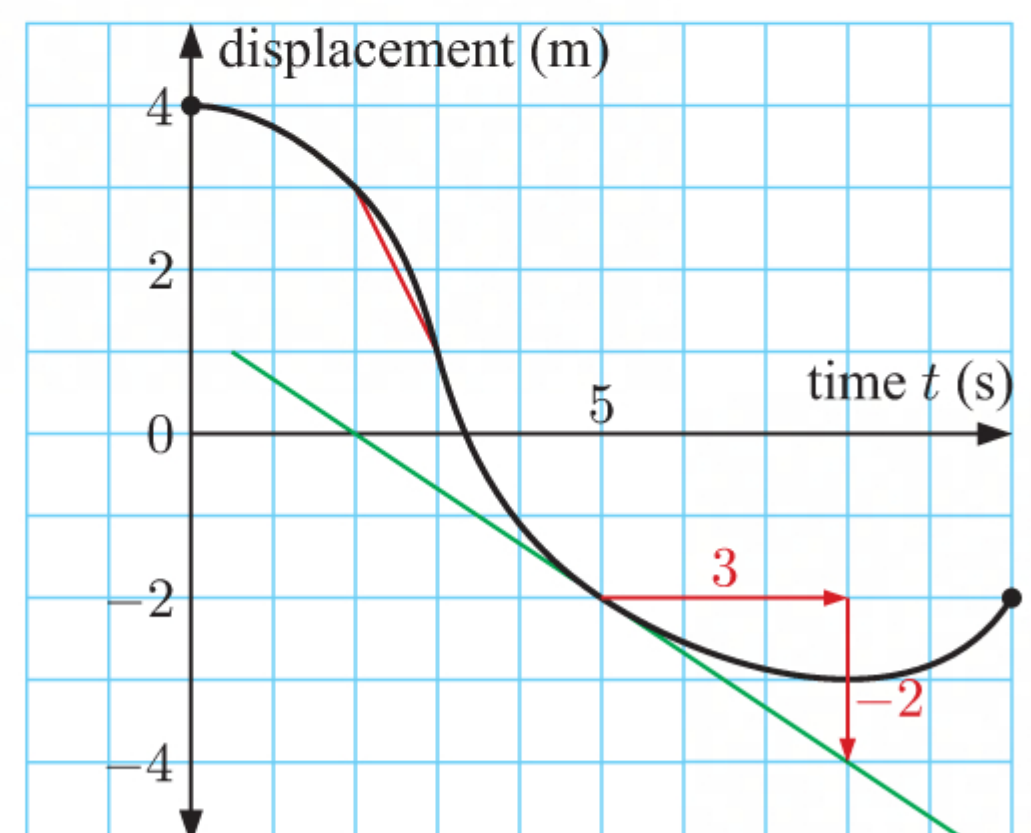


EXERCISE 18B.1

- 1 a i** At $t = 2$ seconds, the displacement is 3 m.
ii At $t = 8$ seconds, the displacement is -3 m.

b average velocity $= \frac{s(3) - s(2)}{3 - 2}$
 $= \frac{1 - 3}{1}$
 $= -2 \text{ m s}^{-1}$

- c** The gradient of the tangent at $t = 5$ seconds is $-\frac{2}{3}$.
 \therefore the instantaneous velocity at $t = 5$ seconds is $-\frac{2}{3} \text{ m s}^{-1}$.



2 $s(t) = t^2 - 6t + 1 \text{ m}, t \geq 0 \text{ s}$

a $s(1) = (1)^2 - 6(1) + 1$ $s(3) = (3)^2 - 6(3) + 1$
 $= 1 - 6 + 1$ $= 9 - 18 + 1$
 $= -4 \text{ m}$ $= -8 \text{ m}$

$$\begin{aligned} \text{average velocity} &= \frac{s(3) - s(1)}{3 - 1} \\ &= \frac{-8 - (-4)}{2} \\ &= \frac{-4}{2} \\ &= -2 \text{ m s}^{-1} \end{aligned}$$

b $v(t) = s'(t) = 2t - 6 \text{ m s}^{-1}$

c i $v(1) = 2(1) - 6$
 $= 2 - 6$
 $= -4 \text{ m s}^{-1}$

\therefore the instantaneous velocity at $t = 1$ second is -4 m s^{-1} .

ii $v(5) = 2(5) - 6$
 $= 10 - 6$
 $= 4 \text{ m s}^{-1}$

\therefore the instantaneous velocity at $t = 5$ seconds is 4 m s^{-1} .

3 a At time $t = 0$ seconds, the displacement of the object is -2 cm .

\therefore the object is initially 2 cm to the left of the origin.

b The displacement of the object is 0 cm at time $t = 6$ seconds.

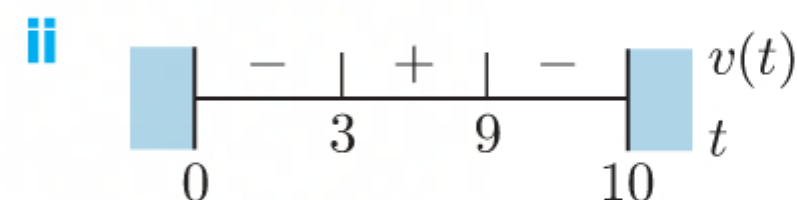
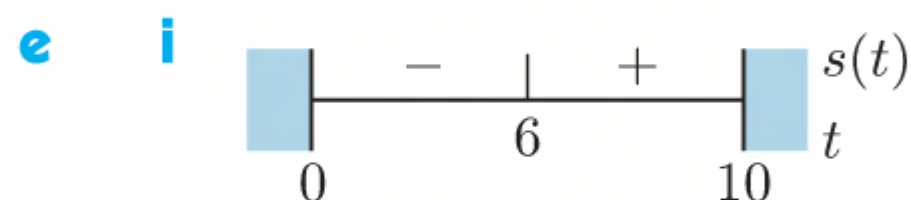
\therefore the object is at the origin when $t = 6$ seconds.

c At time $t = 5$ seconds, the object has negative displacement, but this value is increasing.

\therefore the object is moving to the right when $t = 5$ seconds.

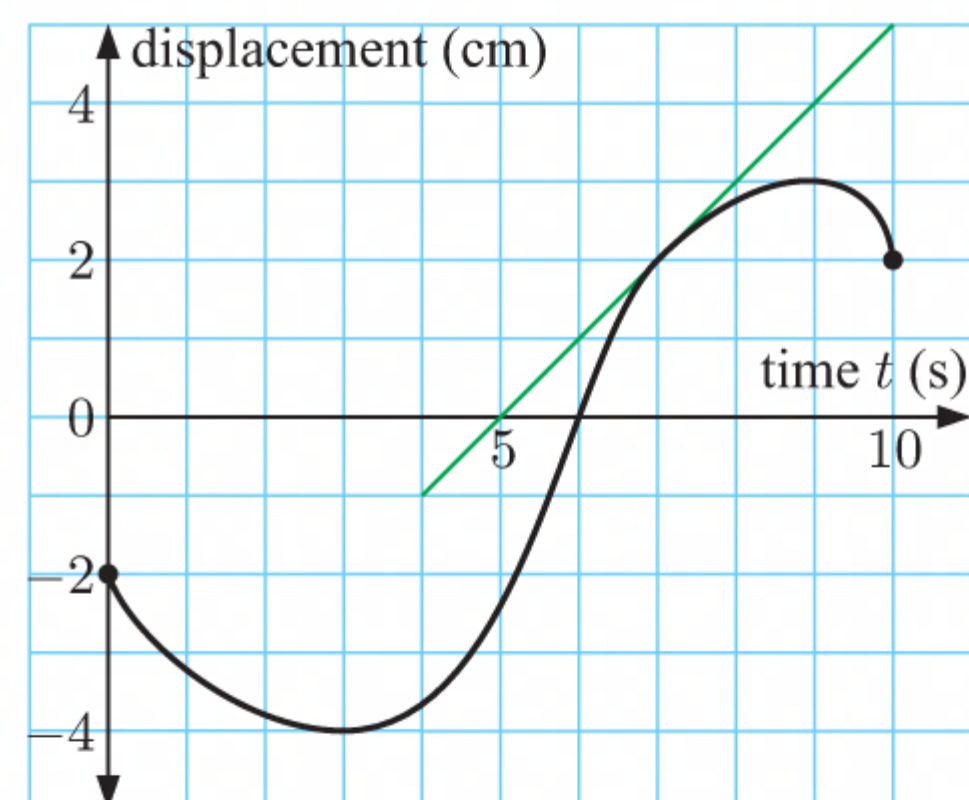
d The object changes direction at the turning points of the displacement graph.

\therefore the object changes direction at times $t = 3$ seconds and $t = 9$ seconds.



f The gradient of the tangent at $t = 7$ is 1 .

\therefore the instantaneous velocity at $t = 7$ seconds is 1 cm s^{-1} .



4 $s(t) = 2\sqrt{t} + 3 \text{ cm}, t \geq 0 \text{ s}$

a $s(1) = 2\sqrt{1} + 3 = 2 + 3 = 5 \text{ cm}$
 $s(4) = 2\sqrt{4} + 3 = 4 + 3 = 7 \text{ cm}$

average velocity $= \frac{s(4) - s(1)}{4 - 1}$
 $= \frac{7 - 5}{3}$
 $= \frac{2}{3} \text{ cm s}^{-1}$

b $s(0) = 2\sqrt{0} + 3 = 3 \text{ cm}$

\therefore the initial position of the object is 3 cm to the right of the origin.

d i $v(4) = \frac{1}{\sqrt{4}} = \frac{1}{2} \text{ cm s}^{-1}$

\therefore the instantaneous velocity at $t = 4$ seconds is $\frac{1}{2} \text{ cm s}^{-1}$.

c $s(t) = 2t^{\frac{1}{2}} + 3$
 $\therefore v(t) = s'(t) = t^{-\frac{1}{2}} = \frac{1}{\sqrt{t}} \text{ cm s}^{-1}$

ii $v(16) = \frac{1}{\sqrt{16}} = \frac{1}{4} \text{ cm s}^{-1}$

\therefore the instantaneous velocity at $t = 16$ seconds is $\frac{1}{4} \text{ cm s}^{-1}$.

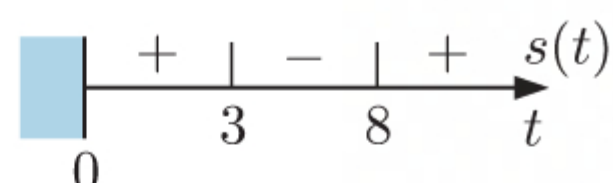
5 $s(t) = t^3 - 11t^2 + 24t \text{ m}, t \geq 0 \text{ s}$

a $v(t) = s'(t) = 3t^2 - 22t + 24 \text{ m s}^{-1}$

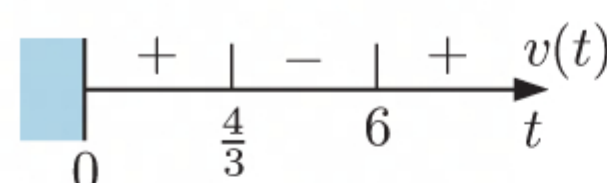
b $s(0) = 0 \text{ m}, v(0) = 24 \text{ m s}^{-1}$

\therefore the object is initially at the origin, moving to the right at 24 m s^{-1} .

c $s(t)$ has sign diagram:



$v(t)$ has sign diagram:



d $s(t) = 0$ when $t = 0, 3$, or 8 {from **c**}

\therefore the object is at O at $t = 0$ seconds, 3 seconds, and 8 seconds.

e The object changes direction when $v(t)$ changes sign.

This occurs when $t = \frac{4}{3}$ and $t = 6$ {from **c**}

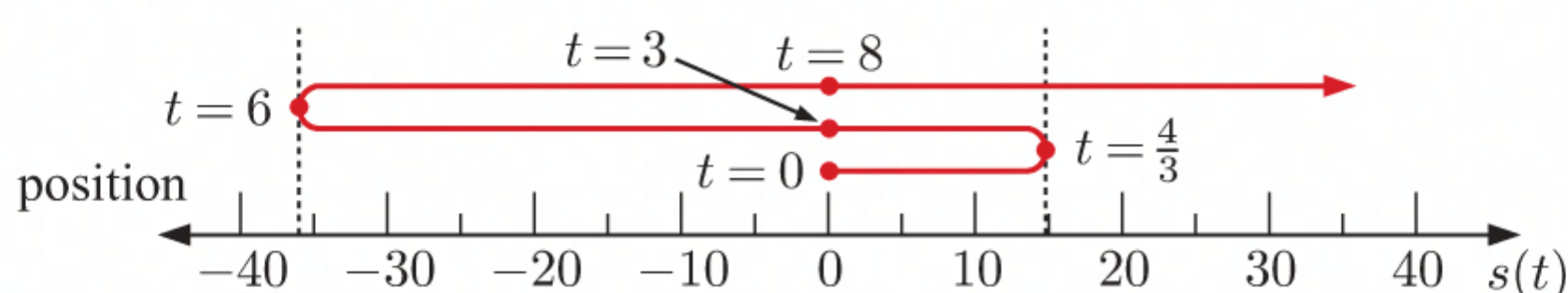
$s(\frac{4}{3}) = (\frac{4}{3})^3 - 11(\frac{4}{3})^2 + 24(\frac{4}{3})$
 $\approx 14.8 \text{ m}$

$s(6) = (6)^3 - 11(6)^2 + 24(6)$
 $= -36 \text{ m}$

\therefore the object changes direction at $t = \frac{4}{3}$ seconds when it is about 14.8 m to the right of the origin, and at $t = 6$ seconds, when it is 36 m to the left of the origin.

f The object starts at O, and moves towards the right at 24 m s^{-1} . Its velocity is decreasing. After $\frac{4}{3}$ seconds, when it is about 14.8 m to the right of O, it changes direction and moves to the left, passing O after 3 seconds. After 6 seconds, when it is 36 m to the left of O, it changes direction again and moves towards the right, passing O once more after 8 seconds.

g



6 $s(t) = bt - 4.9t^2$ m, $t \geq 0$ s

a $v(t) = s'(t) = b - 9.8t$

$$\therefore v(0) = b - 9.8(0) \\ = b \text{ m s}^{-1}$$

\therefore the initial velocity is $b \text{ m s}^{-1}$ upwards.

b i The shell reaches its maximum height after 7.1 seconds.

\therefore the velocity of the shell at $t = 7.1$ seconds is zero.

$$\therefore v(7.1) = 0$$

$$\therefore b - 9.8(7.1) = 0$$

$$\therefore b - 69.58 = 0$$

$$\therefore b = 69.58$$

\therefore the initial velocity of the shell is 69.58 m s^{-1} upwards.

ii $s(t) = 69.58t - 4.9t^2$

The shell reaches its maximum height after 7.1 seconds.

$$s(7.1) = 69.58(7.1) - 4.9(7.1)^2 \\ \approx 247 \text{ m}$$

\therefore the shell reached a maximum height of about 247 m.

EXERCISE 18B.2

1 Total distance travelled

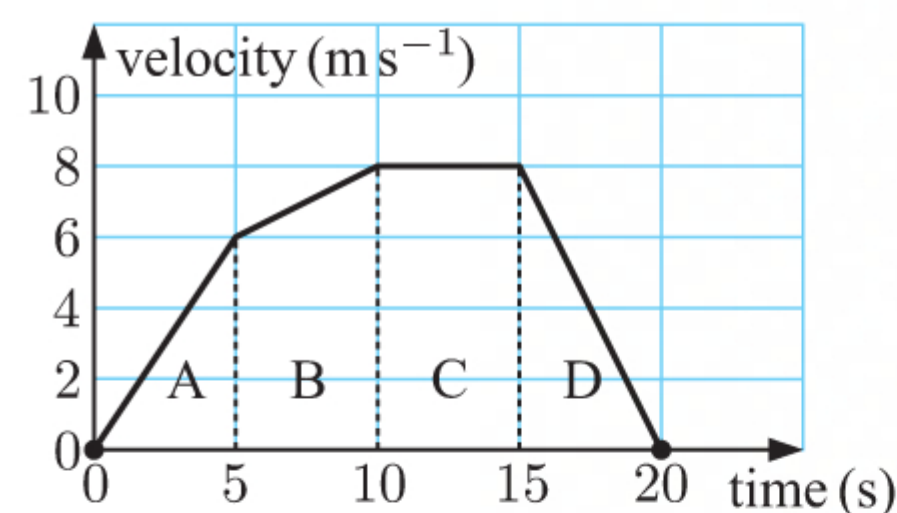
= total area under the graph

= area A + area B + area C + area D

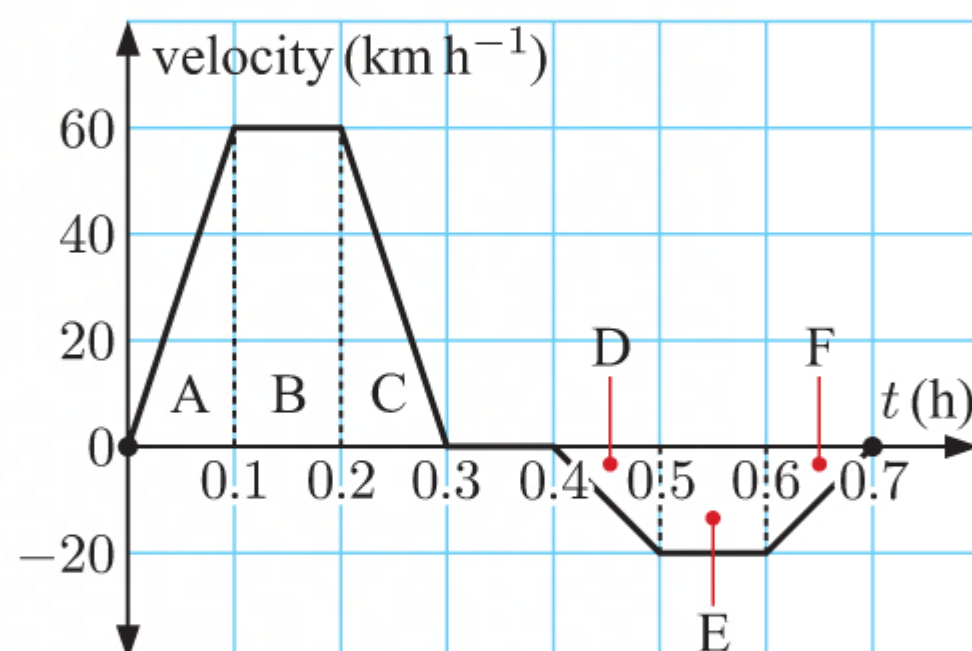
$$= \frac{1}{2}(5)(6) + \left(\frac{6+8}{2}\right)(5) + (5)(8) + \frac{1}{2}(5)(8)$$

$$= 15 + 35 + 40 + 20$$

$$= 110 \text{ m}$$



2



a i When the graph is above the t -axis, the car is travelling forwards.

ii When the graph is below the t -axis, the car is travelling backwards (in the opposite direction).

b Total distance travelled

= total area between the graph and the t -axis

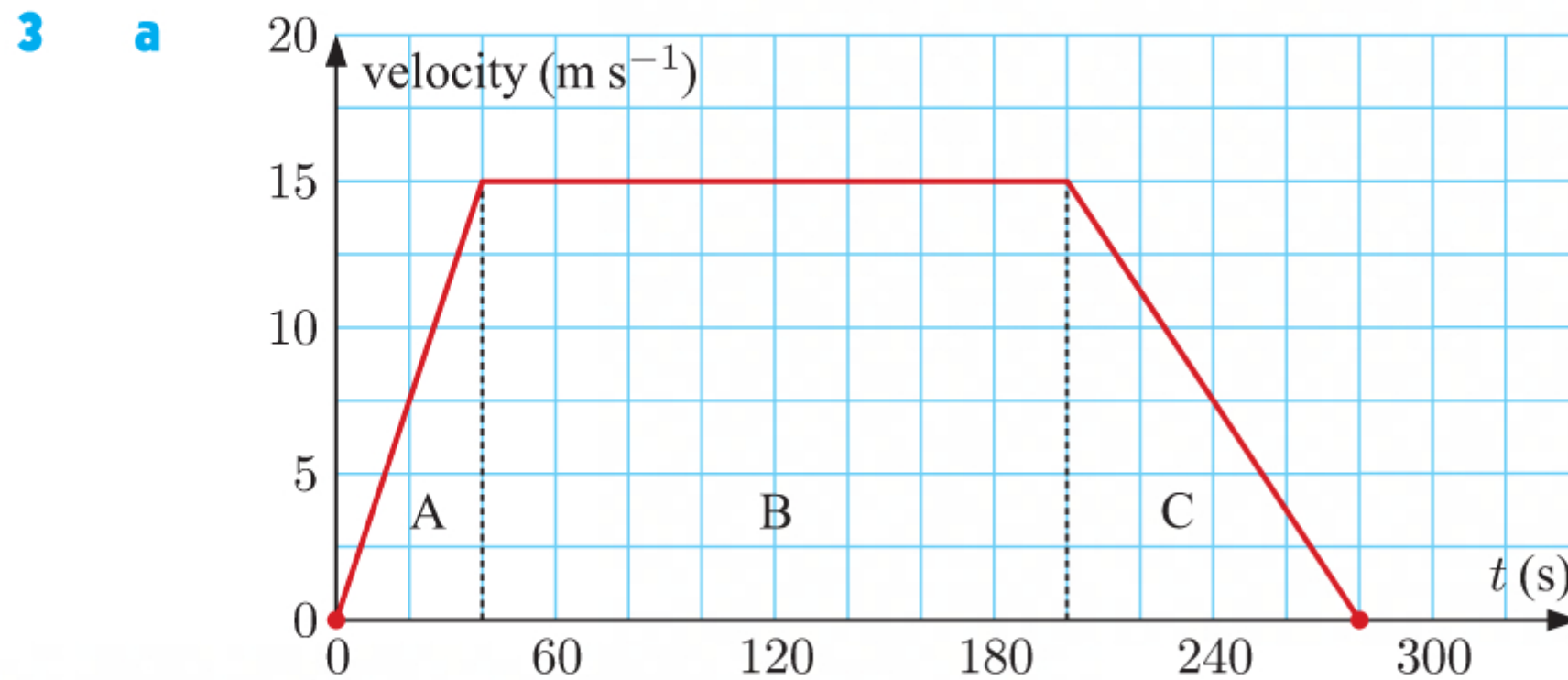
= area A + area B + area C + area D + area E + area F

$$= \frac{1}{2}(0.1)(60) + (0.1)(60) + \frac{1}{2}(0.1)(60) + \frac{1}{2}(0.1)(20) + (0.1)(20) + \frac{1}{2}(0.1)(20)$$

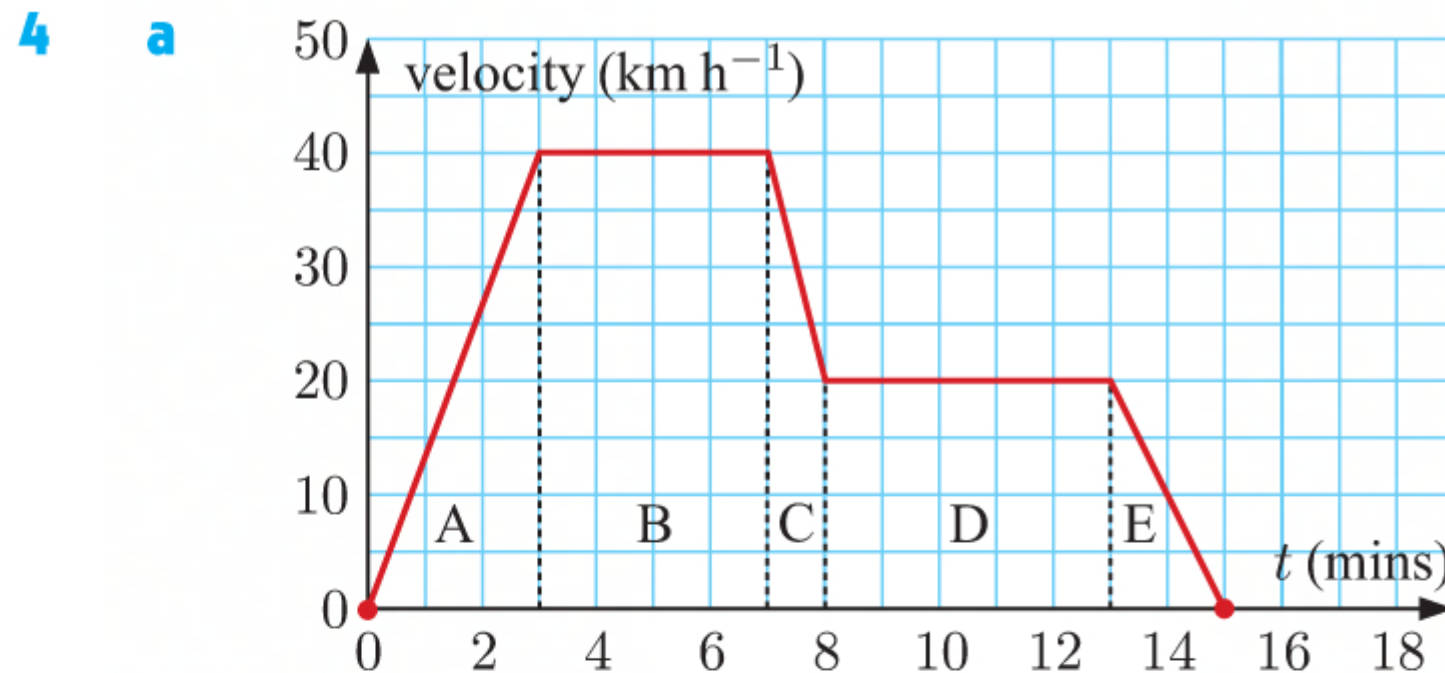
$$= 3 + 6 + 3 + 1 + 2 + 1$$

$$= 16 \text{ km}$$

- c Displacement = forward distance travelled – backward distance travelled
 $= \text{area A} + \text{area B} + \text{area C} - \text{area D} - \text{area E} - \text{area F}$
 $= 3 + 6 + 3 - 1 - 2 - 1$
 $= 8 \text{ km from the starting point (on positive side)}$



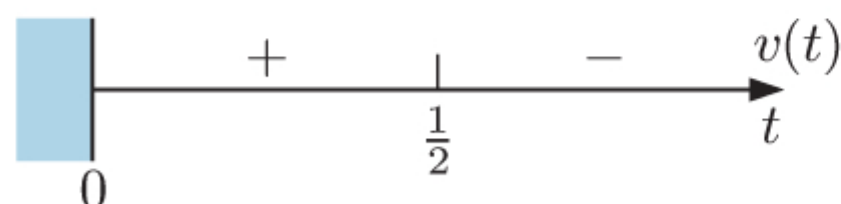
- b Total distance travelled = total area under graph
 $= \text{area A} + \text{area B} + \text{area C}$
 $= \frac{1}{2}(40)(15) + (160)(15) + \frac{1}{2}(80)(15)$
 $= 300 + 2400 + 600$
 $= 3300 \text{ m}$
 $= 3.3 \text{ km}$



- b Total distance travelled
 $= \text{total area under graph}$
 $= \text{area A} + \text{area B} + \text{area C} + \text{area D} + \text{area E}$
 $= \frac{1}{2}\left(\frac{3}{60}\right)(40) + \left(\frac{4}{60}\right)(40) + \left(\frac{40+20}{2}\right)\left(\frac{1}{60}\right) + \left(\frac{5}{60}\right)(20) + \frac{1}{2}\left(\frac{2}{60}\right)(20) \quad \{t \text{ min} = \frac{t}{60} \text{ hours}\}$
 $= 1 + \frac{8}{3} + \frac{1}{2} + \frac{5}{3} + \frac{1}{3}$
 $= 6\frac{1}{6} \text{ km}$

- 5 a $v(t) = s'(t) = 1 - 2t$

\therefore the sign diagram of v is:

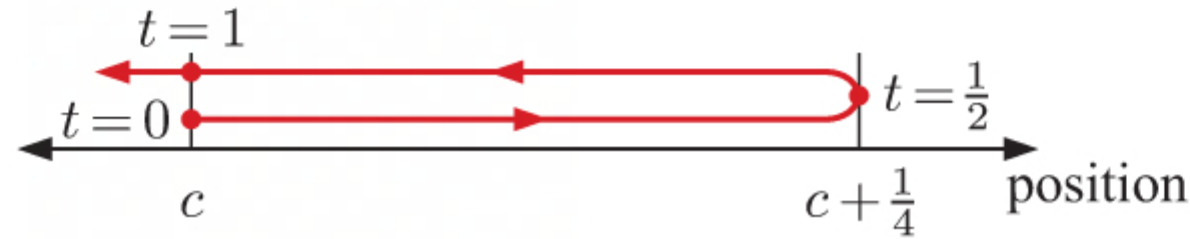


Since the sign changes, the particle changes direction at $t = \frac{1}{2}$ second.

$$\begin{aligned} \text{b } s(t) &= \int (1 - 2t) dt \\ &= t - t^2 + c \end{aligned}$$

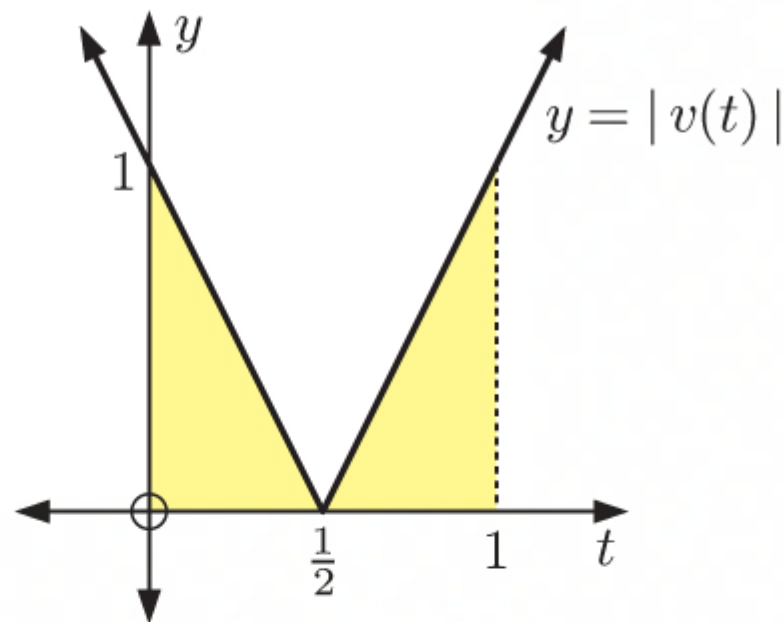
$$\begin{aligned} \text{Hence } s(0) &= c & s\left(\frac{1}{2}\right) &= \frac{1}{2} - \frac{1}{4} + c & s(1) &= 1 - 1 + c \\ & & &= c + \frac{1}{4} & &= c \end{aligned}$$

Motion diagram:



$$\begin{aligned} \therefore \text{ total distance travelled} &= \left(c + \frac{1}{4} - c\right) + \left(c + \frac{1}{4} - c\right) \\ &= \frac{1}{2} \text{ cm} \end{aligned}$$

$$\text{Now, } |v(t)| = |1 - 2t|$$



$$\begin{aligned} \int_0^1 |v(t)| dt &= \frac{1}{2} \left(\frac{1}{2}\right) (1) + \frac{1}{2} \left(\frac{1}{2}\right) (1) \\ &= \frac{1}{4} + \frac{1}{4} \\ &= \frac{1}{2} \quad \checkmark \end{aligned}$$

$$\begin{aligned} \text{c Displacement} &= \text{final position} - \text{original position} \\ &= s(1) - s(0) \\ &= c - c \\ &= 0 \text{ cm} \end{aligned}$$

So, the particle returned to its original position after one second.

$$\text{6 a } v(t) = s'(t) = t^2 - t - 2$$

$$\begin{aligned} \text{Now } s(t) &= \int (t^2 - t - 2) dt \\ &= \frac{1}{3}t^3 - \frac{1}{2}t^2 - 2t + c \end{aligned}$$

The particle is initially at the origin.

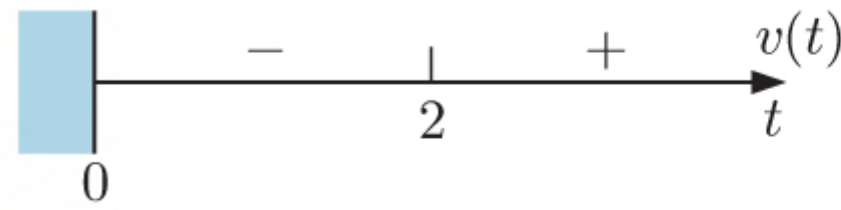
$$\therefore s(0) = 0$$

$$\therefore c = 0$$

$$\therefore s(t) = \frac{1}{3}t^3 - \frac{1}{2}t^2 - 2t \text{ cm}$$

$$\begin{aligned} \text{b } v(t) &= t^2 - t - 2 \\ &= (t+1)(t-2) \end{aligned}$$

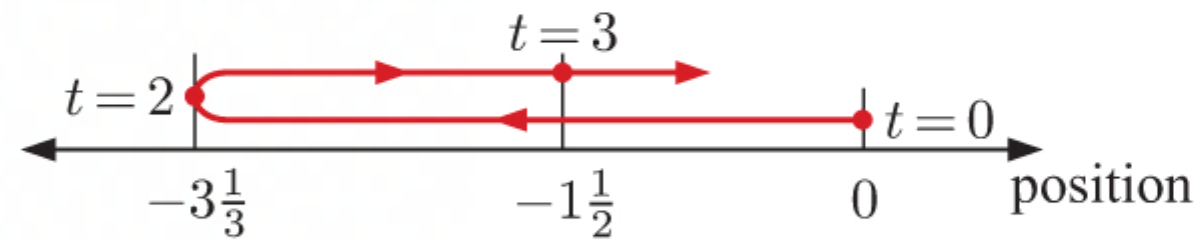
\therefore the sign diagram of v is:



Since the sign changes, the particle changes direction at $t = 2$ seconds.

$$\begin{aligned} \text{Hence } s(0) &= 0 & s(2) &= \frac{8}{3} - 2 - 4 & s(3) &= 9 - \frac{9}{2} - 6 \\ & & &= -3\frac{1}{3} & &= -1\frac{1}{2} \end{aligned}$$

Motion diagram:



$$\begin{aligned} \therefore \text{ total distance travelled} &= (0 - (-3\frac{1}{3})) + (-1\frac{1}{2} - (-3\frac{1}{3})) \\ &= 5\frac{1}{6} \text{ cm} \end{aligned}$$

c Displacement = final position – original position

$$\begin{aligned} &= s(3) - s(0) \\ &= -1\frac{1}{2} - 0 \\ &= -1\frac{1}{2} \end{aligned}$$

So, the particle's displacement is $1\frac{1}{2}$ cm left of its starting position.

$$7 \quad \text{a } v(t) = s'(t) = 29.4 - 9.8t$$

$$\begin{aligned} \therefore s(t) &= \int (29.4 - 9.8t) dt \\ &= 29.4t - 4.9t^2 + c \end{aligned}$$

The ball is initially 1 metre above ground level.

$$\therefore s(0) = 1$$

$$\therefore c = 1$$

$$\therefore s(t) = 29.4t - 4.9t^2 + 1 \text{ m}$$

b The maximum height reached by the ball occurs when its velocity equals 0.

$$\therefore 29.4 - 9.8t = 0$$

$$\therefore 9.8t = 29.4$$

$$\therefore t = 3$$

So, the maximum height is reached at $t = 3$ seconds.

$$\begin{aligned} s(3) &= 29.4(3) - 4.9(3)^2 + 1 \\ &= 45.1 \text{ m} \end{aligned}$$

\therefore the maximum height reached by the ball is 45.1 m.

8 a $v(t) = s'(t) = 32 + 4t$

$$\begin{aligned}\therefore s(t) &= \int (32 + 4t) dt \\ &= 32t + 2t^2 + c\end{aligned}$$

Now $s(0) = 16$

$$\therefore c = 16$$

$$\therefore s(t) = 32t + 2t^2 + 16 \text{ m}$$

b The moving object changes direction when $v(t) = 0$

$$\therefore 32 + 4t = 0$$

$$\therefore t = -8$$

But $t \geq 0$, so there is no change of direction.

$$\therefore \text{displacement} = \text{total distance travelled} = s(\tau) - s(0)$$

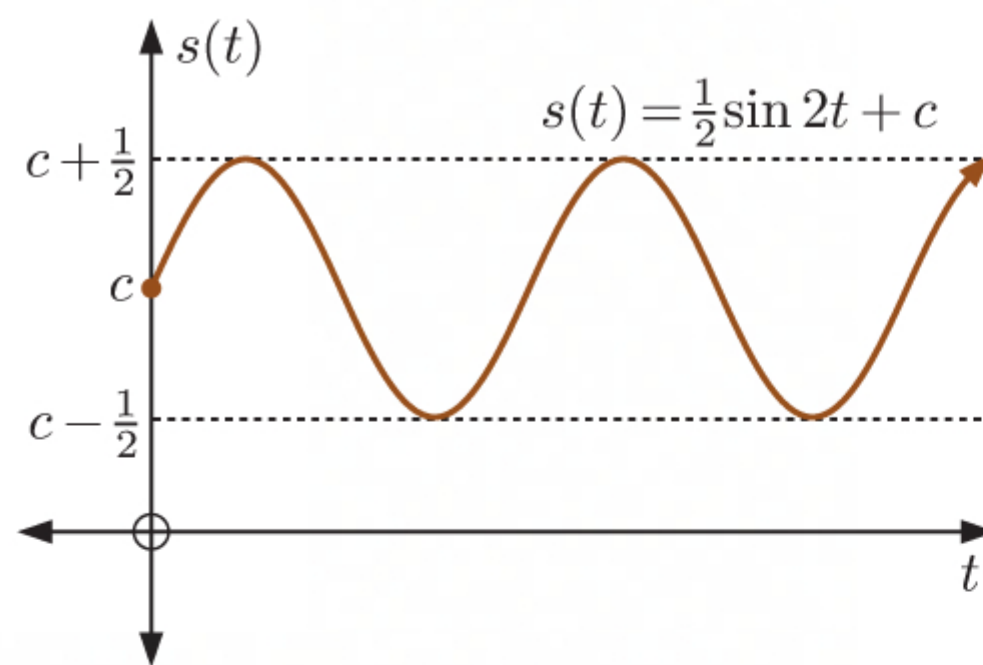
$$= \int_0^\tau (32 + 4t) dt$$

c Total distance travelled in first 4 seconds $= \int_0^4 (32 + 4t) dt$

$$\begin{aligned}&= [32t + 2t^2]_0^4 \\ &= (128 + 32) - 0 \\ &= 160 \text{ m}\end{aligned}$$

9 a $v(t) = s'(t) = \cos 2t$

$$\begin{aligned}\therefore s(t) &= \int \cos 2t dt \\ &= \frac{1}{2} \sin 2t + c\end{aligned}$$



The graph shows that the particle oscillates between positions $c + \frac{1}{2}$ and $c - \frac{1}{2}$.

$$\begin{aligned}\text{Distance} &= (c + \frac{1}{2}) - (c - \frac{1}{2}) \\ &= 1 \text{ m}\end{aligned}$$

b $s(\frac{\pi}{4}) = 1, \therefore \frac{1}{2} \sin \frac{\pi}{2} + c = 1$

$$\therefore \frac{1}{2}(1) + c = 1$$

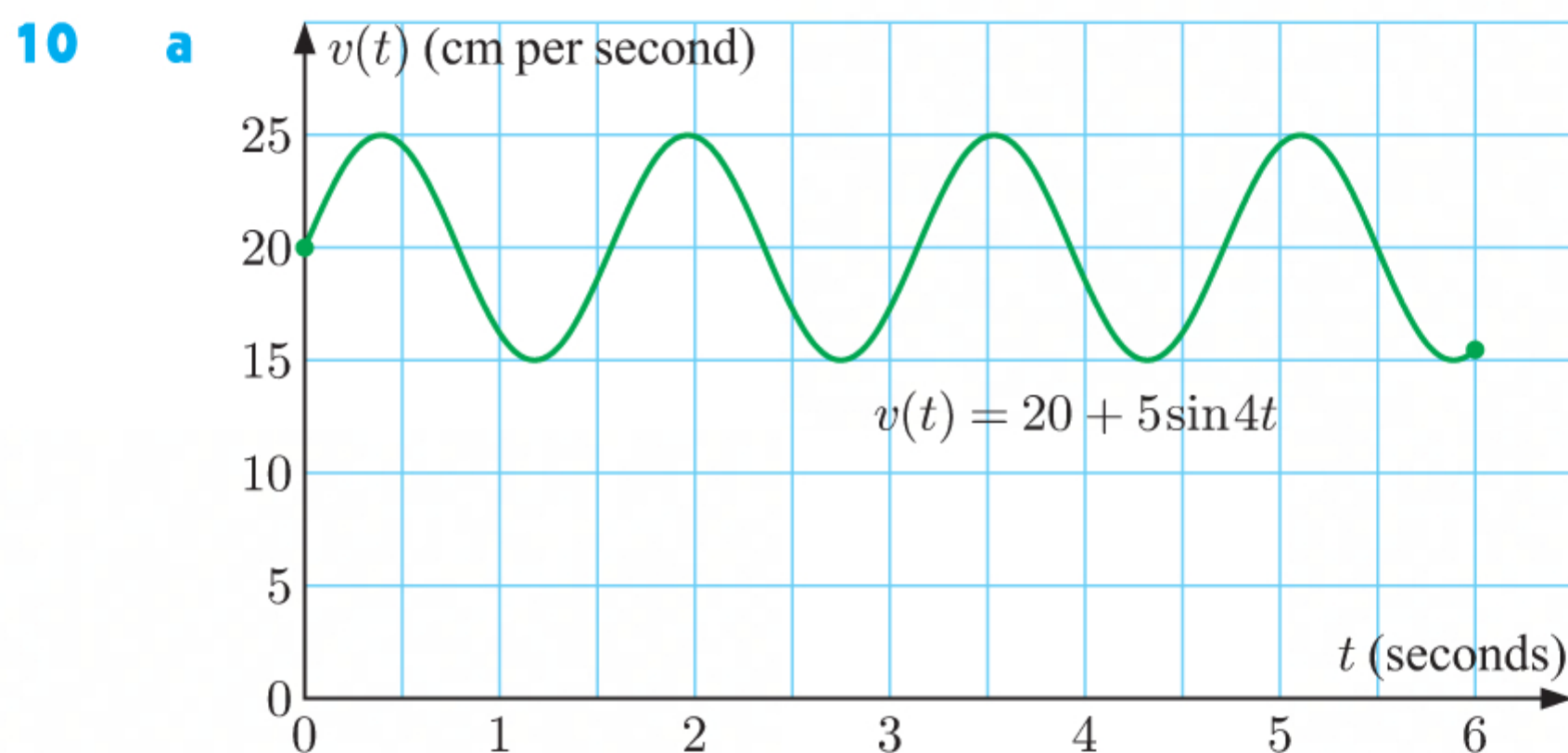
$$\therefore c = \frac{1}{2}$$

$$\therefore s(t) = \frac{1}{2} \sin 2t + \frac{1}{2}$$

$$\therefore s(\frac{\pi}{3}) = \frac{1}{2} \sin \frac{2\pi}{3} + \frac{1}{2}$$

$$= \frac{\sqrt{3}}{4} + \frac{1}{2}$$

$$= \frac{\sqrt{3} + 2}{4} \text{ m}$$



b $v(4.5) = 20 + 5 \sin(4 \times 4.5)$
 $= 20 + 5 \sin 18$
 $\approx 16.2 \text{ cm s}^{-1}$

\therefore the pendulum's velocity after 4.5 seconds is about 16.2 cm s^{-1} .

c Distance travelled by the tip of the pendulum in the first 2 seconds

$$\begin{aligned} &= \int_0^2 |v(t)| dt \\ &= \int_0^2 v(t) dt \\ &= \int_0^2 (20 + 5 \sin 4t) dt \\ &= \left[20t - \frac{5}{4} \cos 4t \right]_0^2 \\ &= \left(40 - \frac{5}{4} \cos 8 \right) - \left(0 - \frac{5}{4} \right) \\ &\approx 41.4 \text{ cm} \end{aligned}$$

11 a $v(t) = s'(t) = -4 + t^{\frac{1}{2}}$
 $\therefore s(t) = \int (-4 + t^{\frac{1}{2}}) dt$
 $= -4t + \frac{2}{3} t^{\frac{3}{2}} + c$
 $s(0) = 0, \therefore c = 0$
 $\therefore s(t) = -4t + \frac{2}{3} t^{\frac{3}{2}} \text{ m}$

b The object changes direction when $v(t) = 0$
 $\therefore -4 + \sqrt{t} = 0$
 $\therefore \sqrt{t} = 4$
 $\therefore t = 16$

\therefore the object changes direction at $t = 16$ seconds.

c Change in displacement $= s(30) - s(0)$
 $= -4(30) + \frac{2}{3}(30)^{\frac{3}{2}} - 0$
 $\approx -10.5 \text{ m}$

\therefore after the first 30 seconds the particle is about 10.5 m to the left of the origin.

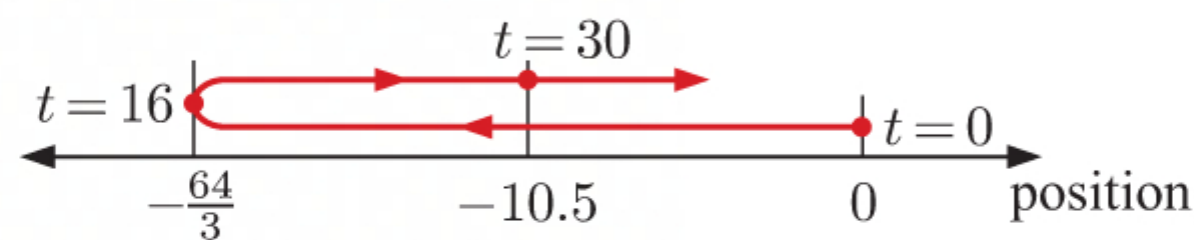
d $s(0) = 0$

$$s(16) = -4(16) + \frac{2}{3}(16)^{\frac{3}{2}}$$

$$= -\frac{64}{3}$$

$$s(30) \approx -10.5$$

Motion diagram:



$$\therefore \text{total distance travelled} \approx (0 - (-\frac{64}{3})) + (-10.5 - (-\frac{64}{3}))$$

$$\approx 32.2 \text{ m}$$

12 a $v(t) = 10\sqrt{t} \text{ m s}^{-1}$

i $v(1) = 10\sqrt{1}$
 $= 10$

So the motorcyclist's velocity after 1 second is 10 m s^{-1} .

ii $v(2) = 10\sqrt{2}$

So the motorcyclist's velocity after 2 seconds is $10\sqrt{2} \text{ m s}^{-1}$.

b $s(t) = \int v(t) dt$
 $= \int 10\sqrt{t} dt$
 $= \int 10t^{\frac{1}{2}} dt$
 $= \frac{20}{3}t^{\frac{3}{2}} + c \text{ m}$

We assume that $s(0) = 0$, $c = 0$

$$\therefore s(t) = \frac{20}{3}t^{\frac{3}{2}} \text{ m}$$

d i $v(t) = 20$ when $10\sqrt{t} = 20$
 $\therefore \sqrt{t} = 2$
 $\therefore t = 4$

It will take 4 seconds for the motorcyclist to reach a speed of 20 m s^{-1} .

ii Distance travelled in first 4 seconds $= \int_0^4 v(t) dt$
 $= \left[\frac{20}{3}t^{\frac{3}{2}} \right]_0^4$
 $= \frac{20}{3}(4^{\frac{3}{2}}) - 0$
 $= 53\frac{1}{3} \text{ m}$

\therefore yes, the motorcyclist has given himself enough distance as he only needs $53\frac{1}{3} \text{ m}$ to reach the required speed.

c $\int_0^2 v(t) dt = \left[\frac{20}{3}t^{\frac{3}{2}} \right]_0^2$
 $= \frac{20}{3}(2^{\frac{3}{2}}) - 0$
 ≈ 18.9

The motorcyclist travels about 18.9 m in the first 2 seconds.

13 a $v(t) = -54(1 - e^{-\frac{t}{6}}) \text{ m s}^{-1}$

$$\int_0^{15} |v(t)| dt = \int_0^{15} \left| -54(1 - e^{-\frac{t}{6}}) \right| dt$$

$$\approx 513 \quad \{\text{using technology}\}$$

\therefore the skydiver travels a total distance of about 513 m in the first 15 seconds.

Math Deg Norm1 ab/c Real

$$\int_0^{15} \left| -54 \left(1 - e^{-\frac{x}{6}} \right) \right| dx$$

512.5955396

JUMP DELETE ▶ MAT MATH

b $v(t) = e^{-t} \cos 16t \text{ cm s}^{-1}$

$$\int_0^{10} |v(t)| dt = \int_0^{10} |e^{-t} \cos 16t| dt$$

$$\approx 0.637$$

\therefore the mass on the spring travels a total distance of about 0.637 cm in the first 10 seconds.

Math Rad Norm1 ab/c Real

$$\int_0^{10} |e^{-x} \cos (16x)| dx$$

0.6369870327

JUMP DELETE ▶ MAT MATH

EXERCISE 18C

1 a $v(t) = 10t - t^2 \text{ cm s}^{-1}, \quad t \geq 0 \text{ s}$

$$\begin{aligned} v(2) &= 10(2) - (2)^2 \\ &= 20 - 4 \\ &= 16 \text{ cm s}^{-1} \end{aligned}$$

\therefore the velocity of the particle at $t = 2$ seconds is 16 cm s^{-1} .

b average acceleration $= \frac{v(3) - v(1)}{3 - 1}$

$$\begin{aligned} &= \frac{(10(3) - (3)^2) - (10(1) - (1)^2)}{3 - 1} \\ &= \frac{(30 - 9) - (10 - 1)}{2} \\ &= \frac{21 - 9}{2} \\ &= 6 \text{ cm s}^{-2} \end{aligned}$$

\therefore the average acceleration from $t = 1$ to $t = 3$ seconds is 6 cm s^{-2} .

c $a(t) = v'(t) = 10 - 2t \text{ cm s}^{-2}$

d $a(3) = 10 - 2(3)$

$$= 4 \text{ cm s}^{-2}$$

\therefore the instantaneous acceleration of the particle at $t = 3$ seconds is 4 cm s^{-2} .

2 a $s(t) = t^3 - t^2 - 5 \text{ m}, \quad t \geq 0 \text{ s}$

$$\therefore v(t) = s'(t) = 3t^2 - 2t \text{ m s}^{-1}$$

$$\therefore a(t) = v'(t) = 6t - 2 \text{ m s}^{-2}$$

$$\begin{aligned} \therefore s(2) &= 2^3 - 2^2 - 5 & v(2) &= 3(2)^2 - 2(2) & a(2) &= 6(2) - 2 \\ &= 8 - 4 - 5 & &= 12 - 4 & &= 12 - 2 \\ &= -1 \text{ m} & &= 8 \text{ m s}^{-1} & &= 10 \text{ m s}^{-2} \end{aligned}$$

At $t = 2$ seconds, the object has displacement -1 m , velocity 8 m s^{-1} , and acceleration 10 m s^{-2} .

b $a(t) = 0$ when $6t - 2 = 0$

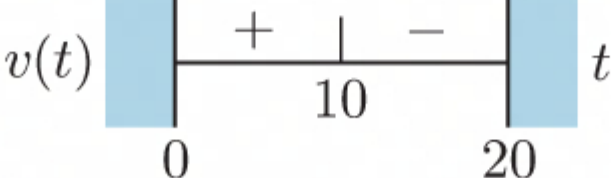
$$\therefore 6t = 2$$

$$\therefore t = \frac{1}{3}$$


\therefore the object has zero acceleration at $t = \frac{1}{3}$ seconds.

3 a $s(t) = 98t - 4.9t^2 \text{ m}$

$\therefore v(t) = s'(t) = 98 - 9.8t \text{ m s}^{-1}$ which has sign diagram:



$\therefore a(t) = v'(t) = -9.8 \text{ m s}^{-2}$ which has sign diagram:



b $s(0) = 0 \text{ m} \quad v(0) = 98 \text{ m s}^{-1}$

The stone is initially 0 m above the ground, moving upward with velocity 98 m s^{-1} .

c $s(5) = 98(5) - 4.9(5)^2 \quad v(5) = 98 - 9.8(5) \quad a(5) = -9.8 \text{ m s}^{-2}$
 $= 367.5 \text{ m} \quad = 49 \text{ m s}^{-1}$

At $t = 5$ seconds, the stone is 367.5 m above the ground and moving upward at 49 m s^{-1} . It has acceleration -9.8 m s^{-2} .

$$\begin{aligned} s(12) &= 98(12) - 4.9(12)^2 & v(12) &= 98 - 9.8(12) & a(12) &= -9.8 \text{ m s}^{-2} \\ &= 470.4 \text{ m} & &= -19.6 \text{ m s}^{-1} \end{aligned}$$

At $t = 12$ seconds, the stone is 470.4 m above the ground and moving downward at 19.6 m s^{-1} . It has acceleration -9.8 m s^{-2} .

d The maximum height reached by the stone occurs when its upward velocity equals 0.

$$\therefore 98 - 9.8t = 0$$

$$\therefore 9.8t = 98$$

$$\therefore t = 10$$

So, the maximum height is reached at $t = 10$ seconds.

$$\begin{aligned} s(10) &= 98(10) - 4.9(10)^2 \\ &= 490 \text{ m} \end{aligned}$$

\therefore the maximum height reached by the stone is 490 m .

- e The stone is on the ground when its displacement equals 0.

$$\therefore 98t - 4.9t^2 = 0$$

$$\therefore t(98 - 4.9t) = 0$$

$$\therefore t = 0 \text{ or } 98 - 4.9t = 0$$

$$4.9t = 98$$

$$t = 20$$

After it is fired from the catapult, it takes 20 seconds for the stone to hit the ground.

4 a $s(t) = 100t + 200e^{-\frac{t}{5}} \text{ cm}$

$$\therefore v(t) = 100 + 200\left(-\frac{1}{5}\right)e^{-\frac{t}{5}} \quad \{v(t) = s'(t)\}$$

$$= 100 - 40e^{-\frac{t}{5}} \text{ cm s}^{-1}$$

and $a(t) = -40\left(-\frac{1}{5}\right)e^{-\frac{t}{5}} \quad \{a(t) = v'(t)\}$

$$= 8e^{-\frac{t}{5}} \text{ cm s}^{-2}$$

b When $t = 0$, $s(0) = 200 \text{ cm}$
 $v(0) = 60 \text{ cm s}^{-1}$
 $a(0) = 8 \text{ cm s}^{-2}$

\therefore the particle is initially 200 cm to the right of the origin, moving to the right at 60 cm s^{-1} , and has acceleration 8 cm s^{-2} .

c As $t \rightarrow \infty$, $v(t) \rightarrow 100$
 \therefore the velocity of P approaches 100 cm s^{-1} as $t \rightarrow \infty$.

d As $t \rightarrow \infty$, $a(t) \rightarrow 0$
 \therefore the acceleration of P approaches 0 cm s^{-2} as $t \rightarrow \infty$.

5 a $s(t) = t - \ln(2t + 1) \text{ cm}$

$$\therefore s(0) = 0 - \ln(2(0) + 1)$$

$$= -\ln(1)$$

$$= 0$$

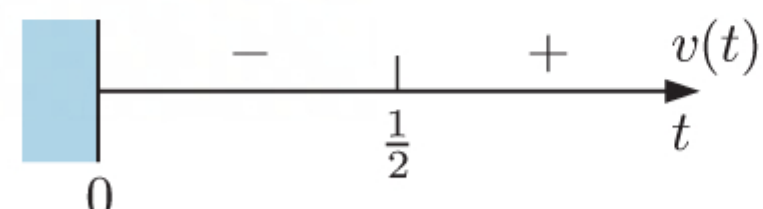
\therefore the object is initially at the origin. ✓

b $v(t) = 1 - \frac{2}{2t+1} \text{ cm s}^{-1} \quad \{v(t) = s'(t)\}$

c $v(t) = 1 - \frac{2}{2t+1}$

$$= \frac{2t+1-2}{2t+1}$$

$$= \frac{2t-1}{2t+1} \text{ which has sign diagram:}$$



- i The object is moving to the right when the velocity is positive.

\therefore the object is moving to the right for $t > \frac{1}{2}$ second.

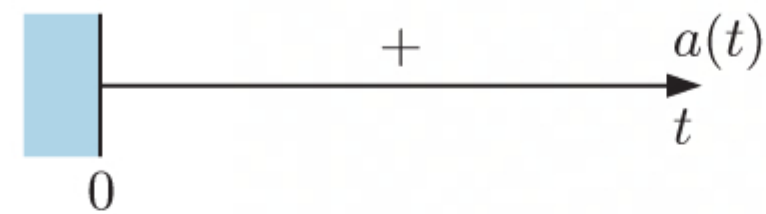
- ii The object is moving to the left when the velocity is negative.

\therefore the object is moving to the left for $0 \leq t < \frac{1}{2}$ second.

d $v(t) = 1 - 2(2t + 1)^{-1} \text{ cm s}^{-1}$

$\therefore a(t) = 2(2t + 1)^{-2}(2) \quad \{\text{chain rule}\}$

$= \frac{4}{(2t + 1)^2} \text{ cm s}^{-2} \quad \text{which has sign diagram:}$



\therefore the object's acceleration is positive for all $t \geq 0$.

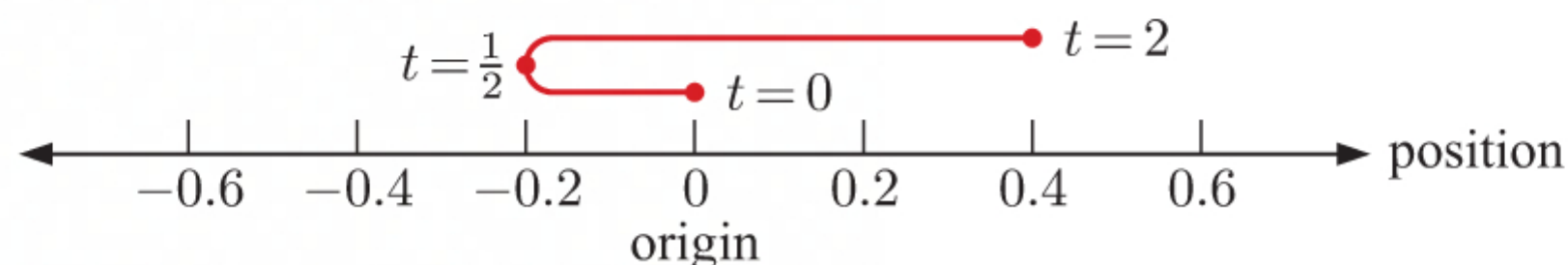
e $a(2) = \frac{4}{(2(2) + 1)^2}$
 $= \frac{4}{25} \text{ cm s}^{-2}$

\therefore the acceleration of the object after 2 seconds is $\frac{4}{25} \text{ cm s}^{-2}$.

f $v(t)$ changes sign when $t = \frac{1}{2}$, so this is where the object changes direction.

$$\begin{aligned} s\left(\frac{1}{2}\right) &= \frac{1}{2} - \ln\left(2\left(\frac{1}{2}\right) + 1\right) & \text{and} & & s(2) &= 2 - \ln(2(2) + 1) \\ &= \frac{1}{2} - \ln 2 \text{ cm} & & & &= 2 - \ln 5 \text{ cm} \end{aligned}$$

The motion diagram of P is:



\therefore the total distance travelled by the object in the first 2 seconds

$$\begin{aligned} &= \left(0 - \left(\frac{1}{2} - \ln 2\right)\right) + \left(2 - \ln 5 - \left(\frac{1}{2} - \ln 2\right)\right) \\ &= -\frac{1}{2} + \ln 2 + 2 - \ln 5 - \frac{1}{2} + \ln 2 \\ &= 1 + 2 \ln 2 - \ln 5 \\ &= 1 + \ln 4 - \ln 5 \\ &= 1 + \ln\left(\frac{4}{5}\right) \approx 0.777 \text{ cm} \end{aligned}$$

6 $v(t) = 50 - 10e^{-0.5t} \text{ m s}^{-1}$

a $v(0) = 50 - 10e^0$
 $= 40$

So, the initial velocity is 40 m s^{-1} .

b $v(3) = 50 - 10e^{-0.5(3)}$
 ≈ 47.8

So, the velocity after 3 seconds is about 47.8 m s^{-1} .

c $v(t) = 45$ when $50 - 10e^{-0.5t} = 45$
 $\therefore -10e^{-0.5t} = -5$
 $\therefore e^{-0.5t} = \frac{1}{2}$
 $\therefore -0.5t = \ln\left(\frac{1}{2}\right)$
 $\therefore t = 2 \ln 2 \approx 1.39$

So, it will take about 1.39 seconds for the particle's velocity to reach 45 m s^{-1} .

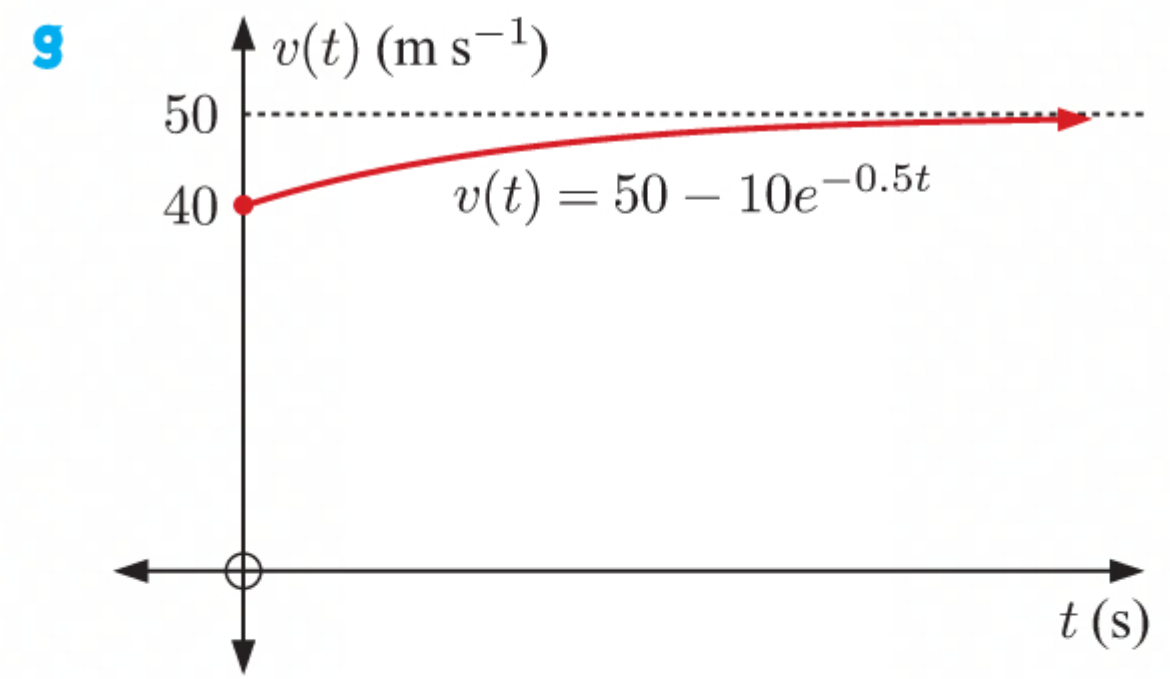
d As $t \rightarrow \infty$, $10e^{-0.5t} \rightarrow 0$ from above,
 thus $v(t) \rightarrow 50 \text{ m s}^{-1}$ from below.

e $a(t) = v'(t)$
 $= 5e^{-0.5t}$

And as $e^x > 0$ for all x ,

then $a(t) = 5e^{-0.5t} > 0$ for all t . ✓

$$\begin{aligned}
 \text{f } a(t) = 2 \quad \text{when} \quad 5e^{-0.5t} &= 2 \\
 \therefore e^{-0.5t} &= \frac{2}{5} \\
 \therefore -0.5t &= \ln\left(\frac{2}{5}\right) \\
 \therefore -0.5t &= -\ln\left(\frac{5}{2}\right) \\
 \therefore t &= 2\ln\left(\frac{5}{2}\right) \text{ s}
 \end{aligned}$$



h The particle does not change direction.

$$\begin{aligned}
 \therefore \text{total distance travelled in first 3 seconds} &= \int_0^3 v(t) \, dt \\
 &= \int_0^3 (50 - 10e^{-0.5t}) \, dt \\
 &= [50t + 20e^{-0.5t}]_0^3 \\
 &= 150 + 20e^{-1.5} - (20) \\
 &\approx 134 \text{ m}
 \end{aligned}$$

7 a

$$\begin{aligned}
 v(t) &= \int a(t) \, dt \\
 &= \int \left(\frac{t}{10} - 3\right) \, dt \\
 &= \frac{t^2}{20} - 3t + c
 \end{aligned}$$

Now $v(0) = 45$
 $\therefore c = 45$
 $\therefore v(t) = \frac{t^2}{20} - 3t + 45 \text{ m s}^{-1}$

b The train does not change direction.

$$\begin{aligned}
 \therefore \text{distance travelled} &= \int_0^{60} v(t) \, dt \\
 &= \int_0^{60} \left(\frac{t^2}{20} - 3t + 45\right) \, dt \\
 &= \left[\frac{t^3}{60} - \frac{3}{2}t^2 + 45t\right]_0^{60} \\
 &= (3600 - 5400 + 2700) - 0 \\
 &= 900
 \end{aligned}$$

The train travels a total of 900 m in the first 60 seconds.

8 a

$$\begin{aligned}
 v(t) &= \int a(t) \, dt \\
 &= \int 4e^{-\frac{t}{20}} \, dt \\
 &= -80e^{-\frac{t}{20}} + c
 \end{aligned}$$

Now $v(0) = 20$
 $\therefore -80 + c = 20$
 $\therefore c = 100$
 $\therefore v(t) = 100 - 80e^{-\frac{t}{20}} \text{ m s}^{-1}$

As $t \rightarrow \infty$, $80e^{-\frac{t}{20}} \rightarrow 0$,
 and thus $v(t) \rightarrow 100 \text{ m s}^{-1}$.

b The object does not change direction.

$$\begin{aligned}
 \therefore \text{total distance travelled in first 10 seconds} &= \int_0^{10} v(t) \, dt \\
 &= \int_0^{10} (100 - 80e^{-\frac{t}{20}}) \, dt \\
 &= \left[100t + 1600e^{-\frac{t}{20}}\right]_0^{10} \\
 &= (1000 + 1600e^{-\frac{1}{2}}) - 1600 \\
 &\approx 370 \text{ m}
 \end{aligned}$$

$$9 \quad a \quad v(t) = \int a(t) dt$$

$$= \int 2(t+1)^{-3} dt$$

$$= -(t+1)^{-2} + c$$

At $t = 0$ the particle is stationary

$$\therefore v(0) = 0$$

$$\therefore -1^{-2} + c = 0$$

$$\therefore c = 1$$

$$\therefore v(t) = -\frac{1}{(t+1)^2} + 1 \text{ m s}^{-1}$$

$$b \quad s(t) = \int v(t) dt$$

$$= \int (-(t+1)^{-2} + 1) dt$$

$$= (t+1)^{-1} + t + c$$

At $t = 0$ the particle is at the origin

$$\therefore s(0) = 0$$

$$\therefore 1^{-1} + 0 + c = 0$$

$$\therefore c = -1$$

$$\therefore s(t) = \frac{1}{t+1} + t - 1 \text{ m}$$

$$c \quad a(2) = \frac{2}{(2+1)^3}$$

$$= \frac{2}{27} \text{ m s}^{-2}$$

$$v(2) = -\frac{1}{(2+1)^2} + 1$$

$$= -\frac{1}{9} + 1$$

$$= \frac{8}{9} \text{ m s}^{-1}$$

$$s(2) = \frac{1}{2+1} + 2 - 1$$

$$= \frac{1}{3} + 1$$

$$= \frac{4}{3} \text{ m}$$

\therefore at $t = 2$, the particle is $\frac{4}{3}$ metres to the right of the origin, moving to the right at $\frac{8}{9} \text{ m s}^{-1}$, and accelerating at $\frac{2}{27} \text{ m s}^{-2}$.

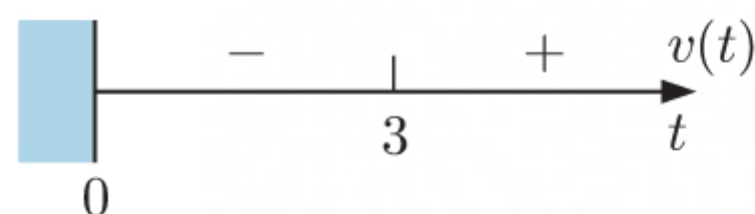
EXERCISE 18D

$$1 \quad a \quad s(t) = t^2 - 6t + 7 \text{ m}$$

$$\therefore v(t) = 2t - 6 \quad \{v(t) = s'(t)\}$$

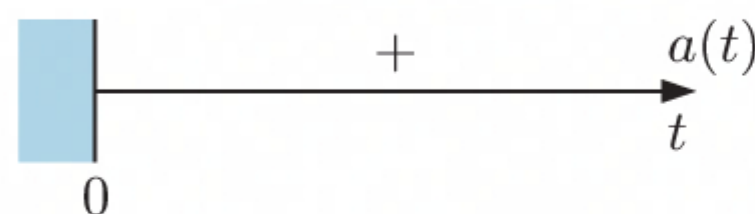
$$= 2(t - 3) \text{ m s}^{-1}$$

which has sign diagram:



and $a(t) = 2 \text{ m s}^{-2} \quad \{a(t) = v'(t)\}$

which has sign diagram:



$$b \quad \text{When } t = 0, \quad s(0) = 7 \text{ m}$$

$$v(0) = -6 \text{ m s}^{-1}$$

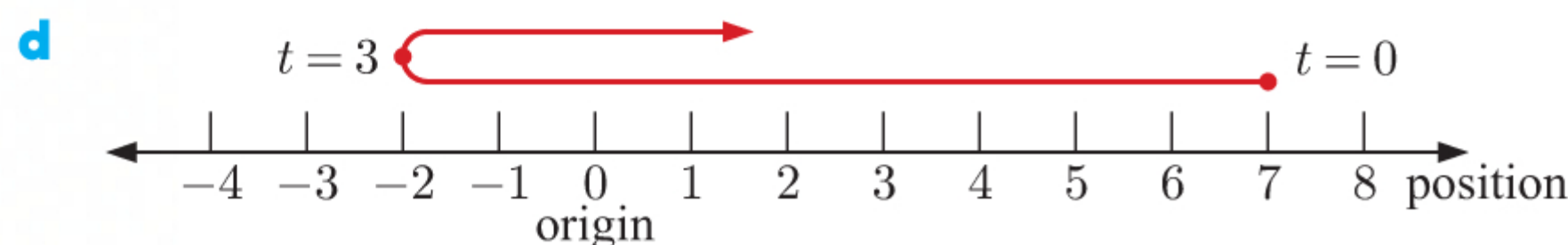
$$a(0) = 2 \text{ m s}^{-2}$$

\therefore the object is initially 7 m to the right of O, moving to the left at 6 m s^{-1} , with acceleration 2 m s^{-2} .

c $v(t)$ changes sign when $t = 3$, so this is when the object changes direction.

$$s(3) = 9 - 18 + 7 = -2$$

So, the object changes direction when it is 2 m to the left of O.



- e The object's speed is decreasing when $v(t)$ and $a(t)$ have opposite signs. This occurs for $0 \leq t \leq 3$.

2 $s(t) = 1.2 + 28.1t - 4.9t^2$ m

- a When the ball was first released, $t = 0$.

$$s(0) = 1.2$$

So, the ball was released 1.2 m above ground level.

- b $s'(t) = 28.1 - 9.8t$ m s⁻¹ which is the instantaneous velocity of the ball t seconds after being released.

- c $s'(t) = 0$ when $28.1 - 9.8t = 0$

$$\therefore 28.1 = 9.8t$$

$$\therefore t = \frac{28.1}{9.8}$$

So, the ball has reached its maximum height after $\frac{28.1}{9.8}$ seconds, and is instantaneously at rest.

$$s\left(\frac{28.1}{9.8}\right) = 1.2 + 28.1\left(\frac{28.1}{9.8}\right) - 4.9\left(\frac{28.1}{9.8}\right)^2$$

$$\approx 41.5$$

So, the maximum height reached by the ball is about 41.5 m.

- d $s'(t) = 28.1 - 9.8t$ m s⁻¹

i $s'(0) = 28.1$

The ball's speed when released is 28.1 m s⁻¹.

ii $s'(2) = 28.1 - 9.8(2)$
 $= 8.5$

The ball's speed at $t = 2$ seconds is 8.5 m s⁻¹.

iii $s'(5) = 28.1 - 9.8(5)$
 $= -20.9$

The ball's speed at $t = 5$ seconds is 20.9 m s⁻¹.

3 a $s(t) = 12t - 2t^3 - 1$ cm

$$\therefore v(t) = 12 - 6t^2 \text{ cm s}^{-1} \quad \{v(t) = s'(t)\}$$

$$\therefore a(t) = -12t \text{ cm s}^{-2} \quad \{a(t) = v'(t)\}$$

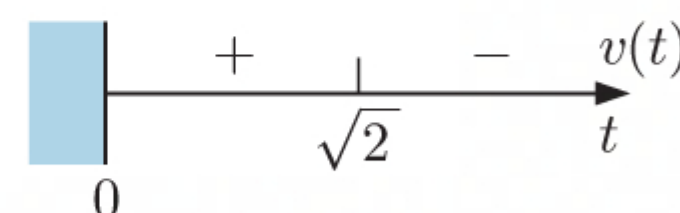
b $s(0) = -1$ cm, $v(0) = 12$ cm s⁻¹, $a(0) = 0$ cm s⁻²

The particle is initially 1 cm to the left of the origin, travelling to the right at a constant speed of 12 cm s⁻¹.

c $v(t) = 12 - 6t^2$

$$= 6(2 - t^2)$$

$$= 6(\sqrt{2} + t)(\sqrt{2} - t) \text{ cm s}^{-1} \quad \text{which has sign diagram:}$$



$v(t)$ changes sign when $t = \sqrt{2}$ seconds, so this is when the particle changes direction.

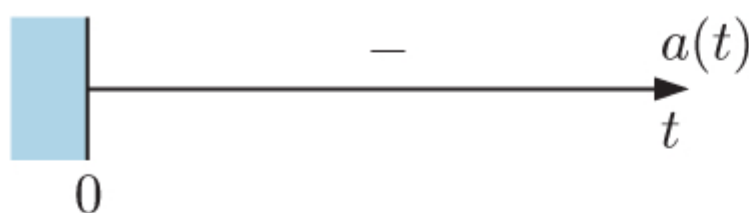
$$s(\sqrt{2}) = 12(\sqrt{2}) - 2(\sqrt{2})^3 - 1$$

$$= 12\sqrt{2} - 4\sqrt{2} - 1$$

$$= 8\sqrt{2} - 1$$

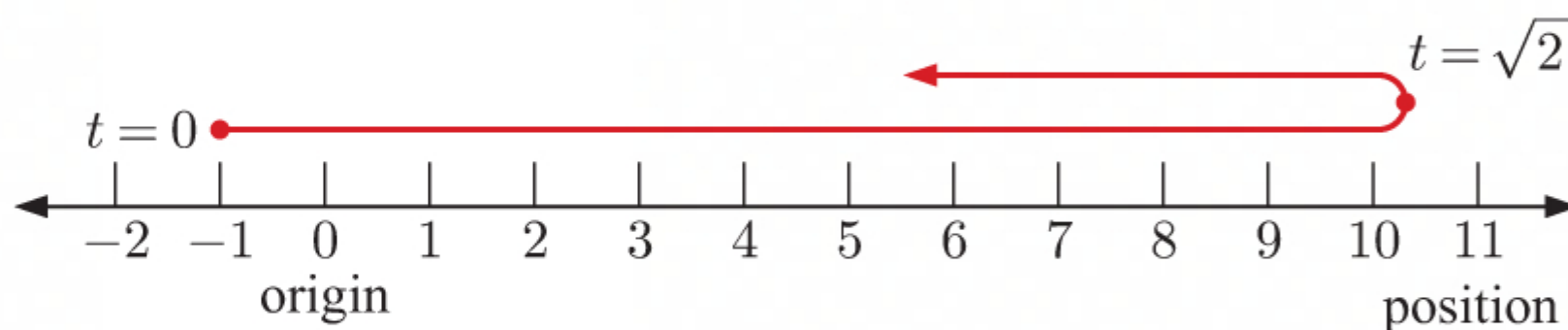
So, the particle changes direction when it is $(8\sqrt{2} - 1)$ cm to the right of O.

d $a(t) = -12t$ has sign diagram:



- i** The particle's speed is increasing when $v(t)$ and $a(t)$ have the same sign. This occurs for $t \geq \sqrt{2}$.
- ii** The particle's velocity is never increasing.

e

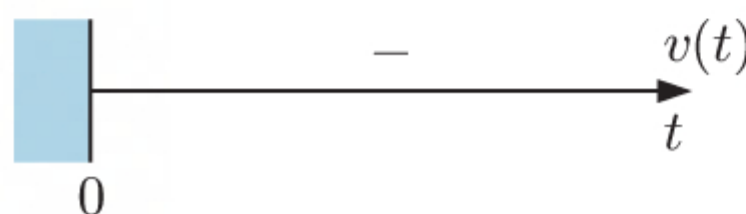


4 a $s(t) = 4 - \sqrt{t+1} = 4 - (t+1)^{\frac{1}{2}} \text{ m}$

$$\therefore v(t) = -\frac{1}{2}(t+1)^{-\frac{1}{2}} \quad \{v(t) = s'(t)\}$$

$$= -\frac{1}{2\sqrt{t+1}} \text{ m s}^{-1}$$

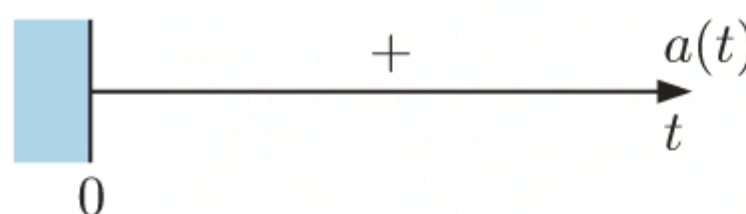
which has sign diagram:



and $a(t) = \frac{1}{4}(t+1)^{-\frac{3}{2}} \quad \{a(t) = v'(t)\}$

$$= \frac{1}{4(t+1)^{\frac{3}{2}}} \text{ m s}^{-2}$$

which has sign diagram:



b $s(0) = 3 \text{ m}, \quad v(0) = -\frac{1}{2} \text{ m s}^{-1}, \quad a(0) = \frac{1}{4} \text{ m s}^{-2}$

Initially, the particle is 3 m to the right of O, moving to the left at $\frac{1}{2} \text{ m s}^{-1}$ with acceleration $\frac{1}{4} \text{ m s}^{-2}$.

c $s(3) = 4 - \sqrt{4} = 2 \text{ m}, \quad v(3) = -\frac{1}{2\sqrt{4}} = -\frac{1}{4} \text{ m s}^{-1}, \quad a(3) = \frac{1}{4(4)^{\frac{3}{2}}} = \frac{1}{32} \text{ m s}^{-2}$

After 3 seconds, the particle is 2 m to the right of O, moving to the left at $\frac{1}{4} \text{ m s}^{-1}$, with acceleration $\frac{1}{32} \text{ m s}^{-2}$.

d The particle's speed is continuously decreasing.

5 a When the device reaches the water,

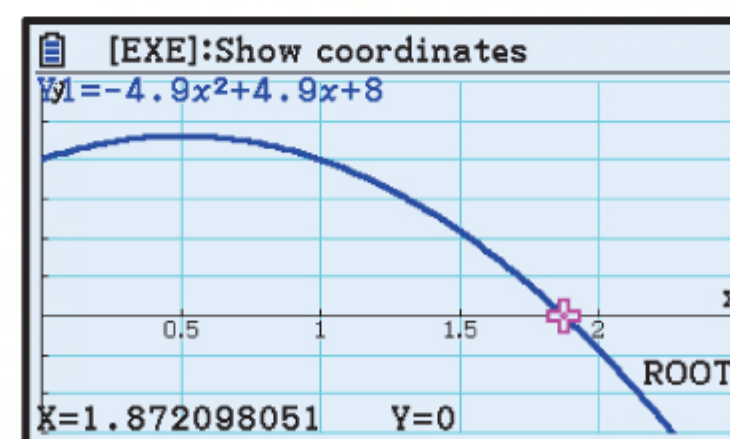
$$s(t) = 0$$

$$\therefore -4.9t^2 + 4.9t + 8 = 0$$

$$\therefore t \approx 1.87 \quad \{\text{using technology, } t > 0\}$$

So, the device takes approximately 1.87 seconds to reach the water.

$$\therefore k \approx 1.87$$

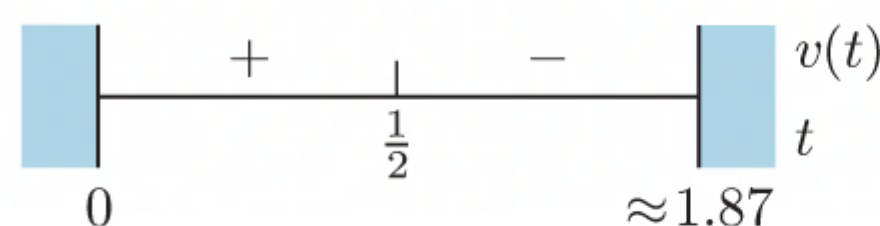


b $s(t) = -4.9t^2 + 4.9t + 8 \text{ m}$

$\therefore v(t) = -9.8t + 4.9 \text{ m s}^{-1} \quad \{v(t) = s'(t)\}$

$= 4.9(1 - 2t)$

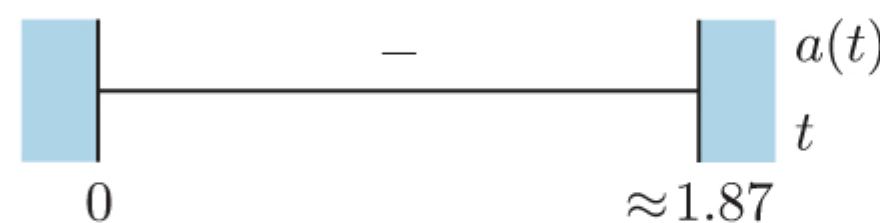
which has sign diagram:



and $a(t) = -9.8 \text{ m s}^{-2}$

$\{a(t) = v'(t)\}$

which has sign diagram:



c **i** Looking at the sign diagrams for $v(t)$ and $a(t)$, $v(0.2) > 0$ and $a(0.2) < 0$

$\therefore v(0.2)$ and $a(0.2)$ have opposite sign.

\therefore the speed of the device is decreasing after 0.2 seconds.

ii $v(1) < 0$ and $a(1) < 0$

$\therefore v(1)$ and $a(1)$ have the same sign.

\therefore the speed of the device is increasing after 1 second.

6 **a** $x(t) = 1 - 2 \cos t \text{ cm}$

$\therefore v(t) = 2 \sin t \text{ cm s}^{-1} \quad \{v(t) = x'(t)\}$

$\therefore a(t) = 2 \cos t \text{ cm s}^{-2} \quad \{a(t) = v'(t)\}$

When $t = 0$, $x(0) = 1 - 2 \cos 0 = -1 \text{ cm}$

$v(0) = 2 \sin 0 = 0 \text{ cm s}^{-1}$

$a(0) = 2 \cos 0 = 2 \text{ cm s}^{-2}$

\therefore P is initially 1 cm to the left of the origin, instantaneously at rest, and accelerating at 2 cm s^{-2} .

b $x\left(\frac{\pi}{4}\right) = 1 - 2 \cos \frac{\pi}{4} = 1 - \sqrt{2} = -(\sqrt{2} - 1) \text{ cm}$, $v\left(\frac{\pi}{4}\right) = 2 \sin \frac{\pi}{4} = \sqrt{2} \text{ cm s}^{-1}$,

$a\left(\frac{\pi}{4}\right) = 2 \cos \frac{\pi}{4} = \sqrt{2} \text{ cm s}^{-2}$

\therefore at $t = \frac{\pi}{4}$ seconds, the particle is $(\sqrt{2} - 1) \text{ cm}$ left of O, moving to the right at $\sqrt{2} \text{ cm s}^{-1}$, with acceleration $\sqrt{2} \text{ cm s}^{-2}$.

c The particle reverses direction when $v(t) = 0$

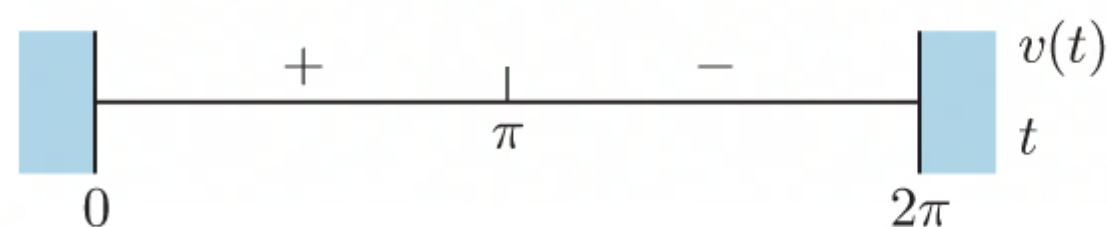
$\therefore 2 \sin t = 0$

$\therefore t = \pi \quad \{0 < t < 2\pi\}$

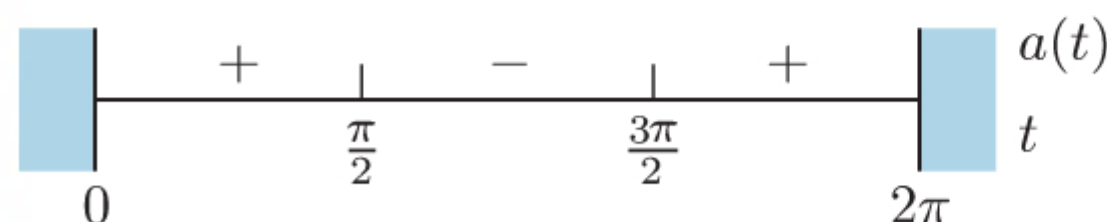
$x(\pi) = 1 - 2 \cos \pi = 3 \text{ cm}$

\therefore the particle reverses direction at $t = \pi$ seconds, 3 cm to the right of the origin.

d $v(t) = 2 \sin t$ has sign diagram:



$a(t) = 2 \cos t$ has sign diagram:



The particle's speed is increasing when $v(t)$ and $a(t)$ have the same sign.

This occurs for $0 \leq t \leq \frac{\pi}{2}$ seconds and $\pi \leq t \leq \frac{3\pi}{2}$ seconds.

7 a $s(t) = 8 \sin \frac{t}{2} \text{ m}$

i $s(3) = 8 \sin \frac{3}{2}$
 $\approx 7.98 > 0$

\therefore after 3 seconds, the dog is to the right of its kennel.

ii $s(7) = 8 \sin \frac{7}{2}$
 $\approx -2.81 < 0$

\therefore after 7 seconds, the dog is to the left of its kennel.

b $s(t) = 8 \sin \frac{t}{2} \text{ m}$

$\therefore v(t) = 8\left(\frac{1}{2}\right) \cos \frac{t}{2} \quad \{v(t) = s'(t)\}$
 $= 4 \cos \frac{t}{2} \text{ m s}^{-1}$

c i $v(4) = 4 \cos 2$
 $\approx -1.66 < 0$

\therefore after 4 seconds, the dog is moving to the left.

ii $v(10) = 4 \cos 5$
 $\approx 1.13 > 0$

\therefore after 10 seconds, the dog is moving to the right.

d $a(t) = 4\left(-\frac{1}{2}\right) \sin \frac{t}{2} \quad \{a(t) = v'(t)\}$
 $= -2 \sin \frac{t}{2} \text{ m s}^{-2}$

e $v(2) = 4 \cos 1 \quad a(2) = -2 \sin 1$
 $\approx 2.16 \text{ m s}^{-1} \quad \approx -1.68 \text{ m s}^{-2}$

$v(2) > 0$ and $a(2) < 0$

$\therefore v(2)$ and $a(2)$ have opposite sign.

\therefore the dog's speed is decreasing after 2 seconds.

f $|v(t)|$ is a maximum when $v'(t) = 0$

$\therefore -4\left(\frac{1}{2}\right) \sin \frac{t}{2} = 0$

$\therefore -2 \sin \frac{t}{2} = 0$

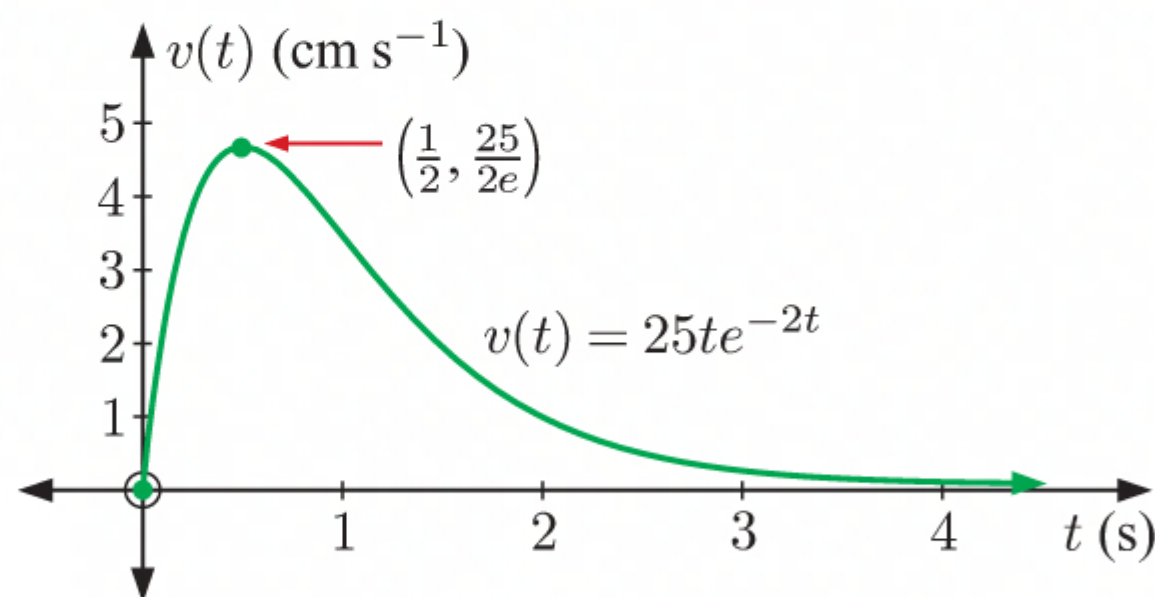
$\therefore \frac{t}{2} = k\pi \quad \{k \in \mathbb{Z}\}$

$\therefore t = 2k\pi$

Now $s(2k\pi) = 8 \sin k\pi$
 $= 0$

\therefore the dog's speed is maximised when it is moving past its kennel.

8 a



b $v(t) = 25te^{-2t} \text{ cm s}^{-1}, \quad t \geq 0$

$\therefore a(t) = v'(t) = 25e^{-2t} + 25t(-2)e^{-2t} \quad \{\text{product rule}\}$
 $= 25e^{-2t} - 50te^{-2t}$
 $= 25(1 - 2t)e^{-2t} \text{ cm s}^{-2}, \quad t \geq 0 \quad \checkmark$

c $a(t) = 25(1 - 2t)e^{-2t}$ has sign diagram:

The acceleration is positive and hence the velocity is increasing for $0 \leq t \leq \frac{1}{2}$ second.

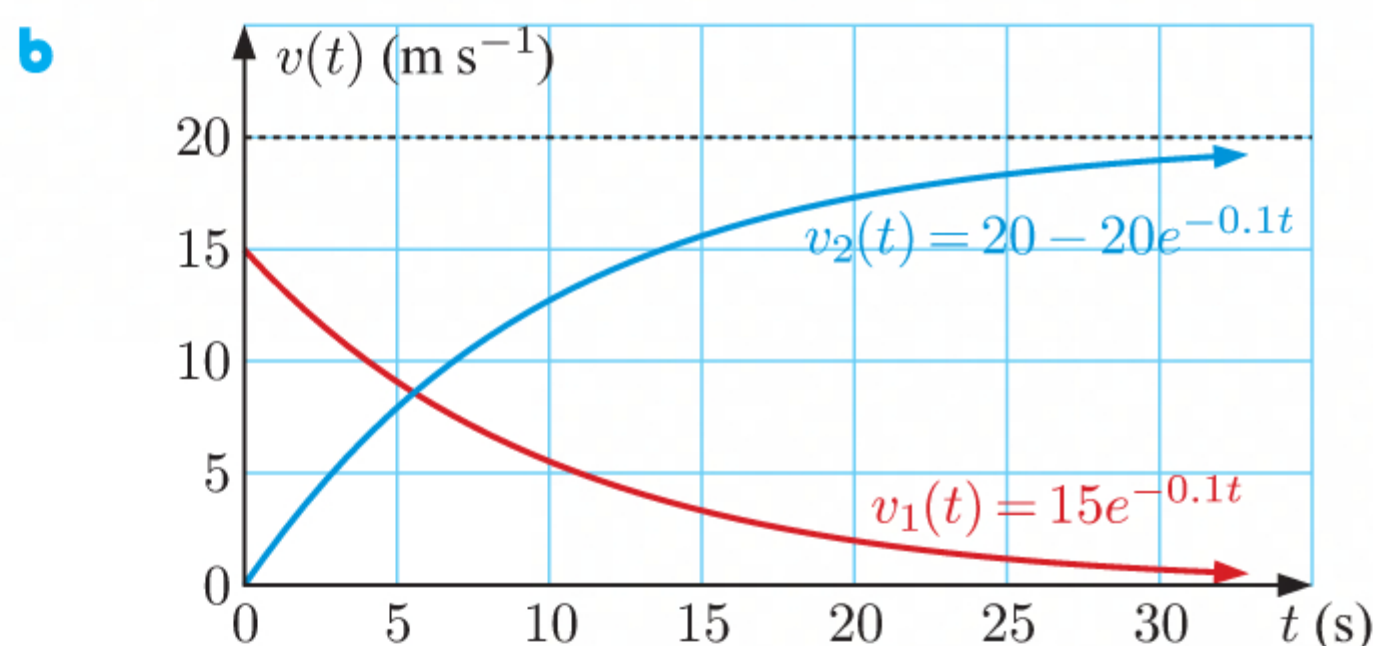
d $v(t) = 25te^{-2t}$ has sign diagram:

The speed of the object is decreasing when $v(t)$ and $a(t)$ have opposite signs. This occurs when $t \geq \frac{1}{2}$ second.

9 Lion: $v_1(t) = 15e^{-0.1t} \text{ m s}^{-1}$

Zebra: $v_2(t) = 20 - 20e^{-0.1t} \text{ m s}^{-1}$

a After 1 second, speed of lion $= v_1(1) = 15e^{-0.1(1)}$
 $\approx 13.6 \text{ m s}^{-1}$
 speed of zebra $= v_2(1) = 20 - 20e^{-0.1(1)}$
 $\approx 1.90 \text{ m s}^{-1}$



As shown by the graph, the lion's speed $v_1(t)$ decreases over time whereas the zebra's speed $v_2(t)$ increases over time.

c $\int_0^3 v_1(t) dt = \int_0^3 15e^{-0.1t} dt$
 $= [-150e^{-0.1t}]_0^3$
 $= -150e^{-0.1(3)} - (-150)$
 $= 150 - 150e^{-0.3}$
 ≈ 38.9

The lion has travelled a total distance of about 38.9 metres in the first 3 seconds.

d $\int_0^3 [v_1(t) - v_2(t)] dt = \int_0^3 [15e^{-0.1t} - (20 - 20e^{-0.1t})] dt$
 $= \int_0^3 (35e^{-0.1t} - 20) dt$
 $= [-350e^{-0.1t} - 20t]_0^3$
 $= (-350e^{-0.3} - 60) - (-350)$
 $= 290 - 350e^{-0.3}$
 ≈ 30.7

In the first 3 seconds, the lion has gained about 30.7 metres on the zebra.

- e** At the time when $v_1(t) = v_2(t)$, the lion and the zebra will be moving at the same speed. Since the lion's speed decreases over time and the zebra's speed increases over time, the zebra will be faster than the lion after that time. So, they will be closest at the point when their speeds are equal.

$$\begin{aligned}
 \text{f } v_1(t) &= v_2(t) \text{ when } 15e^{-0.1t} = 20 - 20e^{-0.1t} \\
 &\therefore 35e^{-0.1t} = 20 \\
 &\therefore e^{-0.1t} = \frac{20}{35} \\
 &\therefore -0.1t = \ln\left(\frac{20}{35}\right) \\
 &\therefore t = -10 \ln\left(\frac{4}{7}\right) \\
 &\therefore t \approx 5.60
 \end{aligned}$$

So, $v_1(t) = v_2(t)$ after about 5.60 seconds.

- g** From **e** and **f**, the lion is closest to the zebra after about 5.60 seconds.

$$\begin{aligned}
 \int_0^{5.60} [v_1(t) - v_2(t)] dt &\approx [-350e^{-0.1t} - 20t]_0^{5.60} \\
 &\approx (-350e^{-0.560} - 20 \times 5.60) - (-350) \\
 &\approx 38.08
 \end{aligned}$$

So, after about 5.60 seconds, the lion has travelled about 38.08 metres more than the zebra has travelled. The zebra was however initially 40 metres ahead of the lion, so at their closest point, the zebra will still be about $40 \text{ m} - 38.08 \text{ m} \approx 1.92 \text{ m}$ ahead of the lion.

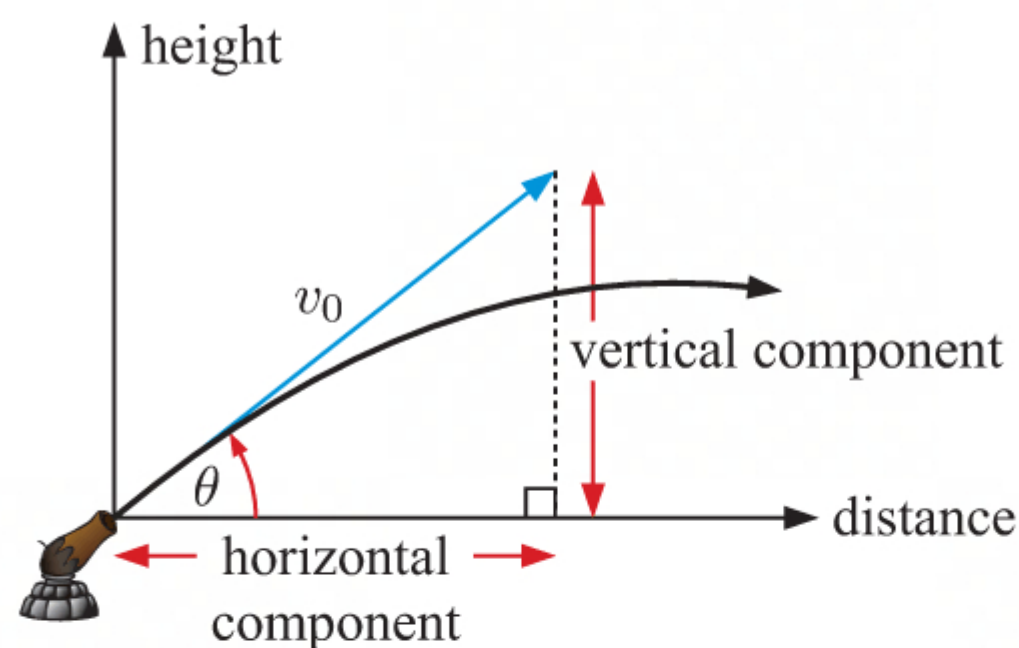
So, the lion did not catch the zebra but was about 1.92 m from the zebra at their closest point.

INVESTIGATION

PROJECTILE MOTION

- 1 a** If the cannon is fired from ground level, the initial vertical height of the cannonball is 0 m.

$$\begin{aligned}
 \text{b } \sin \theta &= \frac{\text{vertical component}}{v_0} \\
 &\therefore \text{vertical component} = v_0 \sin \theta \\
 &\therefore \text{the initial vertical velocity of the} \\
 &\quad \text{cannonball is } v_0 \sin \theta \text{ m s}^{-1}.
 \end{aligned}$$



- c** After the cannonball is fired, the only force acting on the cannonball is the downward force of gravity, which has acceleration 9.8 m s^{-2} .

Since the positive direction is *upwards*, the vertical acceleration of the cannonball is given by $a(t) = -9.8 \text{ m s}^{-2}$.

$$\begin{aligned}
 \text{d } s(t) &= -4.9t^2 + [v_0 \sin \theta]t \\
 &\therefore v(t) = -9.8t + v_0 \sin \theta \quad \{v(t) = s'(t)\} \\
 &\therefore a(t) = -9.8
 \end{aligned}$$

$$s(0) = 0$$

So, the initial vertical height of the cannonball is 0 m.

$$v(0) = v_0 \sin \theta$$

So, the initial vertical velocity of the cannonball is $v_0 \sin \theta \text{ m s}^{-1}$.

$$a(t) = -9.8$$

So, the vertical acceleration of the cannonball is a constant -9.8 m s^{-2} .

All of these satisfy the properties in **a**, **b**, and **c**.

e When the cannonball hits the ground, $s(t) = 0$

$$\therefore -4.9t^2 + [v_0 \sin \theta]t = 0$$

$$\therefore t(v_0 \sin \theta - 4.9t) = 0$$

$$\therefore t = 0 \text{ or } v_0 \sin \theta - 4.9t = 0$$

$$\therefore 4.9t = v_0 \sin \theta$$

$$\therefore t = \frac{v_0 \sin \theta}{4.9}$$

\therefore the cannonball takes $\frac{v_0 \sin \theta}{4.9}$ seconds to hit the ground.

2 a $\cos \theta = \frac{\text{horizontal component}}{v_0}$

$$\therefore \text{horizontal component} = v_0 \cos \theta$$

\therefore the horizontal velocity of the cannonball is $v_0 \cos \theta \text{ m s}^{-1}$.

b Horizontal distance travelled = horizontal velocity \times time of flight

$$= v_0 \cos \theta \text{ m s}^{-1} \times \frac{v_0 \sin \theta}{4.9} \text{ s} \quad \{\text{from 1 e}\}$$

$$= \frac{v_0^2 \cos \theta \sin \theta}{4.9} \text{ m}$$

$$= \frac{v_0^2 \frac{1}{2} \sin 2\theta}{4.9} \text{ m}$$

$$= \frac{v_0^2 \sin 2\theta}{9.8} \text{ m}$$

c i $v_0 = 200 \text{ m s}^{-1}$, $\theta = 20^\circ$

Horizontal distance travelled

$$= \frac{(200)^2 \sin 40^\circ}{9.8}$$

$$\approx 2623.62 \text{ m}$$

ii $v_0 = 200 \text{ m s}^{-1}$, $\theta = 50^\circ$

Horizontal distance travelled

$$= \frac{(200)^2 \sin 100^\circ}{9.8}$$

$$\approx 4019.62 \text{ m}$$

iii $v_0 = 200 \text{ m s}^{-1}$, $\theta = 80^\circ$

$$\text{Horizontal distance travelled} = \frac{(200)^2 \sin 160^\circ}{9.8}$$

$$\approx 1396.00 \text{ m}$$

d $\frac{v_0^2 \sin 2\theta}{9.8}$ is maximised when $\sin 2\theta = 1$
 $\therefore 2\theta = 90^\circ \quad \{0^\circ \leq 2\theta \leq 180^\circ\}$
 $\therefore \theta = 45^\circ$

\therefore the angle which maximises the range of the cannonball is $\theta = 45^\circ$.

REVIEW SET 18A

1 a $s(t) = 12 - 2t \text{ m}$, $0 \leq t \leq 10 \text{ s}$

$$s(0) = 12 - 2(0)$$

$$= 12 \text{ m}$$

\therefore the initial displacement of the object is 12 m to the right of the origin.

$$\begin{aligned} \text{b i } s(1) &= 12 - 2(1) \\ &= 12 - 2 \\ &= 10 \text{ m} \end{aligned}$$

\therefore the displacement of the object at $t = 1$ second is 10 m to the right of the origin.

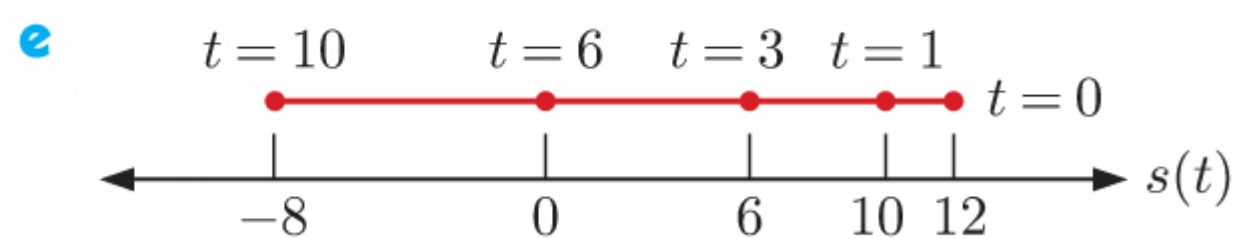
$$\begin{aligned} \text{ii } s(3) &= 12 - 2(3) \\ &= 12 - 6 \\ &= 6 \text{ m} \end{aligned}$$

\therefore the displacement of the object at $t = 3$ seconds is 6 m to the right of the origin.

$$\begin{aligned} \text{c The object is at the origin when } s(t) &= 0 \\ \therefore 12 - 2t &= 0 \\ \therefore 2t &= 12 \\ \therefore t &= 6 \end{aligned}$$

\therefore the object reaches the origin at $t = 6$ seconds.

d No, the displacement function is linear, so it has no turning points.



$$\text{2 a } s(t) = 2t^2 + t - 5 \text{ cm, } t \geq 0 \text{ s}$$

$$\begin{aligned} s(1) &= 2(1)^2 + 1 - 5 & s(5) &= 2(5)^2 + 5 - 5 \\ &= 2 + 1 - 5 & &= 50 + 5 - 5 \\ &= -2 \text{ cm} & &= 50 \text{ cm} \end{aligned}$$

$$\begin{aligned} \text{average velocity} &= \frac{s(5) - s(1)}{5 - 1} \\ &= \frac{50 - (-2)}{5 - 1} \\ &= \frac{52}{4} \\ &= 13 \text{ cm s}^{-1} \end{aligned}$$

\therefore the average velocity from $t = 1$ to $t = 5$ seconds is 13 cm s^{-1} .

$$\text{b } v(t) = s'(t) = 4t + 1 \text{ cm s}^{-1}$$

$$\begin{aligned} \text{i } v(2) &= 4(2) + 1 \\ &= 8 + 1 \\ &= 9 \text{ cm s}^{-1} \end{aligned}$$

\therefore the instantaneous velocity at $t = 2$ seconds is 9 cm s^{-1} .

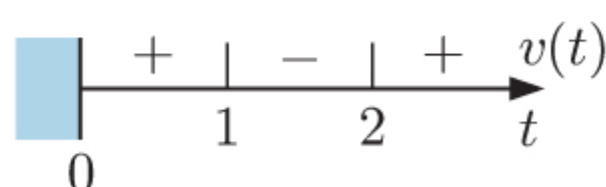
$$\begin{aligned} \text{ii } v(4) &= 4(4) + 1 \\ &= 16 + 1 \\ &= 17 \text{ cm s}^{-1} \end{aligned}$$

\therefore the instantaneous velocity at $t = 4$ seconds is 17 cm s^{-1} .

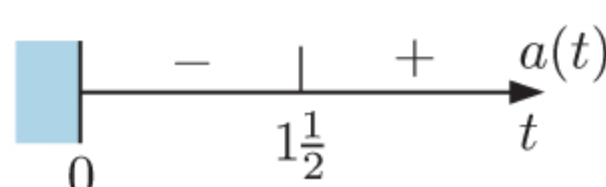
$$\text{c } a(t) = v'(t) = 4 \text{ cm s}^{-2}$$

$$\begin{aligned} \text{3 a } s(t) &= 2t^3 - 9t^2 + 12t - 5 \text{ cm, } t \geq 0 \text{ s} \\ \therefore v(t) &= 6t^2 - 18t + 12 \text{ cm s}^{-1} & \{v(t) = s'(t)\} \\ \therefore a(t) &= 12t - 18 \text{ cm s}^{-2} & \{a(t) = v'(t)\} \end{aligned}$$

$v(t)$ has sign diagram:



$a(t)$ has sign diagram:



b $s(0) = -5 \text{ cm}$
 $v(0) = 12 \text{ cm s}^{-1}$
 $a(0) = -18 \text{ cm s}^{-2}$

The particle P is initially 5 cm to the left of the origin, moving to the right at 12 cm s^{-1} , with acceleration -18 cm s^{-2} (decreasing speed).

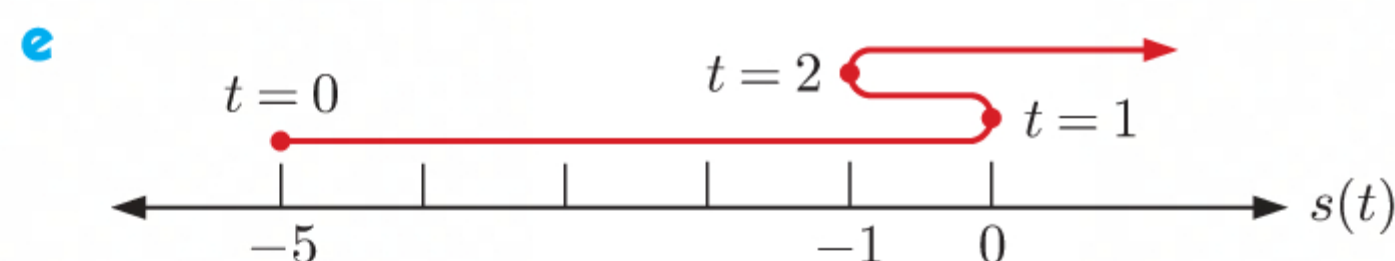
c $s(2) = 2(2)^3 - 9(2)^2 + 12(2) - 5$ $v(2) = 6(2)^2 - 18(2) + 12$
 $= 16 - 36 + 24 - 5$ $= 24 - 36 + 12$
 $= -1 \text{ cm}$ $= 0 \text{ cm s}^{-1}$
 $a(2) = 12(2) - 18$
 $= 24 - 18$
 $= 6 \text{ cm s}^{-2}$

At $t = 2$, the particle is 1 cm to the left of the origin, is instantaneously stationary, and is beginning to accelerate.

- d** The particle changes direction when $v(t)$ changes sign.
 From the sign diagram in **a**, this occurs at $t = 1$ second and $t = 2$ seconds.

$$\begin{array}{ll} s(1) = 2 - 9 + 12 - 5 & s(2) = 2(2)^3 - 9(2)^2 + 12(2) - 5 \\ = 0 & = 16 - 36 + 24 - 5 \\ & = -1 \end{array}$$

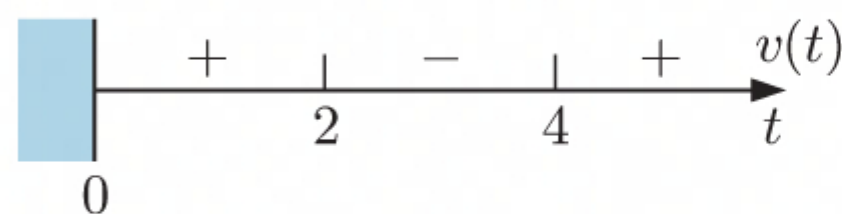
\therefore the particle changes direction at $t = 1$ second when it is at the origin, and at $t = 2$ seconds when it is 1 cm to the left of the origin.



- f** The particle's speed is increasing when $v(t)$ and $a(t)$ have the same sign.
 From the sign diagrams in **a**, this occurs for $1 \leq t \leq 1\frac{1}{2}$ and $t \geq 2$ seconds.

4 a $v(t) = s'(t) = t^2 - 6t + 8$
 $= (t - 2)(t - 4)$

\therefore the sign diagram of $v(t)$ is:



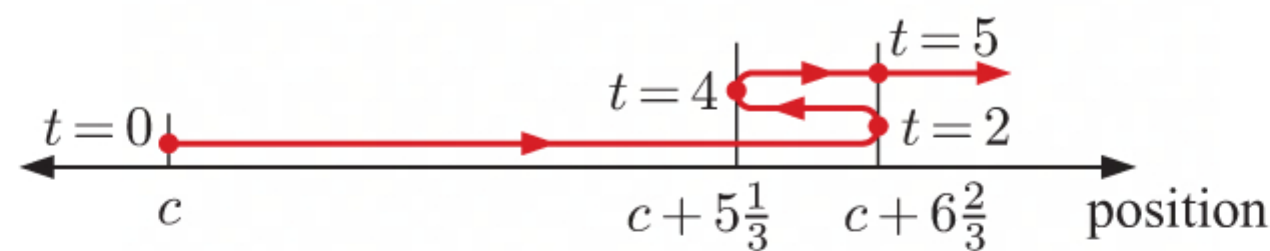
- b** Since the signs change, the particle reverses direction at $t = 2$ and $t = 4$ seconds.

Now $s(t) = \int v(t) dt$
 $= \int (t^2 - 6t + 8) dt$
 $= \frac{1}{3}t^3 - 3t^2 + 8t + c$

Hence $s(0) = c$

$$\begin{aligned} s(2) &= \frac{1}{3}(2)^3 - 3(2)^2 + 8(2) + c = \frac{8}{3} - 12 + 16 + c = c + 6\frac{2}{3} \\ s(4) &= \frac{1}{3}(4)^3 - 3(4)^2 + 8(4) + c = \frac{64}{3} - 48 + 32 + c = c + 5\frac{1}{3} \\ s(5) &= \frac{1}{3}(5)^3 - 3(5)^2 + 8(5) + c = \frac{125}{3} - 75 + 40 + c = c + 6\frac{2}{3} \end{aligned}$$

Motion diagram:



The particle initially moves in the positive direction, then at $t = 2$, $6\frac{2}{3}$ m from its starting point, it changes direction. It changes direction again at $t = 4$, $5\frac{1}{3}$ m from its starting point, and at $t = 5$, it is $6\frac{2}{3}$ m from its starting point again.

- c** After 5 seconds, the particle is $6\frac{2}{3}$ metres from its original position.
- d** Total distance travelled $= (c + 6\frac{2}{3} - c) + (c + 6\frac{2}{3} - [c + 5\frac{1}{3}]) + (c + 6\frac{2}{3} - [c + 5\frac{1}{3}])$
 $= 6\frac{2}{3} + 1\frac{1}{3} + 1\frac{1}{3}$
 $= 9\frac{1}{3}$ metres

5 a $v(t) = 2.75 - t + 0.5t^{1.2} \text{ m s}^{-1}, \quad 0 \leq t \leq 6 \text{ s}$

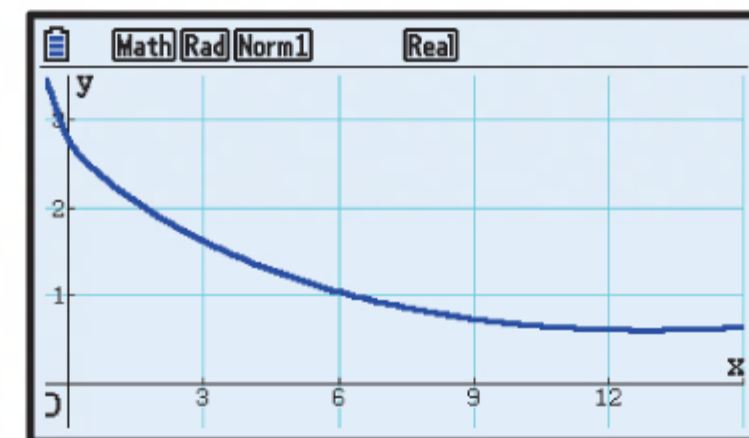
i $v(0) = 2.75 \text{ m s}^{-1}$

\therefore the velocity of the kayak after the kayaker stops paddling is 2.75 m s^{-1} .

ii $v(3) = 2.75 - 3 + 0.5(3)^{1.2}$
 ≈ 1.62

\therefore the velocity of the kayak after 3 seconds is about 1.62 m s^{-1} .

b $v(t) > 0$ for all t {using technology}

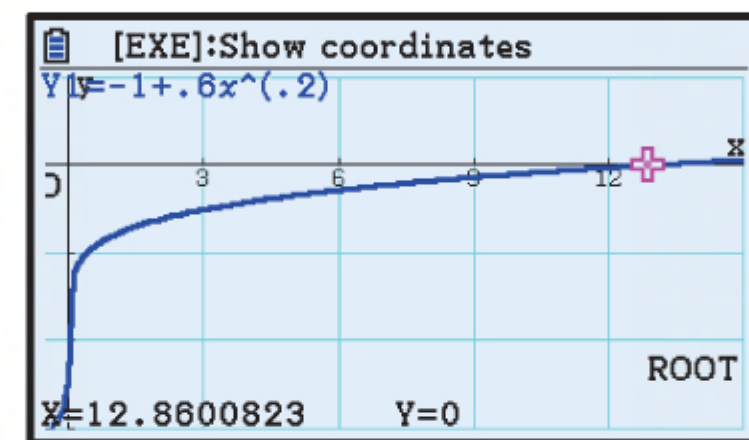


So, $v(t)$ has sign diagram:

$$a(t) = -1 + 0.5(1.2)t^{0.2} \quad \{a(t) = v'(t)\}$$

$$= -1 + 0.6t^{0.2} \text{ m s}^{-2}$$

$a(t) = 0$ when $t \approx 12.9$ {using technology}



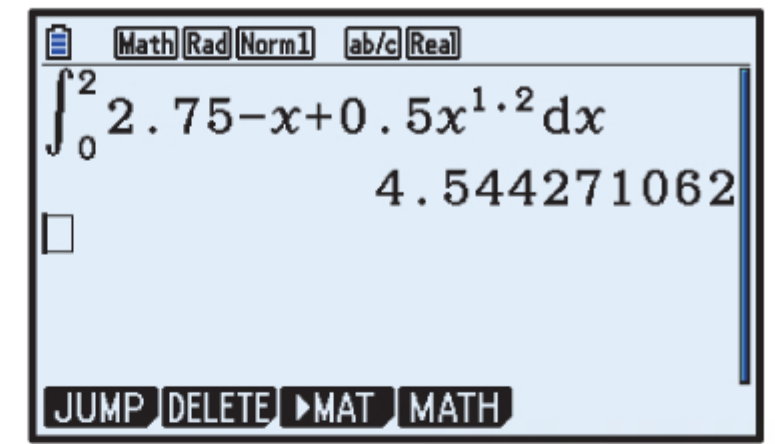
So, $a(t)$ has sign diagram:

Now, our model only considers the interval $0 \leq t \leq 6$ seconds, where $v(t)$ and $a(t)$ have opposite sign.

\therefore the kayak's speed is decreasing during the 6 second period.

$$\begin{aligned} \text{c} \quad \int_0^2 v(t) dt &= \int_0^2 (2.75 - t + 0.5t^{1.2}) dt \\ &\approx 4.54 \quad \{\text{using technology}\} \end{aligned}$$

The kayak travels approximately 4.54 m in the first 2 seconds after the kayaker stops paddling.



$$\begin{aligned} \text{6 a} \quad s(t) &= 15t - \frac{60}{(t+1)^2} \text{ cm}, \quad t \geq 0 \text{ s} \\ &= 15t - 60(t+1)^{-2} \\ \therefore v(t) &= 15 + 120(t+1)^{-3} \quad \{v(t) = s'(t)\} \\ &= 15 + \frac{120}{(t+1)^3} \text{ cm s}^{-1} \\ \therefore a(t) &= -360(t+1)^{-4} \quad \{a(t) = v'(t)\} \\ &= -\frac{360}{(t+1)^4} \text{ cm s}^{-2} \end{aligned}$$

$$\begin{aligned} \text{b} \quad s(3) &= 15(3) - \frac{60}{(3+1)^2} & v(3) &= 15 + \frac{120}{(3+1)^3} & a(3) &= -\frac{360}{(3+1)^4} \\ &= 41.25 \text{ cm} & &\approx 16.9 \text{ cm s}^{-1} & &\approx -1.41 \text{ cm s}^{-2} \end{aligned}$$

So, at time $t = 3$ seconds, the particle is 41.25 cm to the right of O, moving to the right at about 16.9 cm s^{-1} , with decreasing speed ($a(3) \approx -1.41 \text{ cm s}^{-2}$).

c $v(t)$ has sign diagram:

$a(t)$ has sign diagram:

The signs of $v(t)$ and $a(t)$ are never the same.
 \therefore the particle's speed is never increasing.

$$\begin{aligned} \text{7 a} \quad x(t) &= 3 + 2 \sin \pi t \text{ m}, \quad t \geq 0 \text{ s} \\ \therefore v(t) &= 2\pi \cos \pi t \quad \{v(t) = x'(t)\} \\ \therefore a(t) &= -2\pi^2 \sin \pi t \quad \{a(t) = v'(t)\} \\ x(0) &= 3 + 2 \sin 0 & v(0) &= 2\pi \cos 0 & a(0) &= -2\pi^2 \sin 0 \\ &= 3 \text{ m} & &= 2\pi \text{ m s}^{-1} & &= 0 \text{ m s}^{-2} \end{aligned}$$

The object is initially 3 m to the right of the origin, moving to the right at $2\pi \text{ m s}^{-1}$, and has acceleration 0 m s^{-2} .

b $v(t) = 0$ when $2\pi \cos \pi t = 0$

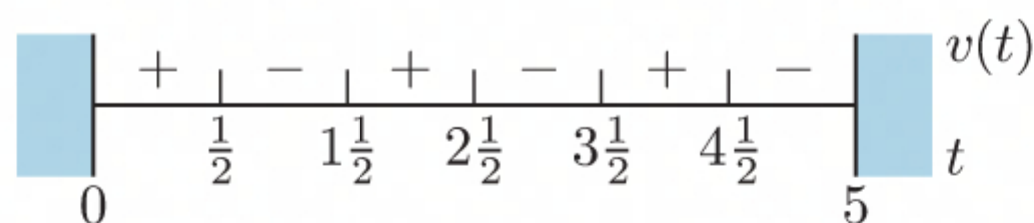
$$\therefore \cos \pi t = 0$$

$$\therefore \pi t = \left(k + \frac{1}{2}\right) \pi, \quad k \in \mathbb{Z}$$

$$\therefore t = k + \frac{1}{2}, \quad k \in \mathbb{Z}$$

$$\therefore t = \frac{1}{2}, 1\frac{1}{2}, 2\frac{1}{2}, 3\frac{1}{2}, 4\frac{1}{2} \text{ seconds} \quad \{0 \leq t \leq 5\}$$

$v(t)$ has sign diagram:

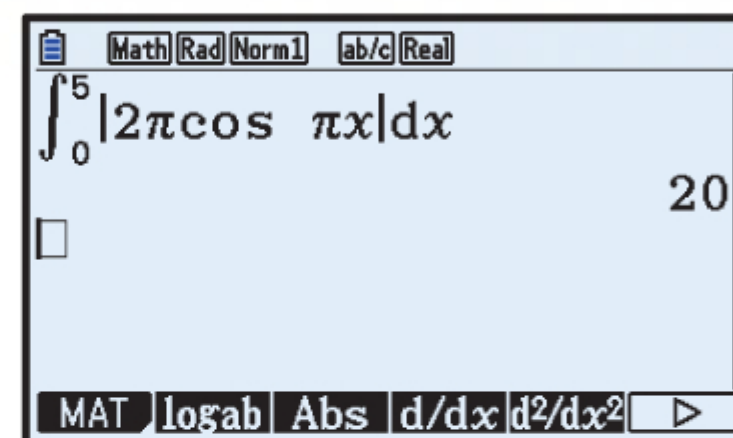


So, the spotlight changes direction at $t = \frac{1}{2}, 1\frac{1}{2}, 2\frac{1}{2}, 3\frac{1}{2}$, and $4\frac{1}{2}$ seconds during the first 5 seconds.

c
$$\int_0^5 |v(t)| dt = \int_0^5 |2\pi \cos \pi t| dt$$

$$= 20 \quad \{\text{using technology}\}$$

\therefore the total distance travelled by the spotlight in the first 5 seconds is 20 m.



8 a $v(t) = s'(t) = 2 \cos 4t \text{ m s}^{-1}$

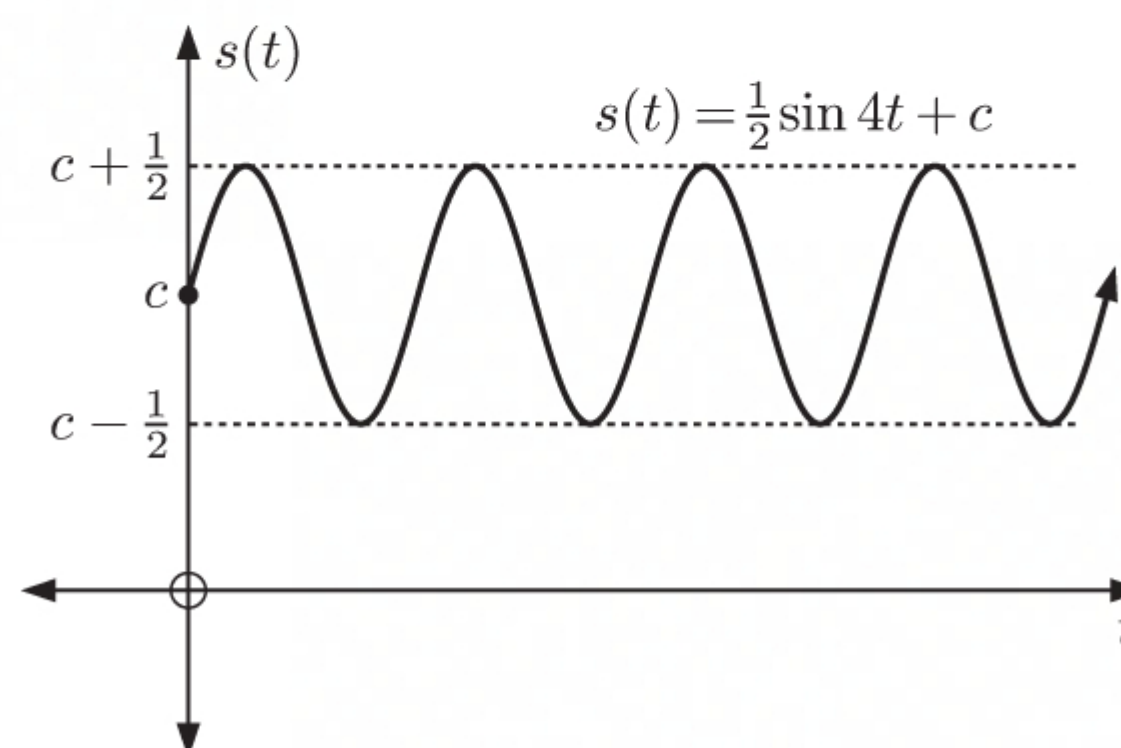
$$\therefore s(t) = \int 2 \cos 4t dt$$

$$= \frac{1}{2} \sin 4t + c \text{ m}$$

The graph shows that the particle oscillates between positions $c + \frac{1}{2}$ and $c - \frac{1}{2}$.

$$\text{Distance} = \left(c + \frac{1}{2}\right) - \left(c - \frac{1}{2}\right)$$

$$= 1 \text{ m}$$



b $s(t) = \frac{1}{2} \sin 4t + c$

$$s\left(\frac{\pi}{12}\right) = 6$$

$$\therefore \frac{1}{2} \sin \frac{\pi}{3} + c = 6$$

$$\therefore \frac{\sqrt{3}}{4} + c = 6$$

$$\therefore c = 6 - \frac{\sqrt{3}}{4}$$

$$s\left(\frac{\pi}{6}\right) = \frac{1}{2} \sin \frac{2\pi}{3} + 6 - \frac{\sqrt{3}}{4}$$

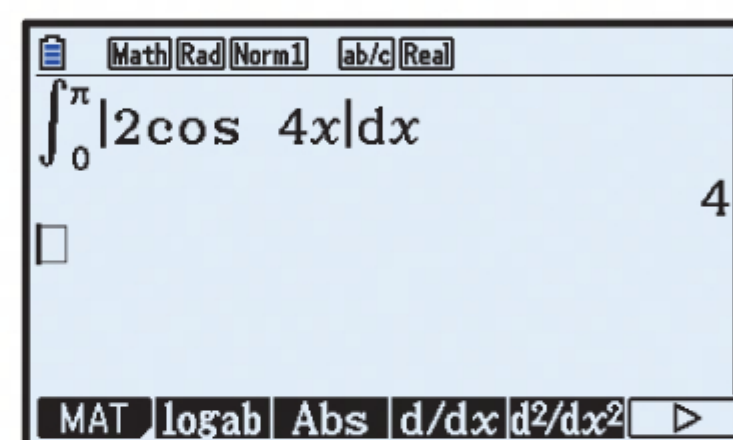
$$= \frac{\sqrt{3}}{4} + 6 - \frac{\sqrt{3}}{4}$$

$$= 6 \text{ m}$$

c
$$\int_0^\pi |v(t)| dt = \int_0^\pi |2 \cos 4t| dt$$

$$= 4 \quad \{\text{using technology}\}$$

\therefore the total distance travelled by the particle in the first π seconds is 4 m.



9 a $a(t) = -2 \text{ m s}^{-2}$

$$\begin{aligned}\therefore v(t) &= \int a(t) dt \\ &= \int -2 dt \\ &= -2t + c \text{ m s}^{-1}\end{aligned}$$

But $v(0) = 65$

$$\therefore c = 65$$

$$\therefore v(t) = -2t + 65 \text{ m s}^{-1}$$

c i $v(t) = 3$ when $-2t + 65 = 3$
 $\therefore 2t = 62$
 $\therefore t = 31$

\therefore it will take 31 seconds for the plane to reduce its speed to 3 m s^{-1} .

ii
$$\begin{aligned}\int_0^{31} v(t) dt &= \int_0^{31} (-2t + 65) dt \\ &= [-t^2 + 65t]_0^{31} \\ &= (-31^2 + 65(31)) - 0 \\ &= -961 + 2015 \\ &= 1054\end{aligned}$$

\therefore the plane will have travelled 1054 m along the runway after 31 seconds.

10 $v(t) = \frac{100}{(t+2)^2} \text{ m s}^{-1}$

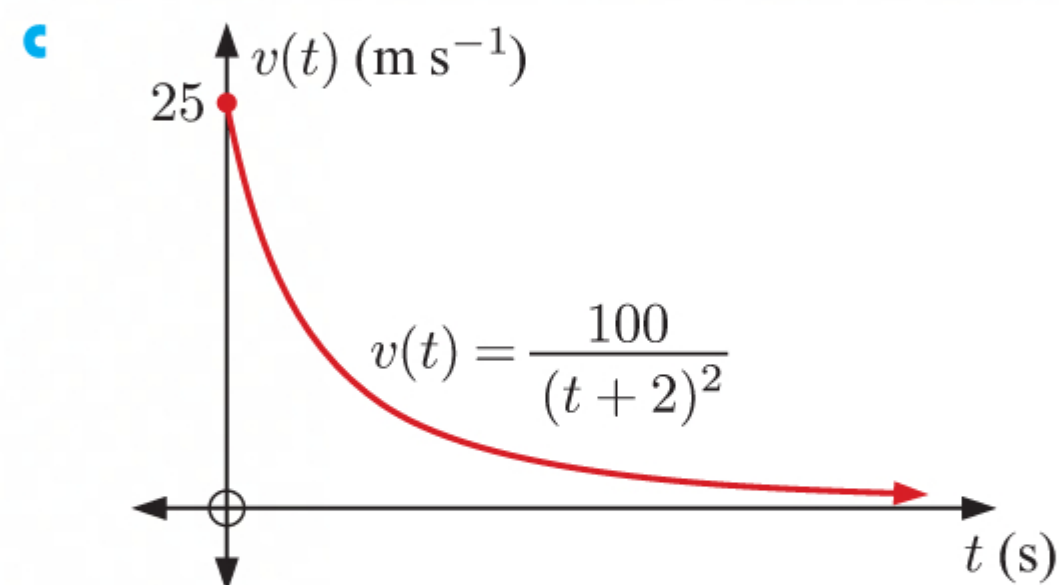
a $v(0) = \frac{100}{2^2} = \frac{100}{4} = 25$

The initial velocity of the boat was 25 m s^{-1} .

b As $t \rightarrow \infty$, $(t+2)^2 \rightarrow \infty$
 $\therefore v(t) \rightarrow 0$ from above

$v(3) = \frac{100}{5^2} = \frac{100}{25} = 4$

The velocity of the boat after 3 seconds was 4 m s^{-1} .



d
$$\begin{aligned}\int_0^2 v(t) dt &= \int_0^2 100(t+2)^{-2} dt \\ &= [-100(t+2)^{-1}]_0^2 \\ &= -\frac{100}{4} - \left(-\frac{100}{2}\right) \\ &= 25\end{aligned}$$

The boat travels a total distance of 25 m in the first 2 seconds after its engine is turned off.

e Suppose the boat travels 30 m after T seconds, then $\int_0^T (100(t+2)^{-2}) dt = 30$
 $\therefore [-100(t+2)^{-1}]_0^T = 30$
 $\therefore \left(-\frac{100}{T+2}\right) - \left(-\frac{100}{2}\right) = 30$
 $\therefore -\frac{100}{T+2} = -20$
 $\therefore 20T + 40 = 100$
 $\therefore 20T = 60$
 $\therefore T = 3$

So it will take 3 seconds for the boat to travel 30 metres.

REVIEW SET 18B

1 a $s(t) = t^2 + 4t + 1$ m, $t \geq 0$ s

$s(0) = 1$ m

\therefore the object is initially 1 m to the right of the origin.

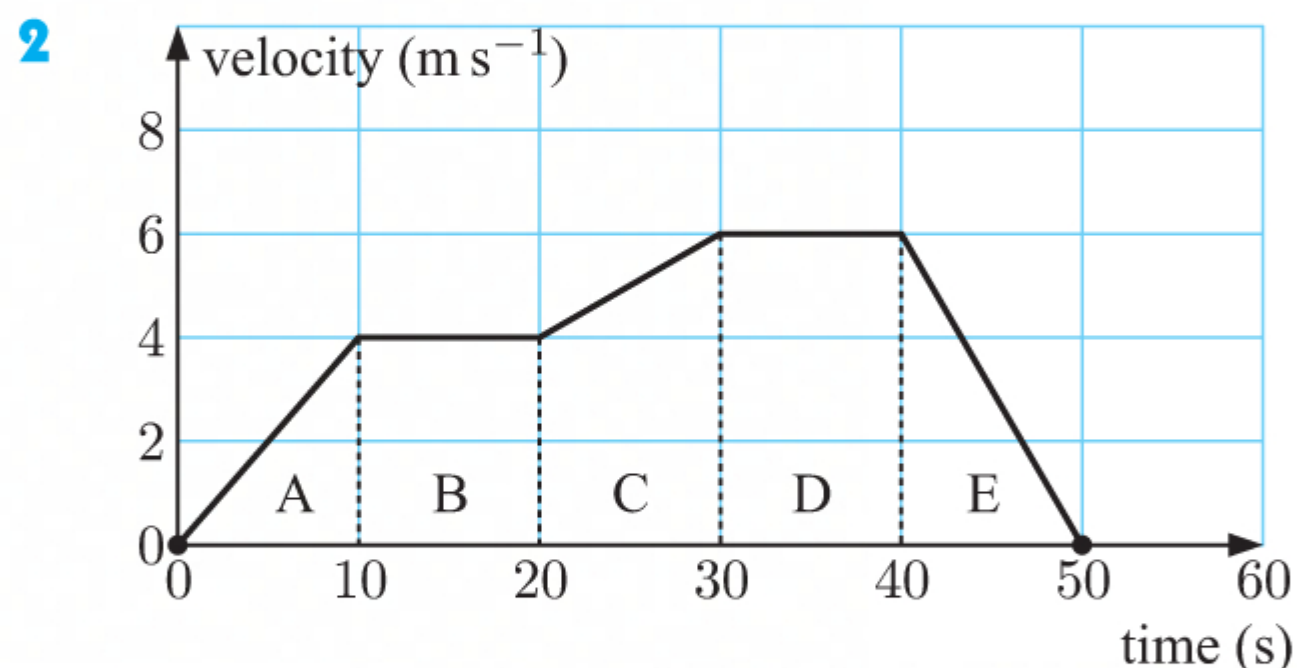
b average velocity $= \frac{s(3) - s(1)}{3 - 1}$
 $= \frac{3^2 + 4(3) + 1 - (1^2 + 4(1) + 1)}{2}$
 $= \frac{22 - 6}{2}$
 $= 8 \text{ m s}^{-1}$

\therefore the average velocity from $t = 1$ to $t = 3$ seconds is 8 m s^{-1} .

c $v(t) = s'(t) = 2t + 4 \text{ m s}^{-1}$

d $v(1) = 2(1) + 4$
 $= 2 + 4$
 $= 6 \text{ m s}^{-1}$

\therefore the instantaneous velocity at $t = 1$ second is 6 m s^{-1} .



Total distance travelled = total area under graph

$= \text{area A} + \text{area B} + \text{area C} + \text{area D} + \text{area E}$
 $= \frac{1}{2}(10)(4) + (10)(4) + \left(\frac{6+4}{2}\right)(10) + (10)(6) + \frac{1}{2}(10)(6)$
 $= 20 + 40 + 50 + 60 + 30$
 $= 200 \text{ metres}$

$$\begin{aligned}
 \mathbf{3} \quad \mathbf{a} \quad v(t) &= \int a(t) dt \\
 &= \int (6t - 30) dt \\
 &= 3t^2 - 30t + c
 \end{aligned}$$

$$\begin{aligned}
 \text{Now } v(0) &= 27 \\
 \therefore c &= 27 \\
 \therefore v(t) &= 3t^2 - 30t + 27 \text{ cm s}^{-1}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad s(t) &= \int v(t) dt \\
 &= \int (3t^2 - 30t + 27) dt \\
 &= t^3 - 15t^2 + 27t + c
 \end{aligned}$$

$$\begin{aligned}
 \text{Now } s(0) &= 0 \\
 \therefore c &= 0 \\
 \therefore s(t) &= t^3 - 15t^2 + 27t \text{ cm} \\
 \therefore s(6) &= 6^3 - 15(6)^2 + 27(6) \\
 &= 216 - 540 + 162 \\
 &= -162
 \end{aligned}$$

\therefore the particle is 162 cm to the left of the origin after 6 seconds.

$$\begin{aligned}
 \mathbf{4} \quad \mathbf{a} \quad x(t) &= 3t - t\sqrt{t} \text{ cm}, \quad t \geq 0 \text{ s} \\
 &= 3t - t^{\frac{3}{2}}
 \end{aligned}$$

$$\begin{aligned}
 \therefore v(t) &= 3 - \frac{3}{2}t^{\frac{1}{2}} \text{ cm s}^{-1} & \{v(t) = x'(t)\} \\
 &= 3 - \frac{3}{2}\sqrt{t} \text{ cm s}^{-1}
 \end{aligned}$$

$$\begin{aligned}
 \therefore a(t) &= -\frac{3}{4}t^{-\frac{1}{2}} \text{ cm s}^{-2} & \{a(t) = v'(t)\} \\
 &= -\frac{3}{4\sqrt{t}} \text{ cm s}^{-2}
 \end{aligned}$$

$v(t)$ has sign diagram:

$a(t)$ has sign diagram:

$$\mathbf{b} \quad x(0) = 0 \text{ cm} \quad v(0) = 3 \text{ cm s}^{-1}$$

The particle is initially at the origin, moving to the right at 3 cm s^{-1} .

$$\begin{aligned}
 \mathbf{c} \quad x(2) &= 3(2) - 2\sqrt{2} & v(2) &= 3 - \frac{3}{2}\sqrt{2} & a(2) &= -\frac{3}{4\sqrt{2}} \\
 &= 6 - 2\sqrt{2} & &\approx 0.879 \text{ cm s}^{-1} & &\approx -0.530 \text{ cm s}^{-2} \\
 &\approx 3.17 \text{ cm}
 \end{aligned}$$

So, at time $t = 2$ seconds, the particle is about 3.17 cm to the right of the origin, travelling to the right at about 0.879 cm s^{-1} , with decreasing speed ($a(2) \approx -0.530 \text{ cm s}^{-2}$).

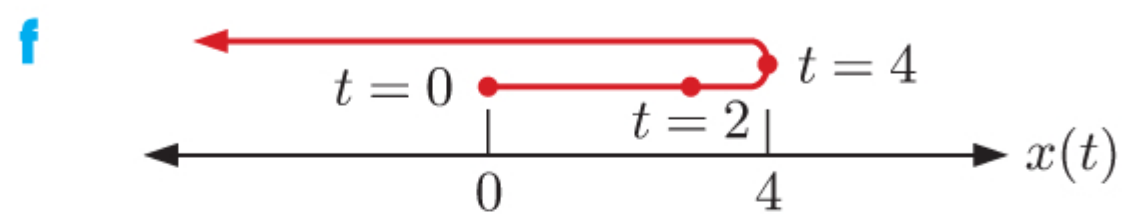
\mathbf{d} The particle reverses direction when the sign of $v(t)$ changes.

This occurs at $t = 4$ seconds.

$$\begin{aligned}
 x(4) &= 3(4) - 4\sqrt{4} \\
 &= 12 - 8 \\
 &= 4 \text{ cm}
 \end{aligned}$$

\therefore the particle reverses direction at $t = 4$ seconds, when it is 4 cm to the right of the origin.

- e** The particle's speed is decreasing when $v(t)$ and $a(t)$ have opposite sign.
From the sign diagrams in **a**, this occurs when $0 \leq t \leq 4$ seconds.



g
$$\begin{aligned} x(6) &= 3(6) - 6\sqrt{6} \\ &= 18 - 6\sqrt{6} \\ &\approx 3.30 \text{ cm} \end{aligned}$$

Total distance travelled in first 6 seconds
 $\approx 4 + (4 - 3.30)$
 $\approx 4.70 \text{ cm}$

5 a
$$\begin{aligned} v(t) &= 4.8t^2 - 0.8t^3 \text{ m s}^{-1}, \quad 0 \leq t \leq 6 \text{ s} \\ a(t) &= 9.6t - 2.4t^2 \text{ m s}^{-2} \quad \{a(t) = v'(t)\} \end{aligned}$$

i
$$\begin{aligned} a(1) &= 9.6 - 2.4 \\ &= 7.2 \text{ m s}^{-2} \end{aligned}$$

\therefore the acceleration of the human cannonball after 1 second is 7.2 m s^{-2} .

iii
$$\begin{aligned} a(4) &= 9.6(4) - 2.4(4)^2 \\ &= 38.4 - 38.4 \\ &= 0 \text{ m s}^{-2} \end{aligned}$$

\therefore the acceleration of the human cannonball after 4 seconds is 0 m s^{-2} .

ii
$$\begin{aligned} a(2) &= 9.6(2) - 2.4(2)^2 \\ &= 19.2 - 9.6 \\ &= 9.6 \text{ m s}^{-2} \end{aligned}$$

\therefore the acceleration of the human cannonball after 2 seconds is 9.6 m s^{-2} .

iv
$$\begin{aligned} a(5) &= 9.6(5) - 2.4(5)^2 \\ &= 48 - 60 \\ &= -12 \text{ m s}^{-2} \end{aligned}$$

\therefore the acceleration of the human cannonball after 5 seconds is -12 m s^{-2} .

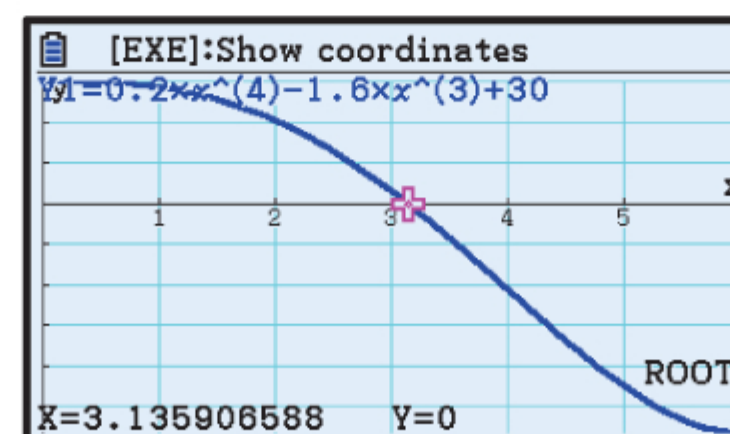
b
$$\begin{aligned} \int_0^3 v(t) dt &= \int_0^3 (4.8t^2 - 0.8t^3) dt \\ &= [1.6t^3 - 0.2t^4]_0^3 \\ &= (1.6(3)^3 - 0.2(3)^4) - 0 \\ &= 27 \end{aligned}$$

The human cannonball travels 27 m in the first 3 seconds.

- c** Suppose the human cannonball has travelled 30 m after T seconds, then

$$\begin{aligned} \int_0^T v(t) dt &= 30 \\ \therefore \int_0^T (4.8t^2 - 0.8t^3) dt &= 30 \\ \therefore [1.6t^3 - 0.2t^4]_0^T &= 30 \\ \therefore 1.6T^3 - 0.2T^4 &= 30 \\ \therefore 0.2T^4 - 1.6T^3 + 30 &= 0 \\ \therefore T &\approx 3.14 \quad \{0 \leq T \leq 6\} \\ &\quad \{\text{using technology}\} \end{aligned}$$

\therefore it takes about 3.14 seconds for the human cannonball to travel 30 m.



6 a $s(t) = 80e^{-\frac{t}{10}} - 40t \text{ m}, \quad t \geq 0 \text{ s}$

$$\therefore v(t) = -8e^{-\frac{t}{10}} - 40 \text{ m s}^{-1} \quad \{v(t) = s'(t)\}$$

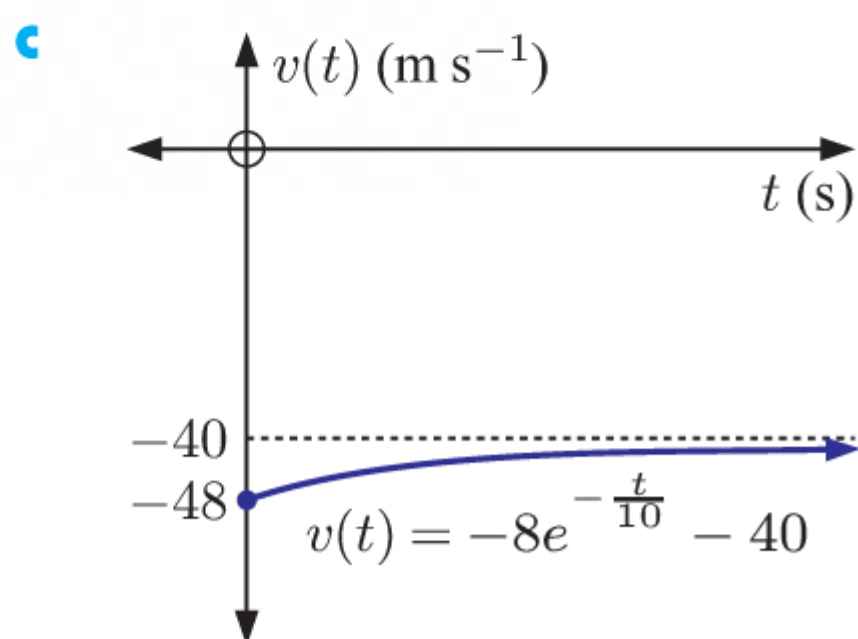
$$\therefore a(t) = \frac{8}{10}e^{-\frac{t}{10}} = \frac{4}{5}e^{-\frac{t}{10}} \text{ m s}^{-2} \quad \{a(t) = v'(t)\}$$

b When $t = 0$, $s(0) = 80 \text{ m}$

$$v(0) = -8 - 40 = -48 \text{ m s}^{-1}$$

$$a(0) = \frac{4}{5} \text{ m s}^{-2}$$

\therefore the particle is initially 80 m to the right of the origin, moving to the left at 48 m s^{-1} with acceleration 0.8 m s^{-2} .



d When $v = -44$, $-8e^{-\frac{t}{10}} - 40 = -44$

$$\therefore -8e^{-\frac{t}{10}} = -4$$

$$\therefore e^{-\frac{t}{10}} = \frac{1}{2}$$

$$\therefore -\frac{t}{10} = \ln\left(\frac{1}{2}\right)$$

$$\therefore t = 10 \ln 2$$

\therefore the particle P has velocity -44 m s^{-1} at $t = 10 \ln 2$ seconds.

7 a $s(t) = 30 + \cos \pi t \text{ cm}, \quad t \geq 0 \text{ s}$

$$\therefore v(t) = -\pi \sin \pi t \text{ cm s}^{-1} \quad \{v(t) = s'(t)\}$$

$$v(0) = -\pi \sin 0 = 0 \text{ cm s}^{-1},$$

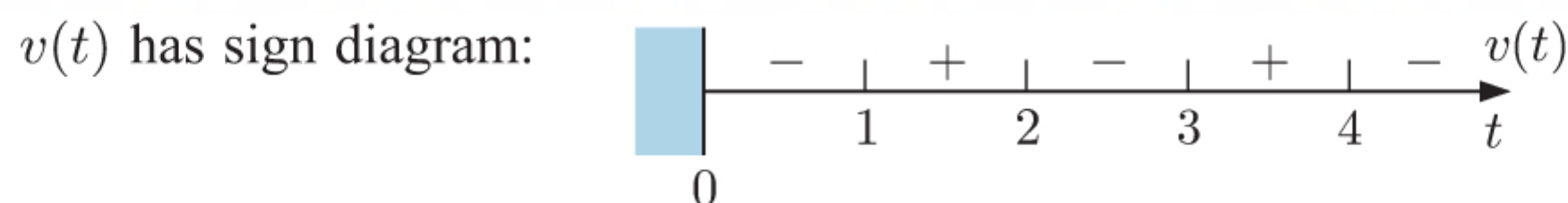
$$v\left(\frac{1}{2}\right) = -\pi \sin \frac{\pi}{2} = -\pi \text{ cm s}^{-1},$$

$$v(1) = -\pi \sin \pi = 0 \text{ cm s}^{-1},$$

$$v\left(1\frac{1}{2}\right) = -\pi \sin \frac{3\pi}{2} = \pi \text{ cm s}^{-1}$$

$$v(2) = -\pi \sin 2\pi = 0 \text{ cm s}^{-1}$$

b The cork is falling when its velocity is negative.



$v(t)$ is negative when $0 \leq t \leq 1$, $2 \leq t \leq 3$, $4 \leq t \leq 5$, and so on.

So, the cork is falling when $2n \leq t \leq 2n + 1$, $n \in \{0, 1, 2, 3, \dots\}$

$$8 \quad a \quad v(t) = \frac{(t^{1.1} + 3t)^{1.5}}{10} \text{ m s}^{-1}$$

$$v(4) = \frac{(4^{1.1} + 3(4))^{1.5}}{10} \\ \approx 6.76 \text{ m s}^{-1}$$

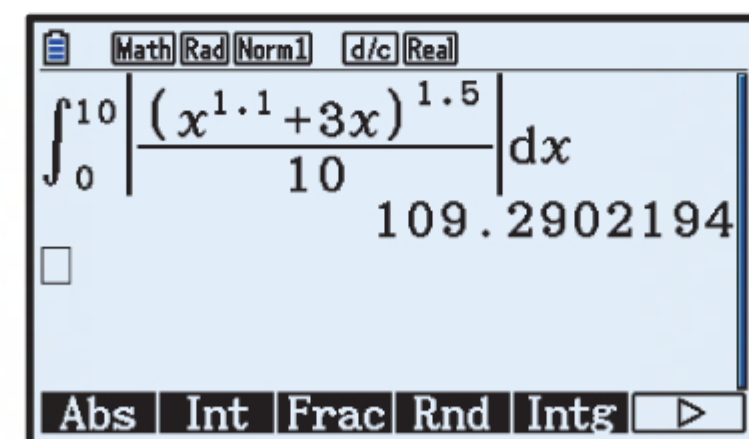
$$b \quad a(t) = v'(t) \\ = \frac{1}{10} \times 1.5(t^{1.1} + 3t)^{0.5}(1.1t^{0.1} + 3) \quad \{\text{chain rule}\} \\ = 0.15(t^{1.1} + 3t)^{0.5}(1.1t^{0.1} + 3) \text{ m s}^{-2}$$

$$c \quad a(2) = 0.15(2^{1.1} + 3(2))^{0.5}(1.1(2)^{0.1} + 3) \\ \approx 1.79 \text{ m s}^{-2}$$

\therefore the acceleration of the skier after 2 seconds is about 1.79 m s^{-2} .

d Total distance travelled in first 10 seconds

$$= \int_0^{10} |v(t)| dt \\ = \int_0^{10} \left| \frac{(t^{1.1} + 3t)^{1.5}}{10} \right| dt \\ \approx 109 \text{ m} \quad \{\text{using technology}\}$$



$$9 \quad a \quad v(t) = -\frac{1}{24}t^3 - \frac{1}{12}t \text{ m s}^{-1}$$

$$s(t) = \int v(t) dt \\ = \int \left(-\frac{1}{24}t^3 - \frac{1}{12}t\right) dt \\ = -\frac{1}{96}t^4 - \frac{1}{24}t^2 + c \text{ m}$$

$$\text{But } s(0) = 2$$

$$\therefore c = 2$$

$$\therefore s(t) = -\frac{1}{96}t^4 - \frac{1}{24}t^2 + 2 \text{ m}$$

b The feather is on the ground when

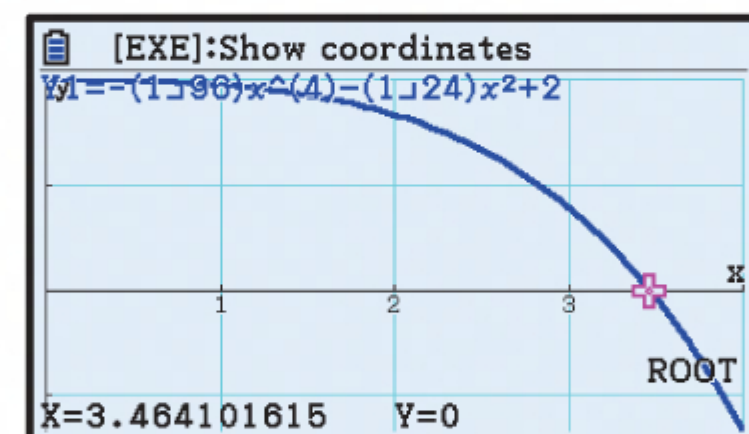
$$s(t) = 0$$

$$\therefore -\frac{1}{96}t^4 - \frac{1}{24}t^2 + 2 = 0$$

$$\therefore t \approx 3.46 \text{ or } -3.46 \quad \{\text{using technology}\}$$

$$\therefore t \approx 3.46 \quad \{t \geq 0\}$$

\therefore it takes about 3.46 seconds for the feather to reach the ground.



$$10 \quad a \quad \text{After 2 seconds, } v_1(2) = 10(1 - e^{-1.25(2)}) \quad v_2(2) = 10.5(1 - e^{-2}) \\ \approx 9.18 \text{ m s}^{-1} \quad \approx 9.08 \text{ m s}^{-1}$$

\therefore Tyson is running faster after 2 seconds.

$$\begin{aligned}
 \text{b } \int_0^5 v_1(t) dt &= \int_0^5 10(1 - e^{-1.25t}) dt \\
 &= \int_0^5 (10 - 10e^{-1.25t}) dt \\
 &= [10t + 8e^{-1.25t}]_0^5 \\
 &\approx (50 + 0.015) - (8) \\
 &\approx 42.0
 \end{aligned}$$

Tyson has travelled about 42.0 m in the first 5 seconds of the race.

$$\begin{aligned}
 \text{c } s_1(t) &= \int v_1(t) dt & s_2(t) &= \int v_2(t) dt \\
 &= \int 10(1 - e^{-1.25t}) dt & &= \int 10.5(1 - e^{-t}) dt \\
 &= \int (10 - 10e^{-1.25t}) dt & &= \int (10.5 - 10.5e^{-t}) dt \\
 &= 10t + 8e^{-1.25t} + c & &= 10.5t + 10.5e^{-t} + c
 \end{aligned}$$

Now $s_1(0) = 0$ Now $s_2(0) = 0$
 $\therefore 0 + 8e^0 + c = 0$ $\therefore 0 + 10.5e^0 + c = 0$
 $\therefore 8 + c = 0$ $\therefore 10.5 + c = 0$
 $\therefore c = -8$ $\therefore c = -10.5$

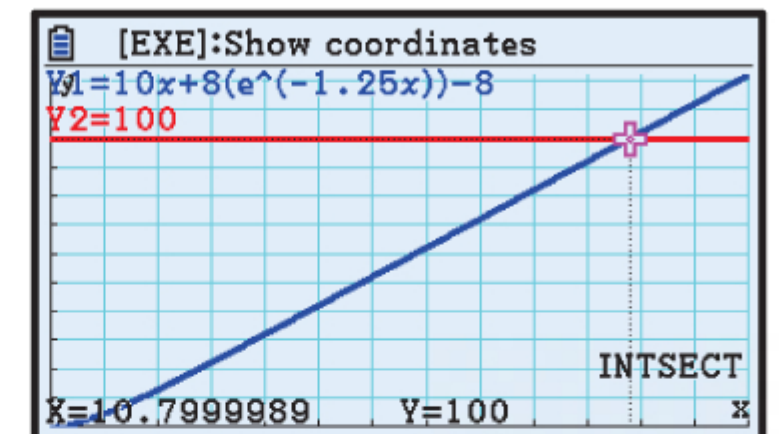
$\therefore s_1(t) = 10t + 8e^{-1.25t} - 8 \text{ m}$ $\therefore s_2(t) = 10.5t + 10.5e^{-t} - 10.5 \text{ m}$

d After 3 seconds, $s_1(3) = 10(3) + 8e^{-1.25(3)} - 8 \approx 22.2 \text{ m}$
 $s_2(3) = 10.5(3) + 10.5e^{-3} - 10.5 \approx 21.5 \text{ m}$
 \therefore Tyson is winning the race after 3 seconds.

e When Tyson completes the race,

$$\begin{aligned}
 s_1(t) &= 100 \\
 \therefore 10t + 8e^{-1.25t} - 8 &= 100 \\
 \therefore t &\approx 10.8 \quad \{\text{using technology}\}
 \end{aligned}$$

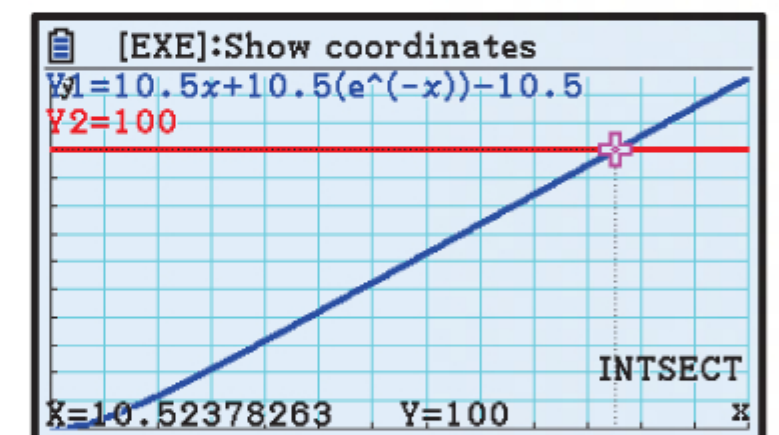
\therefore Tyson completes the race in approximately 10.8 seconds.



f Likewise, when Maurice completes the race,

$$\begin{aligned}
 s_2(t) &= 100 \\
 \therefore 10.5t + 10.5e^{-t} - 10.5 &= 100 \\
 \therefore t &\approx 10.5 \quad \{\text{using technology}\}
 \end{aligned}$$

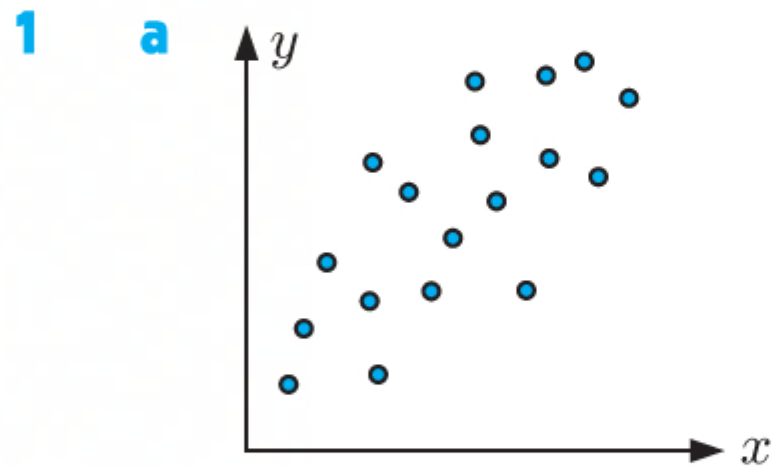
\therefore Maurice completes the race in approximately 10.5 seconds, so Maurice wins the race.



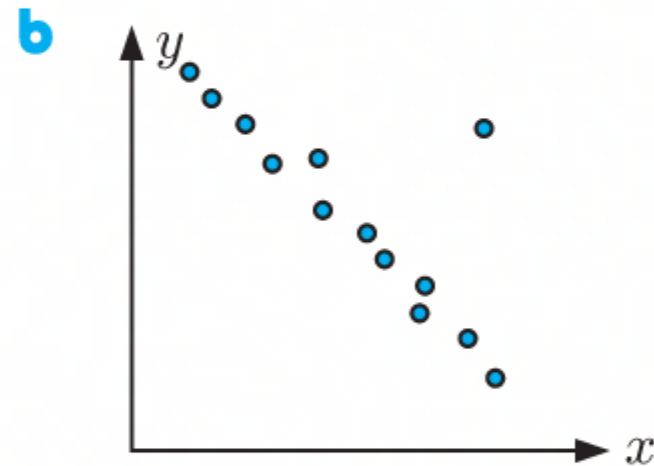
Chapter 19

BIVARIATE STATISTICS

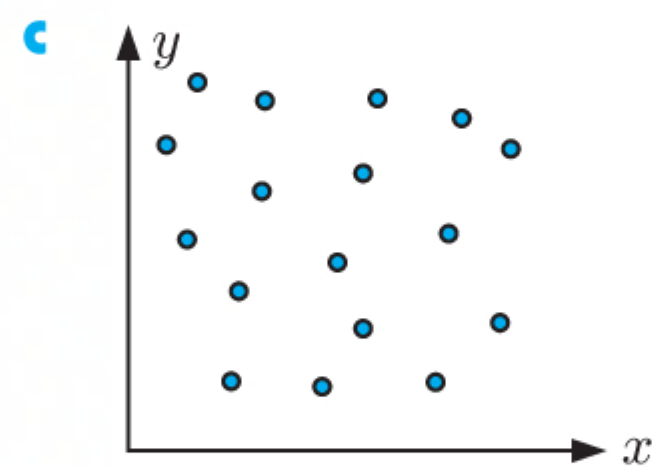
EXERCISE 19A



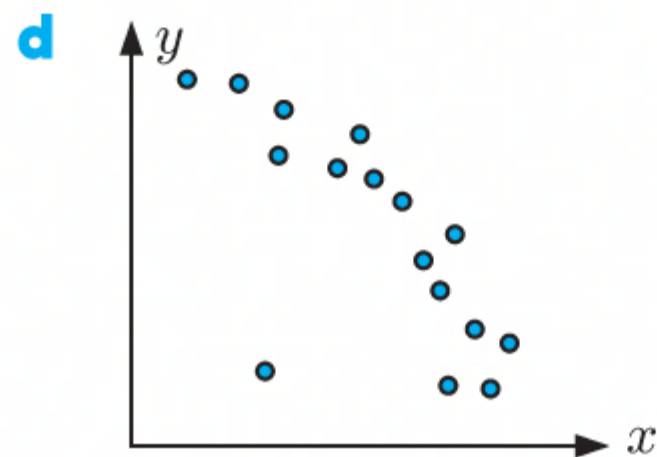
There is a weak, positive, linear correlation with no outliers.



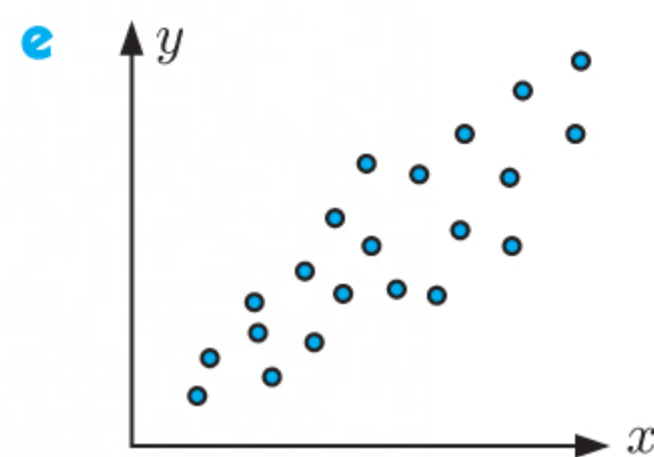
There is a strong, negative, linear correlation with one outlier.



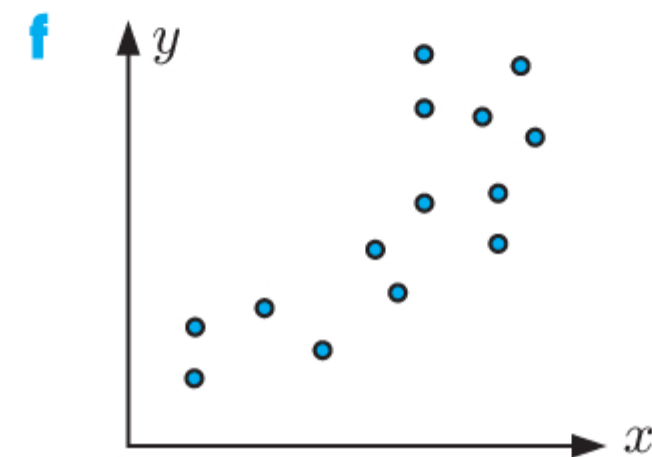
There is no correlation.



There is a strong, negative, non-linear correlation with one outlier.



There is a moderate, positive, linear correlation with no outliers.

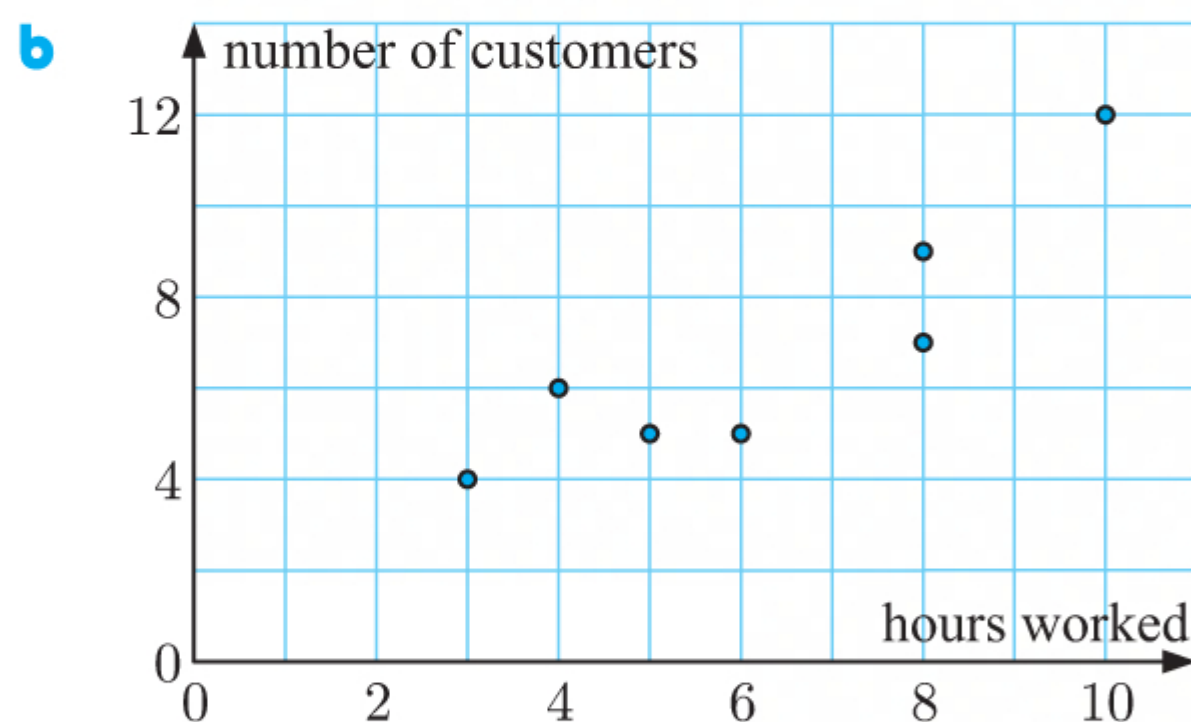


There is a weak, positive, non-linear correlation with no outliers.

2

Day	Mon	Tue	Wed	Thu	Fri	Sat	Sun
Hours worked	8	4	5	10	8	3	6
Number of customers	9	6	5	12	7	4	5

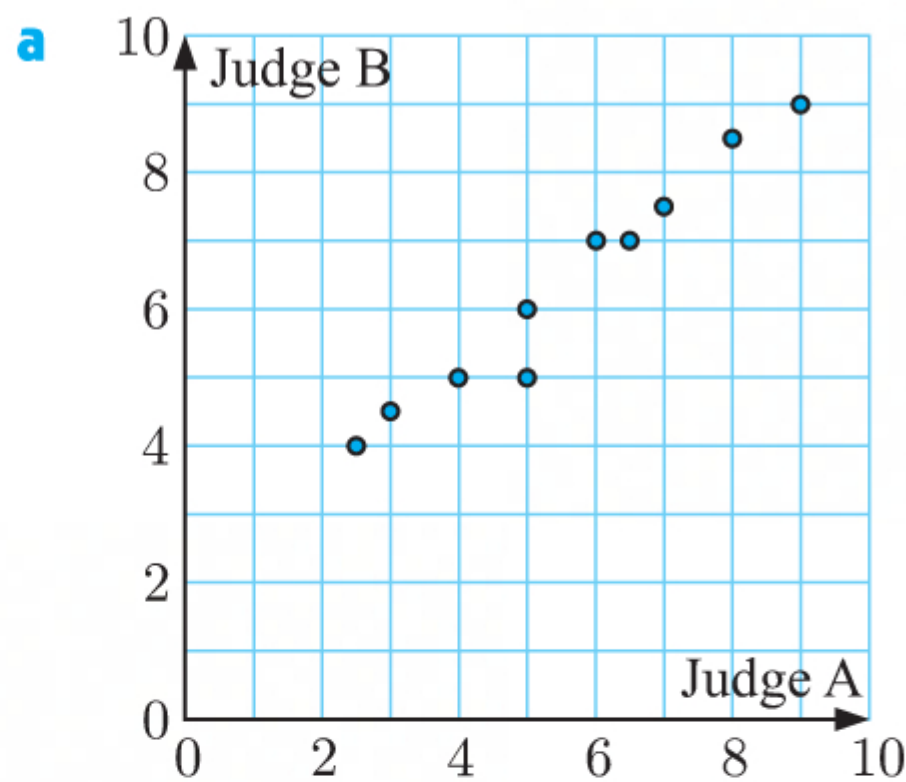
- a** *Hours worked* is the explanatory variable.
Number of customers is the response variable.



- c** **i** Tiffany worked the same number of hours (8 hours) on Monday and Friday.
ii Tiffany had the same number of customers (5 customers) on Wednesday and Sunday.
d The more hours that Tiffany works, the more customers she is likely to have, so we would expect a positive correlation between the variables.

3

Competitor	P	Q	R	S	T	U	V	W	X	Y
Judge A	5	6.5	8	9	4	2.5	7	5	6	3
Judge B	6	7	8.5	9	5	4	7.5	5	7	4.5

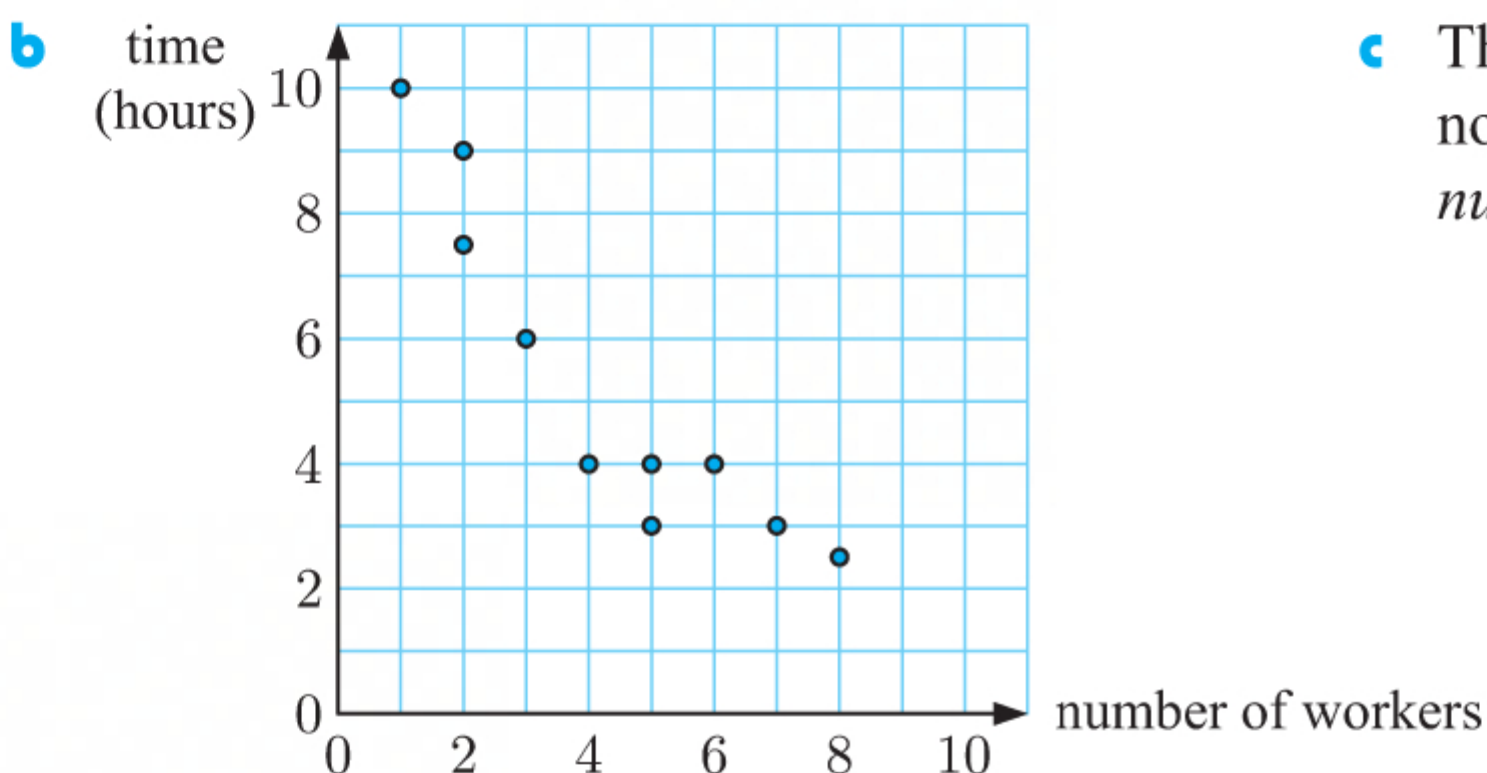


- b There appears to be **strong, positive, linear** correlation between Judge A's scores and Judge B's scores. This means that as Judge A's scores increase, Judge B's scores **increase**.
- c No, an increase in Judge A's scores are not likely to cause an increase in Judge B's scores. It is much more likely that both scores are related to the quality of the ice skaters' performances.

4

Job	A	B	C	D	E	F	G	H	I	J
Number of workers	5	3	8	2	5	6	1	4	2	7
Time (hours)	4	6	2.5	9	3	4	10	4	7.5	3

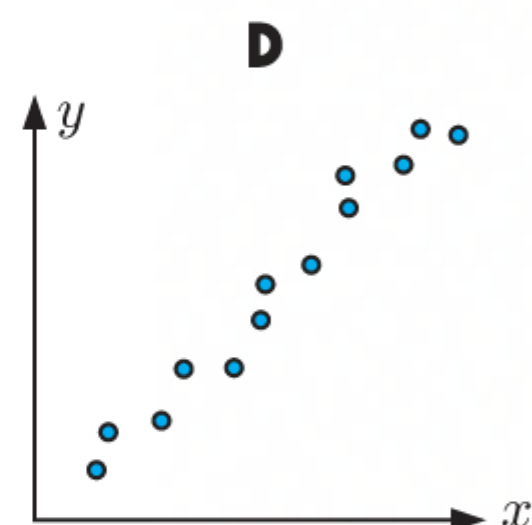
- a
- Job G took the longest to complete (10 hours).
 - Job C involved the most workers (8 workers).



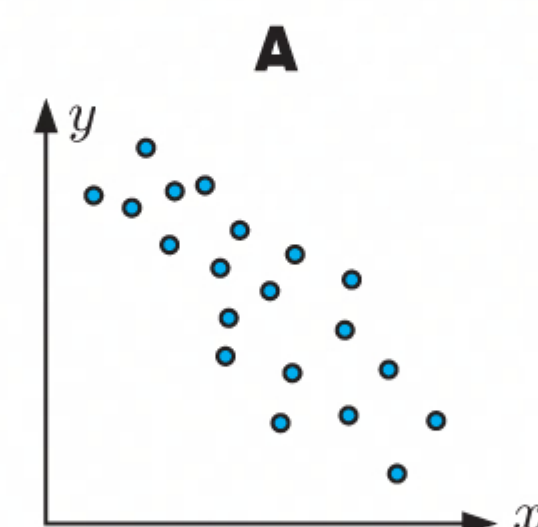
- c There is a strong, negative, non-linear correlation between the *number of workers* and *time*.

5

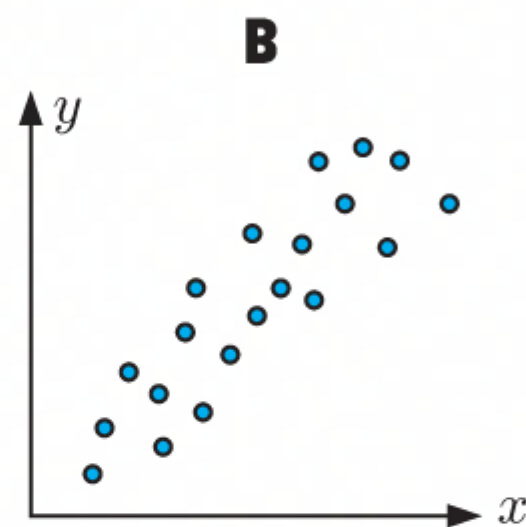
- a x = the number of apples bought by customers
 y = the total cost of apples bought
 We expect strong, positive, linear correlation. This corresponds to **D**.



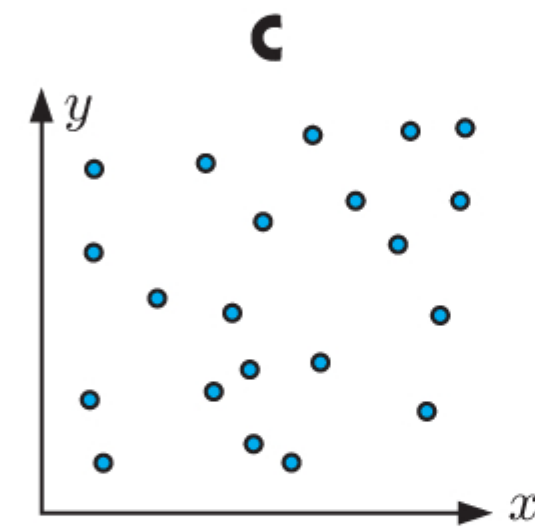
- b x = the number of pushups a student can perform in one minute
 y = the time taken for a student to run 100 metres
 We expect moderate, negative, linear correlation. This corresponds to **A**.



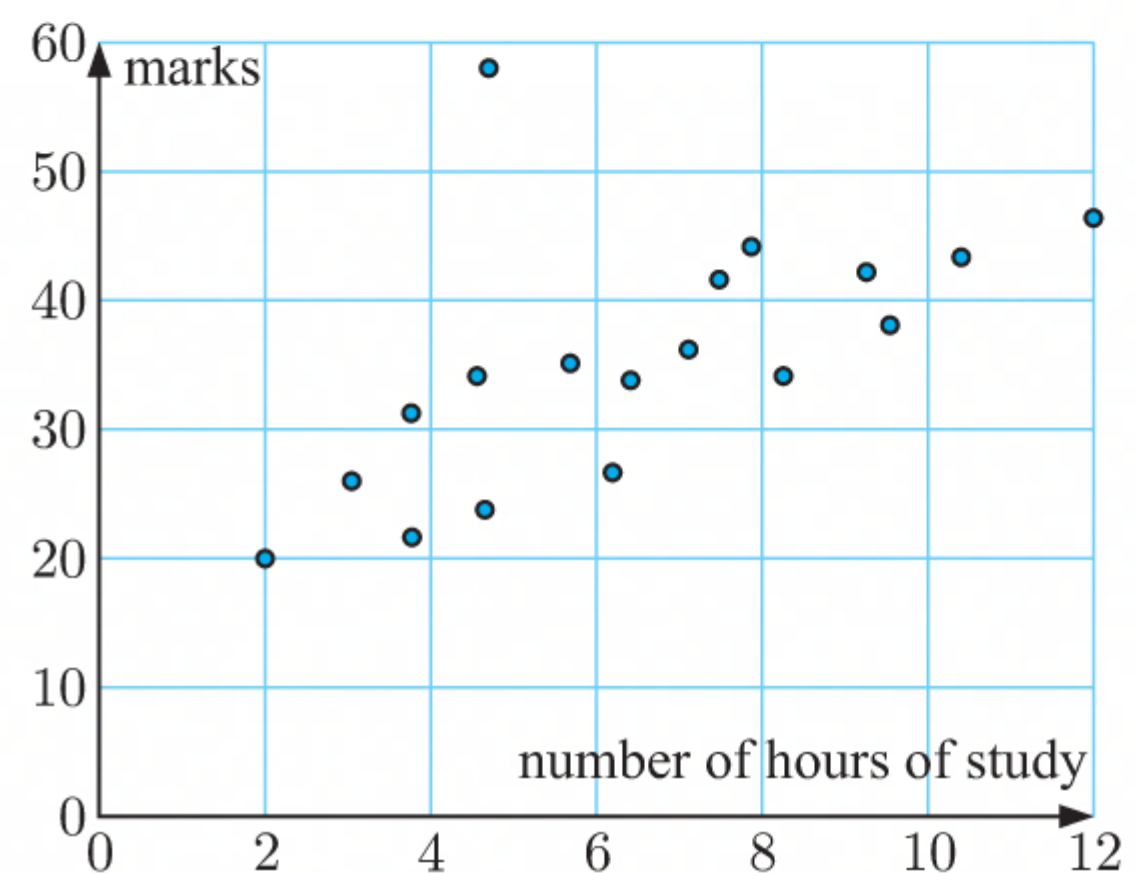
- c** x = the height of a person
 y = the weight of the person
 We expect moderate, positive, linear correlation. This corresponds to **B**.



- d** x = the distance a student travels to school
 y = the height of the student's uncle
 We expect no correlation. This corresponds to **C**.



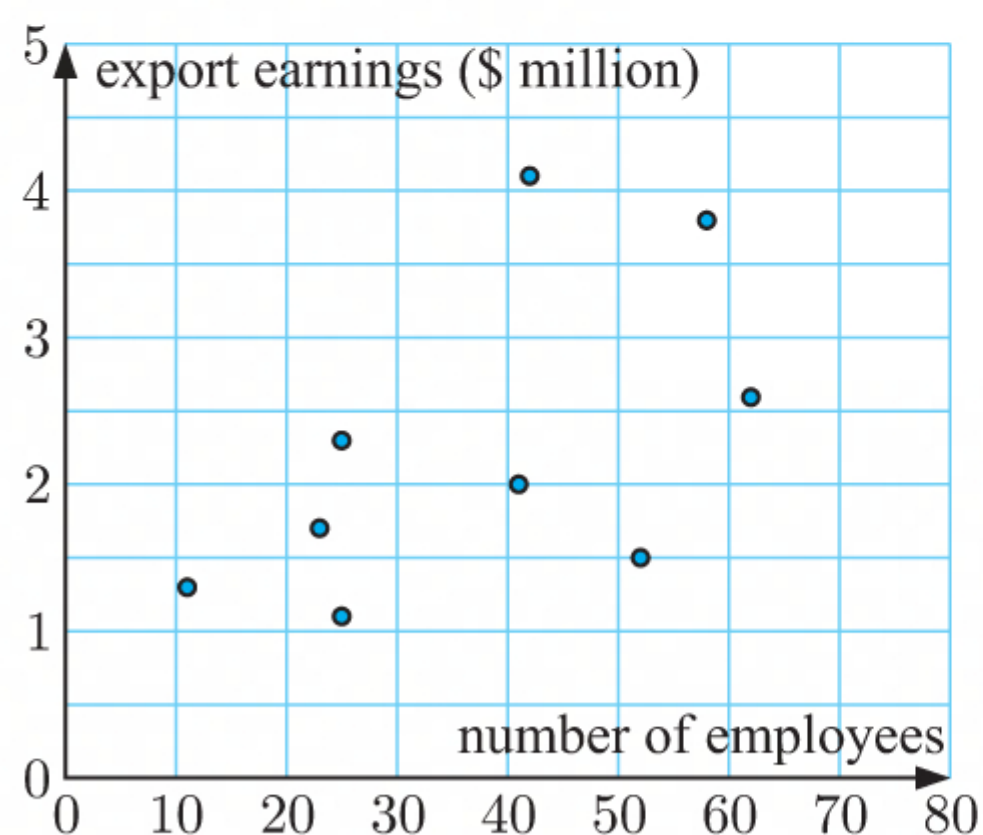
- 6** **a** There is a moderate, positive, linear correlation between the *number of hours of study* and the *marks obtained*.
b As the test is out of 50 marks and the outlier is greater than 50, we can assume it is an error and discard it.
c Yes, this is a causal relationship as spending more time studying for the test is likely to cause a higher mark.

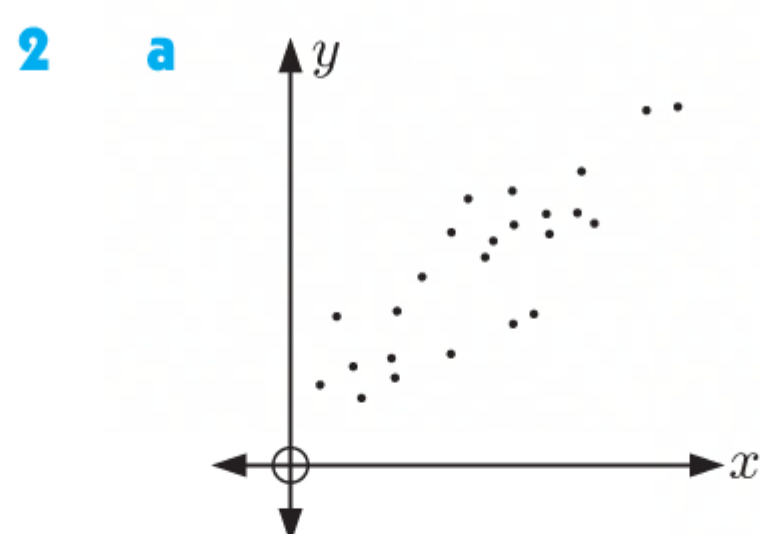


- 7** **a** Not causal, dependent on genetics and/or age.
b Not causal, dependent on the size of the fire.
c Causal, an increase in advertising is likely to cause an increase in sales.
d Causal, the childrens' adult height is determined by the genetics inherited from their parents to a great extent.
e Not causal, dependent on the population of the towns or the number of tourists.

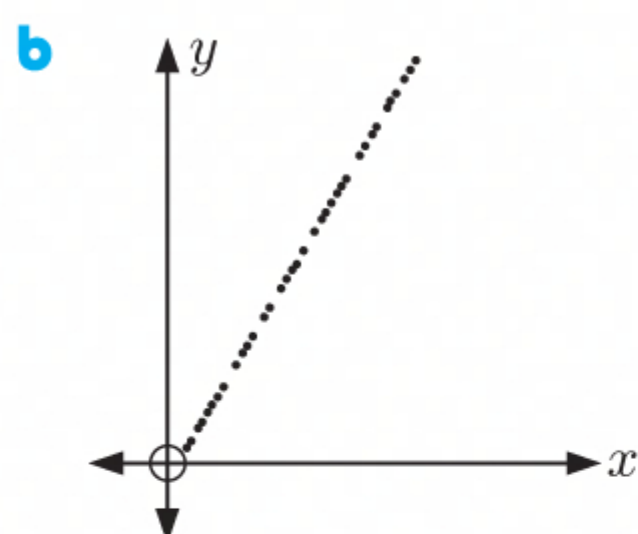
EXERCISE 19B

- 1** $r = 0.556$
 There is a weak, positive correlation between the *number of employees of a company* and its *export earnings*.

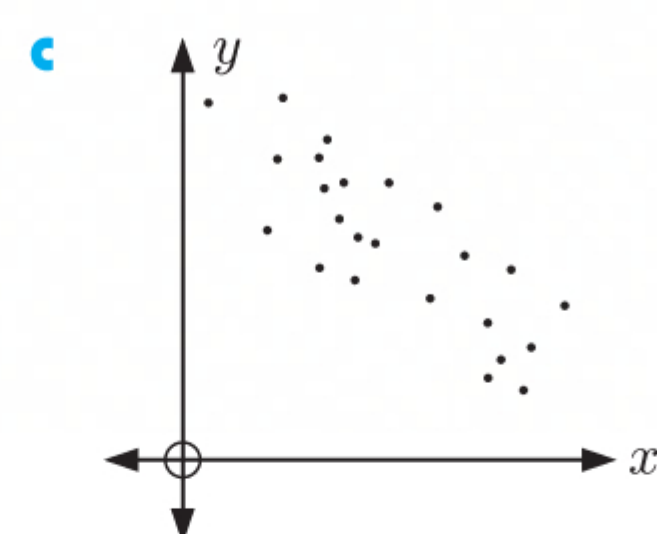




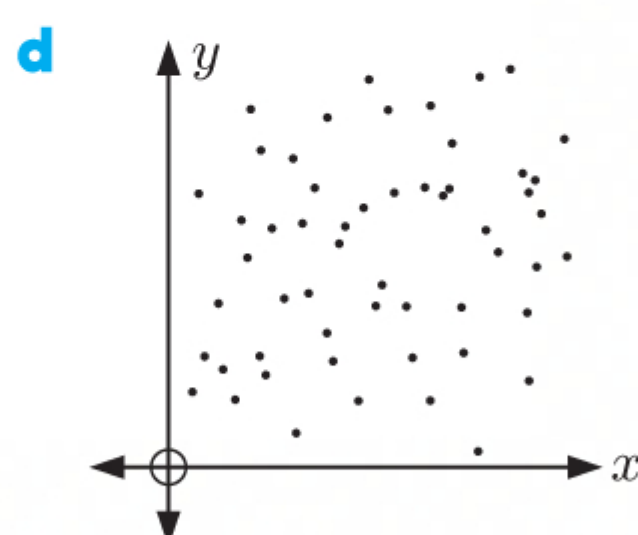
B $r = 0.6$



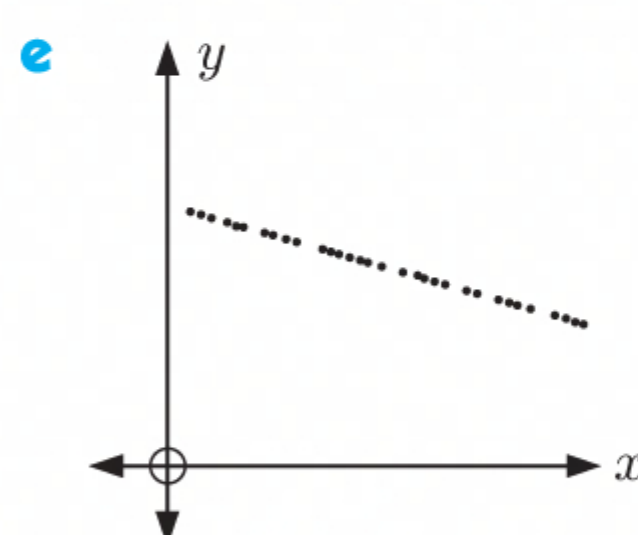
A $r = 1$



D $r = -0.7$



C $r = 0$

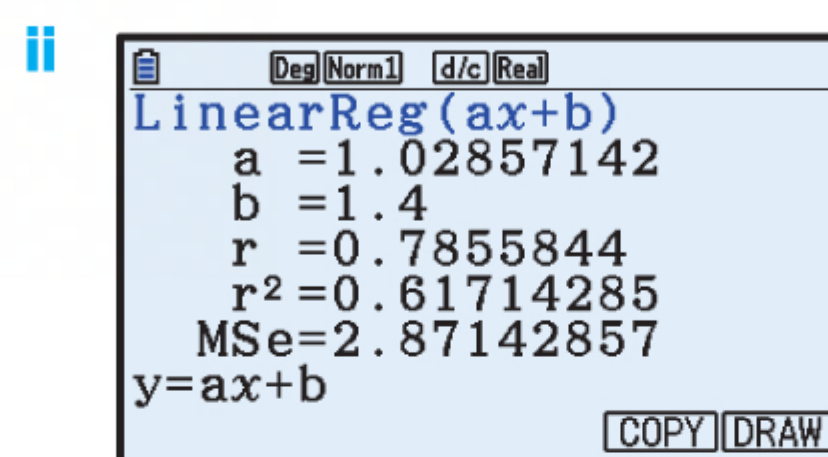
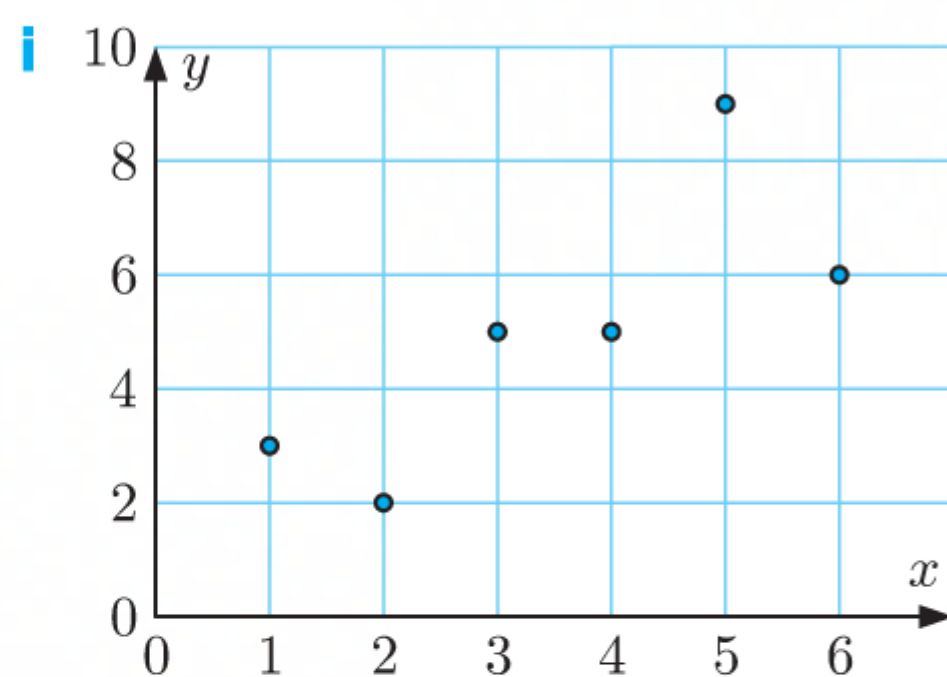
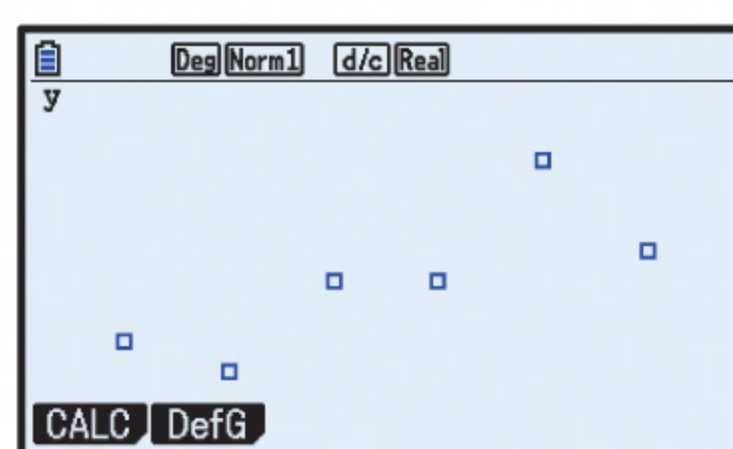


E $r = -1$

3 a

x	1	2	3	4	5	6
y	3	2	5	5	9	6

	List 1	List 2	List 3	List 4
SUB				
1	1	3		
2	2	2		
3	3	5		
4	4	5		



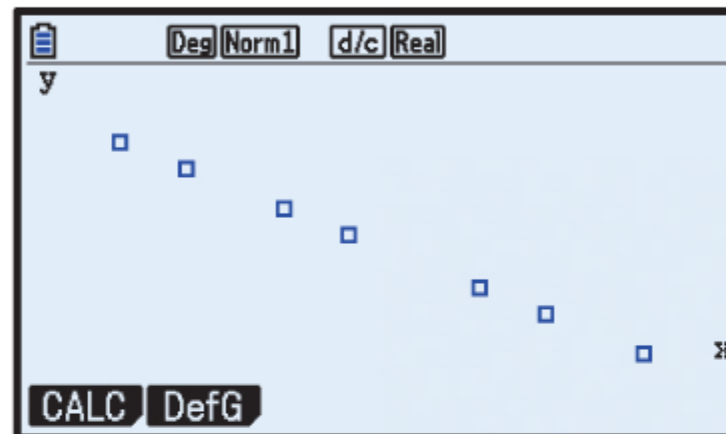
So, $r \approx 0.786$.

iii There is a moderate, positive correlation between x and y .

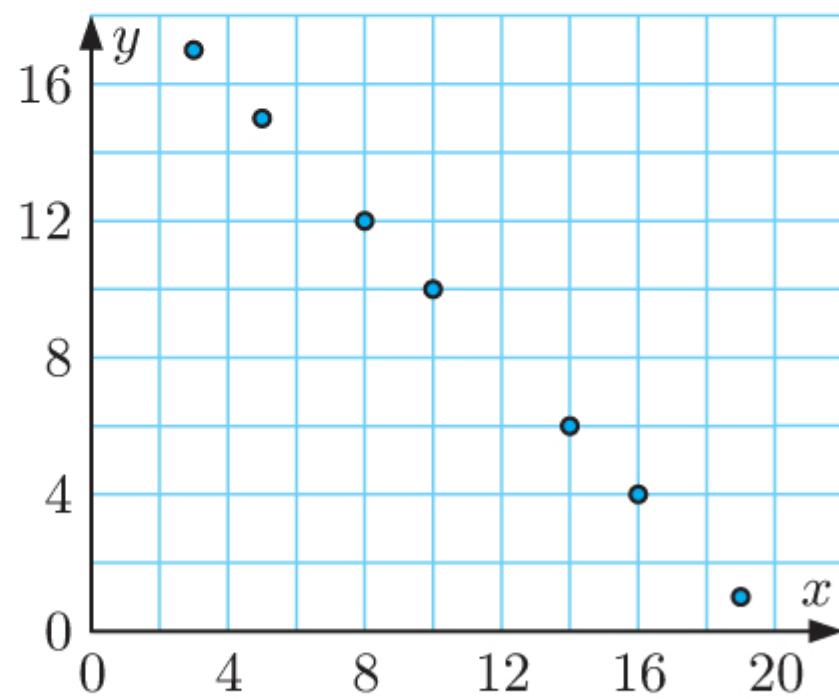
b

x	3	8	5	14	19	10	16
y	17	12	15	6	1	10	4

	List 1	List 2	List 3	List 4
SUB				
1	3	17		
2	8	12		
3	5	15		
4	14	6		



i



ii

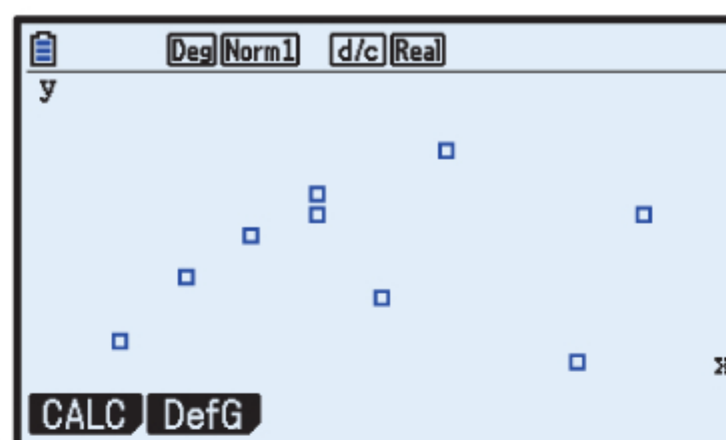
	List 1	List 2	List 3	List 4
SUB				
1	3	17		
2	8	12		
3	5	15		
4	14	6		

So, $r = -1$.iii There is a perfect, negative correlation between x and y .

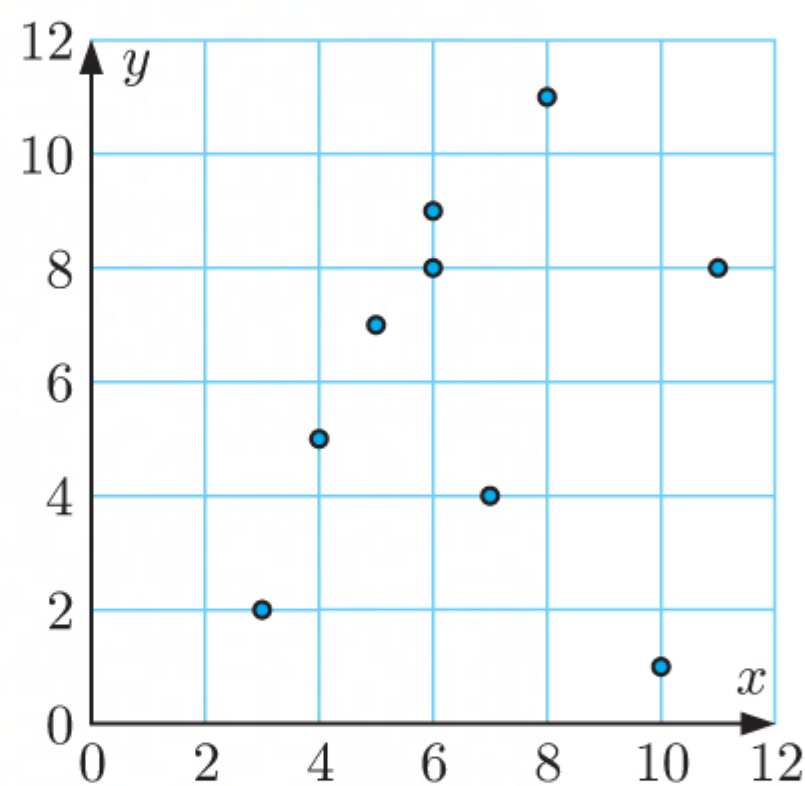
c

x	3	6	11	7	5	6	8	10	4
y	2	8	8	4	7	9	11	1	5

	List 1	List 2	List 3	List 4
SUB				
1	3	2		
2	6	8		
3	11	8		
4	7	4		



i



ii

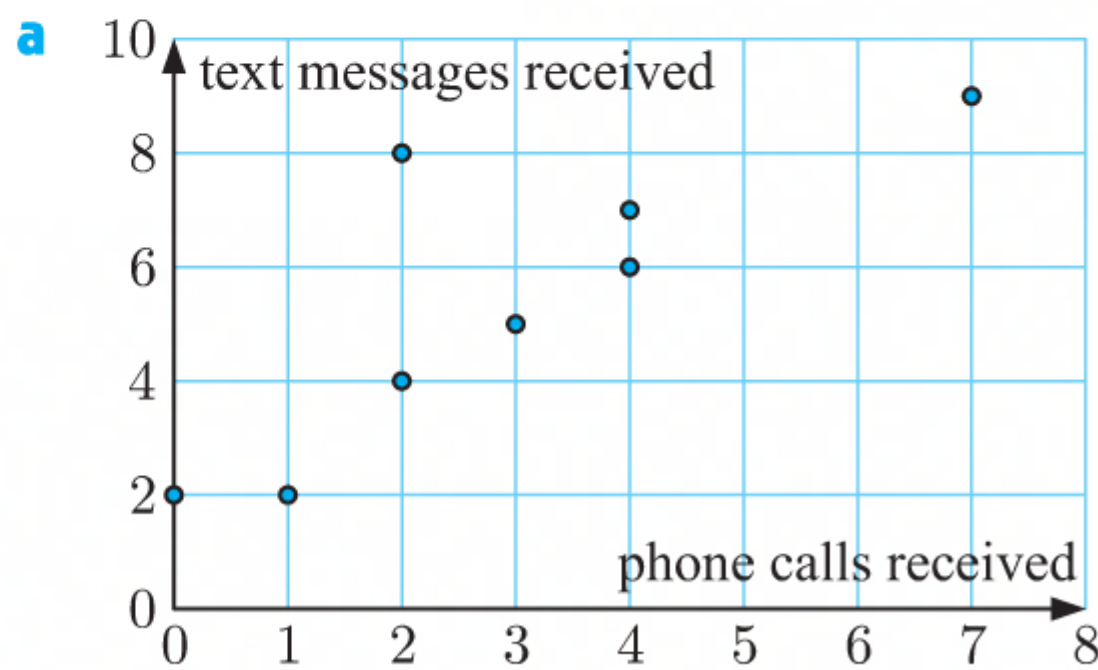
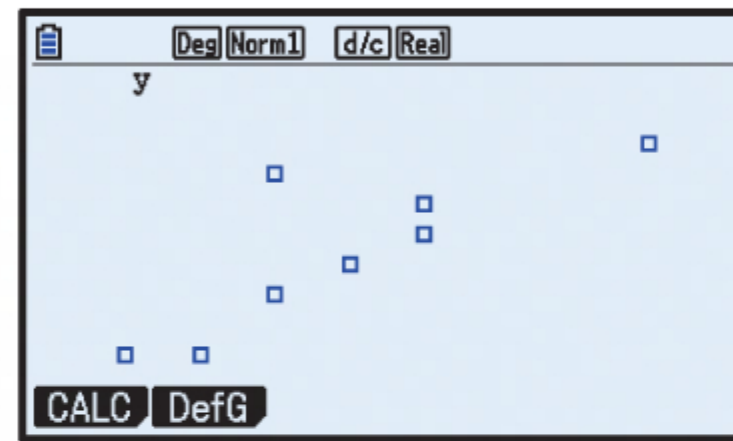
	List 1	List 2	List 3	List 4
SUB				
1	3	2		
2	6	8		
3	11	8		
4	7	4		

So, $r \approx 0.146$.iii There is a very weak, positive correlation between x and y .

4

Student	A	B	C	D	E	F	G	H
Phone calls received	4	7	1	0	3	2	2	4
Text messages received	6	9	2	2	5	8	4	7

	List 1	List 2	List 3	List 4
SUB				
1	4	6		
2	7	9		
3	1	2		
4	0	2		



b

	LinearReg(ax+b)
a	=0.98479087
b	=2.54372623
r	=0.81606077
r ²	=0.66595518
MSe	=2.66539923
y	=ax+b

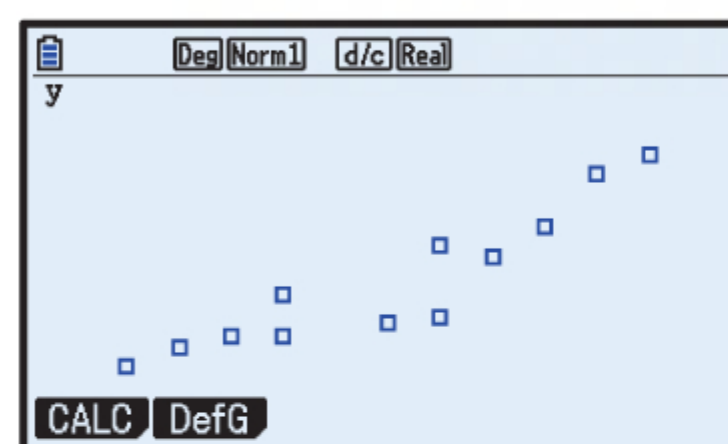
So, $r \approx 0.816$.

- c There is a moderate, positive correlation between *phone calls received* and *text messages received*.
- d Those students who receive several phone calls are also likely to receive several text messages and vice versa.

5

Athlete	A	B	C	D	E	F	G	H	I	J	K	L
Age (years)	12	16	16	18	13	19	11	10	20	17	15	13
Distance thrown (m)	20	35	23	38	27	47	18	15	50	33	22	20

	List 1	List 2	List 3	List 4
SUB				
1	12	20		
2	16	35		
3	16	23		
4	18	38		



	LinearReg(ax+b)
a	=3.28947368
b	=-20.342105
r	=0.91730097
r ²	=0.84144108
MSe	=23.2447368
y	=ax+b

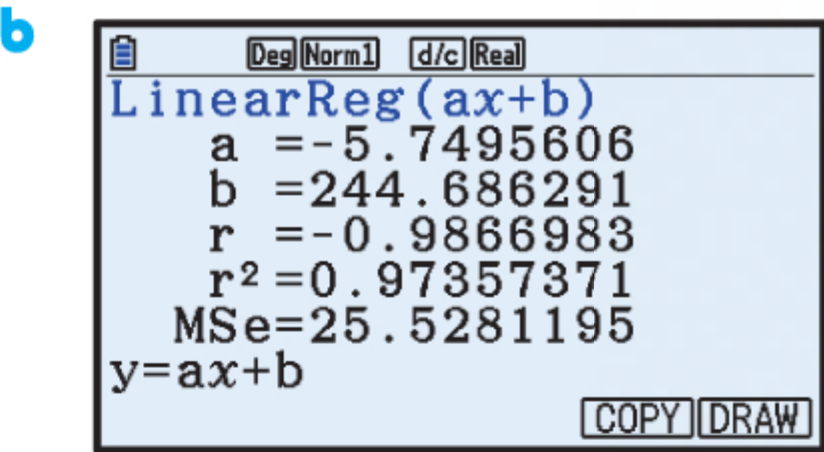
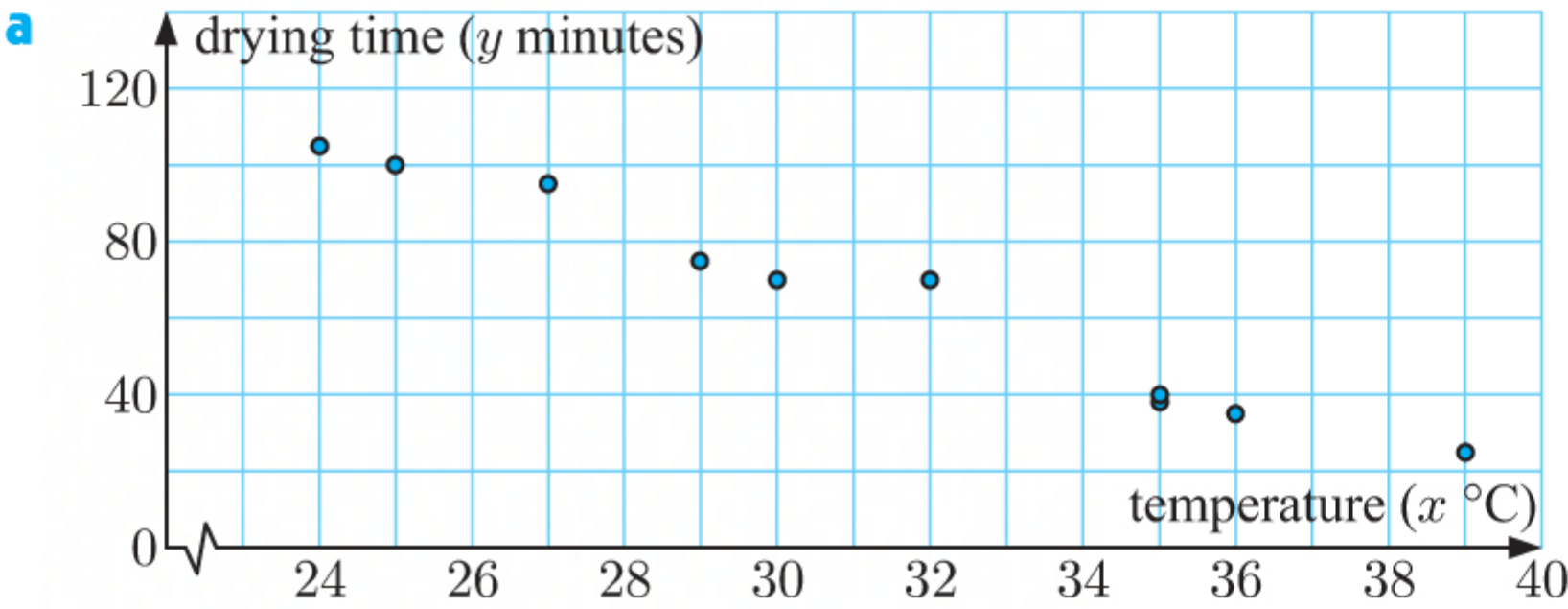
So, $r \approx 0.917$.

- b There is a strong, positive correlation between the *age* of the young athlete and the *distance thrown*. In general, the higher the young athlete's age, the further they can throw a discus.

6

Temperature ($x^{\circ}\text{C}$)	25	32	27	39	35	24	30	36	29	35
Drying time (y minutes)	100	70	95	25	38	105	70	35	75	40

	List 1	List 2	List 3	List 4
SUB				
1	25	100		
2	32	70		
3	27	95		
4	39	25		



c

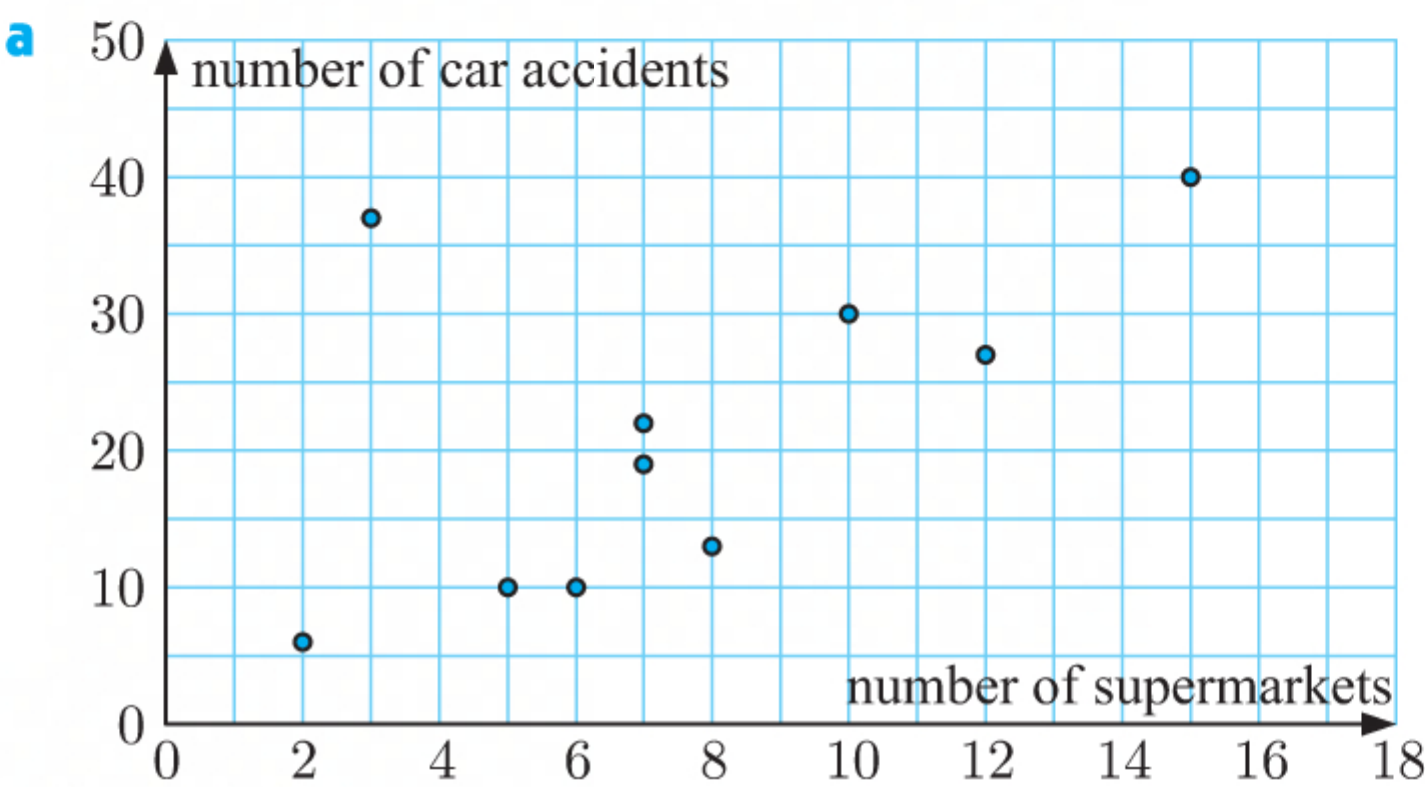
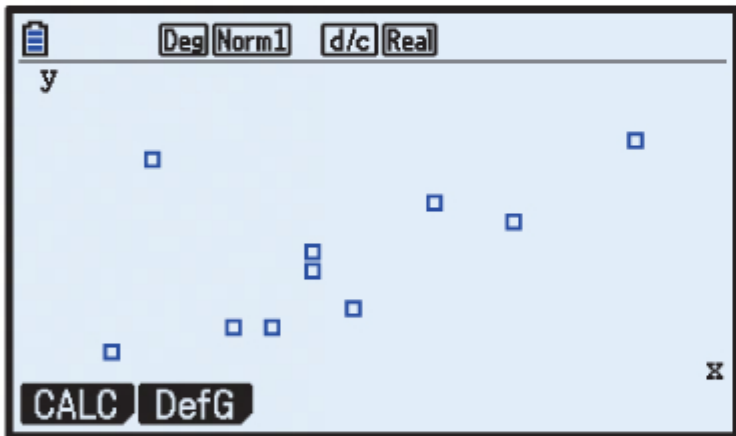
There is a very strong, negative correlation between *temperature* and *drying time*. In general, the higher the temperature, the lower the drying time.

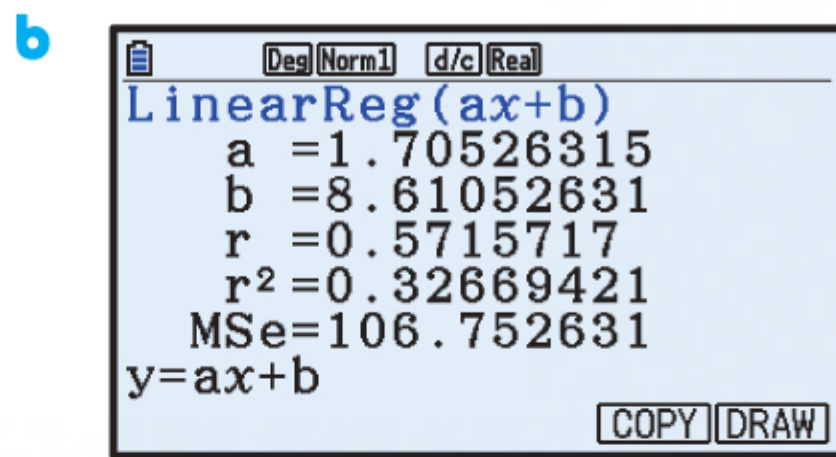
So, $r \approx -0.987$.

7

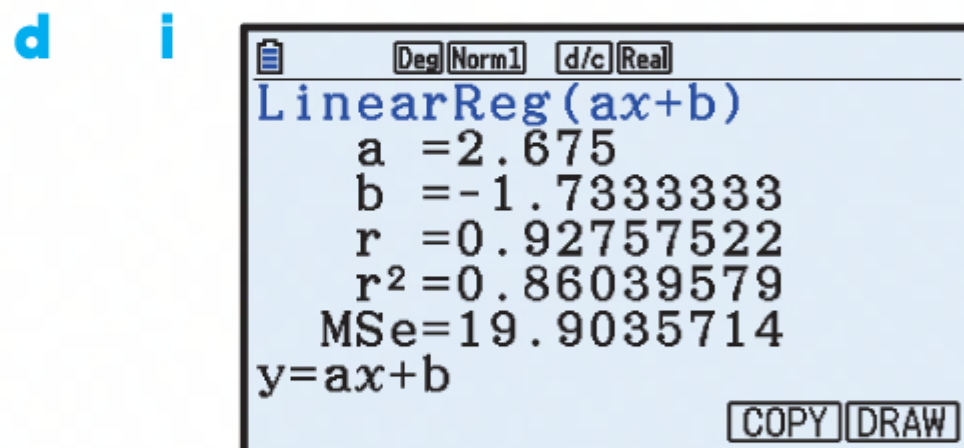
Number of supermarkets	5	8	12	7	6	2	15	10	7	3
Number of car accidents	10	13	27	19	10	6	40	30	22	37

	List 1	List 2	List 3	List 4
SUB				
1	5	10		
2	8	13		
3	12	27		
4	7	19		





So, $r \approx 0.572$.



So, $r \approx 0.928$.

- ii** There is a strong, positive correlation between the *number of supermarkets* and the *number of car accidents*.
- iii** By removing the outlier, the value of r increased significantly.
- e** No, it is not a causal relationship. Both variables depend on the number of people in each town, not on each other.

8

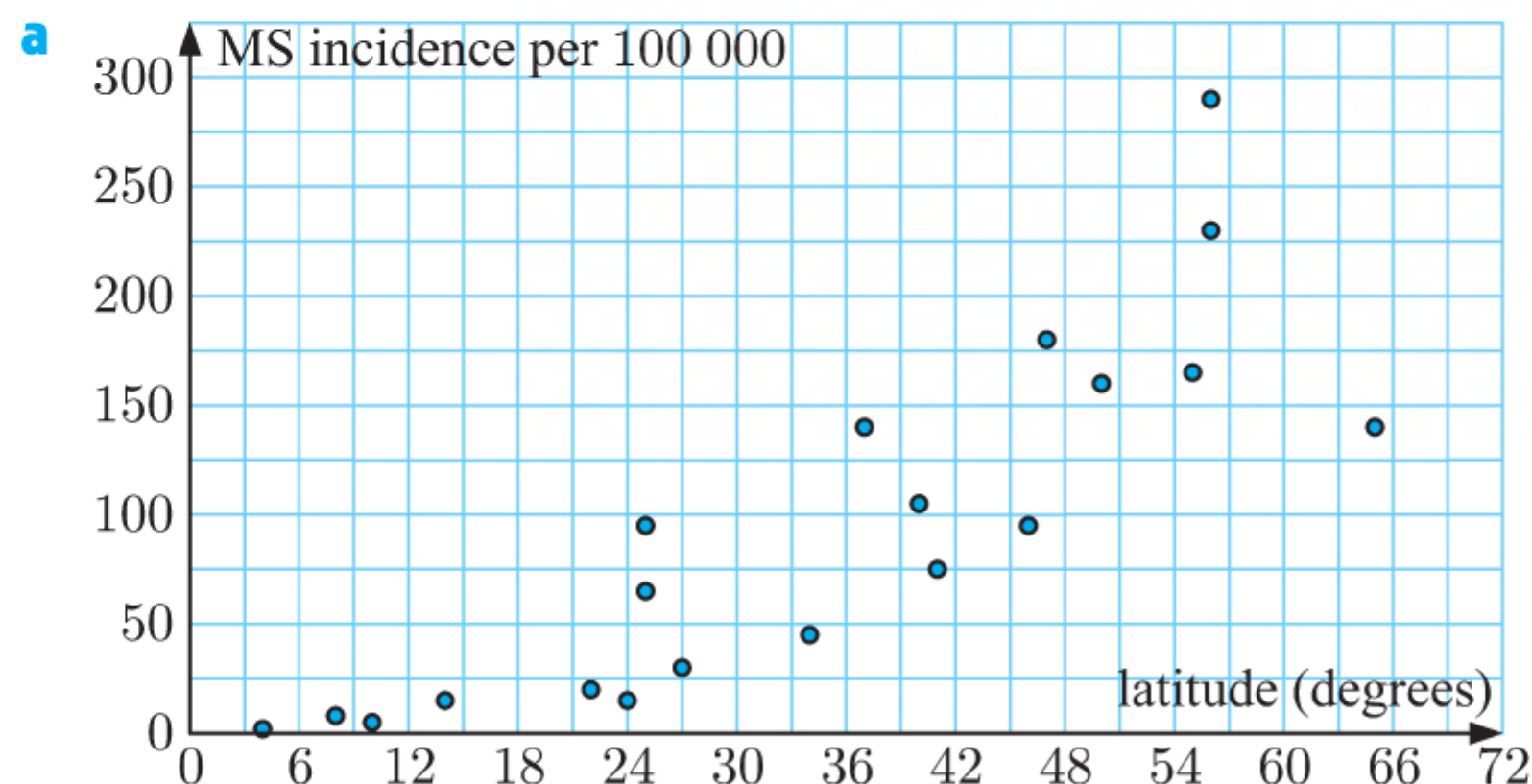
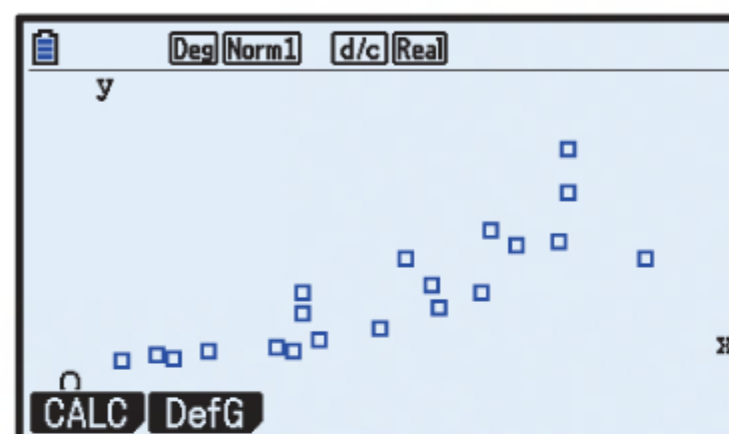
<i>Latitude (degrees)</i>	55	25	41	22	47	37	56	14	34	25
<i>MS incidence per 100 000</i>	165	95	75	20	180	140	230	15	45	65

<i>Latitude (degrees)</i>	27	65	10	24	4	56	46	8	50	40
<i>MS incidence per 100 000</i>	30	140	5	15	2	290	95	8	160	105

	List 1	List 2	List 3	List 4
SUB				
1	55	165		
2	25	95		
3	41	75		
4	22	20		

20

[GRAPH] [CALC] [TEST] [INTR] [DIST] [▶]



b

LinearReg(ax+b)
a = 3.90314994
b = -39.878043
r = 0.84940985
r ² = 0.7214971
MSe = 1972.69789
y = ax + b
COPY DRAW

So, $r \approx 0.849$.

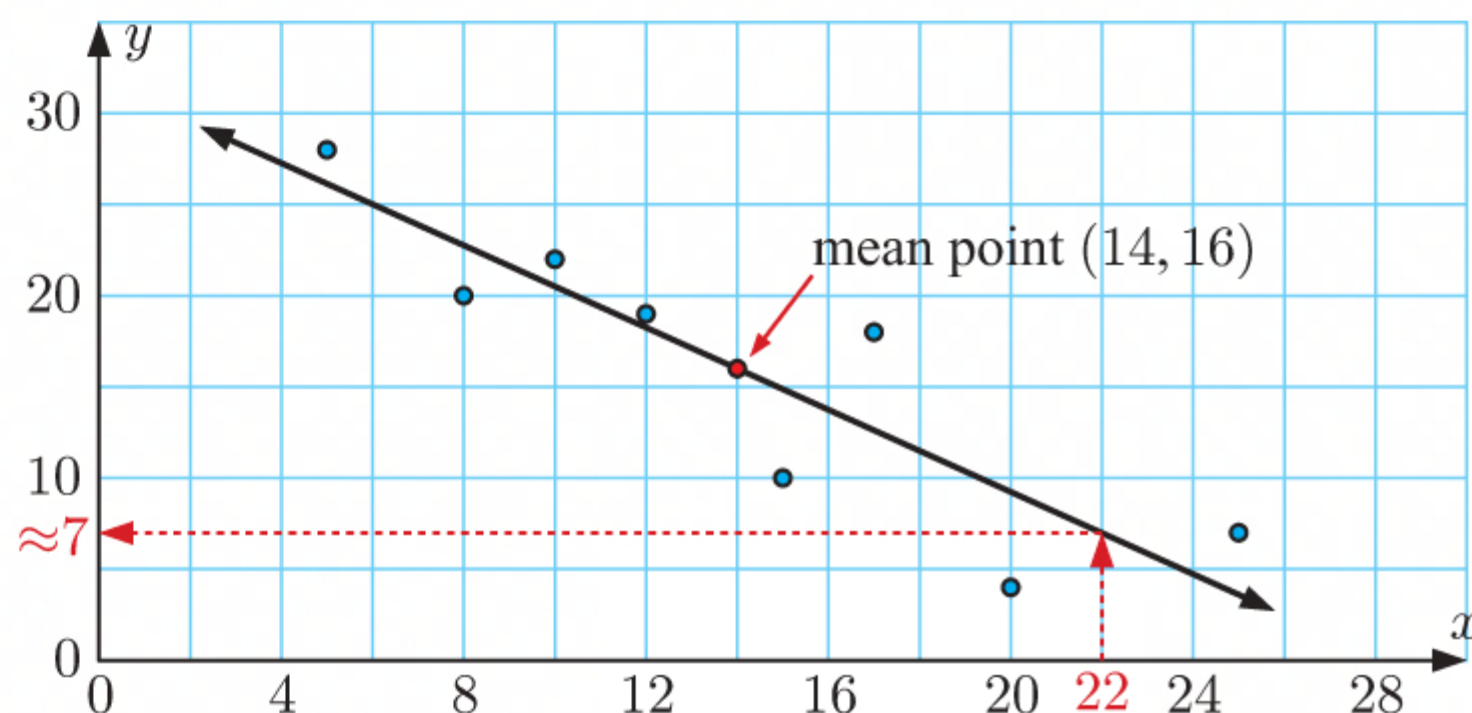
d The incidence of MS is higher near the poles.

c There is a moderate, positive correlation between *latitude* and *MS incidence*.

EXERCISE 19C

1

x	5	12	20	17	10	8	25	15
y	28	19	4	18	22	20	7	10

a, f

b The data appears to be negatively correlated.

c

LinearReg(ax+b)
a = -1.0953947
b = 31.3355263
r = -0.8809647
r ² = 0.77609882
MSe = 17.5389254
y = ax + b
COPY DRAW

So, $r \approx -0.881$.

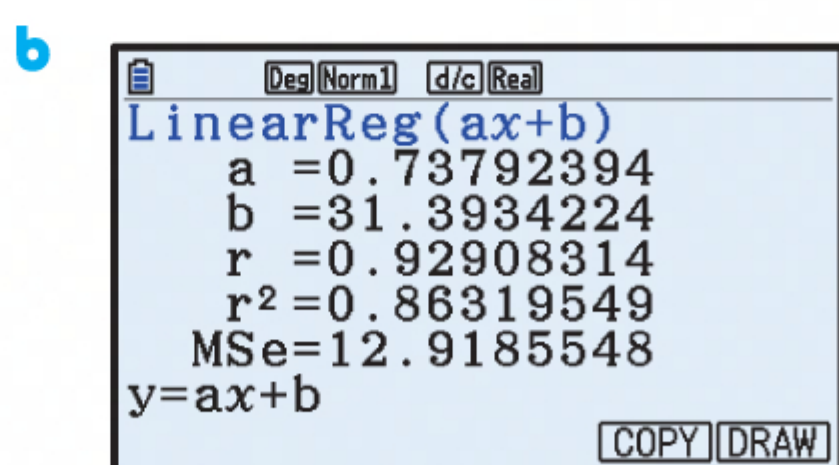
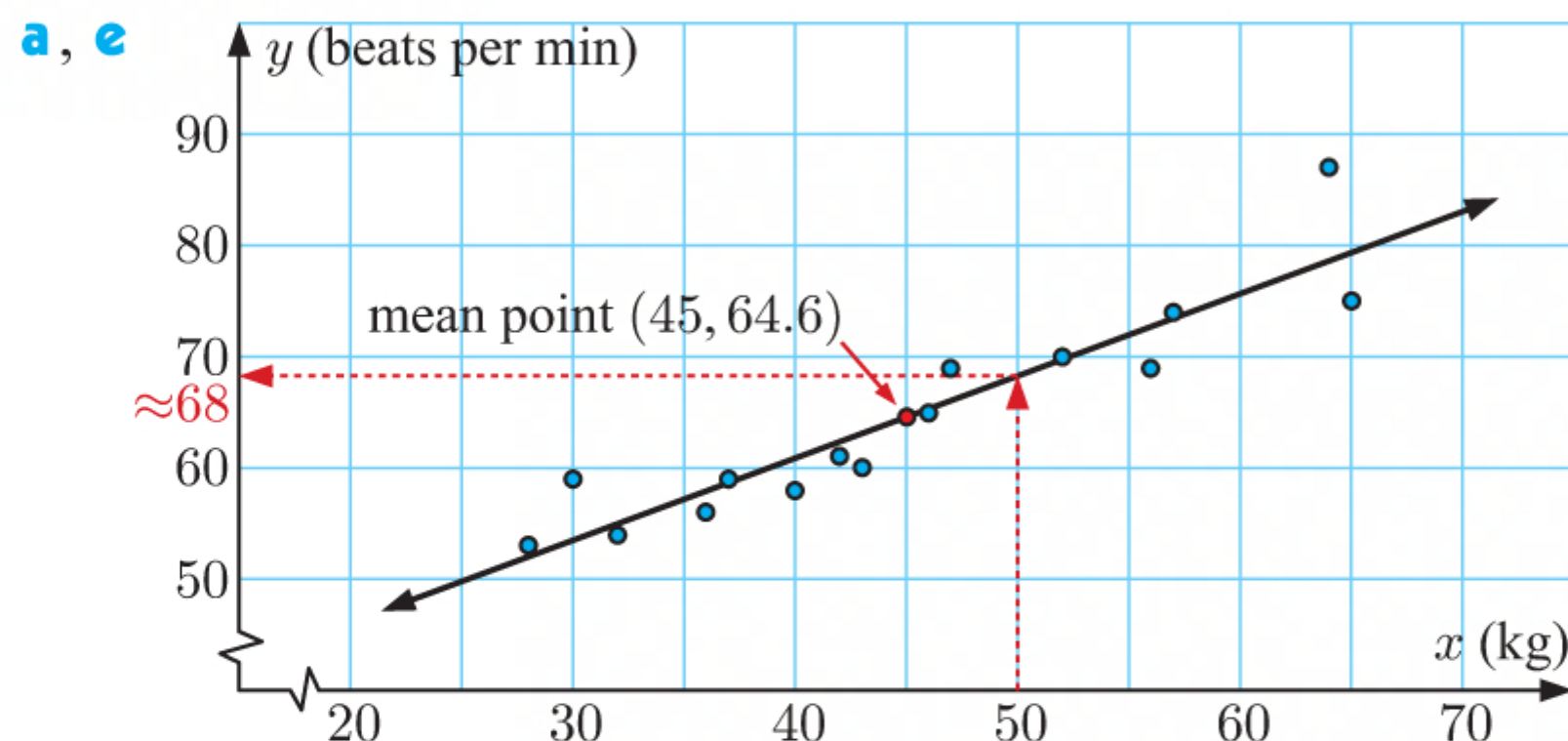
d There is a strong, negative correlation between x and y .

$$\begin{aligned} \bar{x} &= \frac{5 + 12 + 20 + 17 + 10 + 8 + 25 + 15}{8}, & \bar{y} &= \frac{28 + 19 + 4 + 18 + 22 + 20 + 7 + 10}{8} \\ &= 14, & &= 16 \end{aligned}$$

So the mean point is $(14, 16)$.

g When $x = 22$, $y \approx 7$.

2	<i>Weight (x kg)</i>	46	37	32	57	47	64	42	30	52	56	65	43	36	28	40
	<i>Pulse rate (y beats per min)</i>	65	59	54	74	69	87	61	59	70	69	75	60	56	53	58



So, $r \approx 0.929$.

c There is a strong, positive correlation between the *weight* of a student and their *pulse rate*.

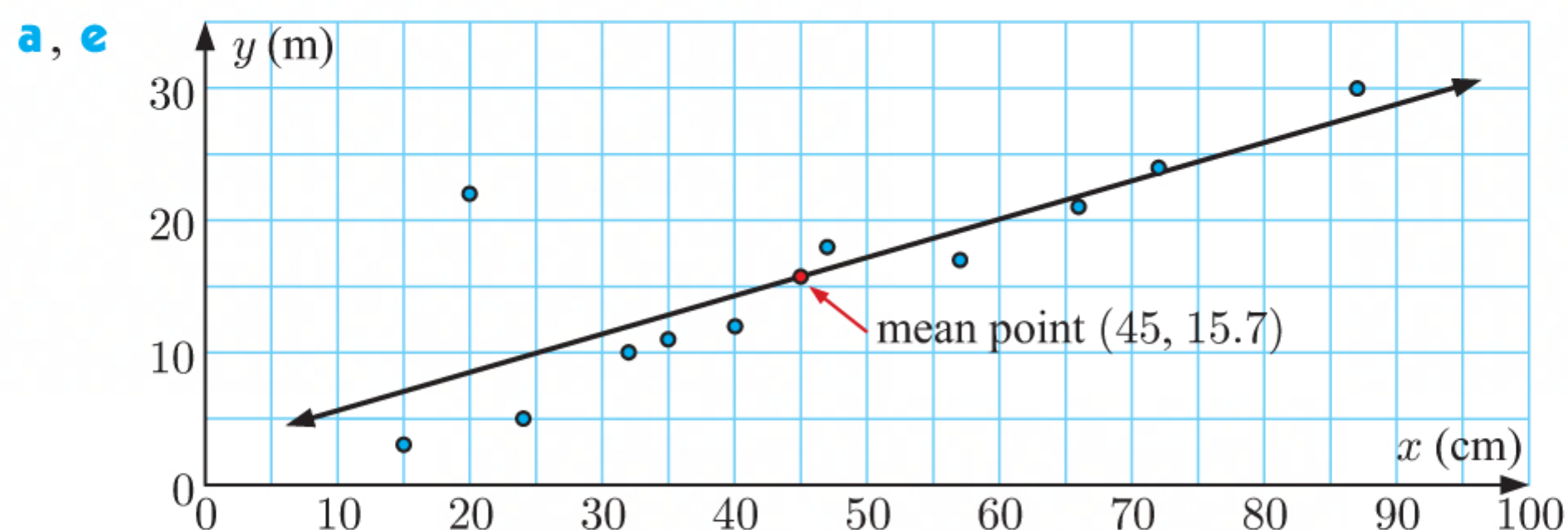
d $\bar{x} = \frac{46 + 37 + \dots + 28 + 40}{15}, \quad \bar{y} = \frac{65 + 59 + \dots + 53 + 58}{15}$
 $= 45 \qquad \qquad \qquad = 64.6$

So the mean point is $(45, 64.6)$.

f When $x = 50$, $y \approx 68$.

A student who weighs 50 kg will have a pulse rate of approximately 68 beats per minute. This is an interpolation, so the estimate is reliable.

3	<i>Trunk width (x cm)</i>	35	47	72	40	15	87	20	66	57	24	32
	<i>Height (y m)</i>	11	18	24	12	3	30	22	21	17	5	10



b $(20, 22)$ is an outlier as it appears separated from the rest of the data.

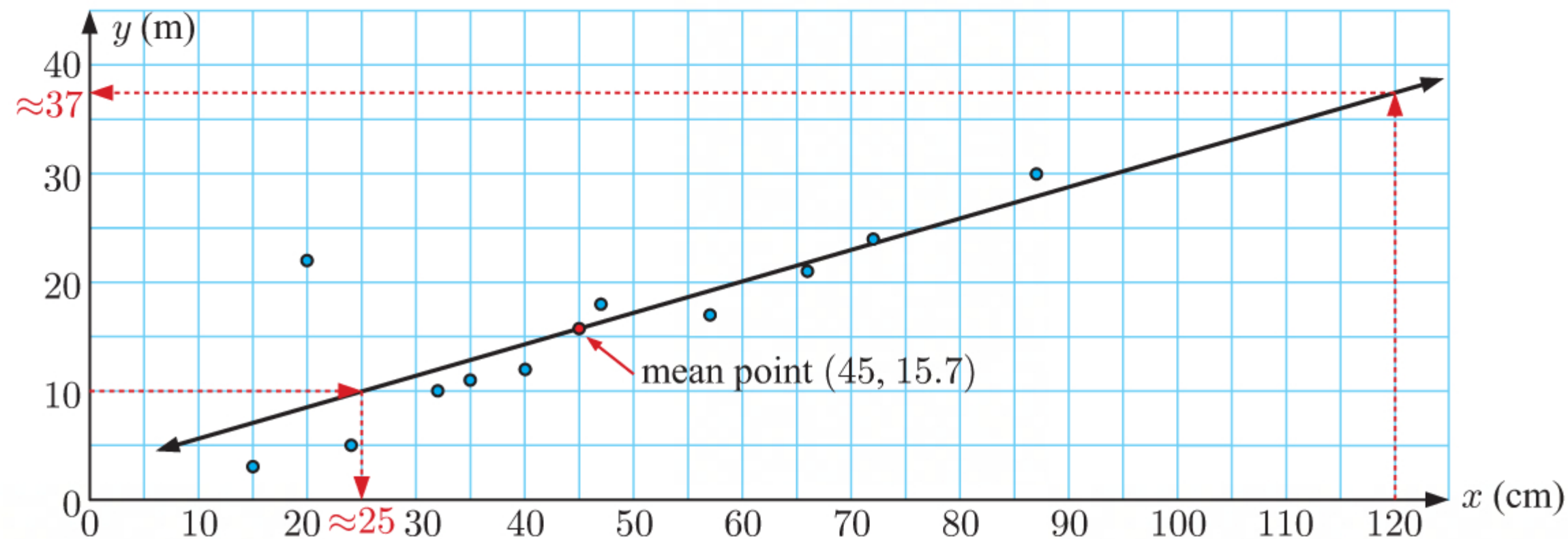
c The tree represented by the outlier would be very tall and thin.

$$\text{d } \bar{x} = \frac{35 + 47 + \dots + 24 + 32}{11}, \quad \bar{y} = \frac{11 + 18 + \dots + 5 + 10}{11}$$

$$= 45 \qquad \qquad \qquad \approx 15.7$$

So the mean point is (45, 15.7).

f We extend the scatter diagram from a to include the value $x = 120$:



When $x = 120$, $y \approx 37$.

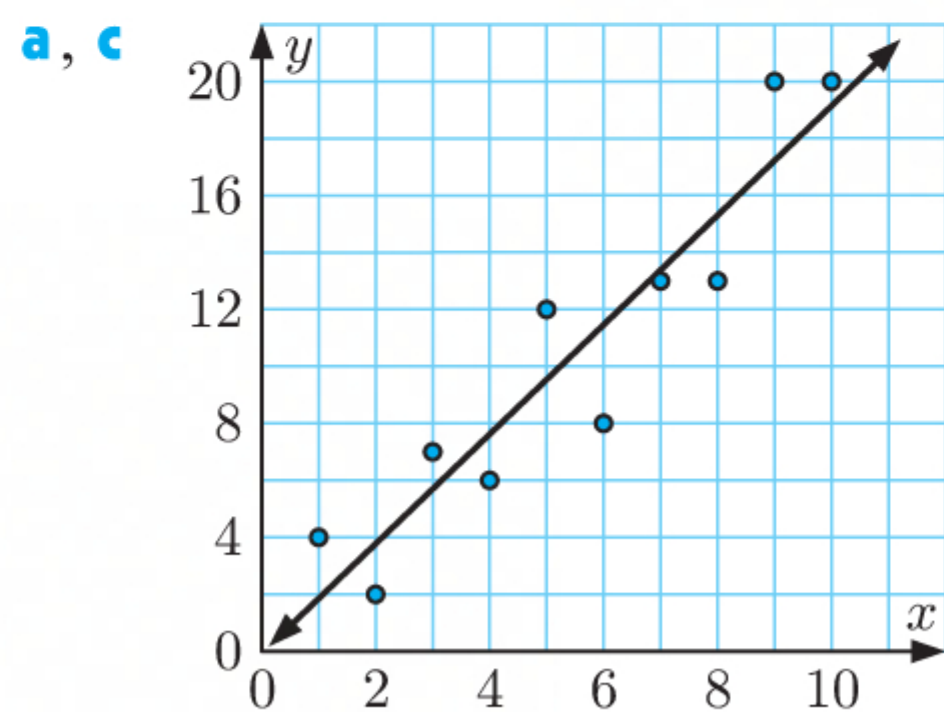
A tree with trunk width 120 cm will have a height of approximately 37 m. This is an extrapolation, so the prediction may not be reliable.

g When $y = 10$, $x \approx 25$.

A tree with height 10 m will have a trunk width of approximately 25 cm. This is an interpolation, so the estimate is reliable.

EXERCISE 19D

1	x	10	4	6	8	9	5	7	1	2	3
	y	20	6	8	13	20	12	13	4	2	7

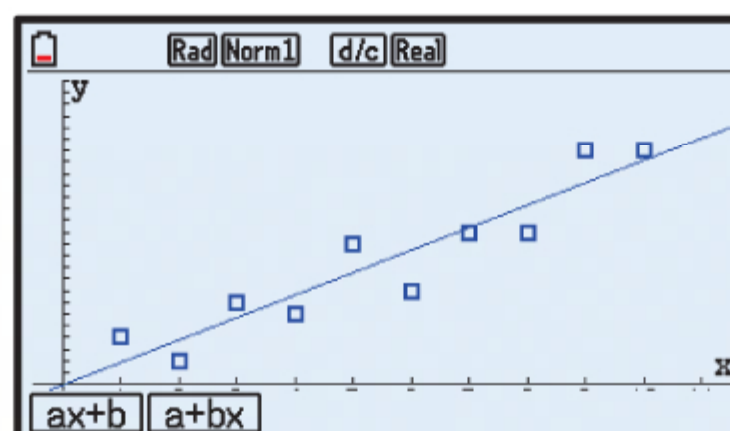


b

```

Des Norm1 d/c Real
LinearReg(ax+b)
a =1.92121212
b =-0.0666666
r =0.93476168
r^2=0.8737794
MSe=5.49848484
y=ax+b
COPY DRAW

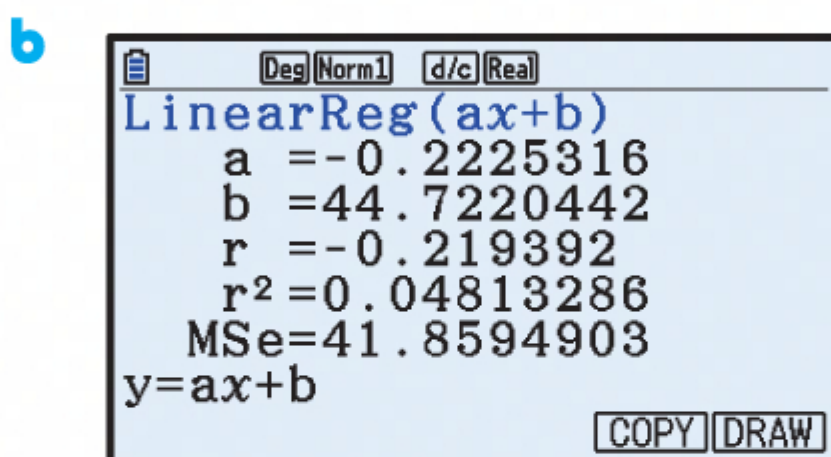
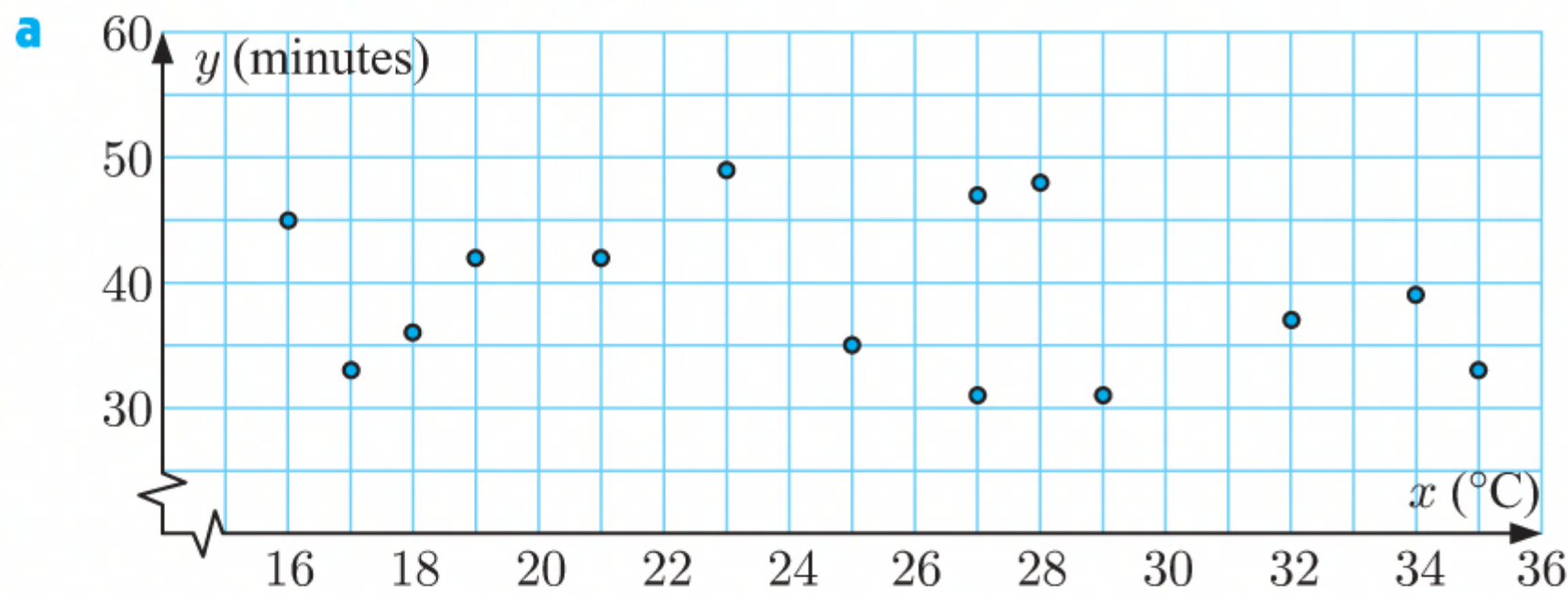
```



Using technology, the least squares regression line is $y \approx 1.92x - 0.0667$.

2

Temperature (x °C)	25	19	23	27	32	35	29	27	21	18	16	17	28	34
Time (y minutes)	35	42	49	31	37	33	31	47	42	36	45	33	48	39



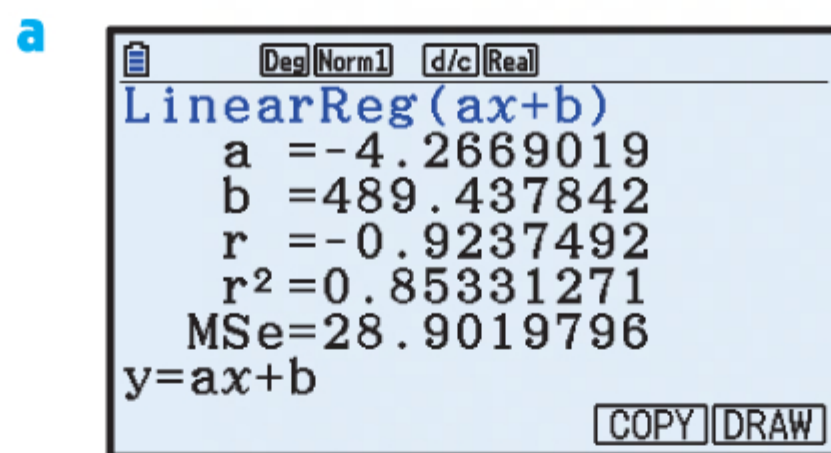
So, $r \approx -0.219$.

- c There is a very weak, negative correlation between *temperature* and *time*.
- d No, it is not reasonable to find a line of best fit for this data as there is almost no correlation.

3

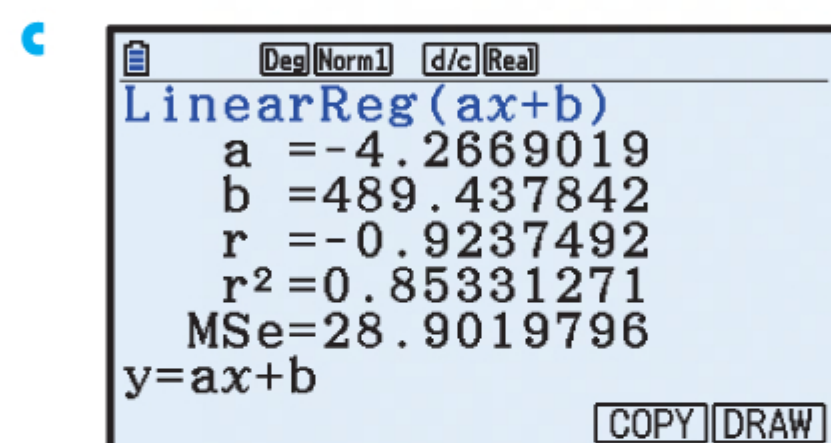
Petrol price (x cents per litre)	105.9	106.9	109.9	104.5	104.9	111.9	110.5	112.9
Number of customers (y)	45	42	25	48	43	15	19	10

Petrol price (x cents per litre)	107.5	108.0	104.9	102.9	110.9	106.9	105.5	109.5
Number of customers (y)	30	23	42	50	12	24	32	17



So, $r \approx -0.924$.

- b There is a strong, negative correlation between *petrol price* and the *number of customers*.



Using technology, the least squares regression line is $y \approx -4.27x + 489$.

- d** The gradient of the least squares regression line ≈ -4.27 . This means that for every cent per litre the petrol price increases by, the number of customers will decrease by approximately 4.27.

e When $x = 115.9$, $y \approx -4.27(115.9) + 489$
 ≈ -5.10

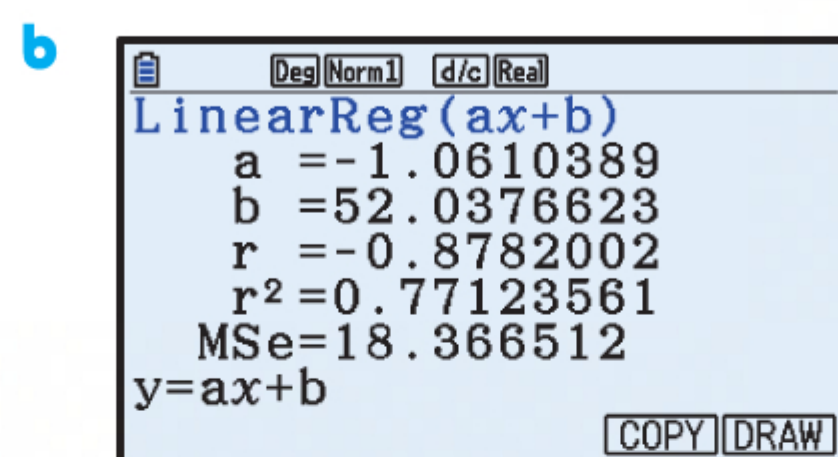
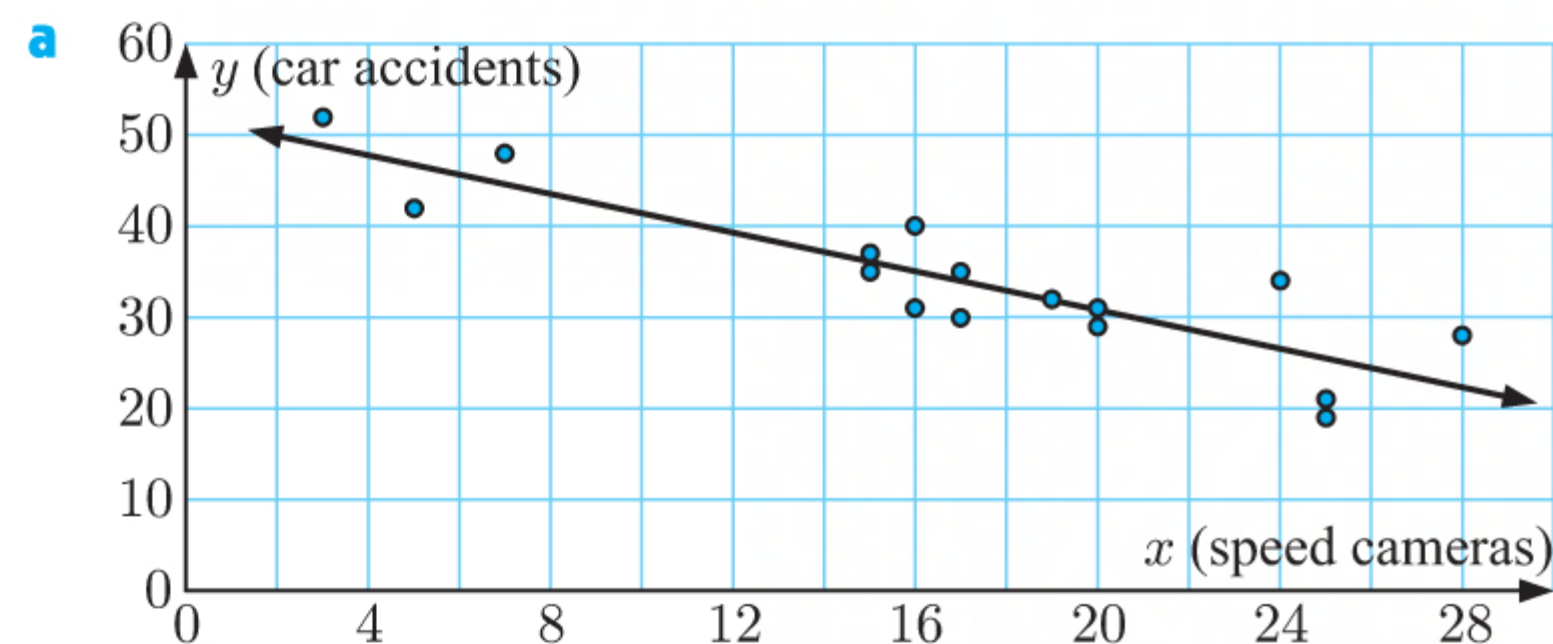
So, when petrol is 115.9 cents per litre, we would expect about -5.10 customers per hour.

f When $y = 40$, $40 \approx -4.27x + 489$
 $\therefore -449 \approx -4.27x$
 $\therefore x \approx 105.3$

So, a petrol station which has 40 customers per hour would sell petrol at approximately 105.3 cents per litre.

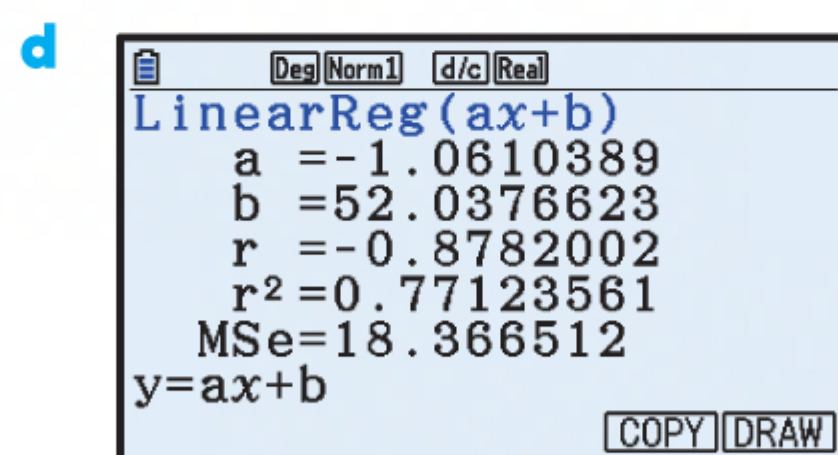
- g** In **e**, it is impossible to have a negative number of customers. This extrapolation is not valid. In **f**, this is an interpolation, so this estimate is likely to be reliable.

4	Number of speed cameras (x)	7	15	20	3	16	17	28	17	24	25	20	5	16	25	15	19
	Number of car accidents (y)	48	35	31	52	40	35	28	30	34	19	29	42	31	21	37	32



So, $r \approx -0.878$.

- c** There is a strong, negative correlation between the *number of speed cameras* and the *number of car accidents*.



Using technology, the least squares regression line is $y \approx -1.06x + 52.0$.

- e The gradient of the least squares regression line ≈ -1.06 . This indicates that for every additional speed camera, the number of car accidents per week decreases by an average of 1.06.

The y -intercept of the least squares regression line ≈ 52.0 . This indicates that if there were no speed cameras in a city, an average of 52.0 car accidents would occur each week.

- f When $x = 10$, $y \approx -1.06(10) + 52.0$
 ≈ 41.4

So, there will be approximately 41.4 car accidents per week in a city with 10 speed cameras.

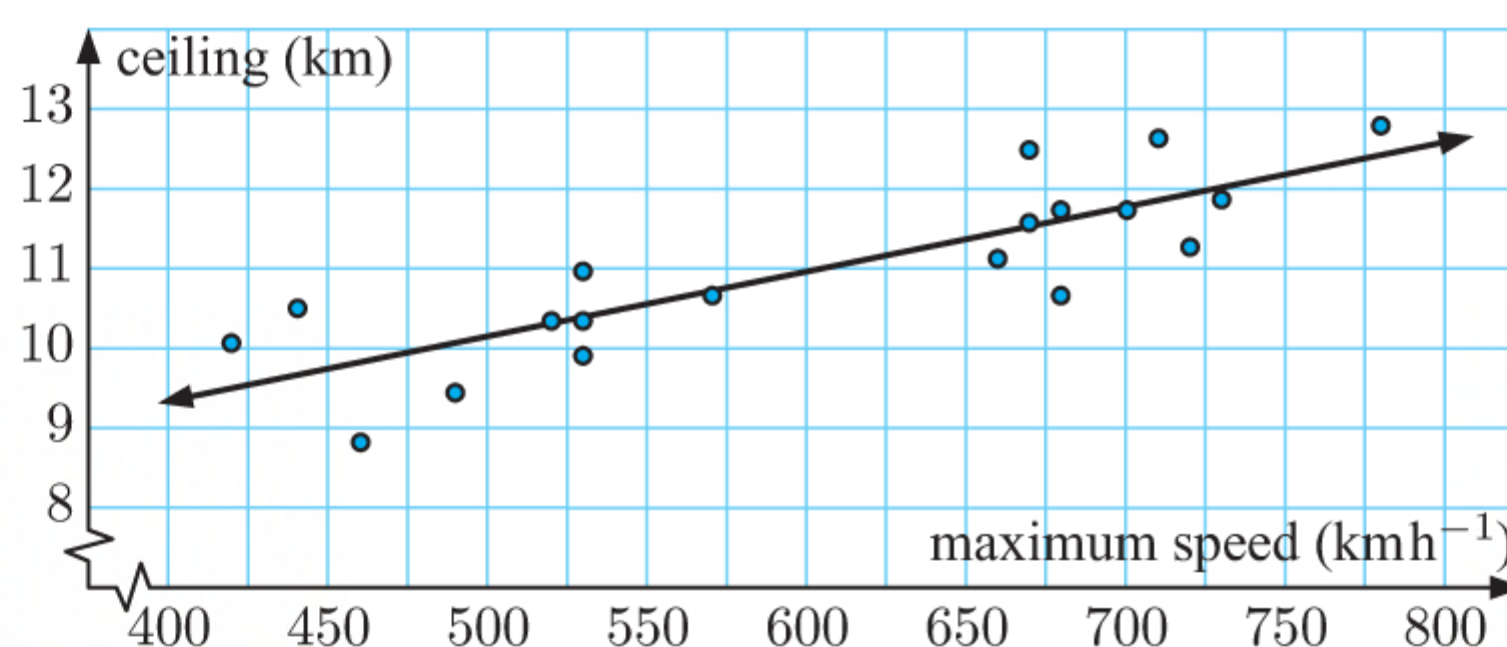
5

Maximum speed	Ceiling
460	8.84
420	10.06
530	10.97
530	9.906
490	9.448
530	10.36
680	11.73

Maximum speed	Ceiling
680	10.66
720	11.27
710	12.64
660	11.12
780	12.80
730	11.88

Maximum speed	Ceiling
670	12.49
570	10.66
440	10.51
670	11.58
700	11.73
520	10.36

a, d



b

Deg Norm1 d/c Real LinearReg(ax+b) $a = 8.1202E-03$ $b = 6.09013455$ $r = 0.84010344$ $r^2 = 0.70577379$ $MSe = 0.36102817$ $y = ax + b$	COPY DRAW
----------------------------------------------------------------------------------------------------------------------------------------------------------------------	-----------

So, $r \approx 0.840$.

- c There is a moderate, positive, linear correlation between *maximum speed* and *ceiling*.

d

Deg Norm1 d/c Real LinearReg(ax+b) $a = 8.1202E-03$ $b = 6.09013455$ $r = 0.84010344$ $r^2 = 0.70577379$ $MSe = 0.36102817$ $y = ax + b$	COPY DRAW
----------------------------------------------------------------------------------------------------------------------------------------------------------------------	-----------

Using technology, the least squares regression line is $y \approx 0.00812x + 6.09$.

- e The gradient of the least squares regression line ≈ 0.00812 . This indicates that for each additional km h^{-1} , the ceiling increases by an average of approximately 0.00812 km or 8.12 m.

f When $x = 600$, $y \approx 0.00812(600) + 6.09$
 ≈ 11.0

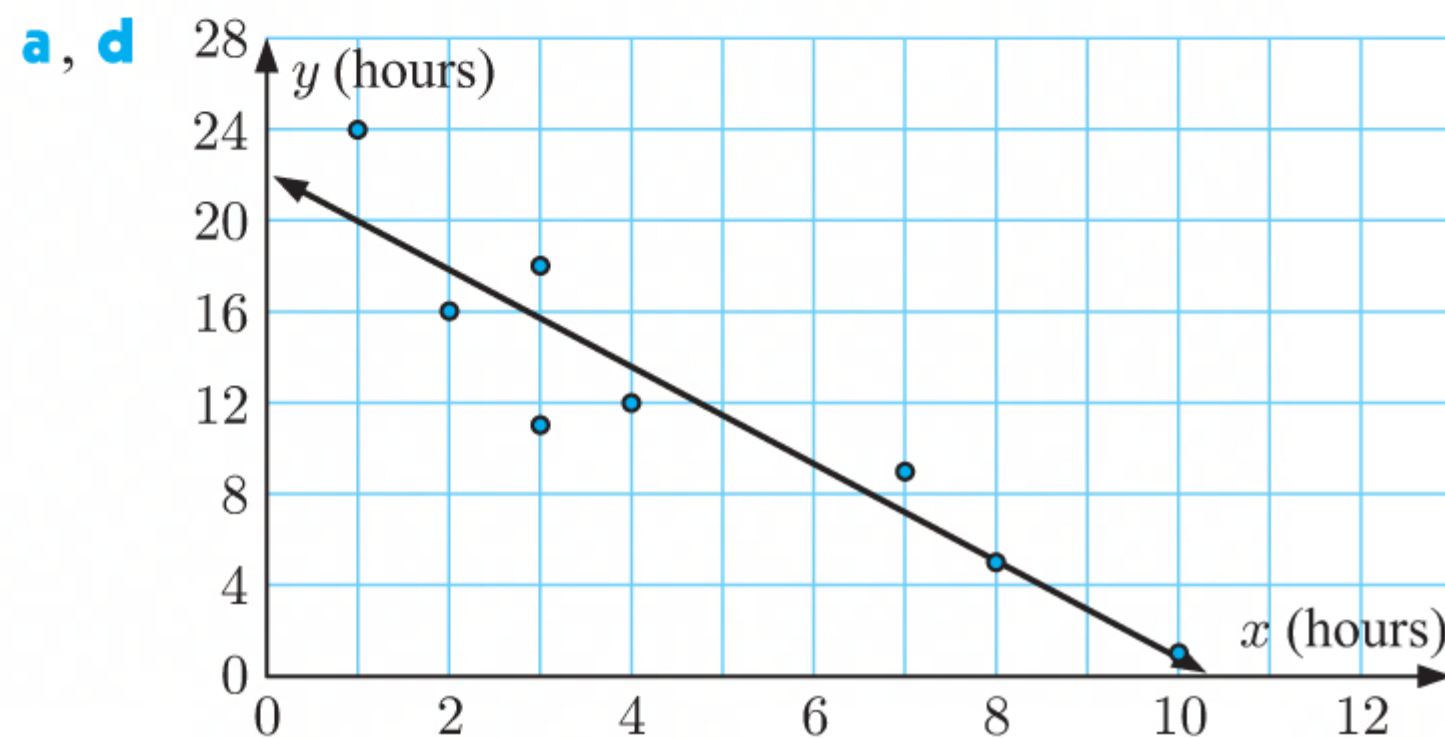
So, a fighter plane with maximum speed 600 km h^{-1} would have a ceiling of approximately 11.0 km.

g When $y = 11$, $11 \approx 0.00812x + 6.09$
 $\therefore 4.91 \approx 0.00812x$
 $\therefore x \approx 605$

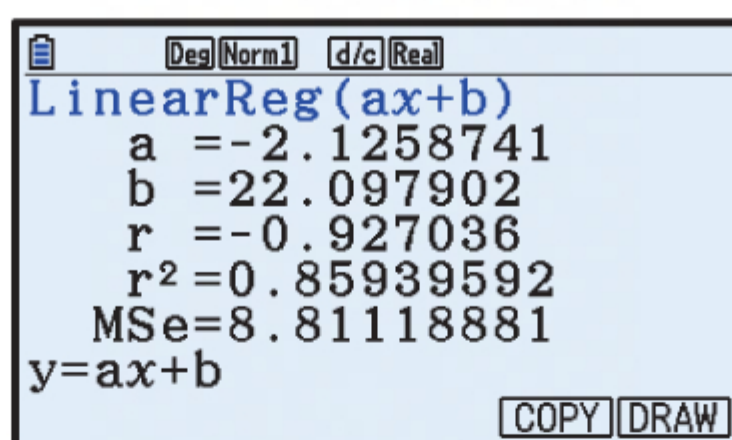
So, a fighter plane with a ceiling of 11 km would have maximum speed of approximately 605 km h^{-1} .

6

Exercise (x hours per week)	4	1	8	7	10	3	3	2
Television (y hours per week)	12	24	5	9	1	18	11	16



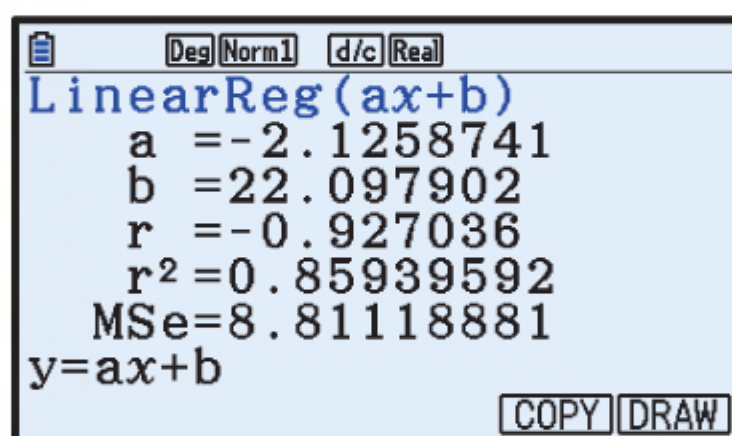
b



So, $r \approx -0.927$.

c There is a strong, negative, linear correlation between *time exercising* and *time watching television*.

d



Using technology, the least squares regression line is $y \approx -2.13x + 22.1$.

e The gradient of the least squares regression line ≈ -2.13 . This indicates that for each additional hour a child exercises each week, the number of hours they spend watching television each week decreases by 2.13.

The y -intercept of the least squares regression line ≈ 22.1 . This indicates that for children who do not spend time exercising, they would watch television for an average of about 22.1 hours per week.

- f** **i** From the table, the student who exercised for 7 hours each week watched 9 hours of television each week.

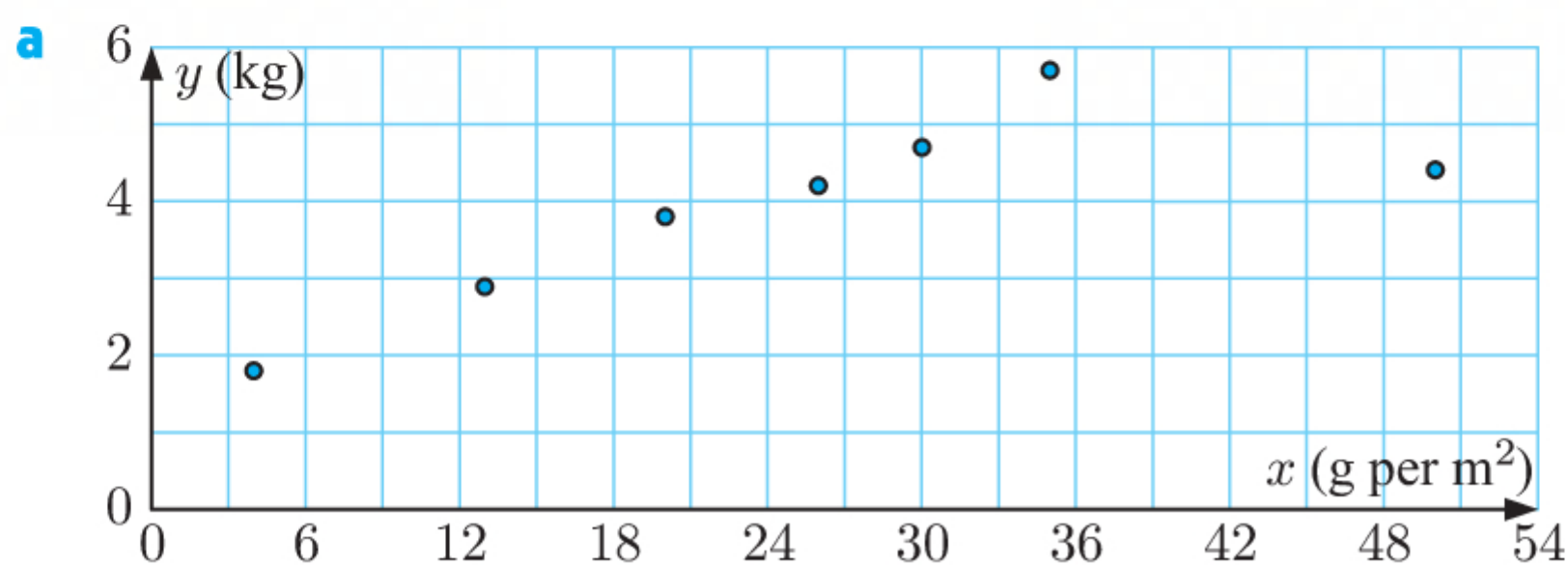
ii When $x = 7$, $y \approx -2.126(7) + 22.1$
 ≈ 7.22

Using the least squares regression line, a child who exercises for 7 hours each week watches approximately 7.22 hours of television each week.

- iii** This particular child spent more time watching television than predicted.

7

Fertiliser (x g per m^2)	4	13	20	26	30	35	50
Yield (y kg)	1.8	2.9	3.8	4.2	4.7	5.7	4.4



(50, 4.4) is the outlier.

- b** **i** The outlier reduces the strength of correlation of the data.
ii The outlier decreases the gradient of the least squares regression line.

c **i**

```

LinearReg(ax+b)
a = 0.06715696
b = 2.22086572
r = 0.79782039
r^2 = 0.63651738
MSe = 0.70037907
y = ax + b
  
```

So, $r \approx 0.798$.

ii

```

LinearReg(ax+b)
a = 0.1187182
b = 1.31734486
r = 0.99257385
r^2 = 0.98520284
MSe = 0.03468082
y = ax + b
  
```

So, $r \approx 0.993$.

d **i**

```

LinearReg(ax+b)
a = 0.06715696
b = 2.22086572
r = 0.79782039
r^2 = 0.63651738
MSe = 0.70037907
y = ax + b
  
```

Using technology, the least squares regression line is $y \approx 0.0672x + 2.22$.

ii

```

LinearReg(ax+b)
a = 0.1187182
b = 1.31734486
r = 0.99257385
r^2 = 0.98520284
MSe = 0.03468082
y = ax + b
  
```

Using technology, the least squares regression line is $y \approx 0.119x + 1.32$.

- e** The regression line which excludes the outlier should be used to estimate the yield when 15 g per m^2 of fertiliser is used. This will be more accurate for an interpolation.
- f** Too much fertiliser often kills the plants. In this case, the outlier should be kept when analysing the data as it is a valid data value. If the outlier is a recording error caused by bad measurement or recording skills, it should be removed before analysing data.

ACTIVITY 2

ANSCOMBE'S QUARTET

1 Data set A:

x	10	8	13	9	11	14	6	4	12	7	5
y	8.04	6.95	7.58	8.81	8.33	9.96	7.24	4.26	10.84	4.82	5.68

\bar{x}	=9
Σx	=99
Σx^2	=1001
σx	=3.16227766
sx	=3.31662479
n	=11

$$\mu_x = 9, \sigma_x \approx 3.16$$

\bar{x}	=7.50090909
Σx	=82.51
Σx^2	=660.1763
σx	=1.93710869
sx	=2.03165673
n	=11

$$\mu_y \approx 7.50, \sigma_y \approx 1.94$$

Data set B:

x	10	8	13	9	11	14	6	4	12	7	5
y	9.14	8.14	8.74	8.77	9.26	8.1	6.13	3.1	9.13	7.26	4.74

\bar{x}	=9
Σx	=99
Σx^2	=1001
σx	=3.16227766
sx	=3.31662479
n	=11

$$\mu_x = 9, \sigma_x \approx 3.16$$

\bar{x}	=7.50090909
Σx	=82.51
Σx^2	=660.1763
σx	=1.93710869
sx	=2.03165673
n	=11

$$\mu_y \approx 7.50, \sigma_y \approx 1.94$$

Data set C:

x	10	8	13	9	11	14	6	4	12	7	5
y	7.46	6.77	12.74	7.11	7.81	8.84	6.08	5.39	8.15	6.42	5.73

\bar{x}	=9
Σx	=99
Σx^2	=1001
σx	=3.16227766
sx	=3.31662479
n	=11

$$\mu_x = 9, \sigma_x \approx 3.16$$

\bar{x}	=7.5
Σx	=82.5
Σx^2	=659.9762
σx	=1.93593294
sx	=2.0304236
n	=11

$$\mu_y = 7.5, \sigma_y \approx 1.94$$

Data set D:

x	8	8	8	8	8	8	8	19	8	8	8
y	6.58	5.76	7.71	8.84	8.47	7.04	5.25	12.5	5.56	7.91	6.89

1-Variable	
\bar{x}	=9
Σx	=99
Σx^2	=1001
σx	=3.16227766
sx	=3.31662479
n	=11

$$\mu_x = 9, \sigma_x \approx 3.16$$

1-Variable	
\bar{x}	=7.50090909
Σx	=82.51
Σx^2	=660.1325
σx	=1.93608064
sx	=2.03057851
n	=11

$$\mu_y \approx 7.50, \sigma_y \approx 1.94$$

- a** In each data set: The mean of x is 9.
The mean of y is 7.5 (or very close to 7.5).
- b** In each data set: The population standard deviation of $x \approx 3.16$.
The population standard deviation of $y \approx 1.94$.

2 Data set A:

LinearReg(ax+b)	
a	=0.5000909
b	=3.0000909
r	=0.81642051
r^2	=0.66654245
MSe	=1.52918777
$y=ax+b$	

The regression line is $y \approx 0.500x + 3.00$.

Data set C:

LinearReg(ax+b)	
a	=0.49972727
b	=3.00245454
r	=0.81628673
r^2	=0.66632404
MSe	=1.52846575
$y=ax+b$	

The regression line is $y \approx 0.500x + 3.00$.

Data set B:

LinearReg(ax+b)	
a	=0.5
b	=3.00090909
r	=0.8162365
r^2	=0.66624203
MSe	=1.53069898
$y=ax+b$	

The regression line is $y \approx 0.5x + 3.00$.

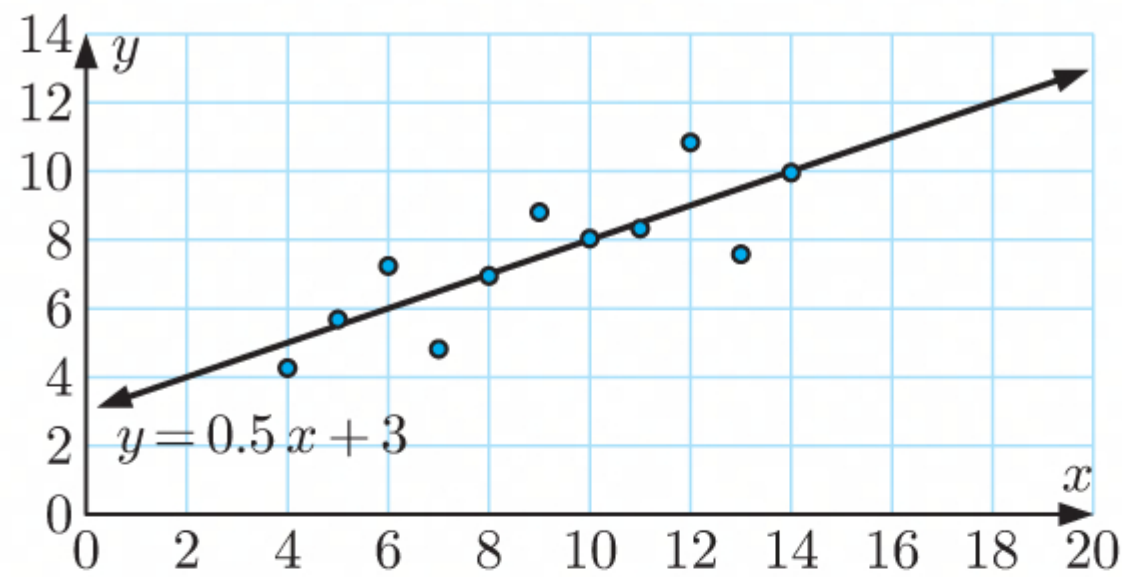
Data set D:

LinearReg(ax+b)	
a	=0.49990909
b	=3.00172727
r	=0.81652143
r^2	=0.66670725
MSe	=1.52694333
$y=ax+b$	

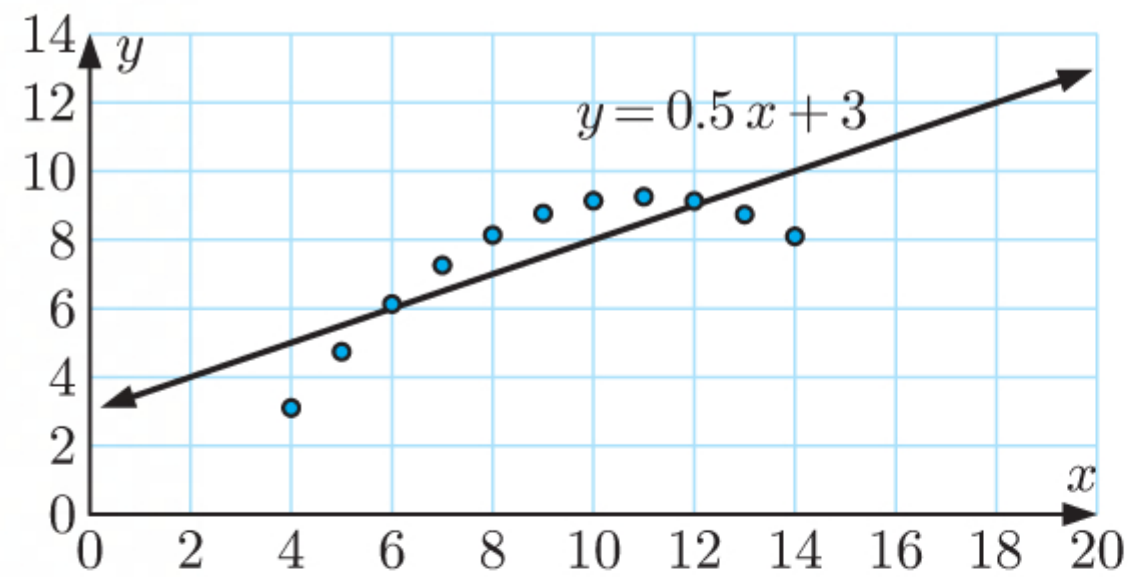
The regression line is $y \approx 0.500x + 3.00$.

The regression lines are almost identical for each data set.

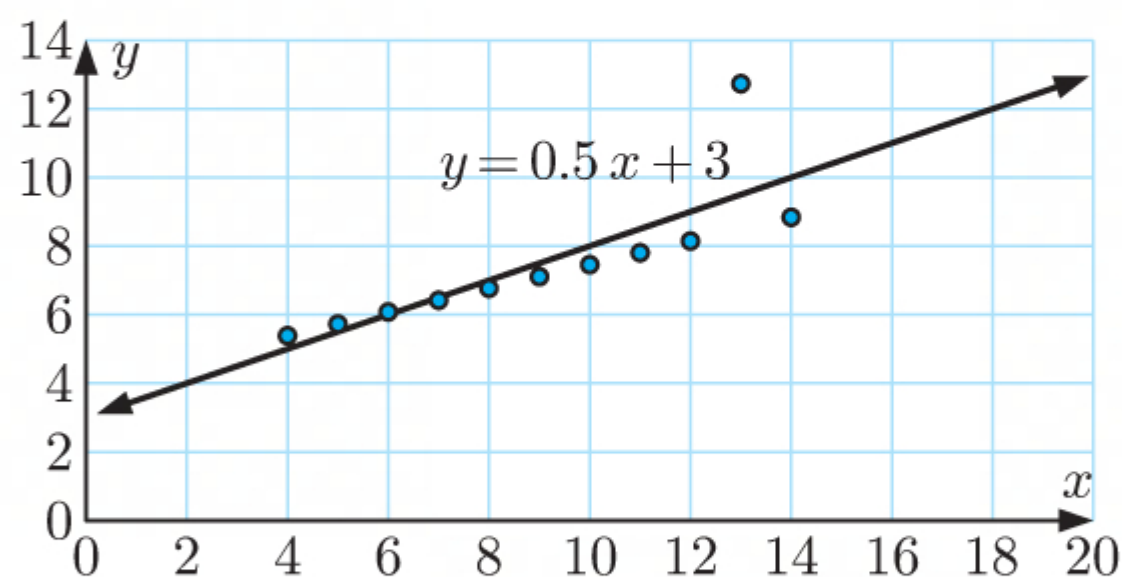
3 Data set A:



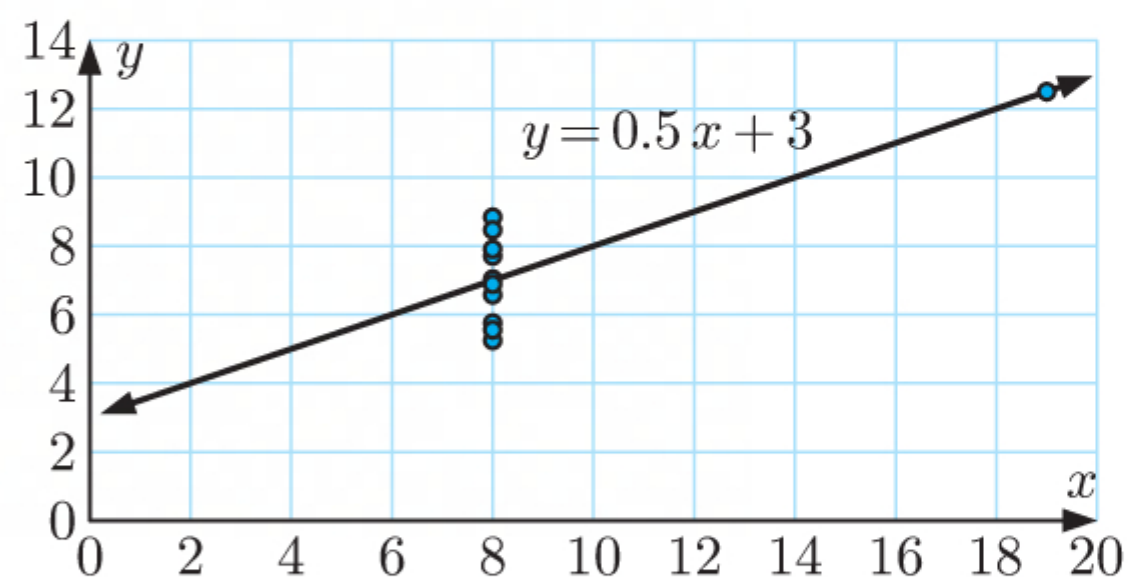
Data set B:



Data set C:



Data set D:



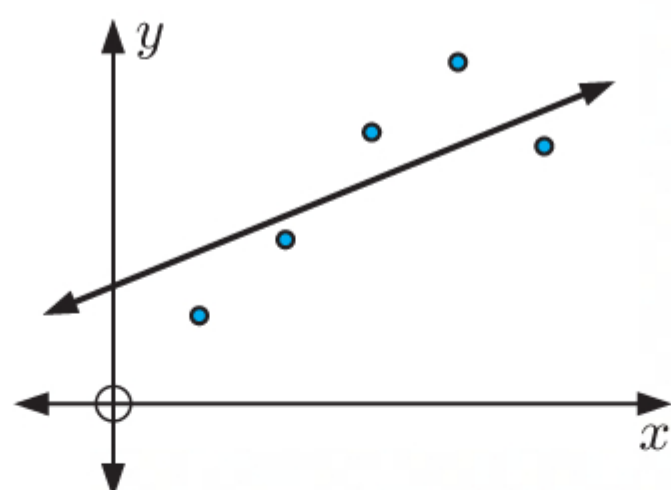
- 4 Each data set has the same mean and standard deviation for both variables, and the same regression line. However, we see that the scatter diagrams for each data set are wildly different from each other. A linear model is not necessarily appropriate for each data set.
- 5 A scatter diagram allows us to see patterns in data that cannot be conveyed with descriptive statistics alone.

ACTIVITY 3

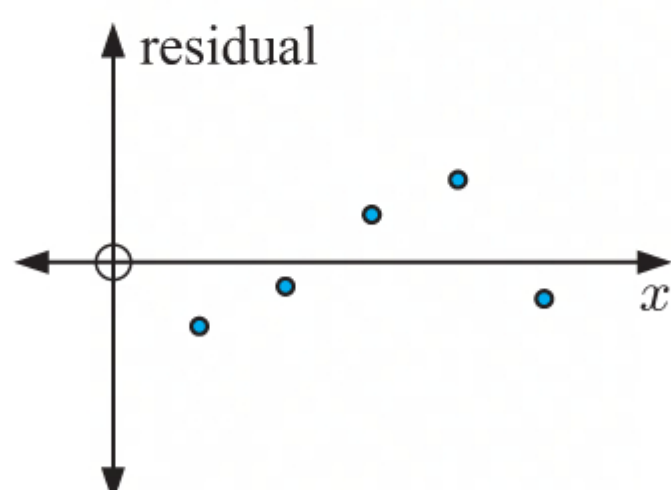
RESIDUAL PLOTS

PART 1: CONSTRUCTING A RESIDUAL PLOT

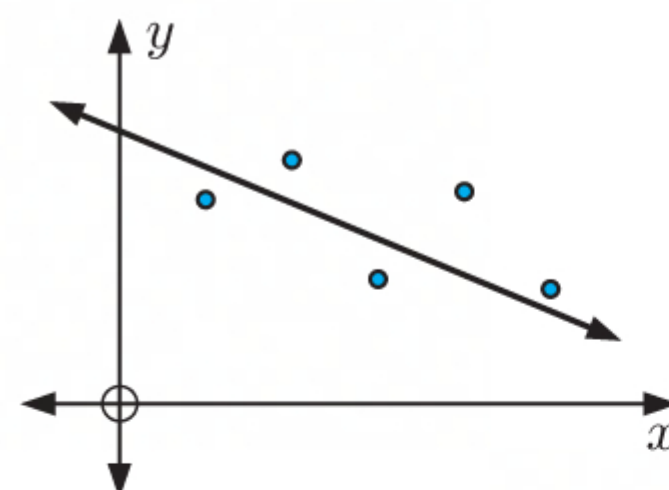
1 a



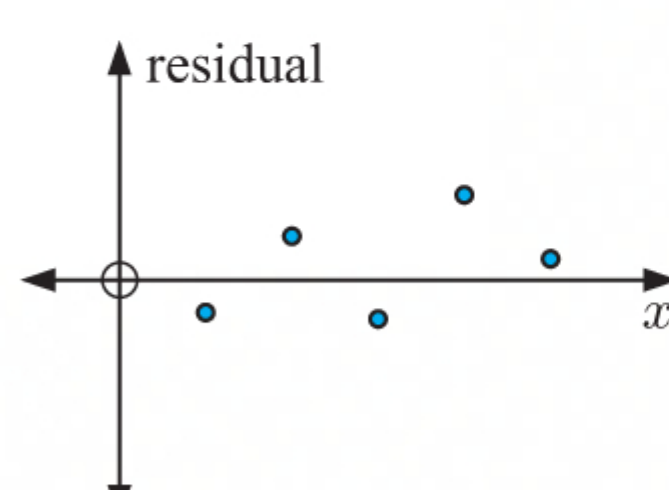
This scatter diagram has 2 data points above the least squares regression line and 3 points below the regression line. This pattern is shown in the residual plot in **B**.

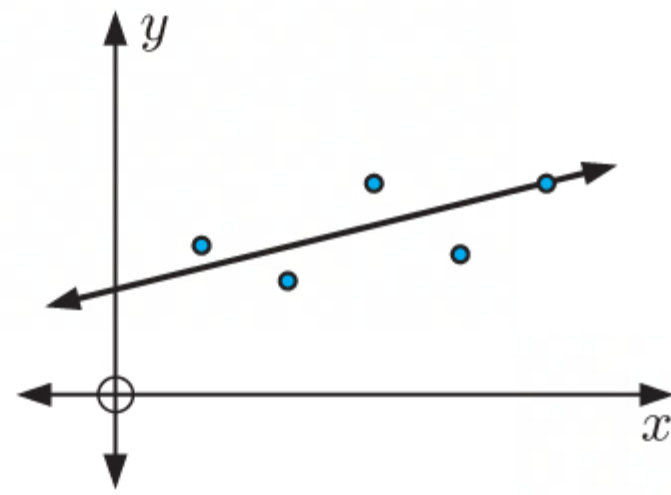


b



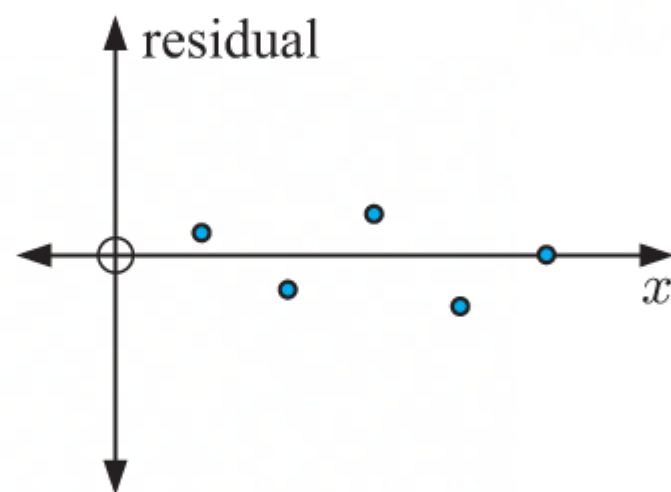
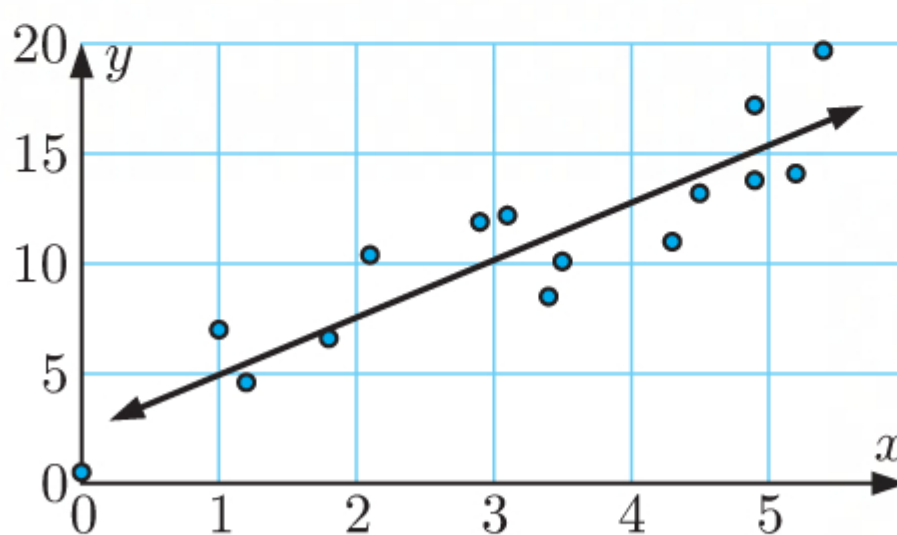
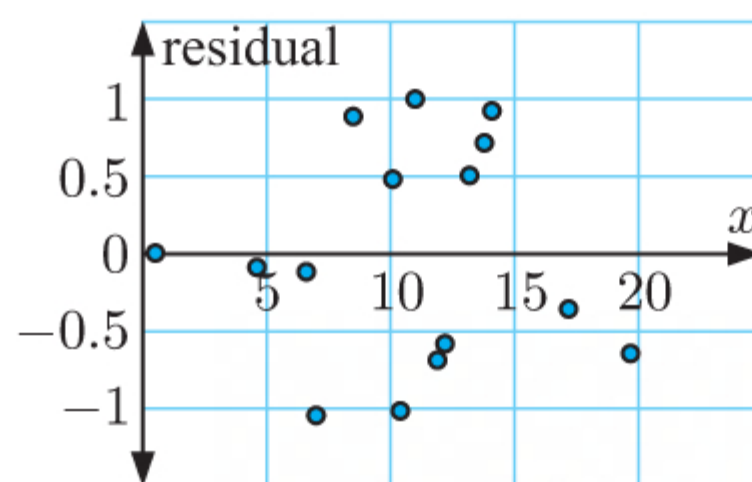
This scatter diagram has 3 data points above the least squares regression line and 2 points below the regression line. This pattern is shown in the residual plot in **C**.



C

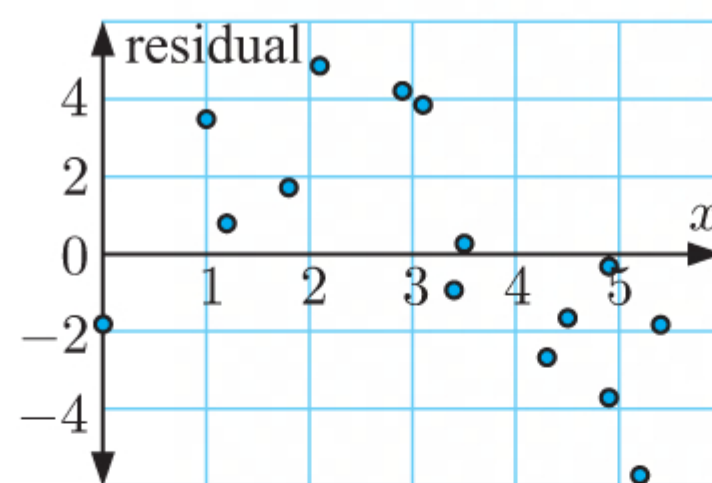
This scatter diagram has 2 data points above the least squares regression line, 2 points below the regression line, and 1 point on the regression line.

This pattern is shown in the residual plot in **A**.

**2****A**

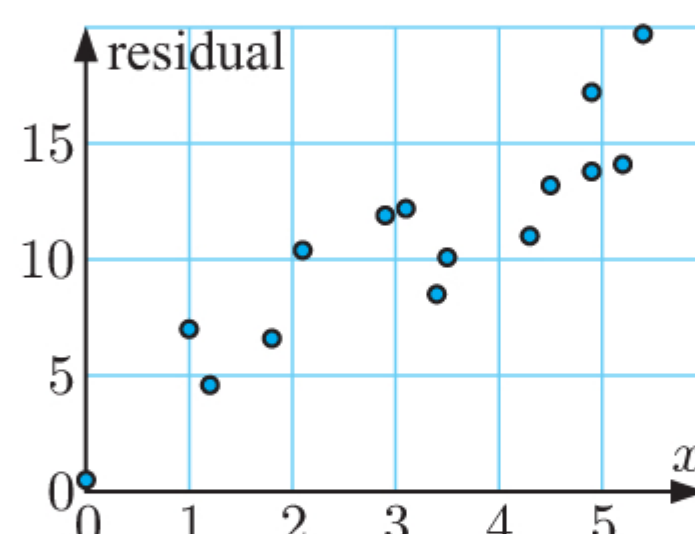
The values on the x -axis do not correspond to those on the scatter diagram.

So **A** is not the correct residual plot.

B

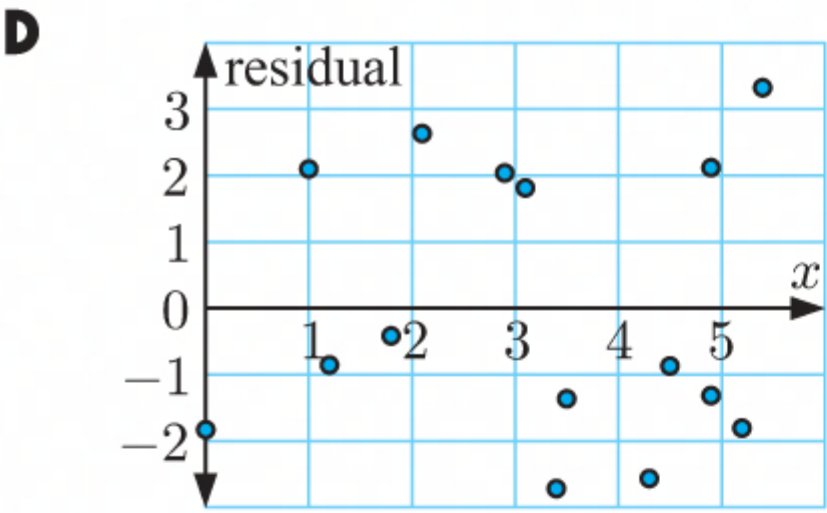
The residuals in this plot indicate that there are values which are more than 4 units from the regression line. The scatter diagram however shows that the values are within about ± 3 of the regression line.

So **B** is not the correct residual plot.

C

The residuals in this plot indicate that all values are above the regression line. The scatter diagram however shows that there are values below the regression line.

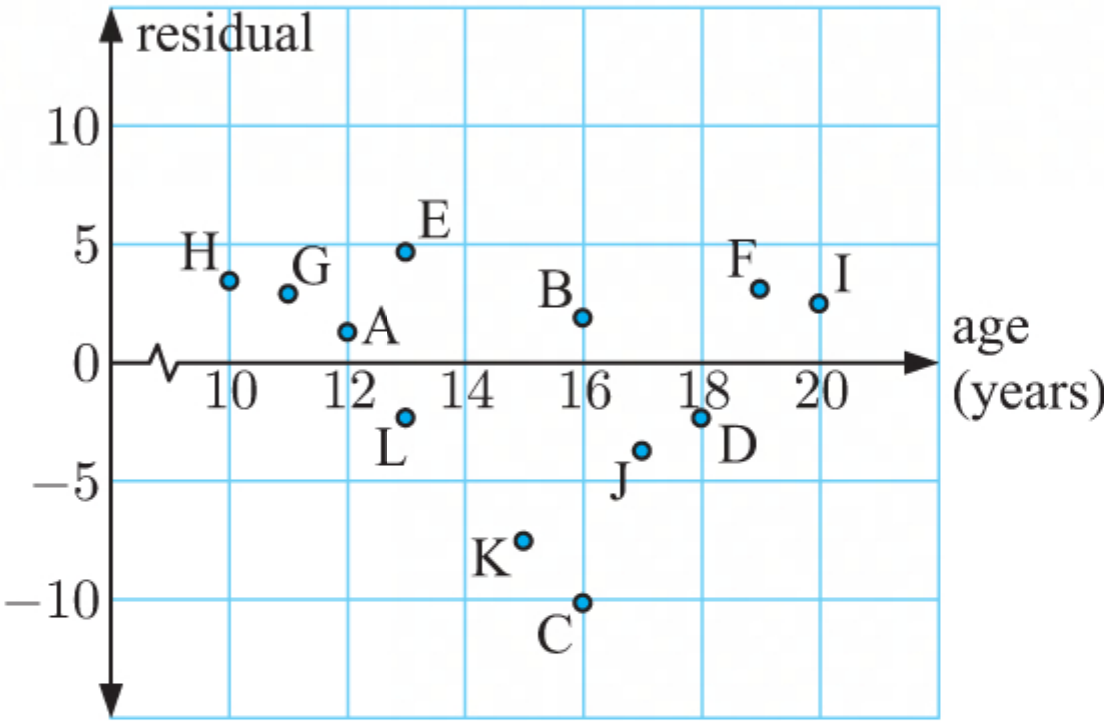
So **C** is not the correct residual plot.



The residuals in this plot indicate that all values are within about ± 3 of the regression line. The scatter diagram shows that the values are within about ± 3 of the regression line.

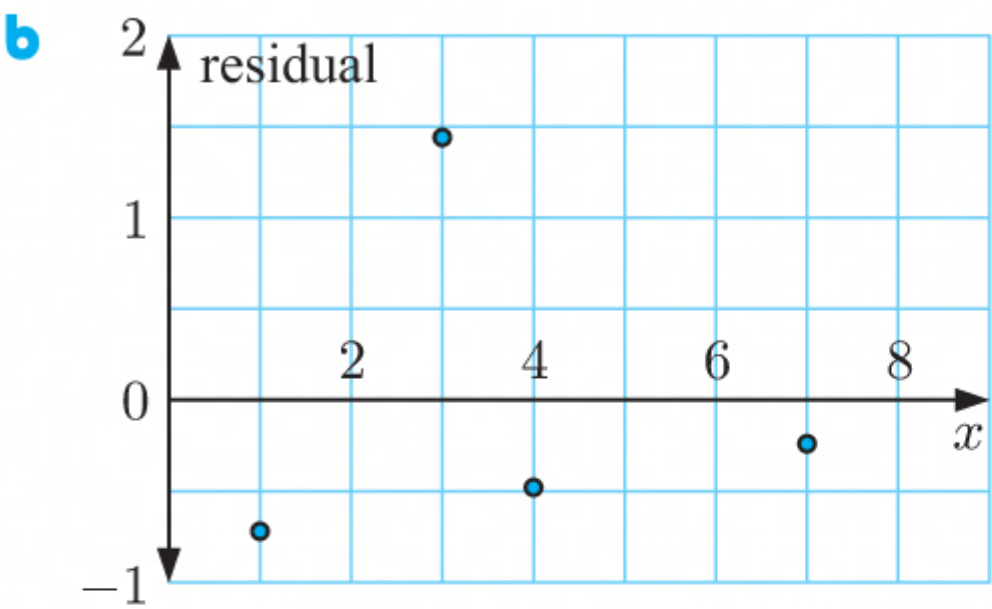
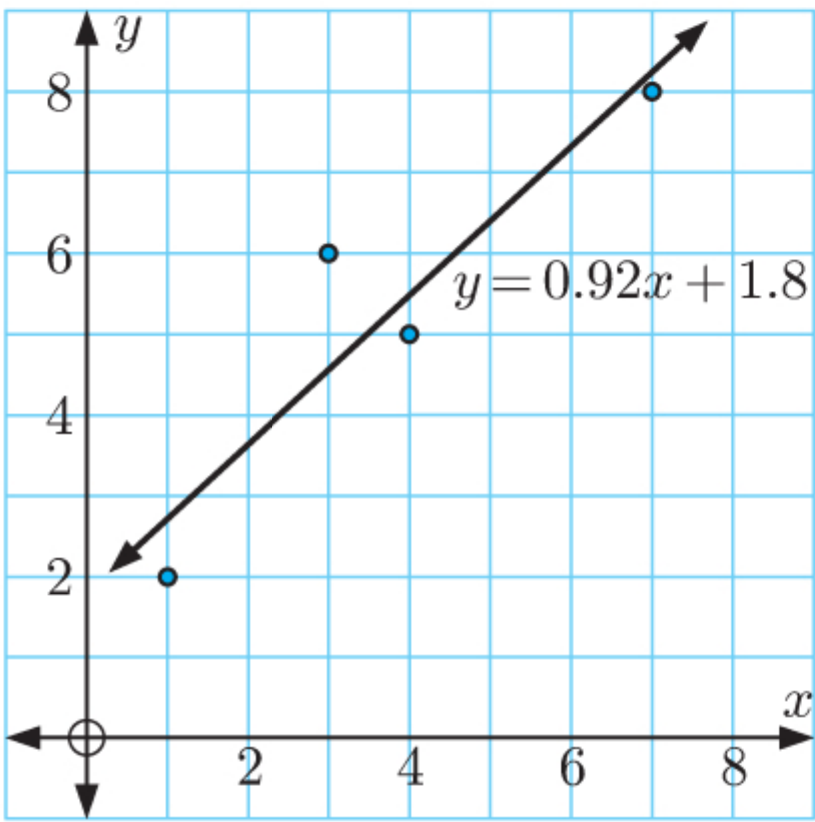
So **D** is the correct residual plot.

- 3**
- a** The athletes corresponding to the points above the x -axis threw the discus further than expected. These were athletes H, G, A, E, B, F, and I.
 - b** The athlete closest to the x -axis is A. So athlete A performed closest to what the linear model predicted.
 - c** No, it is not possible to determine which athlete threw the discus furthest. The residual plot only shows the difference between the actual distance and the predicted distance.



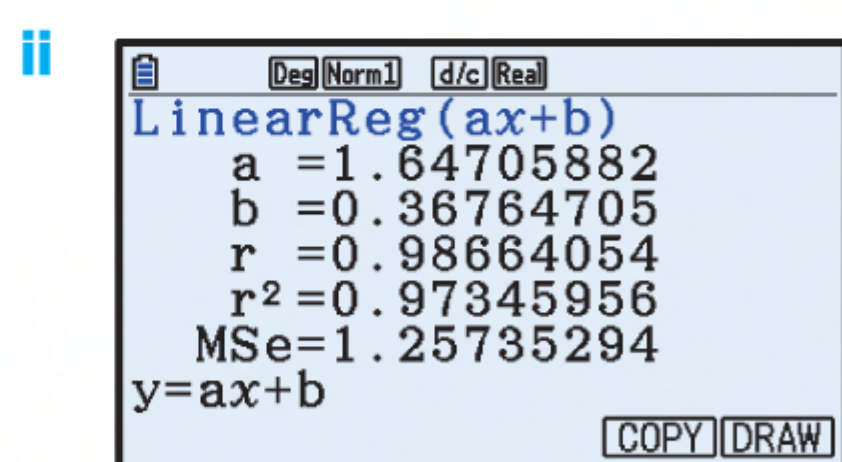
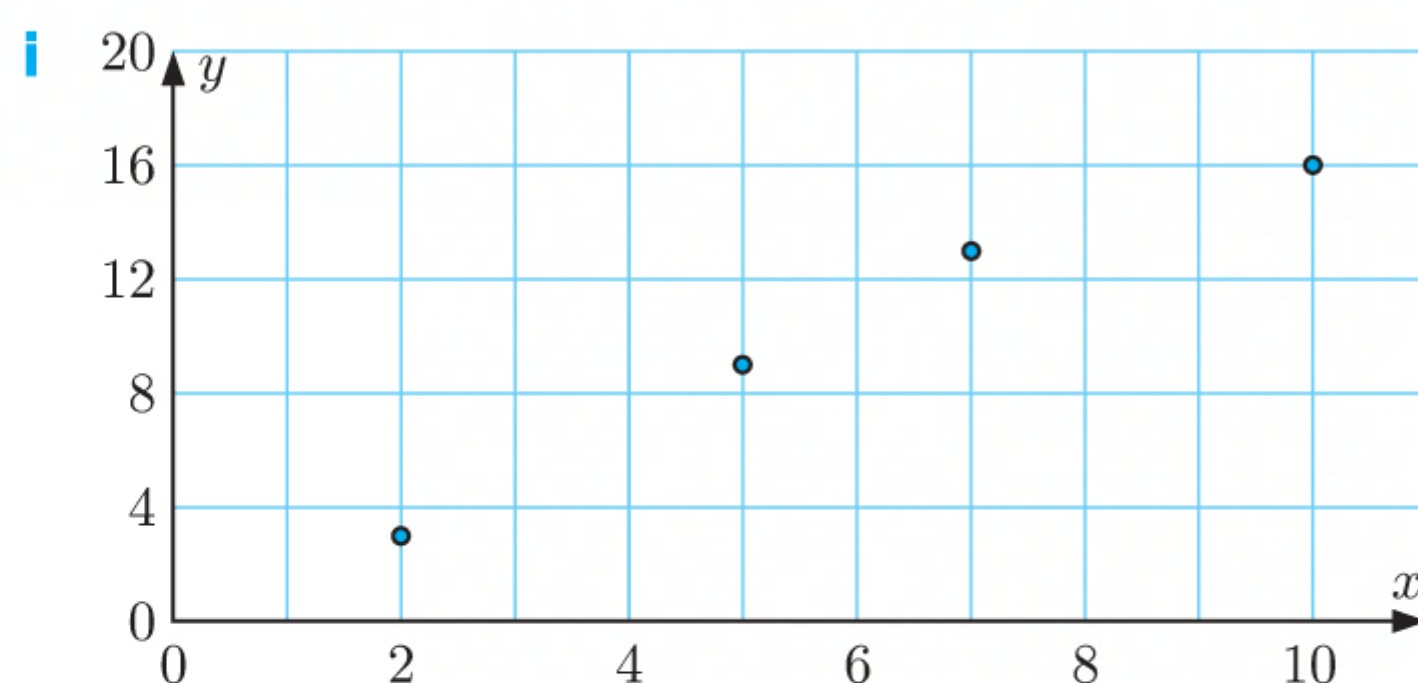
- 4**
- a** When $x = 3$, $y = 0.92(3) + 1.8 = 4.56$
When $x = 4$, $y = 0.92(4) + 1.8 = 5.48$
When $x = 7$, $y = 0.92(7) + 1.8 = 8.24$
So, the table is:

x	y_{obs}	y_{pred}	residual = $y_{\text{obs}} - y_{\text{pred}}$
1	2	2.72	-0.72
3	6	4.56	1.44
4	5	5.48	-0.48
7	8	8.24	-0.24



5 a

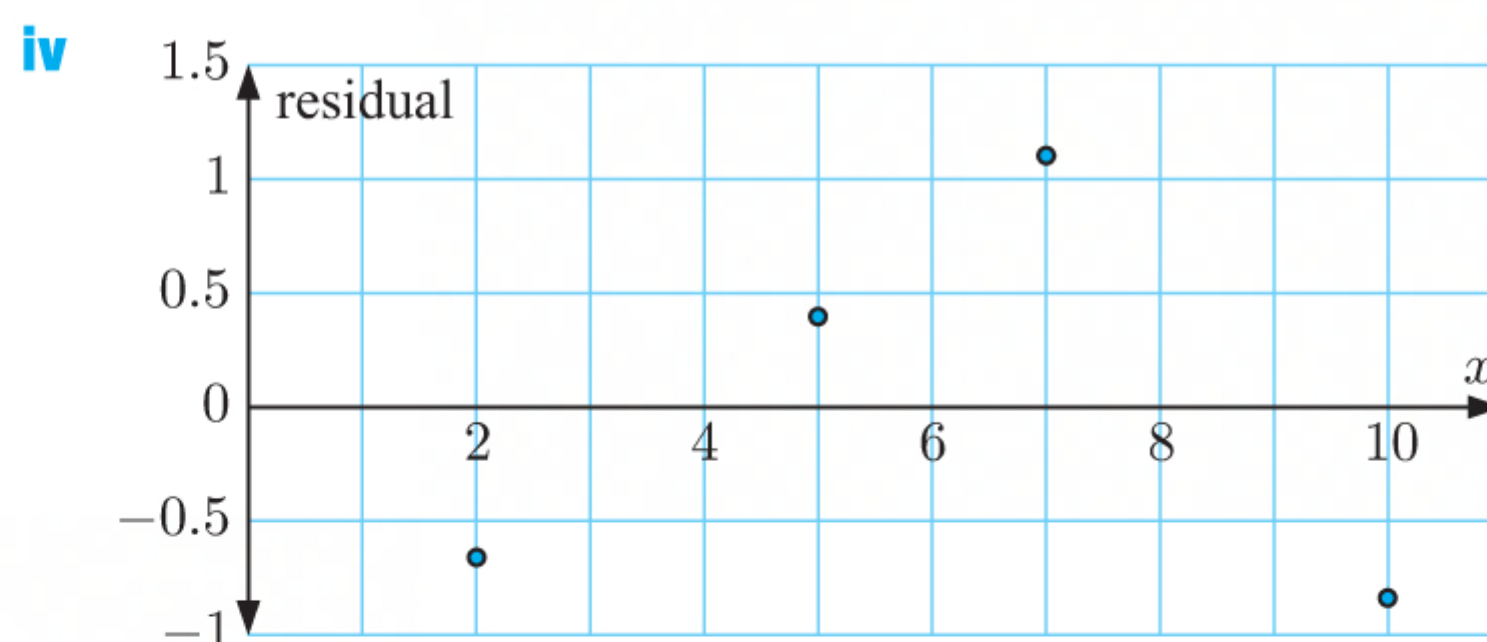
x	2	5	7	10
y	3	9	13	16



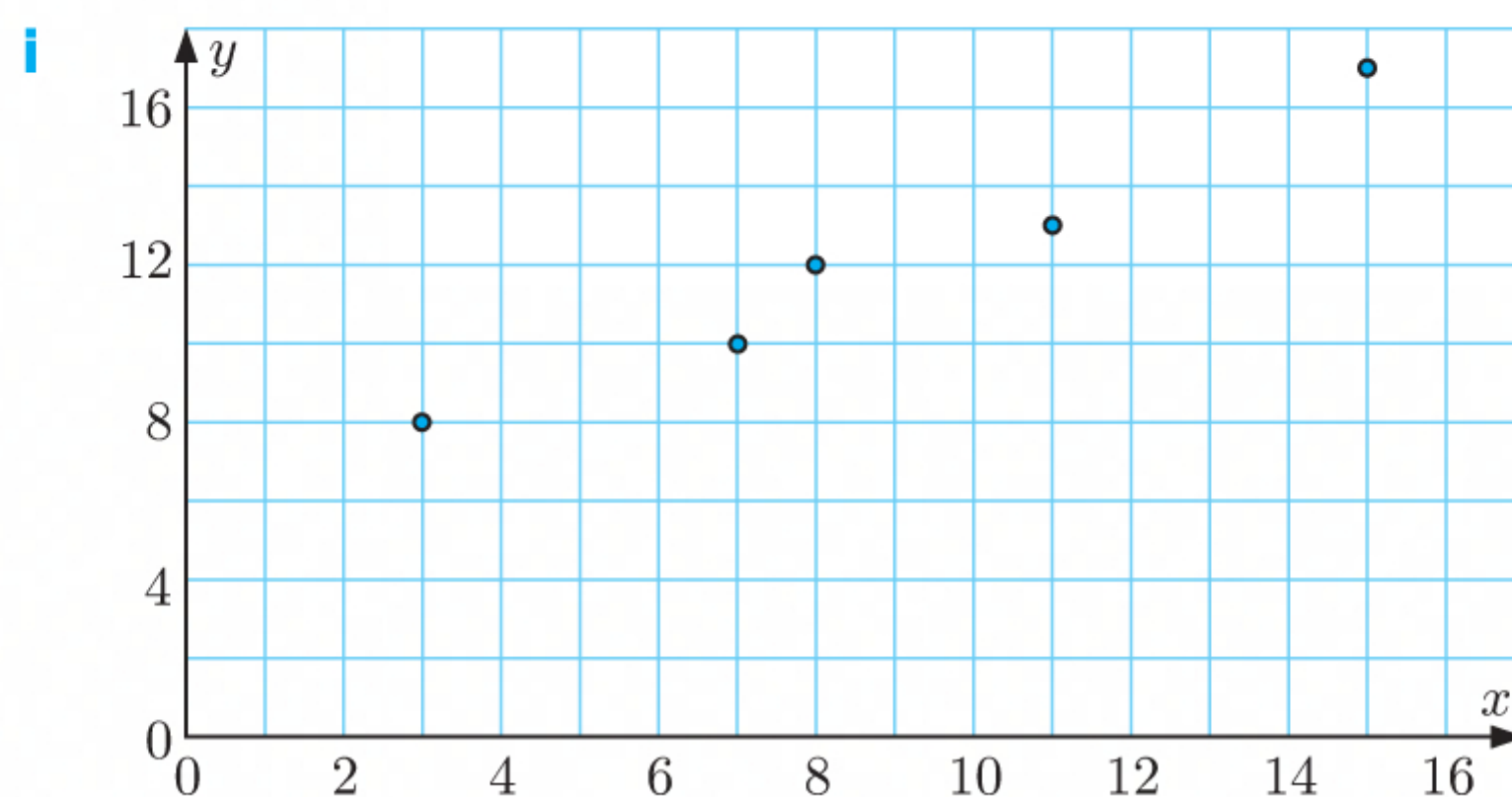
Using technology, the least squares regression line is $y \approx 1.65x + 0.368$.

iii We find y_{pred} for each data point by evaluating $y \approx 1.65x + 0.368$ for each of the x -values.

x	y_{obs}	y_{pred}	residual = $y_{\text{obs}} - y_{\text{pred}}$
2	3	3.66	-0.66
5	9	8.60	0.40
7	13	11.90	1.10
10	16	16.84	-0.84

**b**

x	3	7	8	11	15
y	8	10	12	13	17



ii

LinearReg(ax+b)

a =0.74257425

b =5.46534653

r =0.98416214

r²=0.96857511

MSe=0.48184818

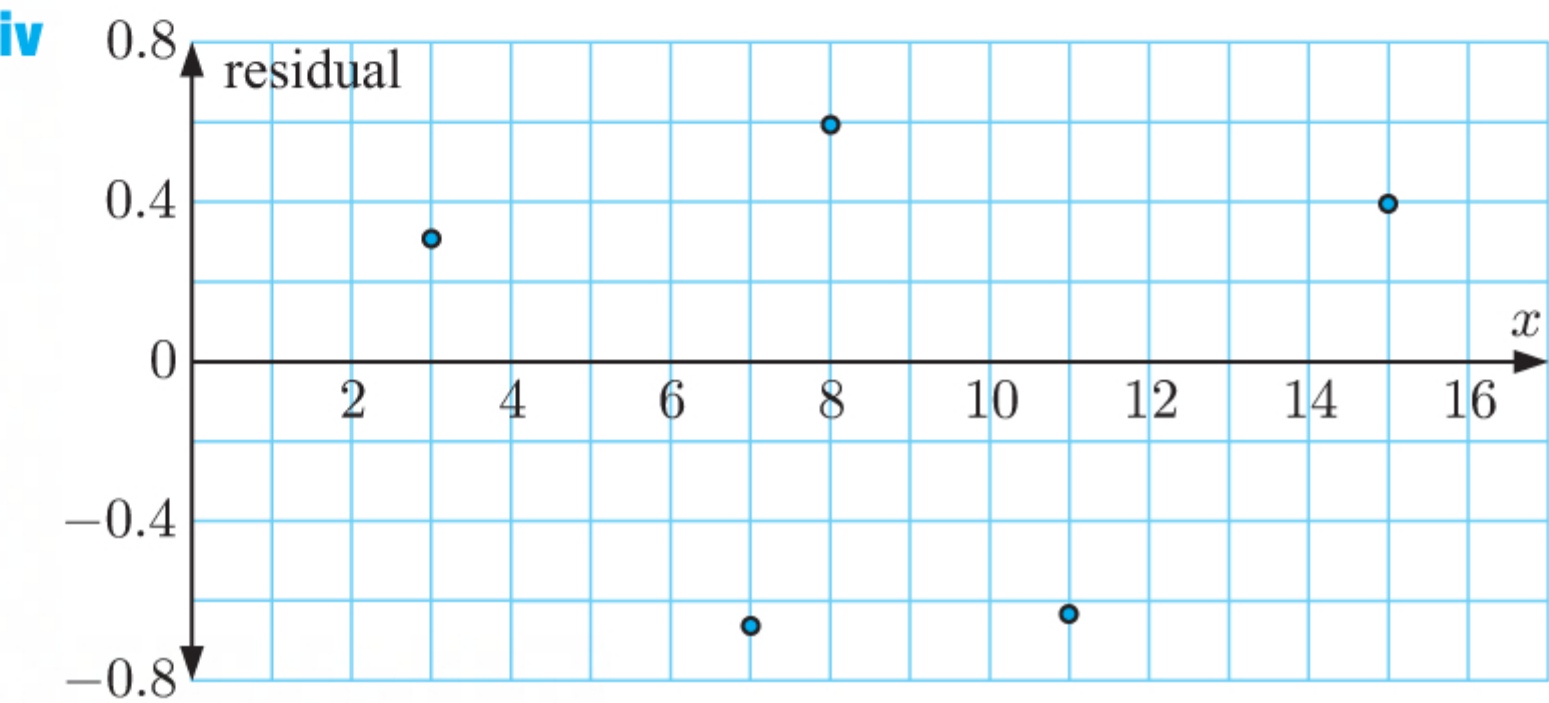
y=ax+b

COPYDRAW

Using technology, the least squares regression line is $y \approx 0.743x + 5.47$.

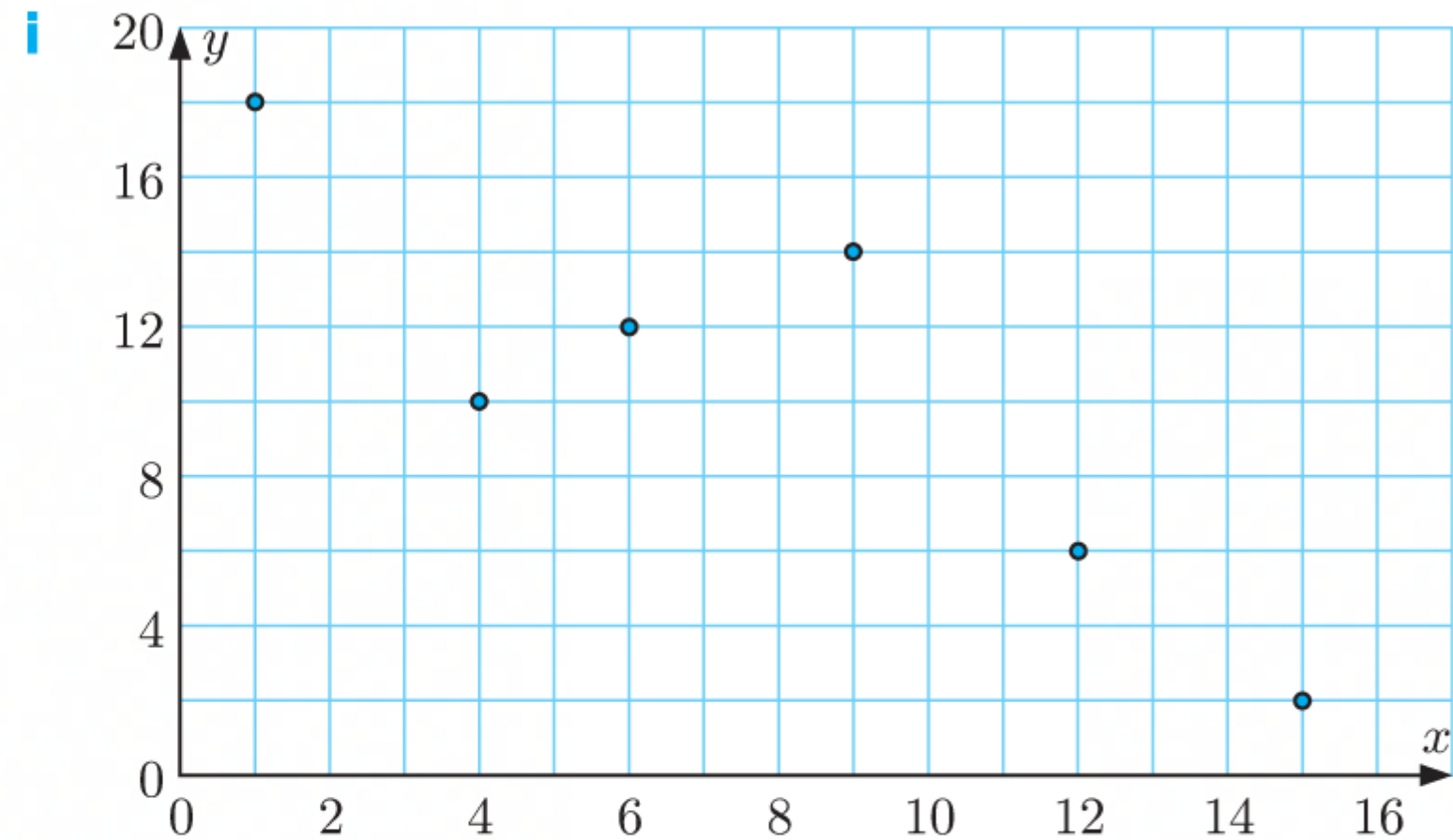
iii We find y_{pred} for each data point by evaluating $y \approx 0.743x + 5.47$ for each of the x -values.

x	y_{obs}	y_{pred}	residual = $y_{\text{obs}} - y_{\text{pred}}$
3	8	7.69	0.31
7	10	10.66	−0.66
8	12	11.41	0.59
11	13	13.63	−0.63
15	17	16.60	0.40



c

x	1	9	6	15	4	12
y	18	14	12	2	10	6



ii

LinearReg(ax+b)

a =-0.9468479

b =17.750309

r =-0.8602832

r²=0.74008728

MSe=10.6131025

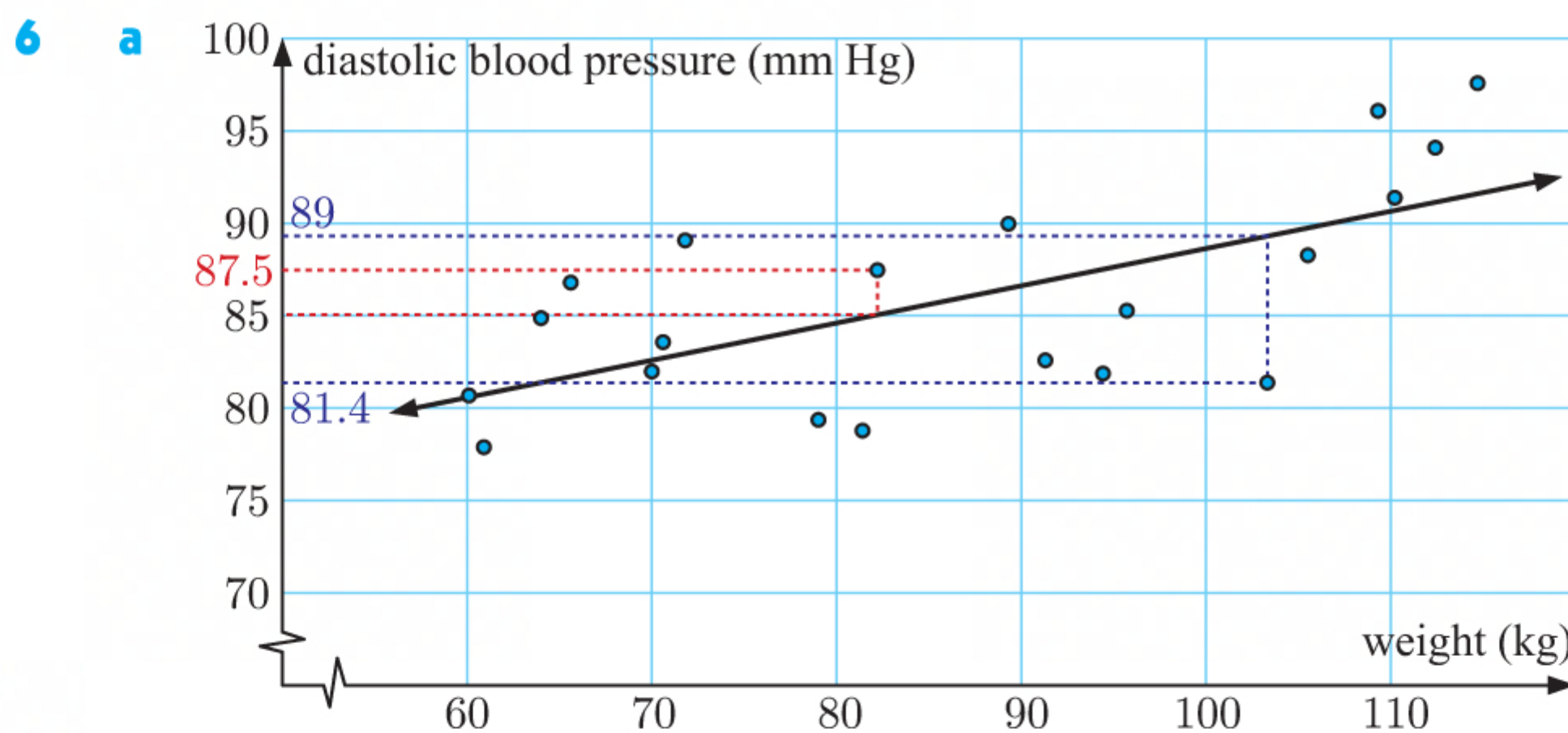
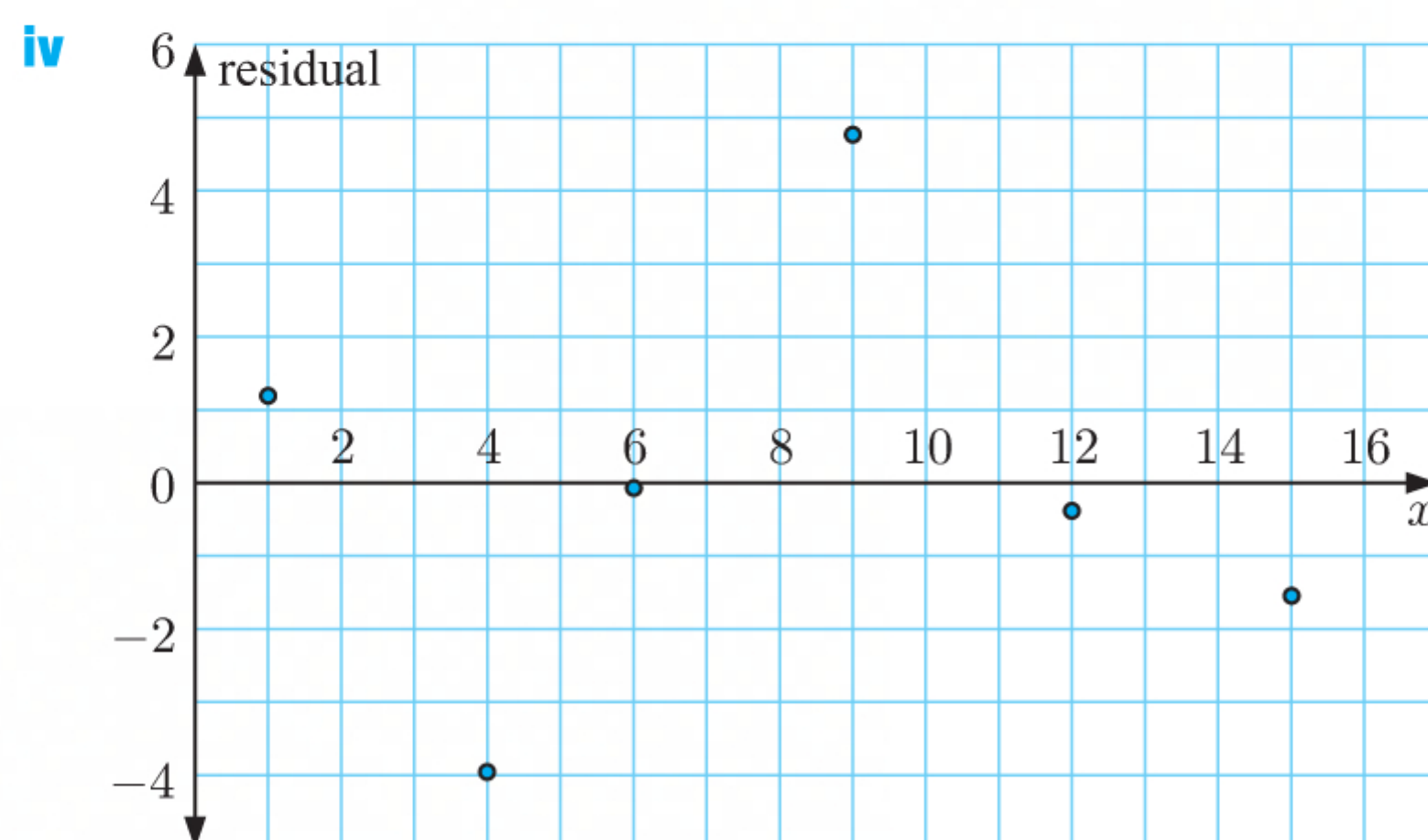
y=ax+b

COPYDRAW

Using technology, the least squares regression line is $y \approx -0.947x + 17.8$.

- iii We find y_{pred} for each data point by evaluating $y \approx -0.947x + 17.8$ for each of the x -values.

x	y_{obs}	y_{pred}	residual = $y_{\text{obs}} - y_{\text{pred}}$
1	18	16.80	1.20
9	14	9.23	4.77
6	12	12.07	-0.07
15	2	3.55	-1.55
4	10	13.96	-3.96
12	6	6.39	-0.39



- b i From the graph, the residual for the point $(82, 87.5)$ is about 2.5.

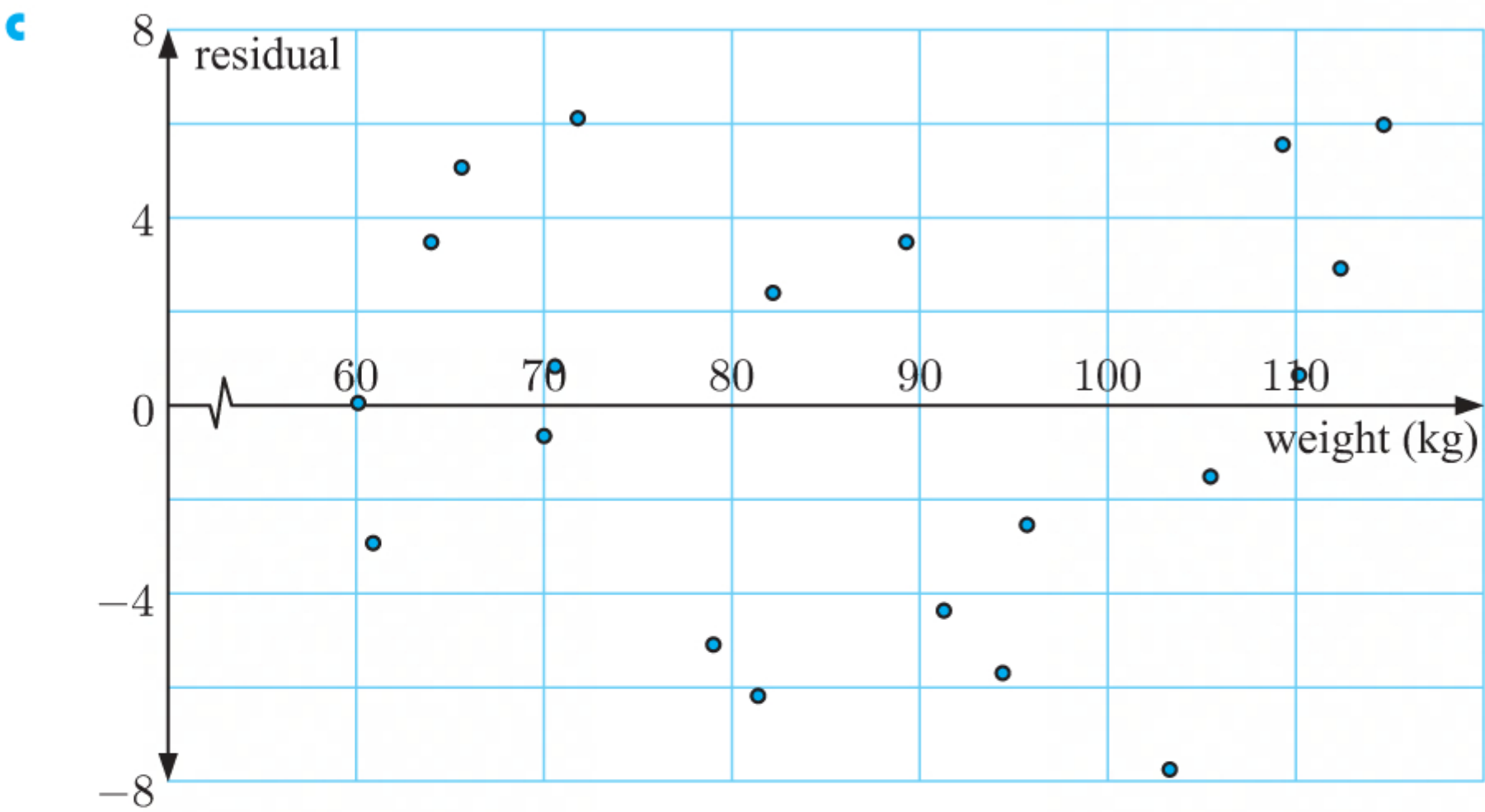
$$\begin{aligned} \text{From the equation, when } x = 82, \quad y_{\text{pred}} &= 68.5 + 0.2 \times 82 \\ &= 84.9 \end{aligned}$$

$$\begin{aligned} \therefore \text{ the residual} &= y_{\text{obs}} - y_{\text{pred}} \\ &= 87.5 - 84.9 \\ &= 2.6 \end{aligned}$$

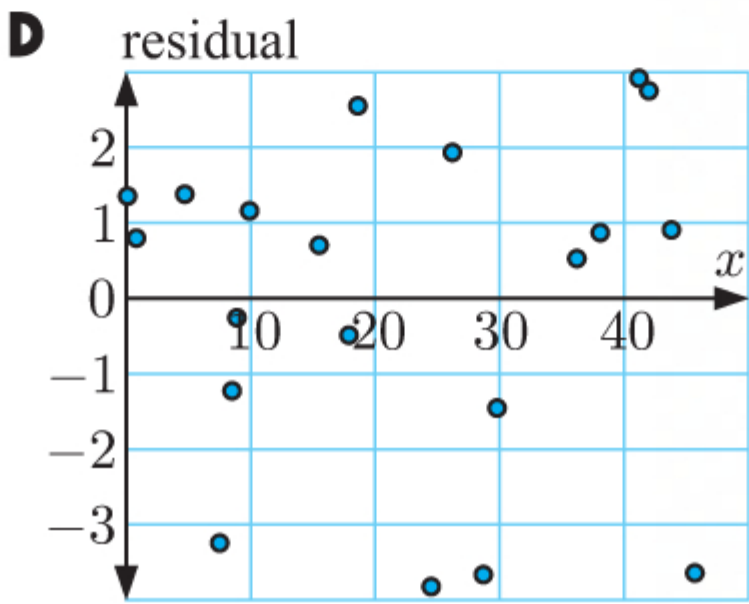
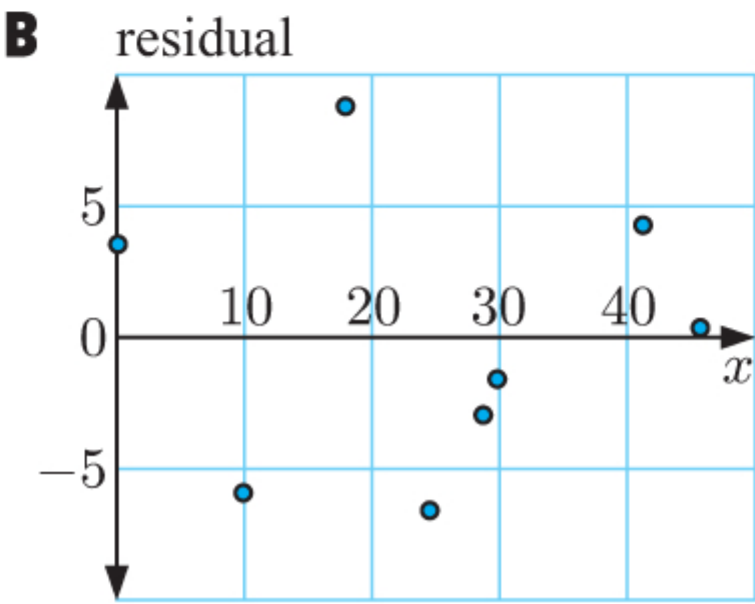
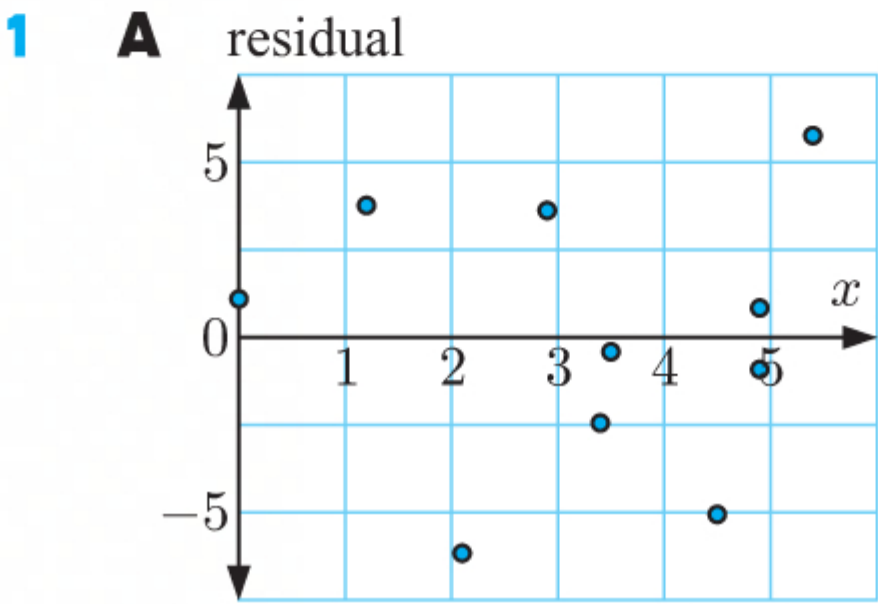
- ii From the graph, the residual for the point $(103.3, 81.4)$ is about -8.

$$\begin{aligned} \text{From the equation, when } x = 103.3, \quad y_{\text{pred}} &= 68.5 + 0.2 \times 103.3 \\ &= 89.16 \end{aligned}$$

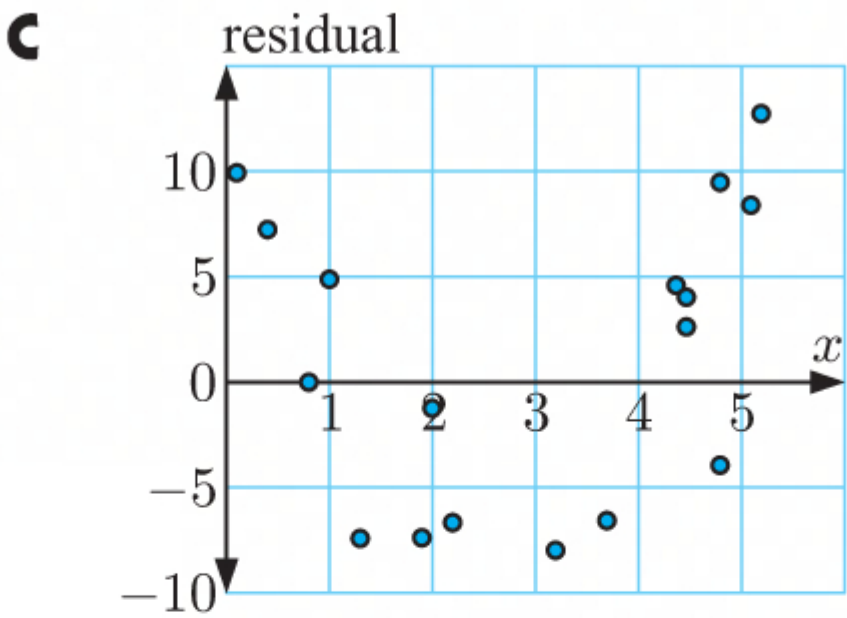
$$\begin{aligned} \therefore \text{ the residual} &= y_{\text{obs}} - y_{\text{pred}} \\ &= 81.4 - 89.16 \\ &= -7.76 \end{aligned}$$



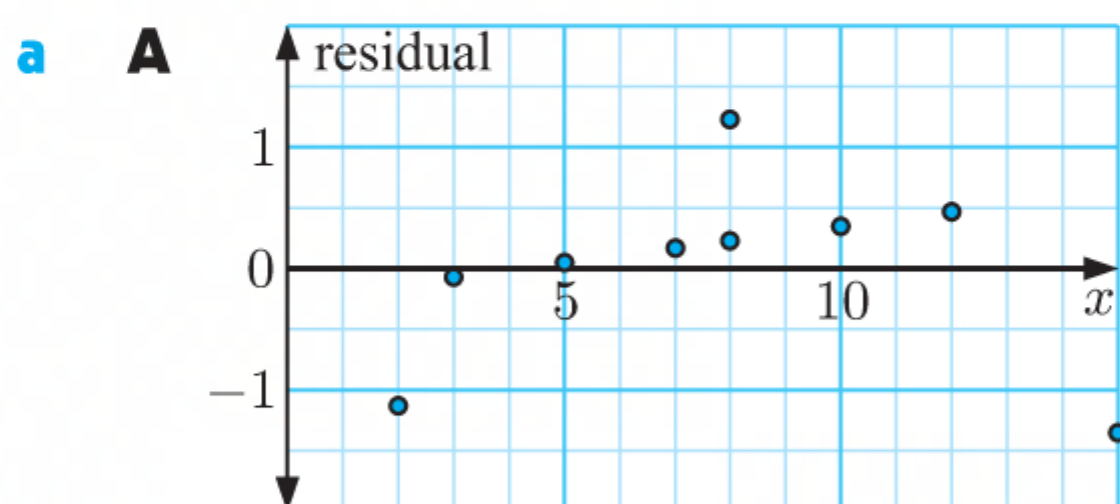
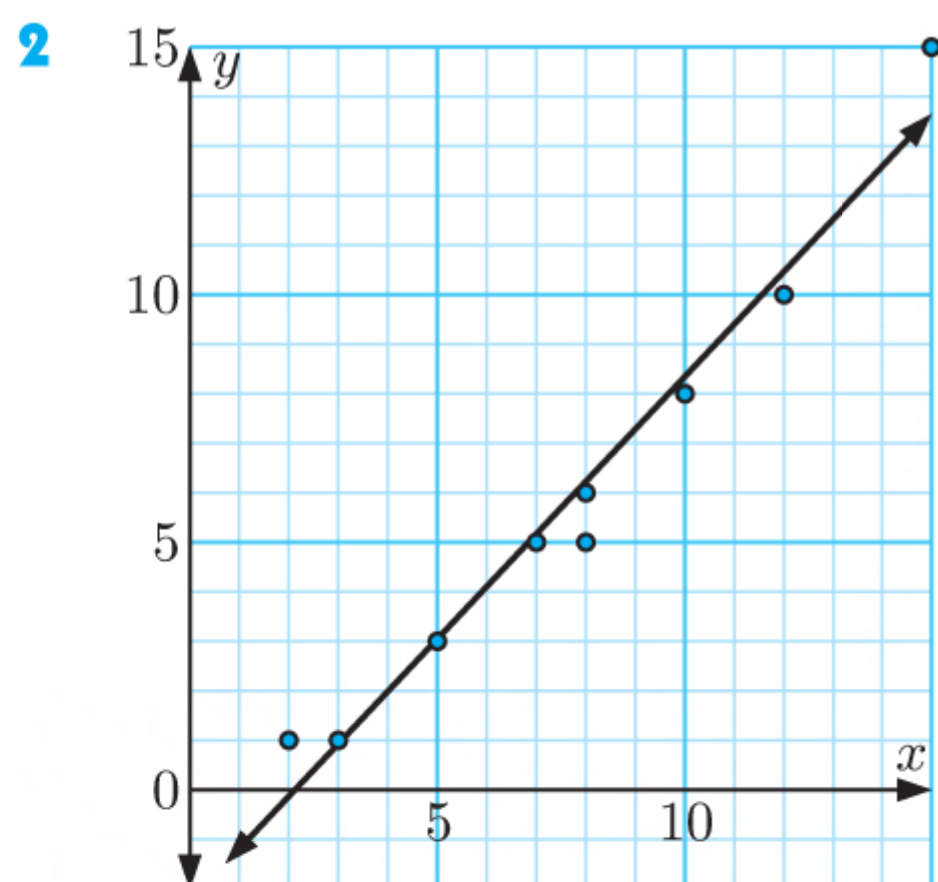
PART 2: ANALYSING RESIDUAL PLOTS



The residual plots for **A**, **B**, and **D** show points randomly scattered about the x -axis, with no obvious pattern.

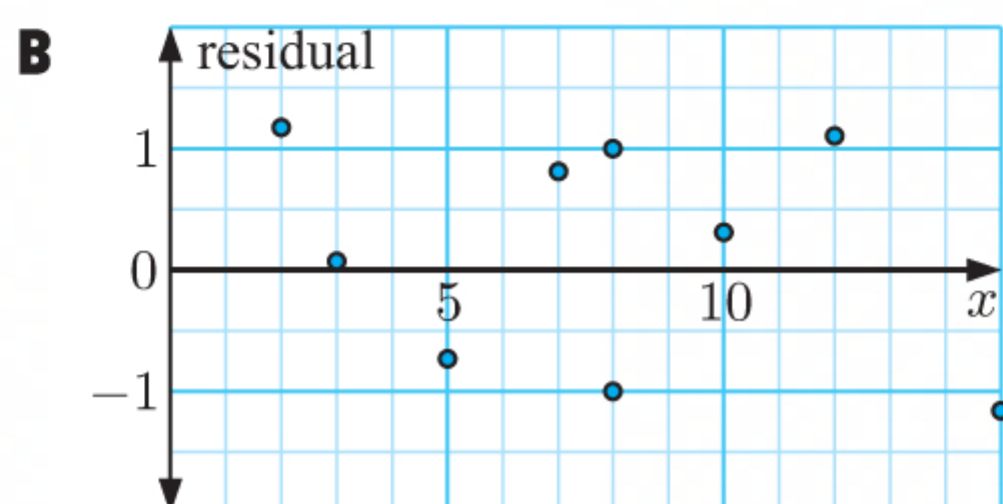


The residual plot for **C** however shows a clear, non-random pattern.
So the residual plot for **C** shows a regression line which is not a good fit for the data.



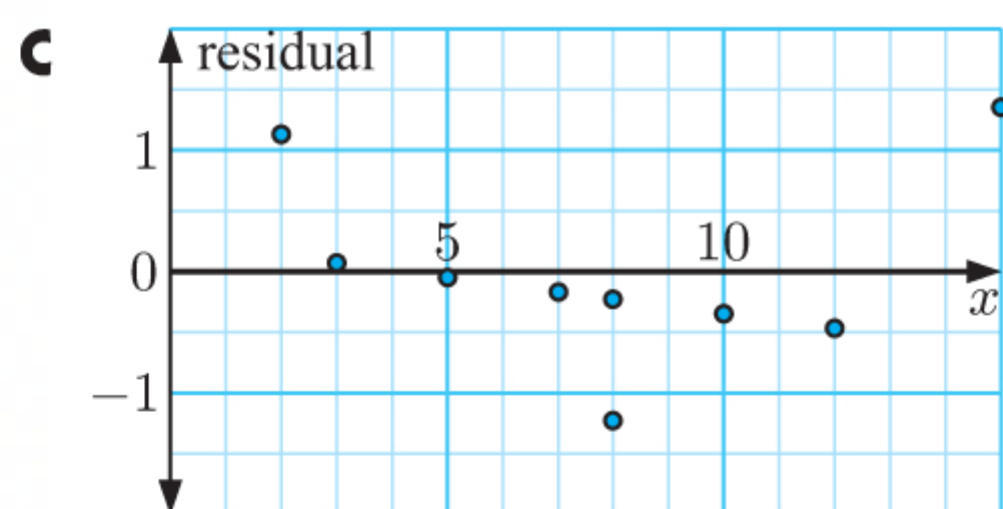
Most of the residuals in this plot are above the x -axis. The scatter diagram however shows that only three data values are above the regression line.

So **A** is not the correct residual plot.



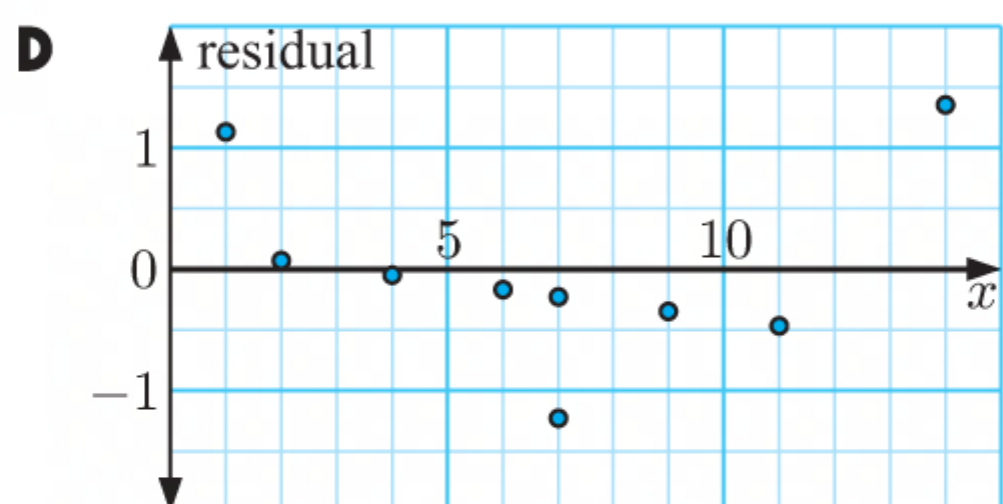
Most of the residuals in this plot are above the x -axis. The scatter diagram however shows that only three data values are above the regression line.

So **B** is not the correct residual plot.



Most of the residuals in this plot are below the x -axis, with only three residuals above it. The scatter diagram shows that only three data values are above the regression line, and the values on the x -axis correspond to those on the scatter diagram.

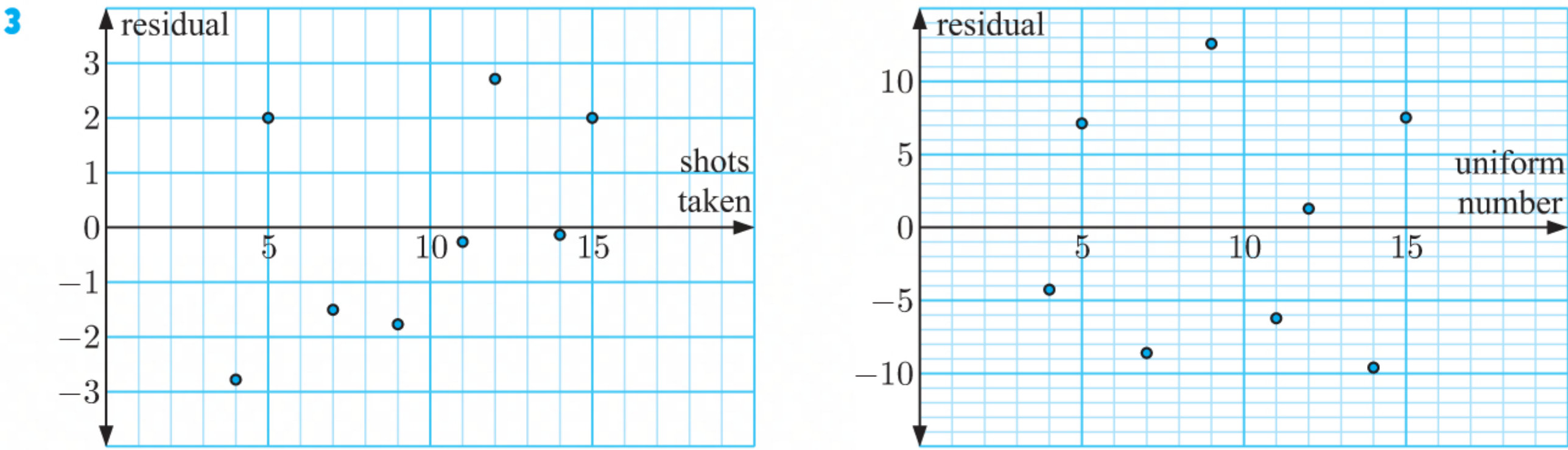
So **C** is the correct residual plot.



This is very similar to residual plot **C**, except the values on the x -axis do not correspond to those on the scatter diagram.

So **D** is not the correct residual plot.

- b** The residual plot does not appear to be random, so a linear model may not be appropriate for the data.

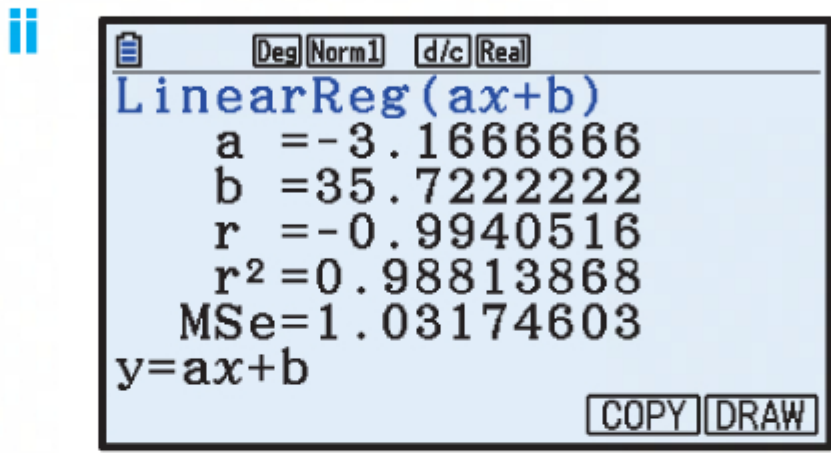
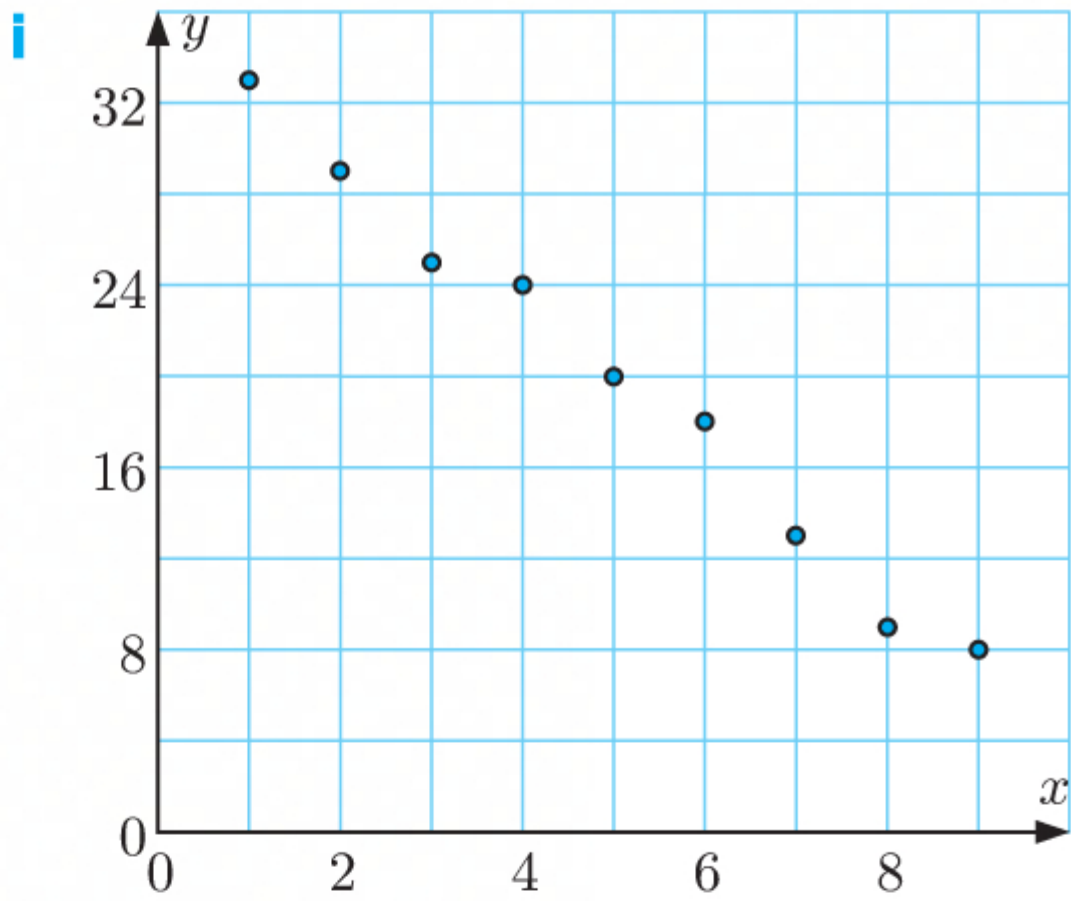


- a
- Yes, the points in both plots appear to be randomly scattered.
- b
- A linear model is most appropriate for the *points scored vs shots taken* data set. The residuals in this plot are generally smaller, which means that the points are generally closer to the least squares regression line.

4

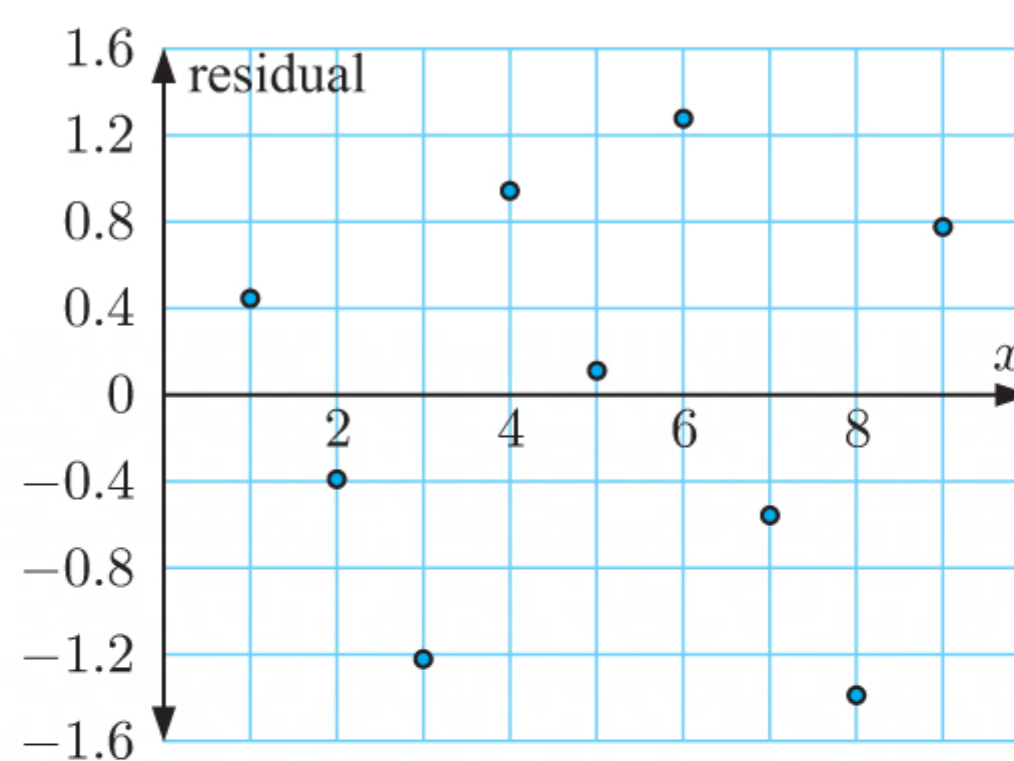
a

x	1	2	3	4	5	6	7	8	9
y	33	29	25	24	20	18	13	9	8



Using technology, the least squares regression line is $y \approx -3.17x + 35.7$, and $r \approx -0.994$.

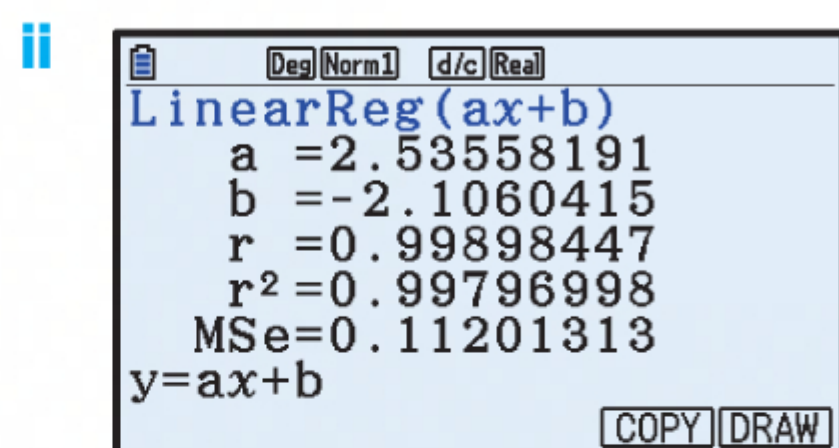
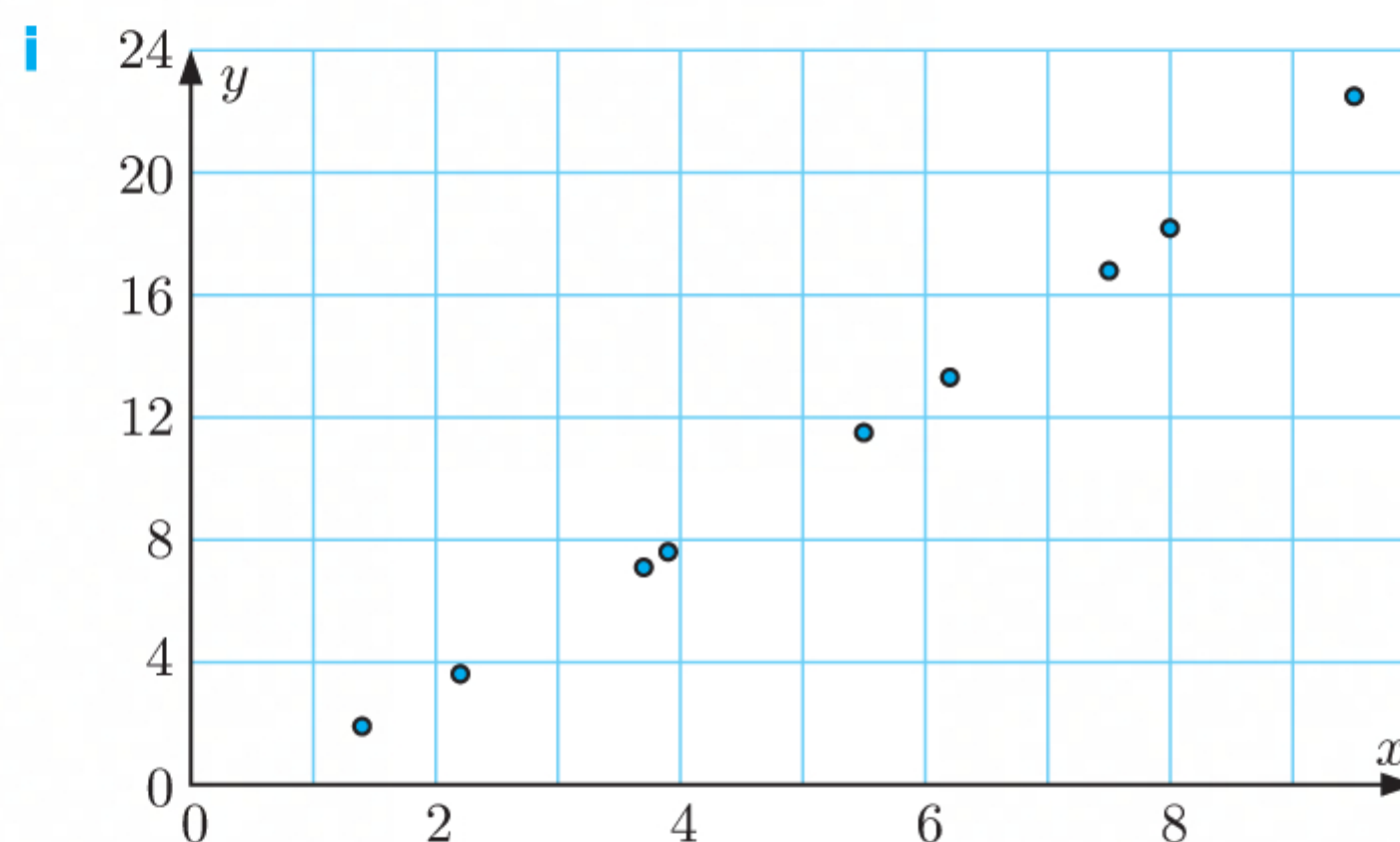
iii Using technology, the residual plot is:



iv There is very strong correlation and the residuals are randomly scattered, so the least squares regression line is appropriate to model the data.

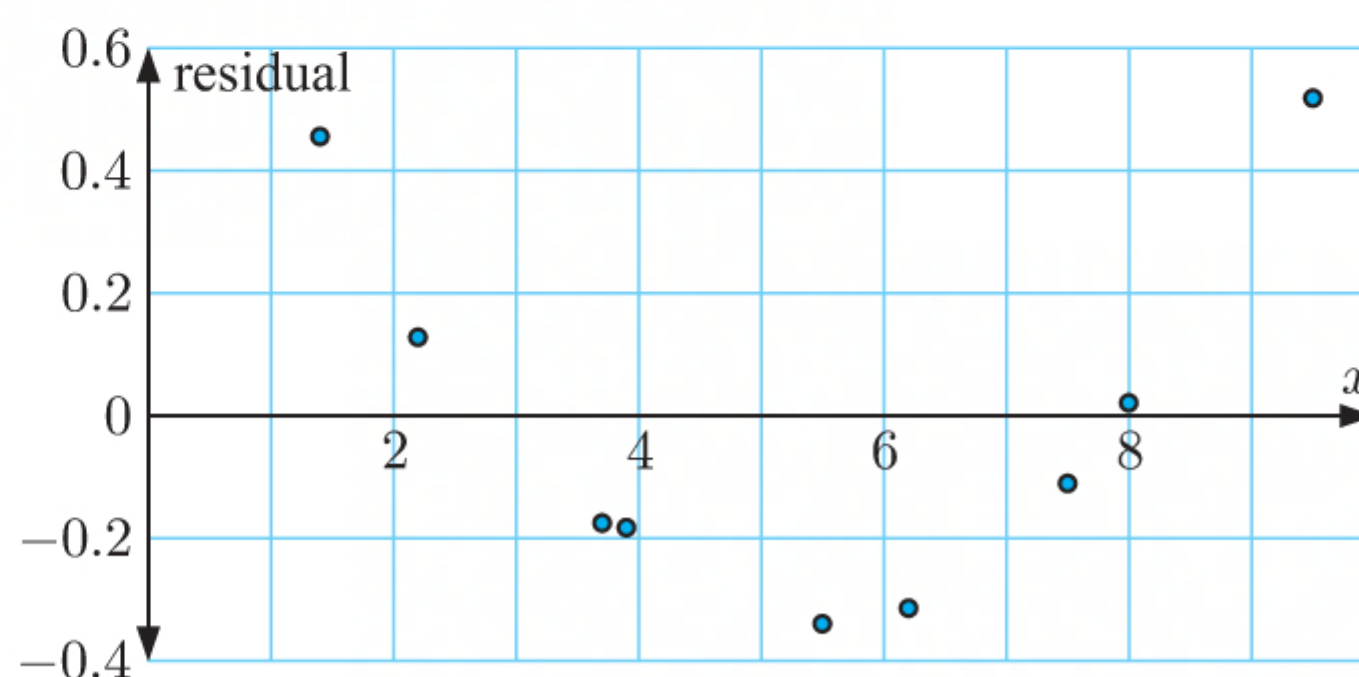
b

x	2.2	3.7	9.5	6.2	1.4	3.9	7.5	8	5.5
y	3.6	7.1	22.5	13.3	1.9	7.6	16.8	18.2	11.5



Using technology, the least squares regression line is $y \approx 2.54x - 2.11$, and $r \approx 0.999$.

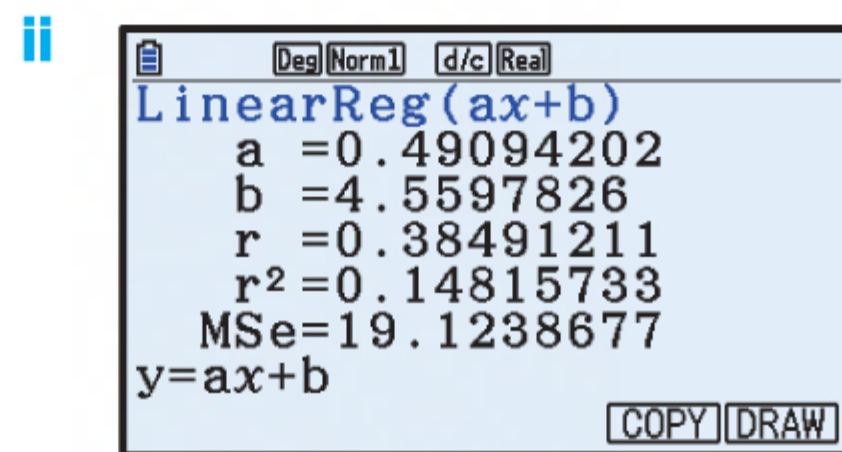
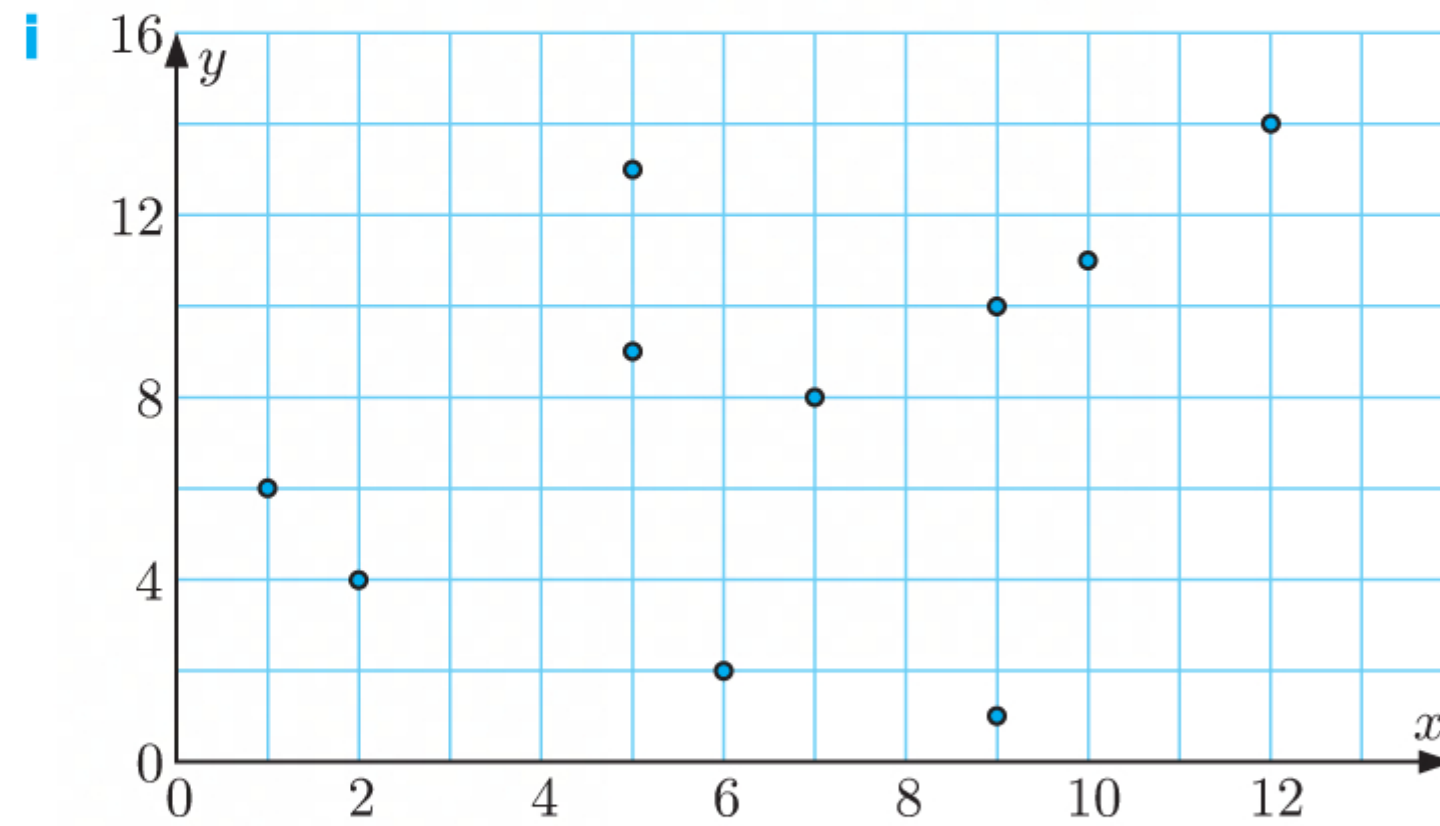
iii Using technology, the residual plot is:



- iv The residual plot shows a clear pattern and does not appear random. So the least squares regression line is not appropriate to model the data.

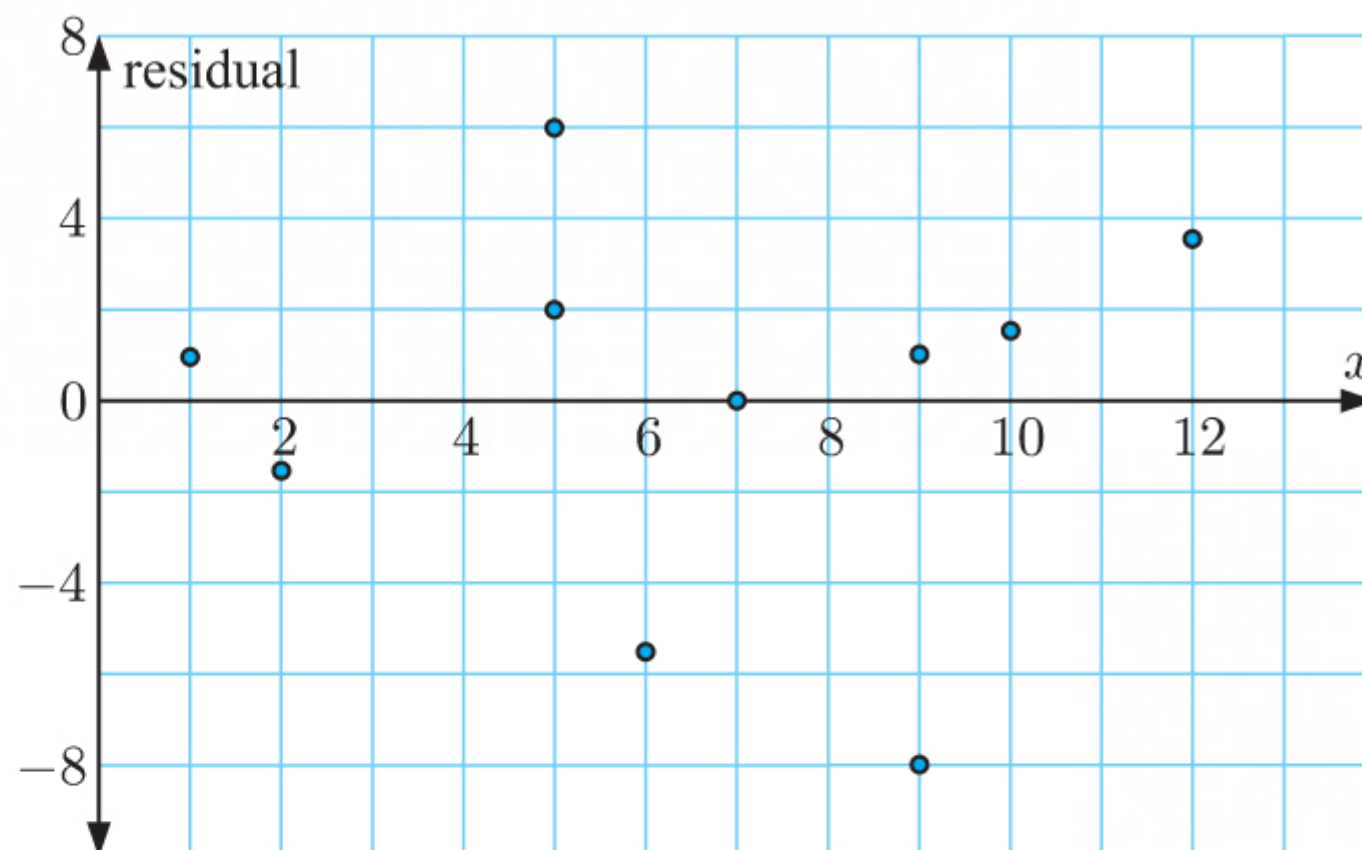
c

x	5	9	1	12	6	5	9	7	2	10
y	13	1	6	14	2	9	10	8	4	11



Using technology, the least squares regression line is $y \approx 0.491x + 4.56$, and $r \approx 0.385$.

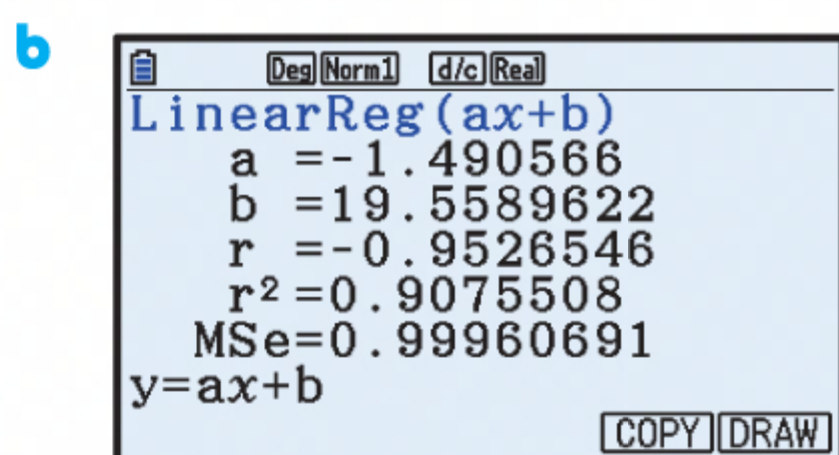
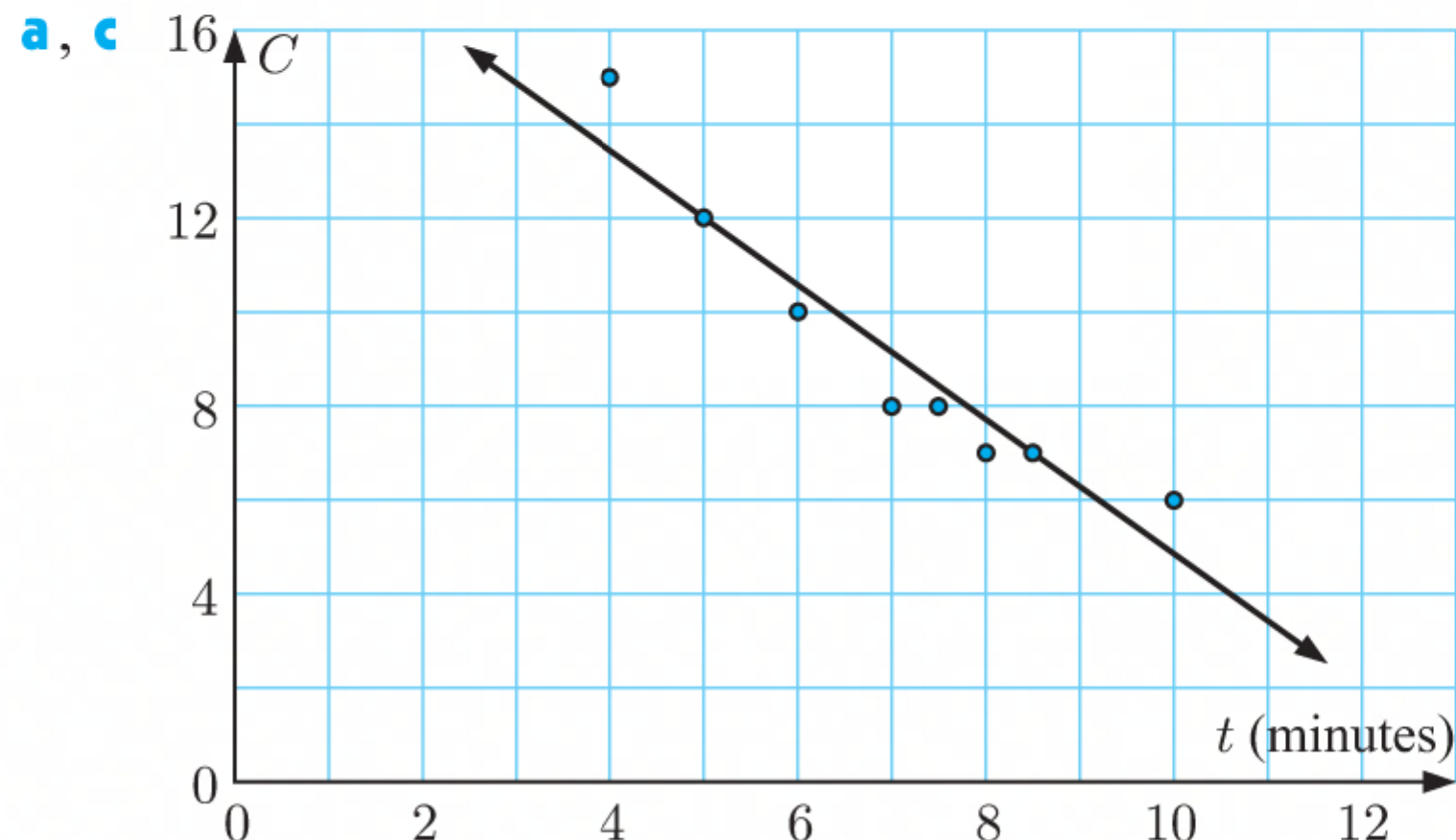
- iii Using technology, the residual plot is:



- iv The value of r is very small which indicates there is very weak correlation. So the least squares regression line is not appropriate to model the data.

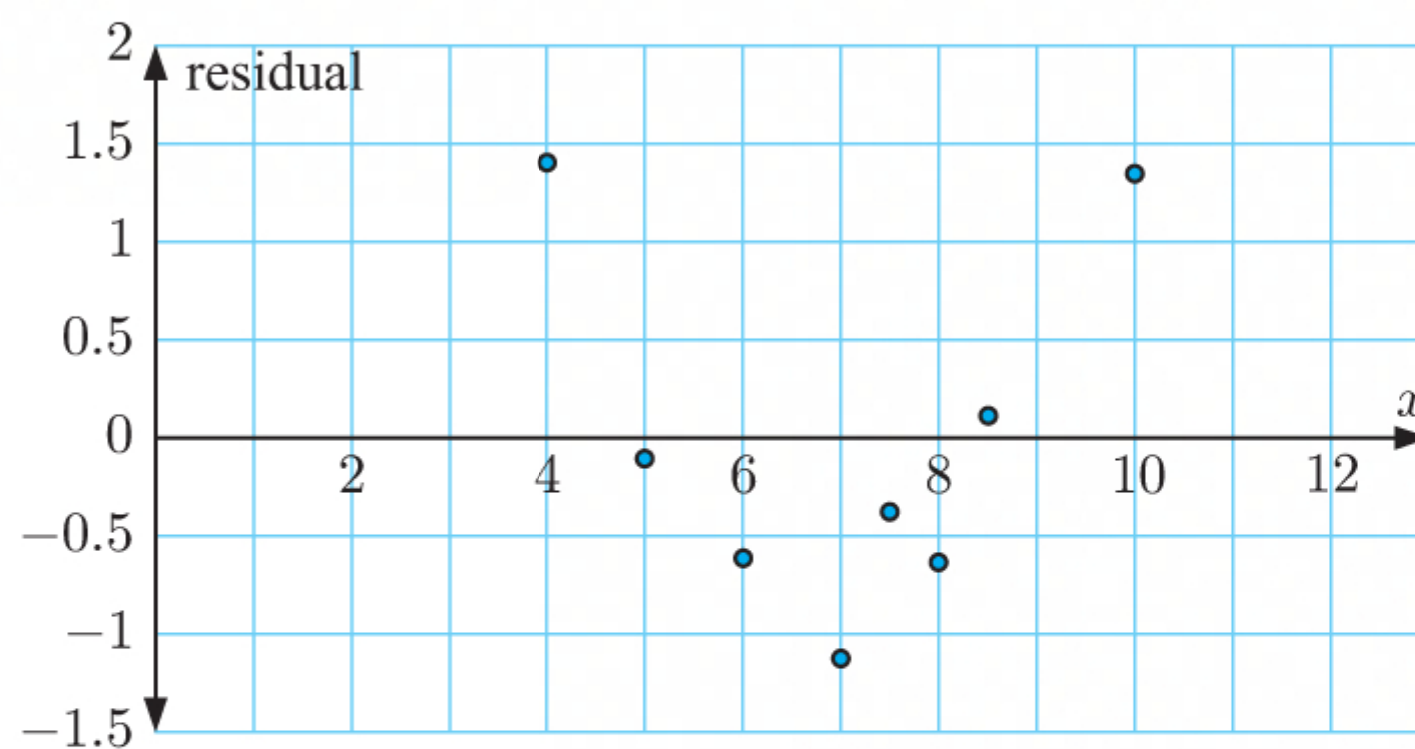
5

Time taken (t minutes)	6	8.5	4	5	8	7.5	10	7
Cranes made (C)	10	7	15	12	7	8	6	8



Using technology, the least squares regression line is $y \approx -1.49x + 19.6$, and $r \approx -0.953$.

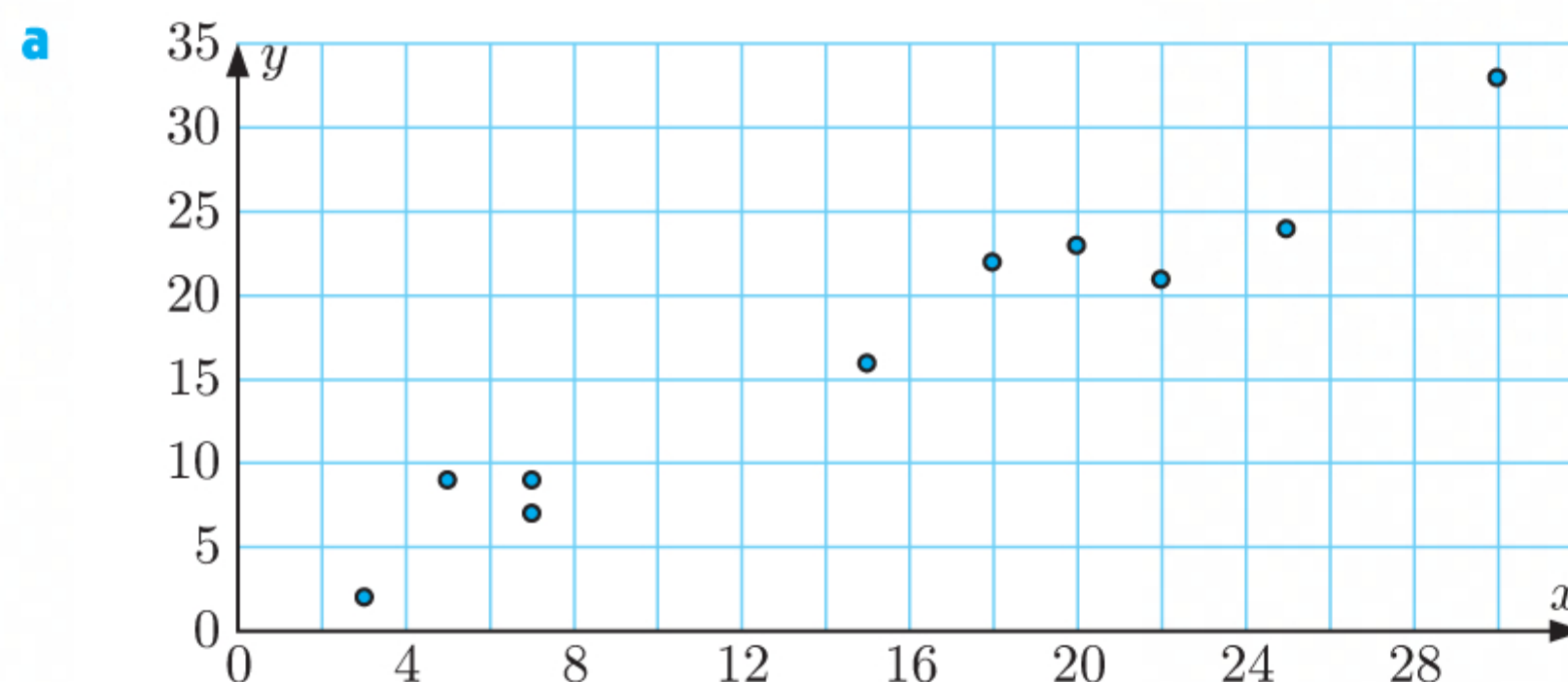
d Using technology, the residual plot is:



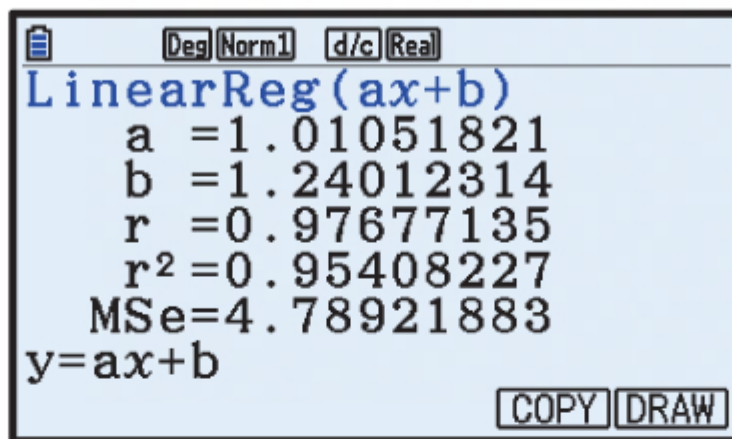
e The residual plot shows a clear pattern, and does not appear random. So the least squares regression line is not appropriate to model the data.

6

Text messages sent (x)	18	3	7	22	15	5	20	30	7	25
Text messages received (y)	22	2	9	21	16	9	23	33	7	24

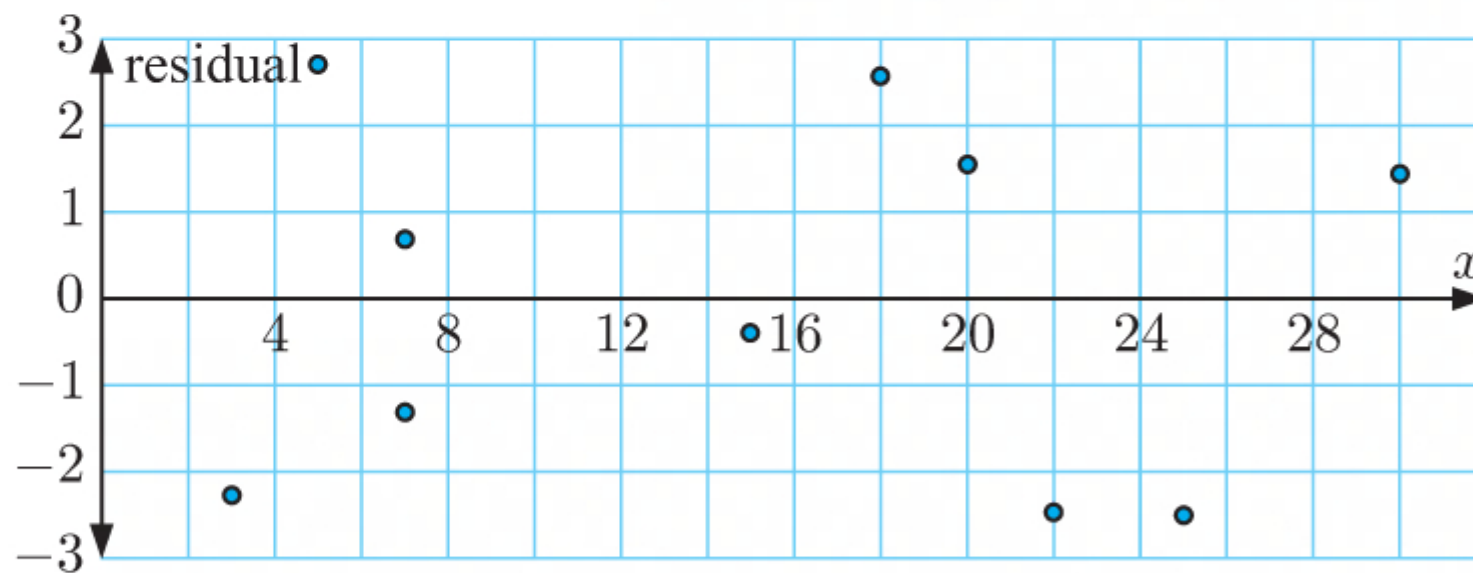


b



Using technology, the least squares regression line is $y \approx 1.01x + 1.24$, and $r \approx 0.977$.

- c There is very strong, positive correlation between *text messages sent* and *text messages received*.
- d Using technology, the residual plot is:



- e There is very strong, positive correlation and the residuals are randomly scattered. So the least squares regression line is appropriate to model the data.

- f i When $y = 10$, $10 \approx 1.01x + 1.24$
 $\therefore 8.76 \approx 1.01x$
 $\therefore x \approx 8.67$
 ≈ 9 {rounded to nearest integer}

So, we estimate Ted sent about 9 text messages.

- ii As the estimate is an interpolation with strongly correlated data, it is fairly reliable.

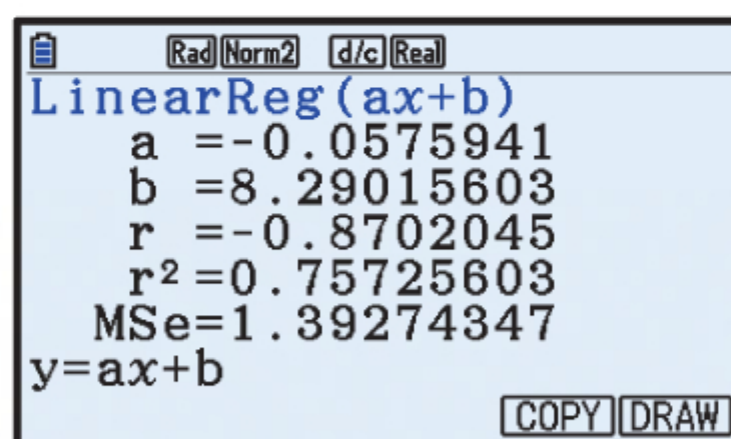
EXERCISE 19E

1	Time on homemade meals (x hours)	3.5	6.0	4.0	8.5	7.0	2.5	9.0	7.0	4.0	7.5
	Money on fast food (\$ y)	85	0	60	0	27	100	15	40	59	29

- a It is appropriate to use the regression line of x against y since the y variable, money spent on fast food, can be measured exactly. The x variable, time spent on homemade meals, will not be measured exactly.

b

	List 1	List 2	List 3	List 4
SUB				
1	3.5	85		
2	6	0		
3	4	60		
4	8.5	0		
				85



The regression line of x against y is $x \approx -0.0576y + 8.29$.

- c** **i** When $y = 45$, $x \approx -0.0576(45) + 8.29$
 ≈ 5.70

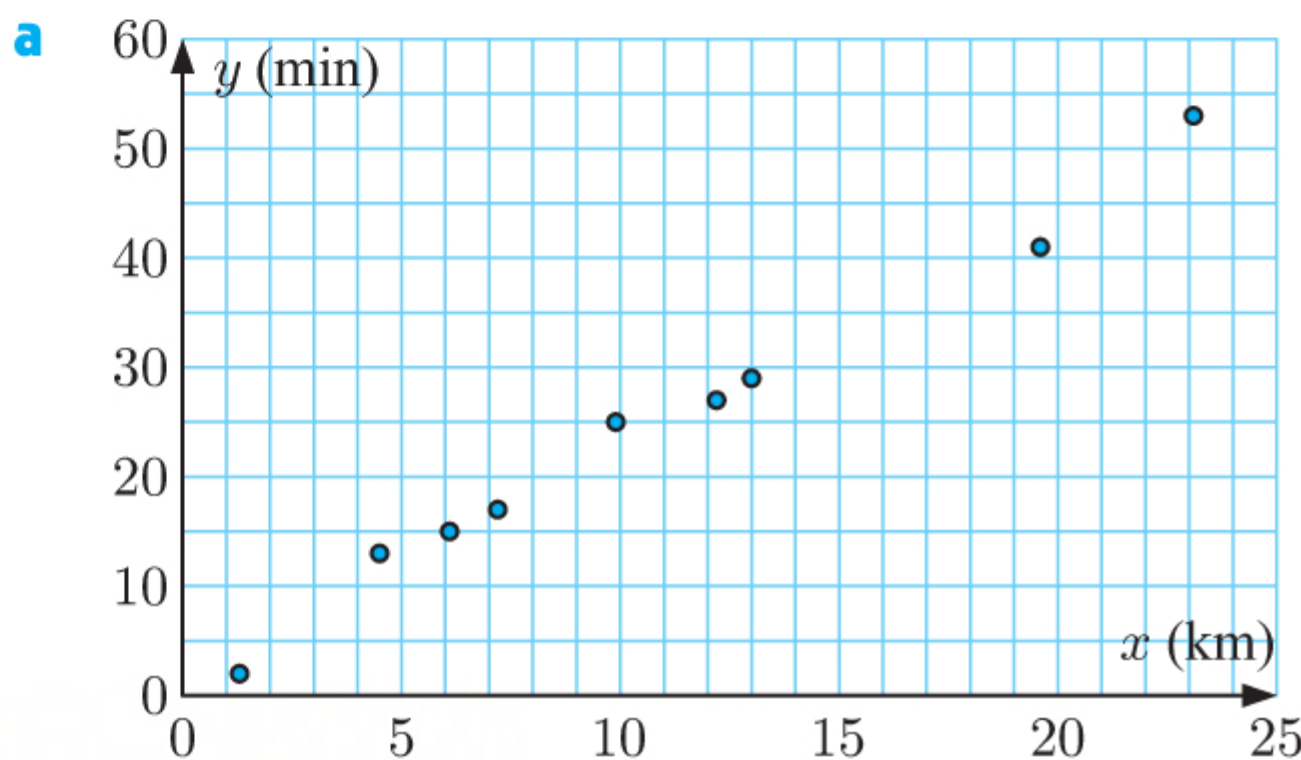
We expect a family that spends \$45 on fast food each week to spend about 5.70 hours each week preparing homemade meals.

- ii** When $x = 5$, $5 \approx -0.0576y + 8.29$
 $\therefore 0.0576y \approx 3.29$
 $\therefore y \approx 57.13$

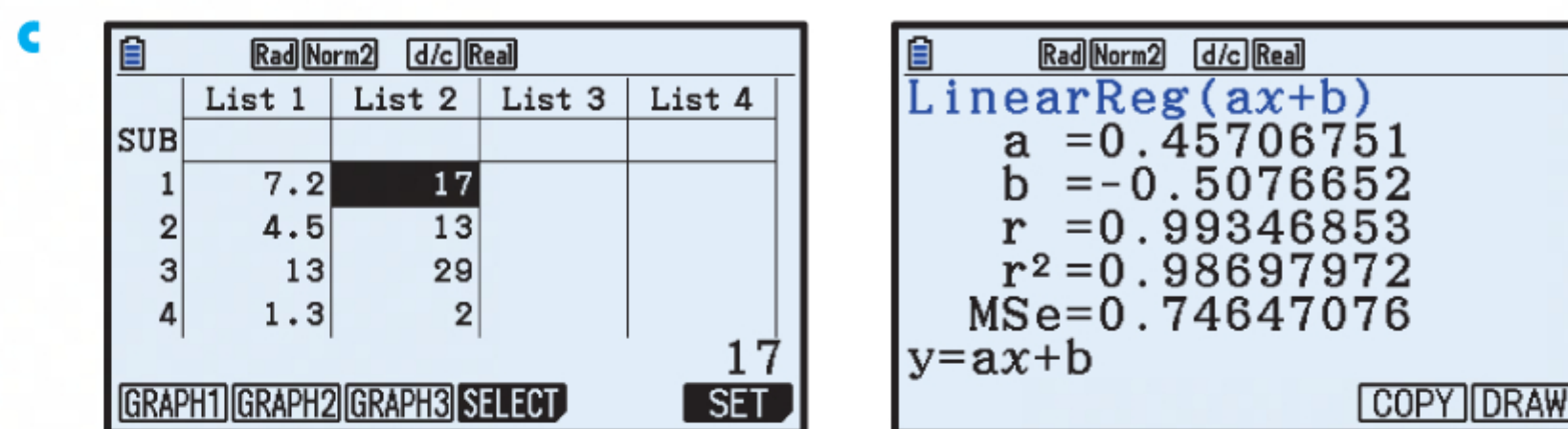
We expect a family that spends 5 hours each week preparing homemade meals to spend about \$57.13 on fast food each week.

2

<i>Distance from school (x km)</i>	7.2	4.5	13	1.3	9.9	12.2	19.6	6.1	23.1
<i>Time to travel to school (y min)</i>	17	13	29	2	25	27	41	15	53



- b** We should use the regression line of x against y , since a student's time taken to travel to school can be more precisely measured than their distance from school.



The regression line of x against y is $x \approx 0.457y - 0.508$.

When $x = 15$, $15 \approx 0.457y - 0.508$

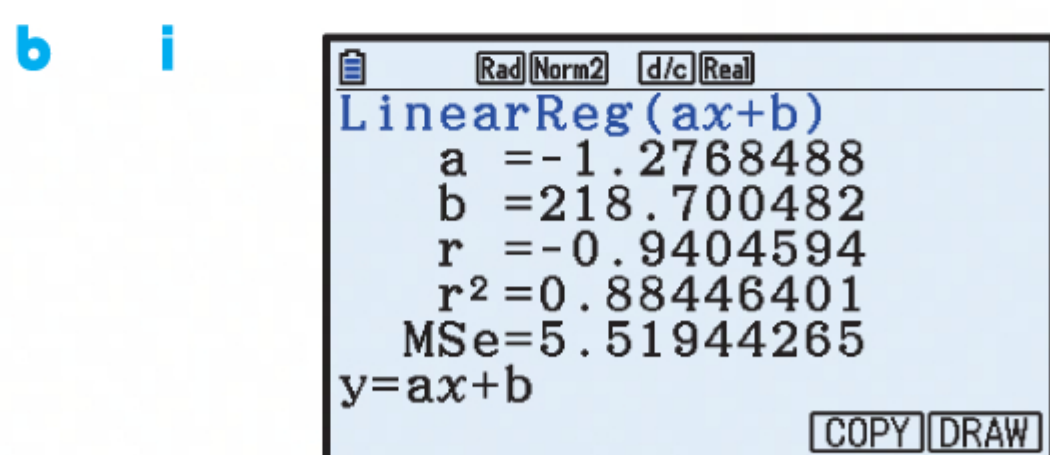
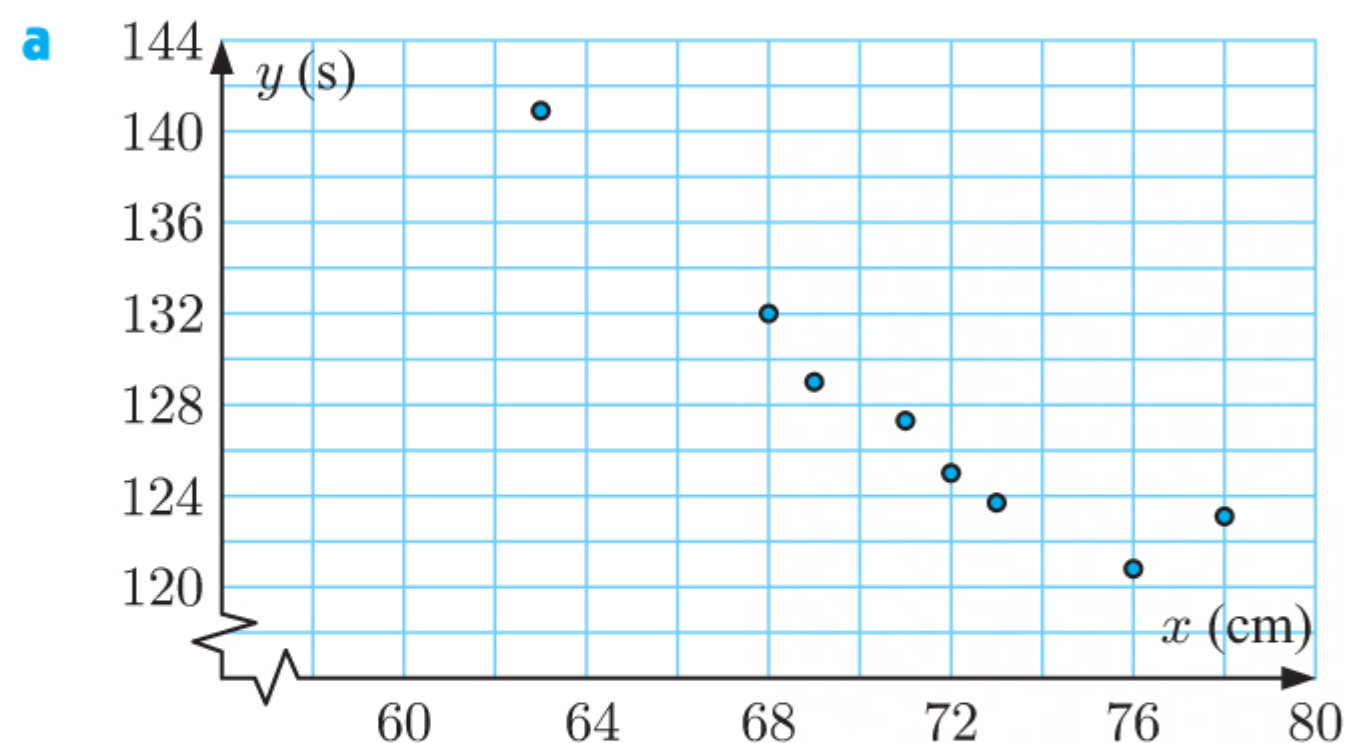
$$\therefore 0.457y \approx 15.508$$

$$\therefore y \approx 33.9$$

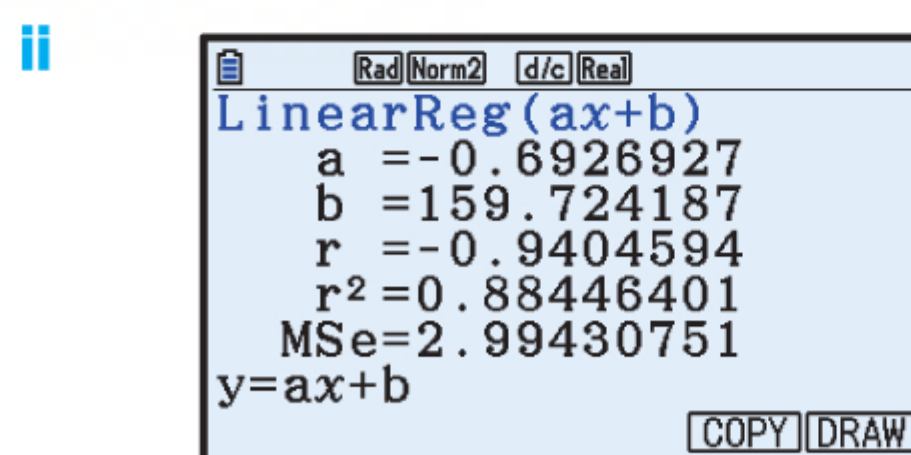
We expect a student who lives 15 km from school will have a travel time of about 33.9 minutes.

- d** The estimate in **c** is an interpolation, so it is likely to be reliable.

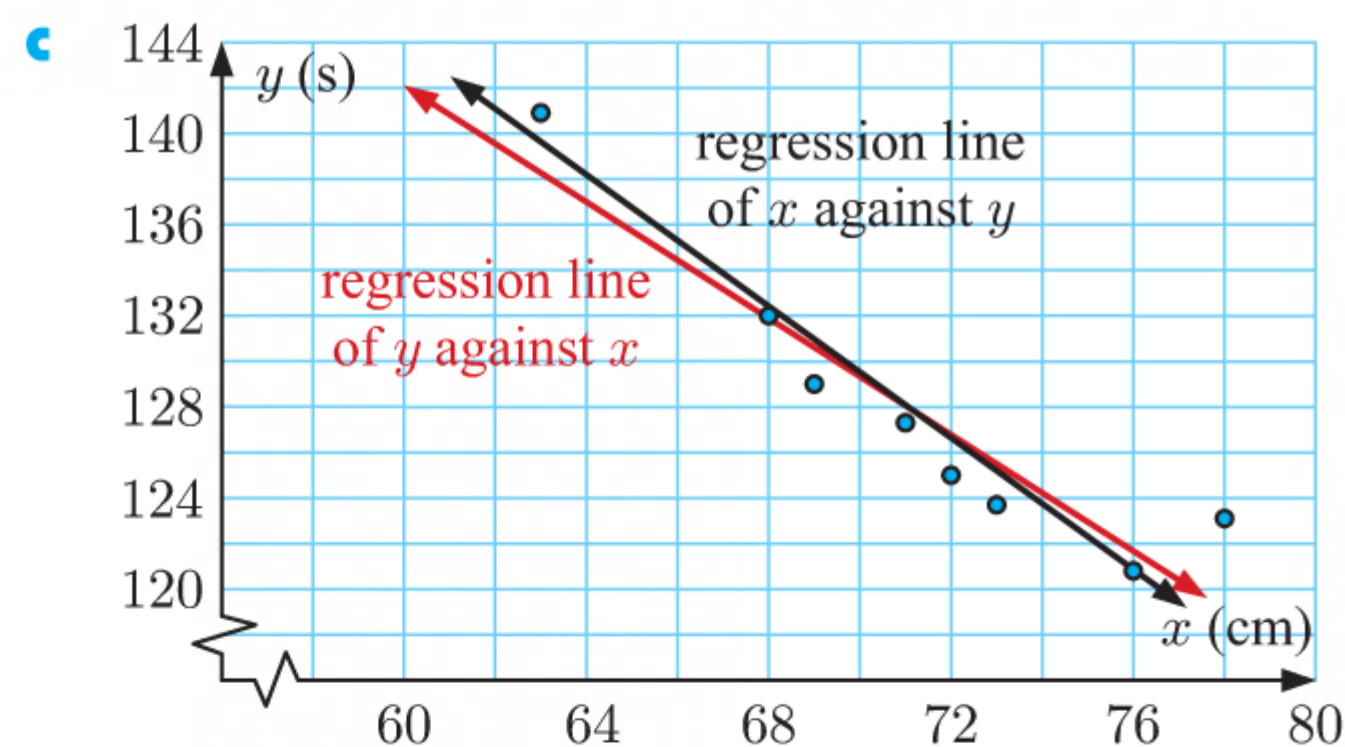
3	<i>Length of arm (x cm)</i>	78	73	71	68	76	72	63	69
	<i>Breaststroke (y seconds)</i>	123.1	123.7	127.3	132.0	120.8	125.0	140.9	129.0



The regression line of y against x is
 $y \approx -1.28x + 219$.



The regression line of x against y is
 $x \approx -0.693y + 160$.



The two regression lines are very similar. The regression line of x against y is slightly steeper.

4 a The regression line of y against x is $y = ax + b$ where $a = \frac{\sum(x_i - \bar{x})(y_i - \bar{y})}{\sum(x_i - \bar{x})^2}$.

The regression line of x against y is $x = my + c$ where $m = \frac{\sum(x_i - \bar{x})(y_i - \bar{y})}{\sum(y_i - \bar{y})^2}$.

$$\begin{aligned}
 \therefore am &= \left[\frac{\sum(x_i - \bar{x})(y_i - \bar{y})}{\sum(x_i - \bar{x})^2} \right] \left[\frac{\sum(x_i - \bar{x})(y_i - \bar{y})}{\sum(y_i - \bar{y})^2} \right] \\
 &= \frac{[\sum(x_i - \bar{x})(y_i - \bar{y})]^2}{\sum(x_i - \bar{x})^2 \sum(y_i - \bar{y})^2} \\
 &= r^2 \quad \left\{ r = \frac{\sum(x_i - \bar{x})(y_i - \bar{y})}{\sqrt{\sum(x_i - \bar{x})^2 \sum(y_i - \bar{y})^2}} \right\}
 \end{aligned}$$

b The regression line of y against x is $y = ax + b$.

The regression line of x against y is $y = \frac{1}{m}x - \frac{c}{m}$.

$$\begin{aligned} \text{In the regression of } y \text{ against } x, \text{ when } x = \bar{x}, \quad y &= a\bar{x} + b \\ &= a\bar{x} + \bar{y} - a\bar{x} \\ &= \bar{y} \end{aligned}$$

$$\begin{aligned} \text{and in the regression of } x \text{ against } y, \text{ when } y = \bar{y}, \quad x &= m\bar{y} + c \\ &= m\bar{y} + \bar{x} - m\bar{y} \\ &= \bar{x} \end{aligned}$$

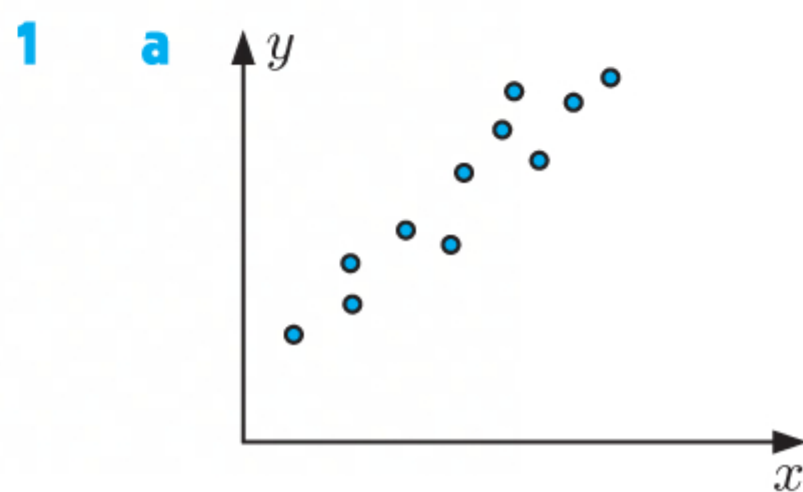
So, both lines pass through the mean point (\bar{x}, \bar{y}) , which means the lines will be the same if and only if their gradients are equal.

\therefore the regression lines will be the same if $a = \frac{1}{m}$

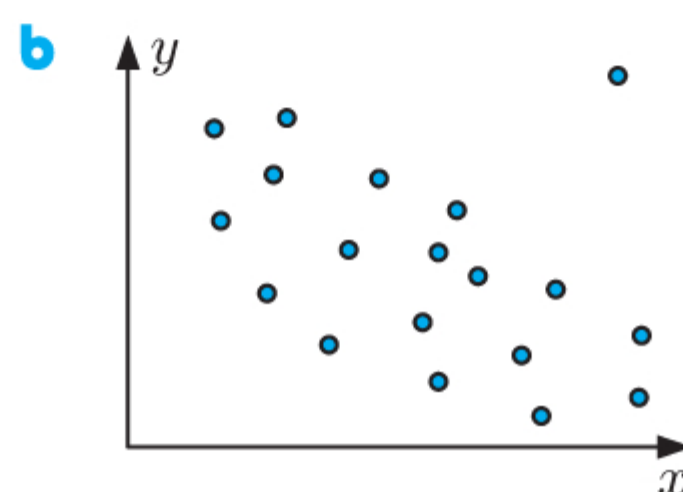
$$\therefore am = 1$$

$$\therefore r^2 = 1 \quad \{\text{using a}\}$$

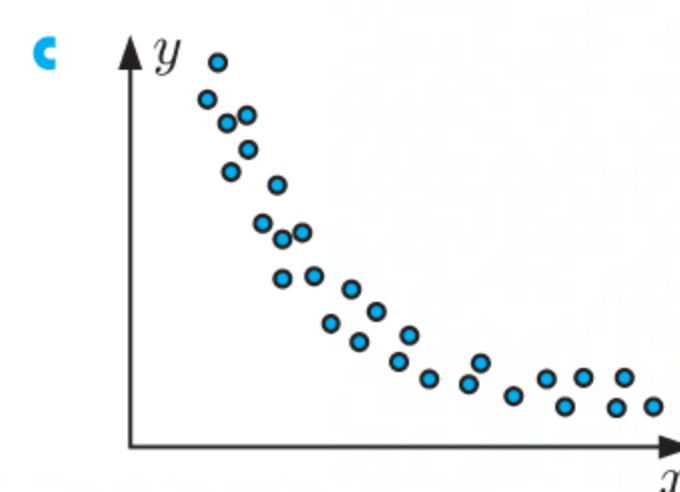
REVIEW SET 19A



There is a strong, positive, linear correlation, with no outliers.



There is a weak, negative, linear correlation, with one outlier.

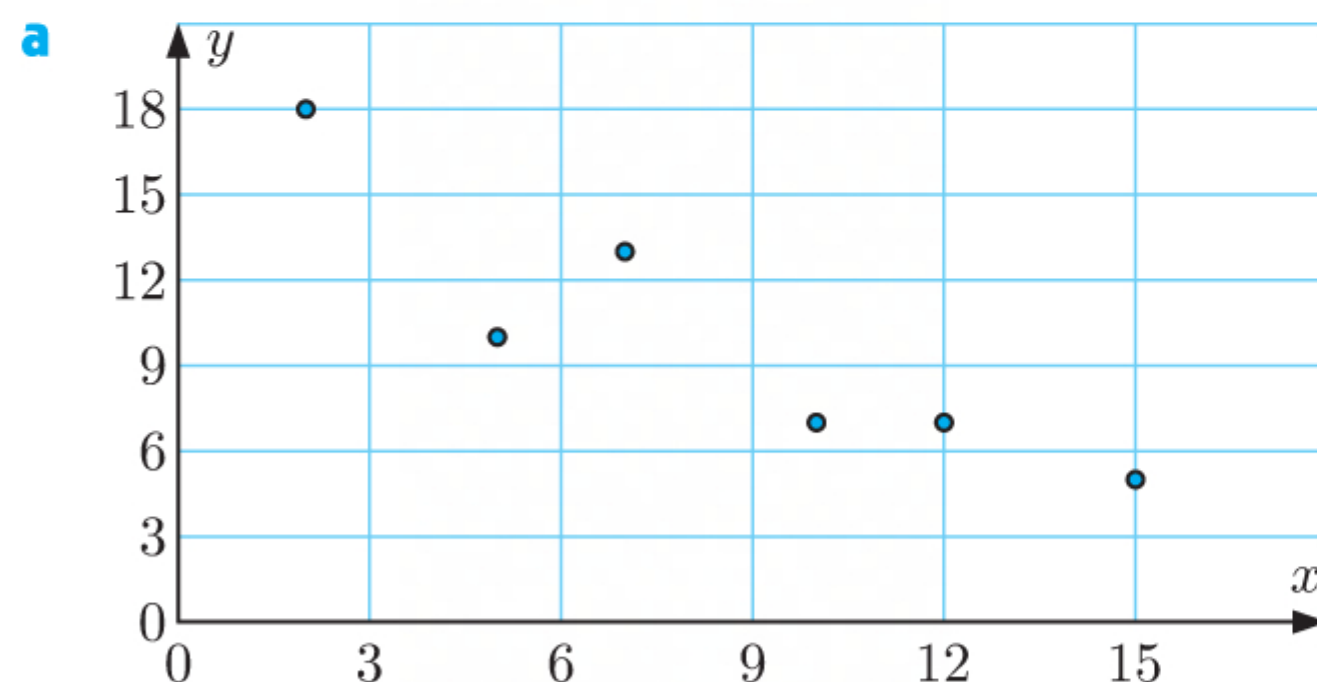


There is a strong, negative, non-linear correlation, with no outliers.

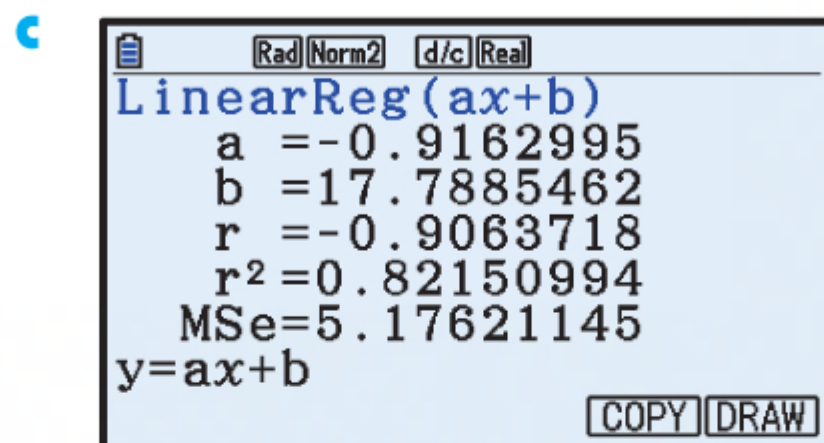
- 2 a** The correlation between water bills and electricity bills is likely to be positive, as a household with a high water bill is also likely to have a high electricity bill, and vice versa.
- b** No, there is not a causal relationship. Both variables mainly depend on the number of occupants in each house.

3

x	2	5	7	10	12	15
y	18	10	13	7	7	5



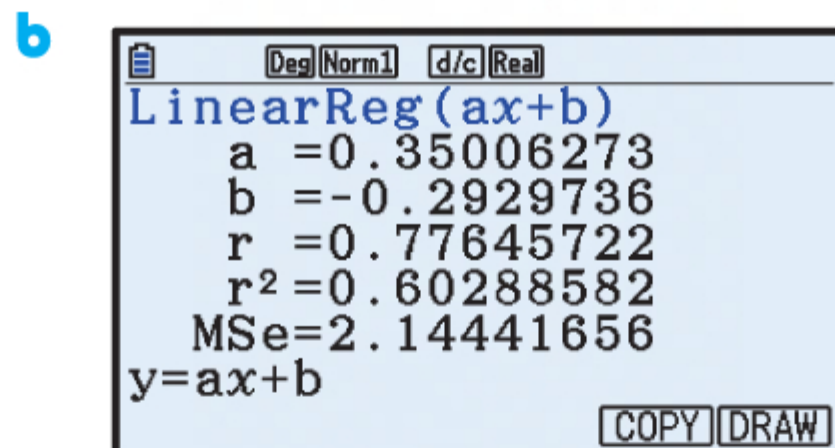
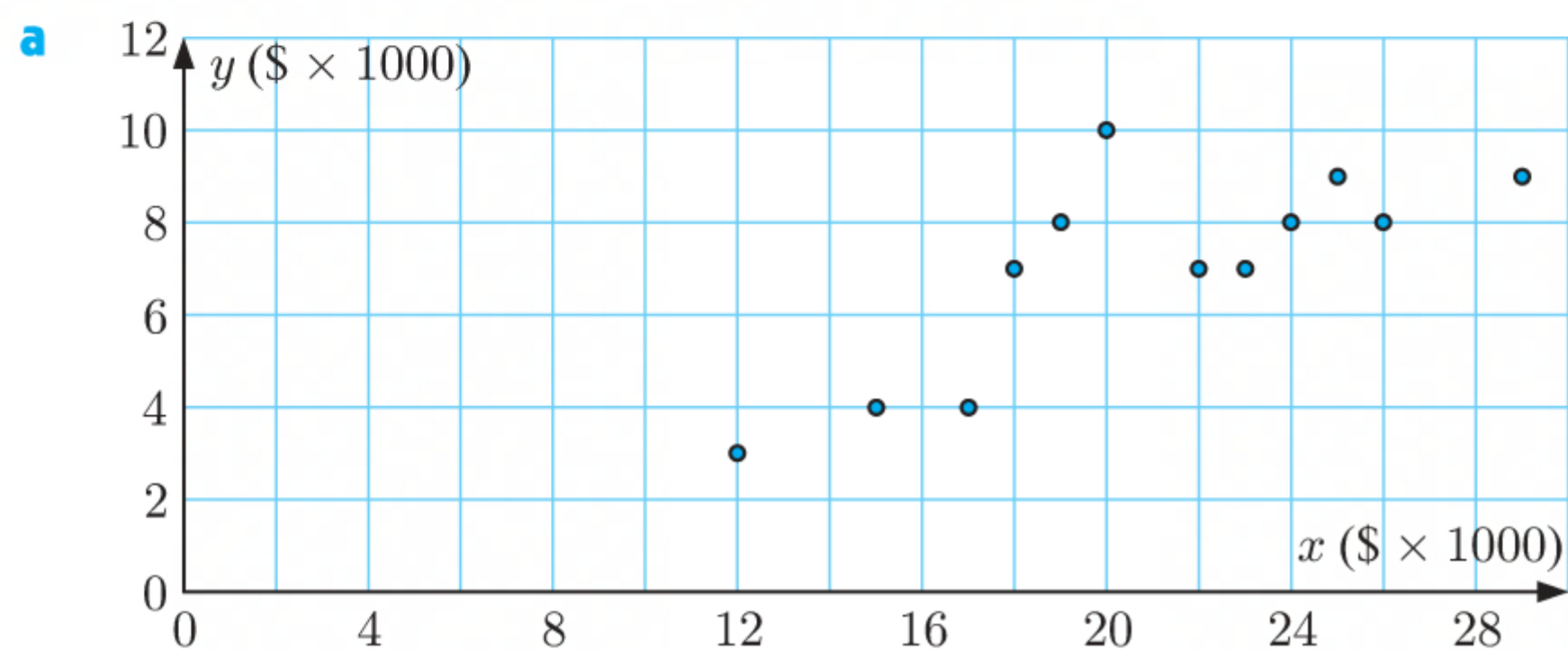
- b** The correlation between the variables appears to be negative.



So, $r \approx -0.906$.

4

<i>Ticket sales</i> ($\$x \times 1000$)	25	22	15	19	12	17	24	20	18	23	29	26
<i>Beverage sales</i> ($\$y \times 1000$)	9	7	4	8	3	4	8	10	7	7	9	8



So, $r \approx 0.776$.

- c** There is a moderate, positive correlation between *ticket sales* and *beverage sales*.

5

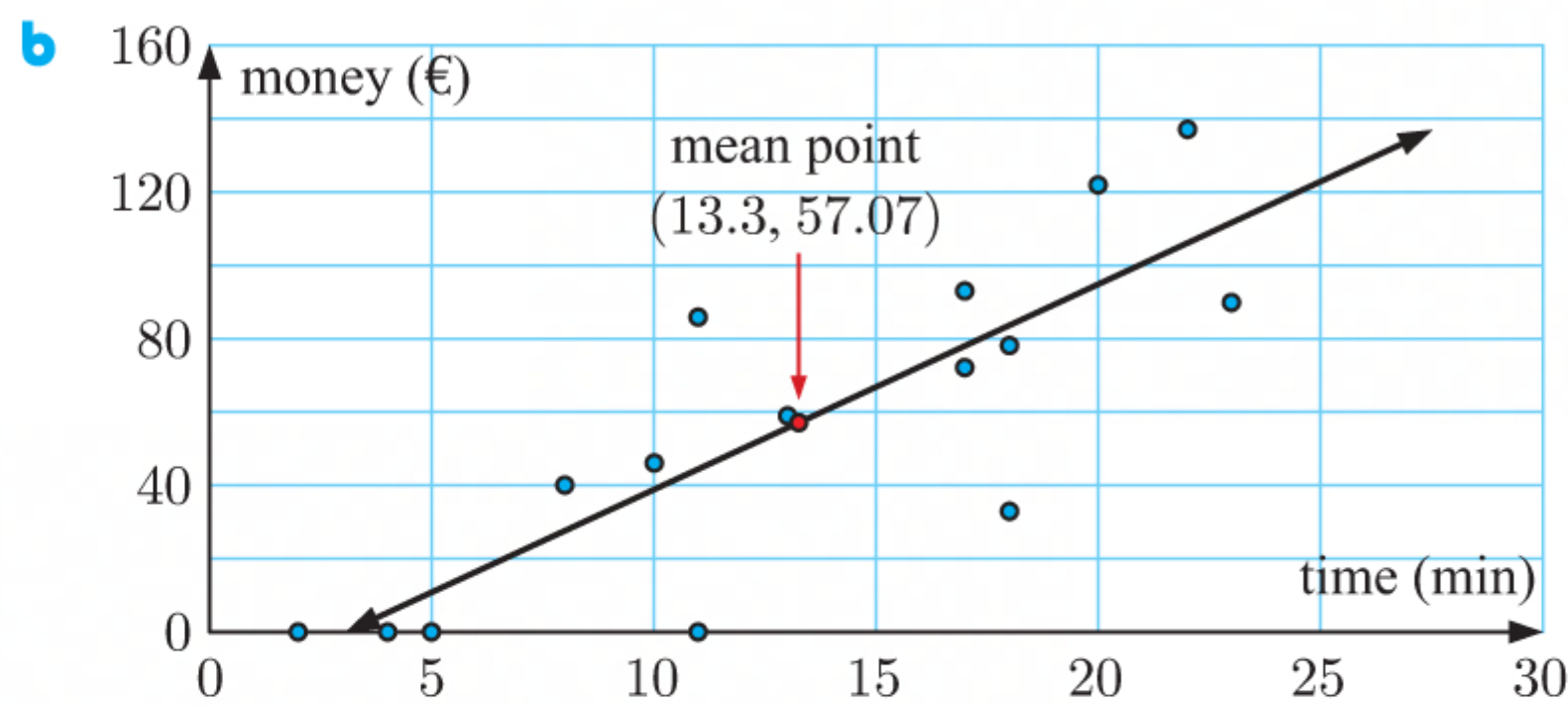
<i>Time</i> (min)	8	18	5	10	17	11	2	13	18	4	11	20	23	22	17
<i>Money</i> (€)	40	78	0	46	72	86	0	59	33	0	0	122	90	137	93

a

$$\begin{aligned}\bar{x} &= \frac{8 + 18 + \dots + 22 + 17}{15} \\ &= \frac{199}{15} \\ &\approx 13.3\end{aligned}$$

$$\begin{aligned}\bar{y} &= \frac{40 + 78 + \dots + 137 + 93}{15} \\ &= \frac{856}{15} \\ &\approx 57.07\end{aligned}$$

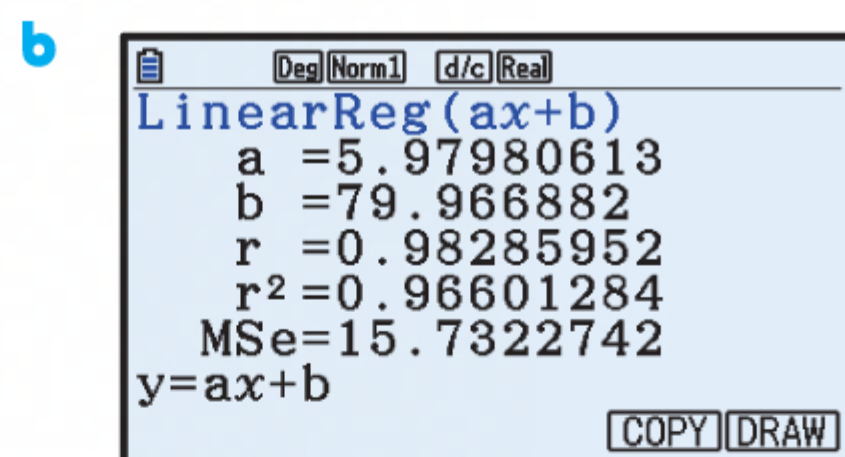
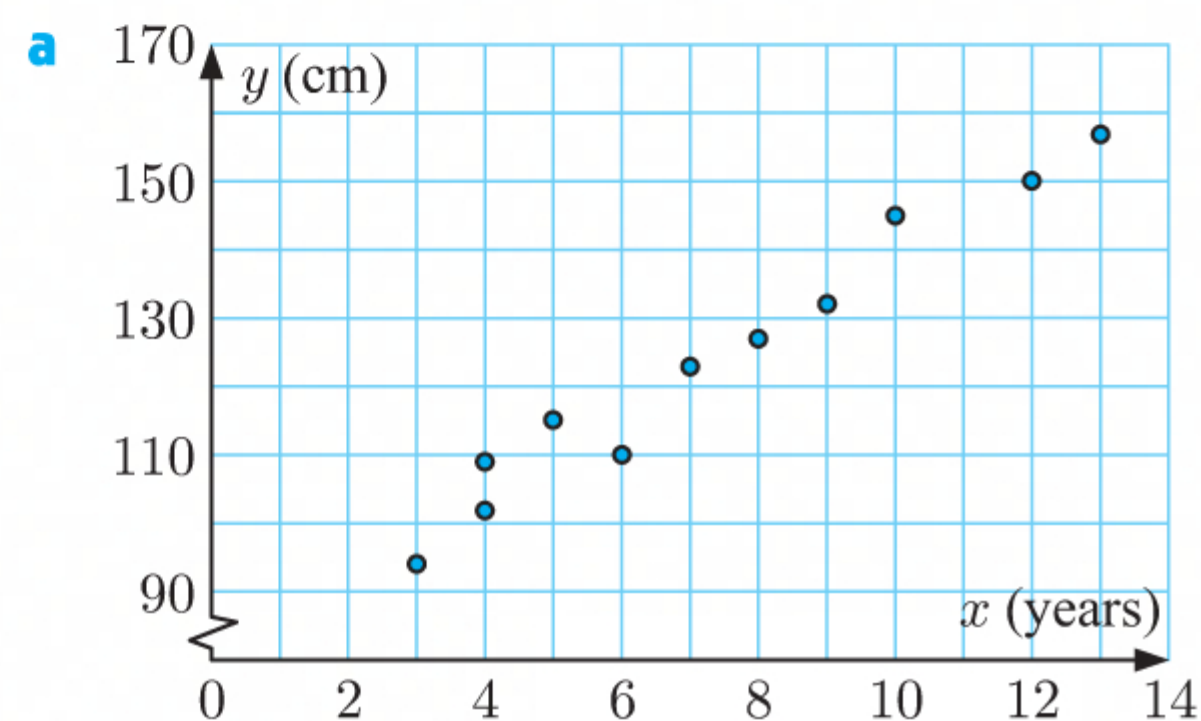
So the mean time is about 13.3 minutes, and the mean spending is about €57.07.



c There is a moderate, positive, linear correlation between *time in the store* and *money spent*.

6

Age (x years)	3	9	7	4	4	12	8	6	5	10	13
Height (y cm)	94	132	123	102	109	150	127	110	115	145	157



Using technology, the least squares regression line is $y \approx 5.98x + 80.0$.

c The gradient of the least squares regression line ≈ 5.98 . This indicates that each year, a child grows taller by an average of 5.98 cm.

d When $x = 5$, $y \approx 5.98(5) + 80.0$
 ≈ 110

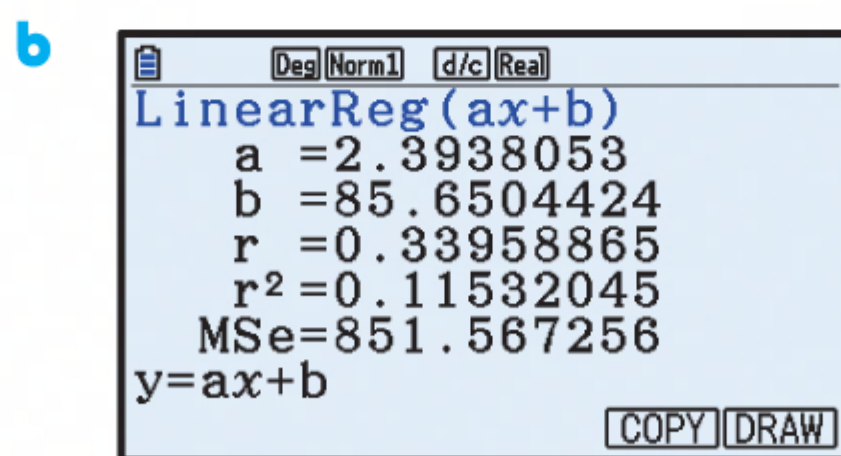
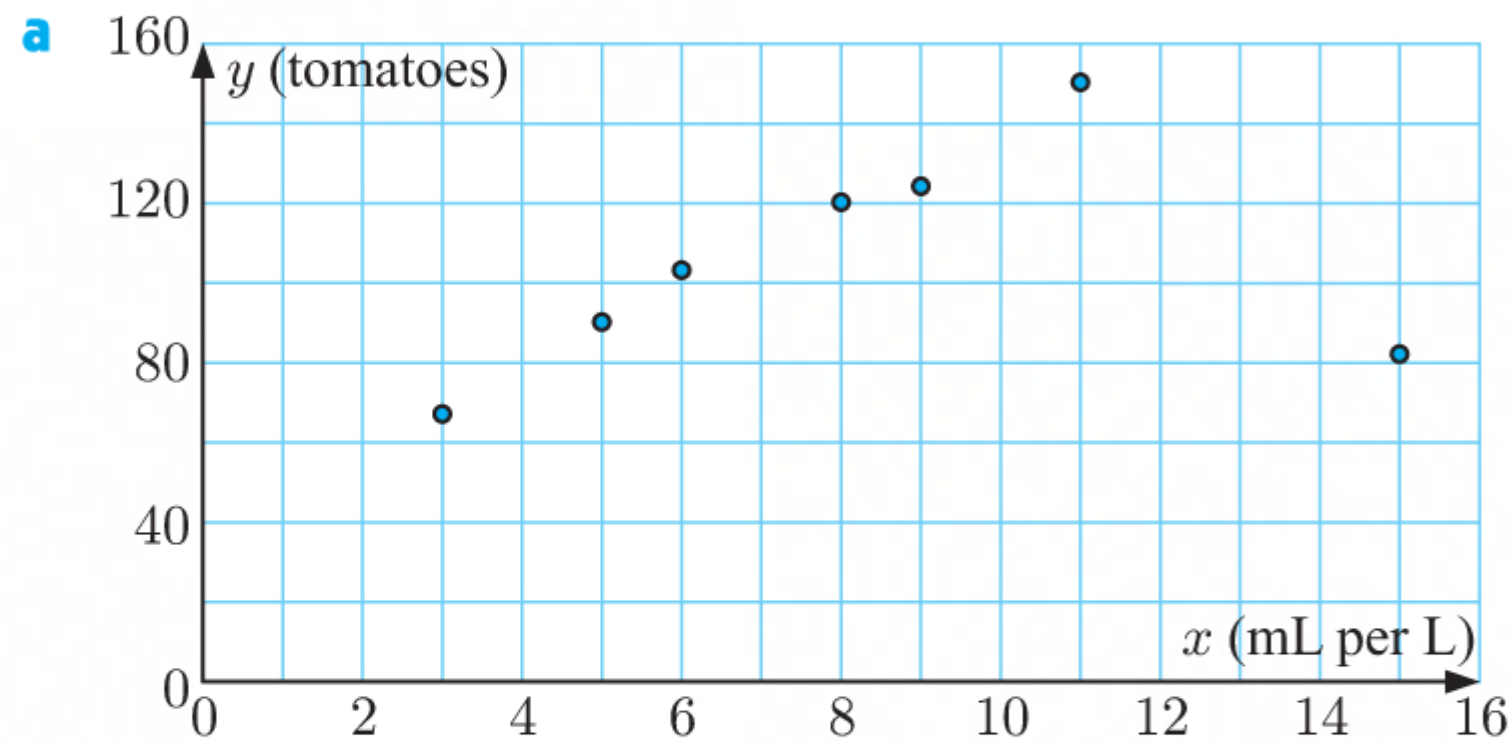
So, a 5 year old child would be approximately 110 cm tall.

e When $y = 140$, $140 \approx 5.98x + 80$
 $5.98x \approx 60$
 $x \approx 10.0$

A child would be expected to reach 140 cm in height at age 10 years.

7

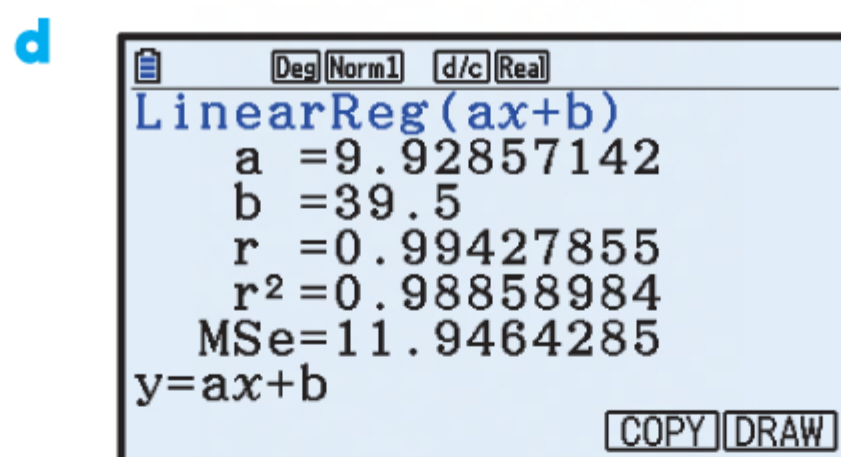
Spray concentration (x mL per L)	3	5	6	8	9	11	15
Yield of tomatoes per bush (y)	67	90	103	120	124	150	82



So, $r \approx 0.340$.

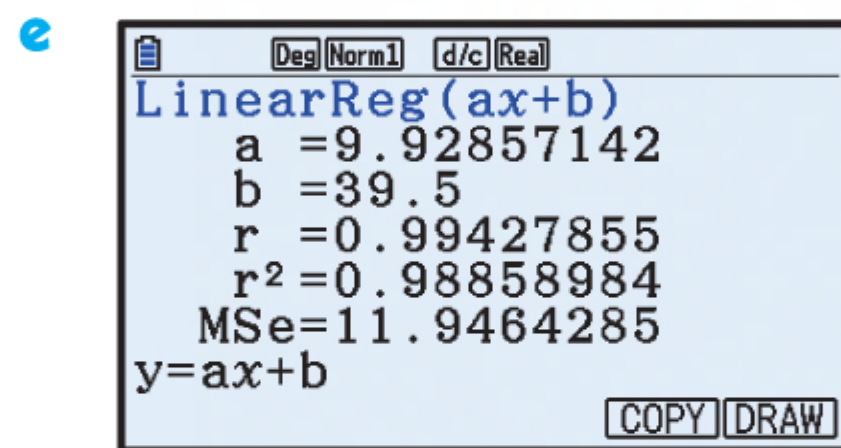
There is a very weak, positive, linear correlation between spray concentration and yield.

c Yes, $(15, 82)$ is an outlier which is affecting the correlation.



So, $r \approx 0.994$.

Yes it is now reasonable to draw a least squares regression line.



Using technology, the least squares regression line is $y \approx 9.93x + 39.5$.

f The gradient of the least squares regression line ≈ 9.93 . This indicates that for every additional mL per L the spray concentration increases, the yield of tomatoes per bush increases on average by 9.93 tomatoes.

The y -intercept of the least squares regression line ≈ 39.5 . This indicates that if the tomato bushes are not sprayed, the average yield per bush is approximately 39.5 tomatoes.

g i When $x = 7$, $y \approx 9.93(7) + 39.5$
 ≈ 109

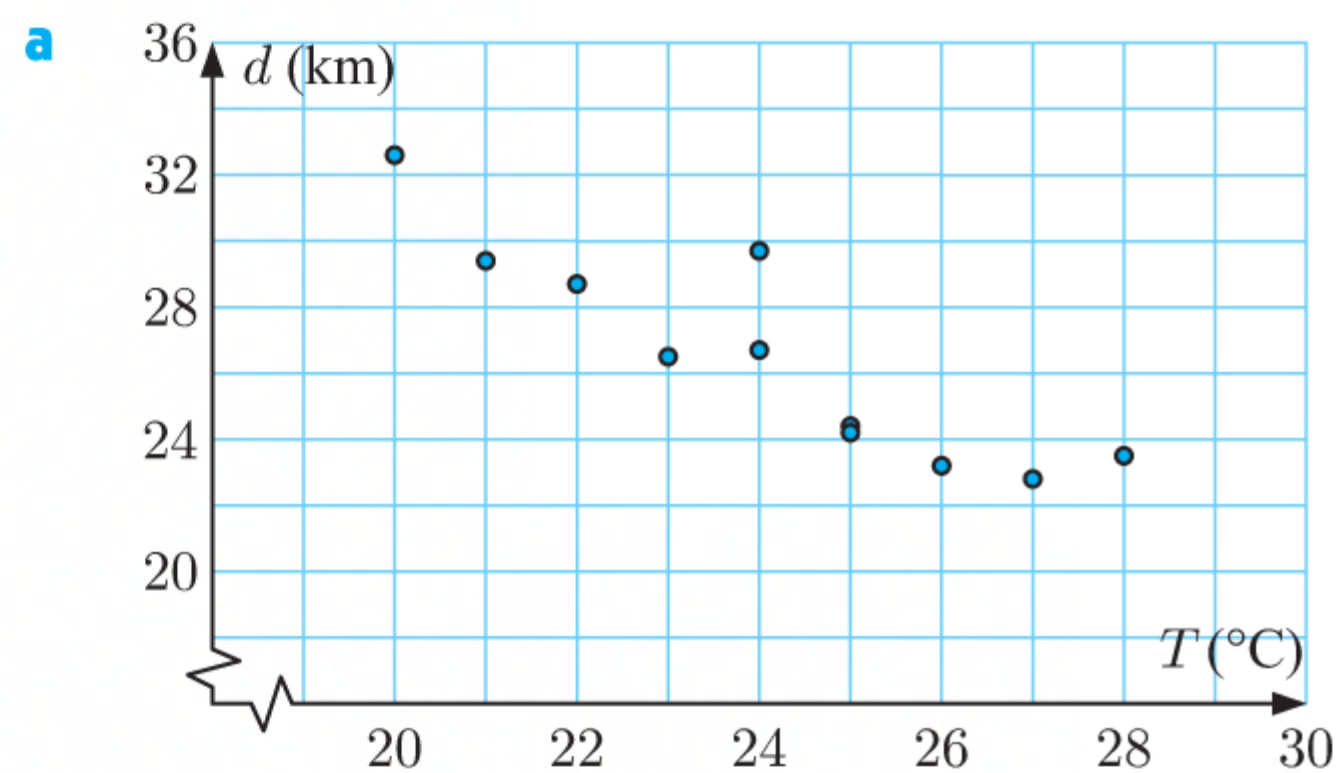
If the spray concentration is 7 mL per L, the yield will be approximately 109 tomatoes per bush.

ii When $y = 200$, $200 \approx 9.93x + 39.5$
 $9.93x \approx 160.5$
 $x \approx 16.2$

If the yield is 200 tomatoes per bush, the spray concentration would be approximately 16.2 mL per L.

- h** In **g i**, this is an interpolation, so this estimate is likely to be reliable.
 In **g ii**, this is an extrapolation, so this estimate may not be reliable.

8	Temperature (T °C)	23	24	25	27	28	20	22	21	25	26	24
	Distance (d km)	26.5	26.7	24.4	22.8	23.5	32.6	28.7	29.4	24.2	23.2	29.7



- b** It is appropriate to use the regression line of T against d since the values for the distance travelled d are more precisely measured than the daily temperature, which Thomas is just estimating.

c

```

Rad(Norm2) d/c(Real)
LinearReg(ax+b)
a = -0.688528
b = 42.3494205
r = -0.8997697
r² = 0.80958564
MSe = 1.28866281
y = ax + b
  
```

COPY DRAW

The regression line of T against d is $T \approx -0.689d + 42.3$.

d When $T = 30$, $30 \approx -0.689d + 42.3$
 $\therefore 0.689d \approx 12.3$
 $\therefore d \approx 17.9$

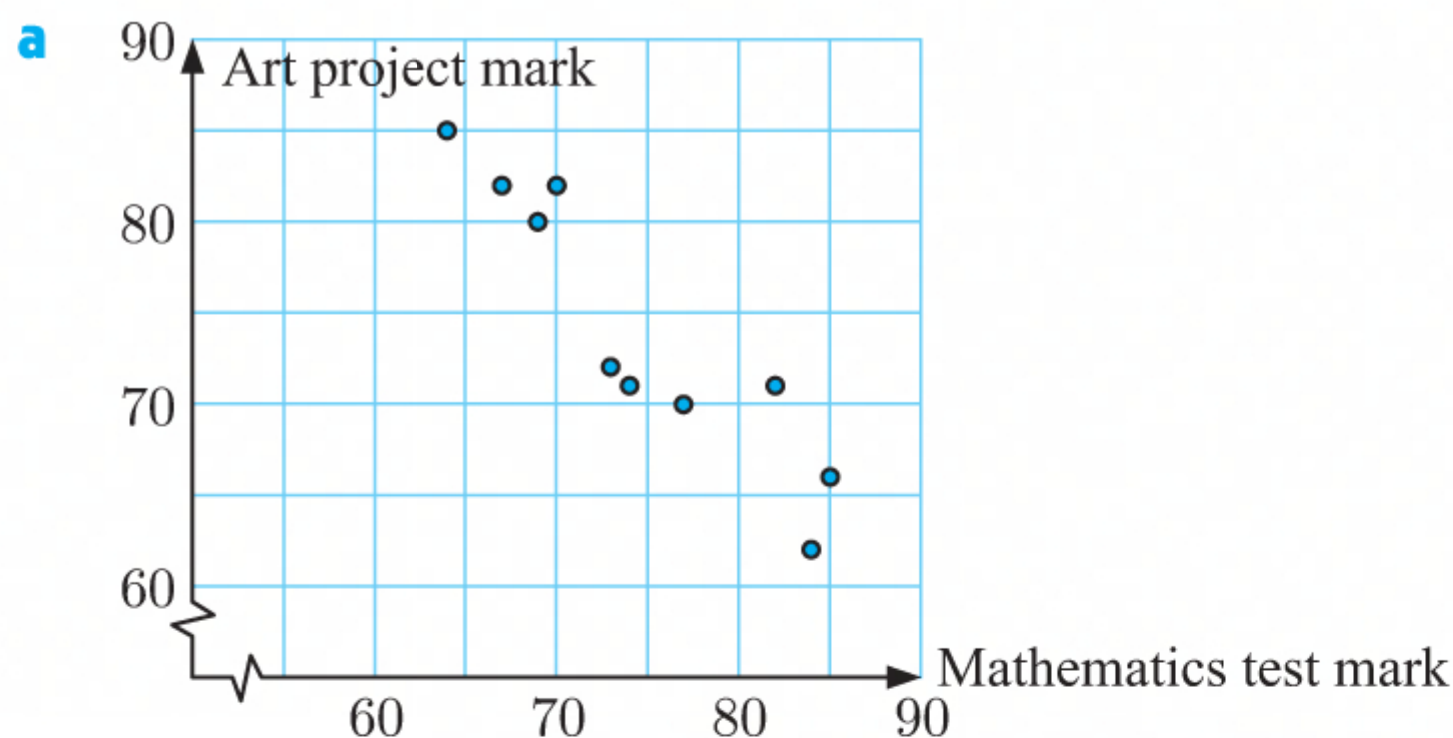
On a 30°C day we would expect Thomas to ride about 17.9 km.

REVIEW SET 19B

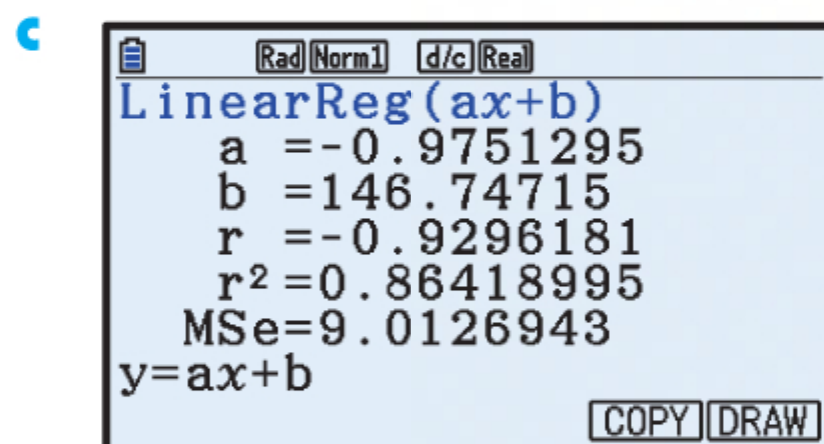
- 1 a The variables are likely to be negatively correlated, as prices increase, the number of tickets sold is likely to decrease.
This is a causal relationship as less people will be able to afford tickets as the prices increase.
- b The variables are likely to be positively correlated, as ice cream sales increase, the number of shark attacks is likely to increase.
This is not a causal relationship as both of these variables are dependent on the time of year.

2

Student	A	B	C	D	E	F	G	H	I	J
Mathematics test	64	67	69	70	73	74	77	82	84	85
Art project	85	82	80	82	72	71	70	71	62	66



- b There is a strong, negative, linear correlation between the Mathematics and Art marks.



So, $r \approx -0.930$.

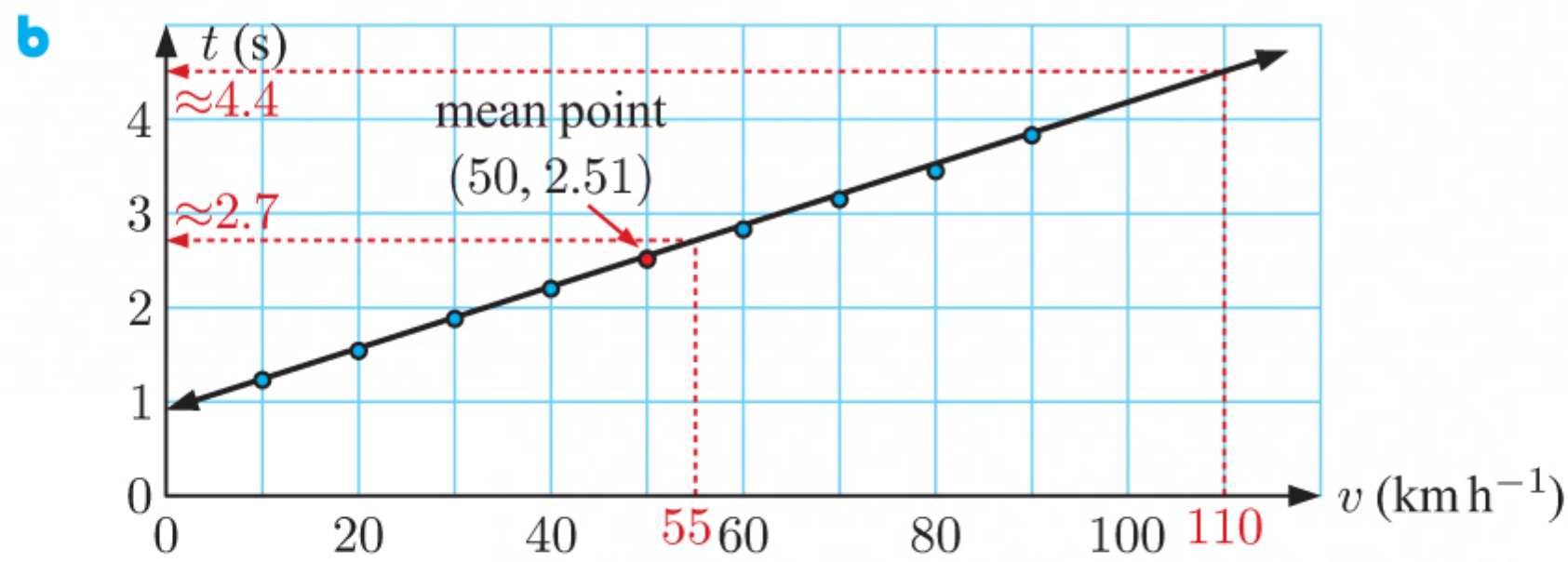
3

Speed (v km h ⁻¹)	10	20	30	40	50	60	70	80	90
Stopping time (t s)	1.23	1.54	1.88	2.20	2.52	2.83	3.15	3.45	3.83

a
$$\bar{v} = \frac{10 + 20 + 30 + \dots + 80 + 90}{9}$$
$$= 50$$

$$\bar{t} = \frac{1.23 + 1.54 + 1.88 + \dots + 3.45 + 3.83}{9}$$
$$\approx 2.51$$

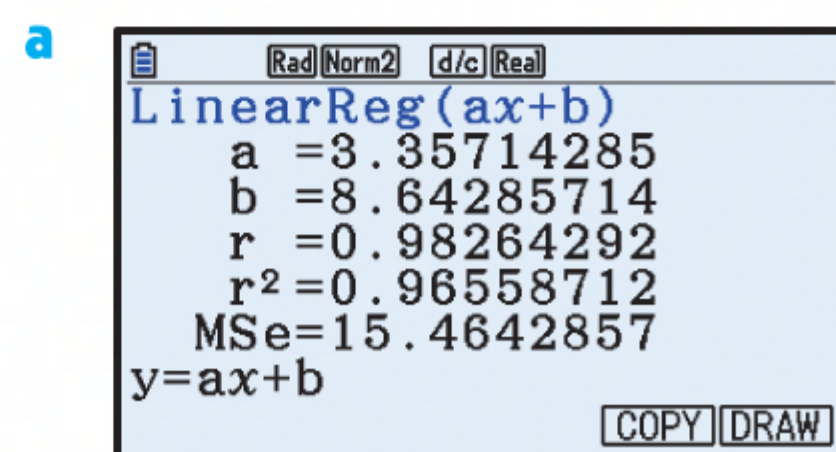
\therefore the mean point (\bar{v}, \bar{t}) is $(50, 2.51)$.



- c**
- i** We estimate that the stopping time for a speed of 55 km h⁻¹ is about 2.7 seconds.
 - ii** We estimate that the stopping time for a speed of 110 km h⁻¹ is about 4.4 seconds.
- d** The estimate in **c i** is more likely to be reliable, since it is an interpolation.

4

x	2	3	6	8	13	16
y	12	17	32	41	50	61



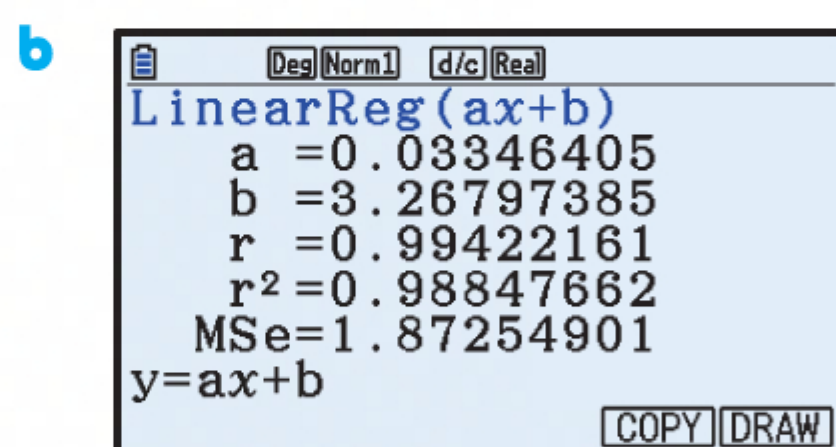
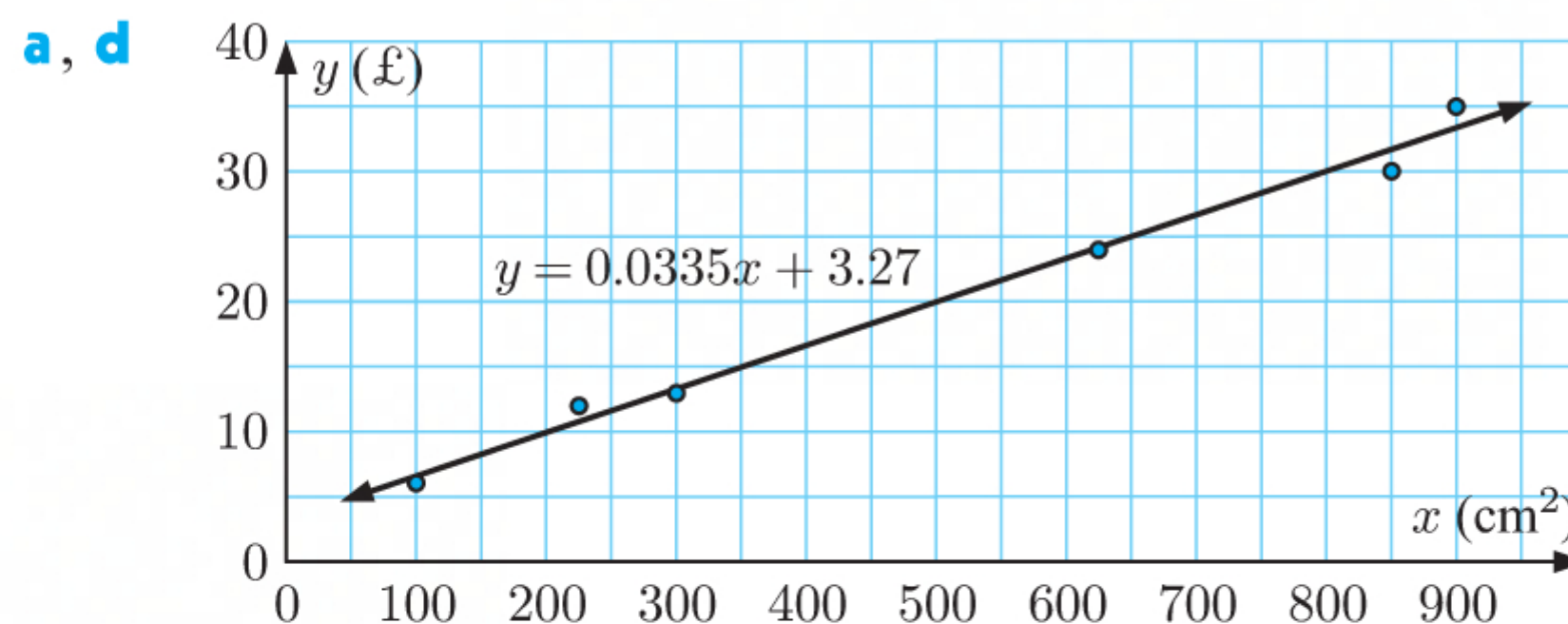
b The regression line is $y \approx 3.36x + 8.64$.

c When $x = 10$, $y \approx 3.36(10) + 8.64$
 ≈ 42.2

So, $r \approx 0.983$.

5

Area (x cm ²)	100	225	300	625	850	900
Price (£ y)	6	12	13	24	30	35



c There is a very strong, positive correlation between the *area* of a canvas and its *price*.

d The regression line is $y \approx 0.0335x + 3.27$.

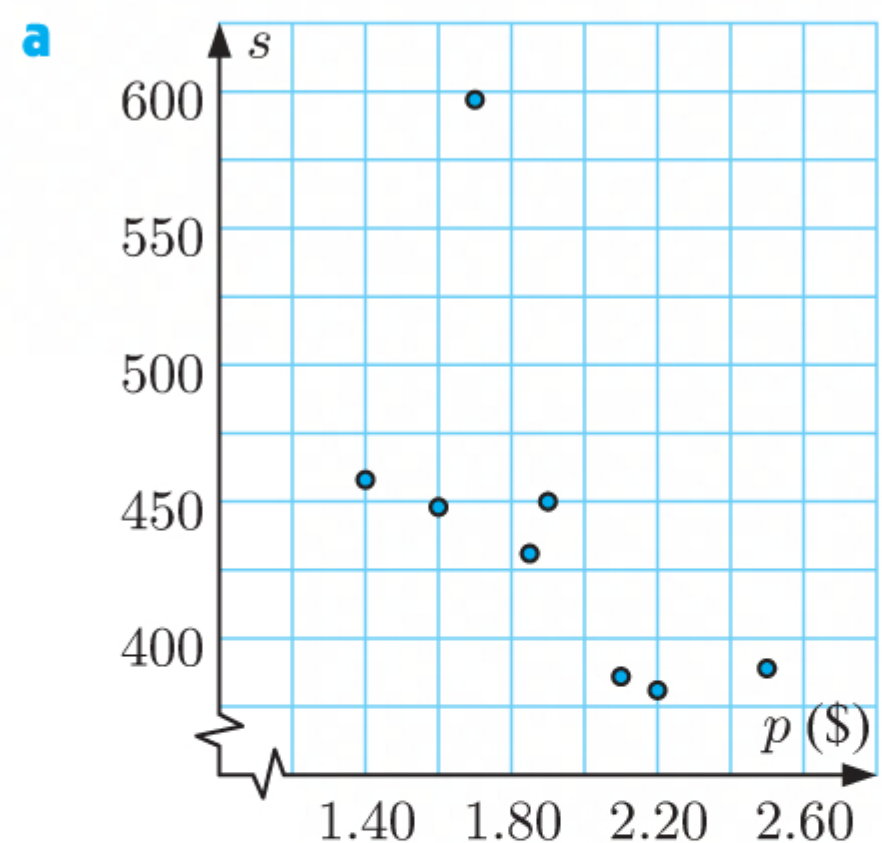
So, $r \approx 0.994$.

e When $x = 1200$, $y \approx 0.0335(1200) + 3.27$
 ≈ 43.42

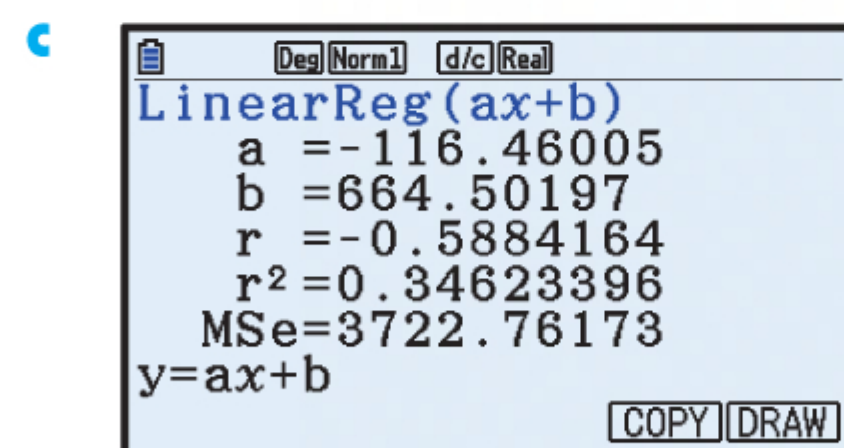
We estimate that a canvas with area 1200 cm² will cost about £43.42. This is an extrapolation however, so it may be unreliable.

6

Price (\$p)	2.50	1.90	1.60	2.10	2.20	1.40	1.70	1.85
Sales (s)	389	450	448	386	381	458	597	431



- b** Yes, the point (1.70, 597) is an outlier. It should not be deleted as there is no evidence that it is a mistake.

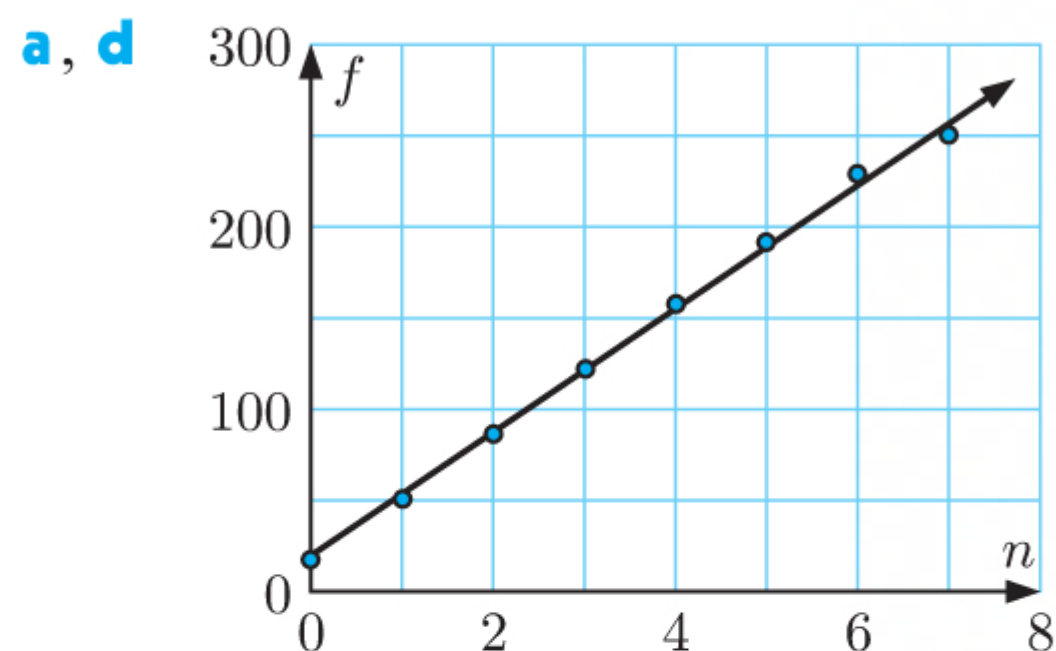


The regression line is $s \approx -116p + 665$.

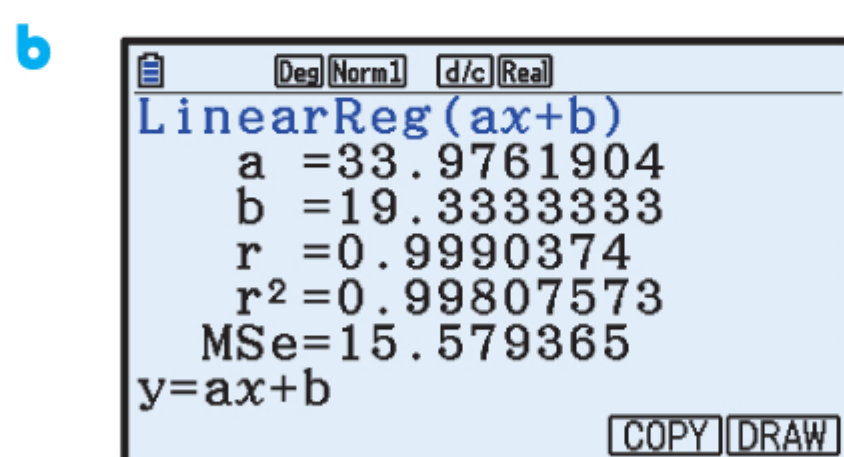
- d** The gradient of the least squares regression line ≈ -116 . This indicates that for every additional dollar the price increases by, the number of sales decreases by 116.
- e** No, the prediction of sales of Supa-fizz if it was priced at 50 cents would not be accurate, as it is an extrapolation well beyond the range of data values given.

7

Number of waterings (n)	0	1	2	3	4	5	6	7
Flowers produced (f)	18	52	86	123	158	191	228	250



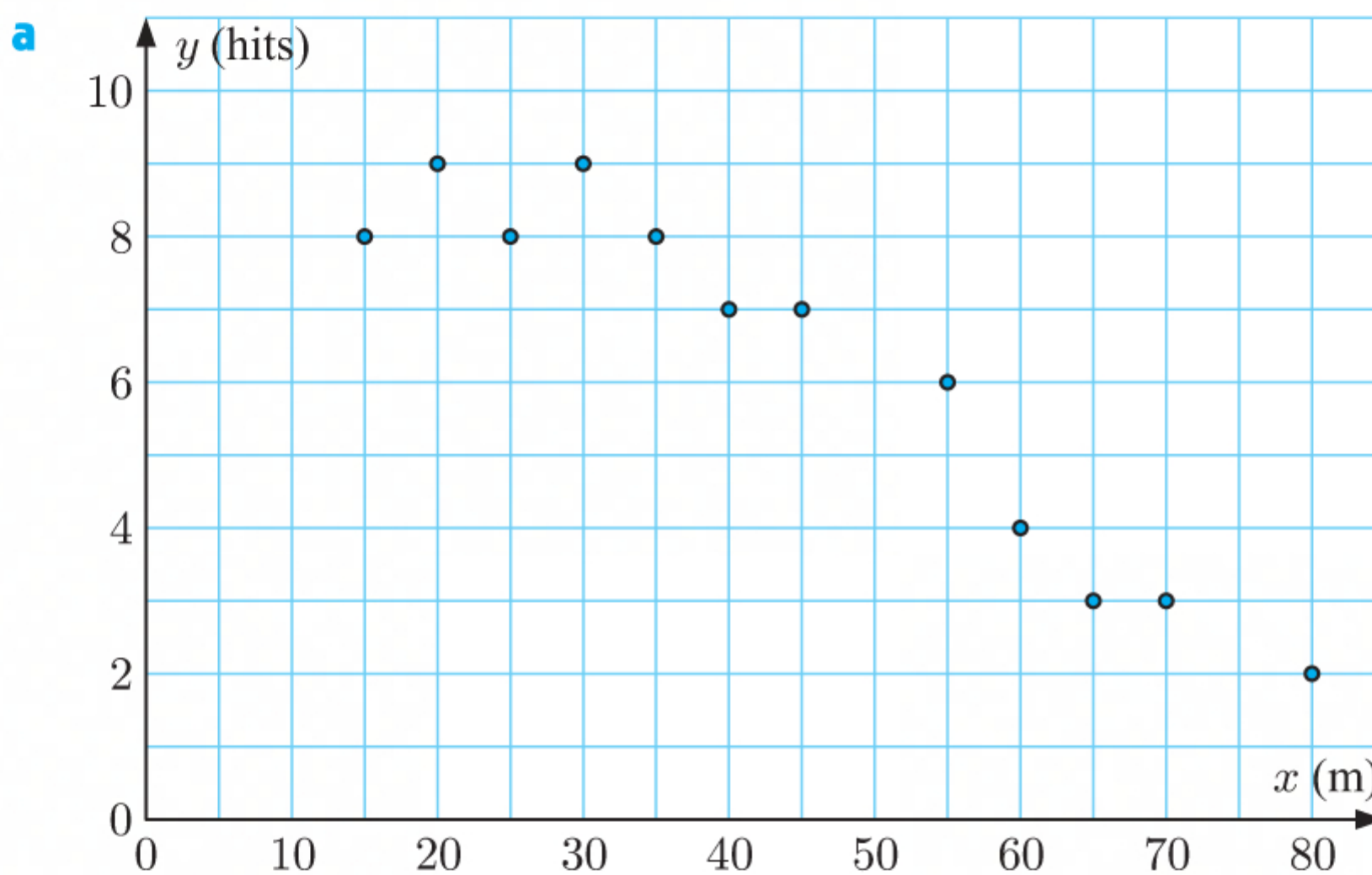
There is a very strong, positive correlation between number of waterings and flowers produced.



The regression line is $f \approx 34.0n + 19.3$.

- c Yes, plants need water to grow. So it is expected that an increase in watering will result in an increase in flower production.
- e i 5 times a fortnight \equiv 2.5 times a week
 When $n = 2.5$, $f \approx 34.0(2.5) + 19.3 \approx 104$
 Violet can expect about 104 flowers from this bed.
- When $n = 10$, $f \approx 34.0(10) + 19.3 \approx 359$
 Violet can expect about 359 flowers from this bed.
- ii The estimate for $n = 2.5$ is reliable as it is an interpolation.
 The estimate for $n = 10$ is unreliable as it is an extrapolation and over-watering could be a problem.

8	Distance from target (x m)	20	25	15	35	40	55	30	45	60	80	65	70
	Hits (y)	9	8	8	8	7	6	9	7	4	2	3	3



- b It is appropriate to use the regression line of x against y , since the number of hits can be counted exactly, while the distance from the target will not be exact.

c

Rad	Norm2	d/c	Real
LinearReg(ax+b)			
a = -7.8947368			
b = 93.6842105			
r = -0.946192			
r ² = 0.89527943			
MSe = 50.7894736			
y = ax + b			
COPY		DRAW	

The regression line of x against y is $x \approx -7.89y + 93.7$.

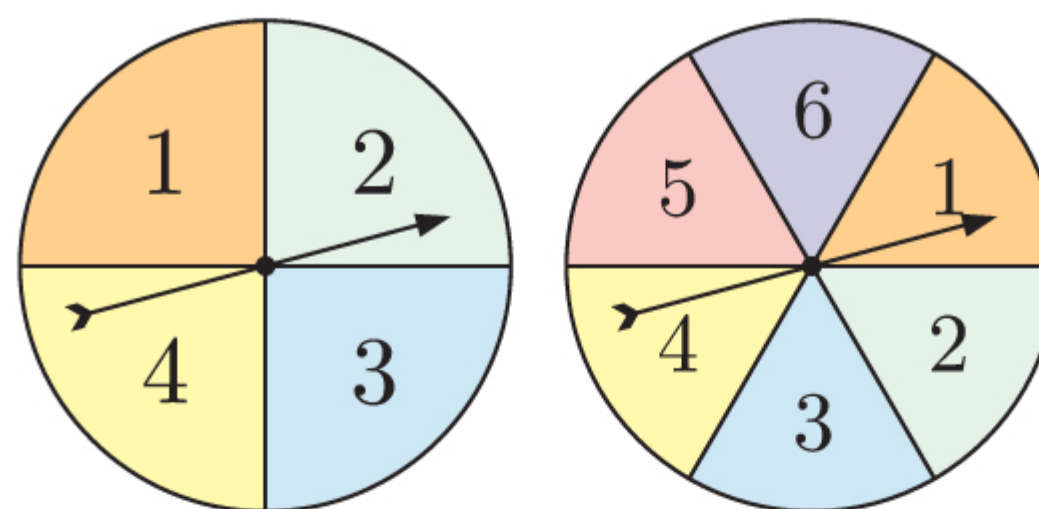
- d When $x = 100$, $100 \approx -7.89y + 93.7$
 $\therefore 7.89y \approx -6.3$
 $\therefore y \approx -0.800$

Out of 10 shots fired at a distance of 100 m, we would expect about -0.8 hits, but it is impossible to make a negative number of shots. This extrapolation is not valid.

DISCRETE RANDOM VARIABLES

- 1**
 - a** The quantity of fat in a sausage is a continuous random variable.
 - b** The mark out of 50 for a geography test is a discrete random variable.
 - c** The weight of a Year 12 student is a continuous random variable.
 - d** The volume of water in a cup of coffee is a continuous random variable.
 - e** The number of trout in a lake is a discrete random variable.
 - f** The number of the hairs on a cat is a discrete random variable.
 - g** The length of a horse's mane is a continuous random variable.
 - h** The height of a skyscraper is a continuous random variable.
- 2**
 - a**
 - i** The random variable X is the height of water in the rain gauge.
 - ii** The variable is a continuous random variable. **iii** $0 \leq X \leq 400$ mm
 - b**
 - i** The random variable X is the stopping distance.
 - ii** The variable is a continuous random variable. **iii** $0 \leq X \leq 50$ m
 - c**
 - i** The random variable X is the number of times that the switch is turned off and on before it fails.
 - ii** The variable is a discrete random variable. **iii** X can be any integer ≥ 1

b $X = 2, 3, 4, 5, 6, 7, 8, 9, \text{ or } 10$



b **i** $X = 5$ **ii** $X = 6$ or 7

Diagram illustrating the greedy algorithm for the Set Cover problem. It shows five columns representing sets A, B, C, and D. Each column contains a 4x1 grid of green checkmarks (✓) and red crosses (✗). A red arrow points from each column to a value X below it. The columns are labeled A, B, C, D from left to right. The values of X are 4, 3, 2, 1, and 0 respectively. The first column (A) has all four elements covered (4 checkmarks). The second column (B) has three elements covered (3 checkmarks). The third column (C) has two elements covered (2 checkmarks). The fourth column (D) has one element covered (1 checkmark). The fifth column (E) has no elements covered (0 checkmarks).

- c** **i** If exactly two devices are accurate, then $X = 2$.
ii If at least two devices are accurate, then 2, 3, or 4 are accurate $\therefore X = 2, 3, \text{ or } 4$.
- 6 a** If 3 coins are tossed then the number of heads X can be 0, 1, 2, or 3.
b Let H represent heads, and T represent tails.
- | | | | |
|-----------|-----------|-----------|-----------|
| HHH | HHT | TTH | TTT |
| ↓ | HTH | THT | ↓ |
| | THH | HTT | |
| $(X = 3)$ | $(X = 2)$ | $(X = 1)$ | $(X = 0)$ |
- c** $P(X = 0) = \frac{1}{8}$, $P(X = 1) = \frac{3}{8}$, $P(X = 2) = \frac{3}{8}$, $P(X = 3) = \frac{1}{8}$
 Since $P(X = 2) \neq P(X = 3)$, the possible values of X are not equally likely to occur.

EXERCISE 20B

1 a i

x	1	2	3	4
$P(X = x)$	0.2	0.4	0.15	0.25

$$\sum_{x=1}^4 P(X = x) = 0.2 + 0.4 + 0.15 + 0.25 = 1$$

Since $\sum_{x=1}^4 P(X = x) = 1$ and $0 \leq P(X = x) \leq 1$ for all x , it is a valid probability distribution.

ii

x	0	1	2	3
$P(X = x)$	0.2	0.3	0.4	0.2

$$\sum_{x=0}^3 P(X = x) = 0.2 + 0.3 + 0.4 + 0.2 = 1.1$$

Since $\sum_{x=0}^3 P(X = x) > 1$, it is not a valid probability distribution.

iii

x	0	1	2	3	4
$P(X = x)$	0.2	0.2	0.2	0.2	0.2

$$\sum_{x=0}^4 P(X = x) = 0.2 + 0.2 + 0.2 + 0.2 + 0.2 = 1$$

Since $\sum_{x=0}^4 P(X = x) = 1$ and $0 \leq P(X = x) \leq 1$ for all x , it is a valid probability distribution.

iv

x	2	3	4	5
$P(X = x)$	0.3	0.4	0.5	-0.2

Since $P(X = 5) = -0.2 < 0$, it is not a valid probability distribution.

- b** X is a uniform random variable for the probability distribution in **a iii**, since $p_i = 0.2$ for each value of i .

2 a

x	0	1	2
$P(X = x)$	0.3	k	0.5

$$\sum_{x=0}^2 P(X = x) = 1$$

$$\therefore 0.3 + k + 0.5 = 1$$

$$\therefore k = 0.2$$

b

x	0	1	2	3
$P(X = x)$	k	$2k$	$3k$	k

$$\sum_{x=0}^3 P(X = x) = 1$$

$$\therefore k + 2k + 3k + k = 1$$

$$\therefore 7k = 1$$

$$\therefore k = \frac{1}{7}$$

3 a

$$\sum_{x=0}^3 P(X = x) = 1$$

$$\therefore 0.1 + 0.25 + 0.45 + a = 1$$

$$\therefore a = 0.2$$

x	0	1	2	3
$P(X = x)$	0.1	0.25	0.45	a

b Since $P(X = 0) \neq P(X = 1)$, the probabilities of each outcome are not all equal, so X is not a uniform discrete random variable.

c Since $P(X = 2)$ is the greatest probability, 2 is the mode of the distribution.

d $P(X \geq 2) = P(X = 2) + P(X = 3)$
 $= 0.45 + 0.2$
 $= 0.65$

4

x	0	1	2	3	4	5
$P(x)$	a	0.3333	0.1088	0.0084	0.0007	0.0000

a $P(2) = 0.1088$ (from table)

b Since this is a probability distribution, $\sum_{i=1}^n P(x_i) = 1$

$$\therefore a + 0.3333 + 0.1088 + 0.0084 + 0.0007 + 0.0000 = 1$$

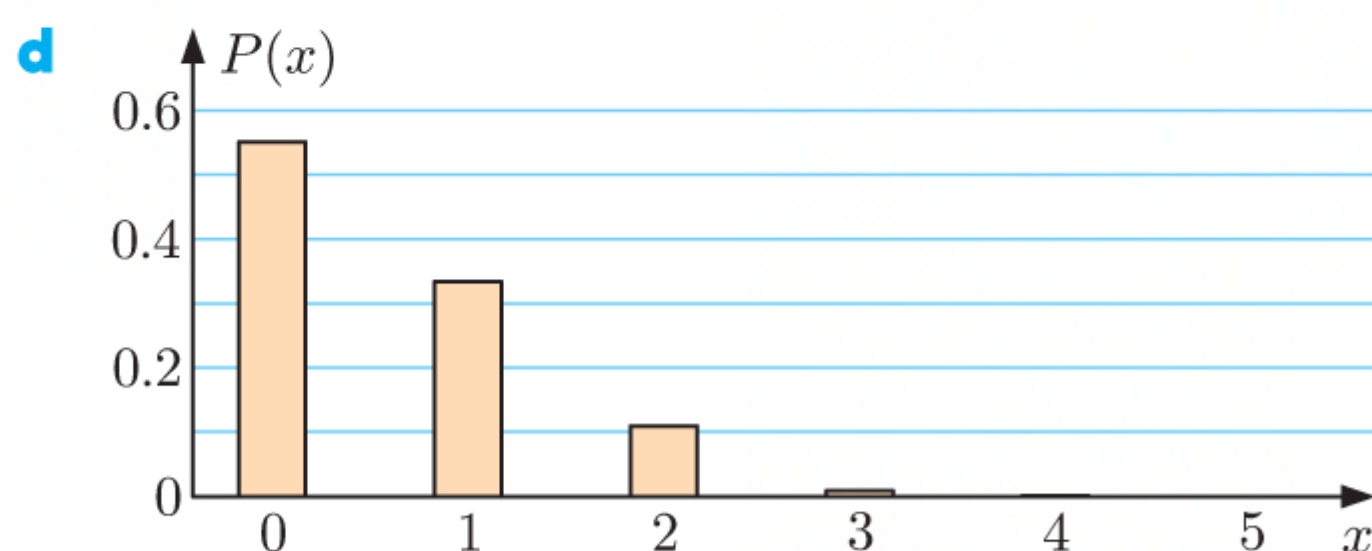
$$\therefore a + 0.4512 = 1$$

$$\therefore a = 0.5488$$

This is the probability that Jason does not hit a home run in a game.

c $P(1) + P(2) + P(3) + P(4) + P(5) = 0.3333 + 0.1088 + 0.0084 + 0.0007 + 0.0000$
 $= 0.4512$

This is the probability that Jason will hit one or more home runs in a game.



e Jason is most likely to score 0 home runs, so this is the mode of the distribution. Using **b**, $P(0) = 0.5488 \geq 0.5$, so the median is 0 home runs.

5

x	0	1	2	3	4
$P(X = x)$	0.68	0.2	0.06	k	0.02

a
$$\sum_{x=0}^4 P(X = x) = 1$$

$$\therefore 0.68 + 0.2 + 0.06 + k + 0.02 = 1$$

$$\therefore k = 0.04$$

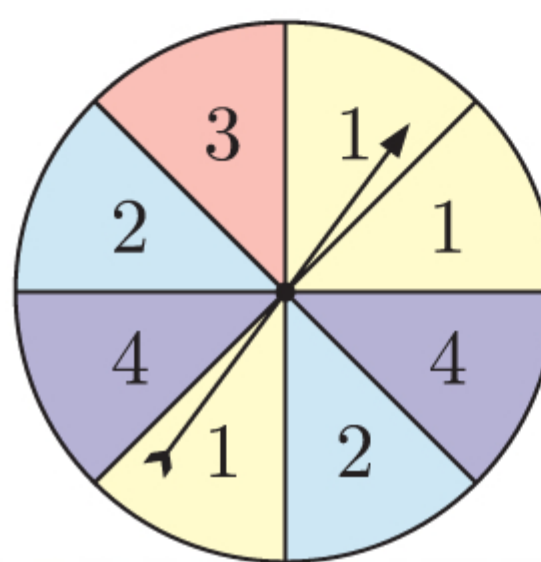
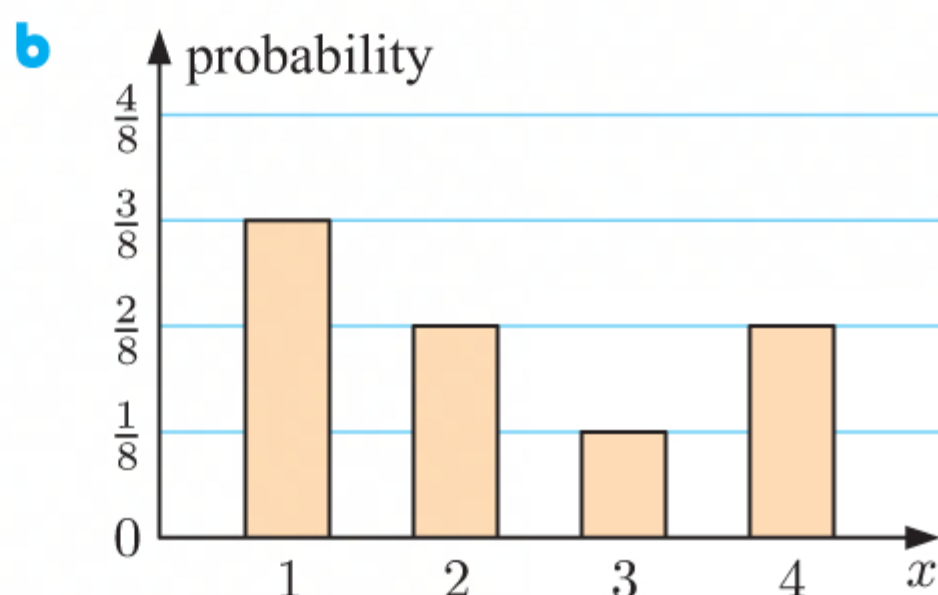
b It is most likely that the number of tyres which needed replacing is 0, so the mode of the distribution is 0 tyres.

c
$$\begin{aligned} P(X > 1) &= P(X = 2) + P(X = 3) + P(X = 4) \\ &= 0.06 + 0.04 + 0.02 \\ &= 0.12 \end{aligned}$$

This is the probability that more than 1 tyre will need replacing on a car being inspected.

6 a

x	1	2	3	4
$P(X = x)$	$\frac{3}{8}$	$\frac{2}{8}$	$\frac{1}{8}$	$\frac{2}{8}$



c The spinner is most likely to land on 1, so this is the mode of the distribution.

$$p_1 = \frac{3}{8} = 0.375$$

$$p_1 + p_2 = \frac{3}{8} + \frac{2}{8} = 0.625$$

Since $p_1 + p_2 \geq 0.5$, the median is 2.

d
$$\begin{aligned} P(X \leq 3) &= P(X = 1) + P(X = 2) + P(X = 3) \\ &= \frac{3}{8} + \frac{2}{8} + \frac{1}{8} \\ &= \frac{6}{8} \\ &= \frac{3}{4} \end{aligned}$$

7 a $X = 1, 2, 3, \text{ or } 4$

b $P(X = 1) = \frac{24}{100} = 0.24$

$$P(X = 2) = \frac{35}{100} = 0.35$$

$$P(X = 3) = \frac{27}{100} = 0.27$$

$$P(X = 4) = \frac{14}{100} = 0.14$$

\therefore the probability table for X is

x	1	2	3	4
$P(X = x)$	0.24	0.35	0.27	0.14

- c** It is most likely for a randomly selected person to have 2 bedrooms in their house, so this is the mode of the distribution.

$$p_1 = 0.24$$

$$p_1 + p_2 = 0.24 + 0.35 = 0.59$$

Since $p_1 + p_2 \geq 0.5$, the median is 2 bedrooms.

- 8 a** $X = 1, 2, 3$, or 4

- b** $P(X = 1) = \frac{12}{25} = 0.48$ \therefore the probability table for X is

$$P(X = 2) = \frac{7}{25} = 0.28$$

$$P(X = 3) = \frac{2}{25} = 0.08$$

$$P(X = 4) = \frac{4}{25} = 0.16$$

x	1	2	3	4
$P(X = x)$	0.48	0.28	0.08	0.16

- c** It is most likely for a randomly selected player to only need one shot to score a goal, so this is the mode of the distribution.

$$p_1 = 0.48$$

$$p_1 + p_2 = 0.48 + 0.28 = 0.76$$

Since $p_1 + p_2 \geq 0.5$, the median is 2 shots.

- 9 a** $P(x) = \frac{x+1}{10}$, $x = 0, 1, 2, 3$

$$\therefore P(0) = \frac{1}{10}, \quad P(1) = \frac{2}{10}, \quad P(2) = \frac{3}{10}, \quad P(3) = \frac{4}{10}$$

$$0 \leq P(x_i) \leq 1 \text{ in each case, and } \sum_{i=1}^n P(x_i) = \frac{1}{10} + \frac{2}{10} + \frac{3}{10} + \frac{4}{10} = 1$$

$\therefore P(x)$ is a valid probability mass function.

- b** $P(x) = \frac{6}{11x}$, $x = 1, 2, 3$

$$\therefore P(1) = \frac{6}{11}, \quad P(2) = \frac{6}{22} = \frac{3}{11}, \quad P(3) = \frac{6}{33} = \frac{2}{11}$$

$$0 \leq P(x_i) \leq 1 \text{ in each case, and } \sum_{i=1}^n P(x_i) = \frac{6}{11} + \frac{3}{11} + \frac{2}{11} = 1$$

$\therefore P(x)$ is a valid probability mass function.

- 10 a** $P(x) = k(x+2)$, $x = 1, 2, 3$

$$\therefore P(1) = 3k, \quad P(2) = 4k, \quad P(3) = 5k$$

Since this is a probability distribution, $\sum_{i=1}^n P(x_i) = 1$

$$\therefore 3k + 4k + 5k = 1$$

$$\therefore 12k = 1$$

$$\therefore k = \frac{1}{12}$$

$$\text{b } P(x) = \frac{k}{x+1}, \quad x = 0, 1, 2, 3$$

$$\therefore P(0) = k, \quad P(1) = \frac{k}{2}, \quad P(2) = \frac{k}{3}, \quad P(3) = \frac{k}{4}$$

$$\text{Since this is a probability distribution, } \sum_{i=1}^n P(x_i) = 1$$

$$\therefore k + \frac{k}{2} + \frac{k}{3} + \frac{k}{4} = 1$$

$$\therefore \frac{12k + 6k + 4k + 3k}{12} = 1$$

$$\therefore \frac{25k}{12} = 1$$

$$\therefore k = \frac{12}{25}$$

$$\text{11 a } P(x) = \frac{4x - x^2}{a}, \quad x = 0, 1, 2, 3$$

$$\therefore P(0) = 0, \quad P(1) = \frac{3}{a}, \quad P(2) = \frac{4}{a}, \quad P(3) = \frac{3}{a}$$

$$\text{Since this is a probability distribution, } \sum_{i=1}^n P(x_i) = 1$$

$$\therefore 0 + \frac{3}{a} + \frac{4}{a} + \frac{3}{a} = 1$$

$$\therefore \frac{10}{a} = 1$$

$$\therefore a = 10$$

$$\text{b } P(X = 1) = P(1) = \frac{3}{a} = \frac{3}{10}$$

$$\text{c } \text{Since } P(X = 2) = P(2) = \frac{4}{10} \text{ is the greatest probability, the mode of the distribution is 2.}$$

$$\text{12 } P(x) = a\left(\frac{1}{3}\right)^{x-1}, \quad x = 1, 2, 3, \dots$$

$$\begin{aligned} \text{a } \sum_{i=1}^{\infty} \left(\frac{1}{3}\right)^{i-1} &= \frac{1}{1 - \frac{1}{3}} \quad \{u_1 = 1, \quad r = \frac{1}{3}\} \\ &= \frac{1}{\frac{2}{3}} \\ &= \frac{3}{2} \end{aligned}$$

$$\text{b } \text{Since this is a probability distribution, } \sum_{i=1}^{\infty} P(x_i) = 1$$

$$\therefore \sum_{i=1}^{\infty} a\left(\frac{1}{3}\right)^{i-1} = 1$$

$$\therefore a \sum_{i=1}^{\infty} \left(\frac{1}{3}\right)^{i-1} = 1$$

$$\therefore a \times \frac{3}{2} = 1 \quad \{\text{using a}\}$$

$$\therefore a = \frac{2}{3}$$

13 $P(x) = a\left(\frac{2}{5}\right)^x, \quad x = 0, 1, 2, 3, \dots$

$$\begin{aligned} \text{Now, } \sum_{i=0}^{\infty} \left(\frac{2}{5}\right)^i &= \frac{1}{1 - \frac{2}{5}} \quad \{u_1 = 1, \quad r = \frac{2}{5}\} \\ &= \frac{1}{\frac{3}{5}} \\ &= \frac{5}{3} \end{aligned}$$

Since this is a probability distribution, $\sum_{i=0}^{\infty} P(x_i) = 1$

$$\therefore \sum_{i=0}^{\infty} a\left(\frac{2}{5}\right)^i = 1$$

$$\therefore a \sum_{i=0}^{\infty} \left(\frac{2}{5}\right)^i = 1$$

$$\therefore a \times \frac{5}{3} = 1$$

$$\therefore a = \frac{3}{5}$$

EXERCISE 20C.1

1 a

x_i	1	2	3
p_i	0.4	0.5	0.1

$$\begin{aligned} E(X) &= \sum_{i=1}^n x_i p_i \\ &= 1(0.4) + 2(0.5) + 3(0.1) \\ &= 1.7 \end{aligned}$$

b

x_i	0	1	2	3	4
p_i	0.1	0.2	0.15	0.2	0.35

$$\begin{aligned} E(X) &= \sum_{i=1}^n x_i p_i \\ &= 0(0.1) + 1(0.2) + 2(0.15) + 3(0.2) + 4(0.35) \\ &= 2.5 \end{aligned}$$

c

x_i	0	2	5	10
p_i	0.2	0.35	0.27	0.18

$$\begin{aligned} E(X) &= \sum_{i=1}^n x_i p_i \\ &= 0(0.2) + 2(0.35) + 5(0.27) + 10(0.18) \\ &= 3.85 \end{aligned}$$

d

x_i	10	15	30	60
p_i	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{12}$	$\frac{1}{3}$

$$\begin{aligned}
 E(X) &= \sum_{i=1}^n x_i p_i \\
 &= 10\left(\frac{1}{4}\right) + 15\left(\frac{1}{3}\right) + 30\left(\frac{1}{12}\right) + 60\left(\frac{1}{3}\right) \\
 &= 30
 \end{aligned}$$

- 2 a** Since this is a probability distribution, $\sum_{i=1}^n P(x_i) = 1$
- $$\therefore \frac{2}{5} + a + \frac{1}{10} = 1$$
- $$\therefore a = \frac{1}{2}$$

x	1	3	5
$P(X = x)$	$\frac{2}{5}$	a	$\frac{1}{10}$

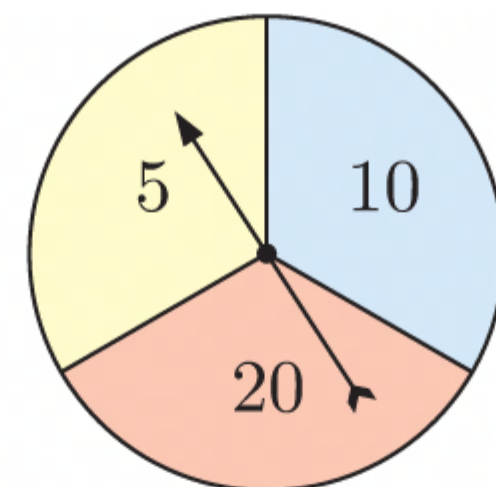
- b** Since $P(X = 3)$ is the greatest probability, 3 is the mode of the distribution.

c $\mu = E(X) = 1\left(\frac{2}{5}\right) + 3\left(\frac{1}{2}\right) + 5\left(\frac{1}{10}\right)$

$$\begin{aligned}
 &= \frac{2}{5} + \frac{3}{2} + \frac{5}{10} \\
 &= \frac{4}{10} + \frac{15}{10} + \frac{5}{10} \\
 &= 2\frac{2}{5}
 \end{aligned}$$

- 3** Each coloured region on the spinner has the same area.
The probability table is:

<i>Number of points</i>	5	10	20
<i>Probability</i>	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$



$$\begin{aligned}
 E(X) &= \sum_{i=1}^n x_i p_i \\
 &= \left(5 \times \frac{1}{3}\right) + \left(10 \times \frac{1}{3}\right) + \left(20 \times \frac{1}{3}\right) \\
 &= \frac{35}{3} \\
 &\approx 11.7 \text{ points}
 \end{aligned}$$

In the long term, we can expect to be awarded an average of about 11.7 points per spin.

4

<i>Number of fish</i>	0	1	2	3
<i>Probability</i>	0.17	0.28	0.36	0.19

$$\begin{aligned}
 E(X) &= \sum_{i=1}^n x_i p_i \\
 &= (0 \times 0.17) + (1 \times 0.28) + (2 \times 0.36) + (3 \times 0.19) \\
 &= 0.28 + 0.72 + 0.57 \\
 &= 1.57 \text{ fish}
 \end{aligned}$$

On average, you would expect Ernie to catch 1.57 fish per trip.

5	<i>Number of books</i>	1	2	3	4	5
	<i>Probability</i>	0.16	0.15	a	0.28	0.16

a Since this is a probability distribution, $\sum_{i=1}^n P(x_i) = 1$

$$\therefore 0.16 + 0.15 + a + 0.28 + 0.16 = 1$$

$$\therefore a = 0.25$$

b Pam is most likely going to borrow 4 books when she visits the library, so this is the mode of the distribution.

c $E(X) = \sum_{i=1}^n x_i p_i$

$$= (1 \times 0.16) + (2 \times 0.15) + (3 \times 0.25) + (4 \times 0.28) + (5 \times 0.16)$$

$$= 0.16 + 0.30 + 0.75 + 1.12 + 0.80$$

$$= 3.13 \text{ books}$$

On average, Pam borrows 3.13 books per visit.

6	<i>Colour</i>	<i>Number of lollies</i>
	Red	4
	Green	6
	White	10

There are 5 red balls, 2 green balls, and 1 white ball, so in total there are $5 + 2 + 1 = 8$ balls.

<i>Number of lollies</i>	4	6	10
<i>Probability</i>	$\frac{5}{8} = 0.625$	$\frac{2}{8} = 0.25$	$\frac{1}{8} = 0.125$

$$E(X) = \sum_{i=1}^n x_i p_i$$

$$= (4 \times 0.625) + (6 \times 0.25) + (10 \times 0.125)$$

$$= 2.5 + 1.5 + 1.25$$

$$= 5.25 \text{ lollies}$$

On average, Lachlan can expect to receive 5.25 lollies.

7 a $P(\text{all ten pins}) = 1 - \frac{1}{3} - \frac{2}{5}$

$$= \frac{15}{15} - \frac{5}{15} - \frac{6}{15}$$

$$= \frac{4}{15}$$

b	<i>Number of pins knocked down</i>	8	9	10
	<i>Probability</i>	$\frac{1}{3}$	$\frac{2}{5}$	$\frac{4}{15}$

$$E(X) = \sum_{i=1}^n x_i p_i$$

$$= \left(8 \times \frac{1}{3}\right) + \left(9 \times \frac{2}{5}\right) + \left(10 \times \frac{4}{15}\right)$$

$$= \frac{40}{15} + \frac{54}{15} + \frac{40}{15}$$

$$= \frac{134}{15}$$

$$\approx 8.93 \text{ pins}$$

On average, Jenna knocks down about 8.93 pins with her first bowl.

- 8 Since this is a probability distribution,

$$\sum_{i=1}^n P(x_i) = 1$$

$$\therefore 0.3 + a + b + 0.2 = 1$$

$$\therefore b = 0.5 - a \quad \dots (*)$$

$$\text{Now, } E(X) = 2.5$$

$$\therefore (1 \times 0.3) + (2 \times a) + (3 \times b) + (4 \times 0.2) = 2.5$$

$$\therefore 0.3 + 2a + 3(0.5 - a) + 0.8 = 2.5 \quad \{\text{using } (*)\}$$

$$\therefore 2a + 1.5 - 3a = 1.4$$

$$\therefore a = 0.1 \text{ and } b = 0.4$$

x	1	2	3	4
$P(X = x)$	0.3	a	b	0.2

- 9 a When Brad's soccer team plays an offensive strategy, $P(\text{draw}) = 1 - 0.3 - 0.55 = 0.15$
 When Brad's soccer team plays a defensive strategy, $P(\text{draw}) = 1 - 0.2 - 0.3 = 0.5$
- b Let X be the number of points awarded per game when Brad's soccer team plays an offensive strategy.

$$\begin{aligned} E(X) &= (3 \times 0.3) + (1 \times 0.15) + (0 \times 0.55) \\ &= 0.9 + 0.15 \\ &= 1.05 \text{ points per game} \end{aligned}$$

<i>Result</i>	W	D	L
<i>Points</i>	3	1	0
<i>Probability</i>	0.3	0.15	0.55

Let Y be the number of points awarded per game when Brad's soccer team plays a defensive strategy.

$$\begin{aligned} E(Y) &= (3 \times 0.2) + (1 \times 0.5) + (0 \times 0.3) \\ &= 0.6 + 0.5 \\ &= 1.1 \text{ points per game} \end{aligned}$$

<i>Result</i>	W	D	L
<i>Points</i>	3	1	0
<i>Probability</i>	0.2	0.5	0.3

- c It is better for the team to play a defensive strategy in the long run as the team is expected to gain more points per game.
- d If 4 points are awarded instead of 3 points for a win:

$$\begin{aligned} E(X) &= (4 \times 0.3) + (1 \times 0.15) + (0 \times 0.55) \\ &= 1.2 + 0.15 \\ &= 1.35 \text{ points per game} \end{aligned}$$

<i>Result</i>	W	D	L
<i>Points</i>	4	1	0
<i>Probability</i>	0.3	0.15	0.55

$$\begin{aligned} \text{and } E(Y) &= (4 \times 0.2) + (1 \times 0.5) + (0 \times 0.3) \\ &= 0.8 + 0.5 \\ &= 1.3 \text{ points per game} \end{aligned}$$

<i>Result</i>	W	D	L
<i>Points</i>	4	1	0
<i>Probability</i>	0.2	0.5	0.3

The team is expected to gain more points per game when they play an offensive strategy. The team should change their strategy.

- 10 a i car park B
 ii car park A
 iii car park B

Car park A

<i>Time</i>	<i>Cost</i>
0 - 1 hour	\$7
1 - 2 hours	\$12
2 - 3 hours	\$15
3 - 4 hours	\$19

Car park B

<i>Time</i>	<i>Cost</i>
0 - 1 hour	\$6.50
1 - 2 hours	\$11
2 - 3 hours	\$16
3 - 4 hours	\$18.50

- b** Let X be the amount Zoe pays for parking.

When Zoe parks her car at car park A:

$$\begin{aligned} E(X) &= (7 \times 0) + (12 \times 0.2) + (15 \times 0.7) + (19 \times 0.1) \\ &= 2.4 + 10.5 + 1.9 \\ &= \$14.80 \end{aligned}$$

When Zoe parks her car at car park B:

$$\begin{aligned} E(X) &= (6.5 \times 0) + (11 \times 0.2) + (16 \times 0.7) + (18.5 \times 0.1) \\ &= 2.2 + 11.2 + 1.85 \\ &= \$15.25 \end{aligned}$$

\therefore Zoe should choose car park A as it has the lower expected cost.

- 11** The probability of the ring not being stolen or lost is $P(\text{ring is safe}) = 1 - 0.0025 - 0.03$
 $= 0.9675$

Let X be the amount the insurance company pays the policy owner.

$$\begin{aligned} E(X) &= (0 \times 0.9675) + (20\,000 \times 0.0025) + (8000 \times 0.03) \\ &= 50 + 240 \\ &= \$290 \text{ per policy} \end{aligned}$$

\therefore the insurance company should charge \$390 per policy to have an expected return of \$100.

EXERCISE 20C.2

- 1** Let X denote the return from one game.

<i>Number</i>	1	2	3	4	5	6
<i>Winnings</i>	\$3	\$1	\$3	\$1	\$3	\$1
<i>Probability</i>	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

$$\begin{aligned} E(X) &= 3\left(\frac{1}{6} \times 3\right) + 3\left(\frac{1}{6} \times 1\right) \\ &= \frac{9}{6} + \frac{3}{6} \\ &= \frac{12}{6} \\ &= 2 \end{aligned}$$

So, \$2 is the expected return.

Since the game costs \$2 to play, the expected gain = expected return – \$2
 $= \$2 - \$2 = \$0$

Since the expected gain is zero, the game is fair.

- 2 a** Let X denote the return from each roll.

$$\begin{aligned} E(X) &= \left(\frac{1}{6} \times 1\right) + \left(\frac{1}{6} \times 2\right) + \left(\frac{1}{6} \times 3\right) + \left(\frac{1}{6} \times 4\right) + \left(\frac{1}{6} \times 5\right) + \left(\frac{1}{6} \times 6\right) \\ &= \frac{1}{6} \times 21 \\ &= \$3.50 \end{aligned}$$

- b** The expected gain is $\$3.50 - \$4 = -\$0.50$

- c** The player should not play many games, as on average he would expect to lose \$0.50 with each roll.

- 3 a** Let X denote the return from each bet.

$$\begin{aligned} E(X) &= \left(\frac{18}{37} \times 2\right) + \left(\frac{19}{37} \times -2\right) \\ &= \frac{36}{37} - \frac{38}{37} \\ &= -\frac{2}{37} \\ &\approx -\$0.05 \end{aligned}$$

b $100 \times -\frac{2}{37} \approx -\5.41

From 100 bets, I would expect to lose about \$5.41.

4

Result	Win
HH	\$10
HT or TH	\$3
TT	\$1

Let X be the gain from each game, and Y be the return from each game.

$$\begin{aligned} E(Y) &= \left(\frac{1}{4} \times 10\right) + \left(\frac{2}{4} \times 3\right) + \left(\frac{1}{4} \times 1\right) \\ &= \frac{10}{4} + \frac{6}{4} + \frac{1}{4} \\ &= \$4.25 \end{aligned}$$

The expected return per game is \$4.25. It costs \$5.00 to play the game.

$$\begin{aligned} \text{So, the expected gain } E(X) &= E(Y) - \$5 \\ &= \$4.25 - \$5.00 \\ &= -\$0.75 \end{aligned}$$

So we expect a loss of \$0.75 per game on average.

5 a i $P(\text{win 5 tokens}) = \frac{6}{20}$ {there are 6 multiples of 3 between 1 and 20}

$$= \frac{3}{10}$$

$$= 0.3$$

ii $P(\text{win 10 tokens}) = \frac{2}{20}$ {there are 2 multiples of 10 between 1 and 20}

$$= \frac{1}{10}$$

$$= 0.1$$

b $E(X) = \left(0 \times \frac{12}{20}\right) + \left(5 \times \frac{6}{20}\right) + \left(10 \times \frac{2}{20}\right)$

$$= \frac{30}{20} + \frac{20}{20}$$

$$= 2\frac{1}{2}$$

$$= 2.5 \text{ tokens}$$

- c** It costs 3 tokens to play the game. So, the expected gain = $2.5 - 3 = -0.5$ tokens. We do not recommend playing the game many times as the player can expect to lose half a token on average per game.

6

a

Disc colour	Black	Blue	Gold
Winnings	\$1	\$5	\$20
Probability	$\frac{10}{15}$	$\frac{4}{15}$	$\frac{1}{15}$

Let X be the return from each game.

$$\begin{aligned} E(X) &= \left(1 \times \frac{10}{15}\right) + \left(5 \times \frac{4}{15}\right) + \left(20 \times \frac{1}{15}\right) \\ &= \frac{10 + 20 + 20}{15} \\ &= \frac{50}{15} \\ &\approx \$3.33 \end{aligned}$$

The expected return per game is \$3.33. It costs \$4.00 to play the game.

$$\begin{aligned} \text{So, the expected gain} &\approx \$3.33 - \$4.00 \\ &\approx -\$0.67 \neq \$0, \text{ so the game is not fair.} \end{aligned}$$

- b** Let the new prize money for selecting the gold disc be \$ x .

Now, for the game to be fair, the expected return must be equal to the cost of each game.

$$\therefore E(X) = \left(1 \times \frac{10}{15}\right) + \left(5 \times \frac{4}{15}\right) + \left(x \times \frac{1}{15}\right) = 4 \quad \{\text{the cost of the game is \$4}\}$$

$$\therefore \frac{10}{15} + \frac{20}{15} + \frac{x}{15} = 4$$

$$\therefore \frac{30+x}{15} = 4$$

$$\therefore 30+x = 60$$

$$\therefore x = 30$$

So, the new prize money for selecting the gold disc is \$30.

7 a

		Die 2					
		1	2	3	4	5	6
Die 1	1	2	3	4	5	6	7
	2	3	4	5	6	7	8
	3	4	5	6	7	8	9
	4	5	6	7	8	9	10
	5	6	7	8	9	10	11
	6	7	8	9	10	11	12

36 possible results

x	2	3	4	5	6	7	8	9	10	11	12
$P(X = x)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

$$\text{So, } P(X \leq 3) = P(X = 2) + P(X = 3) = \frac{1}{36} + \frac{2}{36} = \frac{1}{12}$$

$$P(4 \leq X \leq 6) = P(X = 4) + P(X = 5) + P(X = 6) = \frac{3}{36} + \frac{4}{36} + \frac{5}{36} = \frac{1}{3}$$

$$P(7 \leq X \leq 9) = P(X = 7) + P(X = 8) + P(X = 9) = \frac{6}{36} + \frac{5}{36} + \frac{4}{36} = \frac{5}{12}$$

$$P(X \geq 10) = P(X = 10) + P(X = 11) + P(X = 12) = \frac{3}{36} + \frac{2}{36} + \frac{1}{36} = \frac{1}{6}$$

- b** Let Y be the return on a game. The cost is \$ a , so the expected gain is

$$\begin{aligned} E(Y) - a &= \frac{a}{3} \times [P(X \leq 3) + P(7 \leq X \leq 9)] + 7 \times P(4 \leq X \leq 6) + 21 \times P(X \geq 10) - a \\ &= \frac{a}{3} \times \left(\frac{1}{12} + \frac{5}{12}\right) + \left(7 \times \frac{1}{3}\right) + \left(21 \times \frac{1}{6}\right) - a \quad \{\text{using a}\} \\ &= \frac{a}{6} + \frac{7}{3} + \frac{21}{6} - a \\ &= \frac{35}{6} - \frac{5a}{6} \\ &= \frac{1}{6}(35 - 5a) \text{ dollars, as required.} \end{aligned}$$

- c** The game is fair when the expected gain is 0.

$$\therefore \frac{1}{6}(35 - 5a) = 0$$

$$\therefore 35 - 5a = 0$$

$$\therefore a = 7$$

d If $a = 4$, expected gain $= \frac{1}{6} (35 - 5(4))$
 $= \frac{15}{6} = \$2.50$

So, the people playing would expect to win \$2.50 per game, which means the organisers expect to lose \$2.50 per game.

e If $a = 9$, expected gain $= \frac{1}{6} (35 - 5(9))$
 $= -\frac{10}{6} \approx -\1.33

Expected gain from 2406 games $= -\frac{10}{6} \times 2406$
 $= -\$4010$

\therefore the organisers would expect to gain \$4010.

8 $P(RRR) = \frac{5}{12} \times \frac{4}{11} \times \frac{3}{10} = \frac{60}{1320}$

$P(BBB) = \frac{4}{12} \times \frac{3}{11} \times \frac{2}{10} = \frac{24}{1320}$

$P(GGG) = \frac{3}{12} \times \frac{2}{11} \times \frac{1}{10} = \frac{6}{1320}$

$P(RBG) = P(RGB) = P(BRG) = P(BGR) = P(GRB) = P(GBR) = \frac{5}{12} \times \frac{4}{11} \times \frac{3}{10} = \frac{60}{1320}$

$P(\text{winning}) = P(\text{all the same colour or one of each})$
 $= P(RRR) + P(BBB) + P(GGG) + P(RBG) + P(RGB)$
 $\quad + P(BRG) + P(BGR) + P(GRB) + P(GBR)$
 $= \frac{60}{1320} + \frac{24}{1320} + \frac{6}{1320} + \frac{60}{1320} \times 6$
 $= \frac{60+24+6+360}{1320}$
 $= \frac{450}{1320} = \frac{15}{44}$

The player expects to win $11 \times \frac{15}{44} = \3.75

The organiser makes \$1 when the player loses \$1.

Now, the expected gain for the player = expected win – cost to play

$\therefore -\$1.00 = \$3.75 - \text{cost to play}$

$\therefore \text{cost to play} = \4.75

EXERCISE 20D

- 1
 - a The binomial distribution applies, as tossing a coin has two possible outcomes (a head or a tail) and each toss is independent of every other toss.
 - b The binomial distribution applies, as this is equivalent to tossing one coin 100 times.
 - c The binomial distribution applies as we can draw out a red or a blue marble with the same chances each time.
 - d The binomial distribution does not apply as the result of each draw is dependent upon the results of previous draws.
 - e The binomial distribution does not apply, assuming that ten bolts are drawn without replacement, as we do not have a repetition of independent trials. However, since there is such a large number of bolts in the bin, the trials are approximately independent, so the distribution is approximately binomial.

$$2 \quad a \quad (p + q)^4 = p^4 + 4p^3q + 6p^2q^2 + 4pq^3 + q^4$$

$$\begin{aligned} b \quad i \quad & P(4 \text{ heads}) \\ &= p^4 \\ &= \left(\frac{1}{2}\right)^4 \\ &= \frac{1}{16} \end{aligned}$$

$$\begin{aligned} ii \quad & P(3 \text{ heads}) \\ &= 4p^3q \\ &= 4\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right) \\ &\quad \left\{\text{as } p = q = \frac{1}{2}\right\} \\ &= \frac{1}{4} \end{aligned}$$

$$\begin{aligned} iii \quad & P(2 \text{ heads}) \\ &= 6p^2q^2 \\ &= 6\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^2 \\ &= \frac{3}{8} \end{aligned}$$

$$3 \quad a \quad (p + q)^5 = p^5 + 5p^4q + 10p^3q^2 + 10p^2q^3 + 5pq^4 + q^5$$

$$\begin{aligned} b \quad i \quad & P(4H \text{ and } 1T) \\ &= 5p^4q \\ &= 5\left(\frac{1}{2}\right)^4\left(\frac{1}{2}\right) \\ &= \frac{5}{32} \end{aligned}$$

$$\begin{aligned} ii \quad & P(2H \text{ and } 3T) \\ &= 10p^2q^3 \\ &= 10\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^3 \\ &= \frac{10}{32} \\ &= \frac{5}{16} \end{aligned}$$

$$\begin{aligned} iii \quad & P(\text{HHHHT}) \\ &= \left(\frac{1}{2}\right)^4 \times \frac{1}{2} \\ &= \frac{1}{32} \end{aligned}$$

$$4 \quad a \quad \left(\frac{2}{3} + \frac{1}{3}\right)^4 = \left(\frac{2}{3}\right)^4 + 4\left(\frac{2}{3}\right)^3\left(\frac{1}{3}\right) + 6\left(\frac{2}{3}\right)^2\left(\frac{1}{3}\right)^2 + 4\left(\frac{2}{3}\right)\left(\frac{1}{3}\right)^3 + \left(\frac{1}{3}\right)^4$$

b The probability of getting a strawberry cream is $p = \frac{2}{3}$.

Let X be the number of strawberry creams selected.

$$\begin{aligned} i \quad & P(\text{all strawberry creams}) = P(X = 4) \\ &= \left(\frac{2}{3}\right)^4 \\ &= \frac{16}{81} \end{aligned}$$

$$\begin{aligned} ii \quad & P(\text{two of each}) = P(X = 2) \\ &= 6\left(\frac{2}{3}\right)^2\left(\frac{1}{3}\right)^2 \\ &= \frac{8}{27} \end{aligned}$$

$$\begin{aligned} iii \quad & P(\text{at least 2 strawberry creams}) = P(X \geq 2) \\ &= P(X = 2) + P(X = 3) + P(X = 4) \\ &= \frac{16}{81} + 4\left(\frac{2}{3}\right)^3\left(\frac{1}{3}\right) + \frac{8}{27} \quad \{\text{using } i \text{ and } ii\} \\ &= \frac{8}{9} \end{aligned}$$

$$5 \quad a \quad \left(\frac{3}{4} + \frac{1}{4}\right)^5 = \left(\frac{3}{4}\right)^5 + 5\left(\frac{3}{4}\right)^4\left(\frac{1}{4}\right) + 10\left(\frac{3}{4}\right)^3\left(\frac{1}{4}\right)^2 + 10\left(\frac{3}{4}\right)^2\left(\frac{1}{4}\right)^3 + 5\left(\frac{3}{4}\right)\left(\frac{1}{4}\right)^4 + \left(\frac{1}{4}\right)^5$$

b The probability of getting a “normal” kiwi is $p = \frac{3}{4}$.

Let X be the number of “normal” kiwis selected.

$$\begin{aligned} i \quad & P(2 \text{ “flat backs”}) \\ &= P(3 \text{ “normal” kiwis}) \\ &= P(X = 3) \\ &= 10\left(\frac{3}{4}\right)^3\left(\frac{1}{4}\right)^2 \\ &= \frac{135}{512} \end{aligned}$$

$$\begin{aligned} ii \quad & P(\text{at least 3 “flat backs”}) \\ &= P(\text{at most 2 “normal” kiwis}) \\ &= P(X = 0) + P(X = 1) + P(X = 2) \\ &= \left(\frac{1}{4}\right)^5 + 5\left(\frac{3}{4}\right)\left(\frac{1}{4}\right)^4 + 10\left(\frac{3}{4}\right)^2\left(\frac{1}{4}\right)^3 \\ &= \frac{53}{512} \end{aligned}$$

$$\begin{aligned} iii \quad & P(\text{at most 3 “normal” kiwis}) = P(\text{at most 2 “normal” kiwis}) + P(3 \text{ “normal” kiwis}) \\ &= \frac{135}{512} + \frac{53}{512} \quad \{\text{using } i \text{ and } ii\} \\ &= \frac{47}{128} \end{aligned}$$

- 6 a** $\sum_{x=0}^n P(x) = \sum_{x=0}^n \binom{n}{x} p^x (1-p)^{n-x}$
 $= (p + (1-p))^n \quad \{\text{binomial theorem}\}$
 $= 1^n$
 $= 1$
- b** $\binom{n}{x}, p^x, (1-p)^{n-x} \geq 0$ for all $x = 0, 1, \dots, n$
 $\therefore P(x) \geq 0$ for all x
- Now $\sum_{x=0}^n P(x) = 1$ and $P(x) \geq 0$ for all x
 $\therefore P(x) \leq 1$ for all x
 $\therefore 0 \leq P(x) \leq 1$ for all $x = 0, 1, \dots, n$. ✓
- c** Our answers to **a** and **b** tell us that $P(x)$ is a valid probability distribution.

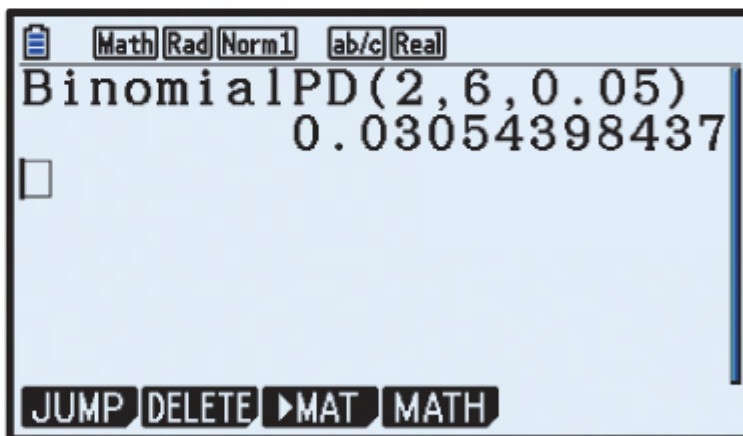
INVESTIGATION 1

THE GRAPH OF A BINOMIAL DISTRIBUTION

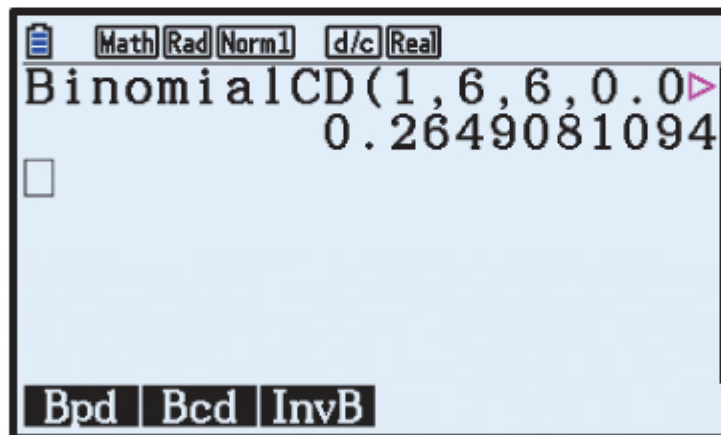
- 1 a** $X \sim B(n, p)$
 When $n = 25$, $p = 0.1$, the mode of X is 2.
- b** The distribution is positively skewed.
- 2** When $p = 0.5$, the distribution is symmetric.
 When $p < 0.5$, the distribution is positively skewed.
 When $p > 0.5$, the distribution is negatively skewed.
- 3** $p = 0.1$, and the value of n is free to change.
 As n increases, the distribution becomes approximately symmetrical.

EXERCISE 20E

- 1** Let X be the number of defective light bulbs.
 $n = 6$, so $X = 0, 1, 2, 3, 4, 5$, or 6 , and $p = 5\% = 0.05$
 $\therefore X \sim B(6, 0.05)$

a 

$$P(X = 2) \approx 0.0305$$

b 

$$P(X \geq 1) \approx 0.265$$

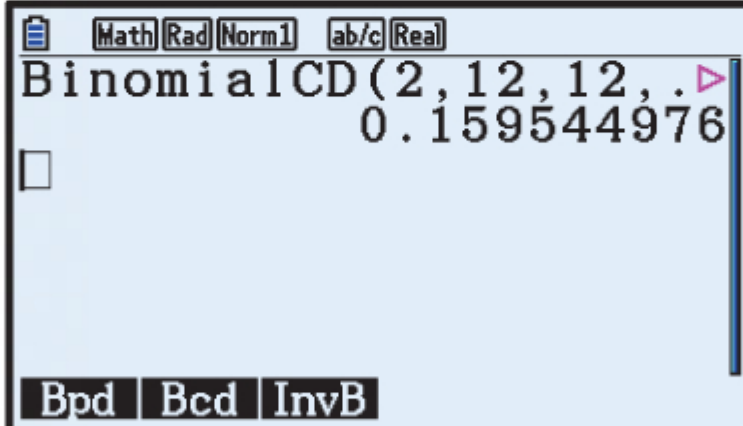
- 2** Let X be the number of faulty items.

$n = 12$, so $X = 0, 1, 2, 3, \dots$, or 12 , and $p = 6\% = 0.06$

$$\therefore X \sim B(12, 0.06)$$

a $P(\text{none will be faulty})$
 $= P(X = 0)$
 $= \binom{12}{0} (0.06)^0 (0.94)^{12}$
 ≈ 0.476

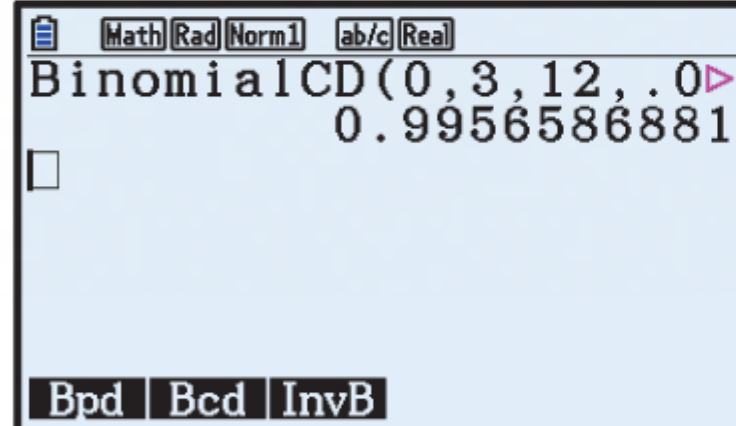
b $P(\text{at most one is faulty})$
 $= P(X \leq 1)$
 $= P(X = 0) + P(X = 1)$
 $\approx 0.476 + \binom{12}{1} (0.06)^1 (0.94)^{11}$
 ≈ 0.840

c 

$$P(\text{at least two are faulty}) = P(X \geq 2)$$

$$\approx 0.160$$

or $P(\text{at least two are faulty})$
 $= 1 - P(\text{at most one is faulty})$
 $\approx 1 - 0.840 \quad \{\text{from b}\}$
 ≈ 0.160

d 

$$P(\text{less than four are faulty}) = P(X < 4)$$

$$= P(X \leq 3)$$

$$\approx 0.996$$

- 3** Let X be the number of times in a week that the bus is on time.

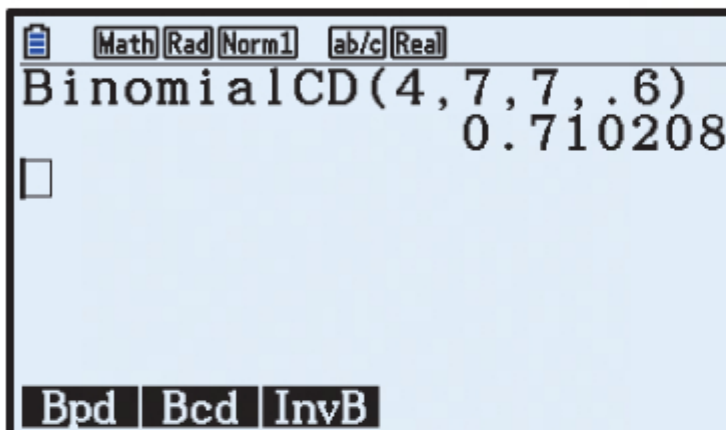
Since it is late 2 in every 5 days, then it is on time 3 in every 5 days, so $p = \frac{3}{5} = 0.6$.

$n = 7$, so $X = 0, 1, 2, 3, 4, 5, 6$, or 7 , and $X \sim B(7, 0.6)$.

a $P(X = 7) = \binom{7}{7} (0.6)^7 (0.4)^0$
 ≈ 0.0280

b $P(\text{on time only on Monday}) = 0.6 \times (0.4)^6$
 ≈ 0.00246

c $P(X = 6) = \binom{7}{6} (0.6)^6 (0.4)$
 ≈ 0.131

d 

$$P(X \geq 4) \approx 0.710$$

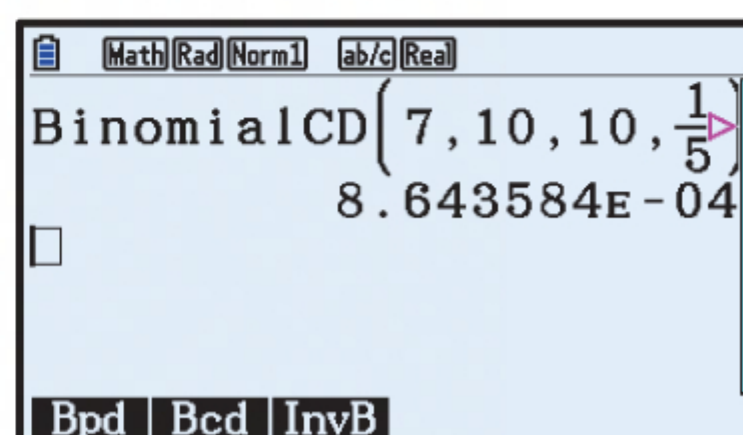
- 4** Let X denote the number of questions Raj answers correctly.

$n = 10$, so $X = 0, 1, 2, \dots$, or 10 and $p = \frac{1}{5}$

$$\therefore X \sim B(10, \frac{1}{5})$$

$$P(\text{Raj passes}) = P(X \geq 7)$$

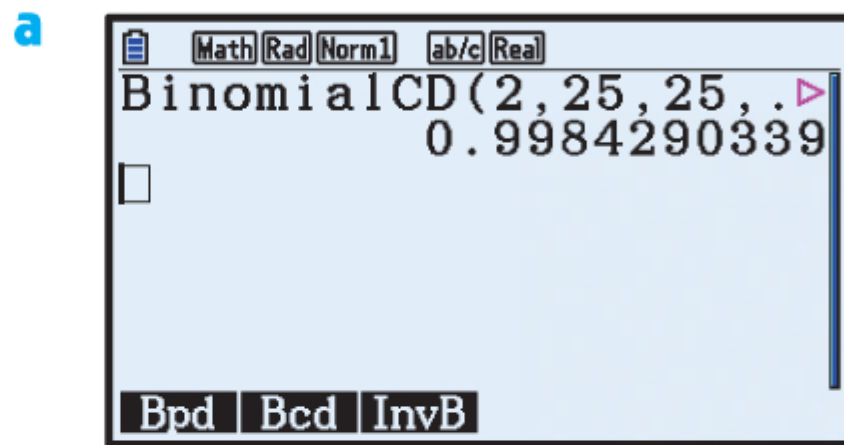
$$\approx 0.000864$$



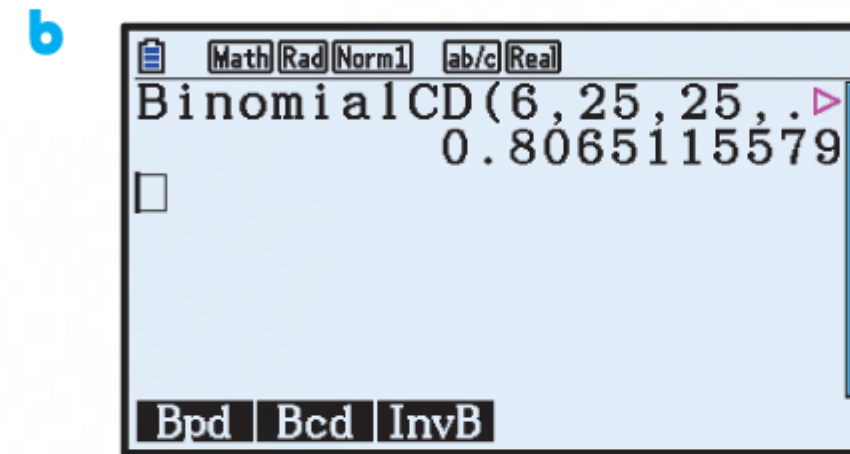
- 5 Let X be the number of students with the flu.

$n = 25$, so $X = 0, 1, 2, 3, \dots$, or 25 , and $p = 0.3$

$$\therefore X \sim B(25, 0.3)$$



$$P(X \geq 2) \approx 0.998$$



$$20\% \text{ of } 25 = 5$$

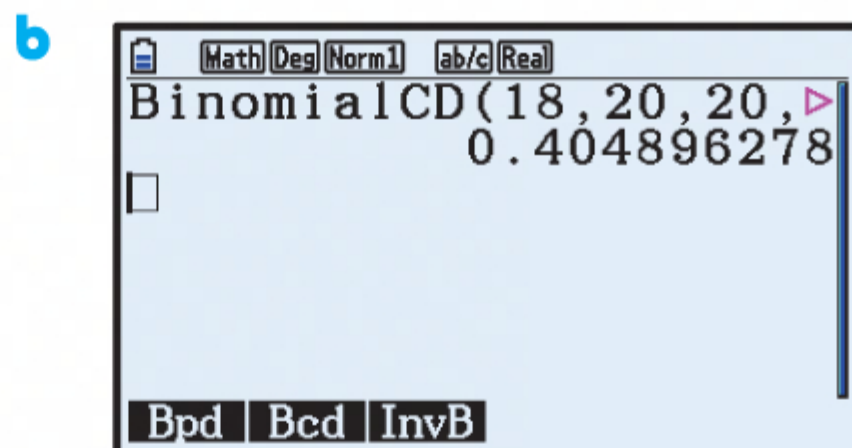
$$\therefore P(\text{test cancelled}) = P(X \geq 6) \\ \approx 0.807$$

- 6 Let X be the number of successful shots from the free throw line.

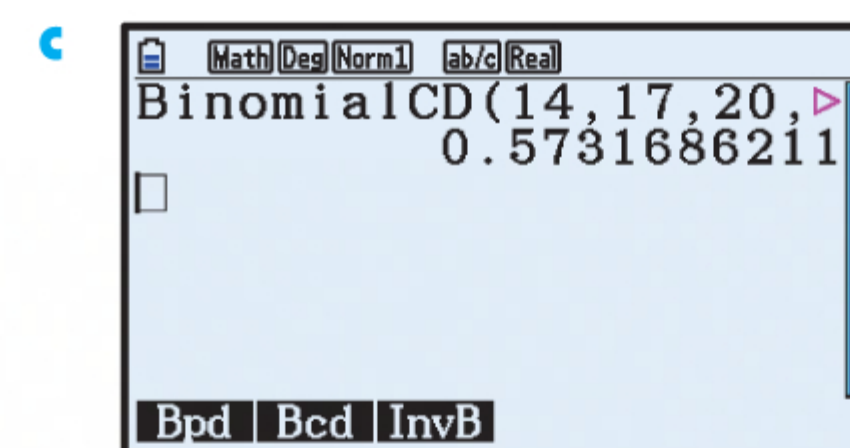
$n = 20$, so $X = 0, 1, 2, 3, \dots$, or 20 , and $p = 85\% = 0.85$

$$\therefore X \sim B(20, 0.85)$$

a
$$P(X = 20) = \binom{20}{20} (0.85)^{20} (0.15)^0 \\ \approx 0.0388$$



$$P(X \geq 18) \approx 0.405$$



$$P(14 \leq X \leq 17) \approx 0.573$$

- 7 For Jelena to win a set of 6 games to 4, she must win 5 of the first 9 games, and then win the 10th game.

Let X be the number of games Jelena wins in the first 9 games.

$n = 9$, so $X = 0, 1, 2, 3, \dots$, or 9

Now, Martina beats Jelena in 2 games out of 3, so the probability of Jelena winning a game is

$$p = 1 - \frac{2}{3} = \frac{1}{3}.$$

$$\therefore X \sim B(9, \frac{1}{3})$$

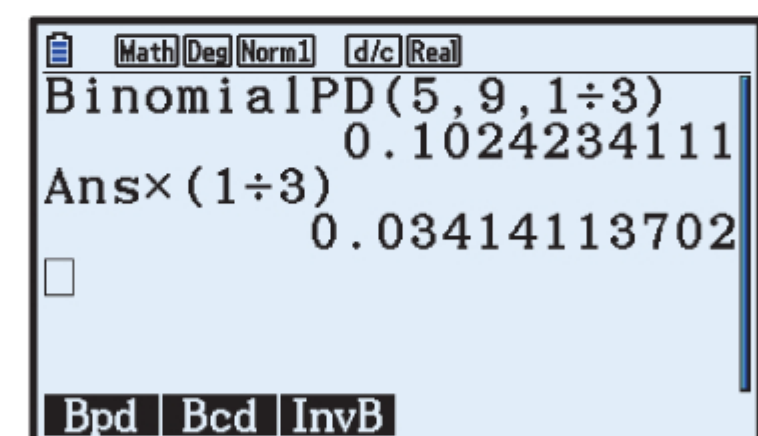
So, $P(\text{J wins 6 games to 4})$

$$= P(\text{J wins 5 of first 9 games}) \times P(\text{J wins 10th game})$$

$$= P(X = 5) \times \frac{1}{3}$$

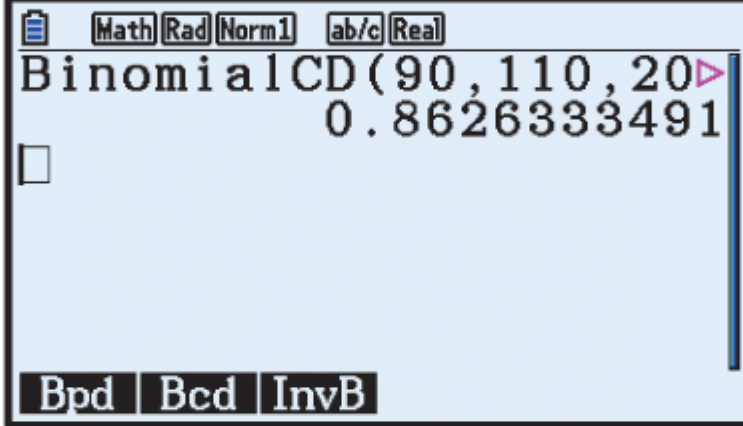
$$\approx 0.1024 \times \frac{1}{3}$$

$$\approx 0.0341$$



- 8 Let X be the number of heads.
 $n = 200$, so $X = 0, 1, 2, 3, \dots$, or 200 , and $p = \frac{1}{2}$
 $\therefore X \sim B(200, \frac{1}{2})$

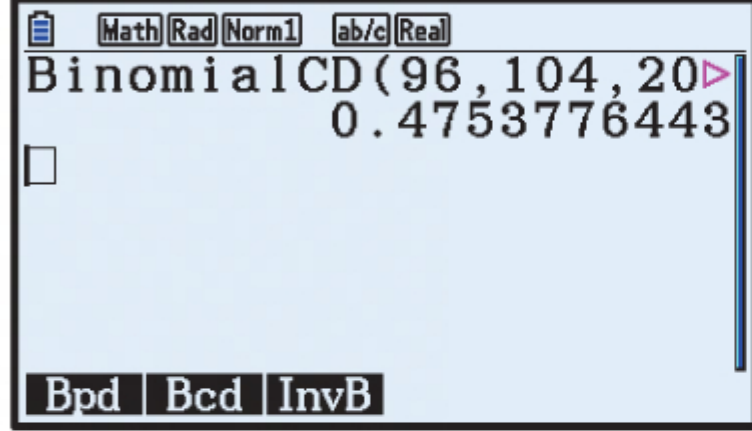
a



BinomialCD(90,110,20)
 0.8626333491

$$P(90 \leq X \leq 110) \approx 0.863$$

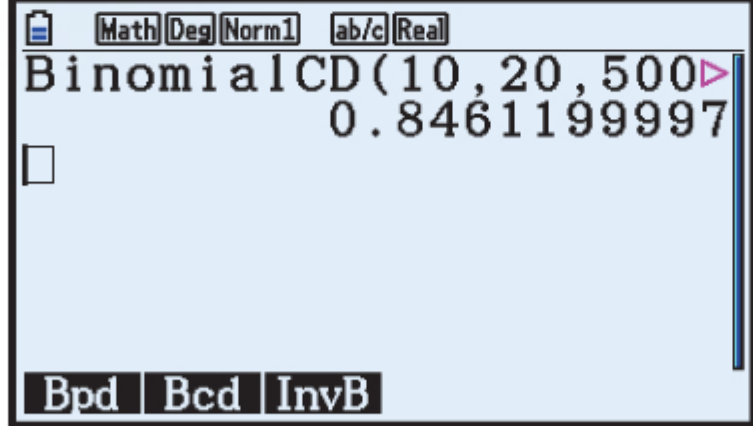
b



BinomialCD(96,104,20)
 0.4753776443

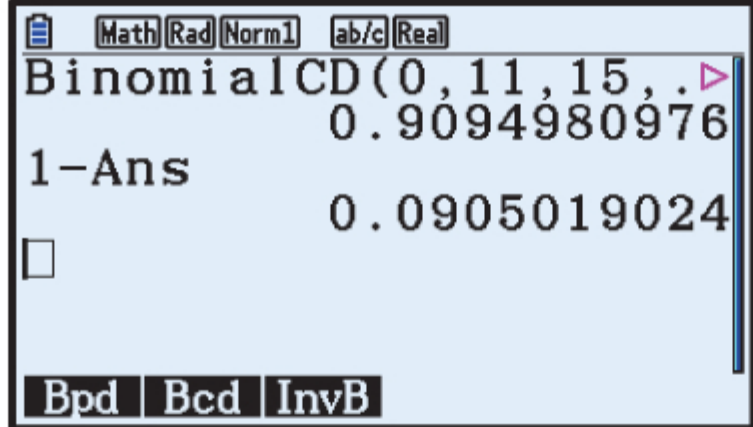
$$P(95 < X < 105) = P(96 \leq X \leq 104) \\ \approx 0.475$$

- 9 a $P(\text{rolling double sixes}) = \frac{1}{6} \times \frac{1}{6} = \frac{1}{36}$
 b Let X be the number of double sixes rolled.
 $n = 500$, so $X = 0, 1, 2, 3, \dots$, or 500 , and $p = \frac{1}{36}$
 $\therefore X \sim B(500, \frac{1}{36})$
 $P(10 \leq X \leq 20) \approx 0.846$
- 10 Let X be the number of traffic lights Shelley has stopped at.
 $n = 15$, so $X = 0, 1, 2, 3, \dots$, or 15 , and $p = 0.6$
 $\therefore X \sim B(15, 0.6)$



BinomialCD(10,20,500)
 0.8461199997

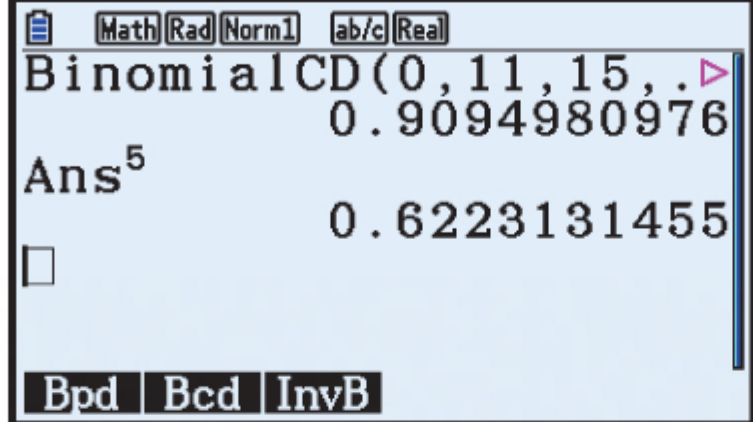
a $P(\text{Shelley will be late}) = P(X > 11)$
 $= 1 - P(X \leq 11)$
 ≈ 0.0905



BinomialCD(0,11,15)
 0.9094980976
 1-Ans
 0.0905019024

b $P(\text{Shelley will be on time}) = P(X \leq 11)$
 ≈ 0.909

$$P(\text{Shelley will be on time all 5 days}) = [P(X \leq 11)]^5 \\ \approx 0.622$$

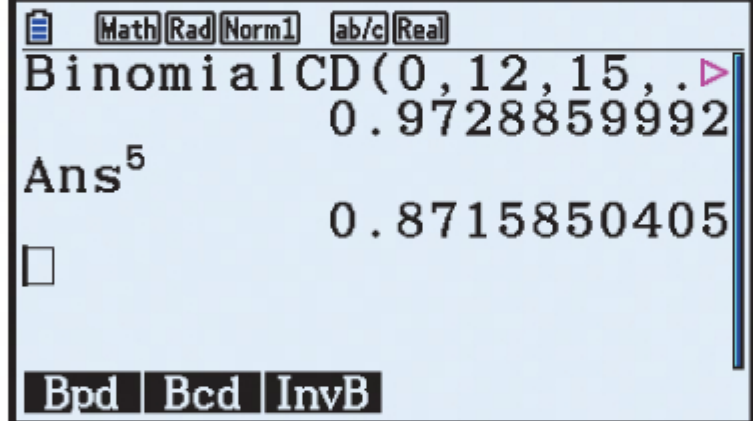


BinomialCD(0,11,15)
 0.9094980976
 Ans^5
 0.6223131455

c $P(\text{Shelley will be on time}) = P(X \leq 12)$
 ≈ 0.973

$$P(\text{Shelley will be on time all 5 days}) = [P(X \leq 12)]^5 \\ \approx 0.872$$

\therefore yes, the probability that Shelley is on time for work each day of a 5 day week is now about 87.2%.



BinomialCD(0,12,15)
 0.9728859992
 Ans^5
 0.8715850405

- 11 Let X be the number of solar components which fail.
 $n = 20$, so $X = 0, 1, 2, 3, \dots$, or 20 , and $p = 0.85$
 $\therefore X \sim B(20, 0.85)$
- a $P(\text{hot water unit fails within one year}) = P(\text{all 20 components fail})$
 $= P(X = 20)$
 $= (0.85)^{20}$
 ≈ 0.0388

- b** $P(\text{hot water unit with } n \text{ components fails within one year}) = (0.85)^n$
 $\therefore P(\text{hot water unit with } n \text{ components is operating after one year}) = 1 - (0.85)^n$
 \therefore we need to find the smallest integer n such that $1 - (0.85)^n \geq 0.98$
 $\therefore (0.85)^n \leq 0.02$
 $\therefore n \log(0.85) \leq \log(0.02)$
 $\therefore n \geq \frac{\log(0.02)}{\log(0.85)} \quad \{\log(0.85) < 0\}$
 $\therefore n \geq 24.1$
 \therefore at least 25 solar components are needed.

INVESTIGATION 2**THE MEAN AND STANDARD DEVIATION OF A BINOMIAL DISTRIBUTION****1** $X \sim B(30, 0.25)$

Consult the graphics calculator instructions by clicking on the icon in the Investigation box if you need help obtaining the result shown.

NORMAL FLOAT AUTO REAL RADIAN MP	
1-Var Stats	
μ	$\bar{x}=7.5$
	$\Sigma x=7.5$
	$\Sigma x^2=61.875$
	$Sx=$
σ	$\sigma x=2.371708245$
	$n=1$
	$\min X=0$
	$\downarrow Q1=6$

2

	$p = 0.1$	$p = 0.25$	$p = 0.5$	$p = 0.7$
$n = 10$	$\mu = 1$ $\sigma \approx 0.9487$	$\mu = 2.5$ $\sigma \approx 1.3693$	$\mu = 5$ $\sigma \approx 1.5811$	$\mu = 7$ $\sigma \approx 1.4491$
$n = 30$	$\mu = 3$ $\sigma \approx 1.6432$	$\mu = 7.5$ $\sigma \approx 2.3717$	$\mu = 15$ $\sigma \approx 2.7386$	$\mu = 21$ $\sigma \approx 2.5100$
$n = 50$	$\mu = 5$ $\sigma \approx 2.1213$	$\mu = 12.5$ $\sigma \approx 3.0619$	$\mu = 25$ $\sigma \approx 3.5355$	$\mu = 35$ $\sigma \approx 3.2404$

3

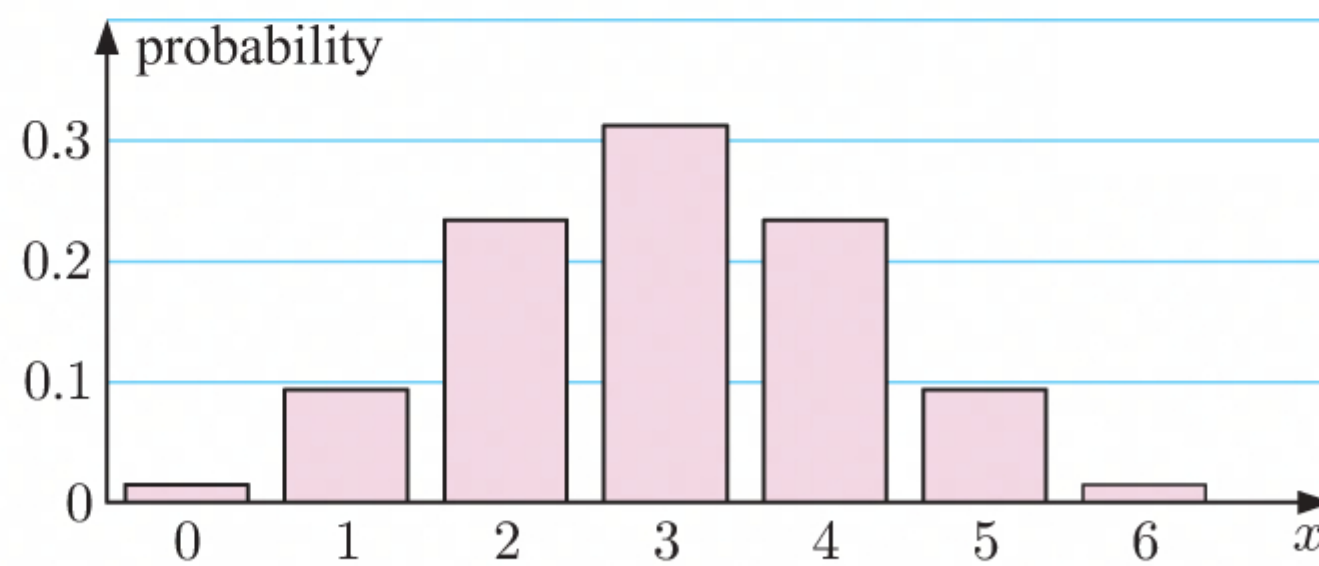
	$p = 0.1$	$p = 0.25$	$p = 0.5$	$p = 0.7$
$n = 10$	$np = 1$ $\sqrt{np(1-p)}$ ≈ 0.9487	$np = 2.5$ $\sqrt{np(1-p)}$ ≈ 1.3693	$np = 5$ $\sqrt{np(1-p)}$ ≈ 1.5811	$np = 7$ $\sqrt{np(1-p)}$ ≈ 1.4491
$n = 30$	$np = 3$ $\sqrt{np(1-p)}$ ≈ 1.6432	$np = 7.5$ $\sqrt{np(1-p)}$ ≈ 2.3717	$np = 15$ $\sqrt{np(1-p)}$ ≈ 2.7386	$np = 21$ $\sqrt{np(1-p)}$ ≈ 2.5100
$n = 50$	$np = 5$ $\sqrt{np(1-p)}$ ≈ 2.1213	$np = 12.5$ $\sqrt{np(1-p)}$ ≈ 3.0619	$np = 25$ $\sqrt{np(1-p)}$ ≈ 3.5355	$np = 35$ $\sqrt{np(1-p)}$ ≈ 3.2404

Our results in **2** and **3** agree with the formulae $\mu = np$ and $\sigma = \sqrt{np(1-p)}$.

EXERCISE 20F**1 a** $X \sim B(6, 0.5)$

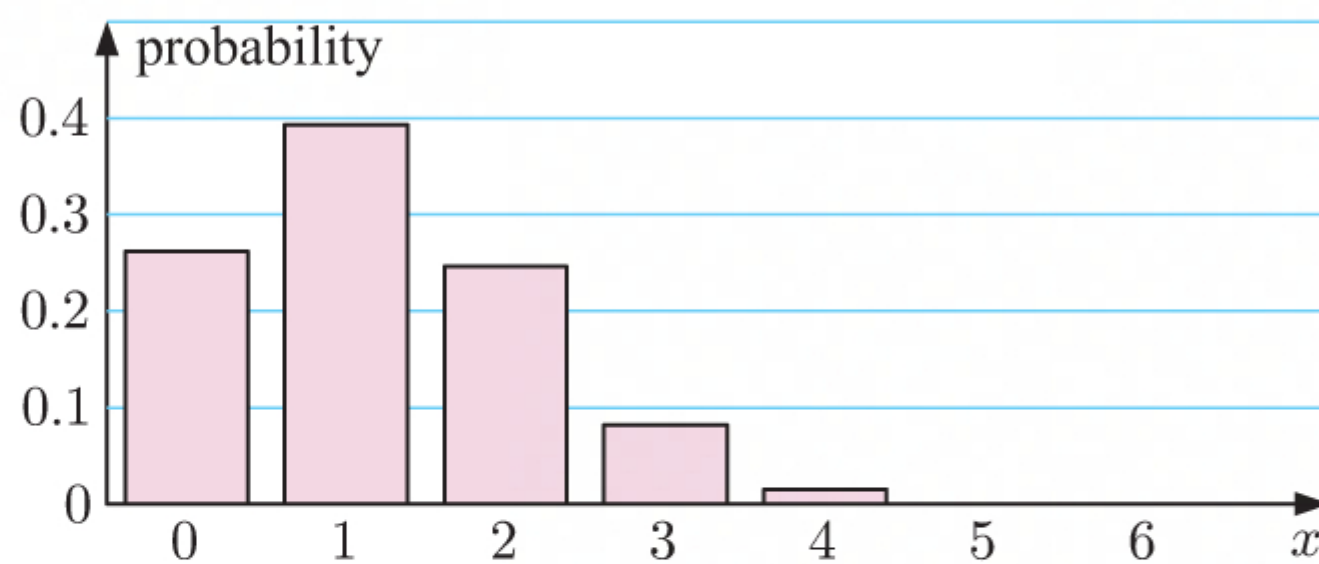
$$\begin{array}{ll}
 \text{i} \quad \mu = np & \sigma = \sqrt{np(1-p)} \\
 & = 6 \times 0.5 \\
 & = 3 \\
 & \sigma = \sqrt{6 \times 0.5 \times 0.5} \\
 & \approx 1.22
 \end{array}$$

x_i	0	1	2	3	4	5	6
$P(x_i)$	0.0156	0.0938	0.2344	0.3125	0.2344	0.0938	0.0156

**iii** The distribution is symmetric.**b** $X \sim B(6, 0.2)$

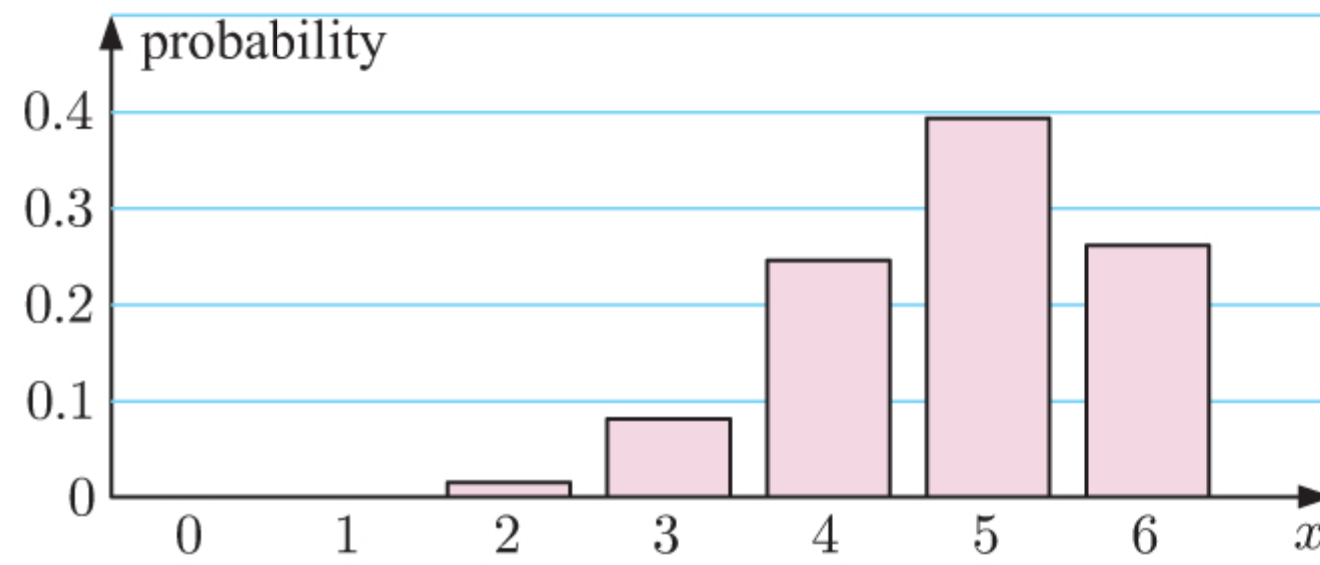
$$\begin{array}{ll}
 \text{i} \quad \mu = np & \sigma = \sqrt{np(1-p)} \\
 & = 6 \times 0.2 \\
 & = 1.2 \\
 & \sigma = \sqrt{6 \times 0.2 \times 0.8} \\
 & \approx 0.980
 \end{array}$$

x_i	0	1	2	3	4	5	6
$P(x_i)$	0.2621	0.3932	0.2458	0.0819	0.0154	0.0015	0.0001

**iii** The distribution is positively skewed.**c** $X \sim B(6, 0.8)$

$$\begin{array}{ll}
 \text{i} \quad \mu = np & \sigma = \sqrt{np(1-p)} \\
 & = 6 \times 0.8 \\
 & = 4.8 \\
 & \sigma = \sqrt{6 \times 0.8 \times 0.2} \\
 & \approx 0.980
 \end{array}$$

ii	x_i	0	1	2	3	4	5	6
	$P(x_i)$	0.0001	0.0015	0.0154	0.0819	0.2458	0.3932	0.2621



iii The distribution is negatively skewed, and is the exact reflection of the distribution in **b**.

2 $X \sim B(10, 0.5)$ mean $\mu = np = 10 \times \frac{1}{2} = 5$ and variance $\sigma^2 = np(1-p) = 10 \times \frac{1}{2} \times \frac{1}{2} = 2.5$

3 a $X \sim B(30, 0.04)$

$$\begin{aligned}\mu_X &= np \\ &= 30 \times 0.04 \\ &= 1.2 \\ \sigma_X &= \sqrt{np(1-p)} \\ &= \sqrt{30 \times 0.04 \times 0.96} \\ &\approx 1.07\end{aligned}$$

b $Y \sim B(30, 0.96)$

$$\begin{aligned}\mu_Y &= np \\ &= 30 \times 0.96 \\ &= 28.8 \\ \sigma_Y &= \sqrt{np(1-p)} \\ &= \sqrt{30 \times 0.96 \times 0.04} \\ &\approx 1.07\end{aligned}$$

4 $X \sim B(30, 0.13)$

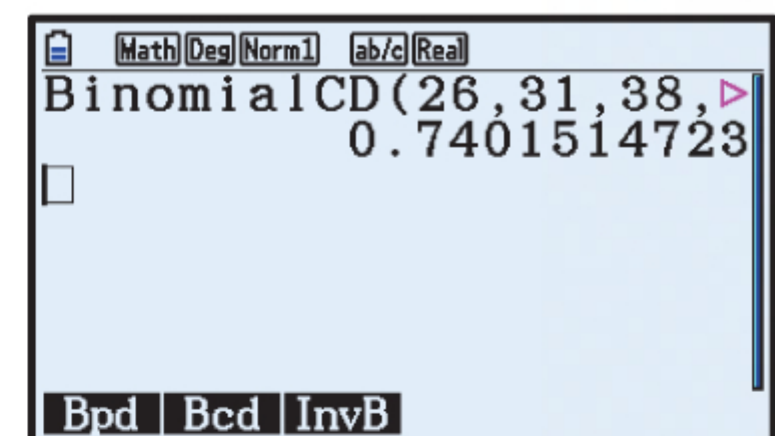
$$\begin{aligned}\mu &= np = 30 \times 0.13 = 3.9 \\ \sigma &= \sqrt{np(1-p)} = \sqrt{30 \times 0.13 \times 0.87} \approx 1.84\end{aligned}$$

5 $X \sim B(38, 0.75)$

a $\mu = np = 38 \times 0.75 = 28.5$ $\sigma = \sqrt{np(1-p)} = \sqrt{38 \times 0.75 \times 0.25} \approx 2.67$

b $\mu - \sigma \approx 28.5 - 2.67 \approx 25.8$ $\mu + \sigma \approx 28.5 + 2.67 \approx 31.2$

$$\therefore P(\mu - \sigma < X < \mu + \sigma) = P(26 \leq X \leq 31) \approx 0.740$$



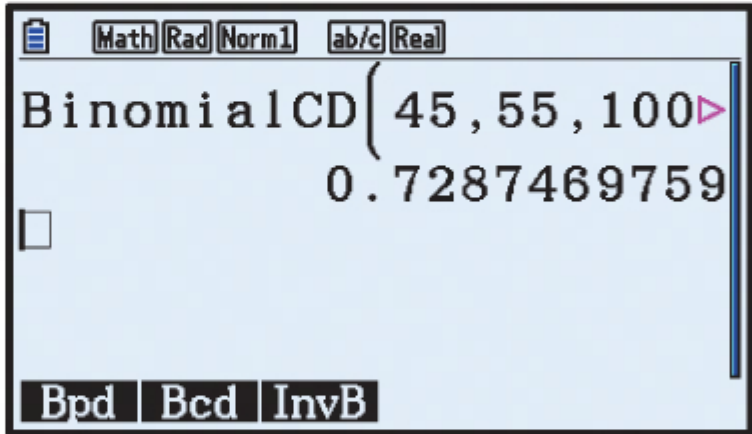
6 $X \sim B(100, \frac{1}{2}), \quad Y \sim B(300, \frac{1}{6})$

a $\mu_X = np = 100 \times \frac{1}{2} = 50$ $\mu_Y = np = 300 \times \frac{1}{6} = 50$

$$\begin{aligned}
 \text{b } \sigma_X &= \sqrt{np(1-p)} & \sigma_Y &= \sqrt{np(1-p)} \\
 &= \sqrt{100 \times \frac{1}{2} \times \frac{1}{2}} & &= \sqrt{300 \times \frac{1}{6} \times \frac{5}{6}} \\
 &= \sqrt{25} & &\approx 6.45 \\
 &= 5
 \end{aligned}$$

- c X is more likely to lie between 45 and 55 inclusive because the standard deviation of X is lower than that of Y , which means there are more values of X which lie close to the mean.

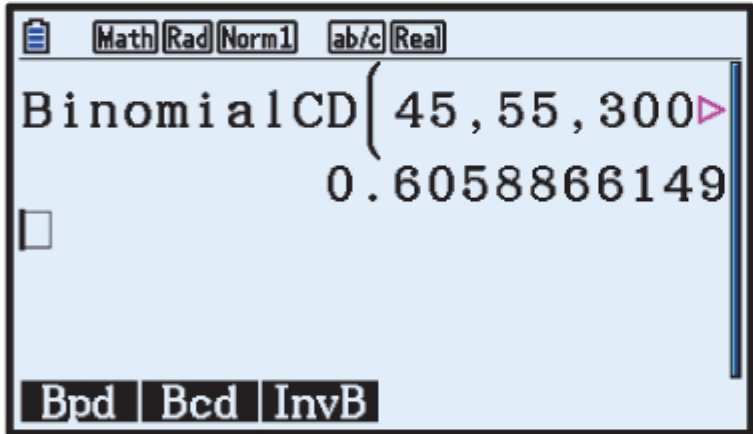
d i



BinomialCD(45, 55, 100) = 0.7287469759

$$P(45 \leq X \leq 55) \approx 0.729$$

ii



BinomialCD(45, 55, 300) = 0.6058866149

$$P(45 \leq Y \leq 55) \approx 0.606$$

REVIEW SET 20A

- 1 a The number of attempts to pass a driving test is a discrete random variable.
 b The length of time before a phone loses its battery charge is a continuous random variable.
 c The number of phone calls made before a salesperson has sold 3 products is a discrete random variable.

2 a i

x	1	2	3
$P(X = x)$	0.6	0.25	0.15

$$\sum_{x=1}^3 P(X = x) = 0.6 + 0.25 + 0.15 = 1$$

Since $\sum_{x=1}^3 P(X = x) = 1$ and $0 \leq P(X = x) \leq 1$ for all x , it is a valid probability distribution.

ii

x	0	2	5	10
$P(X = x)$	0.3	0.5	0.1	0.2

$$\sum p_i = 0.3 + 0.5 + 0.1 + 0.2 = 1.1$$

Since $\sum p_i > 1$, it is not a valid probability distribution.

iii

x	0	1	2	3
$P(X = x)$	0.4	-0.2	0.35	0.45

Since $P(X = 1) = -0.2 < 0$, this is not a valid probability distribution.

iv

x	2	3	4	5
$P(X = x)$	0.25	0.25	0.25	0.25

$$\sum_{x=2}^5 P(X = x) = 0.25 + 0.25 + 0.25 + 0.25 = 1$$

Since $\sum_{x=2}^5 P(X = x) = 1$ and $0 \leq P(X = x) \leq 1$ for all x , it is a valid probability distribution.

v

x	2	3
$P(X = x)$	0.7	0.3

$$\sum_{x=2}^3 P(X = x) = 0.7 + 0.3 = 1$$

Since $\sum_{x=2}^3 P(X = x) = 1$ and $0 \leq P(X = x) \leq 1$ for all x , it is a valid probability distribution.

vi

x	0	1
$P(X = x)$	0.28	0.72

$$\sum_{x=0}^1 P(X = x) = 0.28 + 0.72 = 1$$

Since $\sum_{x=0}^1 P(X = x) = 1$ and $0 \leq P(X = x) \leq 1$ for all x , it is a valid probability distribution.

b The distribution in **a iv** is a uniform discrete random variable because $p_i = 0.25$ for each value of i .

3 a $P(X = x) = \frac{a}{x^2 + 1}$ for $x = 0, 1, 2, 3$

Since this is a probability mass function,

$$\sum_{i=1}^n P(x_i) = 1$$

$$\therefore a + \frac{a}{2} + \frac{a}{5} + \frac{a}{10} = 1$$

$$\therefore 10a + 5a + 2a + a = 10$$

$$\therefore 18a = 10$$

$$\therefore a = \frac{5}{9}$$

x	0	1	2	3
$P(X = x)$	a	$\frac{a}{2}$	$\frac{a}{5}$	$\frac{a}{10}$

b $P(X \geq 1) = 1 - P(X = 0)$

$$= 1 - \frac{5}{9}$$

$$= \frac{4}{9}$$

4

x	0	1	2	3	4
$P(x)$	0.10	0.30	0.45	0.10	k

- a** Since this is a probability distribution, then $\sum_{i=1}^n P(x_i) = 1$
- $$\therefore 0.10 + 0.30 + 0.45 + 0.10 + k = 1$$
- $$\therefore 0.95 + k = 1$$
- $$\therefore k = 0.05$$

b $P(X \geq 3) = P(X = 3) + P(X = 4)$

$$= 0.10 + 0.05$$

$$= 0.15$$

- c** Since $P(X = 2)$ is the greatest probability, 2 is the mode of this distribution.

d $E(X) = \sum_{i=1}^n x_i p_i$

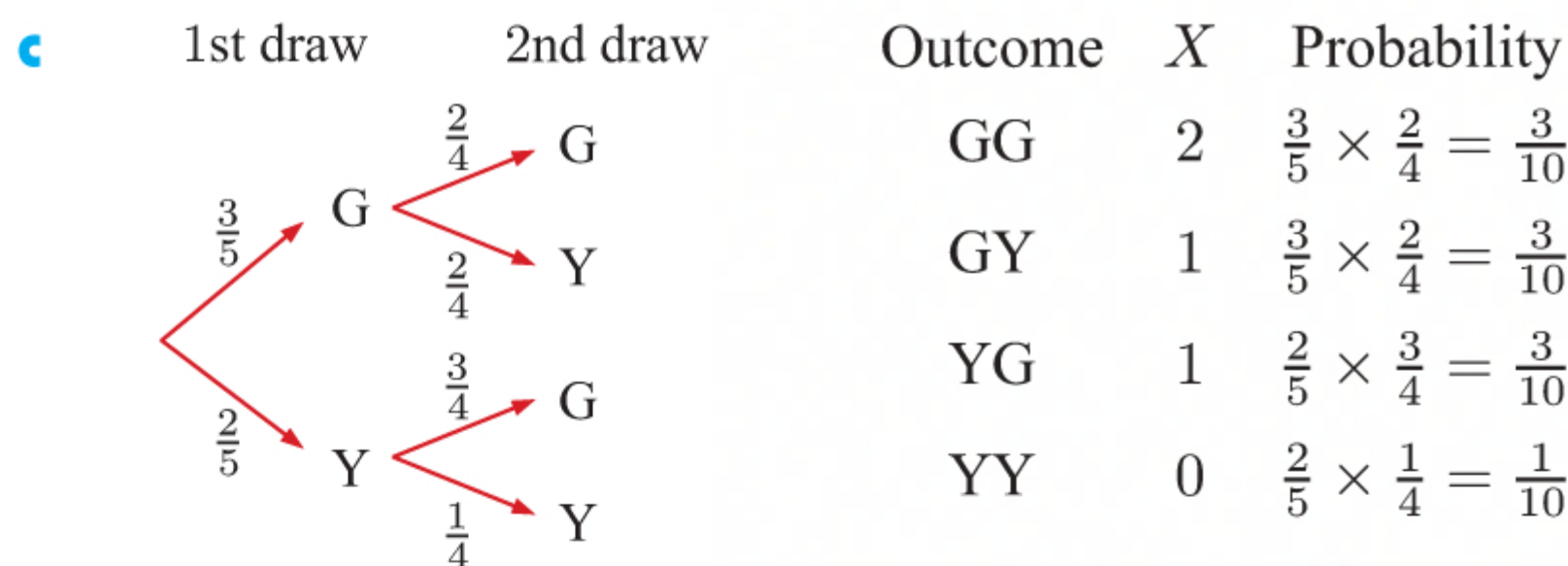
$$= 0(0.10) + 1(0.30) + 2(0.45) + 3(0.10) + 4(0.05)$$

$$= 0 + 0.3 + 0.9 + 0.3 + 0.2$$

$$= 1.7$$

- 5 a** X is a discrete random variable because it has a set of distinct possible values.

b $X = 0, 1$, or 2



x	0	1	2
$P(x)$	$\frac{1}{10}$	$\frac{3}{5}$	$\frac{3}{10}$

d $E(X) = \sum_{i=1}^n x_i p_i$

$$= \left(0 \times \frac{1}{10}\right) + \left(1 \times \frac{3}{5}\right) + \left(2 \times \frac{3}{10}\right)$$

$$= \frac{6}{5}$$

$$= 1.2 \text{ green balls}$$

- 6** X has probability table:

x	1	3	4	6
$P(x)$	$\frac{1}{6}$	$\frac{2}{6}$	$\frac{1}{6}$	$\frac{2}{6}$

$$E(X) = \sum_{i=1}^n x_i p_i$$

$$= 1\left(\frac{1}{6}\right) + 3\left(\frac{2}{6}\right) + 4\left(\frac{1}{6}\right) + 6\left(\frac{2}{6}\right)$$

$$= \frac{23}{6}$$

$$\approx 3.83$$

- 7 a** Let X denote the amount of money Lakshmi wins from one roll.

X has probability table:

x	2	4	6	8	10	12
$P(x)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

$$\begin{aligned} E(X) &= \sum_{i=1}^n x_i p_i \\ &= (2 \times \frac{1}{6}) + (4 \times \frac{1}{6}) + (6 \times \frac{1}{6}) + (8 \times \frac{1}{6}) + (10 \times \frac{1}{6}) + (12 \times \frac{1}{6}) \\ &= 7 \end{aligned}$$

\therefore Lakshmi can expect to win \$7 from one roll of the die.

- b** Expected gain = \$7 - \$8 = -\$1.

So, Lakshmi should not play many games as she would lose \$1 per game in the long run.

- 8** $P(x) = a(x^2 - 8x)$ where $x = 0, 1, 2, 3, \dots, 8$

- a** X has probability table:

x	0	1	2	3	4	5	6	7	8
$P(x)$	0	$-7a$	$-12a$	$-15a$	$-16a$	$-15a$	$-12a$	$-7a$	0

If this is a probability distribution then $\sum_{i=1}^n P(x_i) = 1$

$$\therefore 0 + (-7a) + (-12a) + (-15a) + (-16a) + (-15a) + (-12a) + (-7a) + 0 = 1$$

$$\therefore -84a = 1$$

$$\therefore a = -\frac{1}{84}$$

- b** $E(X) = 0(0) + 1(\frac{7}{84}) + 2(\frac{12}{84}) + 3(\frac{15}{84}) + 4(\frac{16}{84}) + 5(\frac{15}{84}) + 6(\frac{12}{84}) + 7(\frac{7}{84}) + 8(0)$
 $= 4$ marsupials

- 9 a** $(\frac{4}{5} + \frac{1}{5})^5 = (\frac{4}{5})^5 + 5(\frac{4}{5})^4(\frac{1}{5})^1 + 10(\frac{4}{5})^3(\frac{1}{5})^2 + 10(\frac{4}{5})^2(\frac{1}{5})^3 + 5(\frac{4}{5})^1(\frac{1}{5})^4 + (\frac{1}{5})^5$

- b** The probability of Jack scoring a goal is $p = \frac{4}{5}$.

Let X = the number of goals scored, G represents scoring a goal

- i** $P(3 \text{ goals then } 2 \text{ misses})$

$$= P(GGGG'G')$$

$$= (\frac{4}{5})^3 (\frac{1}{5})^2$$

$$= \frac{64}{3125}$$

$$= 0.02048$$

- ii** $P(3 \text{ goals and } 2 \text{ misses})$

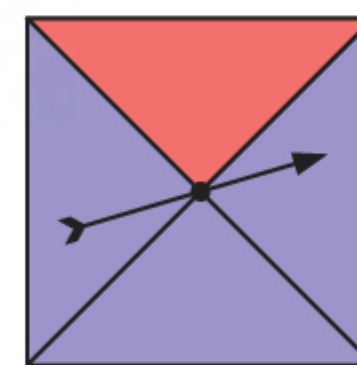
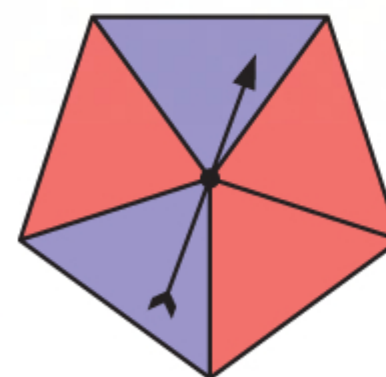
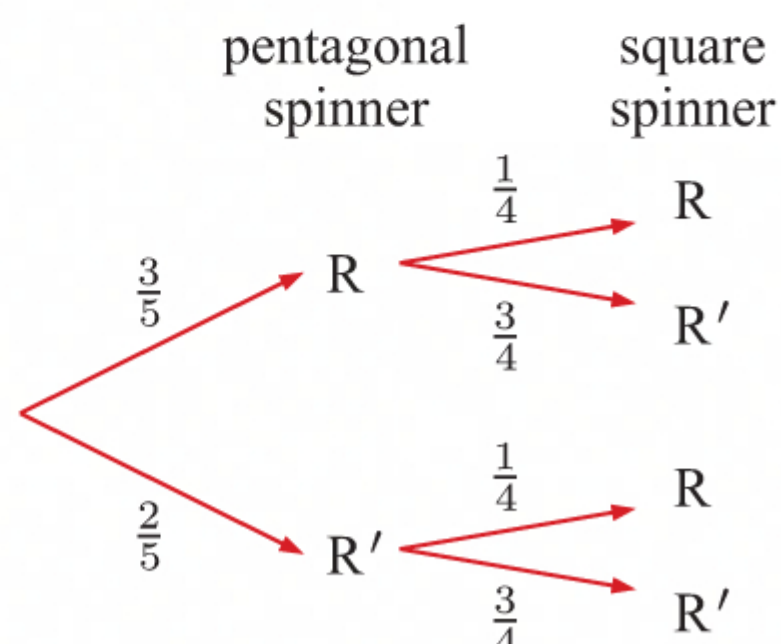
$$= P(X = 3)$$

$$= 10(\frac{4}{5})^3 (\frac{1}{5})^2$$

$$= \frac{128}{625}$$

$$= 0.2048$$

- 10 a**



$$\begin{aligned}
 \text{b } P(\text{exactly one red}) &= P(RR') + P(R'R) \\
 &= \frac{3}{5} \times \frac{3}{4} + \frac{2}{5} \times \frac{1}{4} \\
 &= \frac{9}{20} + \frac{1}{10} \\
 &= \frac{11}{20}
 \end{aligned}$$

$$\text{c } \text{i } X \sim B\left(10, \frac{11}{20}\right)$$

$$\text{ii } n = 10, \quad p = \frac{11}{20}$$

$$P(X = 1) = \binom{10}{1} \left(\frac{11}{20}\right)^1 \left(\frac{9}{20}\right)^9 \approx 0.00416$$

$$P(X = 9) = \binom{10}{9} \left(\frac{11}{20}\right)^9 \left(\frac{9}{20}\right)^1 \approx 0.0207$$

\therefore it is more likely that exactly one red will occur 9 times.

$$11 \quad X \sim B(50, 0.8)$$

$$\begin{aligned}
 \text{a } \mu &= np \\
 &= 50 \times 0.8 \\
 &= 40 \text{ days}
 \end{aligned}$$

$$\begin{aligned}
 \text{b } \sigma &= \sqrt{np(1-p)} \\
 &= \sqrt{50 \times 0.8 \times 0.2} \\
 &\approx 2.83 \text{ days}
 \end{aligned}$$

12 Let X denote the number of players who turn up to a game.

$n = 9$, so $X = 0, 1, 2, 3, \dots$, or 9 , and $p = 75\% = 0.75$

$\therefore X \sim B(9, 0.75)$

$$\begin{aligned}
 \text{a } \text{i } P(X = 9) &= \binom{9}{9} (0.75)^9 (0.25)^0 \\
 &\approx 0.0751
 \end{aligned}$$

$$\begin{aligned}
 \text{ii } & \text{BinomialCD}(0.75, 9, .75) \\
 & 0.165725708
 \end{aligned}$$

$$\begin{aligned}
 P(\text{forfeit}) &= P(X < 6) \\
 &= P(X \leq 5) \\
 &\approx 0.166
 \end{aligned}$$

b The team is expected to forfeit $30 \times 0.1657 \approx 4.97$ games throughout the season.

13 a Let X denote the number of batteries that are defective.

$n = 20$, so $X = 0, 1, 2, 3, \dots$, or 20 , and $p = \frac{3}{100}$

$\therefore X \sim B\left(20, \frac{3}{100}\right)$

$$\begin{aligned}
 \text{i } P(X = 0) &= \binom{20}{0} \left(\frac{3}{100}\right)^0 \left(\frac{97}{100}\right)^{20} \\
 &\approx 0.544
 \end{aligned}$$

$$\begin{aligned}
 \text{ii } P(X \geq 1) &= 1 - P(X = 0) \\
 &\approx 1 - 0.544 \\
 &\approx 0.456
 \end{aligned}$$

$$\text{b } X \sim B\left(n, \frac{3}{100}\right)$$

$$\begin{aligned}
 \text{i } P(X = 0) &= \binom{n}{0} \left(\frac{3}{100}\right)^0 \left(\frac{97}{100}\right)^n \\
 &= (0.97)^n
 \end{aligned}$$

$$\begin{aligned}\text{ii } P(X \geq 1) &= 1 - P(X = 0) \\ &= 1 - (0.97)^n\end{aligned}$$

If $P(X \geq 1) \geq 0.3$ then

$$1 - (0.97)^n \geq 0.3$$

$$\therefore (0.97)^n \leq 0.7$$

$$\therefore n \log(0.97) \leq \log(0.7)$$

$$\therefore n \geq \frac{\log(0.7)}{\log(0.97)} \quad \{\log(0.97) < 0\}$$

$$\therefore n \geq 11.7$$

\therefore the smallest value of n such that $P(X \geq 1) \geq 0.3$ is $n = 12$.

REVIEW SET 20B

1

x	0	1	2	3	4	5
$P(X = x)$	0.07	0.14	k	0.46	0.08	0.02

- a The random variable X represents the number of hits that Sally has in a softball match.
 $X = 0, 1, 2, 3, 4$, or 5

- b i Since this is a probability distribution, $\sum_{i=1}^n P(x_i) = 1$

$$\therefore 0.07 + 0.14 + k + 0.46 + 0.08 + 0.02 = 1$$

$$\therefore k + 0.77 = 1$$

$$\therefore k = 0.23$$

$$\begin{aligned}\text{ii } P(X \geq 2) &= 1 - P(X \leq 1) \\ &= 1 - (0.07 + 0.14) \\ &= 0.79\end{aligned}$$

$$\begin{aligned}\text{iii } P(1 \leq X \leq 3) &= P(X = 1) + P(X = 2) + P(X = 3) \\ &= 0.14 + 0.23 + 0.46 \\ &= 0.83\end{aligned}$$

- c It is most likely that Sally will have 3 hits in one softball match, so 3 hits is the mode.

$$p_1 = 0.07$$

$$p_1 + p_2 = 0.07 + 0.14 = 0.21$$

$$p_1 + p_2 + p_3 = 0.07 + 0.14 + 0.23 = 0.44$$

$$p_1 + p_2 + p_3 + p_4 = 0.07 + 0.14 + 0.23 + 0.46 = 0.9$$

Since $p_1 + p_2 + p_3 + p_4 \geq 0.5$, the median is 3 hits.

2 a $P(x) = \frac{e^x}{1+e}, \quad x = 0, 1$

$$P(0) = \frac{1}{1+e}, \quad P(1) = \frac{e}{1+e}$$

Both of these obey $0 \leq P(x_i) \leq 1$, and $\sum_{i=1}^n P(x_i) = \frac{1}{1+e} + \frac{e}{1+e} = 1$

$\therefore P(x)$ is a valid probability mass function.

b $P(x) = \frac{x^2 + x}{40}, \quad x = 1, 2, 3, 4$

$$P(1) = \frac{1+1}{40} = \frac{2}{40}, \quad P(2) = \frac{4+2}{40} = \frac{6}{40}, \quad P(3) = \frac{9+3}{40} = \frac{12}{40}, \quad P(4) = \frac{16+4}{40} = \frac{20}{40}$$

All of these obey $0 \leq P(x_i) \leq 1$, and $\sum_{i=1}^n P(x_i) = \frac{2}{40} + \frac{6}{40} + \frac{12}{40} + \frac{20}{40} = 1$

$\therefore P(x)$ is a valid probability mass function.

c $P(x) = \log\left(\frac{x+1}{x}\right), \quad x = 1, 2, 3, \dots, 9$

$$P(1) = \log\left(\frac{1+1}{1}\right) = \log 2$$

$$P(2) = \log\left(\frac{2+1}{2}\right) = \log\left(\frac{3}{2}\right)$$

$$P(3) = \log\left(\frac{3+1}{3}\right) = \log\left(\frac{4}{3}\right)$$

$$P(4) = \log\left(\frac{4+1}{4}\right) = \log\left(\frac{5}{4}\right)$$

$$P(5) = \log\left(\frac{5+1}{5}\right) = \log\left(\frac{6}{5}\right)$$

$$P(6) = \log\left(\frac{6+1}{6}\right) = \log\left(\frac{7}{6}\right)$$

$$P(7) = \log\left(\frac{7+1}{7}\right) = \log\left(\frac{8}{7}\right)$$

$$P(8) = \log\left(\frac{8+1}{8}\right) = \log\left(\frac{9}{8}\right)$$

$$P(9) = \log\left(\frac{9+1}{9}\right) = \log\left(\frac{10}{9}\right)$$

$$\begin{aligned} \sum_{i=1}^n P(x_i) &= \log 2 + \log\left(\frac{3}{2}\right) + \log\left(\frac{4}{3}\right) + \log\left(\frac{5}{4}\right) + \log\left(\frac{6}{5}\right) + \log\left(\frac{7}{6}\right) + \log\left(\frac{8}{7}\right) \\ &\quad + \log\left(\frac{9}{8}\right) + \log\left(\frac{10}{9}\right) \\ &= \log\left(\cancel{2} \times \frac{\cancel{2}}{\cancel{2}} \times \frac{\cancel{4}}{\cancel{3}} \times \frac{\cancel{5}}{\cancel{4}} \times \frac{\cancel{6}}{\cancel{5}} \times \frac{\cancel{7}}{\cancel{6}} \times \frac{\cancel{8}}{\cancel{7}} \times \frac{\cancel{9}}{\cancel{8}} \times \frac{10}{\cancel{9}}\right) \quad \{\log A + \log B = \log AB\} \\ &= \log 10 \\ &= 1 \end{aligned}$$

All of these obey $0 \leq P(x_i) \leq 1$, and $\sum_{i=1}^n P(x_i) = 1$

$\therefore P(x)$ is a valid probability mass function.

- 3 a** 2 has the highest probability of occurring, so this is the mode of the distribution.

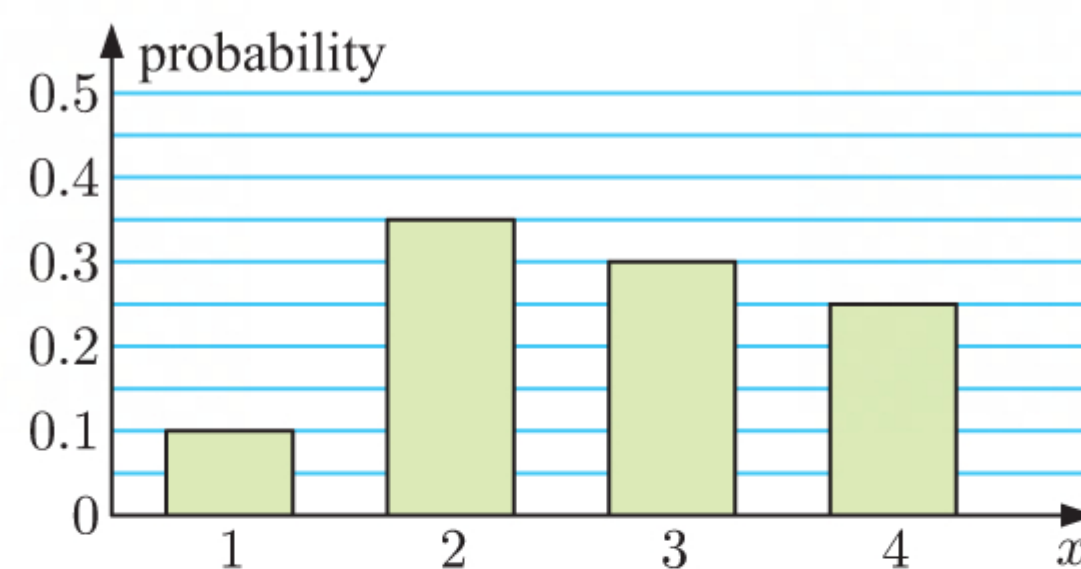
b

$$p_1 = 0.1$$

$$p_1 + p_2 = 0.1 + 0.35 = 0.45$$

$$p_1 + p_2 + p_3 = 0.1 + 0.35 + 0.3 = 0.75$$

Since $p_1 + p_2 + p_3 \geq 0.5$, the median is 3.



c $E(X) = (1 \times 0.1) + (2 \times 0.35) + (3 \times 0.3) + (4 \times 0.25)$
 $= 2.7$

4 $P(X = x) = \frac{1}{3} \times a^{x-1}, \quad x = 1, 2, 3, \dots$

a

$$\sum_{i=1}^{\infty} \frac{1}{3} a^{i-1} = 1$$

$$\therefore \frac{1}{3} \sum_{i=1}^{\infty} a^{i-1} = 1$$

$$\therefore \sum_{i=1}^{\infty} a^{i-1} = 3$$

$$\therefore \frac{1}{1-a} = 3 \quad \{u_1 = 1, r = a\}$$

$$\therefore 1 = 3 - 3a$$

$$\therefore 3a = 2$$

$$\therefore a = \frac{2}{3}$$

b

$$P(X = 1) = \frac{1}{3} \times \left(\frac{2}{3}\right)^0$$

$$= \frac{1}{3}$$

$$P(X = 2) = \frac{1}{3} \times \left(\frac{2}{3}\right)^1$$

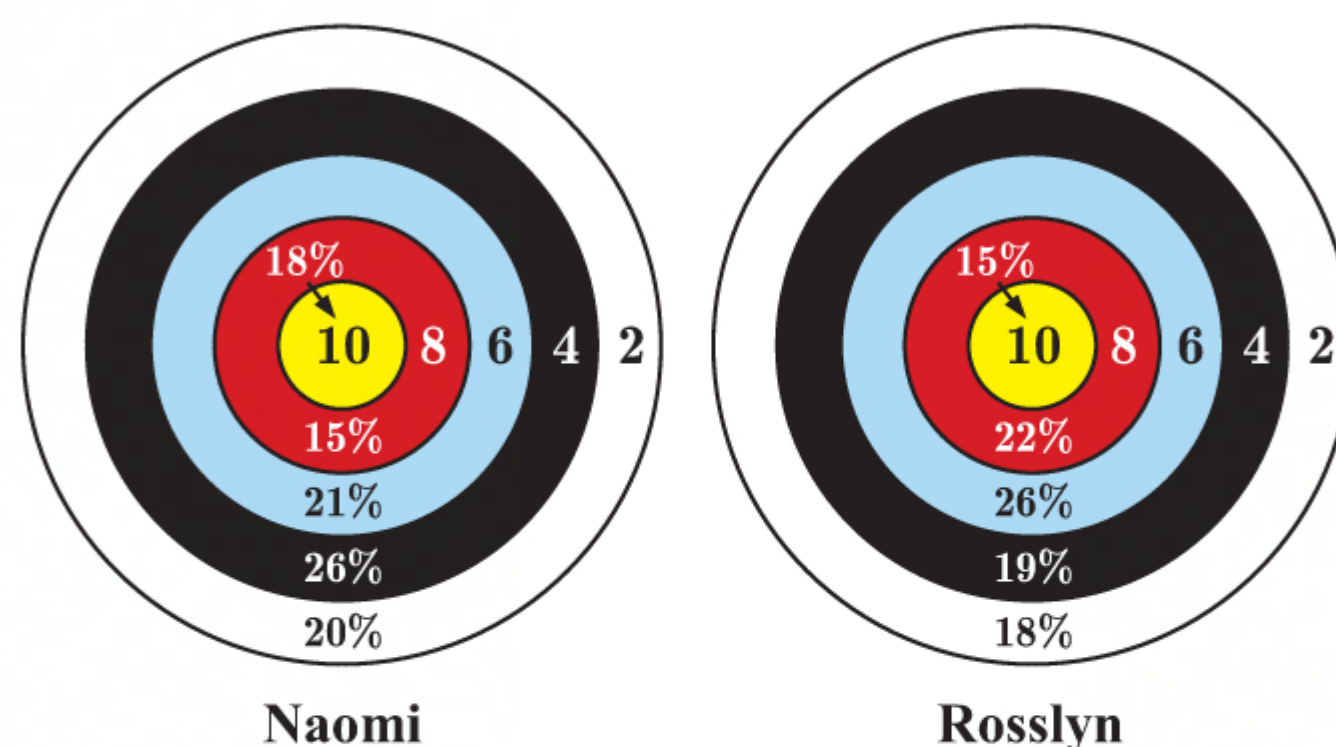
$$= \frac{2}{9}$$

$$\vdots$$

The probabilities after this *decrease* since the common ratio is $\frac{2}{3} < 1$. Since $P(X = 1)$ has the highest probability of occurring, 1 is the mode of the distribution.

- 5** Let N be the number of points Naomi scores per shot.
 Let R be the number of points Rosslyn scores per shot.

a i $P(N = 10) = 0.18$
 $P(R = 10) = 0.15$
 \therefore Naomi is more likely to score 10 points on a single shot.



ii $P(N \geq 6) = P(N = 6) + P(N = 8) + P(N = 10)$
 $= 0.21 + 0.15 + 0.18$
 $= 0.54$

$$P(R \geq 6) = P(R = 6) + P(R = 8) + P(R = 10)$$

$$= 0.26 + 0.22 + 0.15$$

$$= 0.63$$

\therefore Rosslyn is more likely to score at least 6 points.

$$\begin{aligned} \text{b } E(N) &= (2 \times 0.2) + (4 \times 0.26) + (6 \times 0.21) + (8 \times 0.15) + (10 \times 0.18) \\ &= 5.7 \text{ points} \end{aligned}$$

$$\begin{aligned} E(R) &= (2 \times 0.18) + (4 \times 0.19) + (6 \times 0.26) + (8 \times 0.22) + (10 \times 0.15) \\ &= 5.94 \text{ points} \end{aligned}$$

\therefore in the long run, Rosslyn is expected to score more points per shot.

6 Let X denote the number written on the ticket drawn.

$$\text{a i } P(\text{player wins \$3})$$

$$= P(X \text{ is even but not square})$$

$$= \frac{8}{20}$$

$$= \frac{2}{5}$$

$$\text{ii } P(\text{player wins \$6})$$

$$= P(X \text{ is square but not even})$$

$$= \frac{2}{20}$$

$$= \frac{1}{10}$$

$$\text{iii } P(\text{player wins \$9})$$

$$= P(X \text{ is even and square})$$

$$= \frac{2}{20}$$

$$= \frac{1}{10}$$

$$\begin{aligned} \text{b } \text{The expected gain of one game is } E(X) &= \left(0 \times \frac{8}{20}\right) + \left(3 \times \frac{2}{5}\right) + \left(6 \times \frac{1}{10}\right) + \left(9 \times \frac{1}{10}\right) \\ &= \frac{27}{10} \\ &= \$2.70 \text{ per game} \end{aligned}$$

To make the game fair, the game must cost the same as the expected gain, so \$2.70 should be charged each game.

7 Since this is a probability distribution, $\sum p_i = 1$

$$\therefore 0.2 + a + 0.3 + b = 1$$

$$\therefore b = 0.5 - a \quad \dots (*)$$

$$\text{Now, } E(X) = 2.8$$

$$\therefore (1 \times 0.2) + (2 \times a) + (3 \times 0.3) + (4 \times b) = 2.8$$

$$\therefore 0.2 + 2a + 0.9 + 4(0.5 - a) = 2.8 \quad \{\text{using } (*)\}$$

$$\therefore 2a + 2 - 4a = 1.7$$

$$\therefore -2a = -0.3$$

$$\therefore a = 0.15 \text{ and } b = 0.35$$

x_i	1	2	3	4
p_i	0.2	a	0.3	b

8 a Let H be the event that the result is heads.

$$P(H) \times P(H) = 0.64$$

$$\therefore [P(H)]^2 = \frac{64}{100}$$

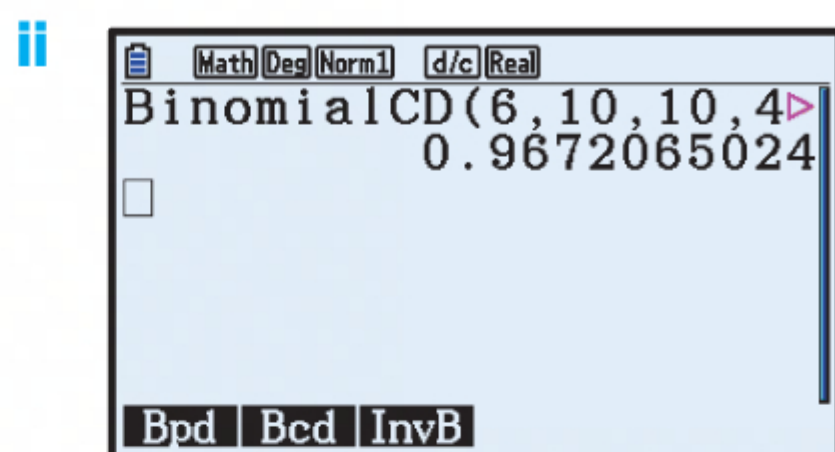
$$\therefore P(H) = \frac{8}{10} \quad \{P(H) \geq 0\}$$

$$= \frac{4}{5}$$

b Let X be the number of heads, $X \sim B\left(10, \frac{4}{5}\right)$.

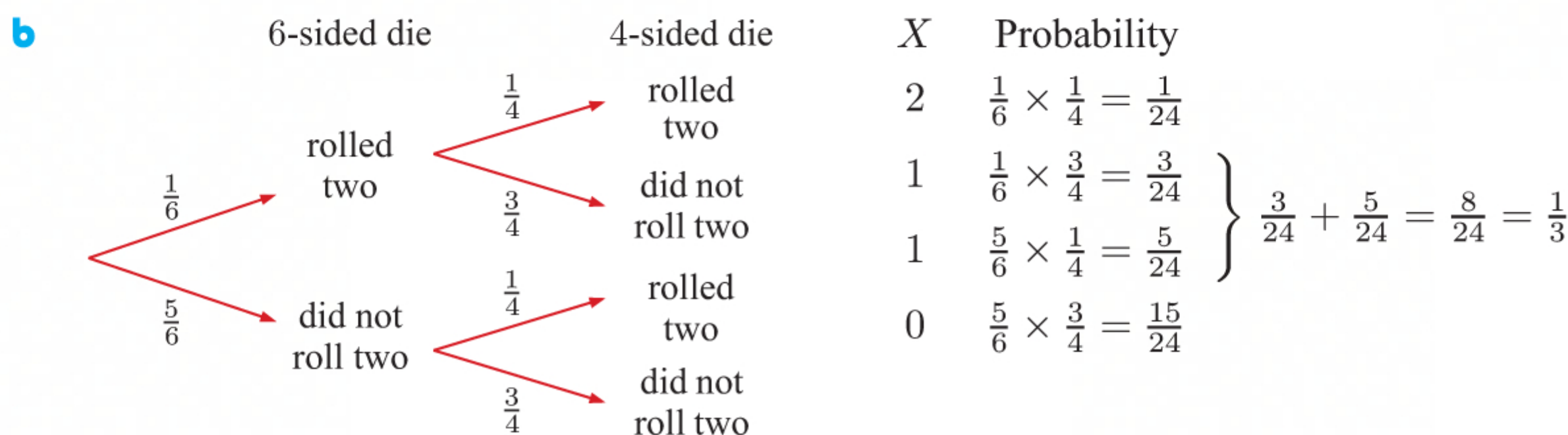
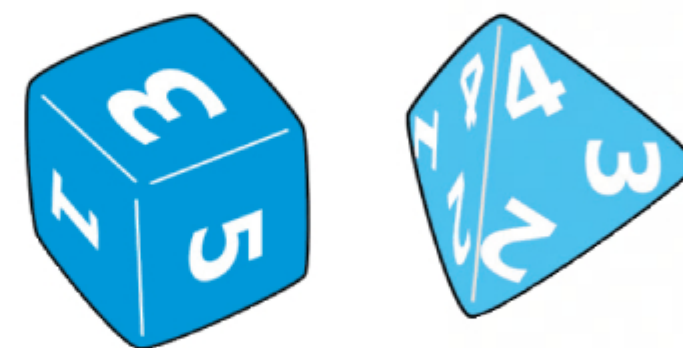
$$\text{i } P(X = 6) = \binom{10}{6} \left(\frac{4}{5}\right)^6 \left(\frac{1}{5}\right)^4$$

$$\approx 0.0881$$



$$P(X \geq 6) \approx 0.967$$

- 9 a X is not a binomial random variable because the probability of rolling a two is not the same for each die.



x	0	1	2
$P(X = x)$	$\frac{15}{24}$	$\frac{1}{3}$	$\frac{1}{24}$

c $E(X) = 0\left(\frac{15}{24}\right) + 1\left(\frac{1}{3}\right) + 2\left(\frac{1}{24}\right) = \frac{5}{12}$

10

x	0	1	2	3	4
$P(X = x)$	0.1	0.2	0.3	0.3	0.1

a $E(X) = 0(0.1) + 1(0.2) + 2(0.3) + 3(0.3) + 4(0.1) = 2.1$

- b If Caleb drops the ball y times, he catches the ball $4 - y$ times.

$$\therefore P(Y = y) = P(X = 4 - y)$$

So, the probability distribution of Y is:

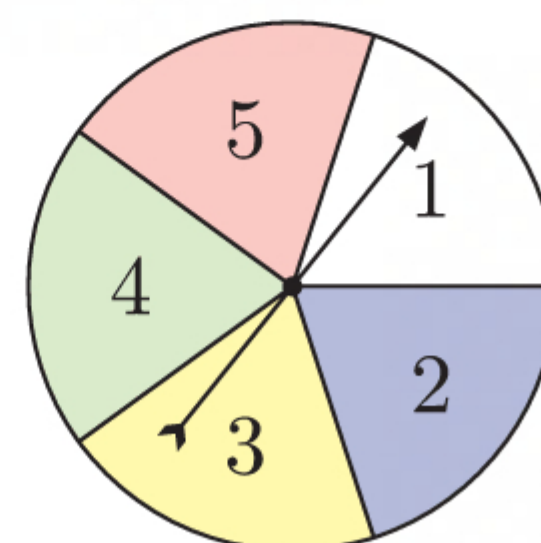
y	0	1	2	3	4
$P(Y = y)$	0.1	0.3	0.3	0.2	0.1

$$\therefore E(Y) = 0(0.1) + 1(0.3) + 2(0.3) + 3(0.2) + 4(0.1) = 1.9$$

- 11 a The probability of success (spinning a 3) is the same for each spin, and the number of spins is fixed.

b $\mu = np$
 $= 20 \times \frac{1}{5}$
 $= 4$

$\sigma = \sqrt{np(1-p)}$
 $= \sqrt{20 \times \frac{1}{5} \times \frac{4}{5}}$
 $= \frac{4}{\sqrt{5}} \approx 1.79$



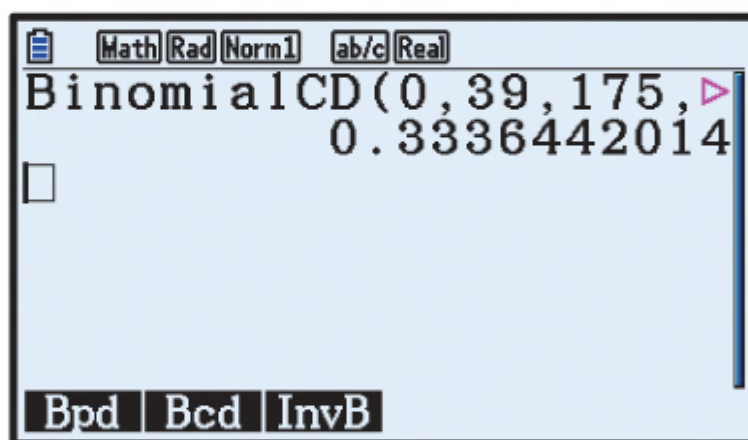
- 12** Let X be the number of visitors who make a voluntary donation upon entry.

$n = 175$, so $X = 0, 1, 2, 3, \dots$, or 175 , and $p = 24\% = 0.24$

$$\therefore X \sim B(175, 0.24)$$

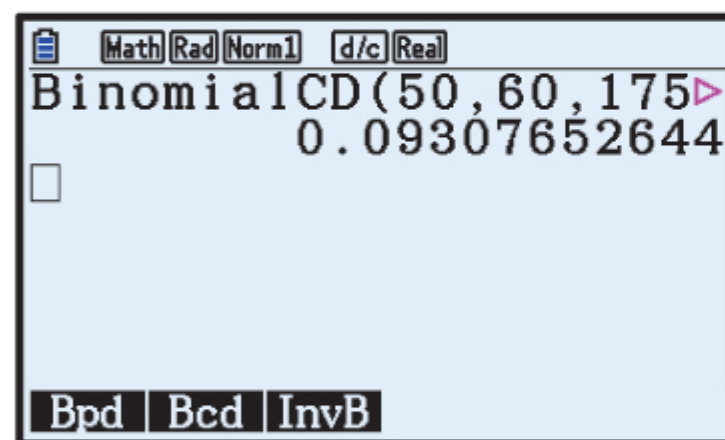
a $E(X) = \mu = np$
 $= 175 \times 0.24$
 $= 42$ donations

b i



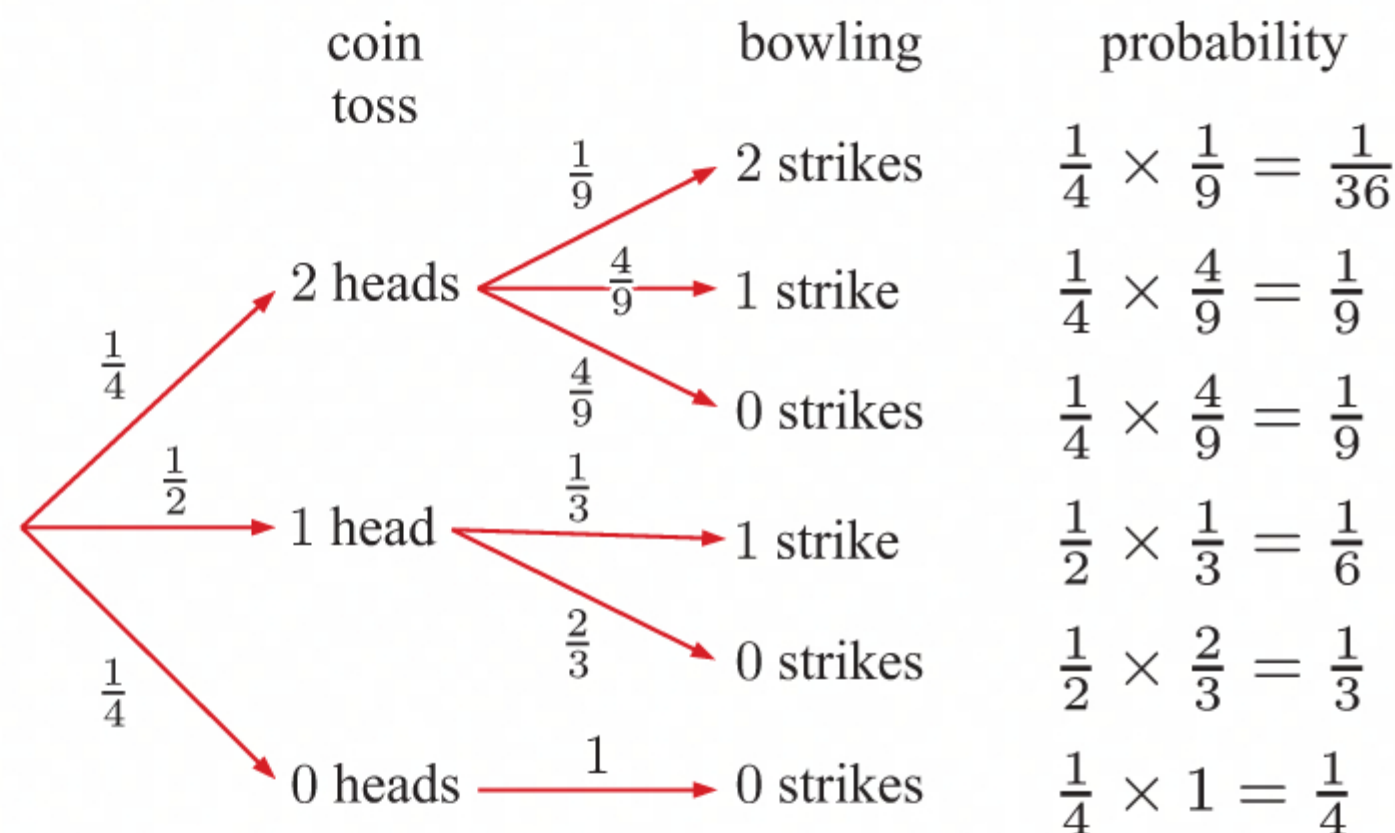
$$P(X < 40) = P(X \leq 39) \\ \approx 0.334$$

ii



$$P(50 \leq X \leq 60) \approx 0.0931$$

13 a



b $P(X = 0) = \frac{1}{9} + \frac{1}{3} + \frac{1}{4} = \frac{25}{36}$ $P(X = 1) = \frac{1}{9} + \frac{1}{6} = \frac{5}{18}$ $P(X = 2) = \frac{1}{36}$

x	0	1	2
$P(X = x)$	$\frac{25}{36}$	$\frac{5}{18}$	$\frac{1}{36}$

c The expected return per game is $E(X) = (0 \times \frac{25}{36}) + (10 \times \frac{5}{18}) + (20 \times \frac{1}{36})$
 $= \frac{10}{3}$ dollars
 $\approx \$3.33$

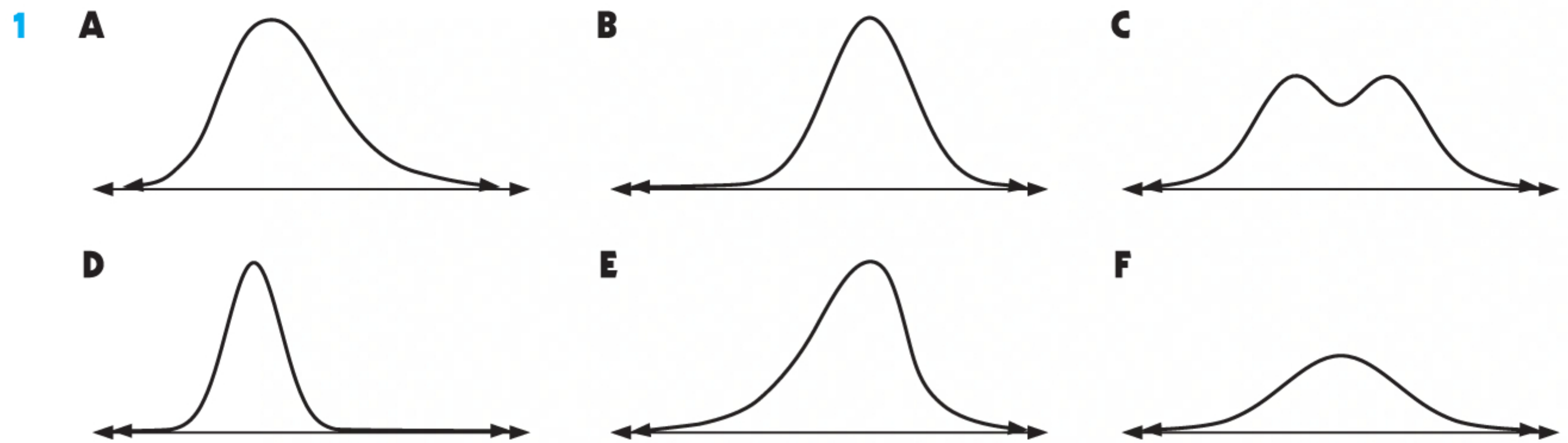
d Suvi's expected gain per game $\approx \$3.33 - \5
 $\approx -\$1.67$

\therefore Suvi should not play the game many times as she is expected to lose \$1.67 per game on average.

Chapter 21

THE NORMAL DISTRIBUTION

EXERCISE 21A.1



Distributions **B**, **D**, and **F** are symmetrical and bell-shaped.

∴ **B**, **D**, and **F** appear to be normally distributed.

- 2 Most measurements in each situation will be centred about the mean, with random variation about the mean explained by some of the factors listed below.

a The diameters may be affected by:

- the type of lathe used
- the steadiness of the woodworker's hand
- the operating speed of the lathe.

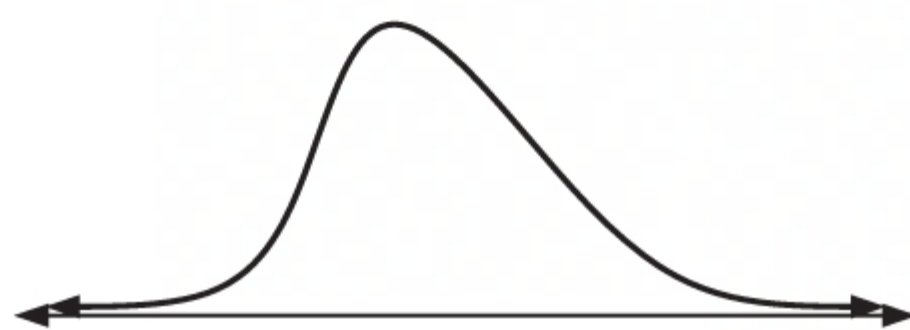
b The scores may be affected by:

- the time spent studying
- natural ability (for example, memory, learning ability)
- general knowledge.

c The times may be affected by:

- weather conditions
- walking speed
- physical fitness
- traffic.

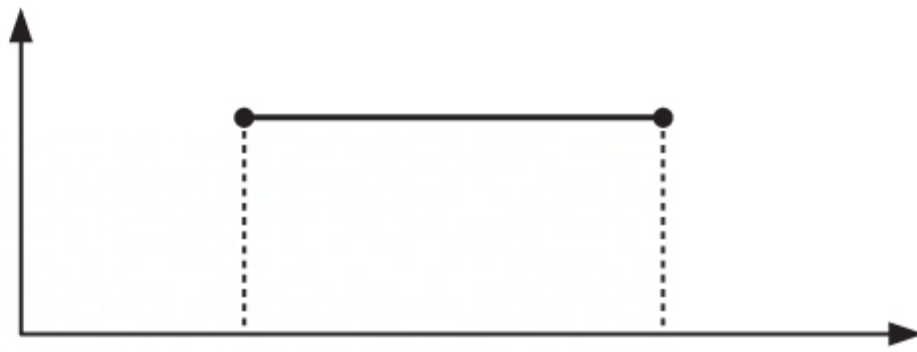
- 3 **a** The variable is not likely to be normally distributed as it is more likely that there would be more people younger than the mean age than there are older. The distribution may be positively skewed.



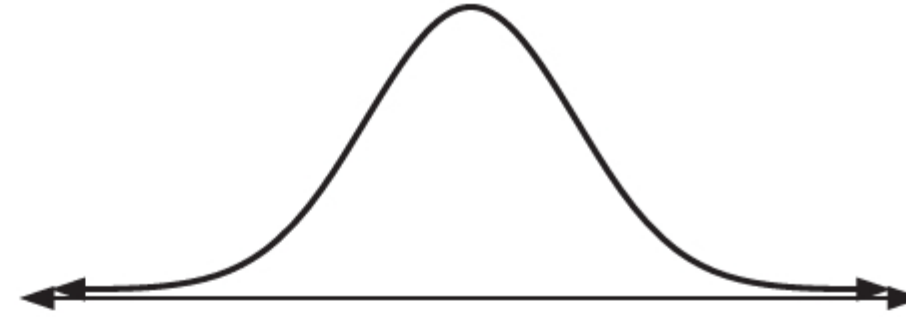
- b** The variable is likely to be normally distributed as the long jumper is likely to jump the same distance consistently, but it will vary due to factors such as the speed at which the long jumper runs before the jump, and the positioning of their body before hitting the sand.



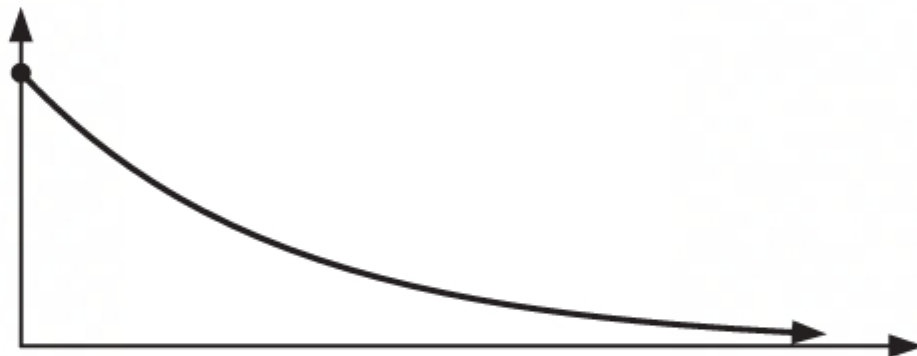
- c The variable is not likely to be normally distributed as each number has the same chance of being drawn. The distribution should be uniform.



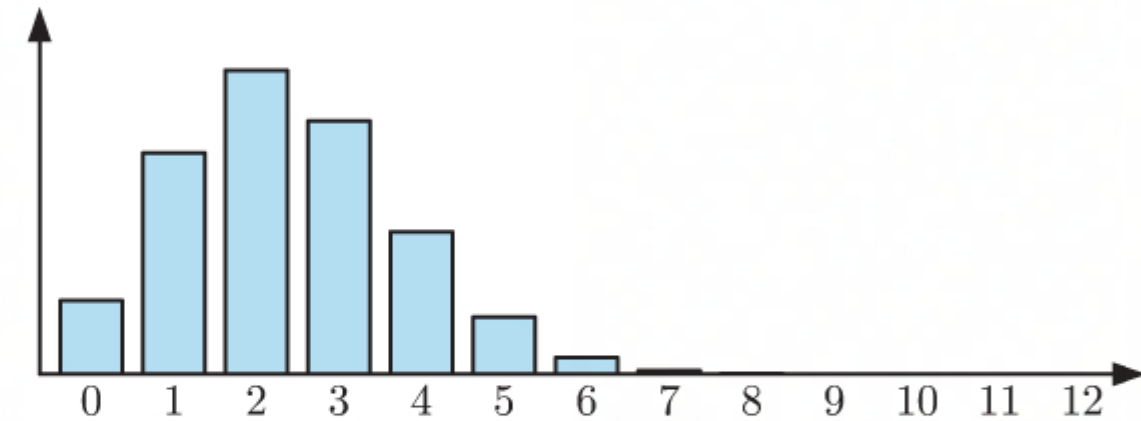
- d The variable is likely to be normally distributed as the lengths of the carrots will be generally centred around the mean, but will vary due to factors such as soil quality, different weather conditions, harvest times, and so on.



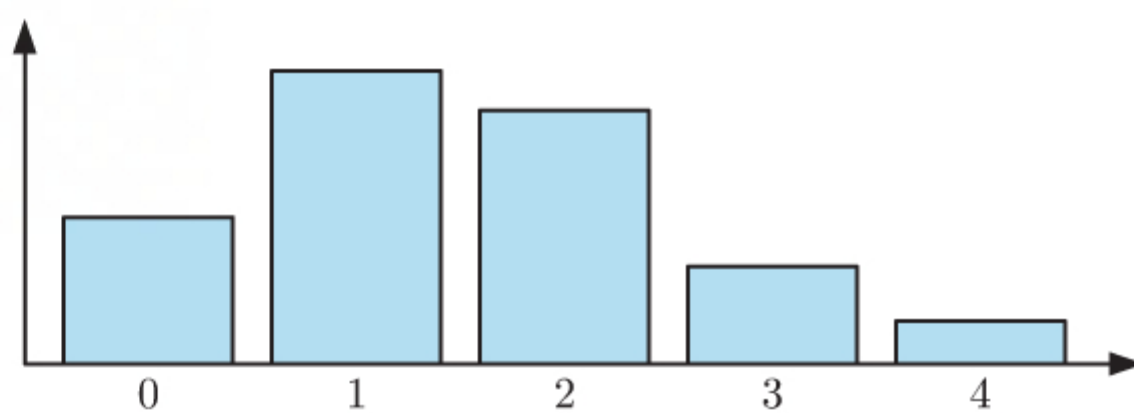
- e The variable is not likely to be normally distributed. People are most likely to be served quite quickly. The distribution is likely to be negatively skewed.



- f The variable is not likely to be normal as it is a discrete variable. Each egg has the same probability of being brown, so the distribution is binomial.



- g The variable is not likely to be normally distributed as it is a discrete variable. Most families will have 0 - 2 children, and there will be much fewer families with more than 2 children. The distribution will be positively skewed.



- h The variable is not likely to be normally distributed as there will tend to be many more shorter buildings than tall buildings in a city. The distribution will be positively skewed.



INVESTIGATION 1

PROPERTIES OF THE NORMAL CURVE

- 1 a μ controls where the centre of the distribution is. As μ changes, the curve is translated horizontally, which is reasonable as μ is the mean and a measure of centre.
 σ controls the shape of the curve. As σ increases, the curve becomes flatter and more spread out, which is reasonable, since σ is the standard deviation and a measure of spread.
- b The curve has a vertical line of symmetry $x = \mu$.
- c The function is never negative. This is important because a probability density function can never be negative.

- d** As $x \rightarrow \pm\infty$, the curve approaches zero from above. The x -axis is a horizontal asymptote.
- e** The area under the curve should remain constant as we change μ and σ , as the area under a probability density function must be 1.

2 a

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

$$\therefore f'(x) = \frac{1}{\sigma\sqrt{2\pi}} \times -\left(\frac{x-\mu}{\sigma}\right) \times \frac{1}{\sigma} \times e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

$$= \frac{\mu-x}{\sigma^3\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

$$= \frac{\mu-x}{\sigma^2} f(x)$$

and $f'(x) = 0$ when $x = \mu$ $\{f(x) > 0\}$

So, $f'(x)$ has sign diagram:

\therefore the stationary point is $(\mu, f(\mu))$ or $\left(\mu, \frac{1}{\sigma\sqrt{2\pi}}\right)$ which is a maximum.

b

$$f'(x) = \frac{\mu-x}{\sigma^2} f(x)$$

$$\therefore f''(x) = -\frac{1}{\sigma^2} f(x) + \frac{\mu-x}{\sigma^2} f'(x) \quad \{\text{product rule}\}$$

$$= -\frac{1}{\sigma^2} f(x) + \frac{\mu-x}{\sigma^2} \left(\frac{\mu-x}{\sigma^2}\right) f(x) \quad \{\text{from a}\}$$

$$= -\frac{1}{\sigma^2} f(x) + \frac{(\mu-x)^2}{\sigma^4} f(x)$$

$$= f(x) \left[\frac{(\mu-x)^2}{\sigma^4} - \frac{1}{\sigma^2} \right]$$

$$f''(x) = 0 \text{ when } \frac{(\mu-x)^2}{\sigma^4} - \frac{1}{\sigma^2} = 0 \quad \{f(x) > 0\}$$

$$\therefore \frac{(\mu-x)^2 - \sigma^2}{\sigma^4} = 0$$

$$\therefore (\mu-x)^2 - \sigma^2 = 0$$

$$\therefore (\mu-x)^2 = \sigma^2$$

$$\therefore \mu-x = \pm\sigma$$

$$\therefore x = \mu \pm \sigma$$

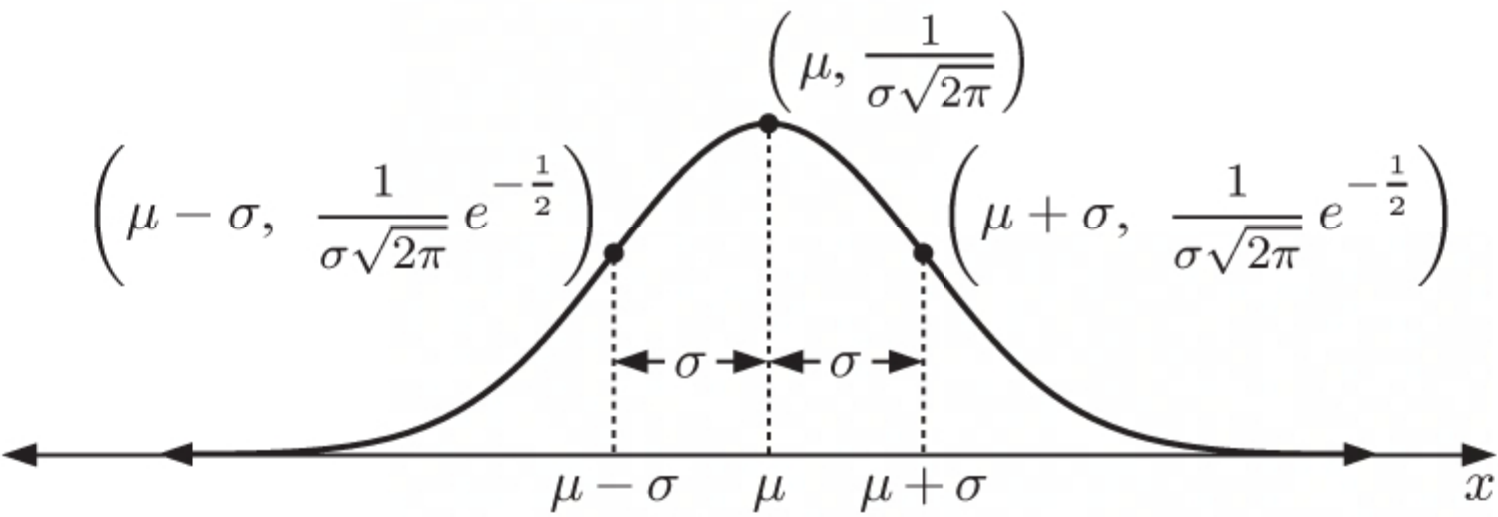
\therefore the inflection points occur at $x = \mu \pm \sigma$

From **a**, the only stationary point is at $x = \mu$, so these are non-stationary inflection points.

$$\text{Now, } f(\mu + \sigma) = f(\mu - \sigma) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}}$$

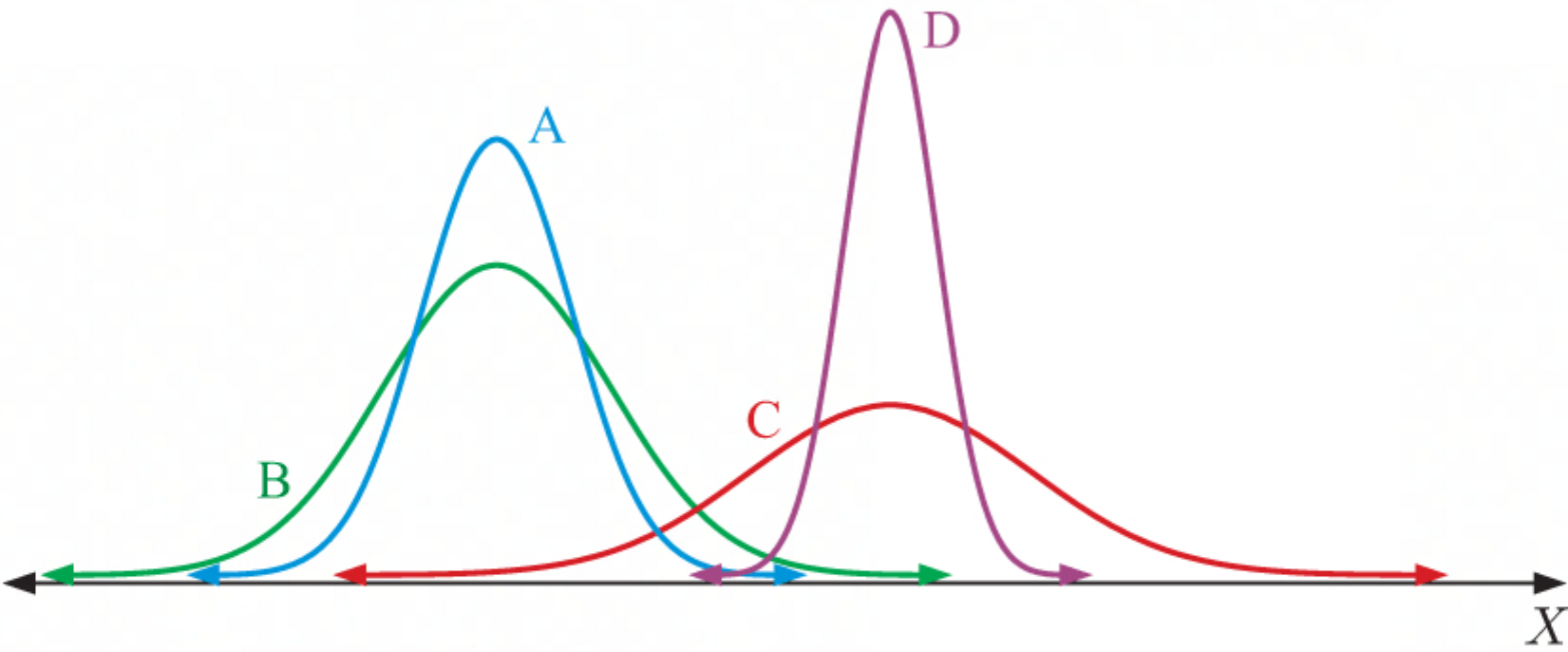
\therefore the non-stationary inflection points are $\left(\mu - \sigma, \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}}\right)$ and $\left(\mu + \sigma, \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}}\right)$.

3



EXERCISE 21A.2

1

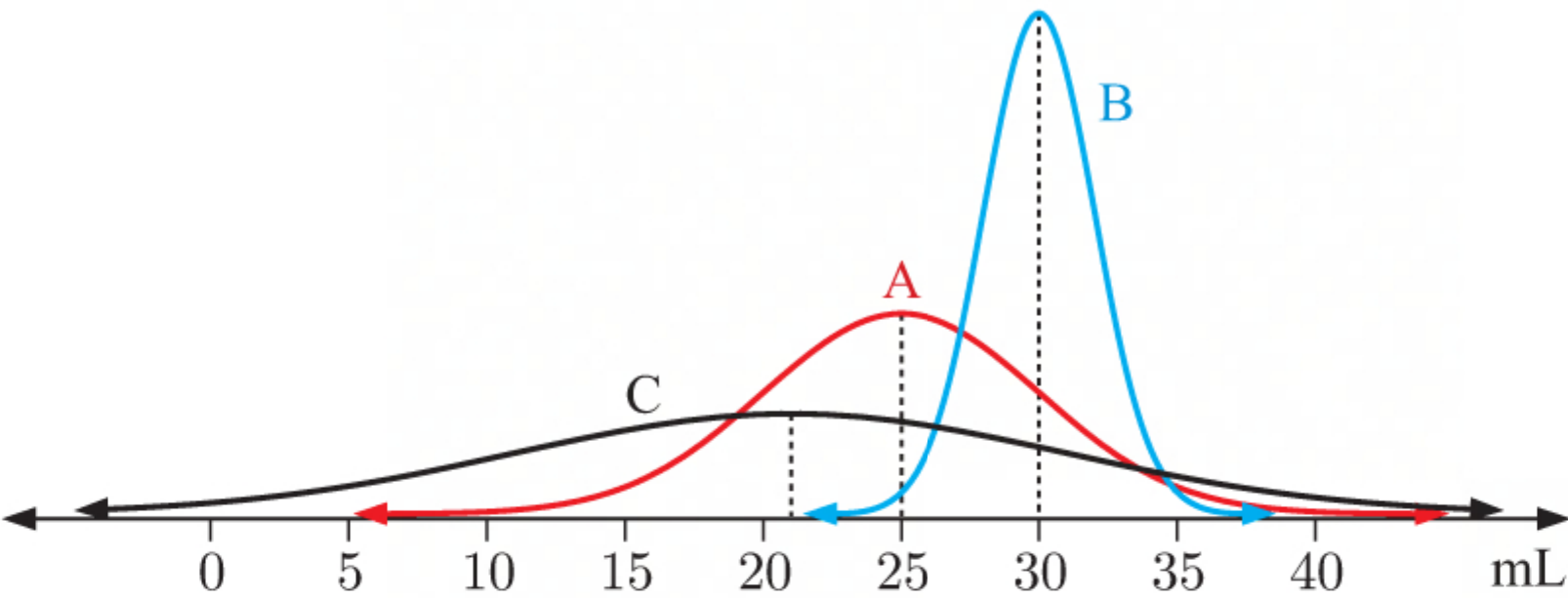


A and B have the same mean, and C and D have the same mean.
The mean of A and B is lower than the mean of C and D.
B has a greater spread, and hence a larger standard deviation than A.
Similarly, C has a larger standard deviation than D.

- a $\mu = 5, \sigma = 2$ corresponds to B
- b $\mu = 15, \sigma = 0.5$ corresponds to D
- c $\mu = 5, \sigma = 1$ corresponds to A
- d $\mu = 15, \sigma = 3$ corresponds to C

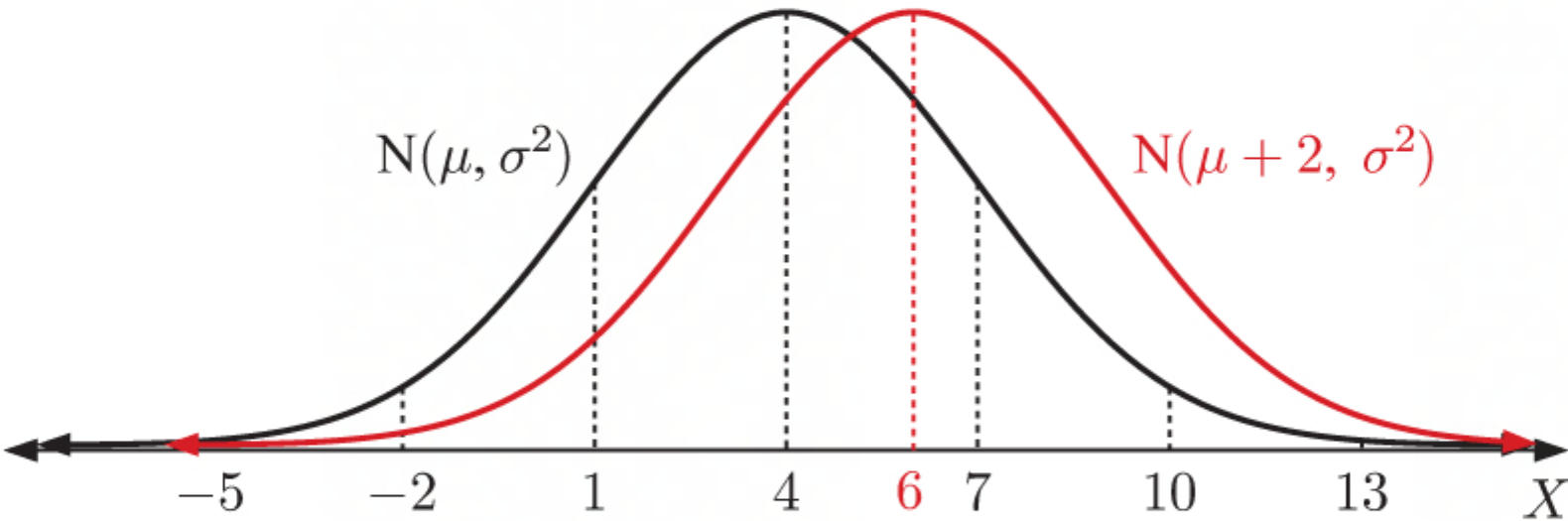
2

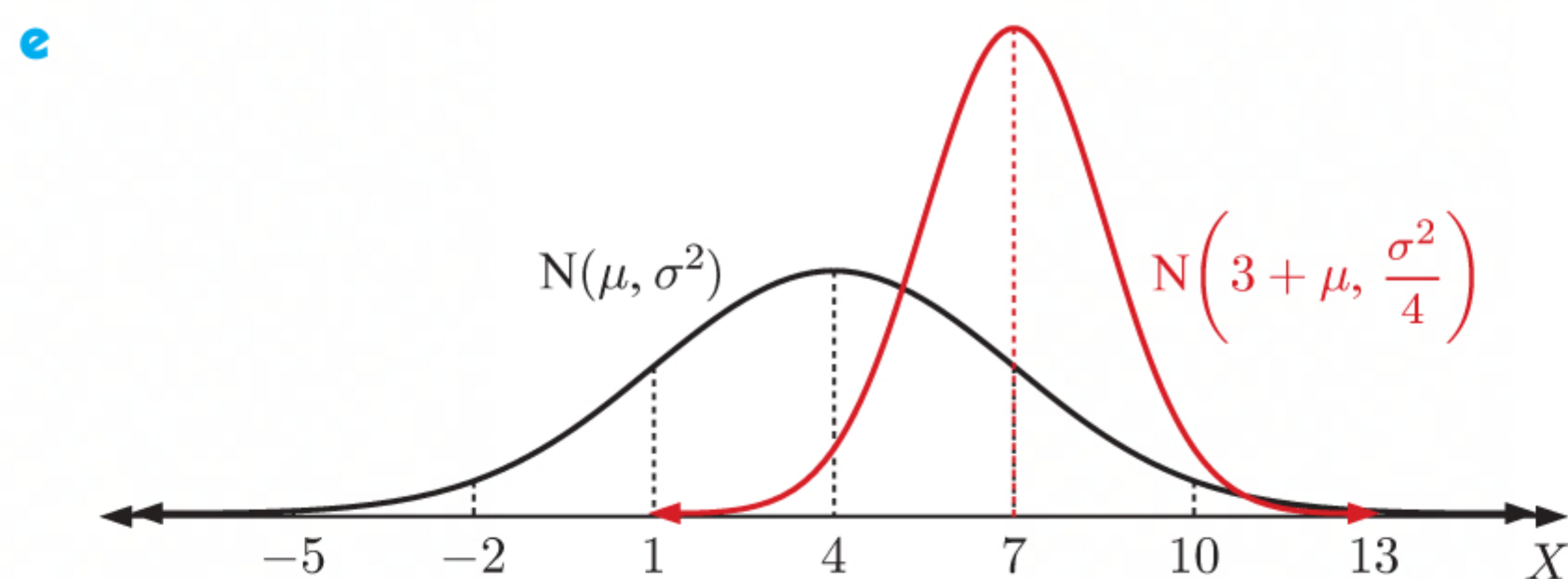
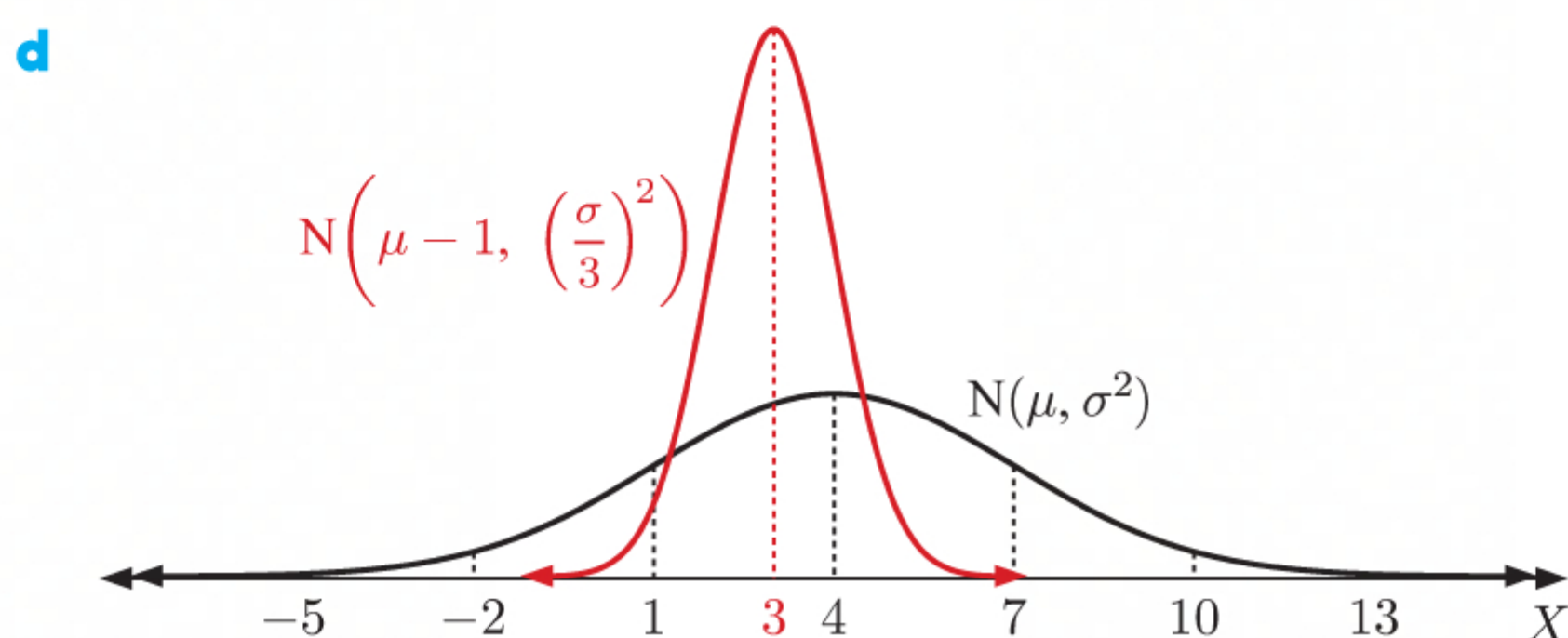
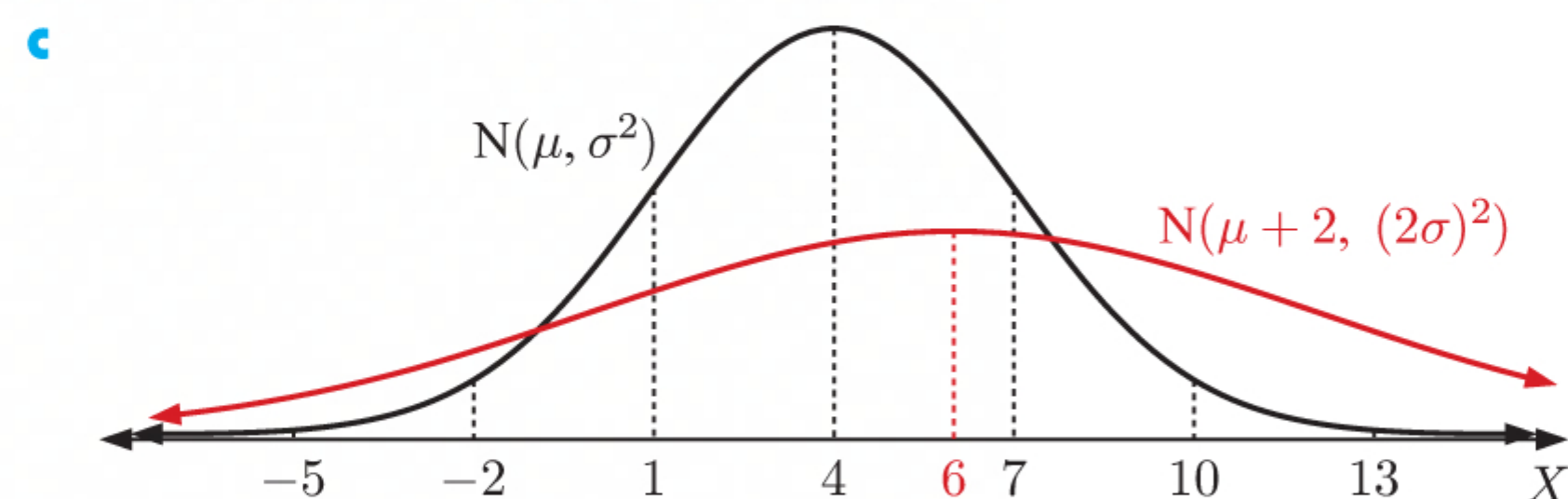
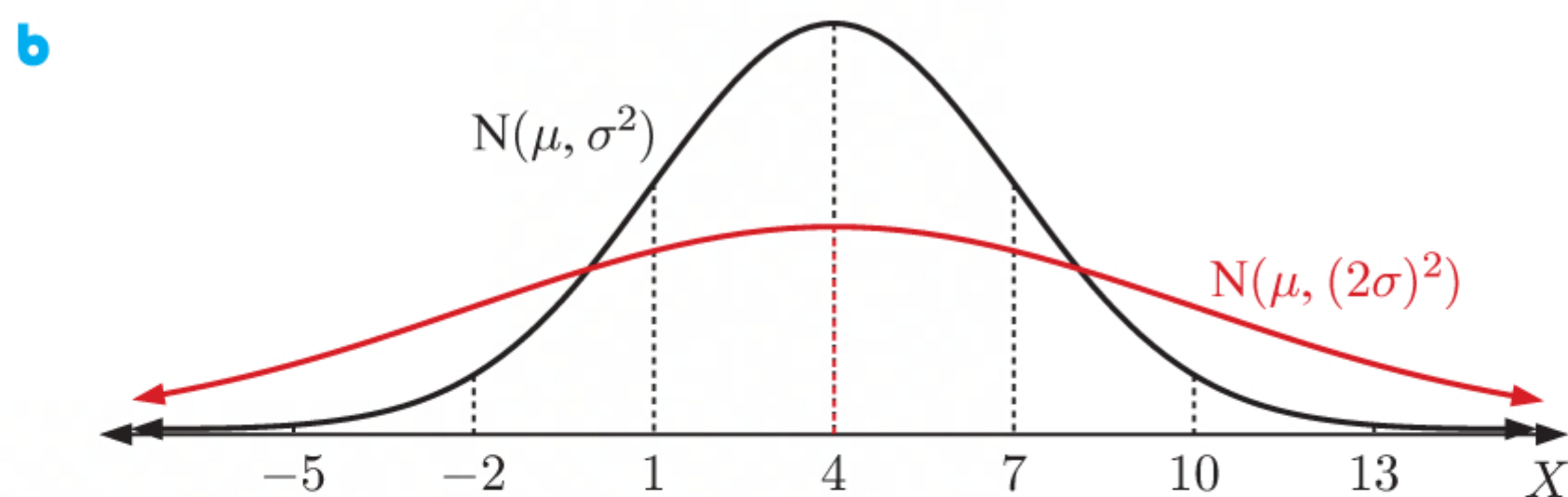
Distribution	Mean (mL)	Standard deviation (mL)
A	25	5
B	30	2
C	21	10



3

a





EXERCISE 21B.1

1 $X \sim N(30, 5^2)$

a **i** The value which is 2 standard deviations above the mean $= 30 + 2 \times 5$
 $= 40$

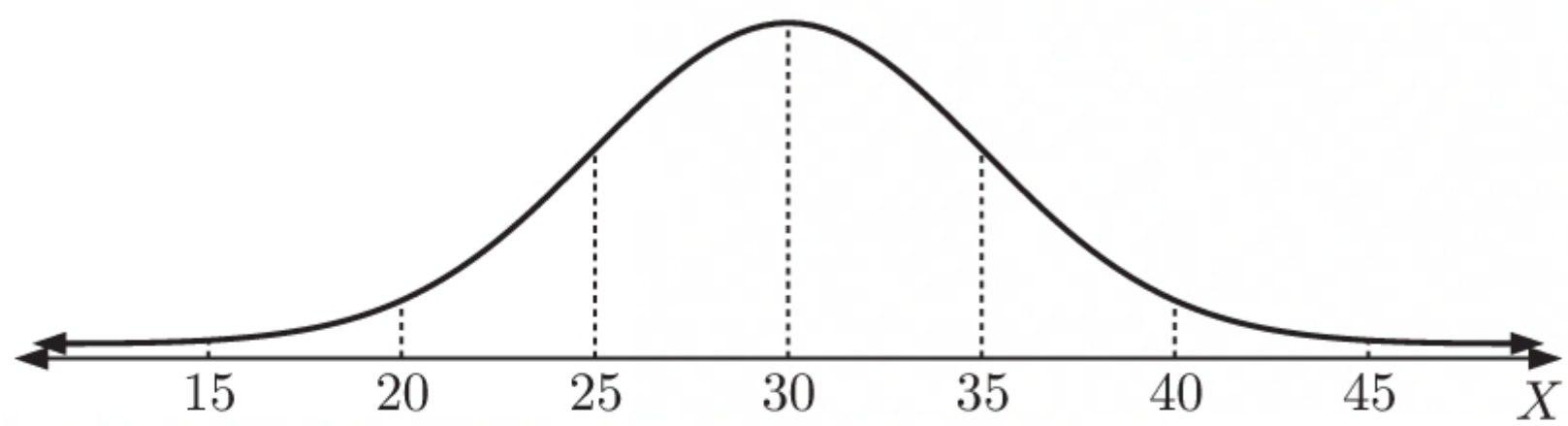
ii The value which is 1 standard deviation below the mean $= 30 - 5$
 $= 25$

b **i** $35 = 30 + 5$
 $\therefore 35$ is 1 standard deviation above the mean.

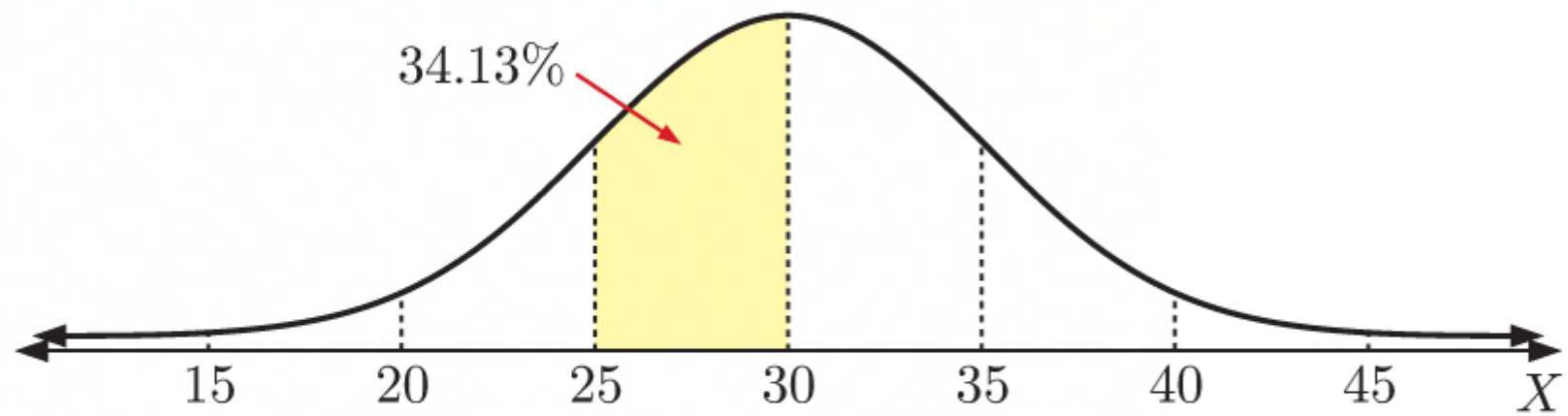
ii $20 = 30 - 2 \times 5$
 $\therefore 20$ is 2 standard deviations below the mean.

iii $45 = 30 + 3 \times 5$
 $\therefore 45$ is 3 standard deviations above the mean.

c

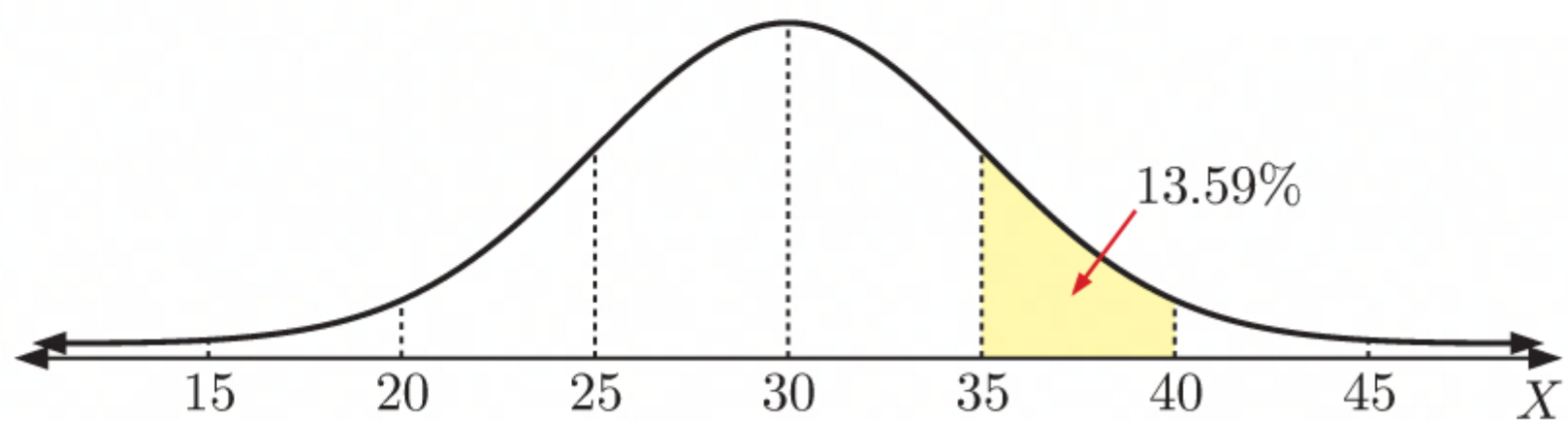


d



About 34.13% of the values of X are between 25 and 30.

e

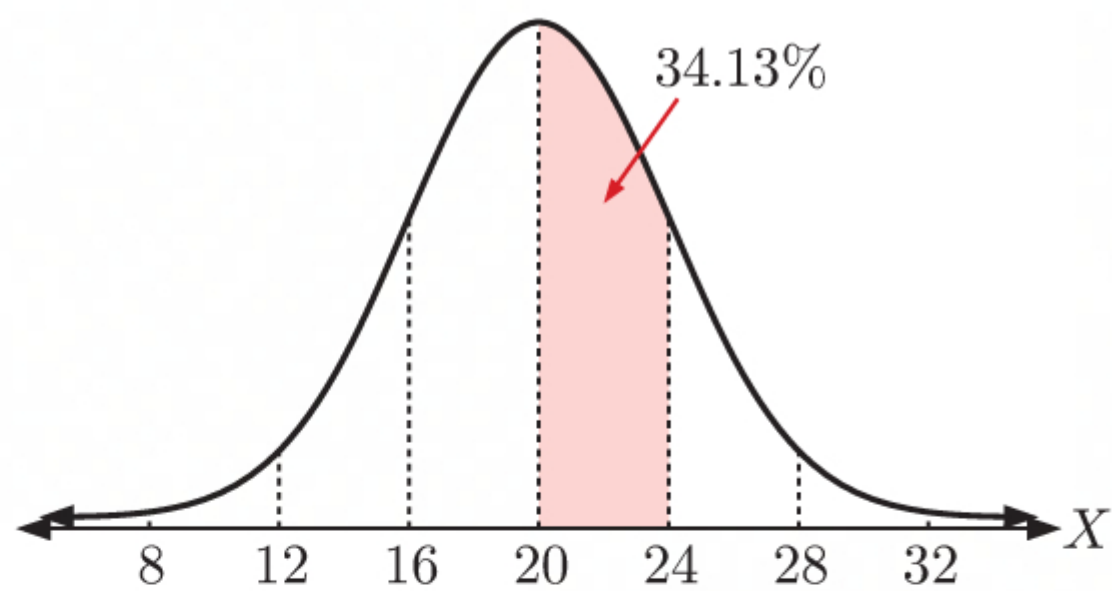


About 13.59% of the values of X are between 35 and 40.

\therefore the probability that a randomly selected member of the population will measure between 35 and 40 is approximately 0.1359.

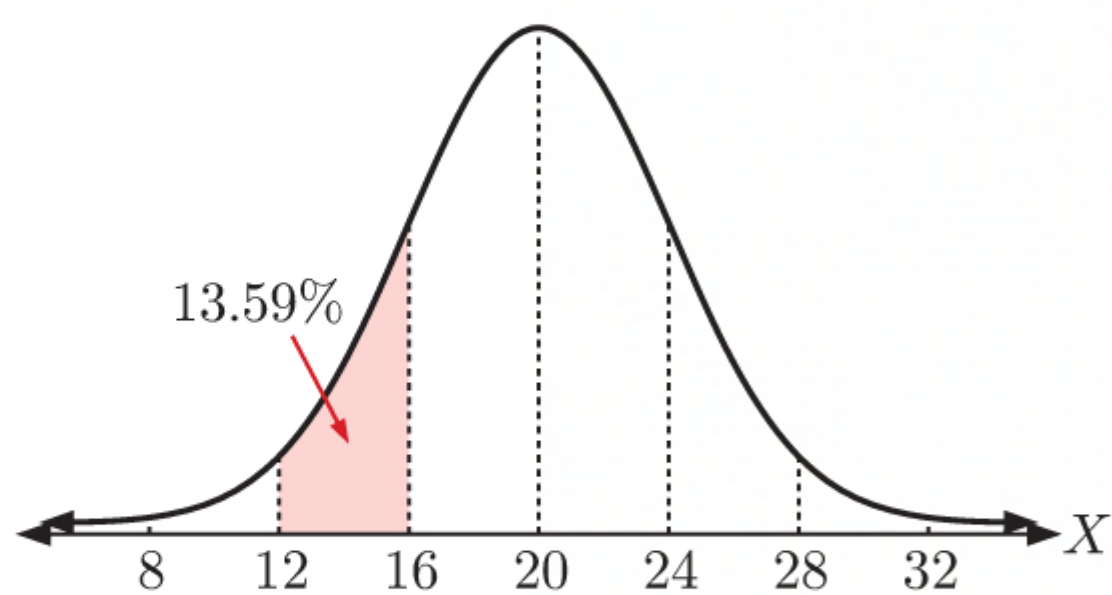
2 a $\mu = 20, \sigma = 4$

b i



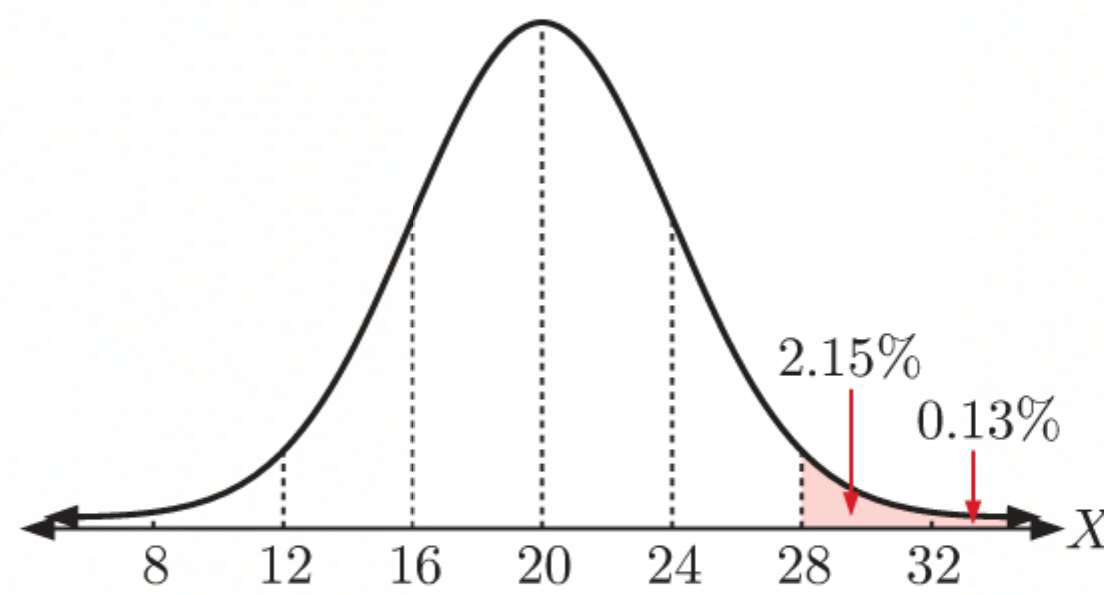
About 34.13% of X values are between 20 and 24.

ii



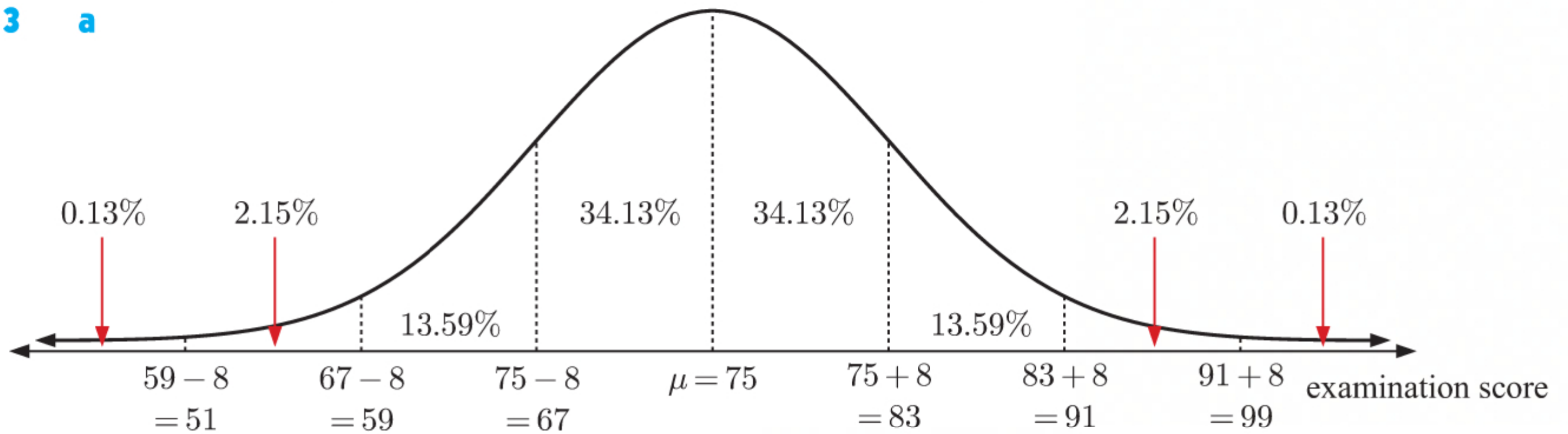
About 13.59% of X values are between 12 and 16.

iii

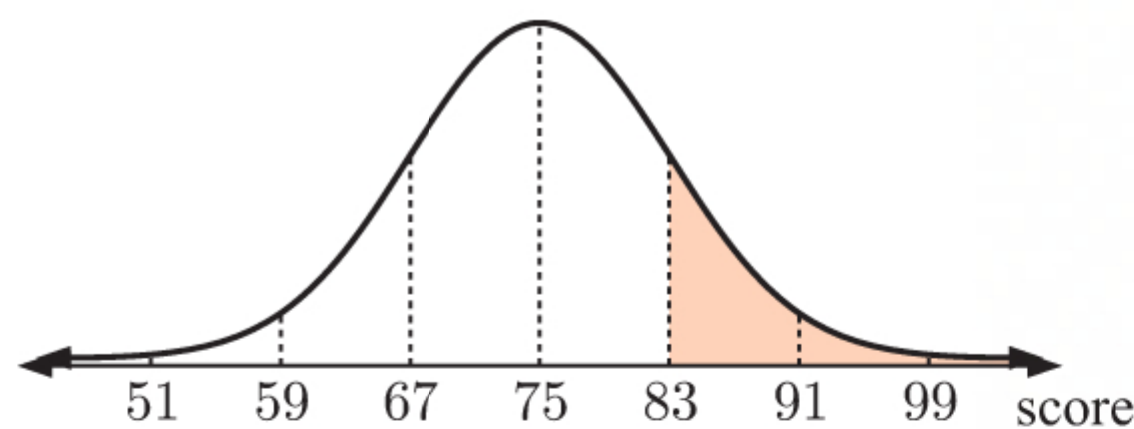


About $2.15\% + 0.13\% = 2.28\%$ of X values are greater than 28.

3 a

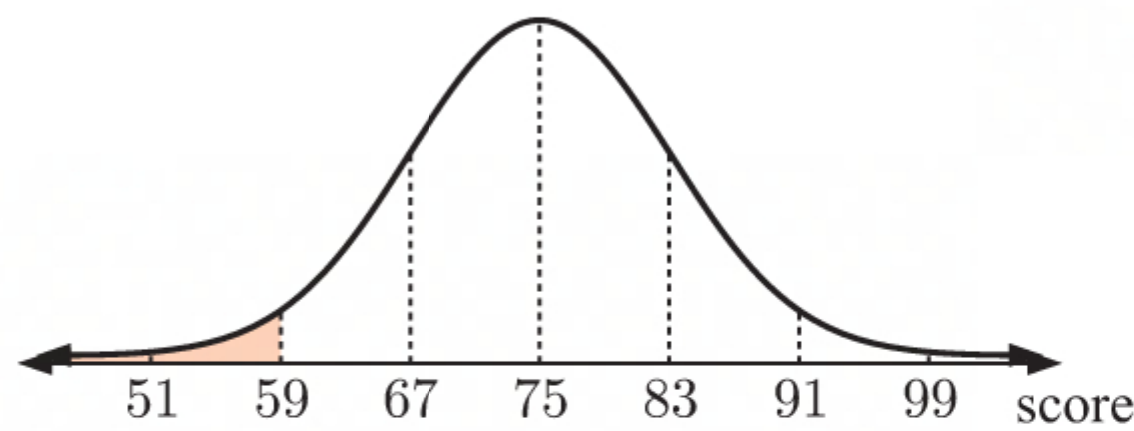


b i



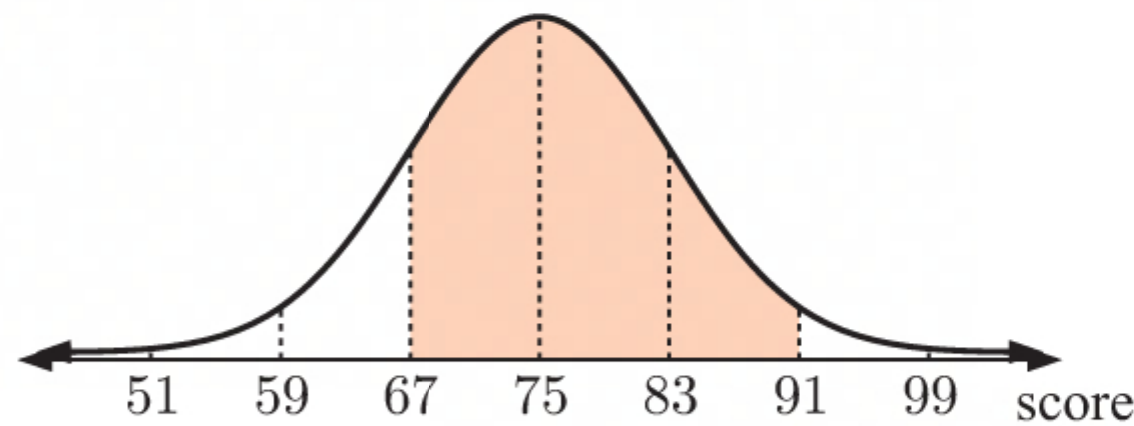
About $13.59\% + 2.15\% + 0.13\% = 15.87\%$ of students would be expected to have scored more than 83.

ii



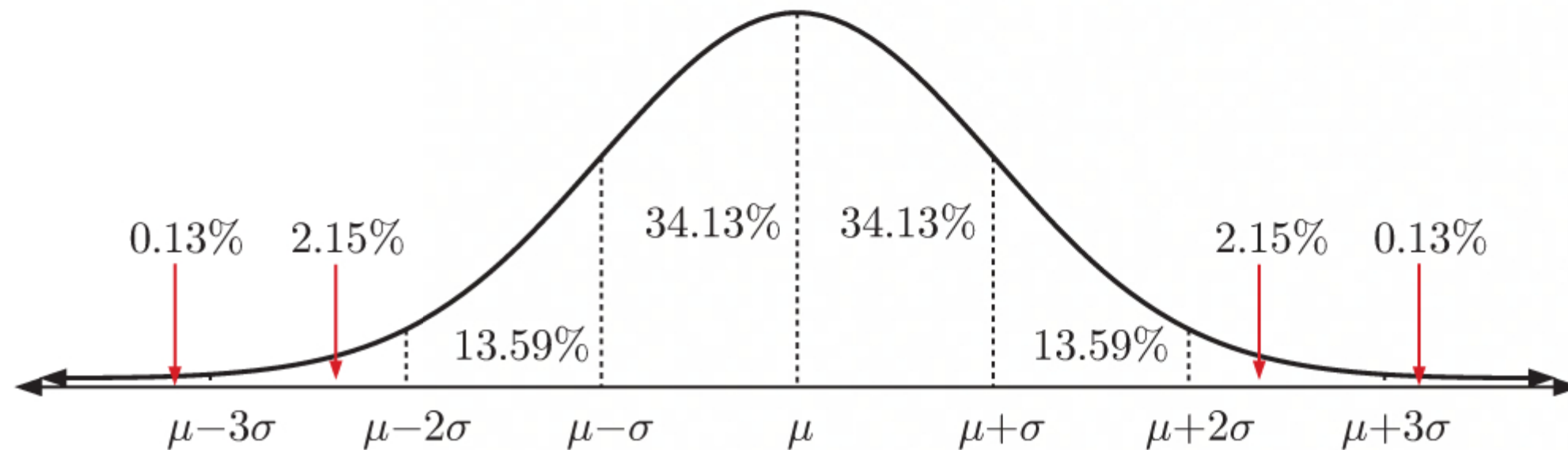
About $2.15\% + 0.13\% = 2.28\%$ of students would be expected to have scored less than 59.

iii



About $34.13\% + 34.13\% + 13.59\% = 81.85\%$ of students would be expected to have scored between 67 and 91.

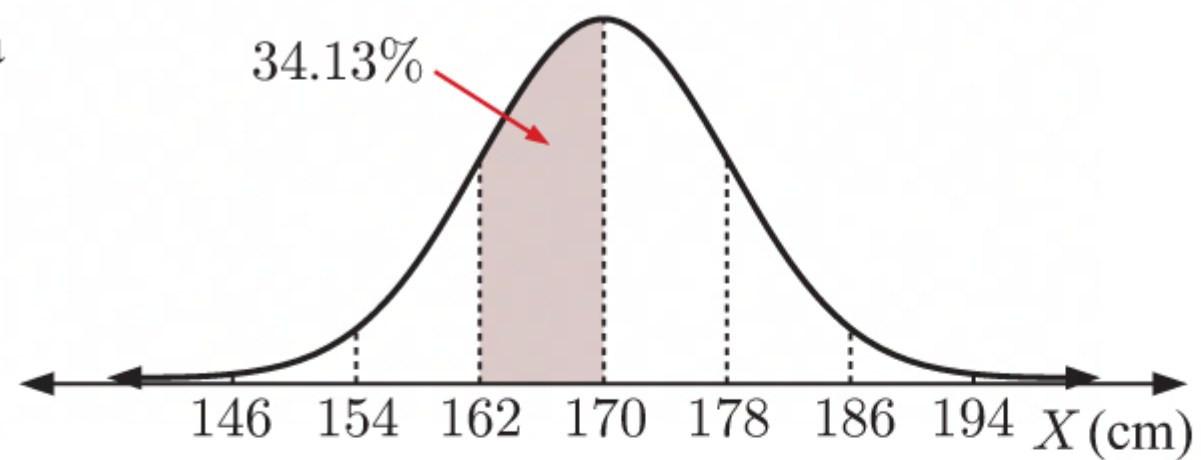
4



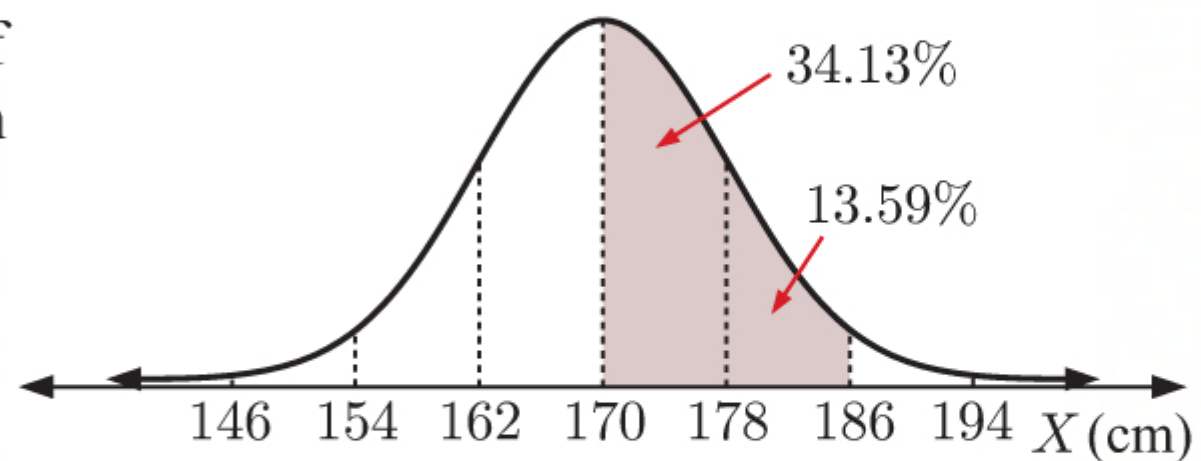
a $P(\text{value between } \mu - \sigma \text{ and } \mu + \sigma)$
 $\approx 0.3413 + 0.3413$
 ≈ 0.6826

b $P(\text{value} > \mu + 2\sigma)$
 $\approx 0.0215 + 0.0013$
 ≈ 0.0228

- 5 a i** About 34.13% of female students have a height between 162 cm and 170 cm.

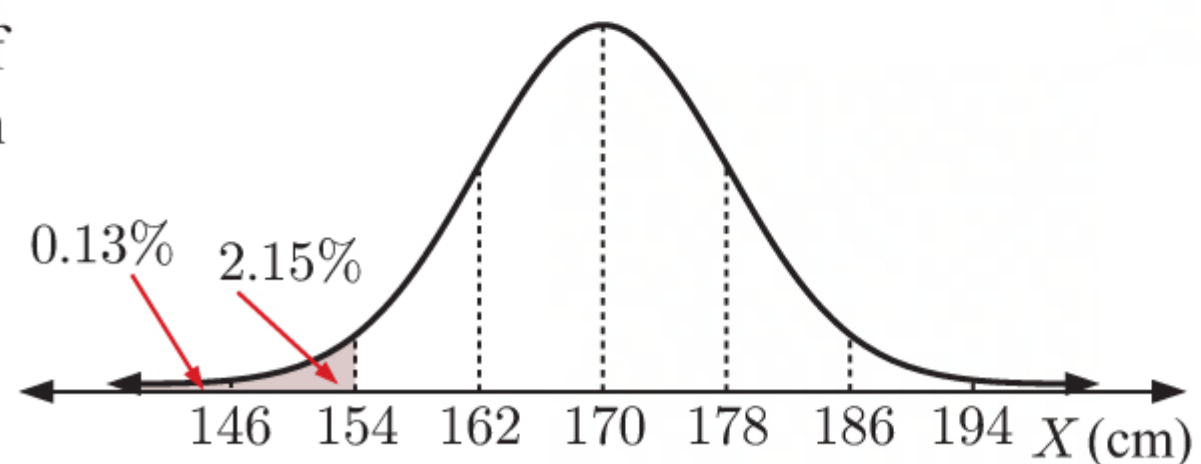


- ii** About $34.13\% + 13.59\% = 47.72\%$ of female students have a height between 170 cm and 186 cm.



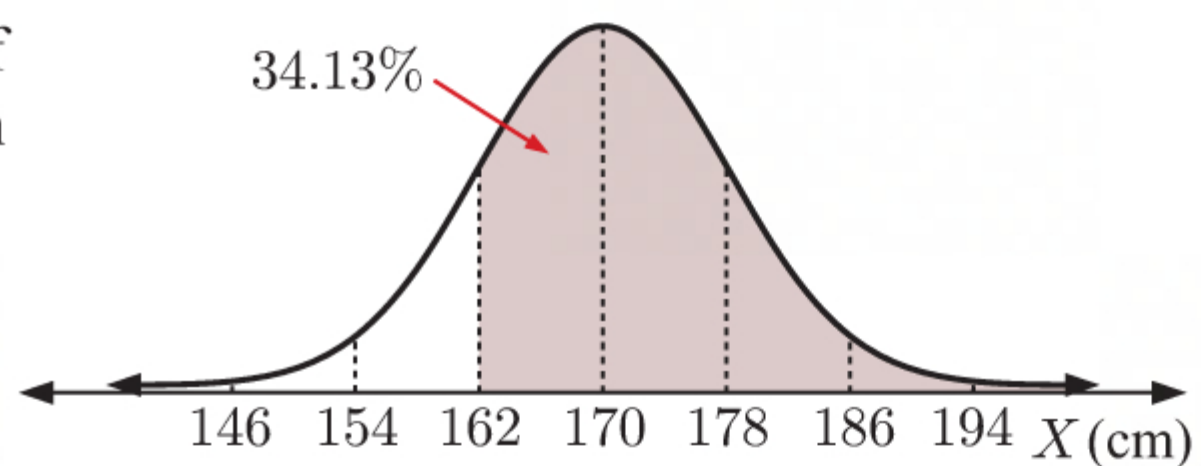
- b i** About $2.15\% + 0.13\% = 2.28\%$ of female students have a height less than 154 cm.

$$\therefore P(\text{height is less than 154 cm}) \approx 0.0228$$

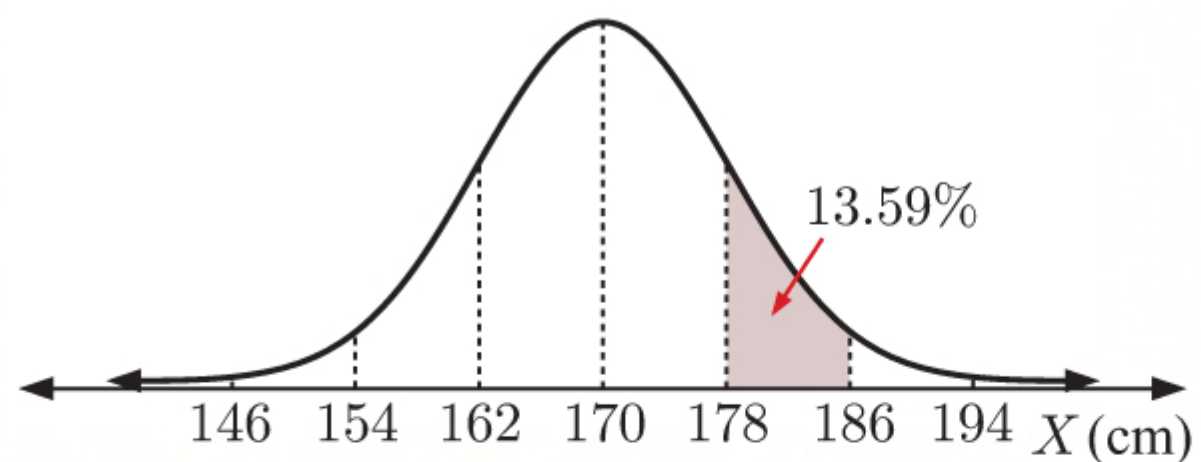


- ii** About $34.13\% + 50\% = 84.13\%$ of female students have a height greater than 162 cm.

$$\therefore P(\text{height is greater than 162 cm}) \approx 0.8413$$



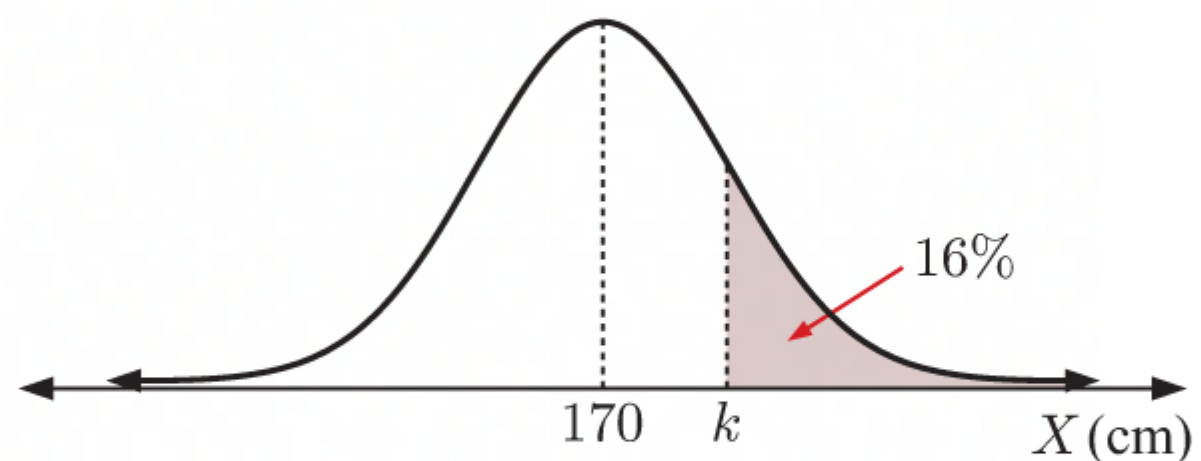
- c** About 13.59% of the female students have a height between 178 cm and 186 cm.
So, we would expect about
13.59% of $500 \approx 68$ students to have a height between 178 cm and 186 cm.



- d** Approximately 16% of data lies more than one standard deviation above the mean.

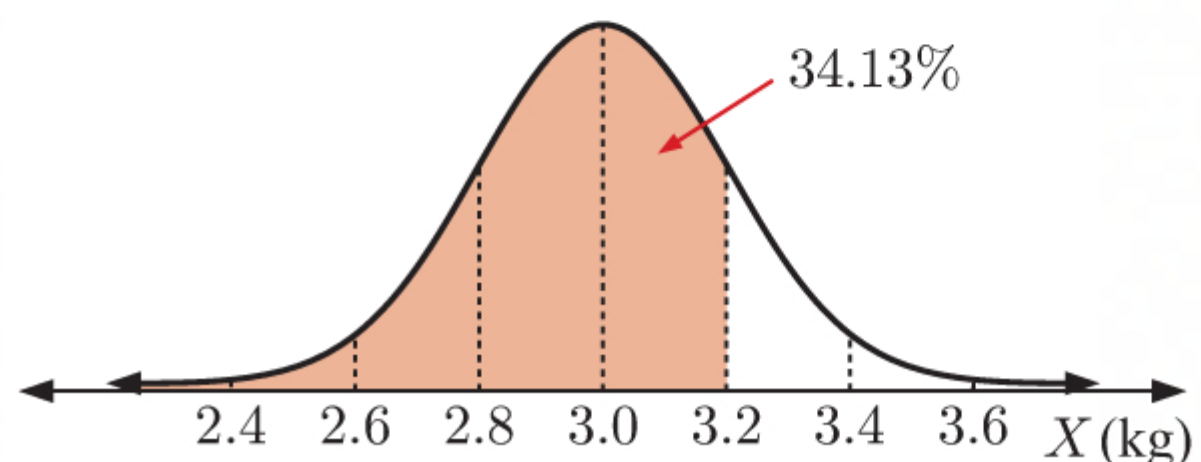
$\therefore k$ is about σ above the mean μ

$$\therefore k \approx 170 + 8 \approx 178$$

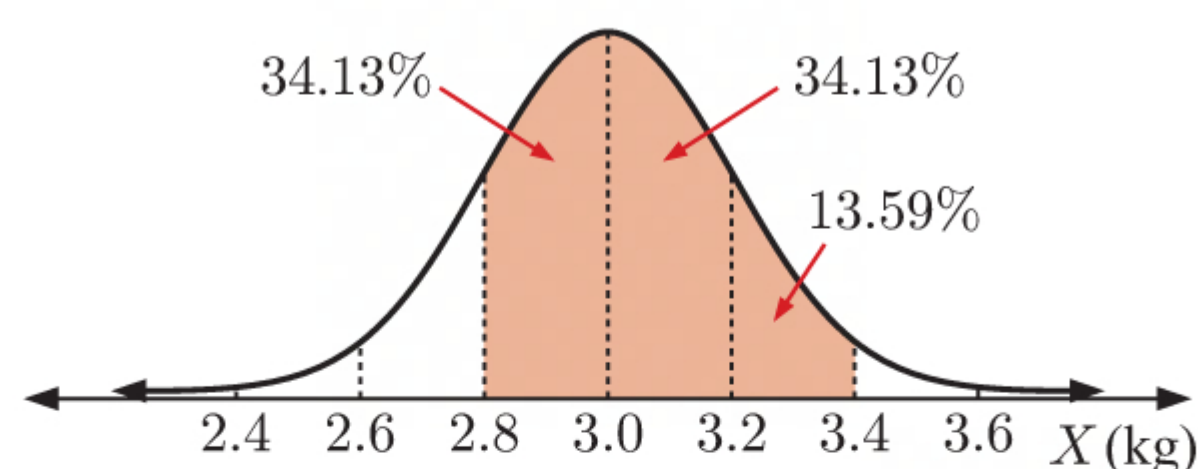


- 6 a** About $50\% + 34.13\% = 84.13\%$ of babies born weighed less than 3.2 kg.

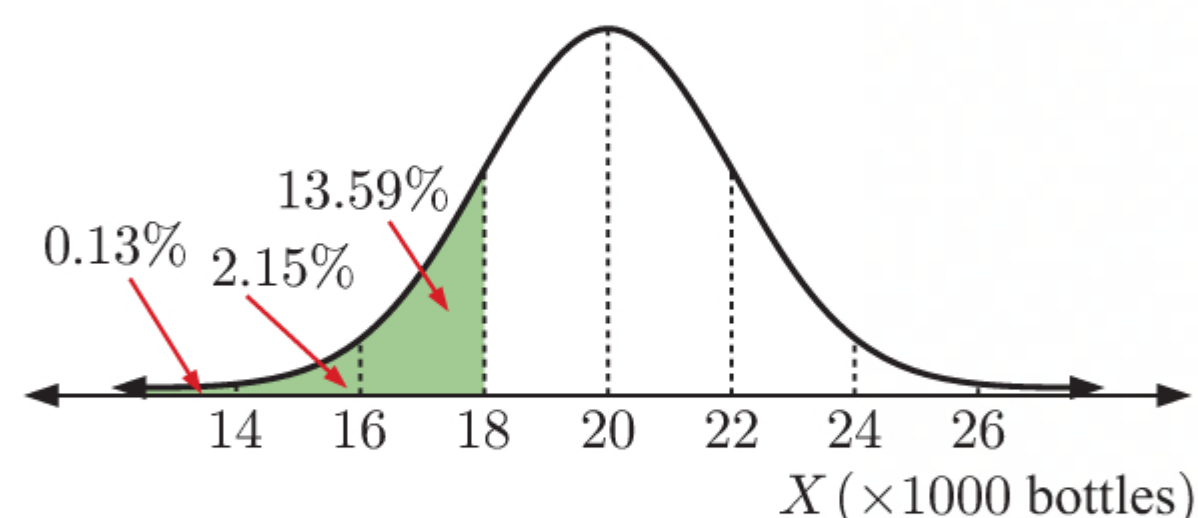
So, about $84.13\% \times 545 \approx 459$ babies born weighed less than 3.2 kg.



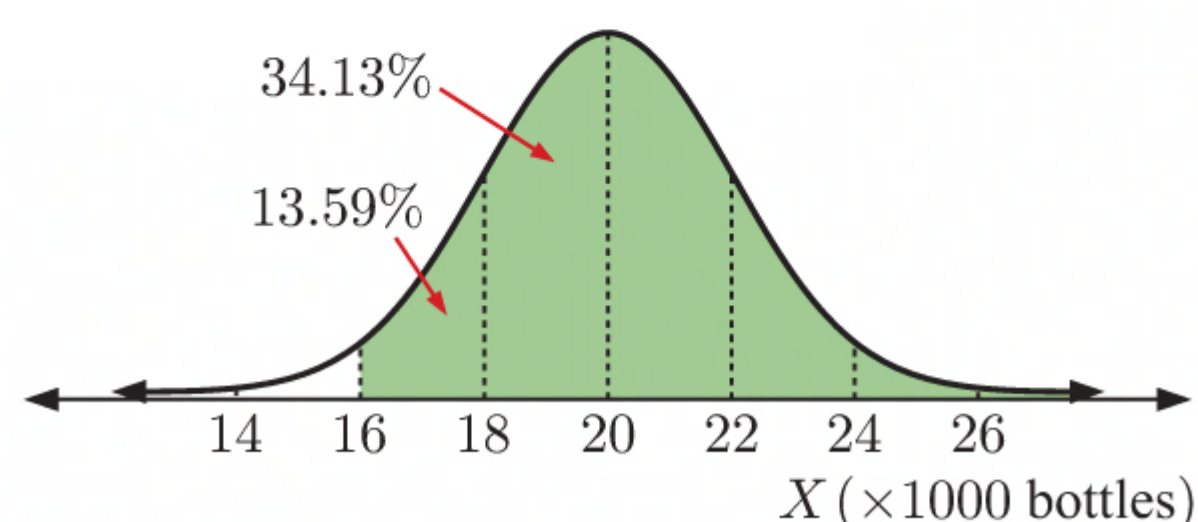
- b** About $34.13\% + 34.13\% + 13.59\%$
 $= 81.85\%$ of babies born weighed
 between 2.8 kg and 3.4 kg.
 So, about $81.85\% \times 545 \approx 446$ babies born
 weighed between 2.8 kg and 3.4 kg.



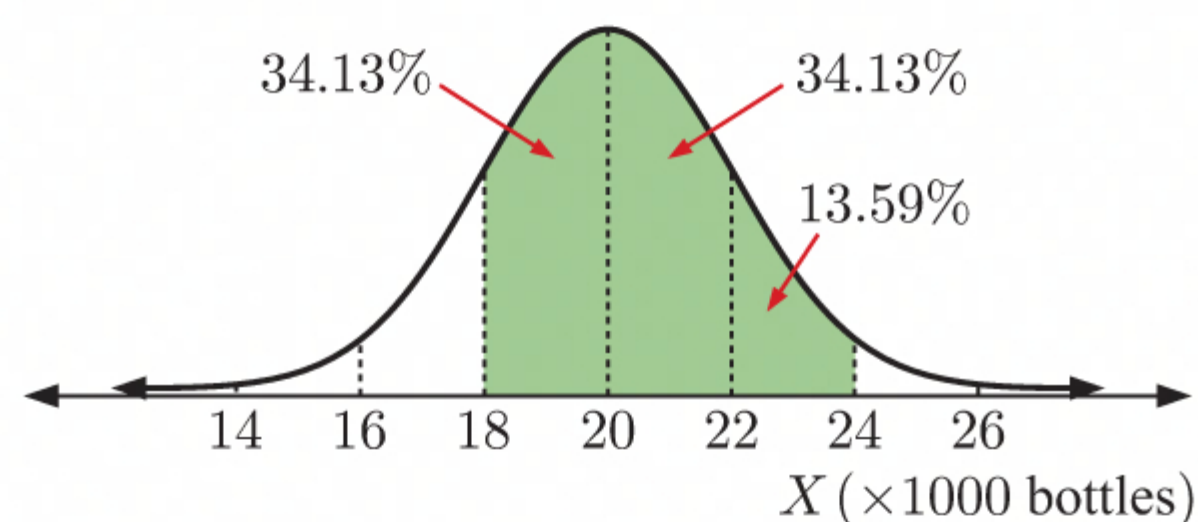
- 7 a** Under 18 000 bottles are filled on about
 $0.13\% + 2.15\% + 13.59\% = 15.87\%$ of days.
 So, under 18 000 bottles are filled on about
 $15.87\% \times 260 \approx 41$ days.



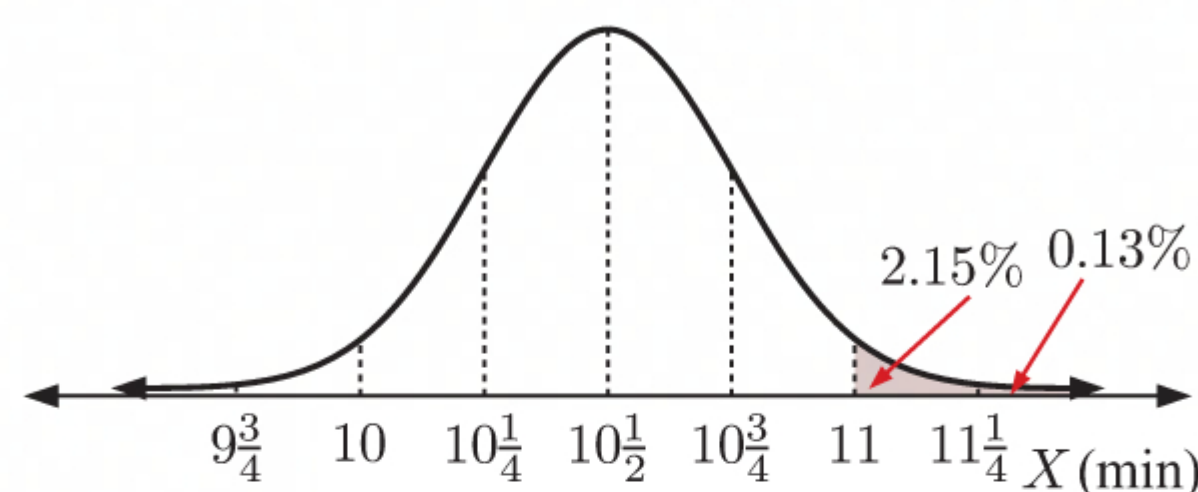
- b** Over 16 000 bottles are filled on about
 $13.59\% + 34.13\% + 50\% = 97.72\%$ of days.
 So, over 16 000 bottles are filled on about
 $97.72\% \times 260 \approx 254$ days.



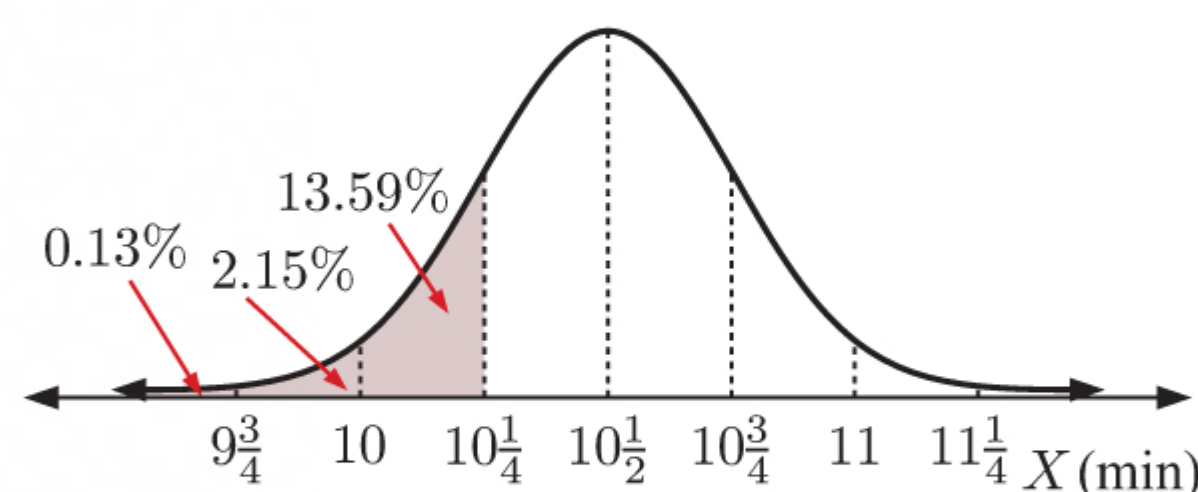
- c** Between 18 000 and 24 000 bottles are filled
 on about $34.13\% + 34.13\% + 13.59\%$
 $= 81.85\%$ of days.
 So, between 18 000 and 24 000 bottles are
 filled on about $81.85\% \times 260 \approx 213$ days.



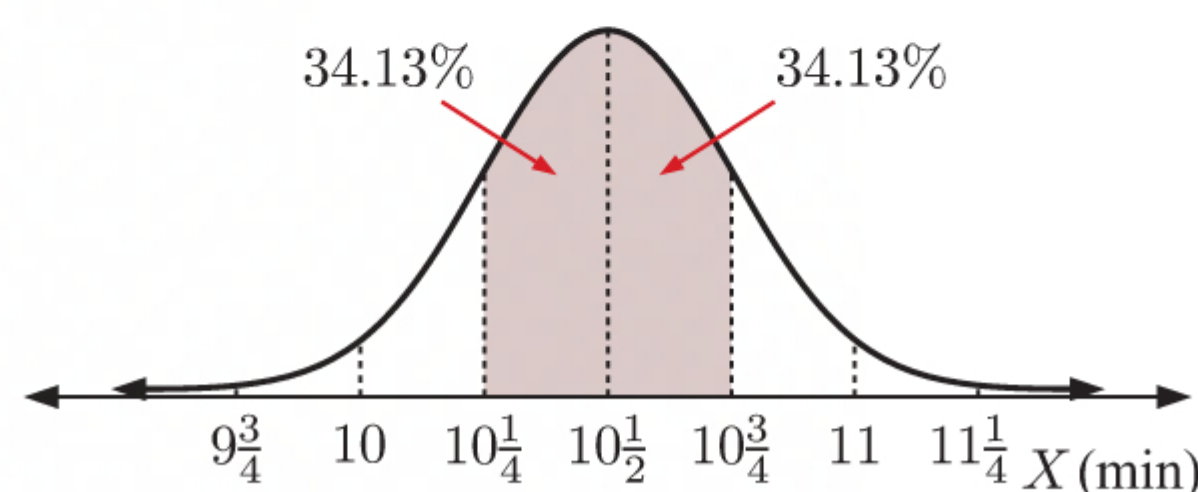
- 8 a** About $2.15\% + 0.13\% = 2.28\%$ of
 competitors completed the race in a time
 longer than 11 minutes.
 So, about $2.28\% \times 200 \approx 5$ competitors
 completed the race in a time longer than
 11 minutes.



- b** About $0.13\% + 2.15\% + 13.59\% = 15.87\%$
 of competitors completed the race in a time
 less than 10 minutes 15 seconds.
 So, about $15.87\% \times 200 \approx 32$ competitors
 completed the race in a time less than
 10 minutes 15 seconds.



- c** About $34.13\% + 34.13\% = 68.26\%$ of
 competitors completed the race in a time
 between 10 minutes 15 seconds and
 10 minutes 45 seconds.
 So, about $68.26\% \times 200 \approx 137$ competitors
 completed the race in a time between
 10 minutes 15 seconds and 10 minutes
 45 seconds.



- 9 a** Approximately 84% of data is more than one standard deviation below the mean.

\therefore 152 grams is about σ below the mean μ

$$\therefore \mu \approx 152 + \sigma \quad \dots (1)$$

Approximately 16% of data is more than one standard deviation above the mean.

\therefore 200 grams is σ above the mean μ

$$\therefore \mu \approx 200 - \sigma \quad \dots (2)$$

Equating (1) and (2) gives: $152 + \sigma \approx 200 - \sigma$

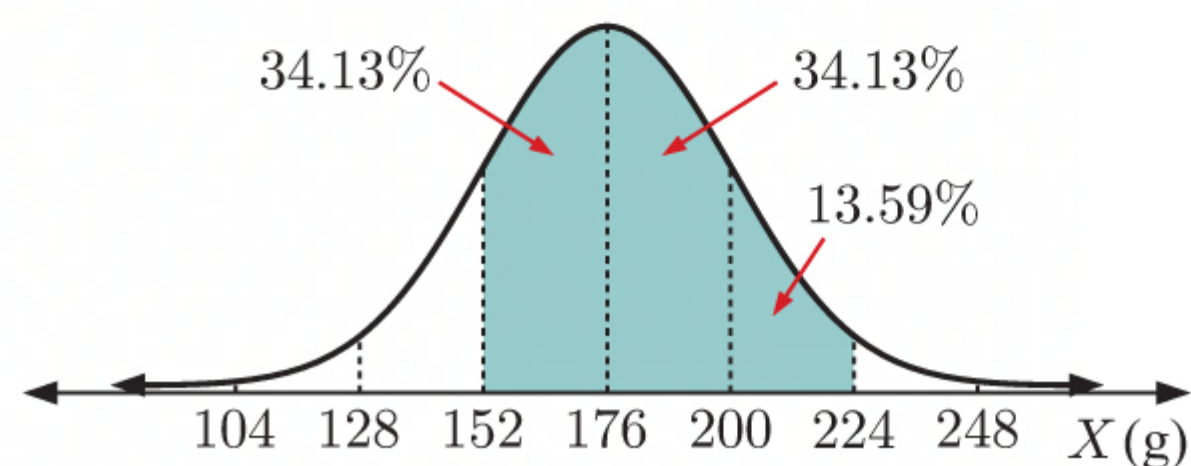
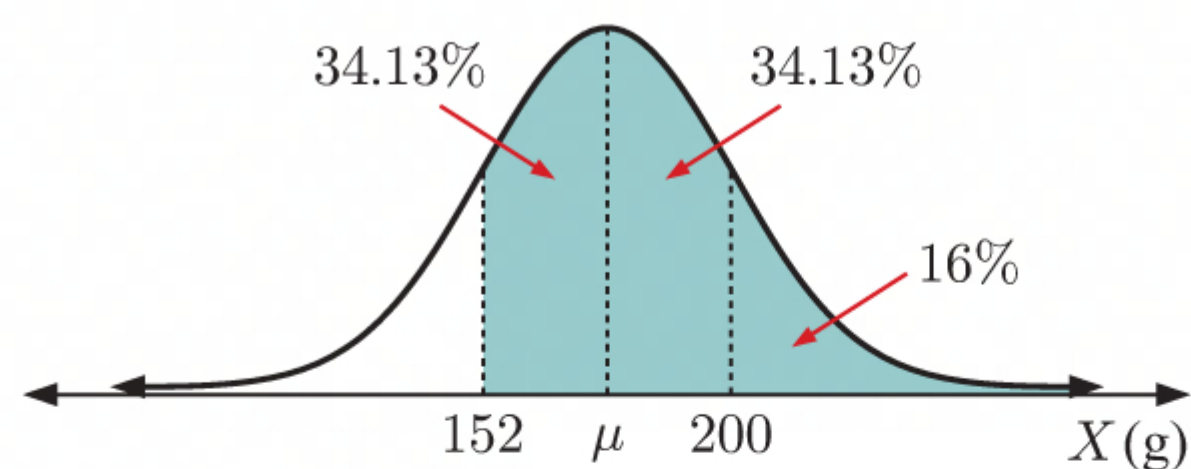
$$\therefore 2\sigma \approx 48$$

$$\therefore \sigma \approx 24$$

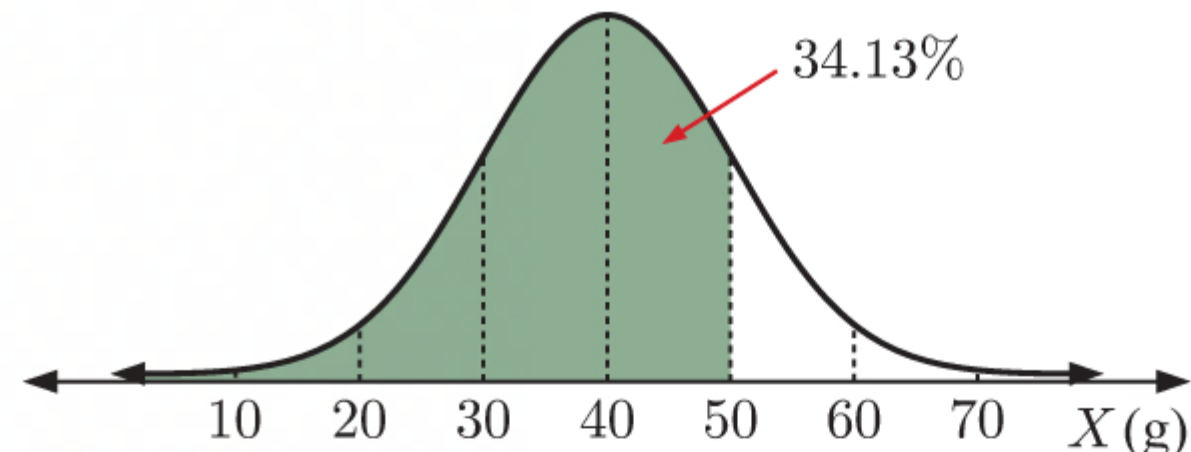
$$\text{and } \mu \approx 200 - 24 \quad \{\text{using (2)}\} \\ \approx 176$$

So, $\mu \approx 176$ grams and $\sigma \approx 24$ grams.

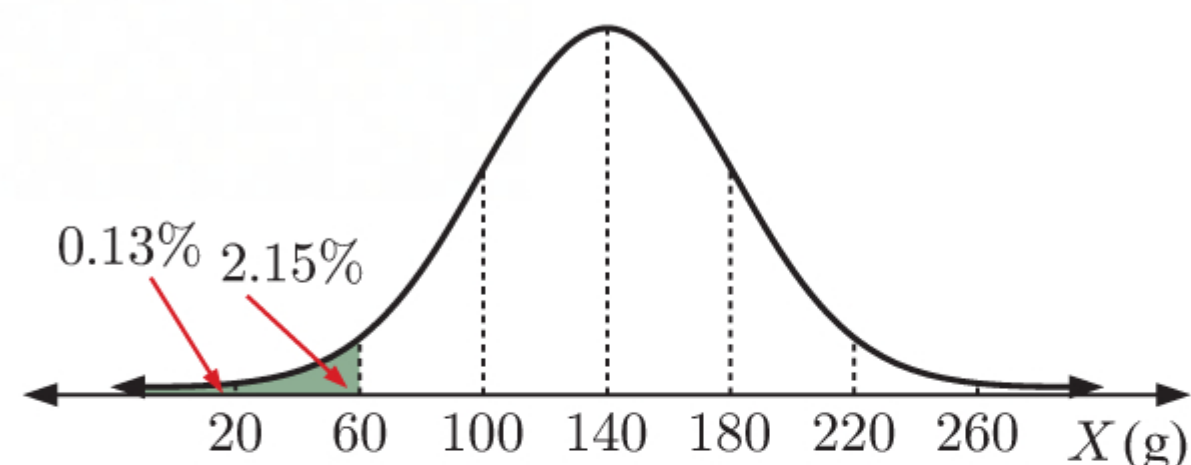
- b** About $34.13\% + 34.13\% + 13.59\%$
 $= 81.85\%$ of the oranges weigh
 between 152 grams and 224 grams.



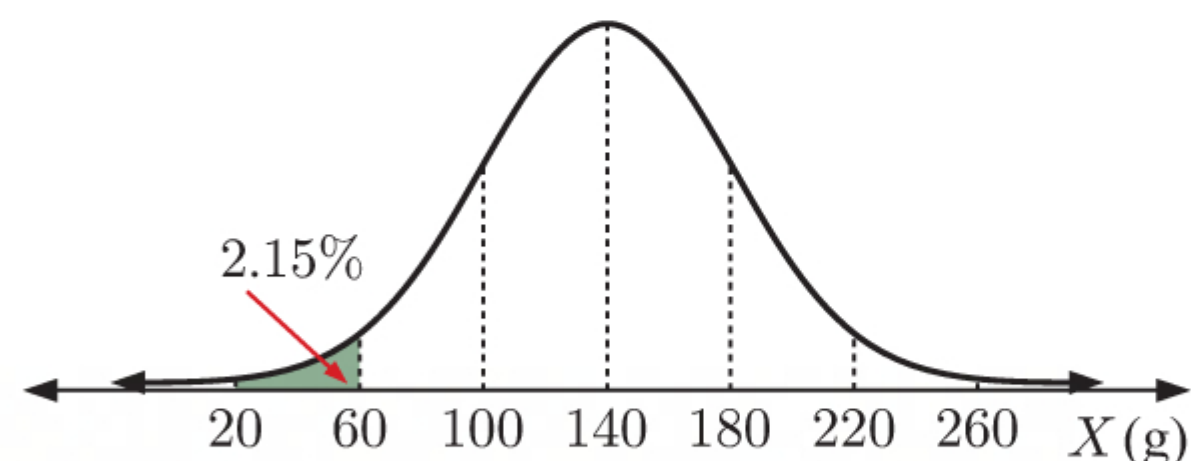
- 10 a i** About $50\% + 34.13\% = 84.13\%$ of radishes grown without fertiliser will have weights less than 50 grams.



- ii** About $0.13\% + 2.15\% = 2.28\%$ of radishes grown with fertiliser will have weights less than 60 grams.

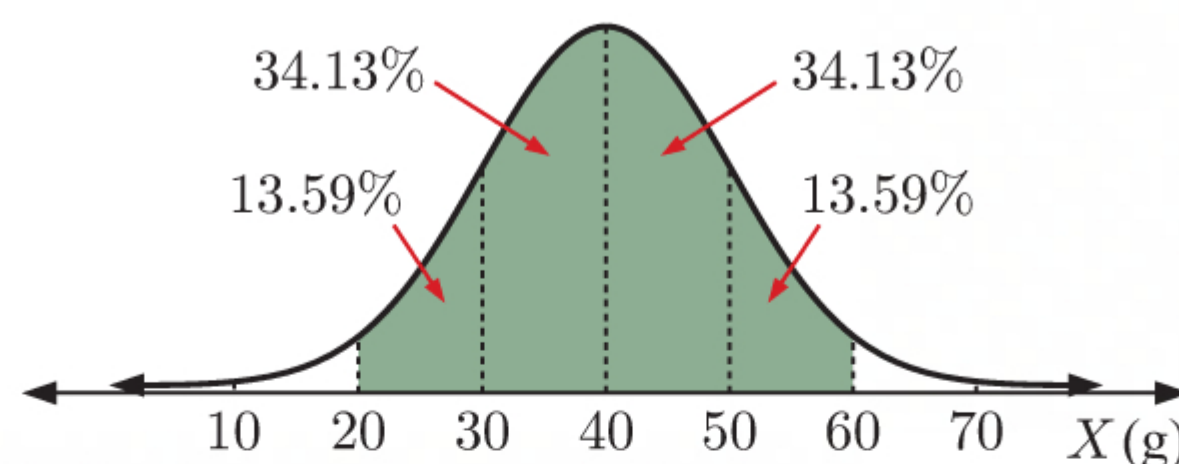


- b i** About 2.15% of radishes grown with fertiliser will have weights between 20 g and 60 g.



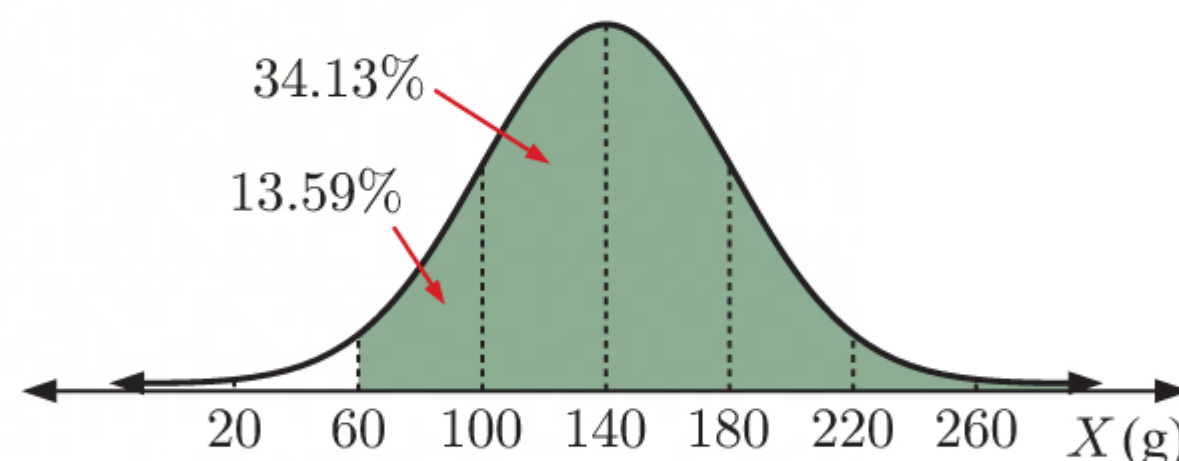
$$\therefore P(\text{radish grown with fertiliser weighs between 20 g and 60 g}) \approx 0.0215$$

- ii About
 $13.59\% + 34.13\% + 34.13\% + 13.59\%$
 $= 95.44\%$
 of radishes grown without fertiliser will
 have weights between 20 g and 60 g.



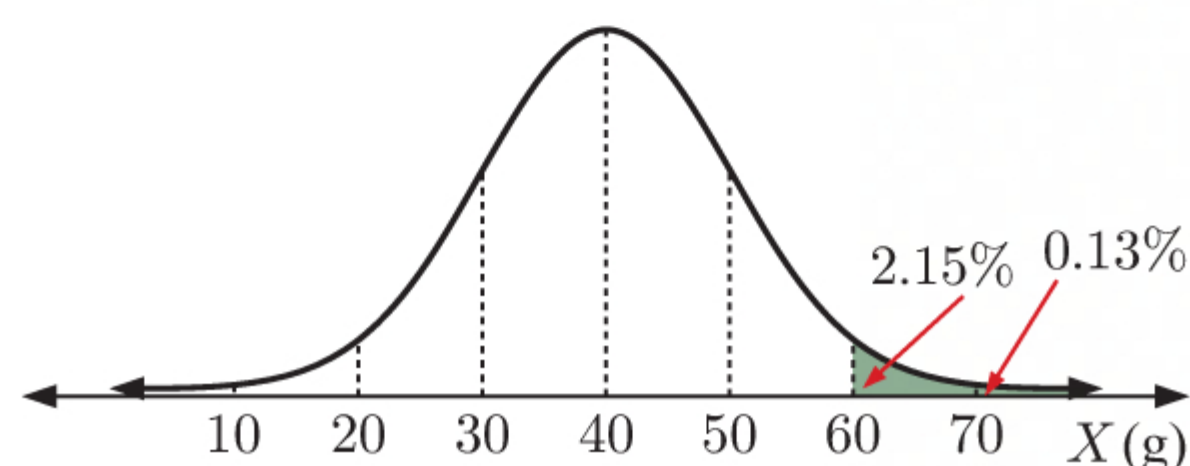
$$\therefore P(\text{radish grown without fertiliser weighs between 20 g and 60 g}) \approx 0.9544$$

- c About $13.59\% + 34.13\% + 50\% = 97.72\%$
 of radishes grown with fertiliser will have
 weights more than 60 g.



$$\therefore P(\text{radish grown with fertiliser weighs more than 60 g}) \approx 0.9772$$

About $2.15\% + 0.13\% = 2.28\%$ of radishes
 grown without fertiliser will have weights
 more than 60 g.

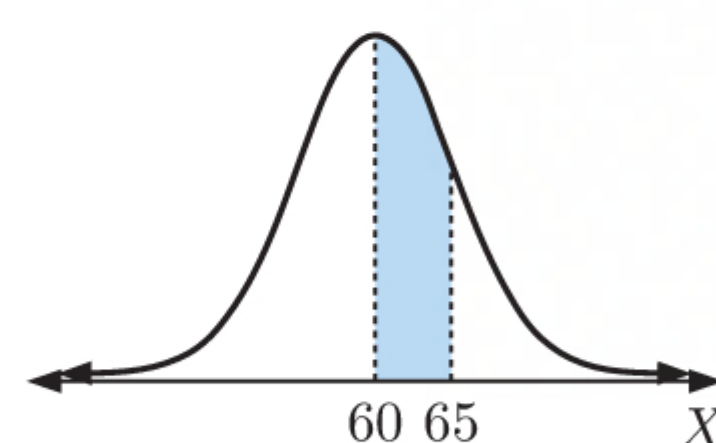
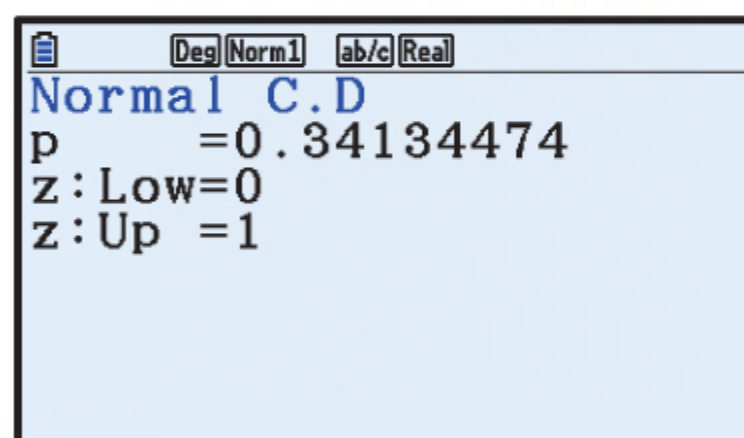
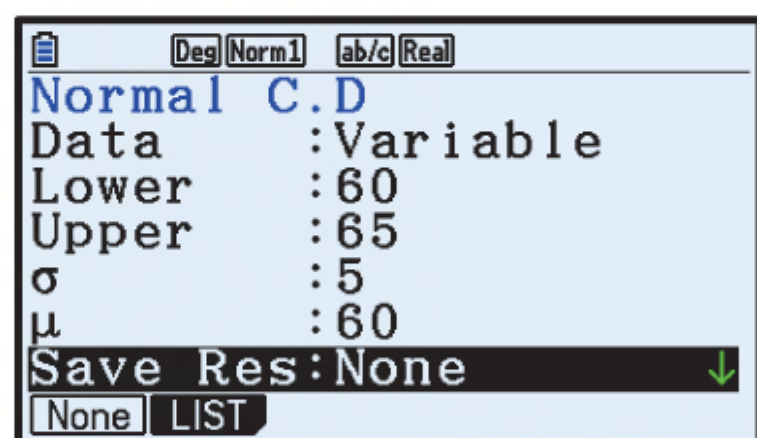


$$\therefore P(\text{radish grown without fertiliser weighs more than 60 g}) \approx 0.0228$$

$$\begin{aligned} &P(\text{both radishes weigh more than 60 g}) \\ &= P(\text{radish grown with fertiliser weighs more than 60 g}) \\ &\quad \times P(\text{radish grown without fertiliser weighs more than 60 g}) \\ &\approx 0.9772 \times 0.0228 \\ &\approx 0.0223 \end{aligned}$$

EXERCISE 21B.2

- 1 a To find $P(60 \leq X \leq 65)$, we set the lower bound to 60 and the upper bound to 65.



$$P(60 \leq X \leq 65) \approx 0.341$$

- b** To find $P(62 \leq X \leq 67)$, we set the lower bound to 62 and the upper bound to 67.

```

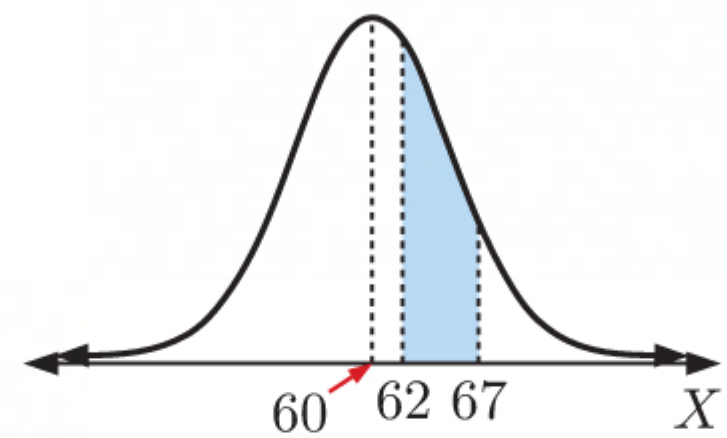
Normal C.D
Data      :Variable
Lower     :62
Upper     :67
σ         :5
μ         :60
Save Res:None
[None] LIST

```

```

Normal C.D
p         =0.26382159
z:Low     =0.4
z:Up      =1.4

```



$$P(62 \leq X \leq 67) \approx 0.264$$

- c** To find $P(X \geq 64)$, we use a very high value such as 10^{99} to represent the upper bound.

```

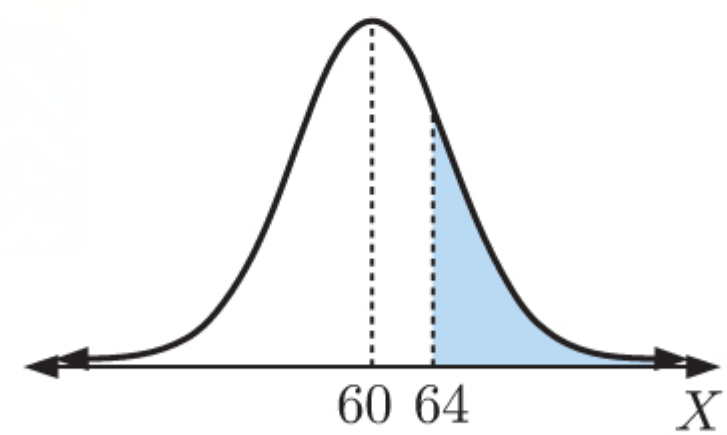
Normal C.D
Data      :Variable
Lower     :64
Upper     :1E+99
σ         :5
μ         :60
Save Res:None
[None] LIST

```

```

Normal C.D
p         =0.21185539
z:Low     =0.8
z:Up      =2E+98

```



$$P(X \geq 64) \approx 0.212$$

- d** To find $P(X \leq 68)$, we use a very low value such as -10^{99} to represent the lower bound.

```

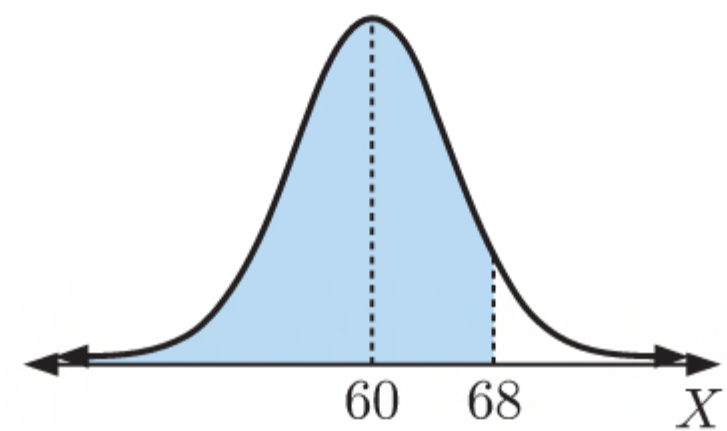
Normal C.D
Data      :Variable
Lower     :-1E+99
Upper     :68
σ         :5
μ         :60
Save Res:None
[None] LIST

```

```

Normal C.D
p         =0.9452007
z:Low     =-2E+98
z:Up      =1.6

```



$$P(X \leq 68) \approx 0.945$$

- e** To find $P(X \leq 61)$, we use a very low value such as -10^{99} to represent the lower bound.

```

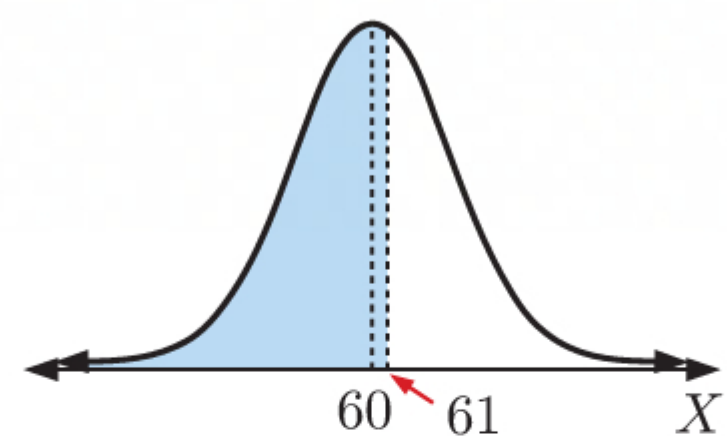
Normal C.D
Data      :Variable
Lower     :-1E+99
Upper     :61
σ         :5
μ         :60
Save Res:None
[None] LIST

```

```

Normal C.D
p         =0.5792597
z:Low     =-2E+98
z:Up      =0.2

```



$$P(X \leq 61) \approx 0.579$$

- f** To find $P(57.5 \leq X \leq 62.5)$, we set the lower bound to 57.5 and the upper bound to 62.5.

```

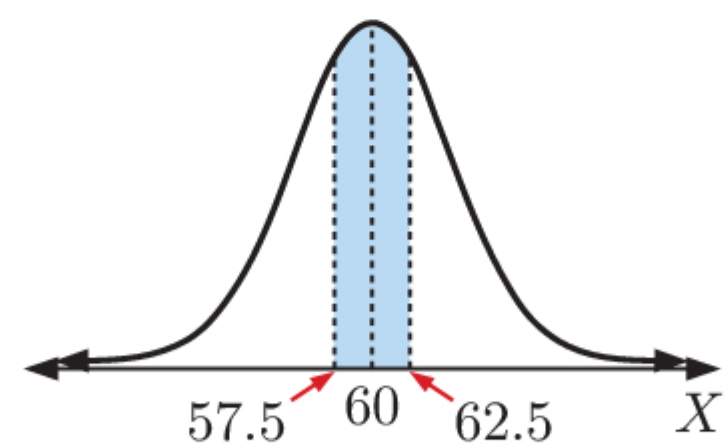
Normal C.D
Data      :Variable
Lower     :57.5
Upper     :62.5
σ         :5
μ         :60
Save Res:None
[None] LIST

```

```

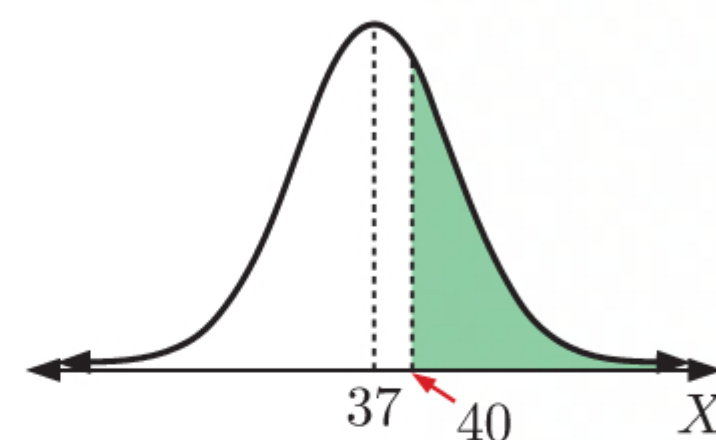
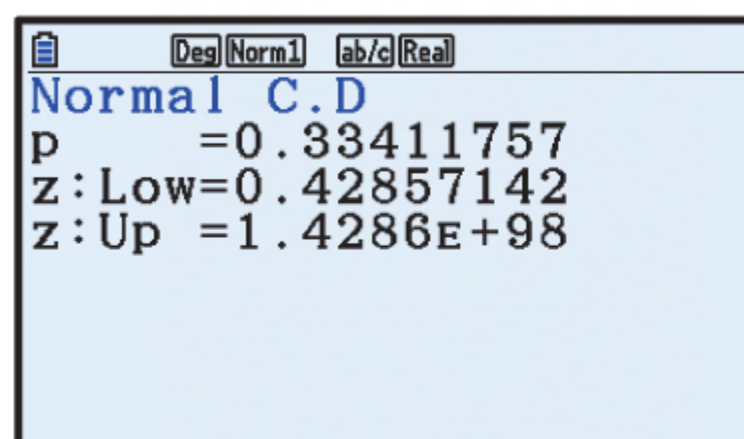
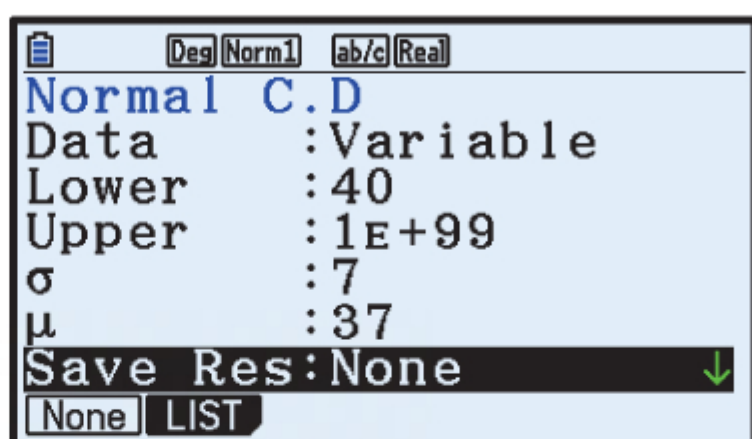
Normal C.D
p         =0.38292492
z:Low     =-0.5
z:Up      =0.5

```



$$P(57.5 \leq X \leq 62.5) \approx 0.383$$

- 2 a To find $P(X > 40)$, we use a very high value such as 10^{99} to represent the upper bound.

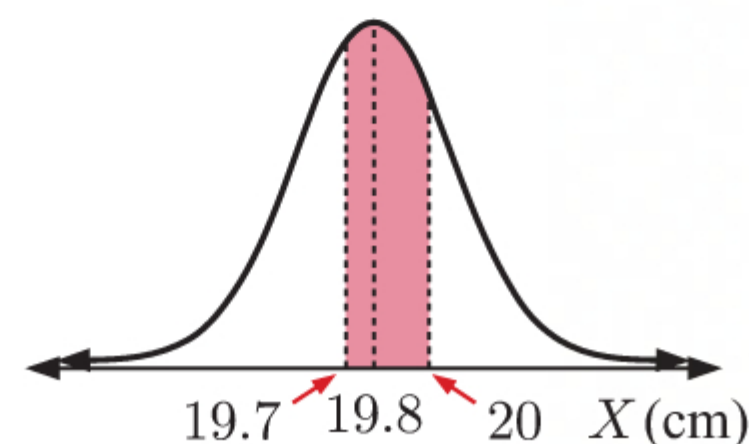
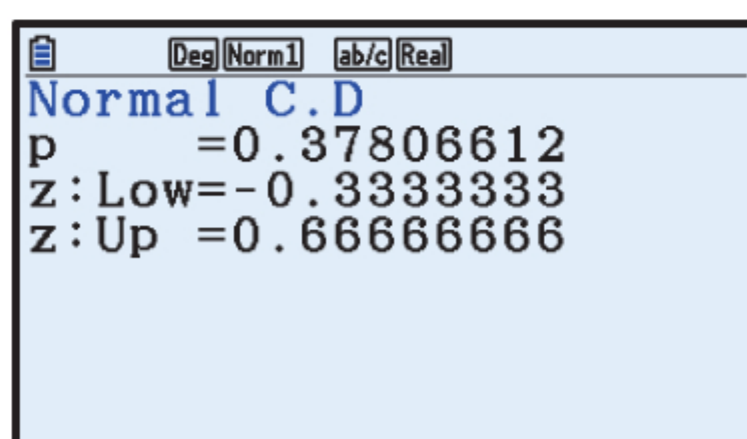
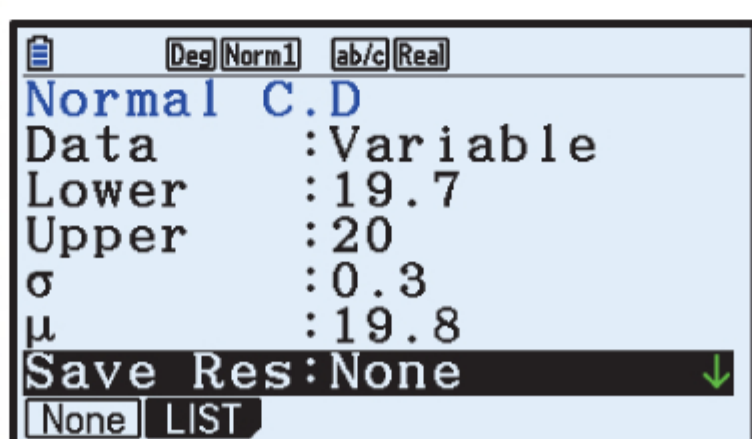


$$P(X > 40) \approx 0.334$$

- b Since the mean of the distribution is 37, then $P(X > 37) = 0.5$

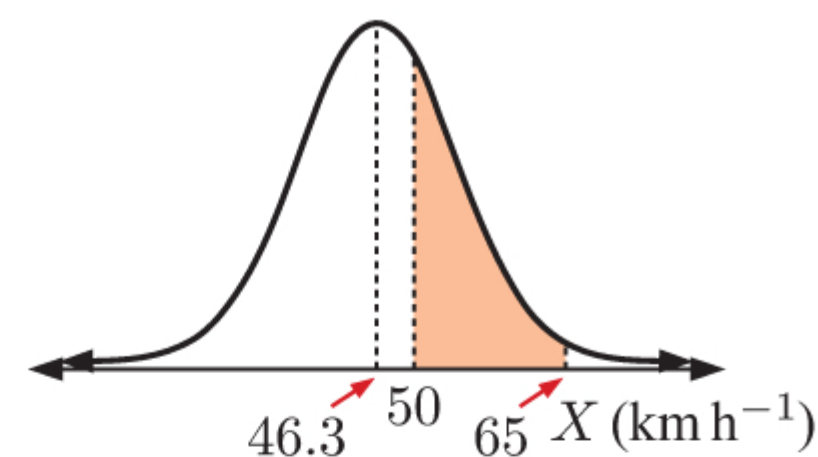
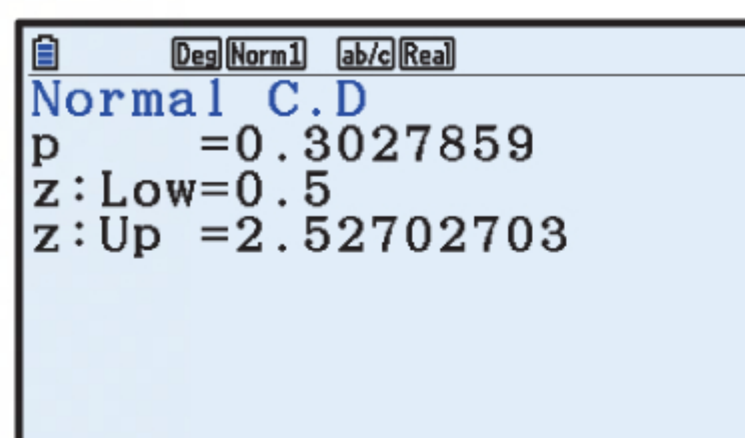
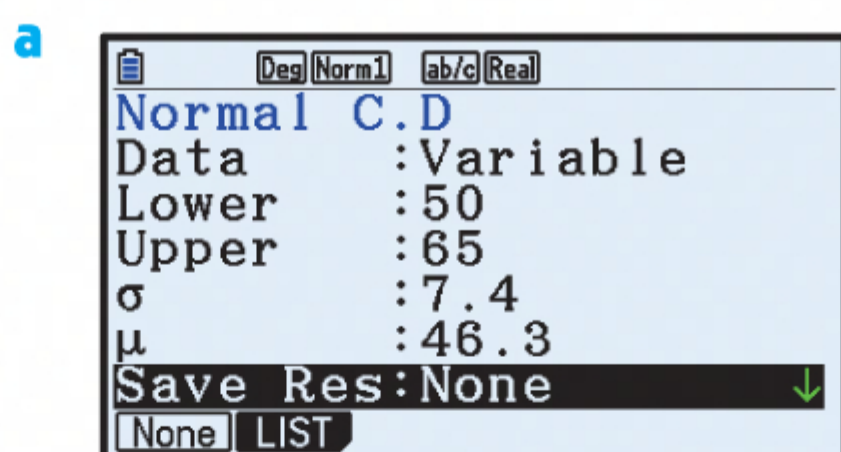
$$\begin{aligned} \therefore P(37 \leq X \leq 40) &= P(X > 37) - P(X > 40) \\ &\approx 0.5 - 0.334 \\ &\approx 0.166 \end{aligned}$$

- 3 Let X cm be the length of a randomly selected bolt.

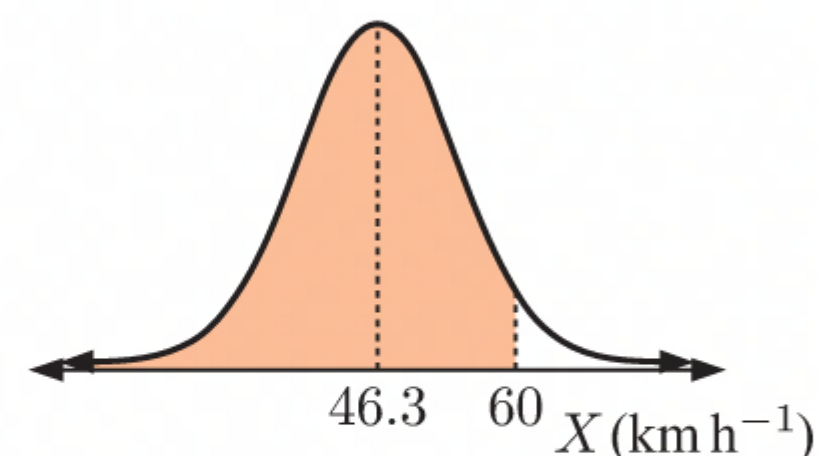
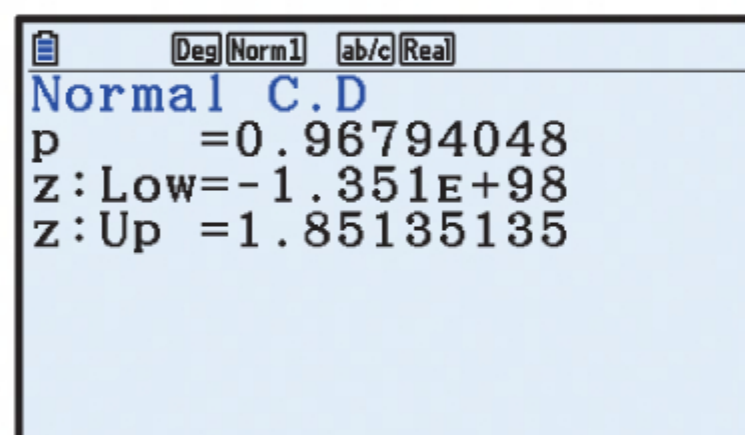
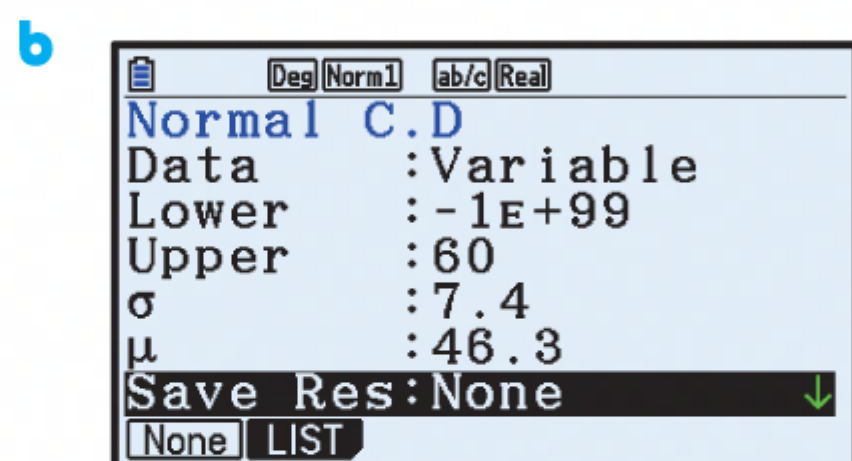


$$P(19.7 < X < 20) \approx 0.378$$

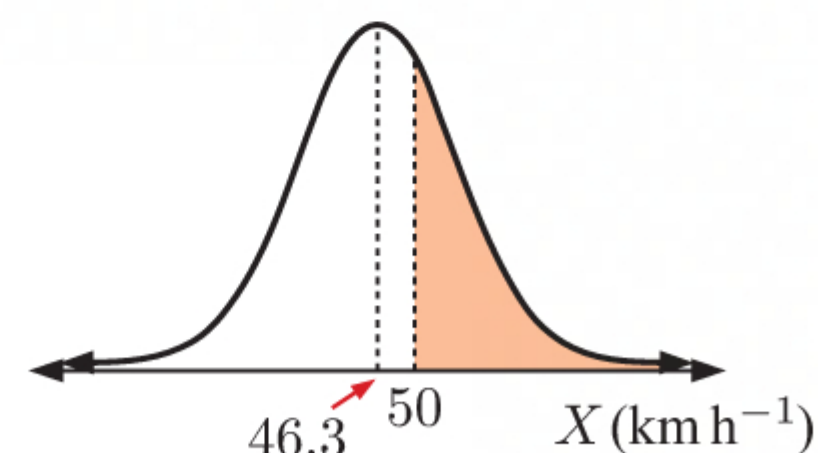
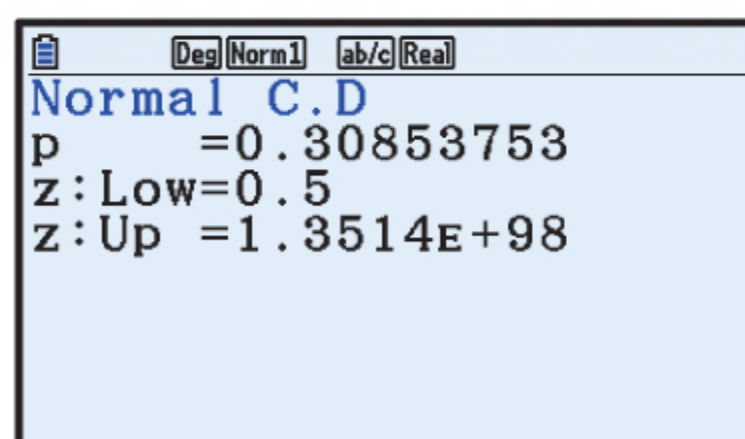
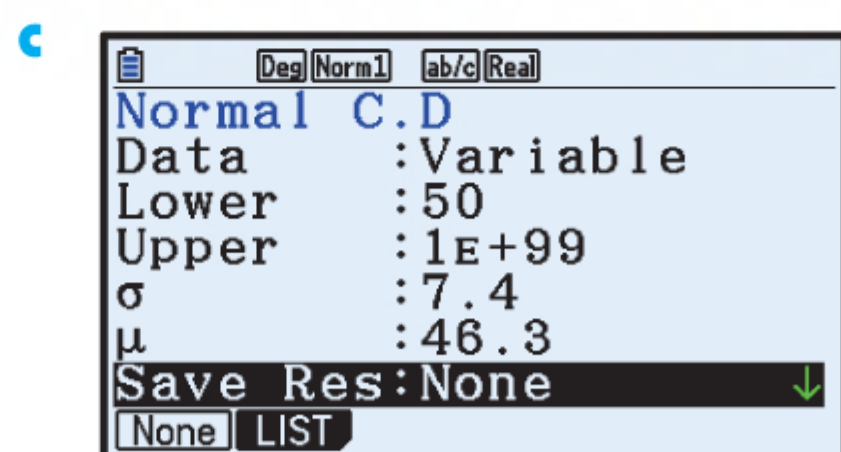
- 4 Let X km h⁻¹ be the speed of a randomly selected car.



$$P(50 < X < 65) \approx 0.303$$



$$P(X < 60) \approx 0.968$$



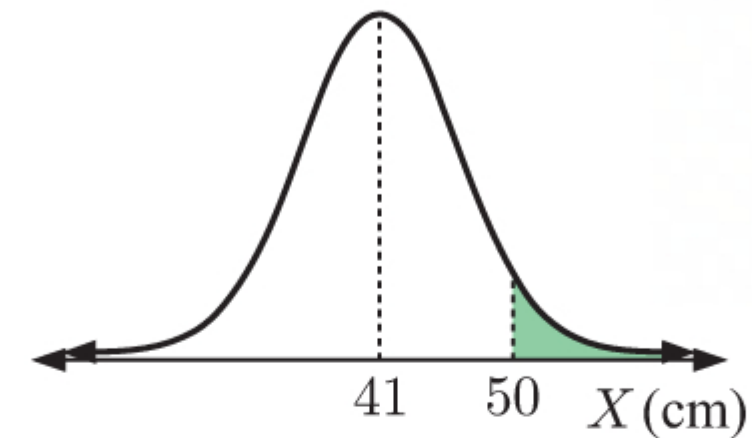
$$P(X > 50) \approx 0.309$$

- 5 Let X cm be the length of a randomly selected eel.

a

Normal C.D
Data : Variable
Lower : 50
Upper : 1E+99
σ : 5.5
μ : 41
Save Res: None
None LIST

Normal C.D
p = 0.05088175
z: Low = 1.63636364
z: Up = 1.8182E+98

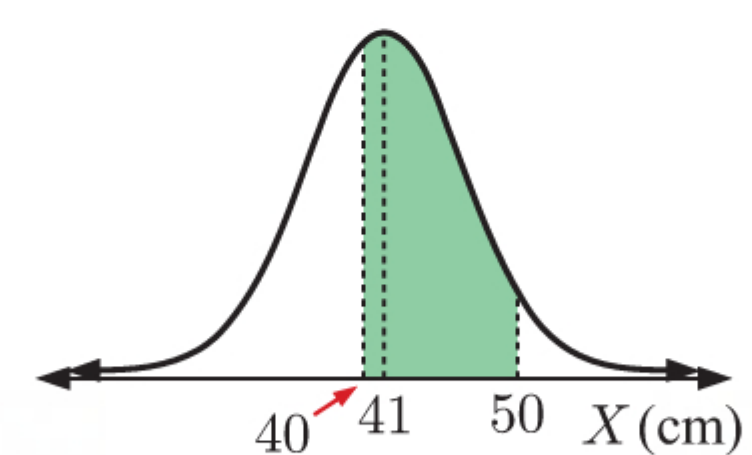


$$P(X \geq 50) \approx 0.0509$$

b

Normal C.D
Data : Variable
Lower : 40
Upper : 50
σ : 5.5
μ : 41
Save Res: None
None LIST

Normal C.D
p = 0.52125554
z: Low = -0.1818181
z: Up = 1.63636364



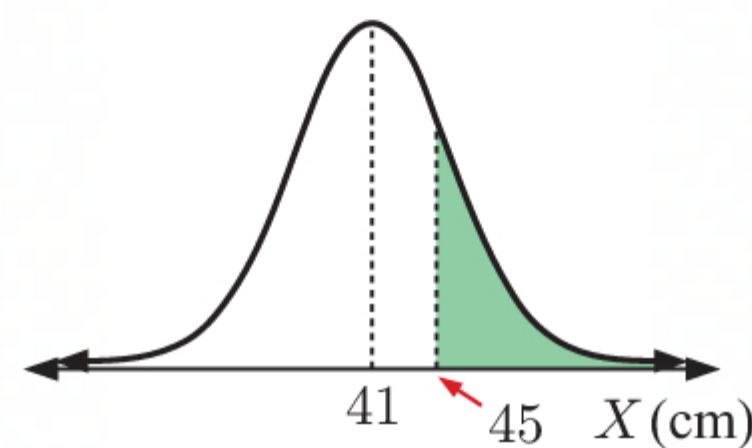
$$P(40 < X < 50) \approx 0.521$$

\therefore about 52.1% of eels measure between 40 cm and 50 cm long.

c

Normal C.D
Data : Variable
Lower : 45
Upper : 1E+99
σ : 5.5
μ : 41
Save Res: None
None LIST

Normal C.D
p = 0.23352945
z: Low = 0.72727272
z: Up = 1.8182E+98



$$P(X \geq 45) \approx 0.234$$

\therefore we would expect about $0.234 \times 200 \approx 47$ eels to measure at least 45 cm in length.

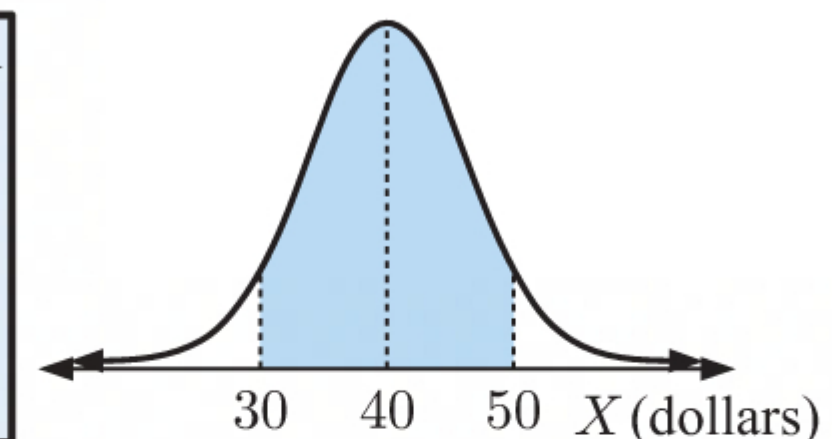
- 6 Let $\$X$ be the amount collected by Max in a randomly selected week.

a

i

Normal C.D
Data : Variable
Lower : 30
Upper : 50
σ : 6
μ : 40
Save Res: None
None LIST

Normal C.D
p = 0.90441929
z: Low = -1.66666667
z: Up = 1.66666667



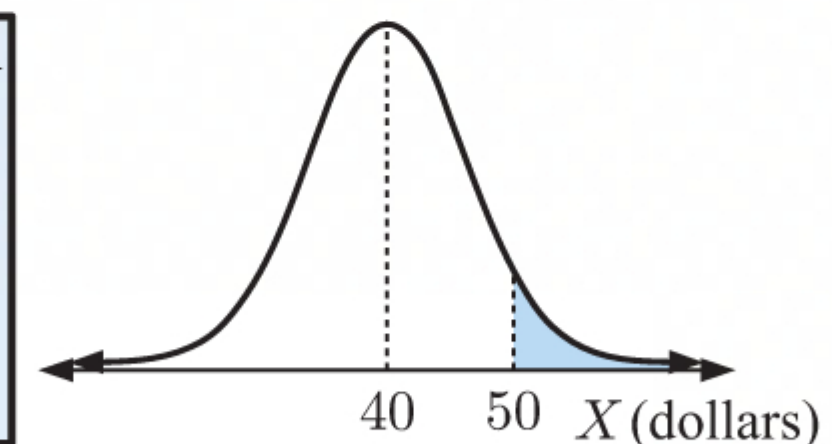
$$P(30 < X < 50) \approx 0.904$$

\therefore Max would expect to collect between \$30 and \$50 in about 90.4% of weeks.

ii

Normal C.D
Data : Variable
Lower : 50
Upper : 1E+99
σ : 6
μ : 40
Save Res: None
None LIST

Normal C.D
p = 0.04779035
z: Low = 1.66666667
z: Up = 1.6667E+98

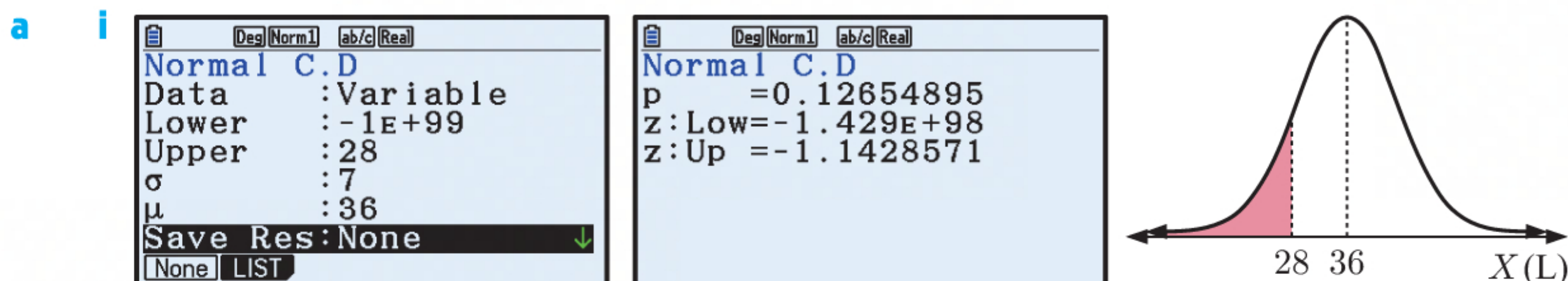


$$P(X \geq 50) \approx 0.0478$$

\therefore Max would expect to collect at least \$50 in about 4.78% of weeks.

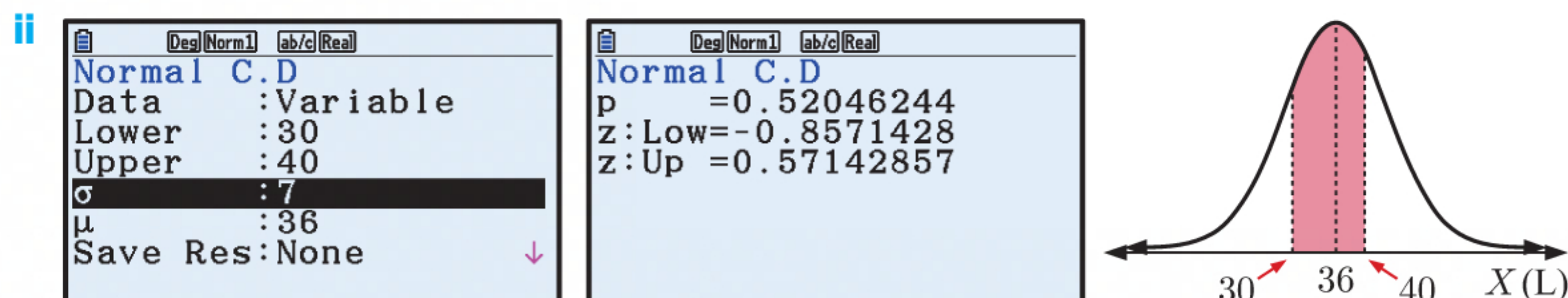
- b There are about 52 weeks in a year, and the average weekly collection is \$40, so in 2 years we would expect Max to collect about $2 \times 52 \times \$40 = \4160 .

7 Let X L be the amount of petrol bought by a randomly selected customer.



$$P(X < 28) \approx 0.127$$

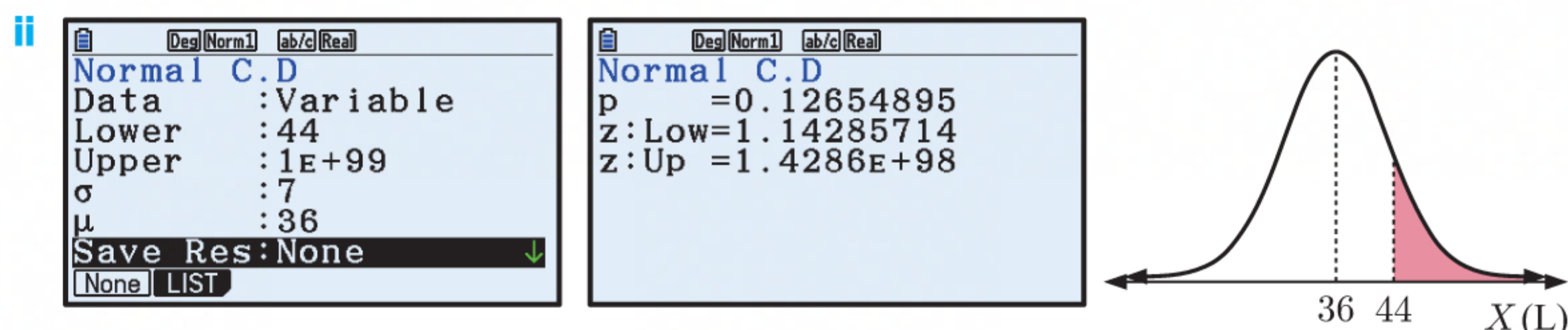
\therefore about 12.7% of customers buy less than 28 L of petrol.



$$P(30 < X < 40) \approx 0.520$$

\therefore about 52.0% of customers buy between 30 L and 40 L of petrol.

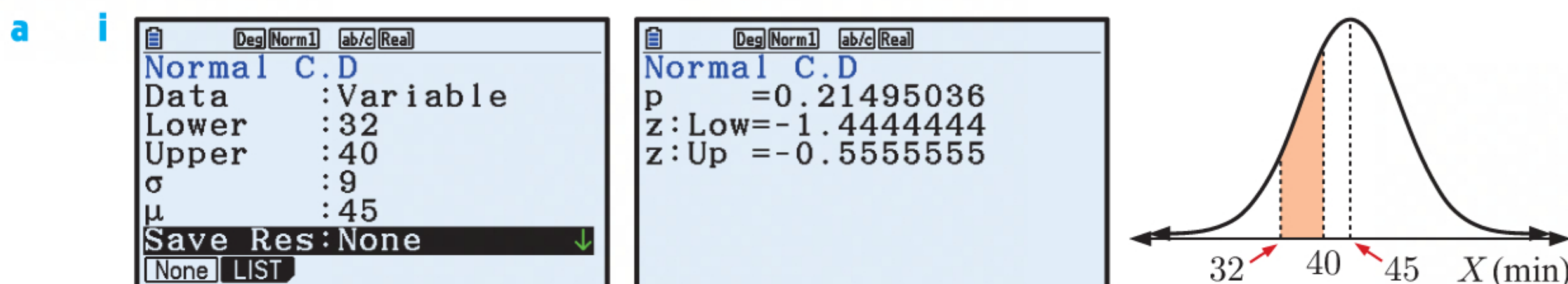
b i We would expect the petrol station to sell about $36 \text{ L} \times 600 = 21.6 \text{ kL}$ of petrol.



$$P(X \geq 44) \approx 0.127$$

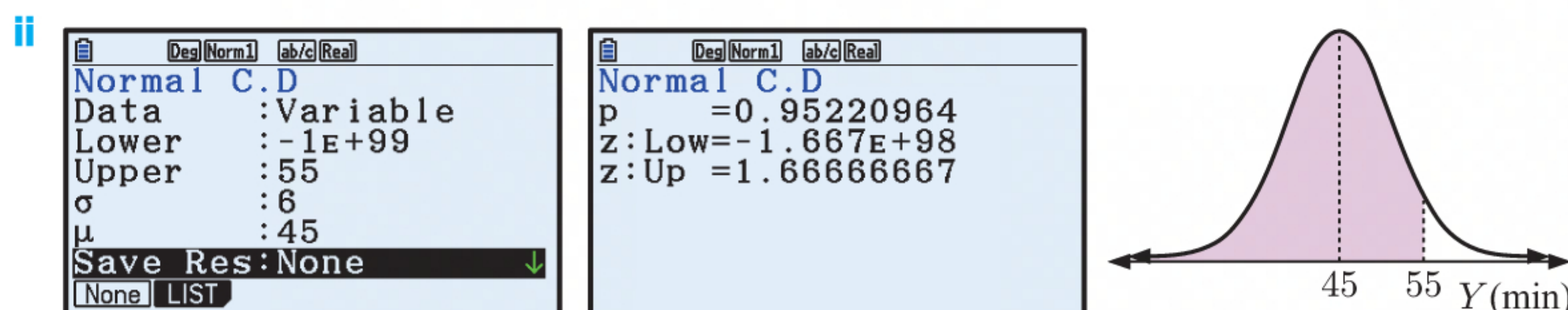
\therefore we would expect about $0.127 \times 600 \approx 76$ customers to buy at least 44 L of petrol.

8 Let X minutes be the amount of time Enrique spends at the gym, and Y minutes be the amount of time Damien spends at the gym.



$$P(32 < X < 40) \approx 0.215$$

\therefore Enrique will spend between 32 and 40 minutes at the gym on about 21.5% of days.



$$P(Y < 55) \approx 0.952$$

\therefore Damien will spend less than 55 minutes at the gym on about 95.2% of days.

- b** **i** Enrique is more likely to spend at least 1 hour at the gym. The mean of both of their times is 45 minutes, but Enrique has a greater standard deviation, and so is more likely to exceed 1 hour.
- ii** Damien is more likely to spend between 40 and 50 minutes at the gym. Damien has the smaller standard deviation and is more likely to stay between 40 and 50 minutes, which is close to the mean of 45 minutes.

c **i** Enrique:

```

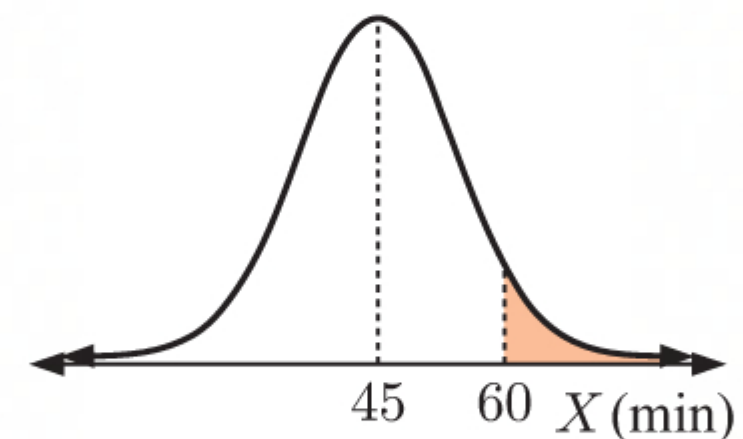
Normal C.D
Data :Variable
Lower :60
Upper :1E+99
σ :9
μ :45
Save Res:None
None LIST

```

```

Normal C.D
p =0.04779035
z:Low=1.66666667
z:Up =1.1111E+98

```



$$P(X > 60) \approx 0.0478$$

Damien:

```

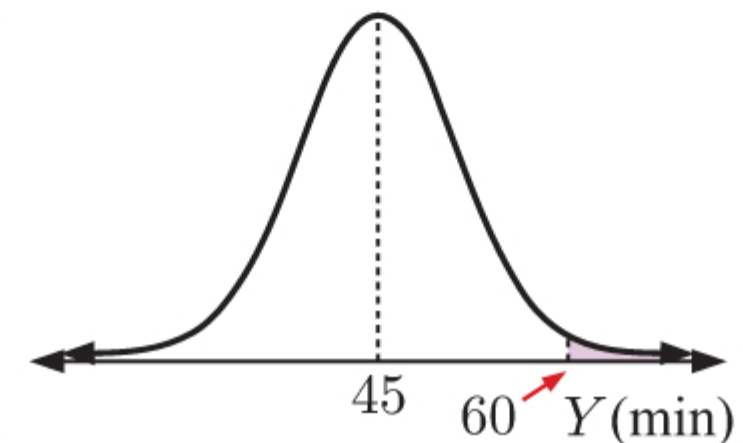
Normal C.D
Data :Variable
Lower :60
Upper :1E+99
σ :6
μ :45
Save Res:None
None LIST

```

```

Normal C.D
p =6.2097E-03
z:Low=2.5
z:Up =1.6667E+98

```



$$P(Y > 60) \approx 0.00621$$

\therefore Enrique is more likely to spend at least 1 hour at the gym.

ii Enrique:

```

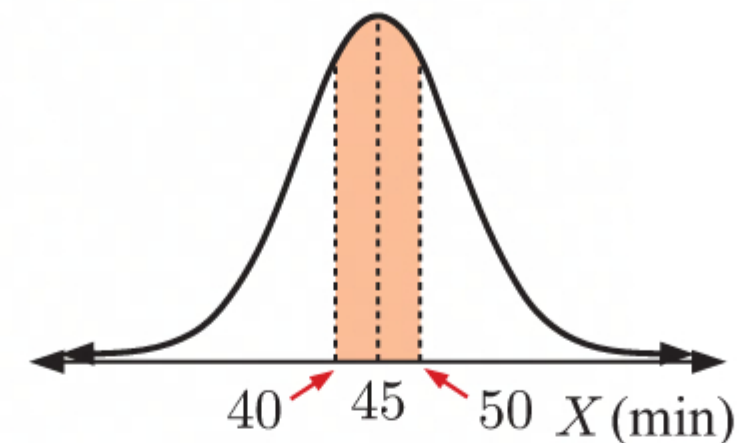
Normal C.D
Data :Variable
Lower :40
Upper :50
σ :9
μ :45
Save Res:None
None LIST

```

```

Normal C.D
p =0.42148527
z:Low=-0.5555555
z:Up =0.55555555

```



$$P(40 < X < 50) \approx 0.421$$

Damien:

```

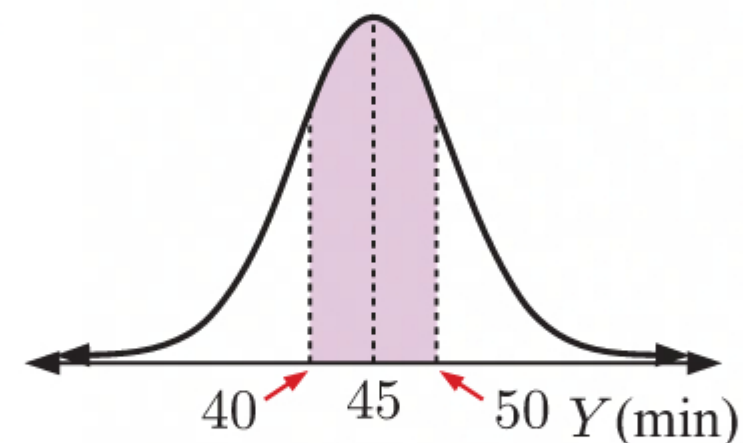
Normal C.D
Data :Variable
Lower :40
Upper :50
σ :6
μ :45
Save Res:None
None LIST

```

```

Normal C.D
p =0.59534323
z:Low=-0.8333333
z:Up =0.83333333

```



$$P(40 < Y < 50) \approx 0.595$$

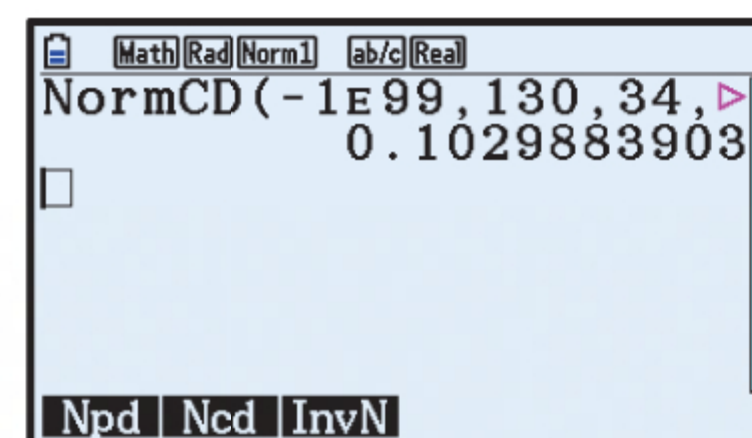
\therefore Damien is more likely to spend between 40 and 50 minutes at the gym.

- 9 a Let X grams be the weight of a randomly selected apple.

$$X \sim N(173, 34^2)$$

$$\therefore P(X < 130) \approx 0.10299 \approx 0.103$$

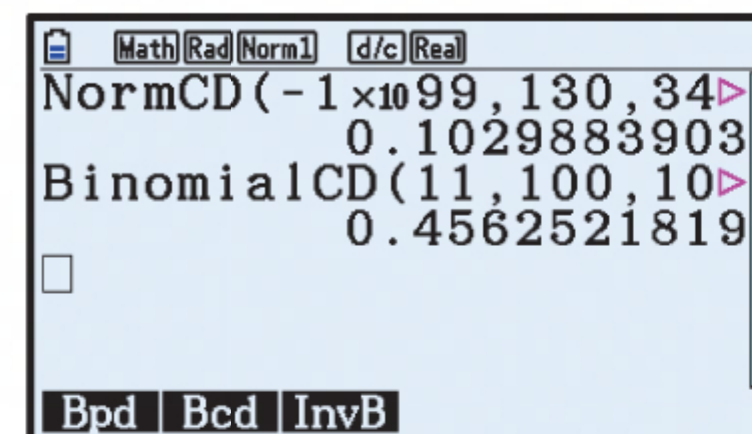
\therefore about 10.3% of apples from the crop were too small to sell.



- b Let Y be the number of apples which were too small to sell.

$$Y \sim B(100, 0.10299)$$

$$\therefore P(Y > 10) = P(Y \geq 11) \approx 0.456$$

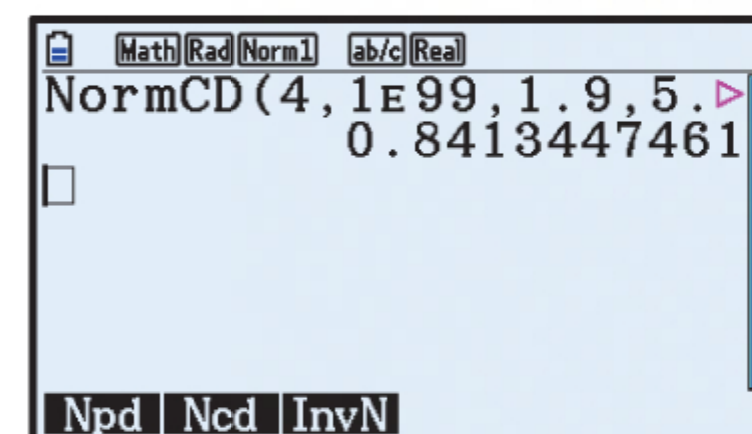


- 10 a Let X units be the drop in blood pressure of a randomly selected patient.

$$X \sim N(5.9, 1.9^2)$$

$$\therefore P(X > 4) \approx 0.84134 \approx 0.841$$

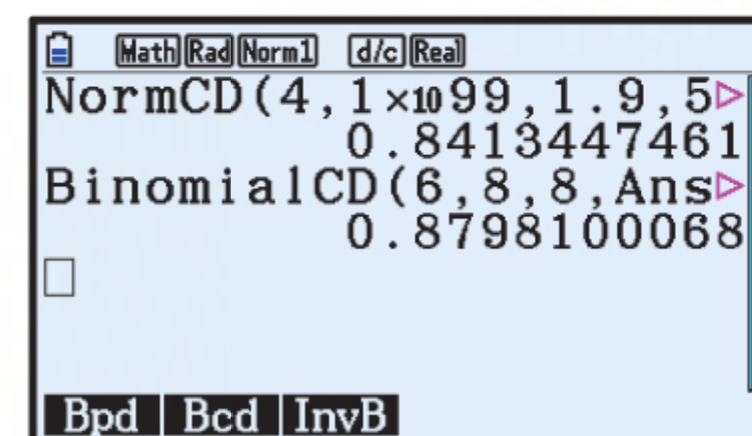
\therefore about 84.1% of patients show a drop of more than 4 units.



- b Let Y be the number of patients with a drop of more than 4 units.

$$Y \sim B(8, 0.84134)$$

$$\therefore P(Y \geq 6) \approx 0.880$$

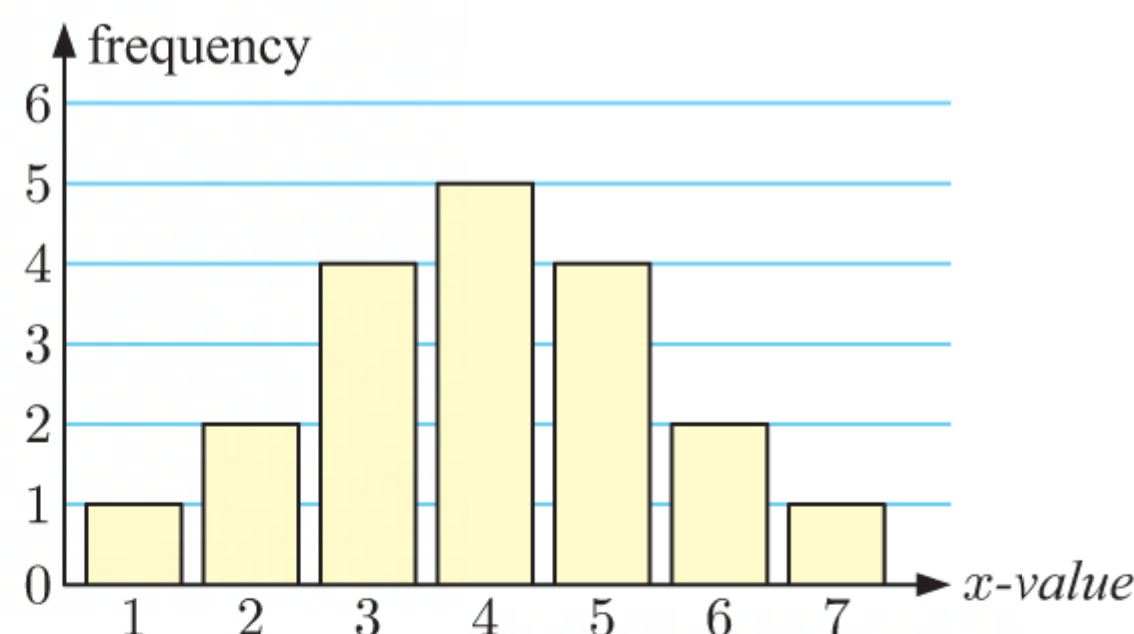


INVESTIGATION 3

z-SCORES

- 1 a

x -value	Frequency
1	1
2	2
3	4
4	5
5	4
6	2
7	1



- b

1-Variable	
\bar{x}	=4
Σx	=76
Σx^2	=346
σx	=1.48678388
sx	=1.52752523
n	=19

So, $\mu = 4$, $\sigma \approx 1.49$

- c We calculate $z = \frac{x - \mu}{\sigma}$ to 3 significant figures.

<i>x</i> -value	1	2	3	4	5	6	7
<i>z</i> -score	−2.02	−1.35	−0.673	0.00	0.673	1.35	2.02

d

<i>z</i> -score	Frequency
−2.02	1
−1.35	2
−0.673	4
0.00	5
0.673	4
1.35	2
2.02	1

\bar{x}	=0
Σx	=0
Σx^2	=19.0000021
σx	=1.0000005
sx	=1.02740239
<i>n</i>	=19

So, the z -scores have mean $\mu = 0$, and standard deviation $\sigma \approx 1$.

- 2 c Both histograms are approximately normally distributed. The histogram of the z -scores appears to be normally distributed with mean 0 and standard deviation 1 for any sample.
- d If the original data is randomly generated from a normal distribution, the z -scores are also normally distributed with mean 0 and standard deviation 1.

If $X \sim N(\mu, \sigma^2)$ and $Z = \frac{X - \mu}{\sigma}$ then $Z \sim N(0, 1^2)$.

EXERCISE 21C.1

- 1 a For English, $z\text{-score} = \frac{48 - 40}{4.4} \approx 1.82$
- For Mandarin, $z\text{-score} = \frac{81 - 60}{9} \approx 2.33$
- For Geography, $z\text{-score} = \frac{84 - 55}{18} \approx 1.61$
- For Biology, $z\text{-score} = \frac{68 - 50}{20} = 0.9$
- For Mathematics, $z\text{-score} = \frac{84 - 50}{15} \approx 2.27$

Subject	Emma's score	μ	σ
English	48	40	4.4
Mandarin	81	60	9
Geography	84	55	18
Biology	68	50	20
Mathematics	84	50	15

- b In order from best to worst: Mandarin, Mathematics, English, Geography, Biology.
- c It is reasonable to compare Emma's performances using z -scores as the scores in each of Emma's classes are normally distributed.

2 a

<i>Subject</i>	<i>Sergio's score</i>	μ	σ
Physics	73%	78%	10.8%
Chemistry	77%	72%	11.6%
Mathematics	76%	74%	10.1%
German	91%	86%	9.6%
Biology	58%	62%	5.2%

For Physics, $z\text{-score} = \frac{73 - 78}{10.8} \approx -0.463$

For Chemistry, $z\text{-score} = \frac{77 - 72}{11.6} \approx 0.431$

For Mathematics, $z\text{-score} = \frac{76 - 74}{10.1} \approx 0.198$

For German, $z\text{-score} = \frac{91 - 86}{9.6} \approx 0.521$

For Biology, $z\text{-score} = \frac{58 - 62}{5.2} \approx -0.769$

b In order from best to worst: German, Chemistry, Mathematics, Physics, Biology.

3

<i>Event</i>	<i>Time (seconds)</i>	μ (seconds)	σ (seconds)
50 m freestyle	32.1	27.8	2.2
100 m backstroke	53.5	58.1	4.3
200 m breaststroke	140.0	143.7	6.4
100 m butterfly	59.6	57.7	5.5

a For 50 m freestyle, $z\text{-score} = \frac{32.1 - 27.8}{2.2} \approx 1.95$

For 100 m backstroke, $z\text{-score} = \frac{53.5 - 58.1}{4.3} \approx -1.07$

For 200 m breaststroke, $z\text{-score} = \frac{140.0 - 143.7}{6.4} \approx -0.578$

For 100 m butterfly, $z\text{-score} = \frac{59.6 - 57.7}{5.5} \approx 0.345$

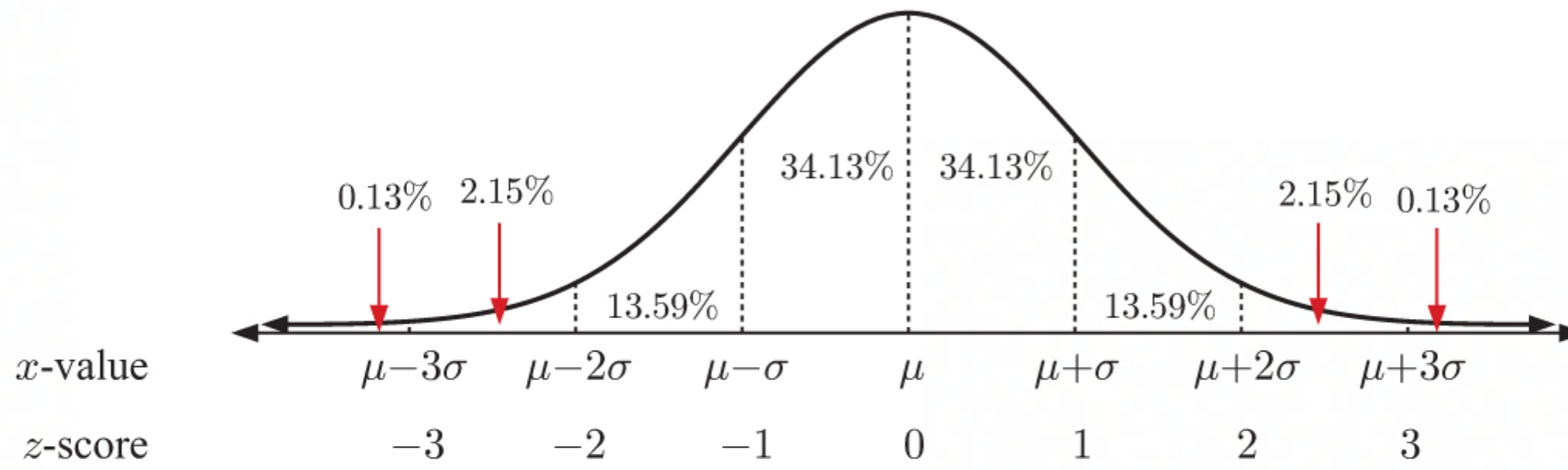
b A lower z -score is better as it indicates that the time is lower, and hence that Frederick swam faster.

c In order from best to worst:

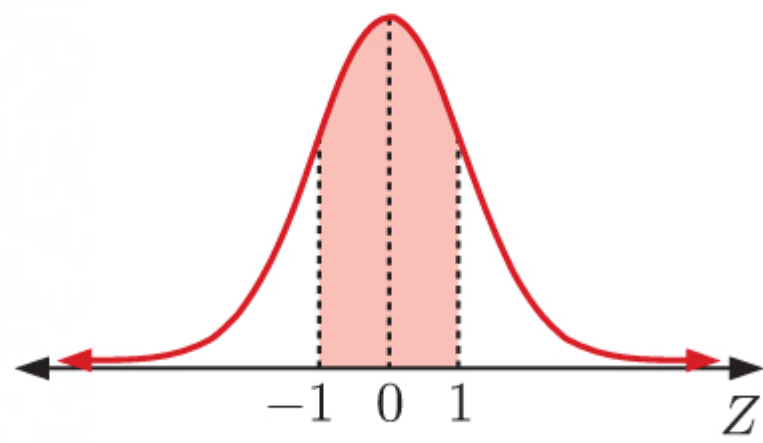
100 m backstroke, 200 m breaststroke, 100 m butterfly, 50 m freestyle.

EXERCISE 21C.2

1

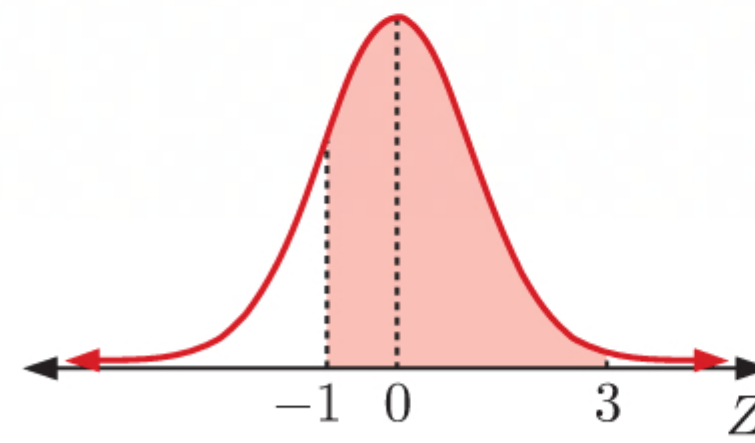


a



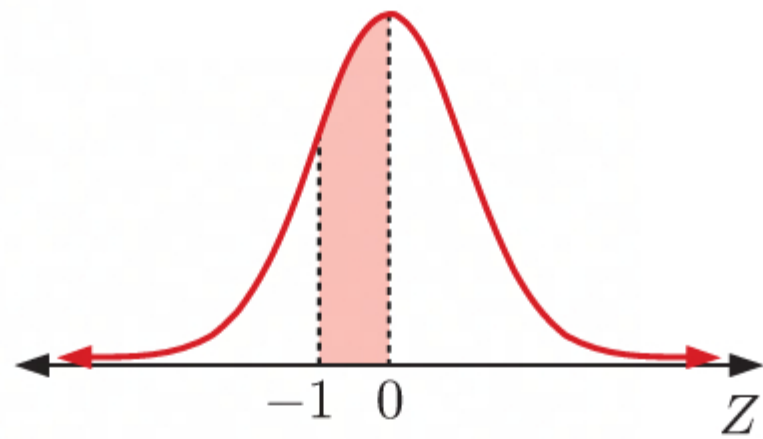
$$\begin{aligned} P(-1 < Z < 1) &\approx 34.13\% + 34.13\% \\ &\approx 68.26\% \\ &\approx 0.683 \end{aligned}$$

b



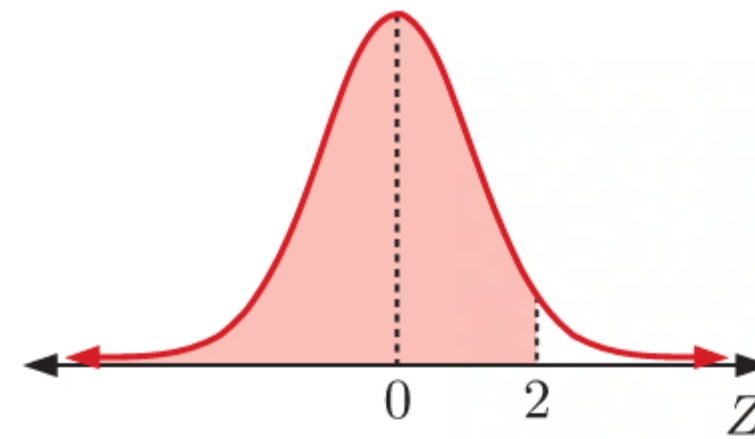
$$\begin{aligned} P(-1 \leq Z \leq 3) &\approx 34.13\% + 34.13\% + 13.59\% + 2.15\% \\ &\approx 84.00\% \\ &\approx 0.840 \end{aligned}$$

c



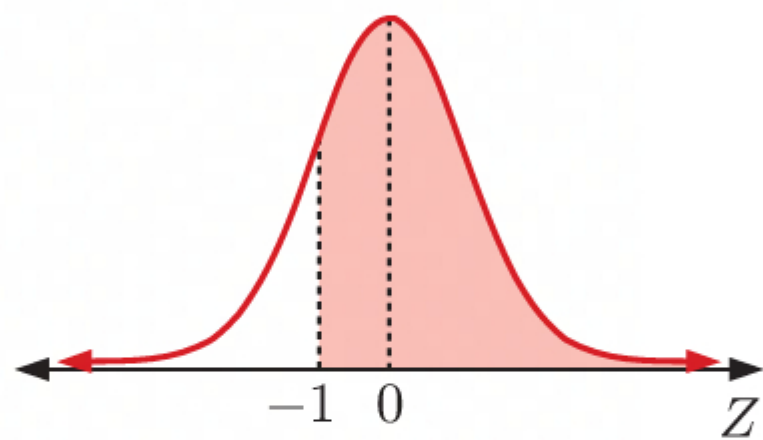
$$\begin{aligned} P(-1 < Z < 0) &\approx 34.13\% \\ &\approx 0.341 \end{aligned}$$

d



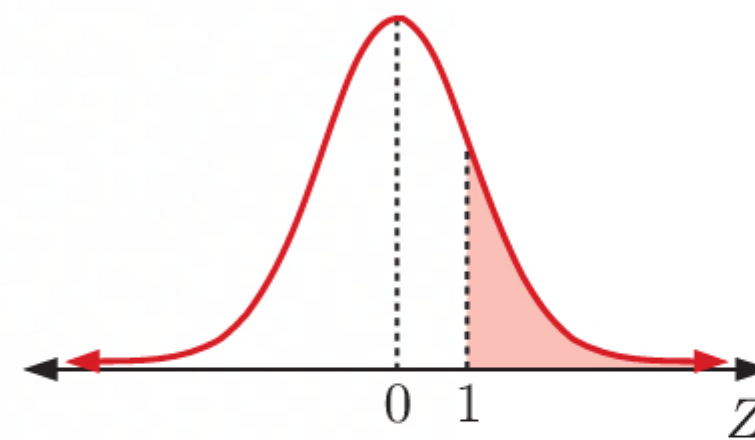
$$\begin{aligned} P(Z < 2) &\approx 50\% + 34.13\% + 13.59\% \\ &\approx 97.72\% \\ &\approx 0.977 \end{aligned}$$

e



$$\begin{aligned} P(-1 < Z) &= P(Z > -1) \\ &\approx 34.13\% + 50\% \\ &\approx 84.13\% \\ &\approx 0.841 \end{aligned}$$

f



$$\begin{aligned} P(Z \geq 1) &\approx 13.59\% + 2.15\% + 0.13\% \\ &\approx 15.87\% \\ &\approx 0.159 \end{aligned}$$

2 a If $P(\mu - \sigma < X < \mu + 2\sigma) = P(a < Z < b)$

$$\text{then } a = \frac{(\mu - \sigma) - \mu}{\sigma} \quad \text{and} \quad b = \frac{(\mu + 2\sigma) - \mu}{\sigma}$$

$$\therefore a = \frac{-\sigma}{\sigma} \qquad \qquad \qquad \therefore b = \frac{2\sigma}{\sigma}$$

$$\qquad \qquad \qquad = -1 \qquad \qquad \qquad \qquad \qquad = 2$$

$$\therefore a = -1, \quad b = 2$$

b If $P(\mu - 0.5\sigma < X < \mu) = P(a < Z < b)$

$$\text{then } a = \frac{(\mu - 0.5\sigma) - \mu}{\sigma} \quad \text{and} \quad b = \frac{\mu - \mu}{\sigma}$$

$$\therefore a = \frac{-0.5\sigma}{\sigma} \qquad \qquad \qquad \therefore b = 0$$

$$\qquad \qquad \qquad = -0.5$$

$$\therefore a = -0.5, \quad b = 0$$

c If $P(0 \leq Z \leq 3) = P(\mu - a\sigma \leq X \leq \mu + b\sigma)$

$$\text{then } \frac{(\mu - a\sigma) - \mu}{\sigma} = 0 \quad \text{and} \quad \frac{(\mu + b\sigma) - \mu}{\sigma} = 3$$

$$\therefore \mu - a\sigma - \mu = 0 \qquad \qquad \qquad \therefore \mu + b\sigma - \mu = 3\sigma$$

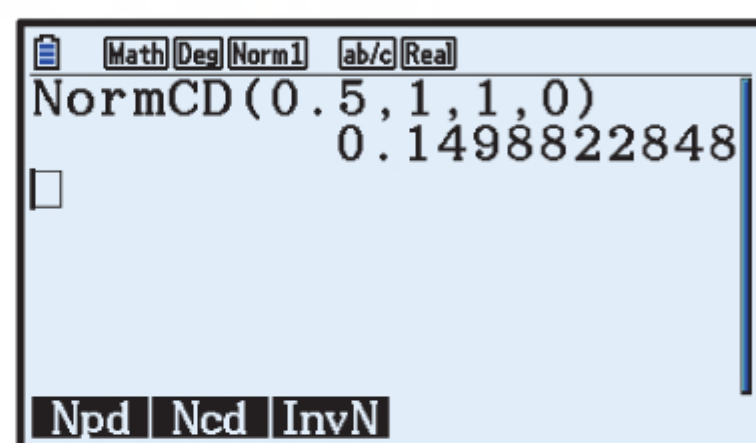
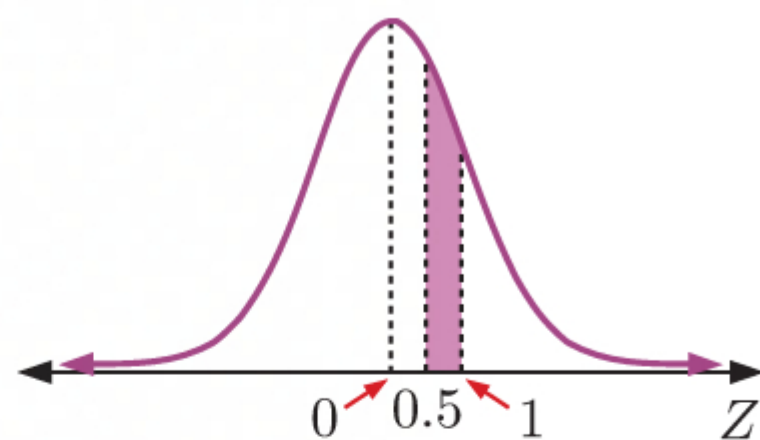
$$\therefore -a\sigma = 0 \qquad \qquad \qquad \therefore b\sigma = 3\sigma$$

$$\therefore a = 0 \qquad \qquad \qquad \therefore b = 3$$

$$\therefore a = 0, \quad b = 3$$

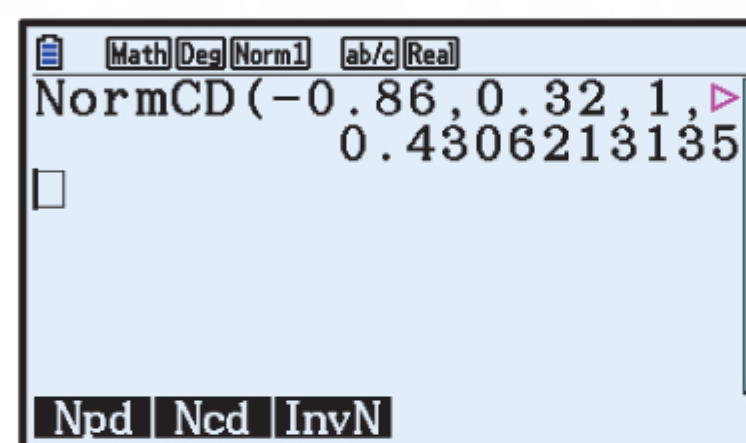
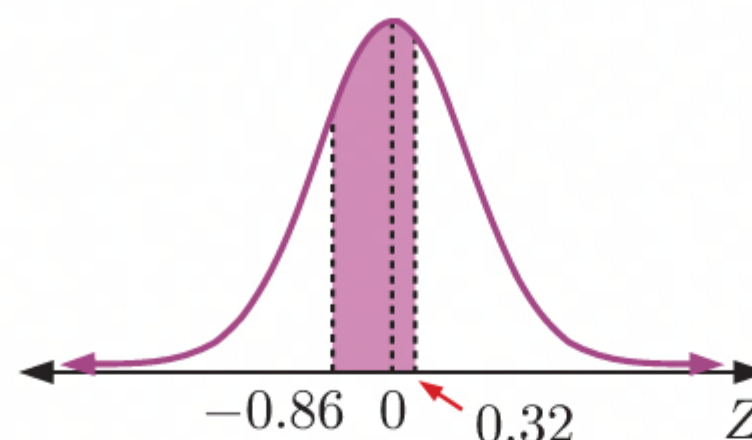
3 $Z \sim N(0, 1^2)$

a



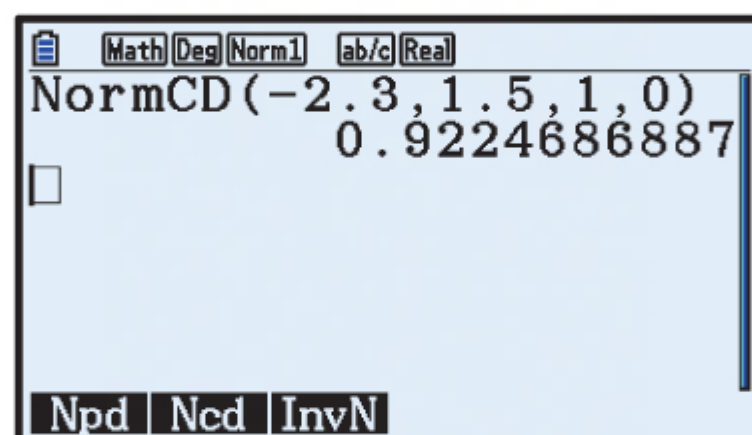
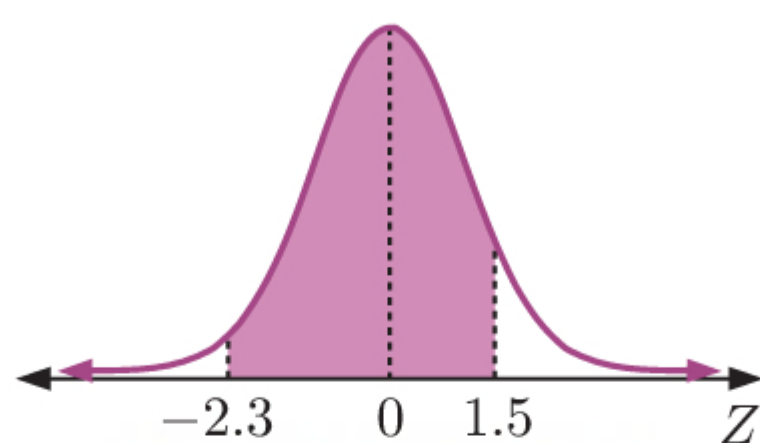
$$P(0.5 \leq Z \leq 1) \approx 0.150$$

b



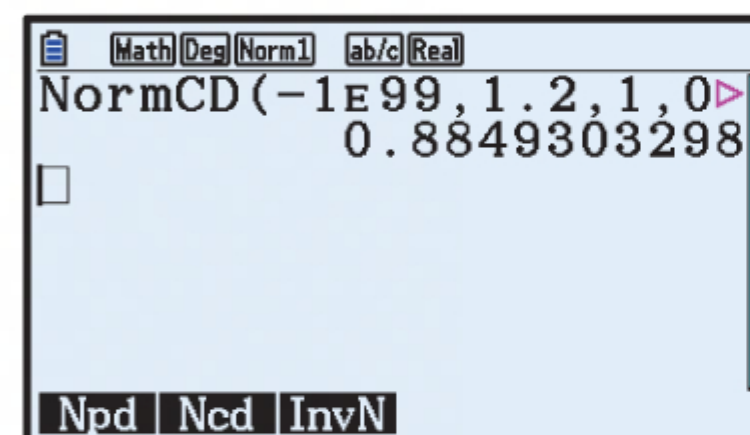
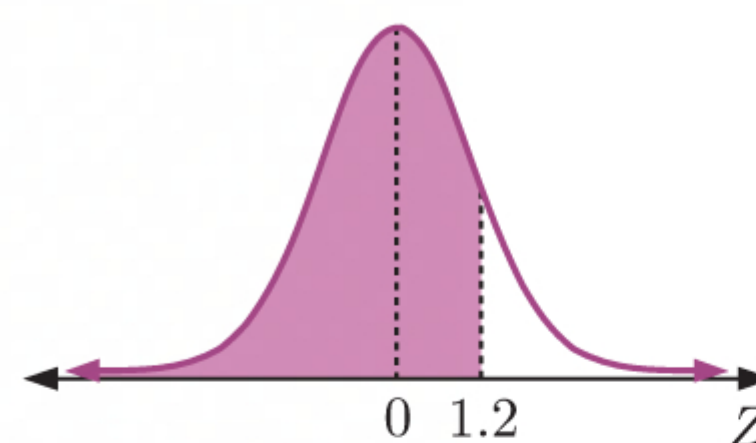
$$P(-0.86 \leq Z \leq 0.32) \approx 0.431$$

c



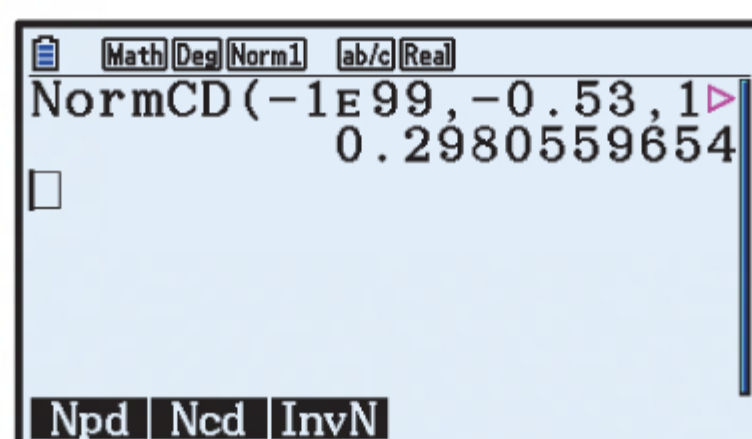
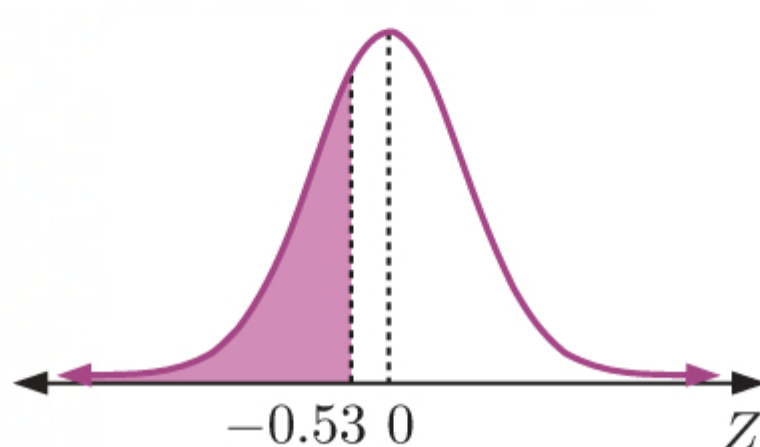
$$P(-2.3 \leq Z \leq 1.5) \approx 0.922$$

d



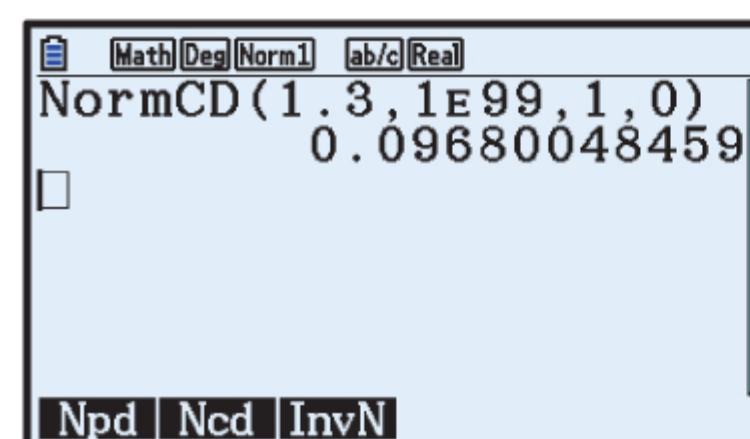
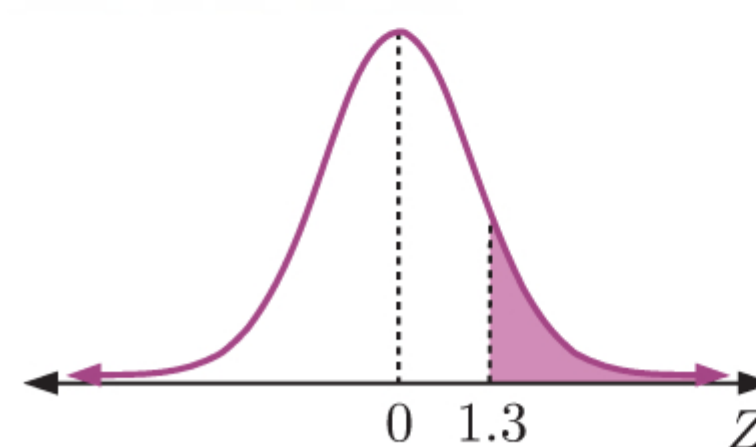
$$P(Z \leq 1.2) \approx 0.885$$

e



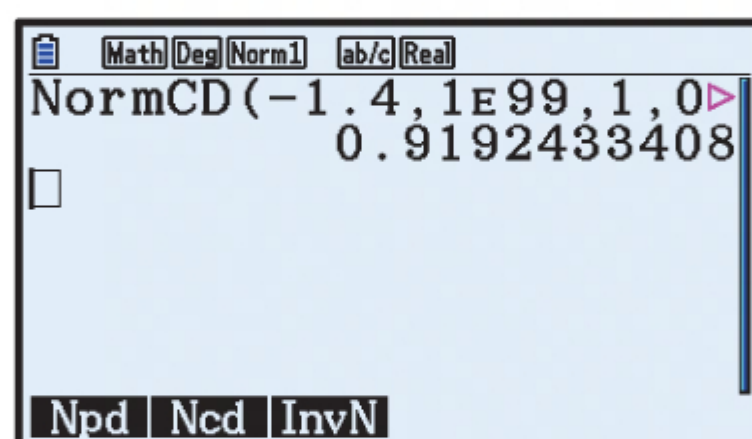
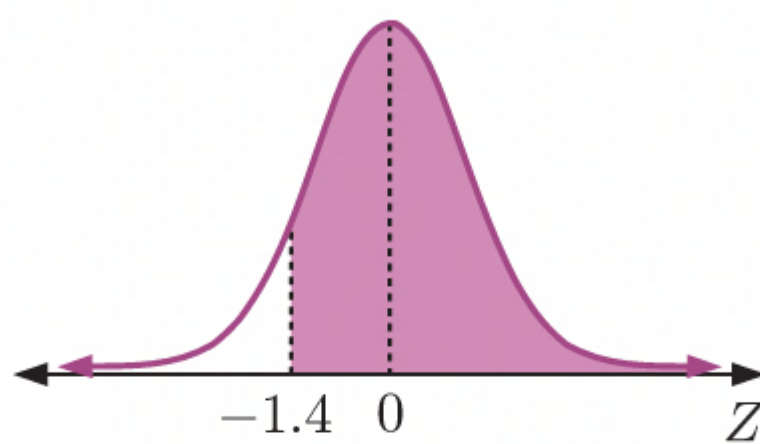
$$P(Z \leq -0.53) \approx 0.298$$

f



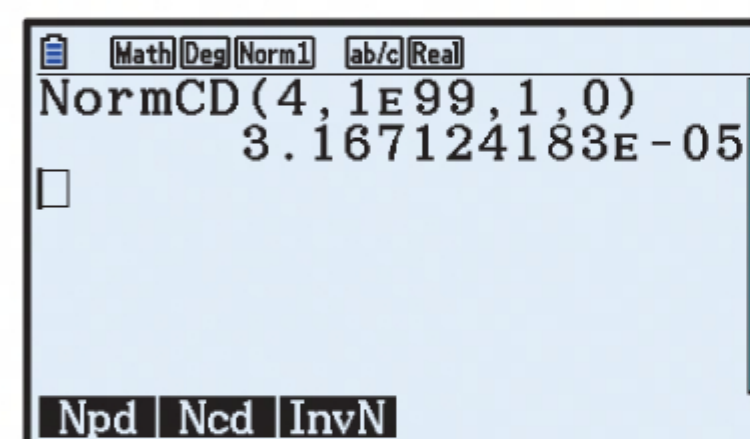
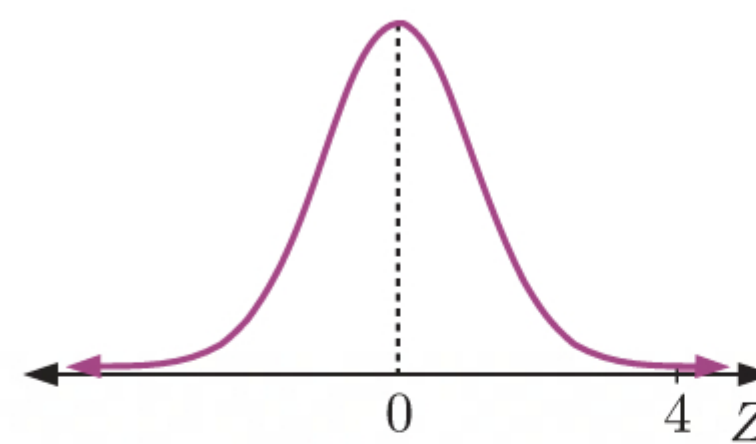
$$P(Z \geq 1.3) \approx 0.0968$$

g

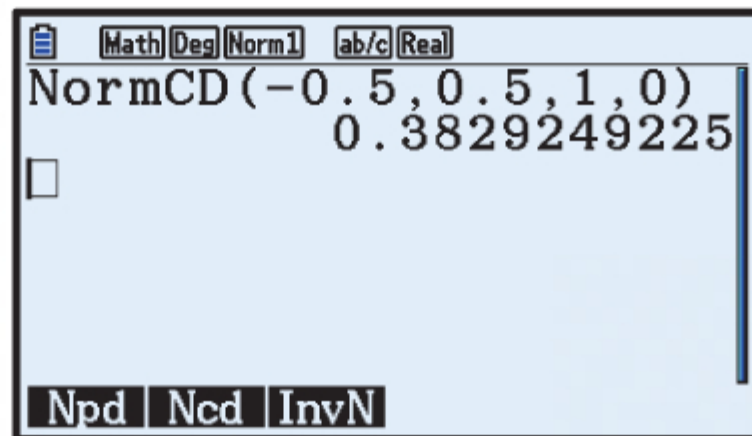
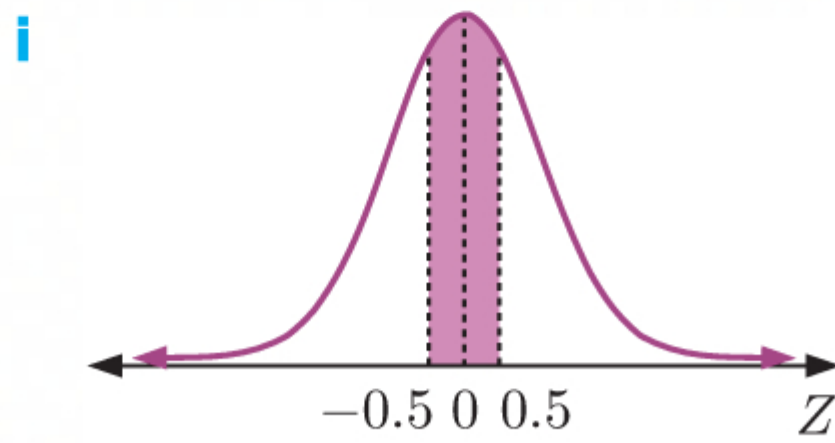


$$P(Z \geq -1.4) \approx 0.919$$

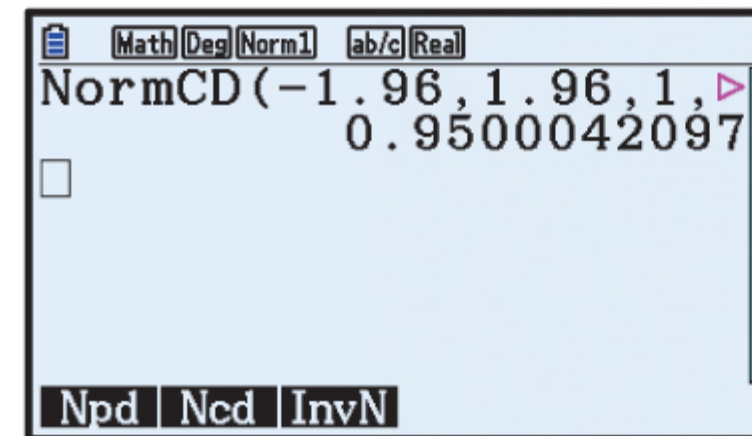
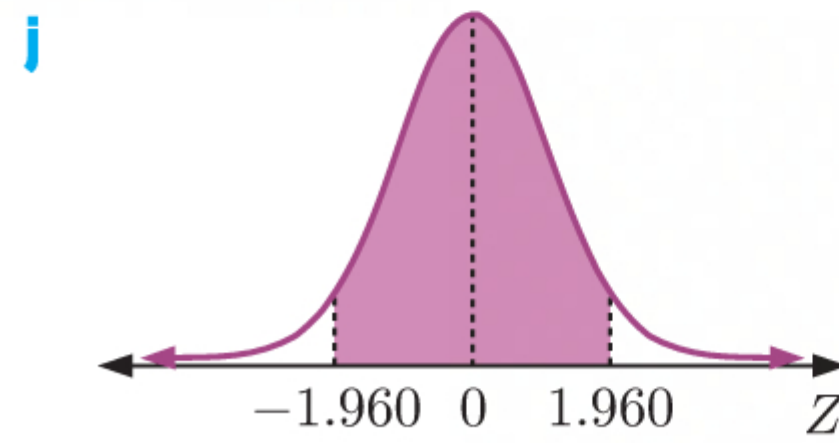
h



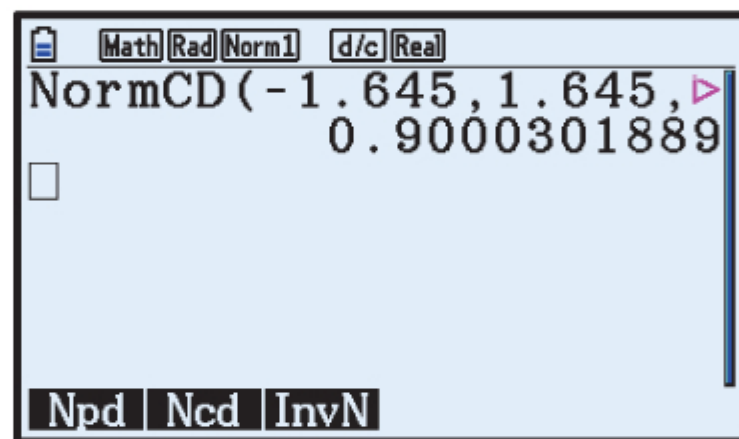
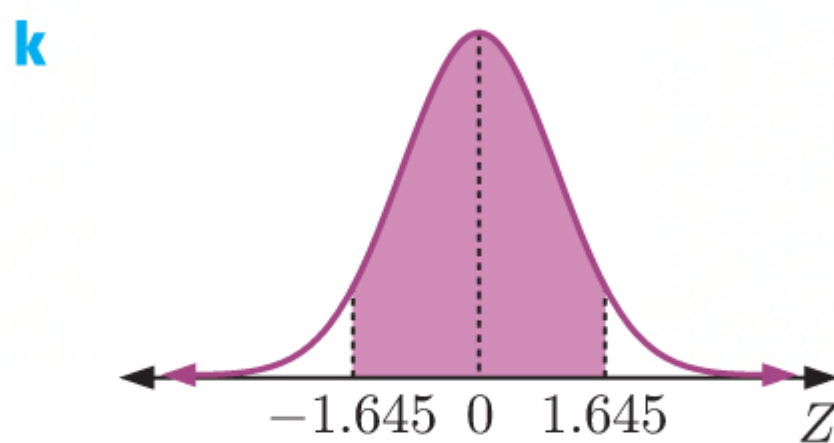
$$P(Z > 4) \approx 0.0000317 \quad (3.17 \times 10^{-5})$$



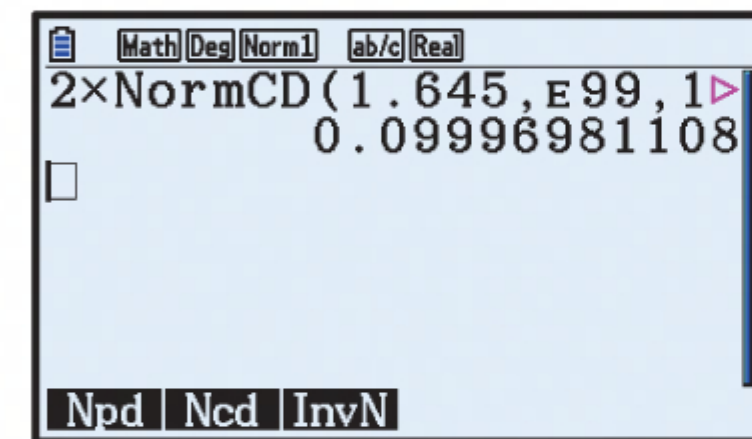
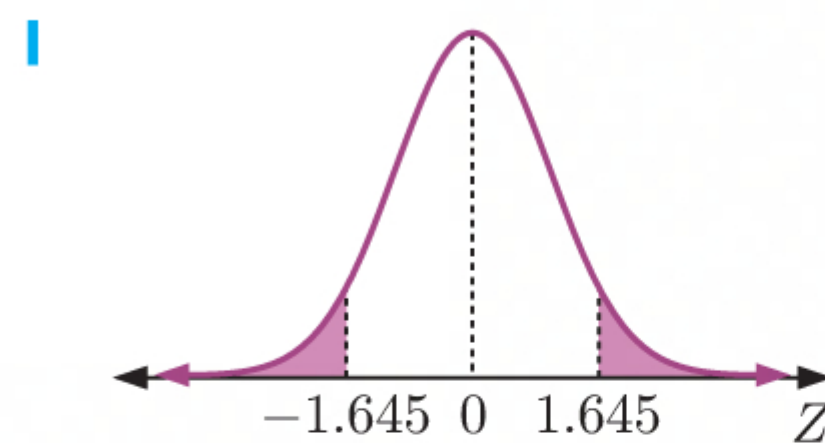
$$P(-0.5 < Z < 0.5) \approx 0.383$$



$$P(-1.960 \leq Z \leq 1.960) \approx 0.950$$



$$P(-1.645 \leq Z \leq 1.645) \approx 0.900$$



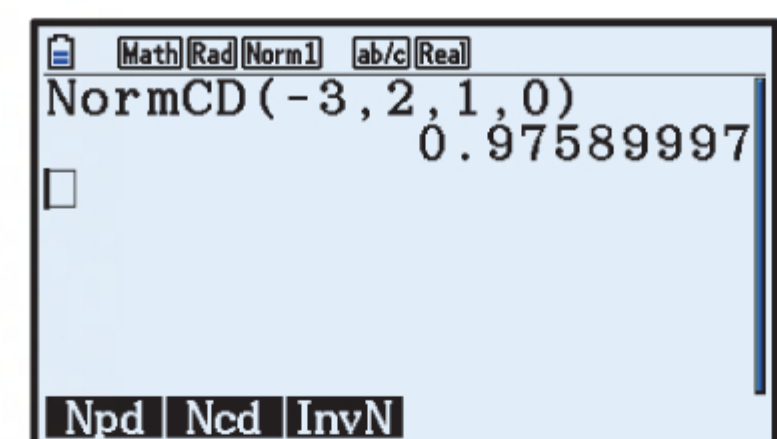
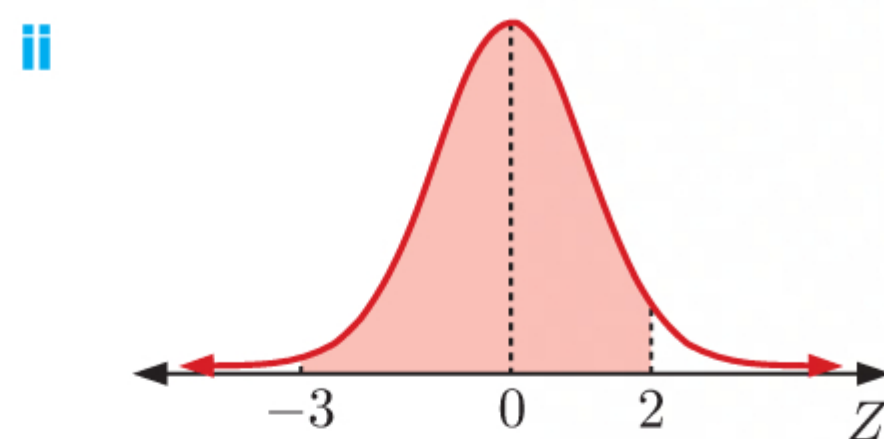
$$\begin{aligned} &P(|Z| > 1.645) \\ &= P(Z < -1.645) + P(Z > 1.645) \\ &= 2 \times P(Z > 1.645) \\ &\approx 0.100 \end{aligned}$$

4 a i Since $X \sim N(\mu, \sigma^2)$, the Z -transformation of X is $Z = \frac{X - \mu}{\sigma}$.

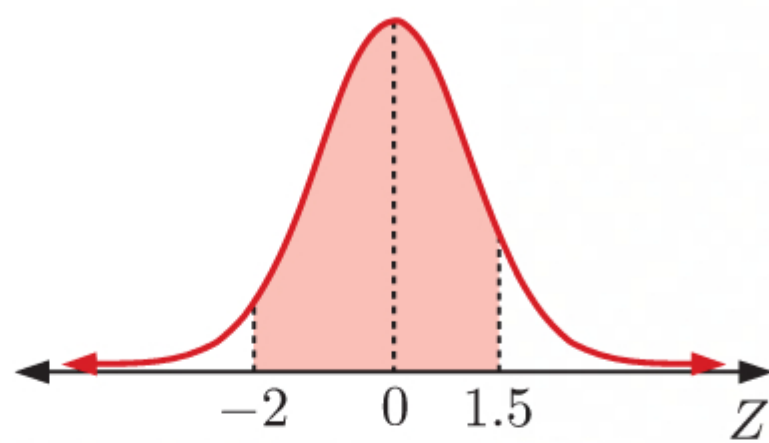
$$\text{Now } P(\mu - 3\sigma < X < \mu + 2\sigma) = P(-3\sigma < X - \mu < 2\sigma)$$

$$= P\left(-3 < \frac{X - \mu}{\sigma} < 2\right)$$

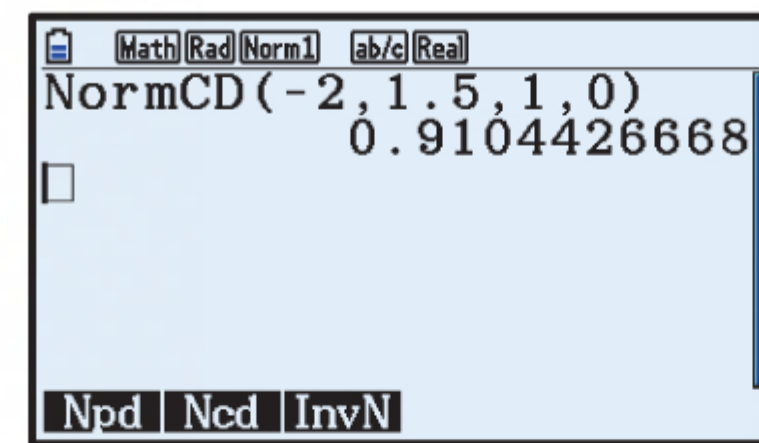
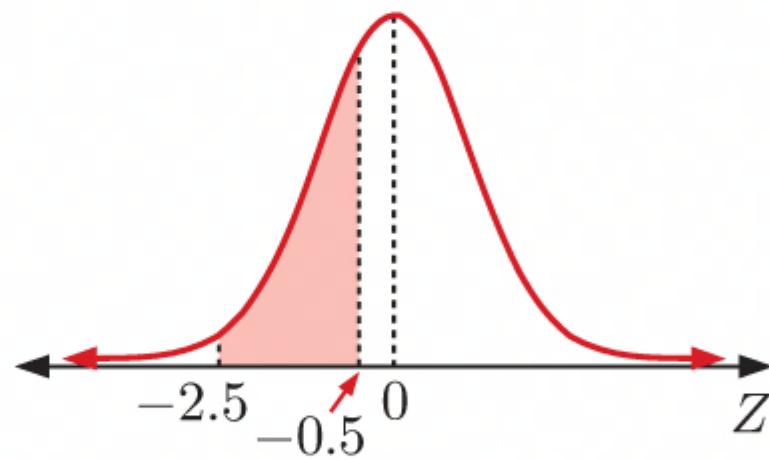
$$= P(-3 < Z < 2)$$



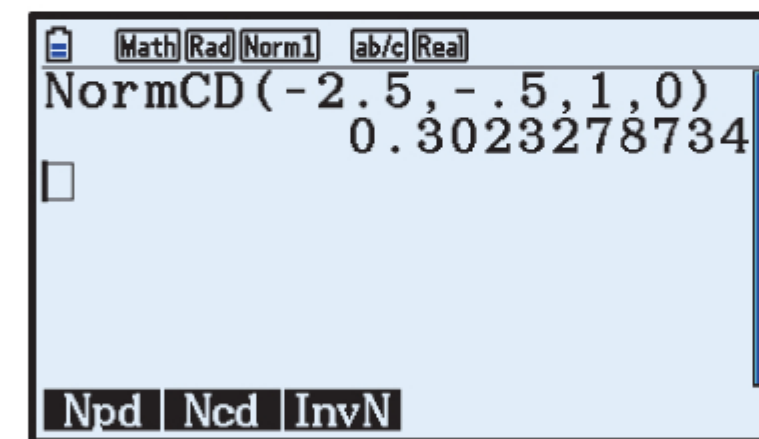
$$\begin{aligned} P(\mu - 3\sigma < X < \mu + 2\sigma) &= P(-3 < Z < 2) \\ &\approx 0.976 \end{aligned}$$

b i

$$P(\mu - 2\sigma < X < \mu + 1.5\sigma) = P(-2 < Z < 1.5) \\ \approx 0.910$$

**ii**

$$P(\mu - 2.5\sigma < X < \mu - 0.5\sigma) = P(-2.5 < Z < -0.5) \\ \approx 0.302$$



5 $X \sim N(58.3, 8.96^2)$

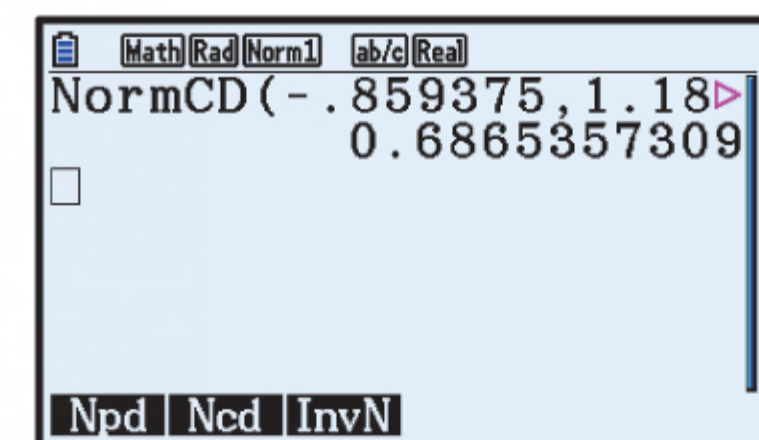
a i

$$z_1 = \frac{50.6 - 58.3}{8.96} = -0.859375 \approx -0.859$$

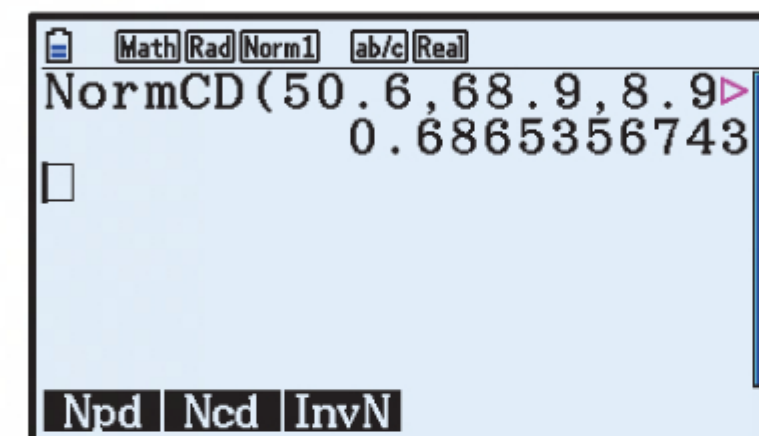
$$z_2 = \frac{68.9 - 58.3}{8.96} = 1.183036 \approx 1.18$$

ii $Z \sim N(0, 1)$

$$P(z_1 \leq Z \leq z_2) \\ = P(-0.859375 \leq Z \leq 1.183036) \\ \approx 0.687$$

**b** Using technology,

$$P(50.6 \leq X \leq 68.9) \approx 0.687 \quad \checkmark$$

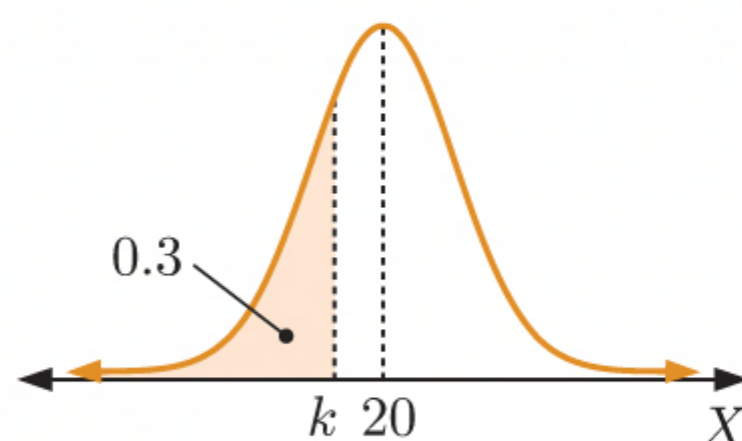


EXERCISE 21D.1**1 a**

Rad Norm1 ab/c Real	
Inverse Normal	
Data	:Variable
Tail	:Left
Area	:0.3
σ	:3
μ	:20
Save Res:None	
None	LIST

Rad Norm1 ab/c Real	
Inverse Normal	
xInv=18.4267985	

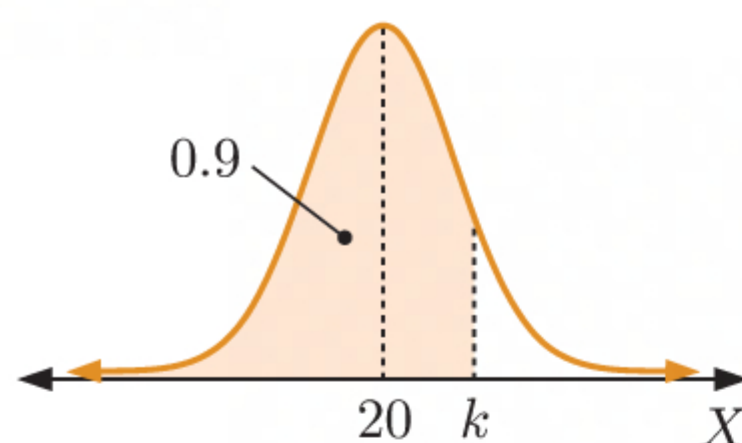
If $P(X \leq k) = 0.3$
 then $k \approx 18.4$

**b**

Rad Norm1 ab/c Real	
Inverse Normal	
Data	:Variable
Tail	:Left
Area	:0.9
σ	:3
μ	:20
Save Res:None	
None	LIST

Rad Norm1 ab/c Real	
Inverse Normal	
xInv=23.8446547	

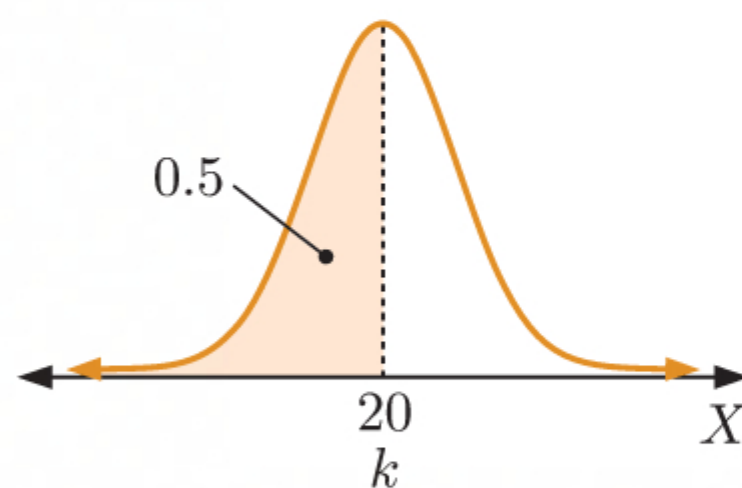
If $P(X \leq k) = 0.9$
 then $k \approx 23.8$

**c**

Rad Norm1 ab/c Real	
Inverse Normal	
Data	:Variable
Tail	:Left
Area	:0.5
σ	:3
μ	:20
Save Res:None	
None	LIST

Rad Norm1 ab/c Real	
Inverse Normal	
xInv=20	

If $P(X \leq k) = 0.5$
 then $k = 20$

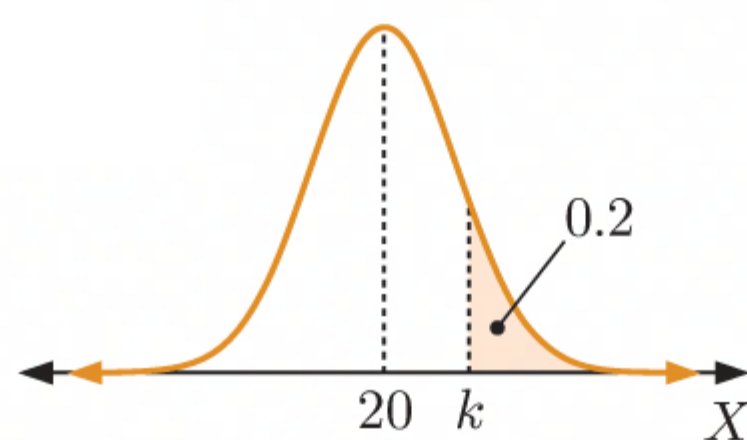


d

Rad Norm1 ab/c Real	
Inverse Normal	
Data	:Variable
Tail	:Right
Area	:0.2
σ	:3
μ	:20
Save Res:None	
None	LIST

Rad Norm1 ab/c Real	
Inverse Normal	
xInv=22.5248637	

If $P(X > k) = 0.2$
then $k \approx 22.5$

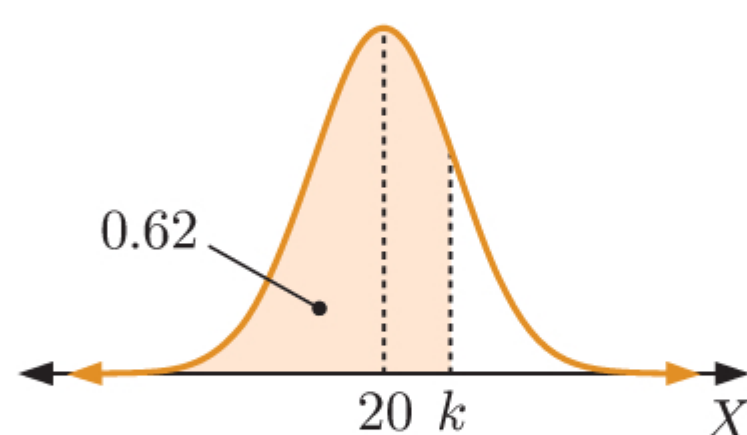


e

Rad Norm1 ab/c Real	
Inverse Normal	
Data	:Variable
Tail	:Left
Area	:0.62
σ	:3
μ	:20
Save Res:None	
None	LIST

Rad Norm1 ab/c Real	
Inverse Normal	
xInv=20.9164424	

If $P(X < k) = 0.62$
then $k \approx 20.9$

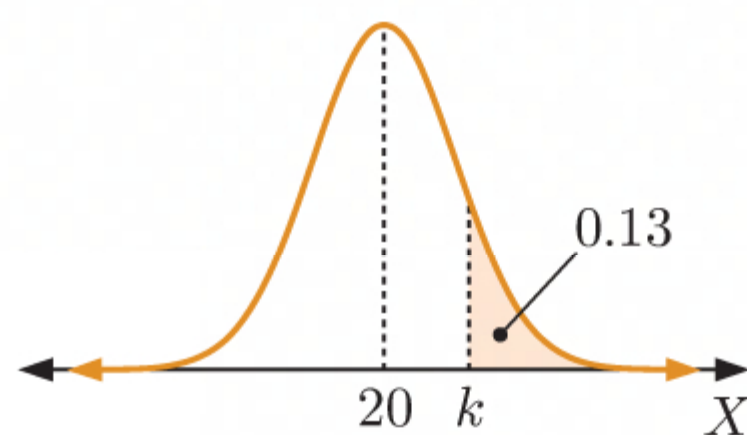


f

Rad Norm1 ab/c Real	
Inverse Normal	
Data	:Variable
Tail	:Right
Area	:0.13
σ	:3
μ	:20
Save Res:None	
None	LIST

Rad Norm1 ab/c Real	
Inverse Normal	
xInv=23.3791734	

If $P(X \geq k) = 0.13$
then $k \approx 23.4$



2

a

Rad Norm1 ab/c Real

Inverse Normal

Data : Variable

Tail : Left

Area : 0.81

σ : 1

μ : 0

Save Res: None

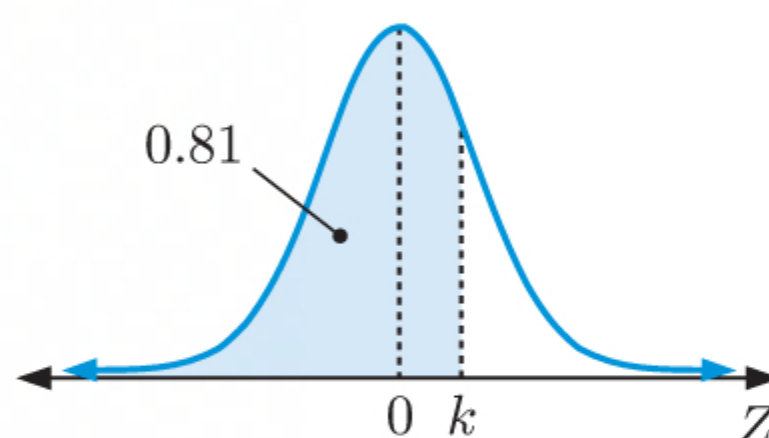
None LIST

Rad Norm1 ab/c Real

Inverse Normal

xInv=0.87789629

If $P(Z \leq k) = 0.81$
then $k \approx 0.878$



b

Rad Norm1 ab/c Real

Inverse Normal

Data : Variable

Tail : Left

Area : 0.58

σ : 1

μ : 0

Save Res: None

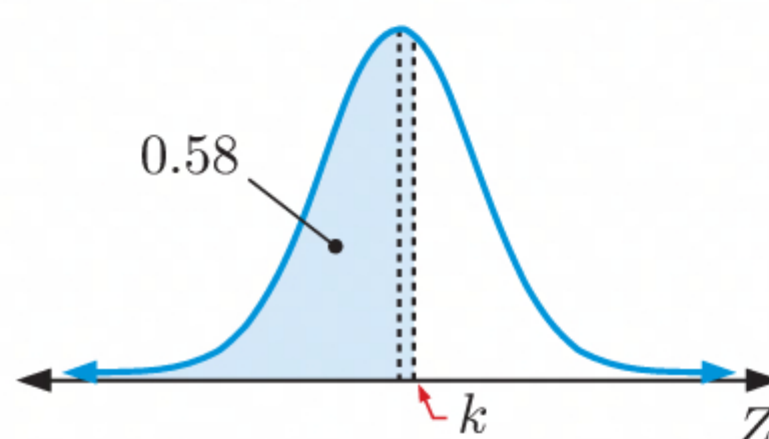
None LIST

Rad Norm1 ab/c Real

Inverse Normal

xInv=0.20189347

If $P(Z \leq k) = 0.58$
then $k \approx 0.202$



c

Rad Norm1 ab/c Real

Inverse Normal

Data : Variable

Tail : Left

Area : 0.17

σ : 1

μ : 0

Save Res: None

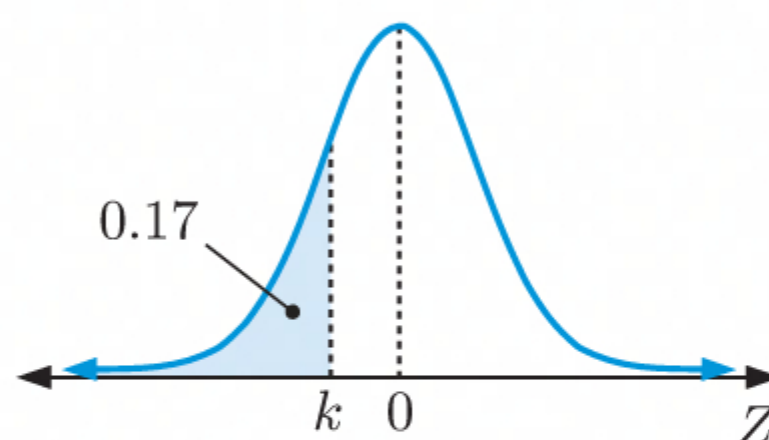
None LIST

Rad Norm1 ab/c Real

Inverse Normal

xInv=-0.9541652

If $P(Z \leq k) = 0.17$
then $k \approx -0.954$



d

Rad Norm1 ab/c Real

Inverse Normal

Data :Variable

Tail :Right

Area :0.95

σ :1

μ :0

Save Res:None

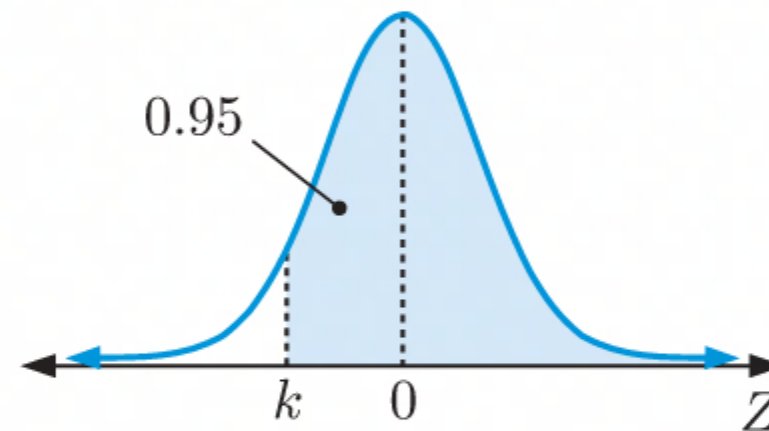
None LIST

Rad Norm1 ab/c Real

Inverse Normal

xInv=-1.6448536

If $P(Z \geq k) = 0.95$
 $\therefore k \approx -1.64$



e

Rad Norm1 ab/c Real

Inverse Normal

Data :Variable

Tail :Right

Area :0.9

σ :1

μ :0

Save Res:None

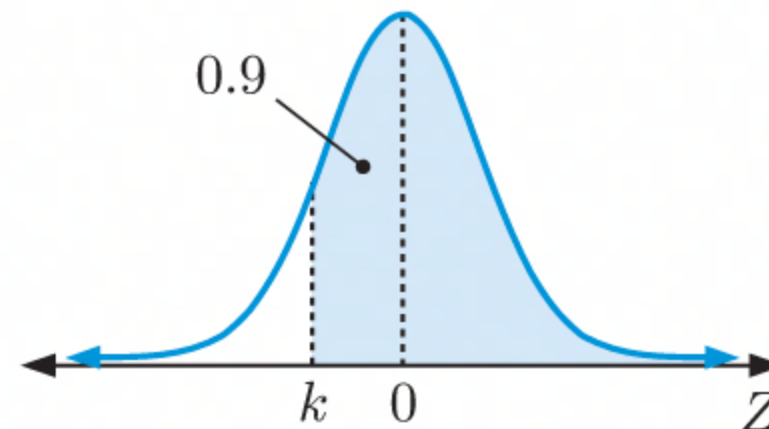
None LIST

Rad Norm1 ab/c Real

Inverse Normal

xInv=-1.2815516

If $P(Z \geq k) = 0.9$
 $\therefore k \approx -1.28$



f

Rad Norm1 ab/c Real

Inverse Normal

Data :Variable

Tail :Right

Area :0.41

σ :1

μ :0

Save Res:None

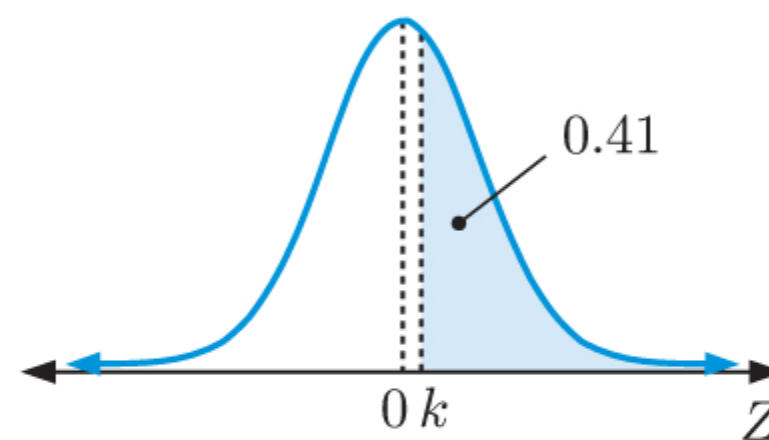
None LIST

Rad Norm1 ab/c Real

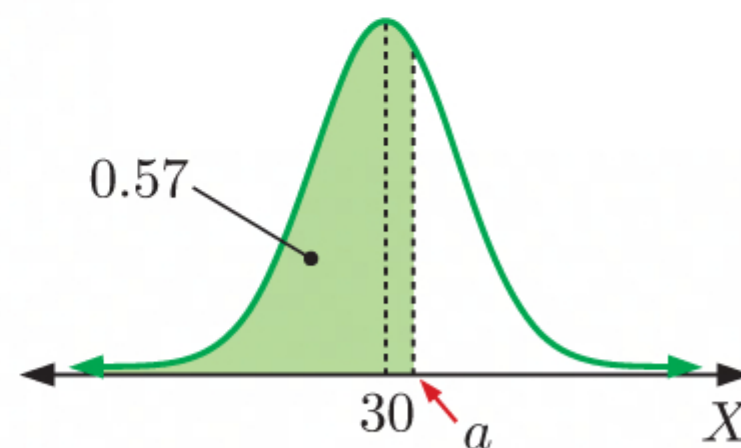
Inverse Normal

xInv=0.22754497

If $P(Z \geq k) = 0.41$
 $\therefore k \approx 0.228$

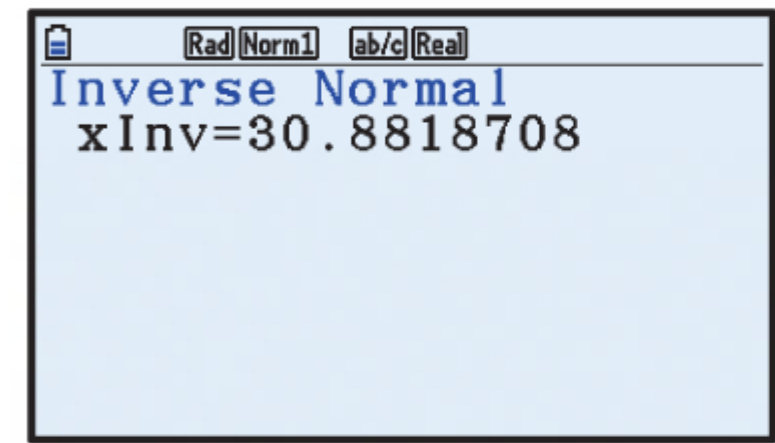
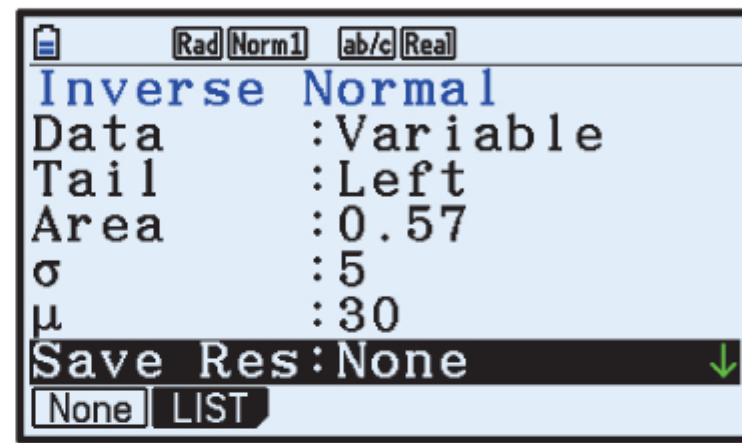


3 a $X \sim N(30, 5^2)$



$\therefore a > 30$

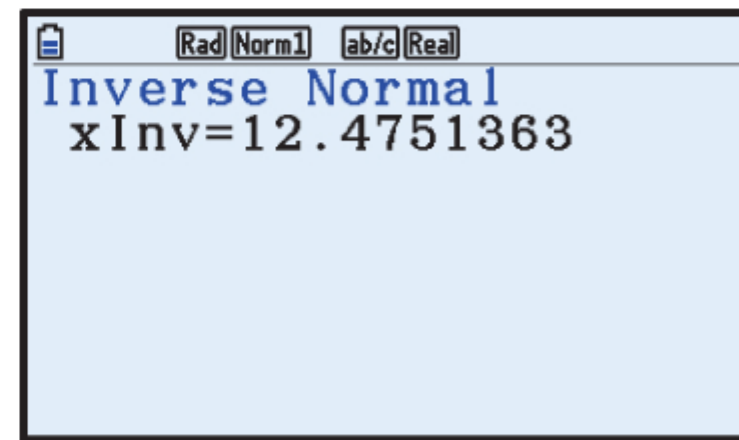
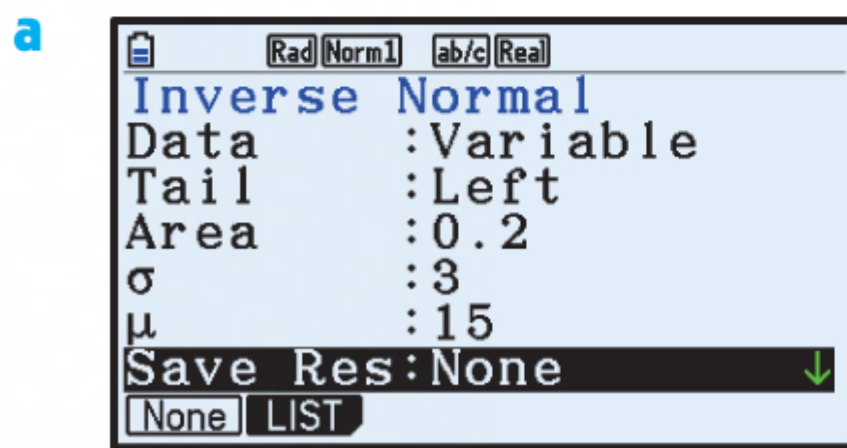
b $P(X \leq a) = 0.57$
 $\therefore a \approx 30.9$



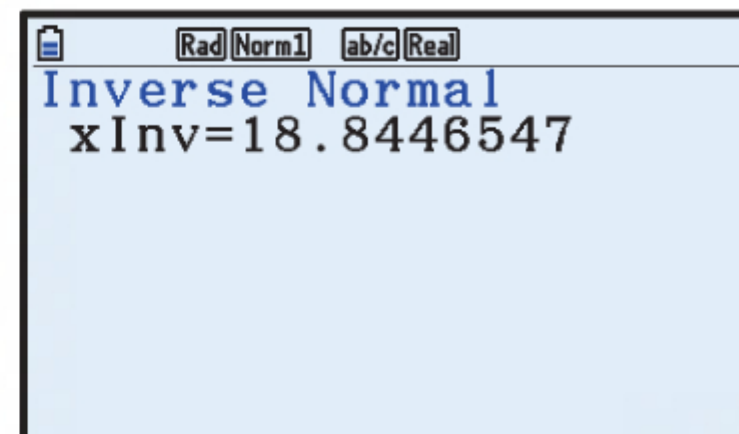
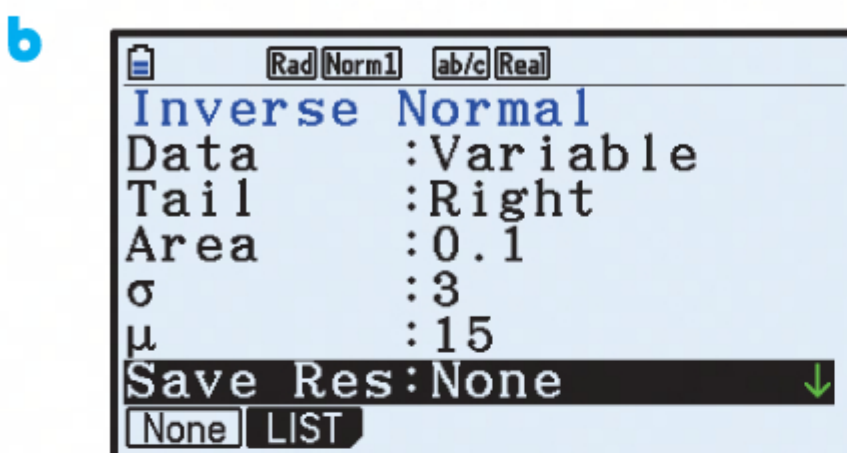
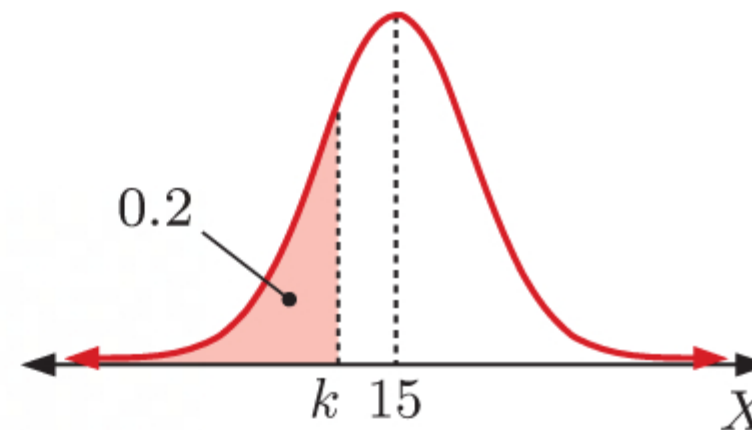
c i $P(X \geq a) = 1 - P(X \leq a)$
 $= 1 - 0.57$
 $= 0.43$

ii $P(30 \leq X \leq a) = P(X \leq a) - P(X \leq 30)$
 $= 0.57 - 0.5$
 $= 0.07$

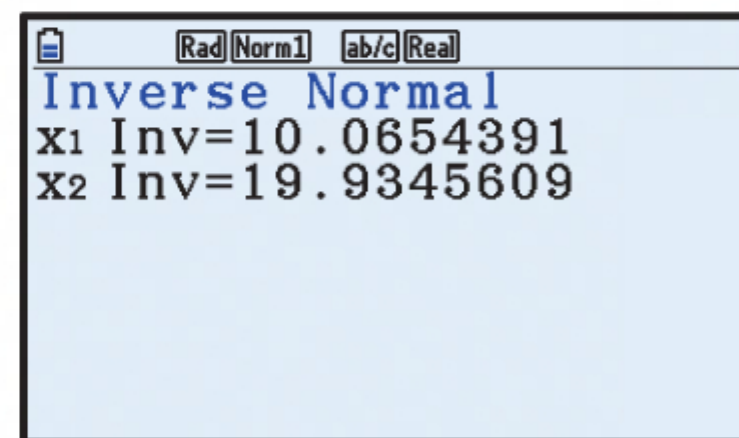
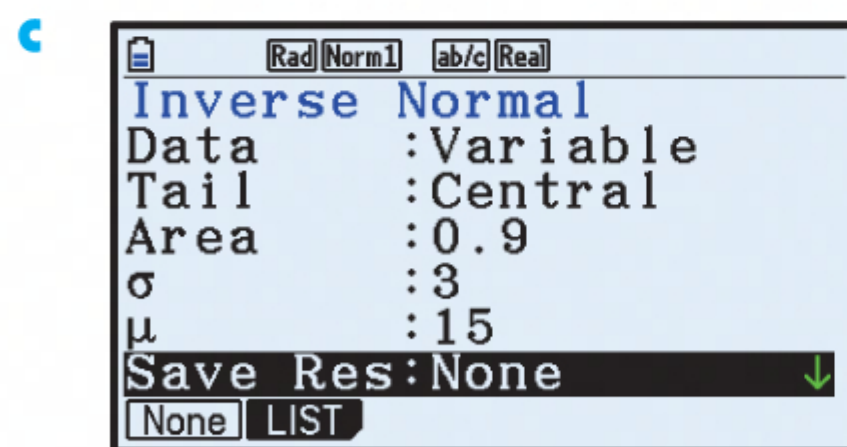
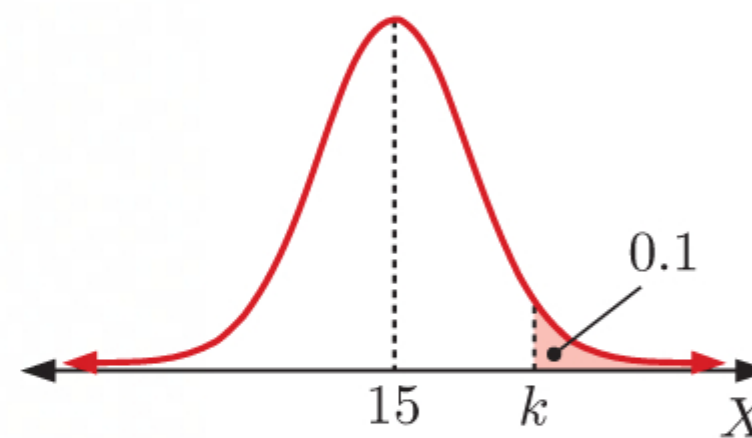
4 $X \sim N(15, 3^2)$



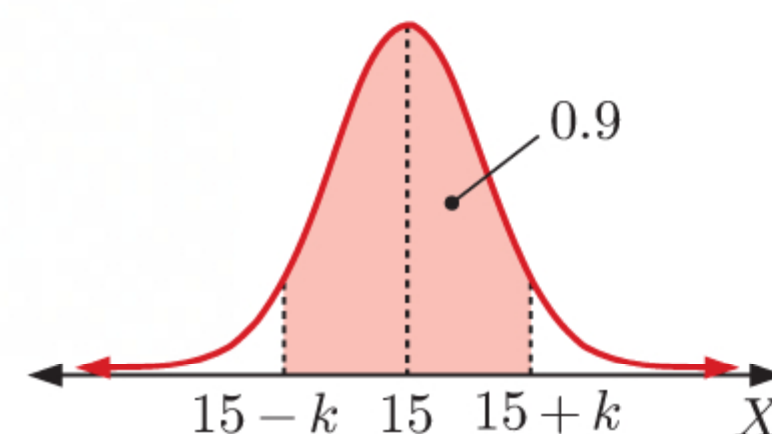
If $P(X < k) = 0.2$
 then $k \approx 12.5$



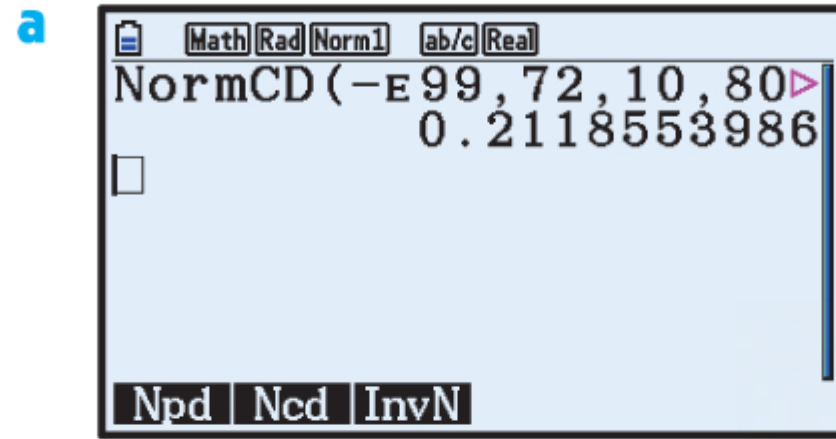
If $P(X > k) = 0.1$
 then $k \approx 18.8$



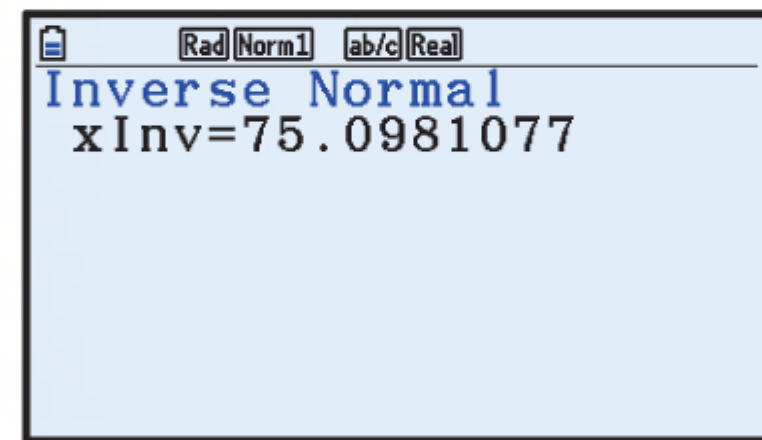
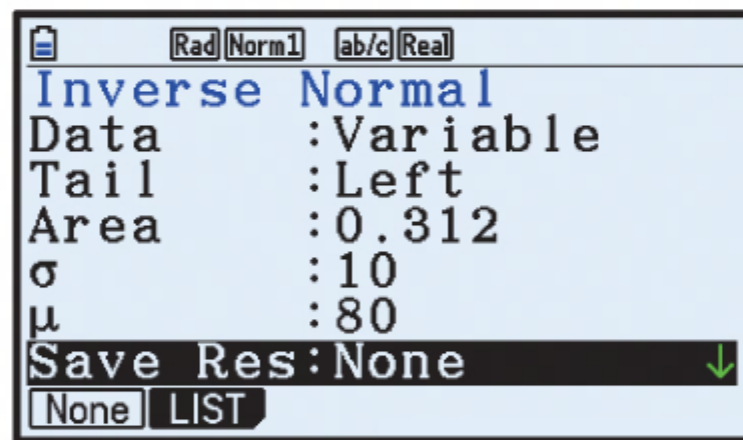
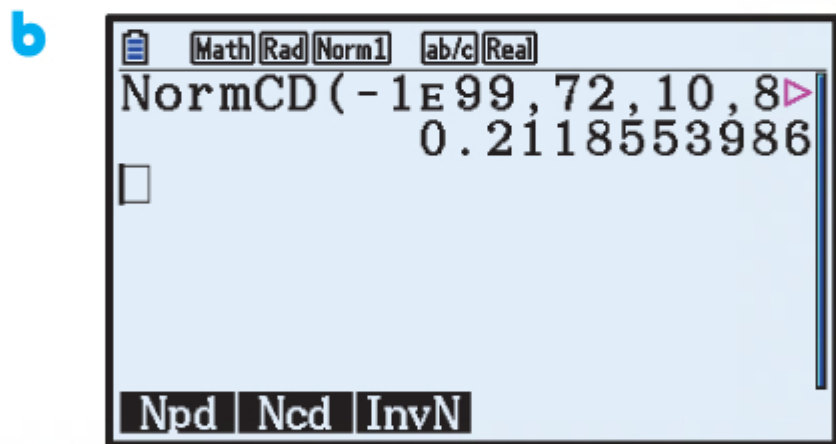
If $P(15 - k < X < 15 + k) = 0.9$
 then $15 - k \approx 10.07$
 or $15 + k \approx 19.93$
 $\therefore k \approx 4.93$



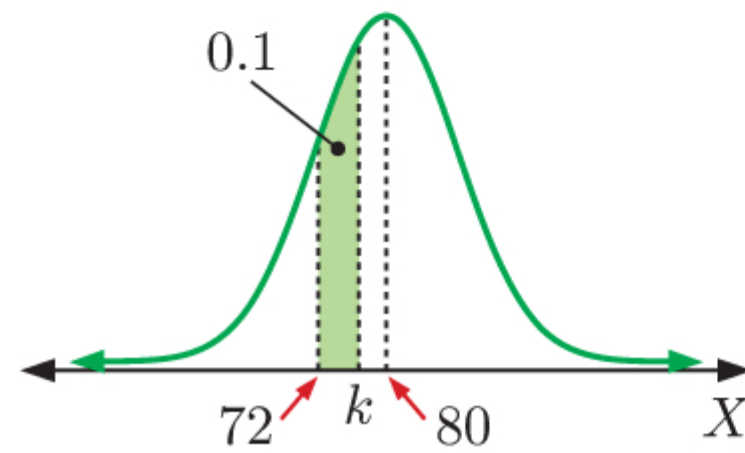
5 $X \sim N(80, 10^2)$



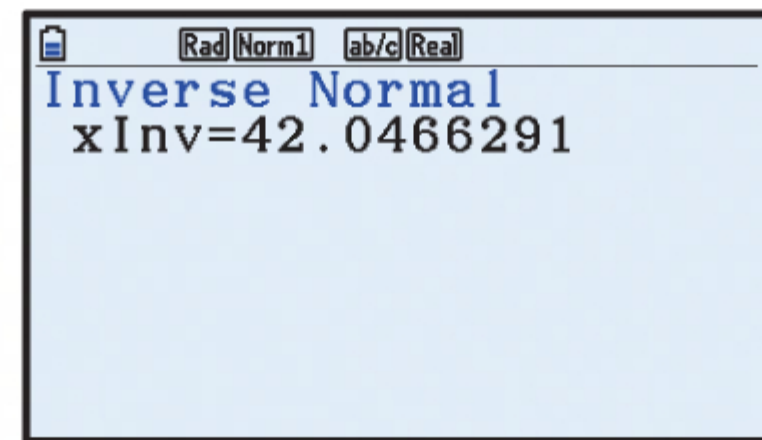
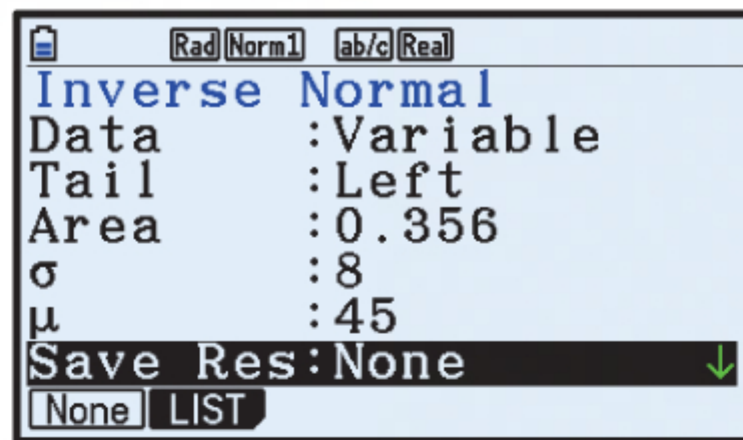
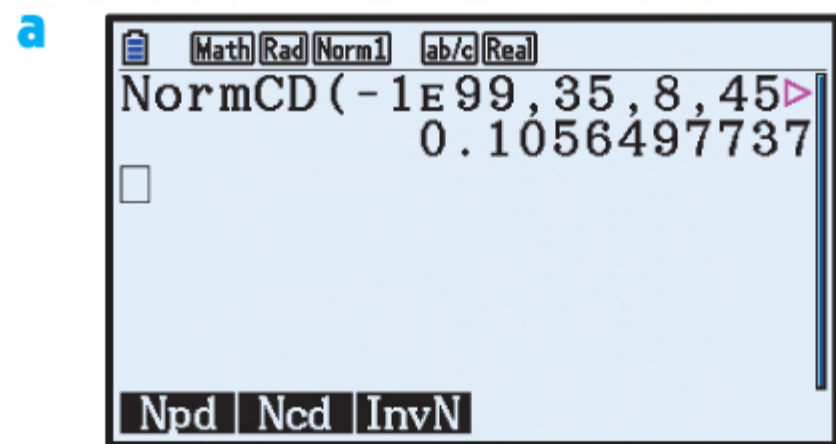
$$P(X \leq 72) \approx 0.212$$



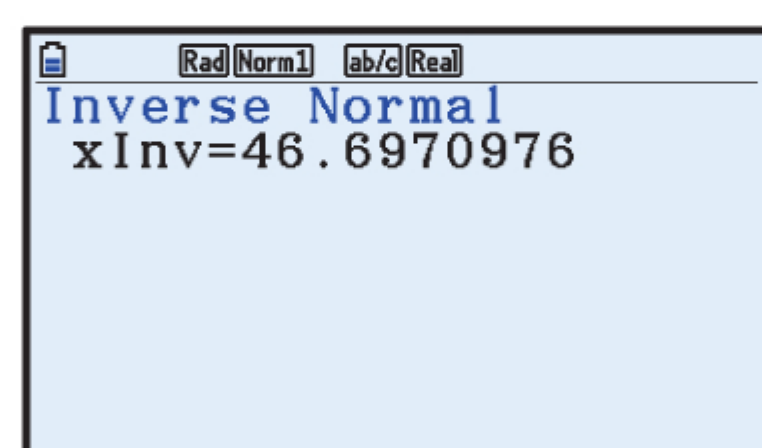
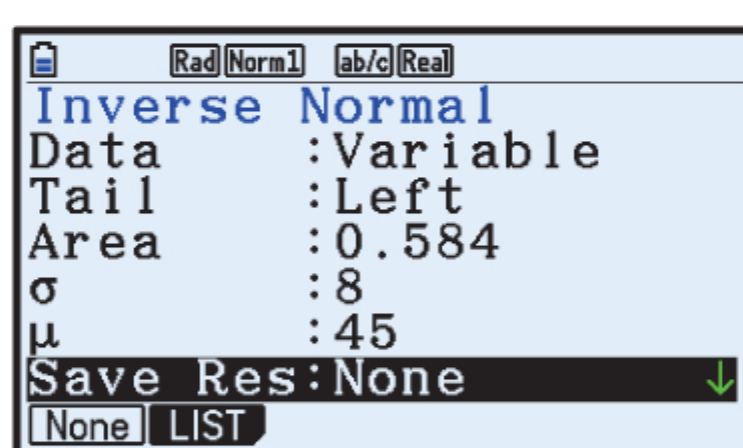
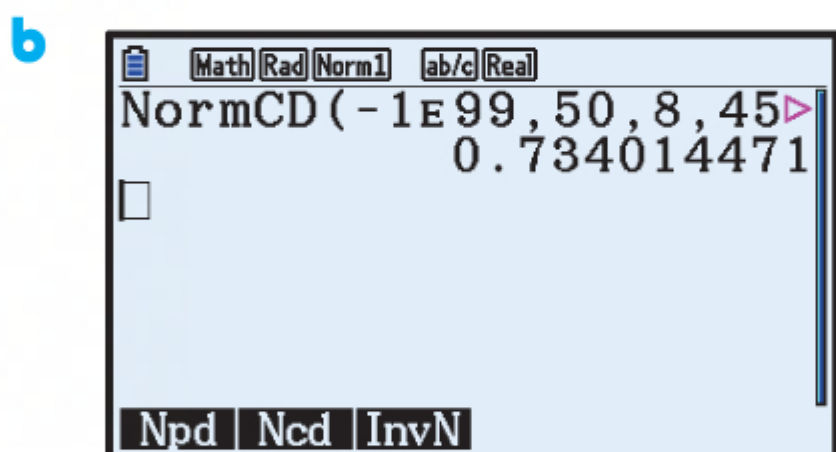
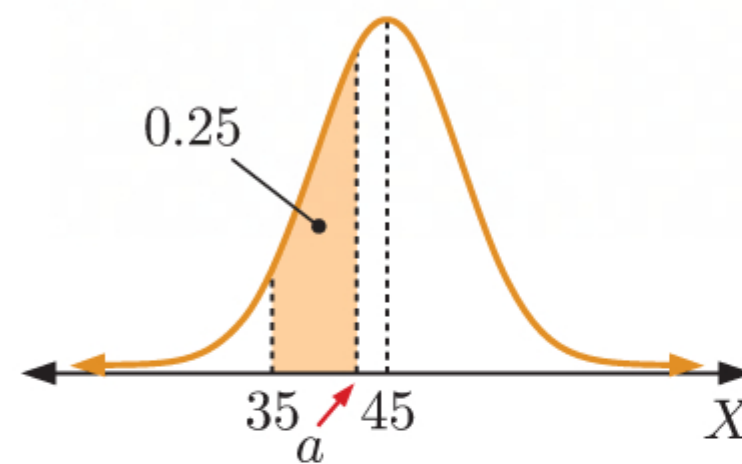
$$\begin{aligned} P(72 \leq X \leq k) &= 0.1 \\ \therefore P(X \leq k) - P(X \leq 72) &= 0.1 \\ \therefore P(X \leq k) - 0.212 &\approx 0.1 \\ \therefore P(X \leq k) &\approx 0.312 \\ \therefore k &\approx 75.1 \end{aligned}$$



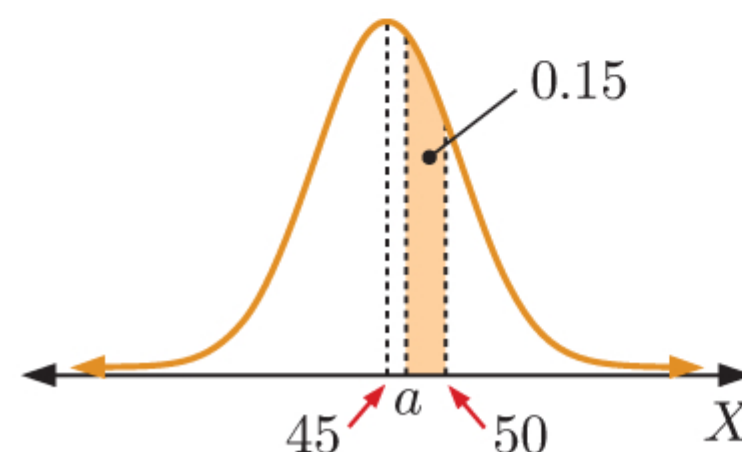
6 $X \sim N(45, 8^2)$

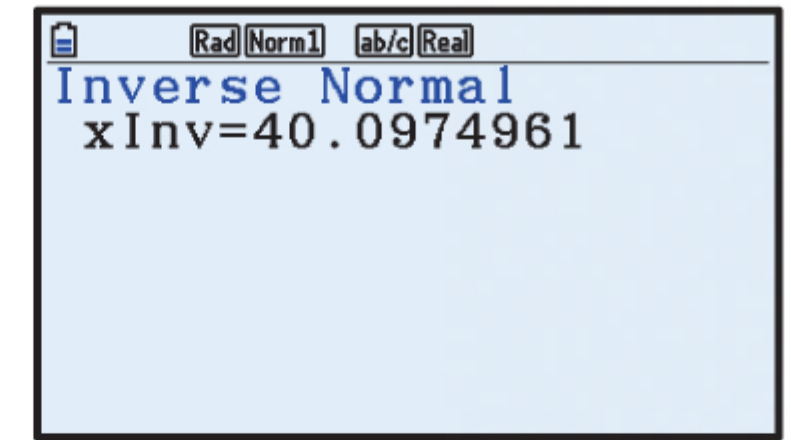
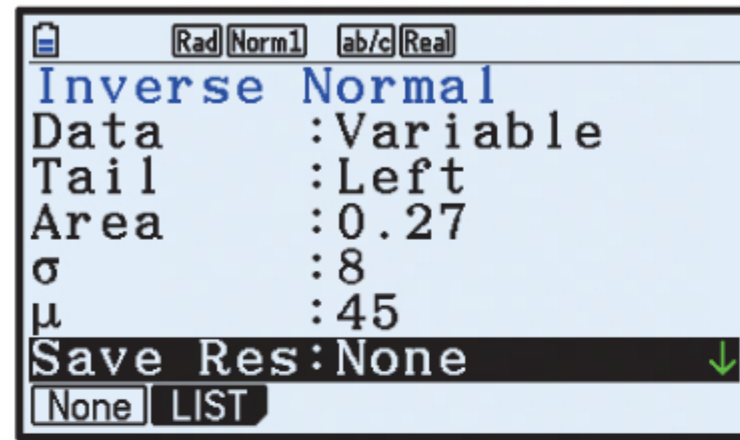
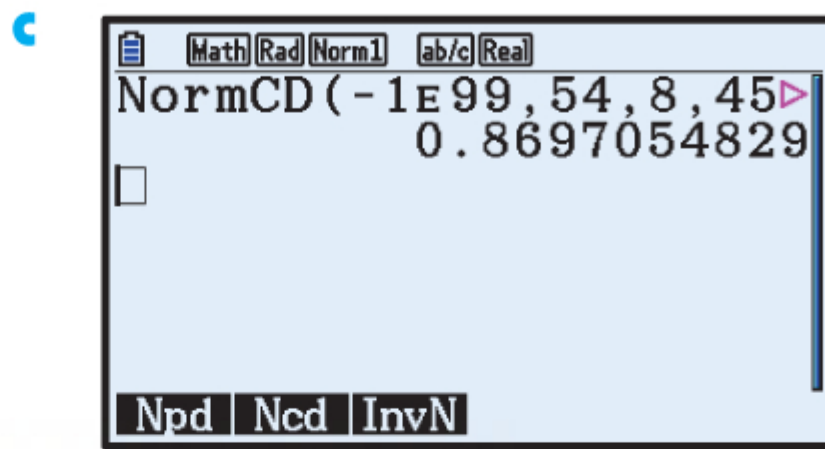


$$\begin{aligned} P(35 \leq X \leq a) &= 0.25 \\ \therefore P(X \leq a) - P(X \leq 35) &= 0.25 \\ \therefore P(X \leq a) - 0.106 &\approx 0.25 \\ \therefore P(X \leq a) &\approx 0.356 \\ \therefore a &\approx 42.0 \end{aligned}$$

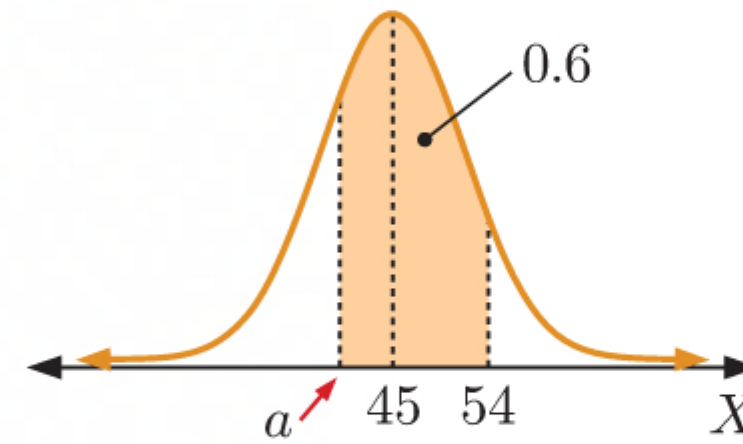


$$\begin{aligned} P(a \leq X \leq 50) &= 0.15 \\ \therefore P(X \leq 50) - P(X \leq a) &= 0.15 \\ \therefore 0.734 - P(X \leq a) &\approx 0.15 \\ \therefore P(X \leq a) &\approx 0.584 \\ \therefore a &\approx 46.7 \end{aligned}$$

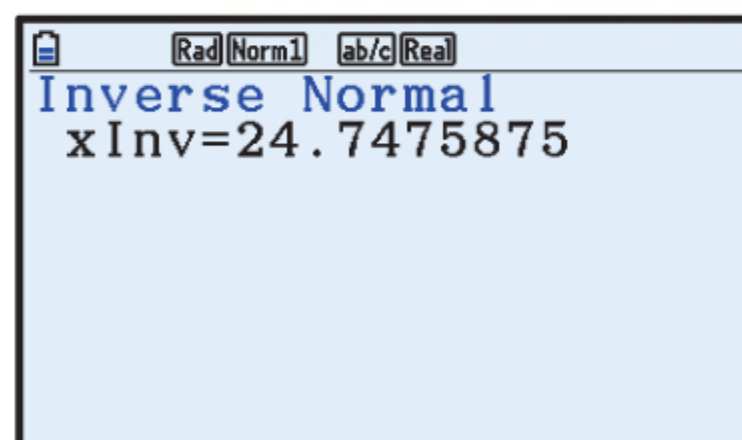
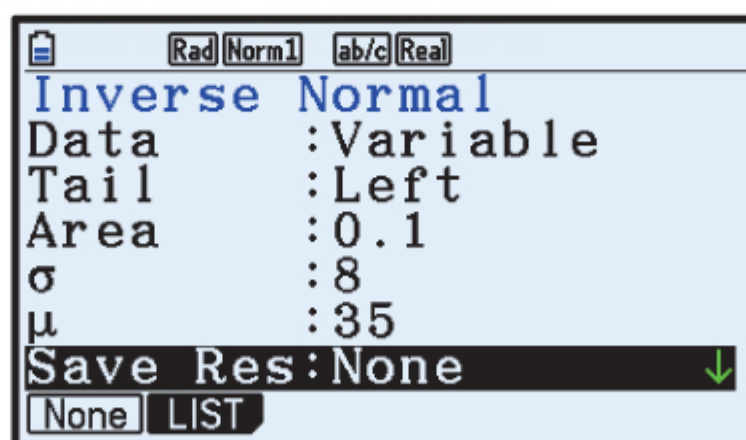




$$\begin{aligned} P(a \leq X \leq 54) &= 0.6 \\ \therefore P(X \leq 54) - P(X \leq a) &= 0.6 \\ \therefore 0.870 - P(X \leq a) &\approx 0.6 \\ \therefore P(X \leq a) &\approx 0.27 \\ \therefore a &\approx 40.1 \end{aligned}$$

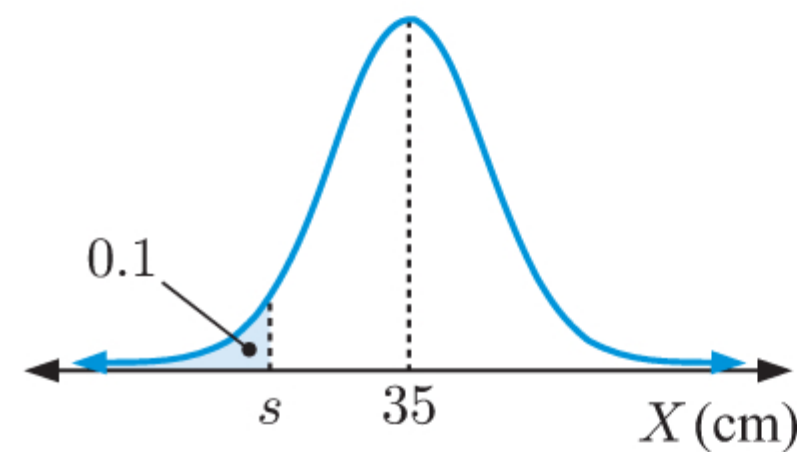


7

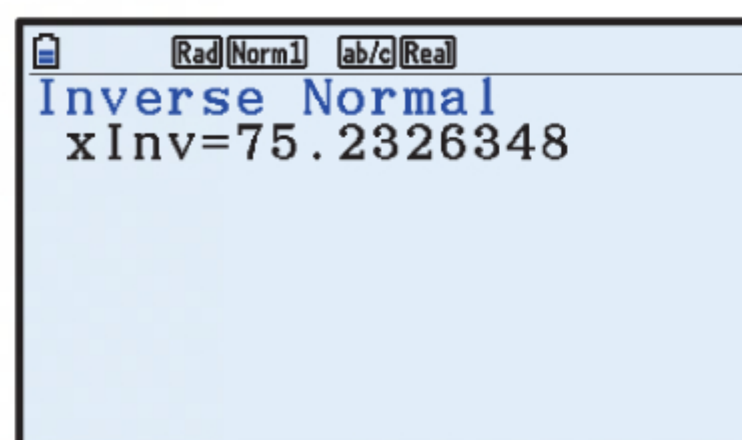
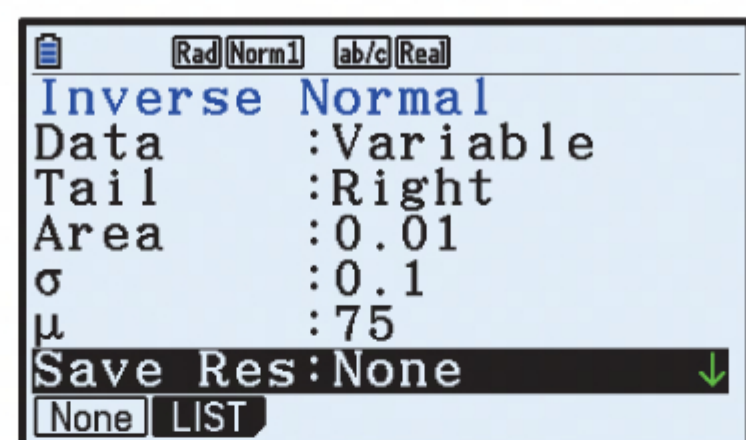


$$\begin{aligned} P(X \leq s) &= 0.1 \\ \therefore s &\approx 24.7 \end{aligned}$$

\therefore the size of the smallest fish that can be harvested is about 24.7 cm.

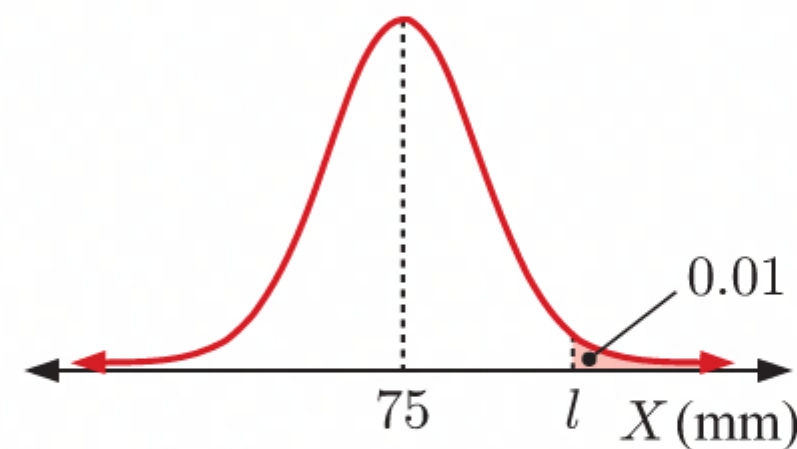


8

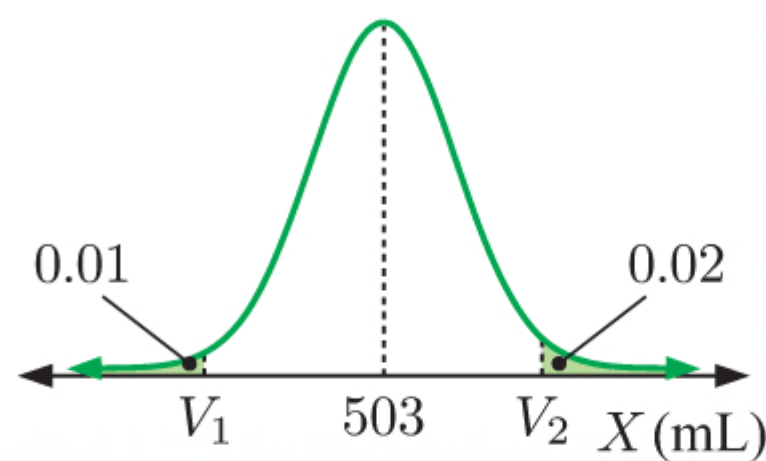


$$\begin{aligned} P(X \geq l) &= 0.01 \\ \therefore l &\approx 75.2 \end{aligned}$$

\therefore the length of the smallest screw to be rejected is about 75.2 mm.

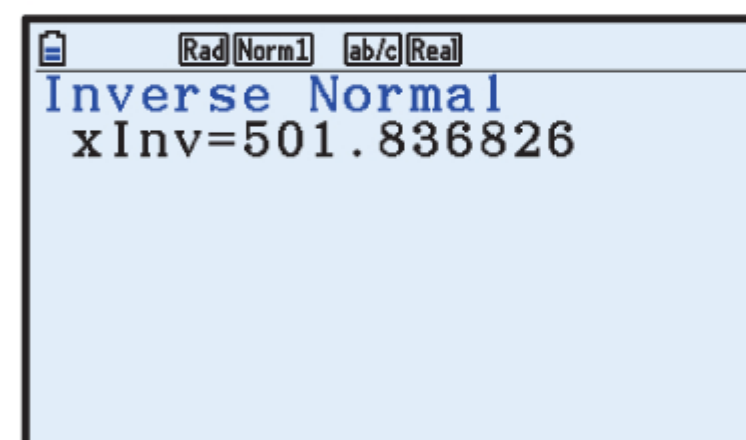
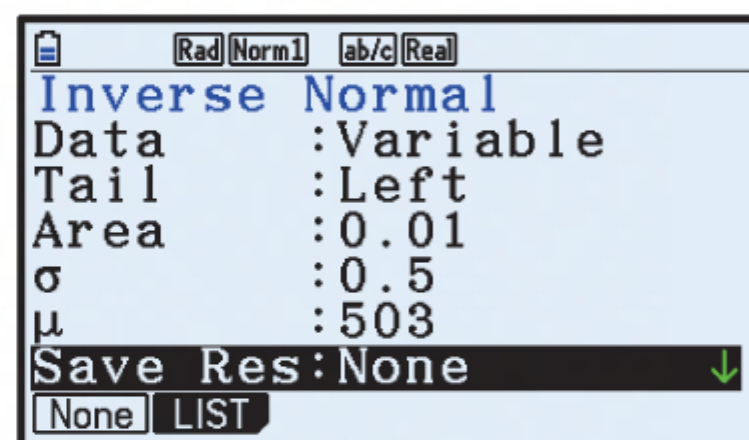


9



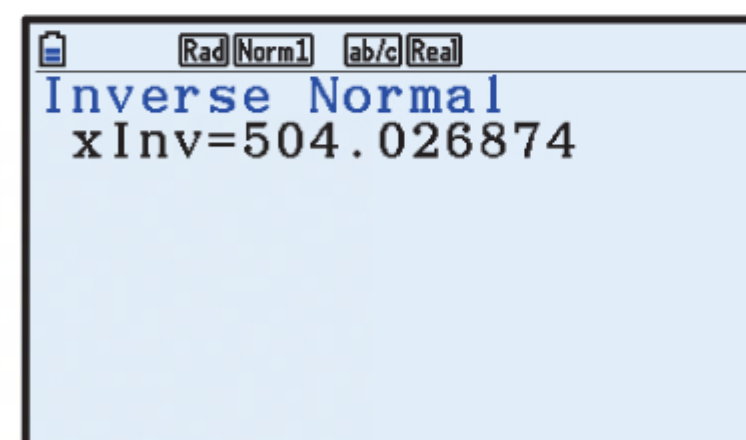
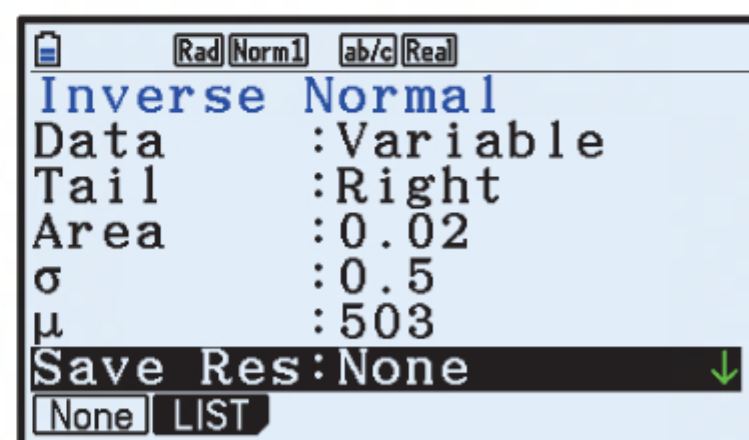
$$P(X \leq V_1) = 0.01$$

$$\therefore V_1 \approx 501.8$$



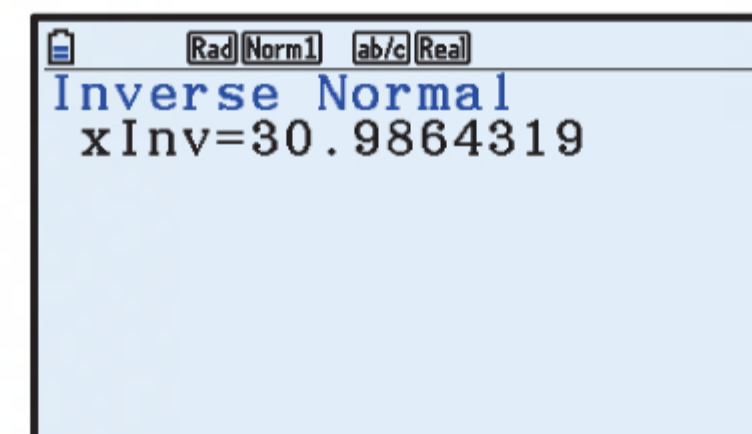
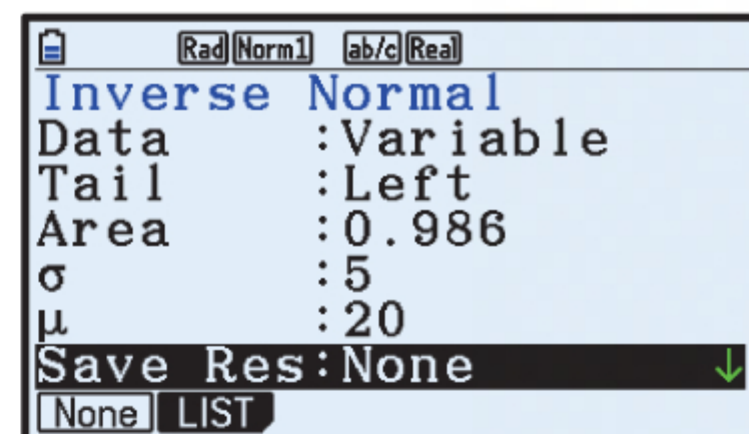
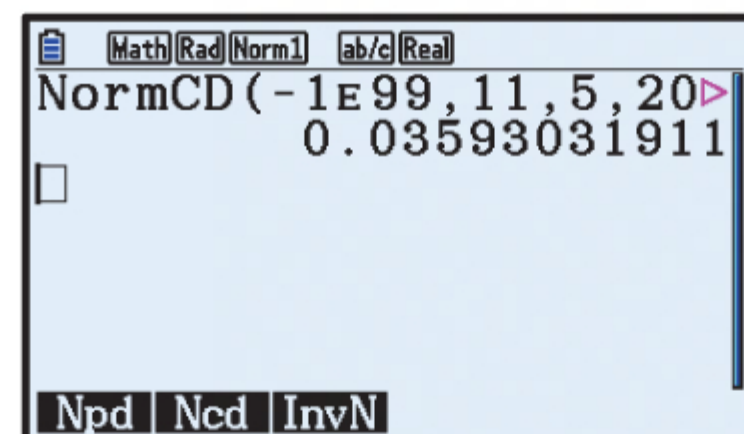
$$P(X \geq V_2) = 0.02$$

$$\therefore V_2 \approx 504.0$$



\therefore the bottles which are kept have volumes ranging from about 501.8 mL to 504.0 mL.

10



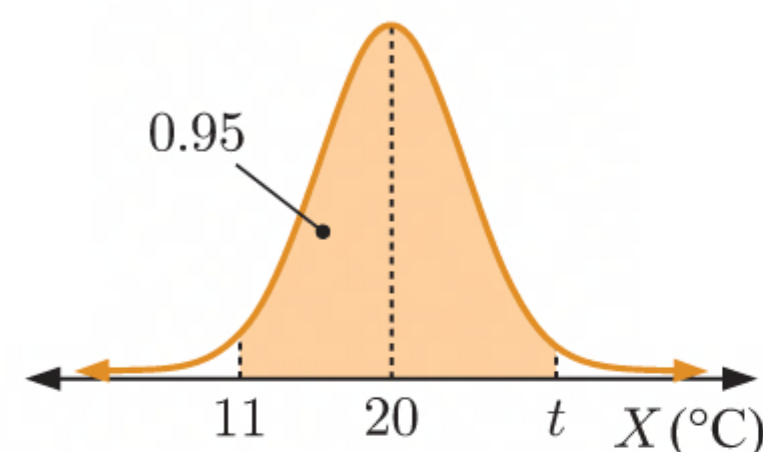
$$P(11 \leq X \leq t) = 0.95$$

$$\therefore P(X \leq t) - P(X \leq 11) = 0.95$$

$$\therefore P(X \leq t) - 0.0359 \approx 0.95$$

$$\therefore P(X \leq t) \approx 0.986$$

$$\therefore t \approx 31.0$$



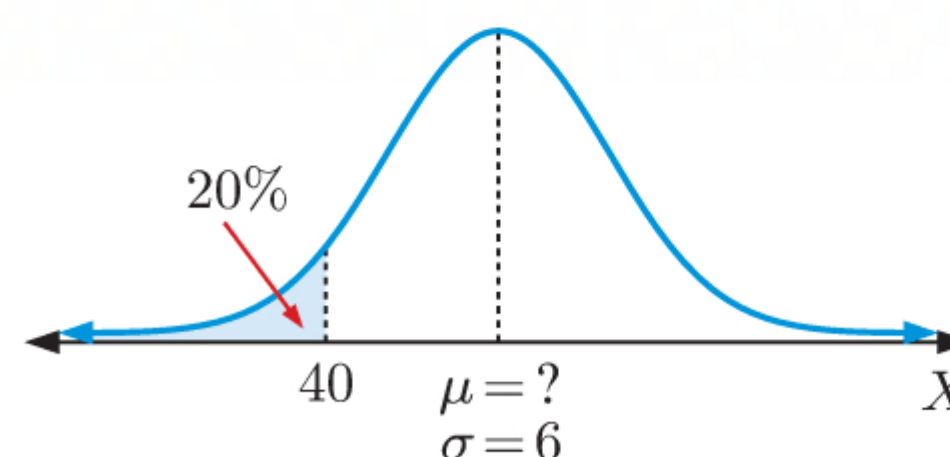
\therefore the upper limit of Abbey's walking temperatures is about 31.0 $^{\circ}\text{C}$.

EXERCISE 21D.2

1 a $P(X < 40) = 0.2$

So, data values which are less than 40 make up only 20% of all values.

\therefore the mean of X would be greater than 40.



b $X \sim N(\mu, 6^2)$

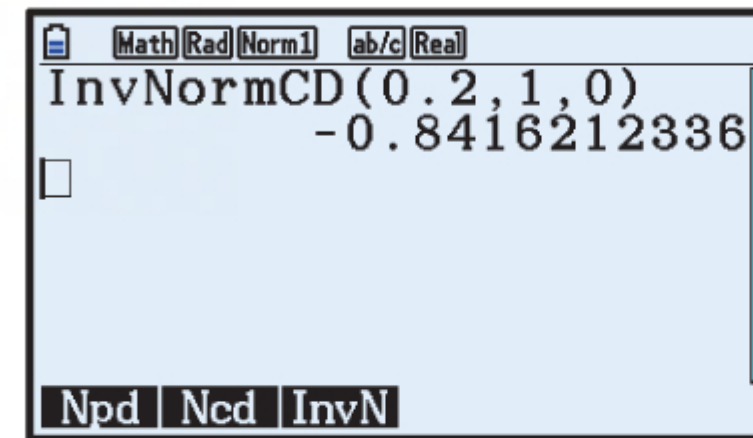
$$P(X < 40) = 0.2$$

$$\therefore P\left(Z \leq \frac{40 - \mu}{6}\right) = 0.2$$

$$\therefore \frac{40 - \mu}{6} \approx -0.8416$$

$$\therefore 40 - \mu \approx -5.0496$$

$$\therefore \mu \approx 45.0$$



2 $X \sim N(15, \sigma^2)$

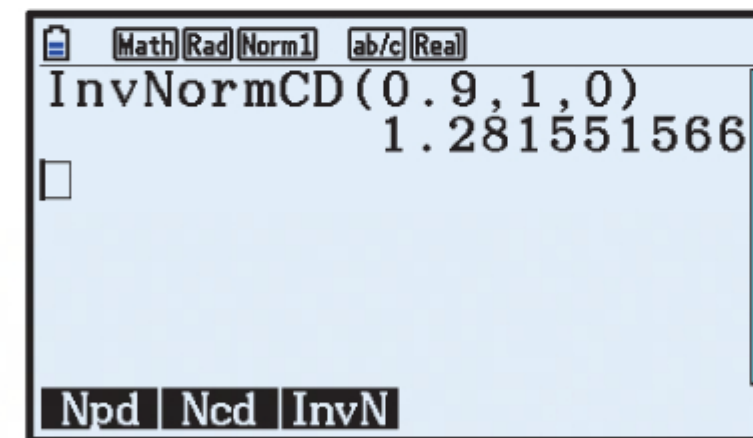
$$P(X > 20) = 0.1$$

$$\therefore P(X < 20) = 0.9$$

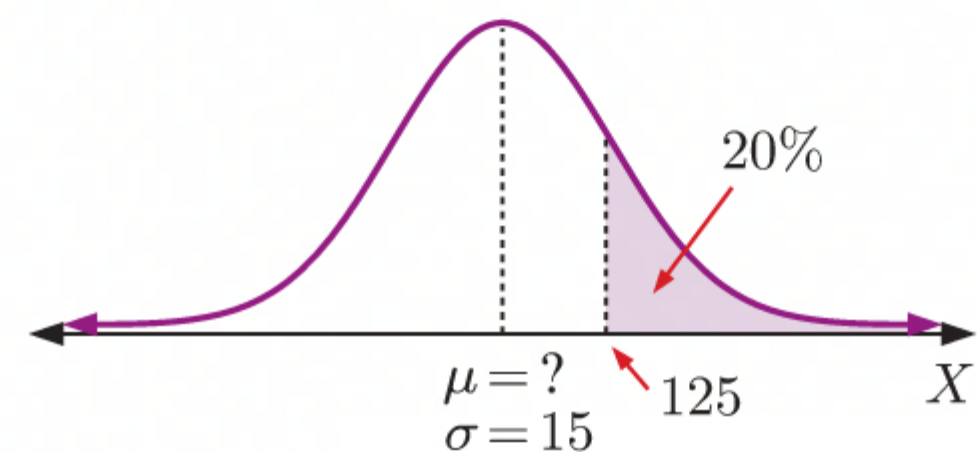
$$\therefore P\left(Z < \frac{20 - 15}{\sigma}\right) = 0.9$$

$$\therefore \frac{5}{\sigma} \approx 1.2816$$

$$\therefore \sigma \approx 3.90$$



- 3** Let the mean IQ of students at the school be μ .
If X is the IQ of a student at the school, then
 $X \sim N(\mu, 15^2)$.



$$P(X \geq 125) = 0.2$$

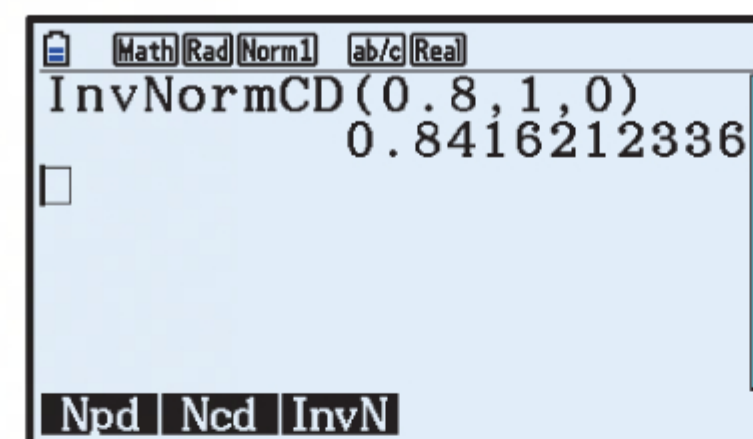
$$\therefore P(X < 125) = 0.8$$

$$\therefore P\left(Z < \frac{125 - \mu}{15}\right) = 0.8$$

$$\therefore \frac{125 - \mu}{15} \approx 0.8416$$

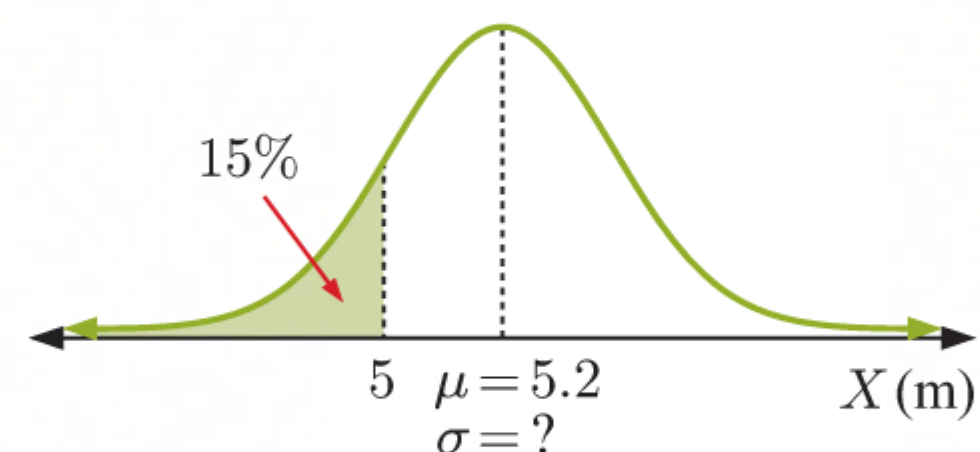
$$\therefore 125 - \mu \approx 12.624$$

$$\therefore \mu \approx 112.4$$



So, the mean IQ of students at the school is approximately 112.

- 4** Let the standard deviation of the distances jumped be σ m.
If X is the distance jumped by the athlete, then
 $X \sim N(5.2, \sigma^2)$.

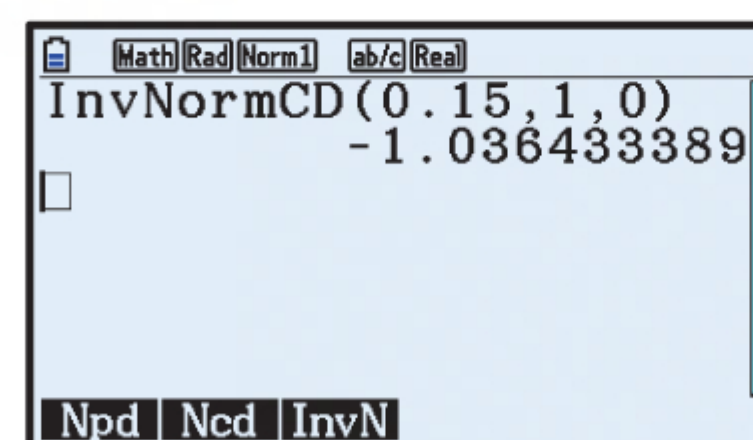


$$P(X < 5) = 0.15$$

$$\therefore P\left(Z < \frac{5 - 5.2}{\sigma}\right) = 0.15$$

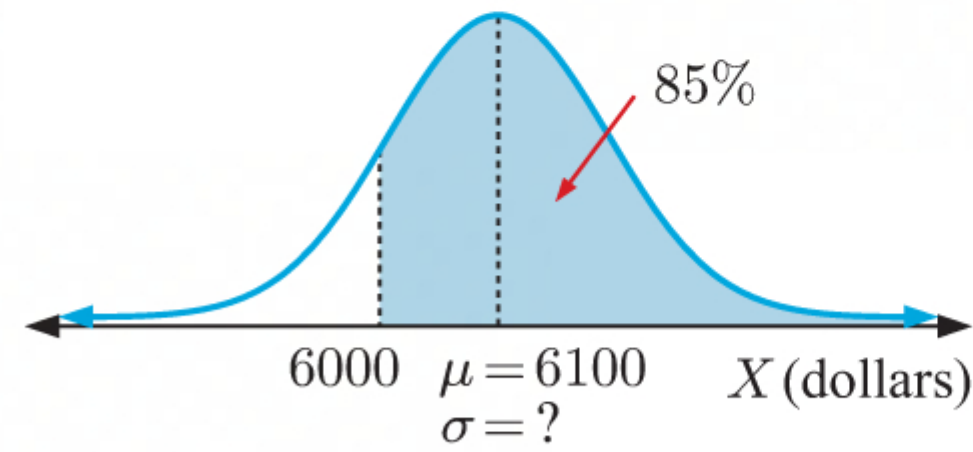
$$\therefore -\frac{0.2}{\sigma} \approx -1.036$$

$$\therefore \sigma \approx 0.193$$

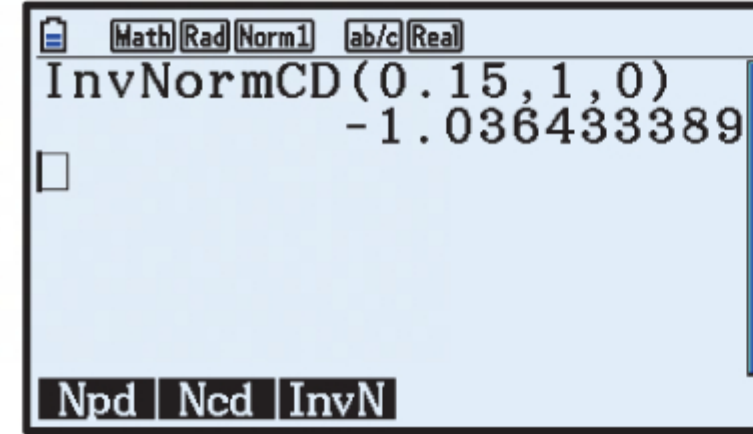


So, the standard deviation of the distances jumped is approximately 0.193 m.

- 5 Let the standard deviation of the weekly income be $\text{€}\sigma$.
 If X denotes the weekly income of the bakery, then $X \sim N(6100, \sigma^2)$.

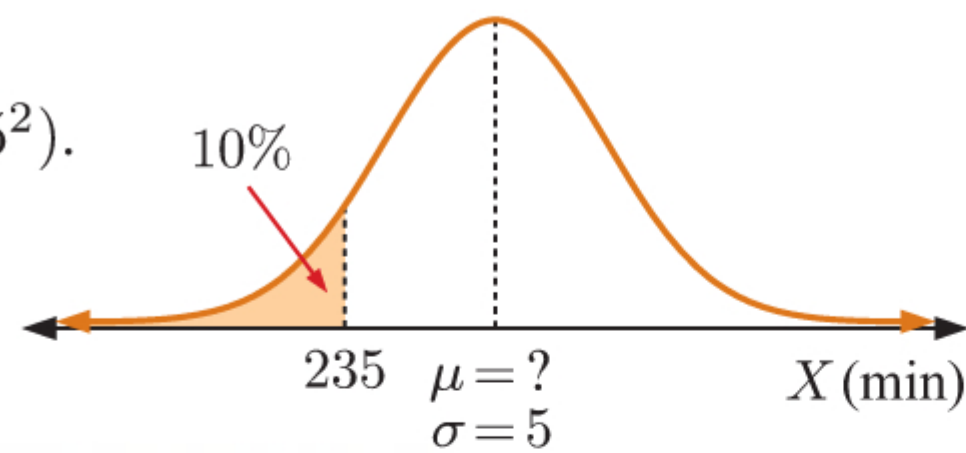


$$\begin{aligned} P(X \geq 6000) &= 0.85 \\ \therefore P(X < 6000) &= 0.15 \\ \therefore P\left(Z < \frac{6000 - 6100}{\sigma}\right) &= 0.15 \\ \therefore \frac{-100}{\sigma} &\approx -1.0364334 \\ \therefore \sigma &\approx 96.48 \end{aligned}$$

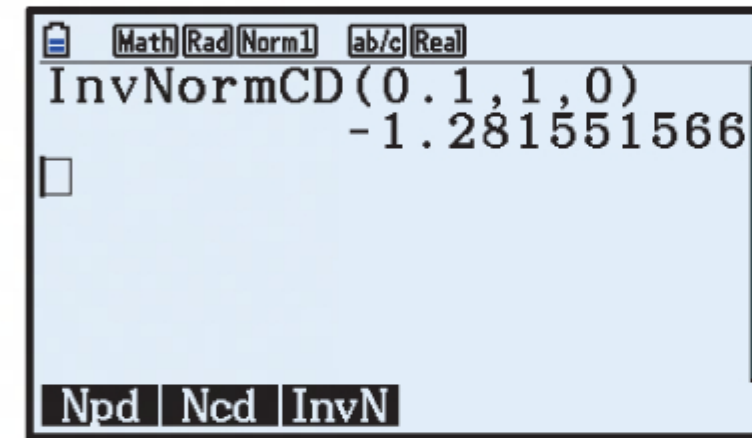


So, the standard deviation is approximately $\text{€}96.48$.

- 6 Let the mean arrival time be μ minutes after midday.
 If X denotes the arrival time of a bus, then $X \sim N(\mu, 5^2)$.
 3:55 pm is $3 \times 60 + 55 = 235$ minutes after midday.



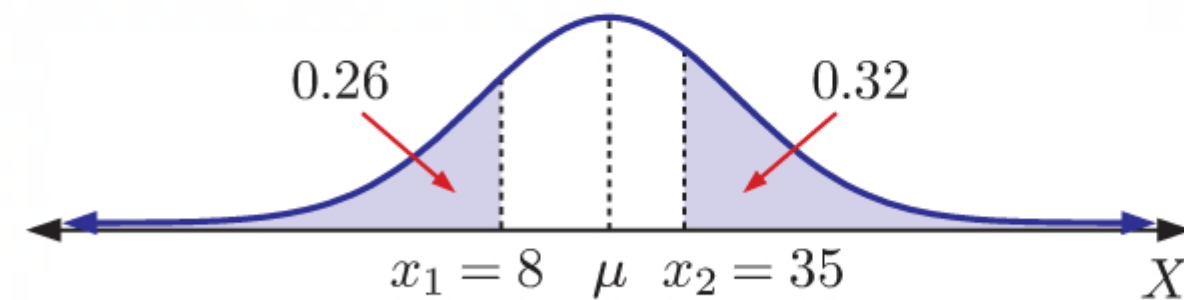
$$\begin{aligned} \text{So, } P(X \leq 235) &= 0.1 \\ \therefore P\left(Z \leq \frac{235 - \mu}{5}\right) &= 0.1 \\ \therefore \frac{235 - \mu}{5} &\approx -1.2815516 \\ \therefore 235 - \mu &\approx -6.407758 \\ \therefore \mu &\approx 241.407758 \end{aligned}$$



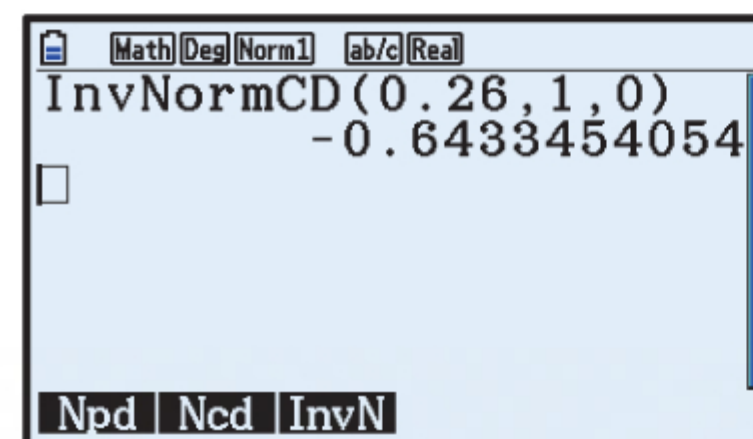
241.407758 minutes $\approx 4 \text{ h } 1 \text{ m } 24 \text{ s}$

\therefore the mean arrival time of buses at the depot is about 4:01:24 pm.

- 7 $X \sim N(\mu, \sigma^2)$ where we have to find μ and σ .
 We start by finding z_1 and z_2 which correspond to $x_1 = 8$ and $x_2 = 35$.

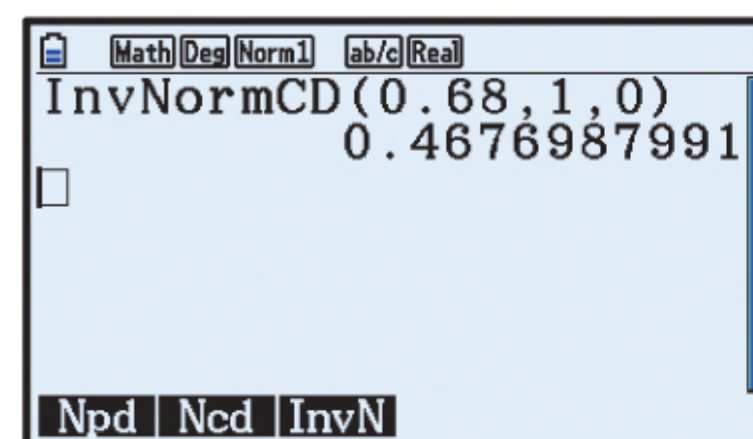


$$\begin{aligned} \text{Now } P(X \leq x_1) &= 0.26 \\ \therefore P\left(Z \leq \frac{8 - \mu}{\sigma}\right) &= 0.26 \\ \therefore z_1 = \frac{8 - \mu}{\sigma} &\approx -0.6433 \\ \therefore 8 - \mu &\approx -0.6433\sigma \quad \dots (1) \end{aligned}$$



$$\begin{aligned} \text{and } P(X \geq x_2) &= 0.32 \\ \therefore P(X < x_2) &= 0.68 \end{aligned}$$

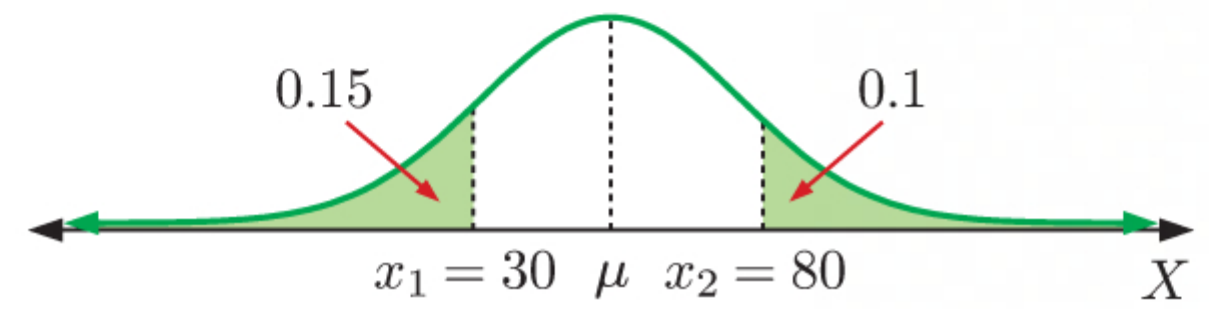
$$\begin{aligned} \therefore P\left(Z < \frac{35 - \mu}{\sigma}\right) &= 0.68 \\ \therefore z_2 = \frac{35 - \mu}{\sigma} &\approx 0.4677 \\ \therefore 35 - \mu &\approx 0.4677\sigma \quad \dots (2) \end{aligned}$$



Solving (1) and (2) simultaneously, $\mu \approx 23.6$ and $\sigma \approx 24.3$.

- 8 a $X \sim N(\mu, \sigma^2)$ where we have to find μ and σ .

We start by finding z_1 and z_2 which correspond to $x_1 = 30$ and $x_2 = 80$.



$$\text{Now } P(X \leq x_1) = 0.15$$

$$\therefore P\left(Z \leq \frac{30 - \mu}{\sigma}\right) = 0.15$$

$$\therefore z_1 = \frac{30 - \mu}{\sigma} \approx -1.0364$$

$$\therefore 30 - \mu \approx -1.0364\sigma \quad \dots (1)$$

$$\text{and } P(X \geq x_2) = 0.1$$

$$\therefore P(X < x_2) = 0.9$$

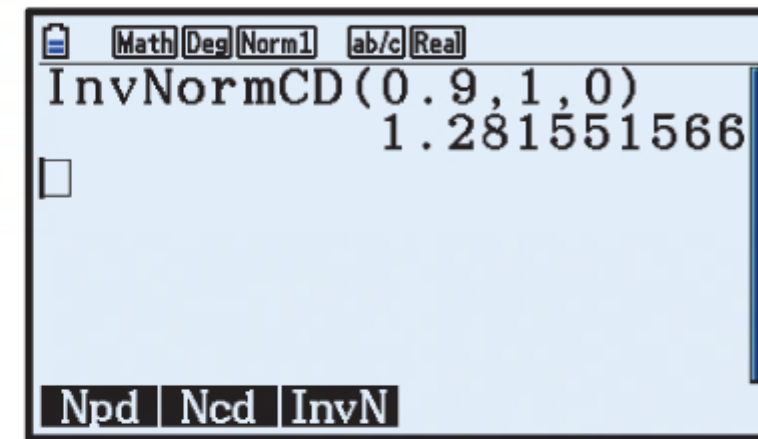
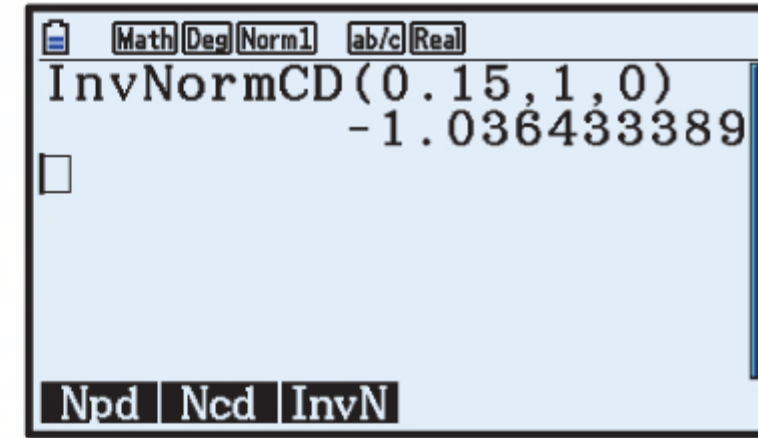
$$\therefore P\left(Z < \frac{80 - \mu}{\sigma}\right) = 0.9$$

$$\therefore z_2 = \frac{80 - \mu}{\sigma} \approx 1.2816$$

$$\therefore 80 - \mu \approx 1.2816\sigma \quad \dots (2)$$

Solving (1) and (2) simultaneously, $\mu \approx 52.35548$ and $\sigma \approx 21.57032$

$$\therefore \mu \approx 52.4 \quad \text{and} \quad \sigma \approx 21.6$$



- b Let X be the result of the Mathematics examination.

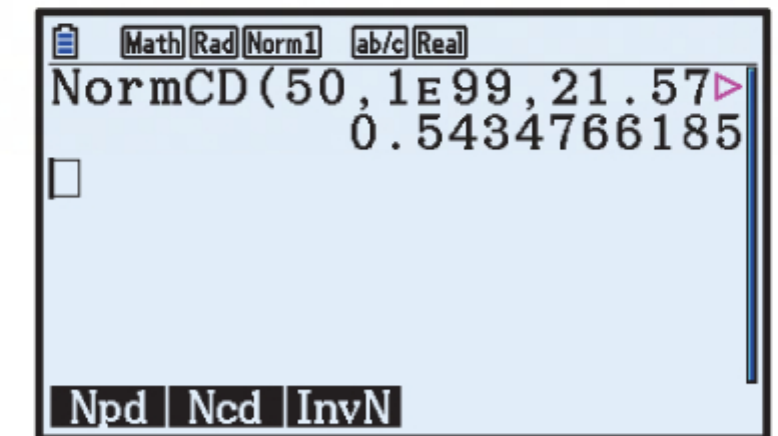
$$X \sim N(\mu, \sigma^2)$$

We know that $P(X \geq 80) = 0.1$ and $P(X \leq 30) = 0.15$.

So, from a, $\mu \approx 52.35548$ and $\sigma \approx 21.57032$.

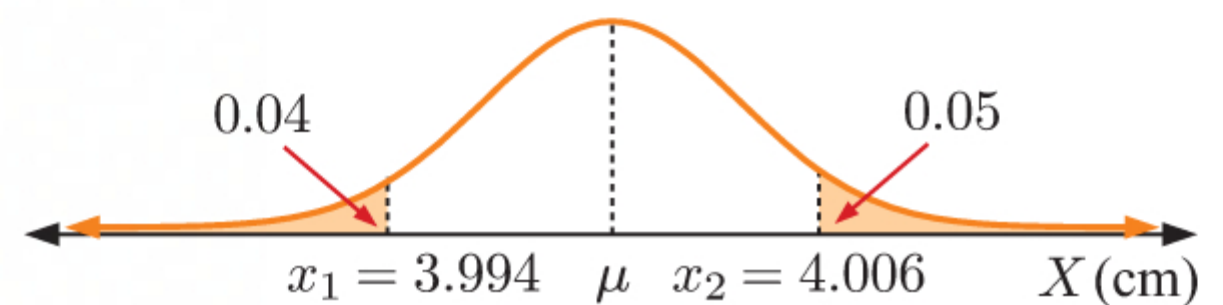
$$P(X > 50) \approx 0.543$$

So, approximately 54.3% of students scored more than 50.



- 9 a $X \sim N(\mu, \sigma^2)$ where we have to find μ and σ .

We start by finding z_1 and z_2 which correspond to $x_1 = 3.994$ and $x_2 = 4.006$.



$$\text{Now } P(X \leq x_1) = 0.04$$

$$\therefore P\left(Z \leq \frac{3.994 - \mu}{\sigma}\right) = 0.04$$

$$\therefore \frac{3.994 - \mu}{\sigma} \approx -1.750686$$

$$\therefore 3.994 - \mu \approx -1.750686\sigma \quad \dots (1)$$

$$\text{and } P(X \geq x_2) = 0.05$$

$$\therefore P(X < x_2) = 0.95$$

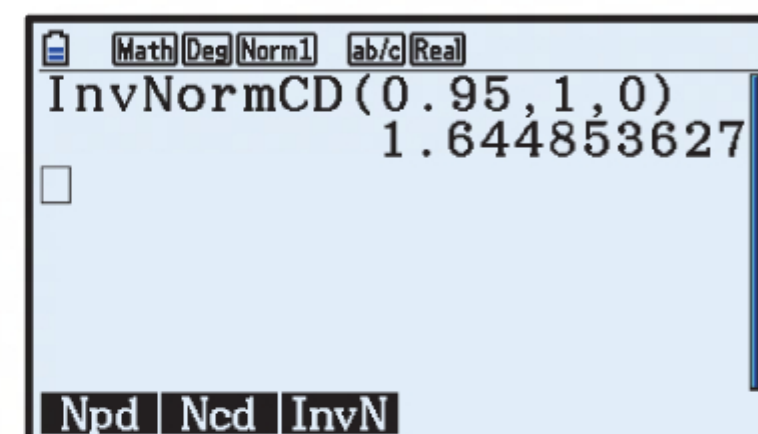
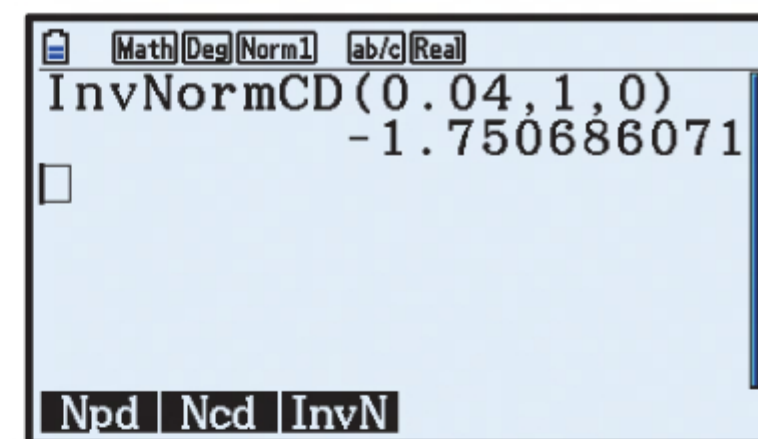
$$\therefore P\left(Z \leq \frac{4.006 - \mu}{\sigma}\right) = 0.95$$

$$\therefore \frac{4.006 - \mu}{\sigma} \approx 1.644854$$

$$\therefore 4.006 - \mu \approx 1.644854\sigma \quad \dots (2)$$

Solving (1) and (2) simultaneously, $\mu \approx 4.000187$ and $\sigma \approx 0.003534$

$$\therefore \mu \approx 4.00 \text{ cm} \quad \text{and} \quad \sigma \approx 0.00353 \text{ cm}$$

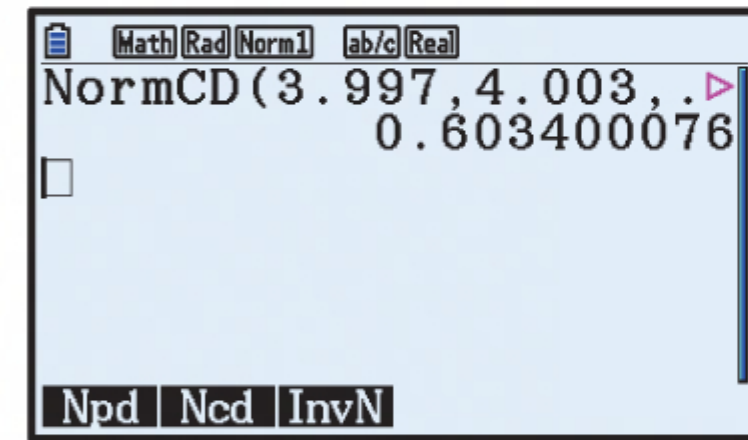


- b** From **a**, $\mu \approx 4.000187$ and $\sigma \approx 0.003534$

$$\therefore X \sim N(4.000187, 0.003534^2)$$

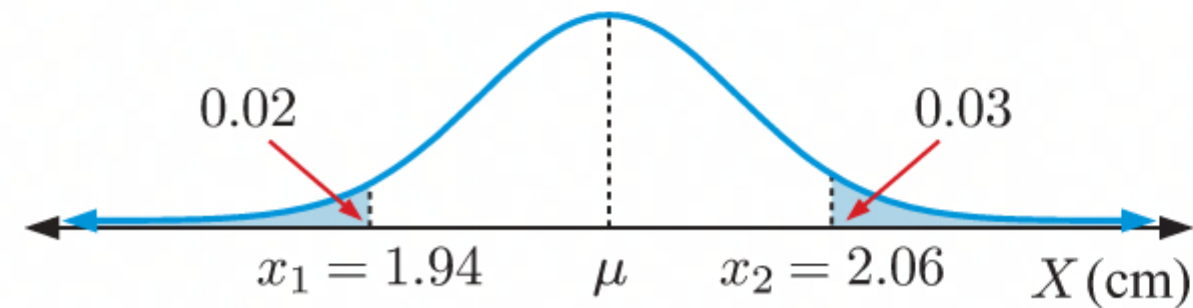
$$\therefore P(3.997 \leq X \leq 4.003) \approx 0.603$$

So, the probability that a randomly chosen piston has diameter between 3.997 cm and 4.003 cm is approximately 0.603.



- 10 a** $X \sim N(\mu, \sigma^2)$ where we have to find μ and σ .

We start by finding z_1 and z_2 which correspond to $x_1 = 1.94$ and $x_2 = 2.06$.



$$\text{Now } P(X < x_1) = 0.02$$

$$\therefore P\left(Z < \frac{1.94 - \mu}{\sigma}\right) = 0.02$$

$$\therefore z_1 = \frac{1.94 - \mu}{\sigma} \approx -2.053749$$

$$\therefore 1.94 - \mu \approx -2.053749\sigma \quad \dots (1)$$

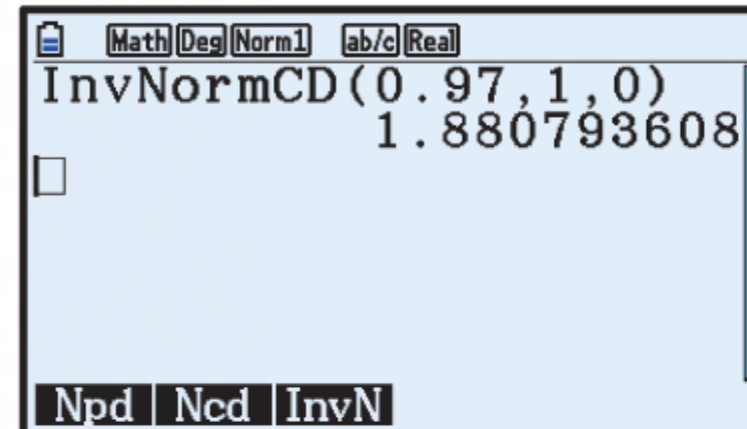
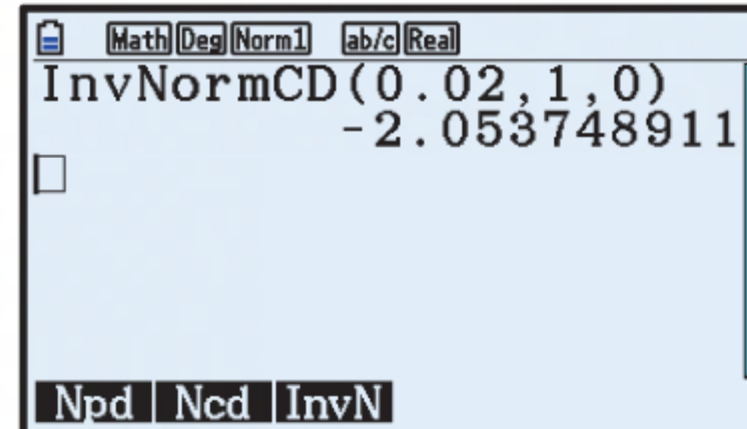
$$\text{and } P(X > x_2) = 0.03$$

$$\therefore P(X < 2.06) = 0.97$$

$$\therefore P\left(Z \leq \frac{2.06 - \mu}{\sigma}\right) = 0.97$$

$$\therefore z_2 = \frac{2.06 - \mu}{\sigma} \approx 1.880794$$

$$2.06 - \mu \approx 1.880794\sigma \quad \dots (2)$$



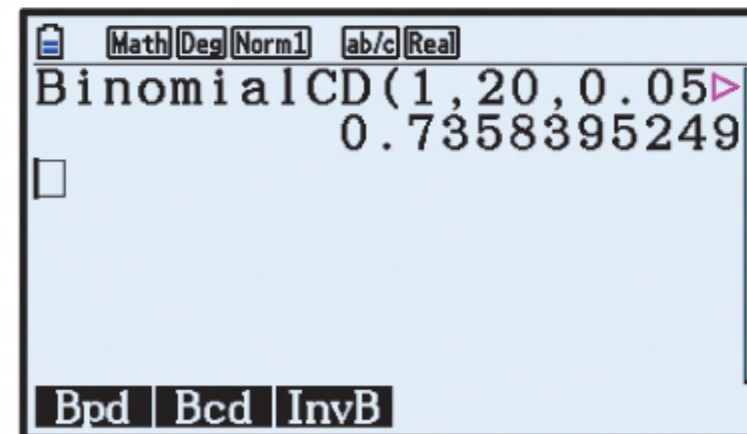
Solving (1) and (2) simultaneously, $\mu \approx 2.002637$ and $\sigma \approx 0.030499$

$$\therefore \mu \approx 2.00 \text{ cm} \quad \text{and} \quad \sigma \approx 0.0305 \text{ cm.}$$

- b** Let Y be the number of tokens which will not operate the machine. This is a binomial situation with the probability $p = 0.02 + 0.03 = 0.05$ of failure to operate and $n = 20$.

So, $Y \sim B(20, 0.05)$.

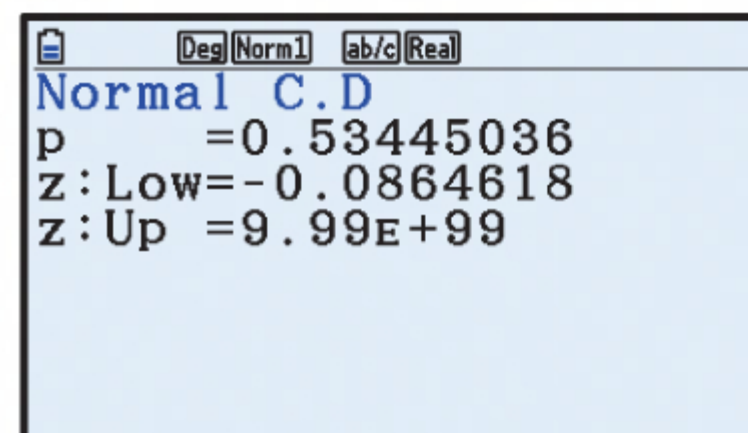
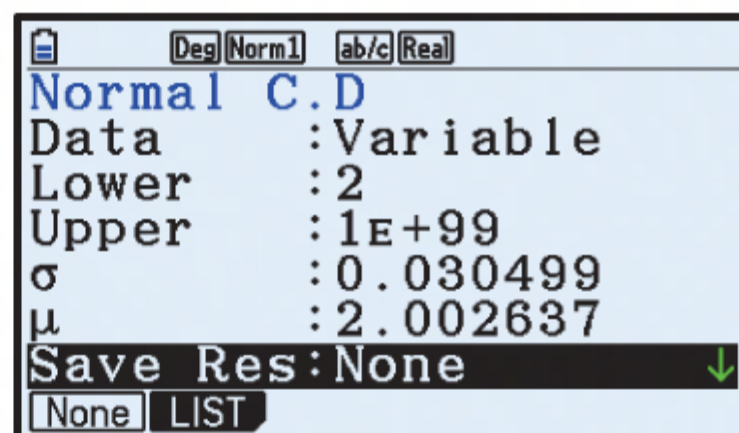
$$\therefore P(\text{at most one will not operate}) = P(Y \leq 1) \approx 0.736$$



- c** $P(X > 2) \approx 0.5345$

The probability of one token being greater than 2 cm is approximately 0.5345.

$$\therefore P(Y = 3) \approx (0.5345)^3 \approx 0.153$$



INVESTIGATION 4**THE NORMAL APPROXIMATION TO THE BINOMIAL DISTRIBUTION**

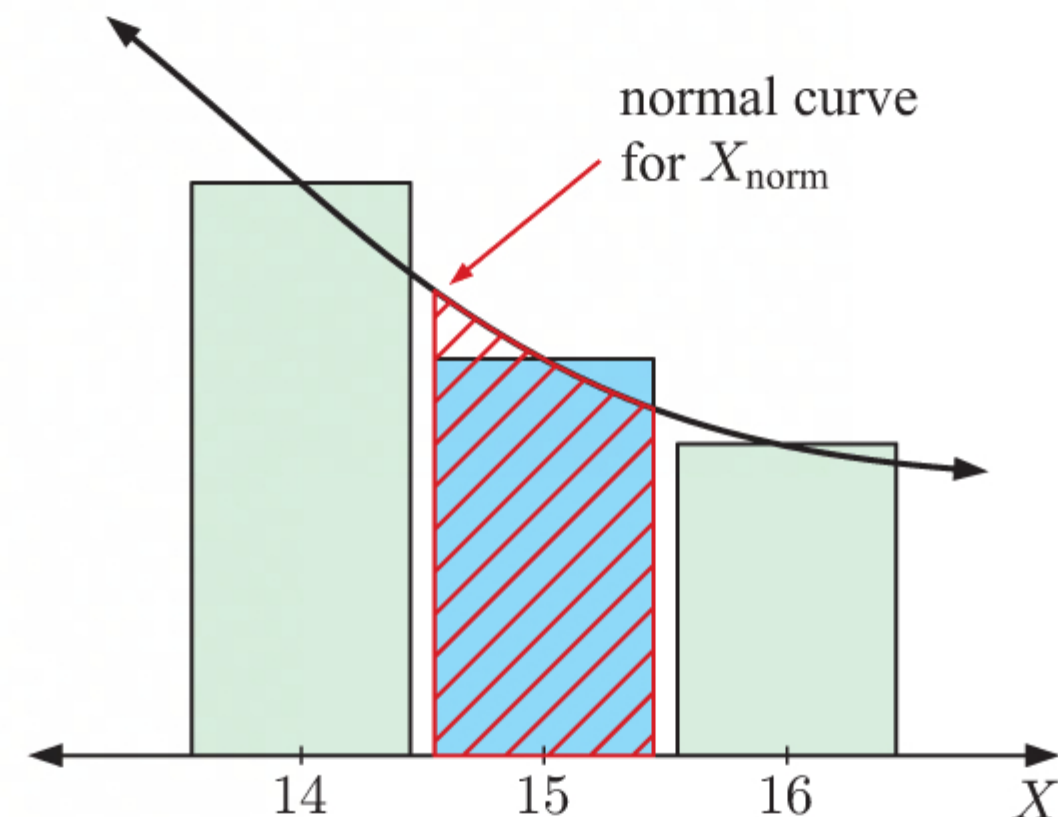
- 1
 - a As n increases, the distribution of X approaches that of the normal distribution.
 - b As n increases, the distribution of X approaches that of the normal distribution for all values of p used.
 - c It is reasonable to approximate the binomial distribution with a normal distribution as long as n is sufficiently large. The distribution should be symmetrical about the most commonly occurring value.
 - d We expect that $\mu = np$ and $\sigma = \sqrt{np(1-p)}$, like that of the binomial distribution.

- 2
 - a

$$\begin{aligned}\mu &= np \\ &= 50 \times 0.2 \\ &= 10\end{aligned}$$

$$\begin{aligned}\sigma &= \sqrt{np(1-p)} \\ &= \sqrt{50 \times 0.2 \times 0.8} \\ &= \sqrt{8} \\ &\approx 2.83\end{aligned}$$

- b The blue shaded area represents $P(X = 15)$.
The red shaded area represents $P(14.5 \leq X_{\text{norm}} \leq 15.5)$.
The two areas are approximately equal.
 $\therefore P(X = 15) \approx P(14.5 \leq X_{\text{norm}} \leq 15.5)$.



- c $X_{\text{norm}} \sim N(10, (\sqrt{8})^2)$
 - i $P(X \leq 10) \approx P(X_{\text{norm}} \leq 10.5)$
 - ii $P(X < 25) = P(X \leq 24) \approx P(X_{\text{norm}} \leq 24.5)$
 - iii $P(10 \leq X < 25) = P(10 \leq X \leq 24) \approx P(9.5 \leq X_{\text{norm}} \leq 24.5)$

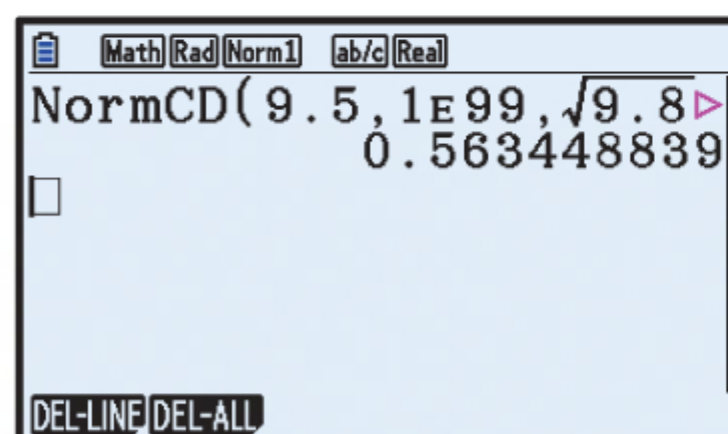
- 3 $X \sim B(500, 0.02)$

$$\begin{aligned}\mu &= np \\ &= 500 \times 0.02 \\ &= 10\end{aligned}$$

$$\begin{aligned}\sigma &= \sqrt{np(1-p)} \\ &= \sqrt{500 \times 0.02 \times 0.98} \\ &= \sqrt{9.8} \\ &\approx 3.13\end{aligned}$$

$$X_{\text{norm}} \sim N(10, (\sqrt{9.8})^2)$$

$$\begin{aligned}P(X \geq 10) &\approx P(X_{\text{norm}} \geq 9.5) \\ &\approx 0.563\end{aligned}$$

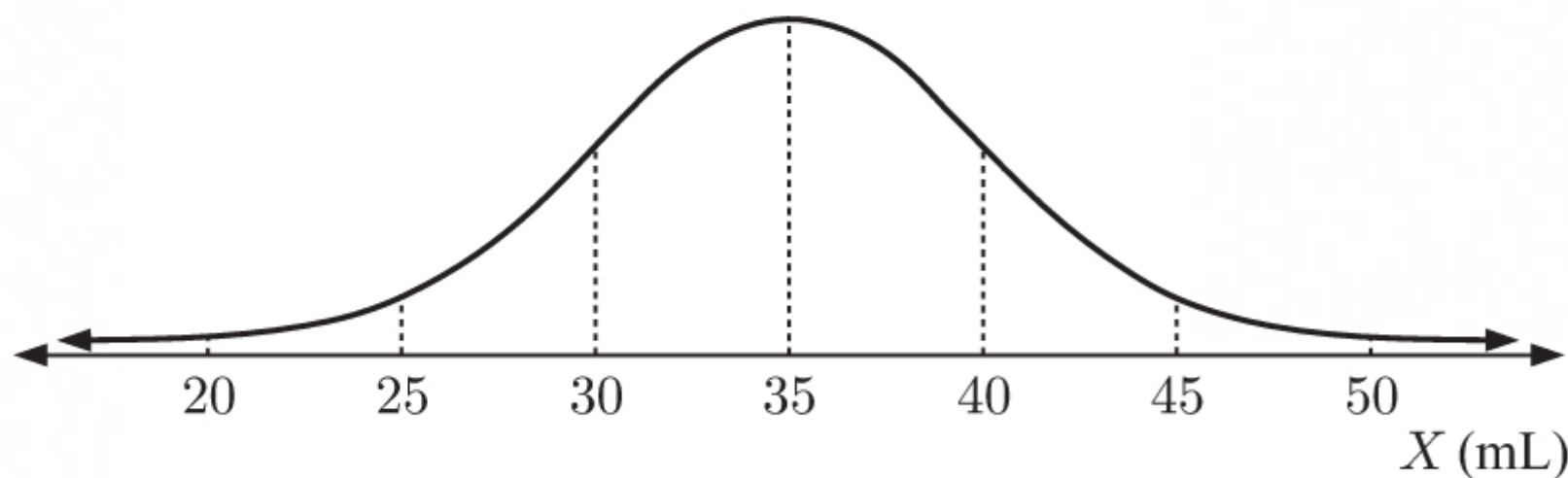


\therefore we estimate that the probability that at least 10 tyres in the sample will be unfit for sale is approximately 0.563.

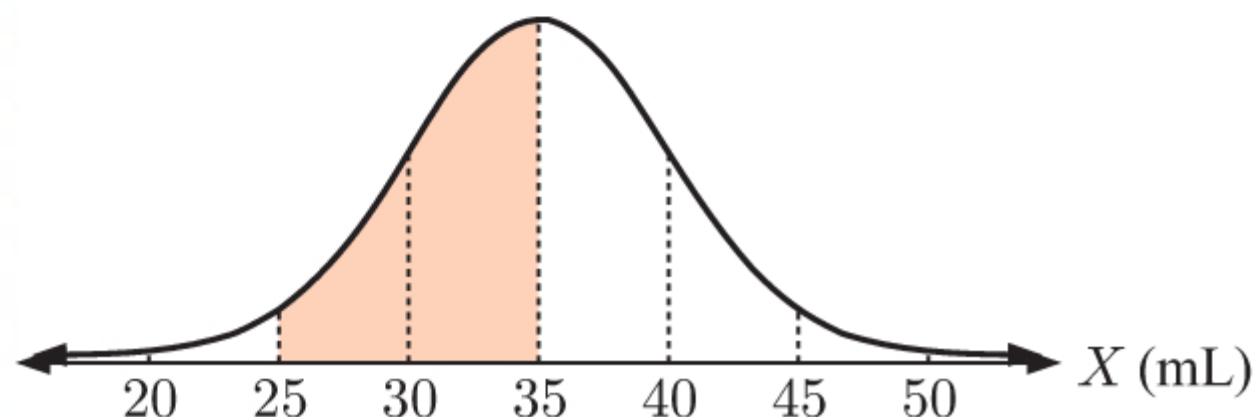
REVIEW SET 21A

- 1 a The distribution of times taken for students to read a novel is likely to be positively skewed, and hence not normal.
- b The mean amount spent on groceries at a supermarket is likely to occur most often with variations around the mean occurring symmetrically as a result of random variation in the prices of items bought and/or the quantities of items bought (for example weights of fruits and vegetables). So, the distribution is likely to be normal.

2 a

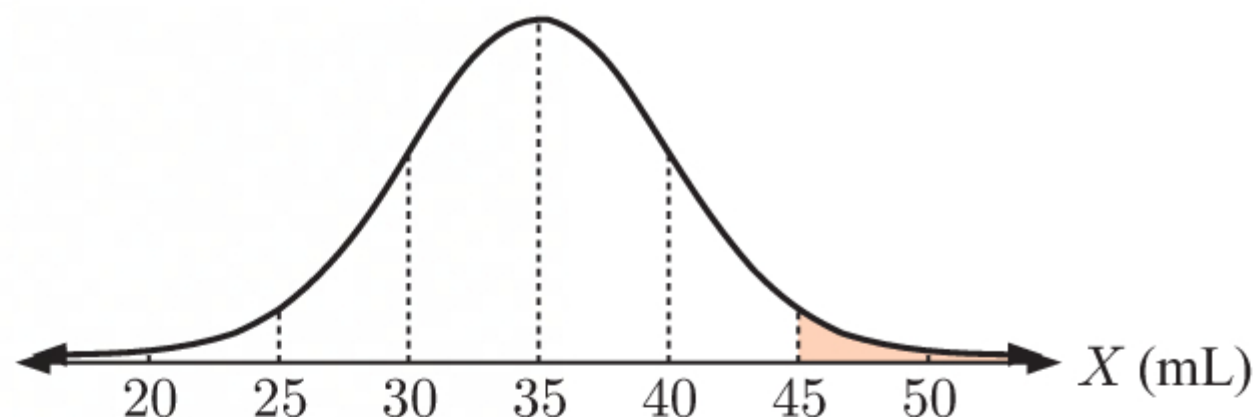


b i



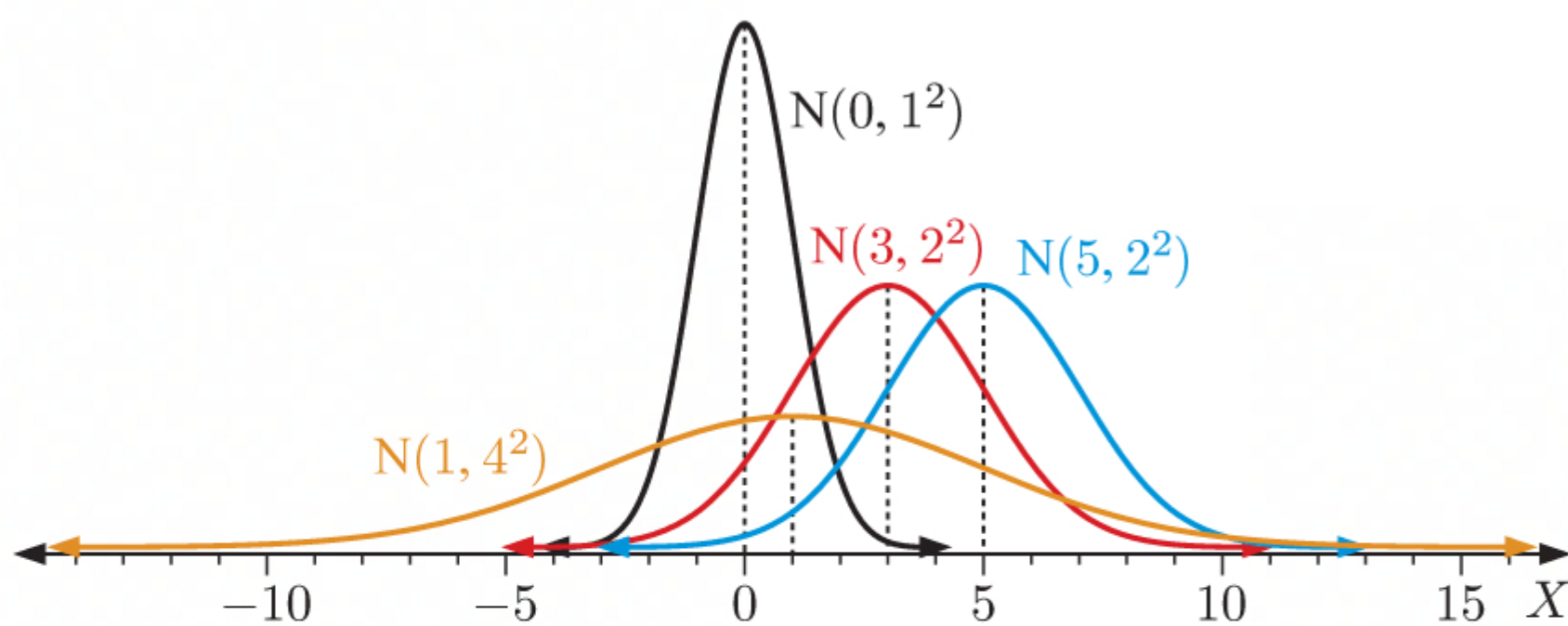
About $13.59\% + 34.13\% = 47.72\%$ of Simon's lemons will produce between 25 mL and 35 mL of juice.

ii



About $2.15\% + 0.13\% = 2.28\%$ of Simon's lemons will produce at least 45 mL of juice.

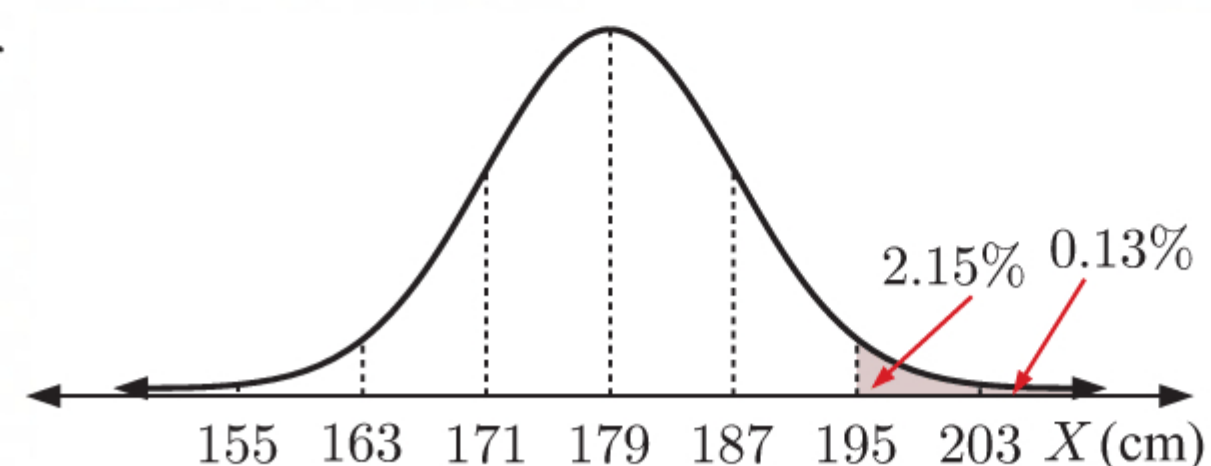
3



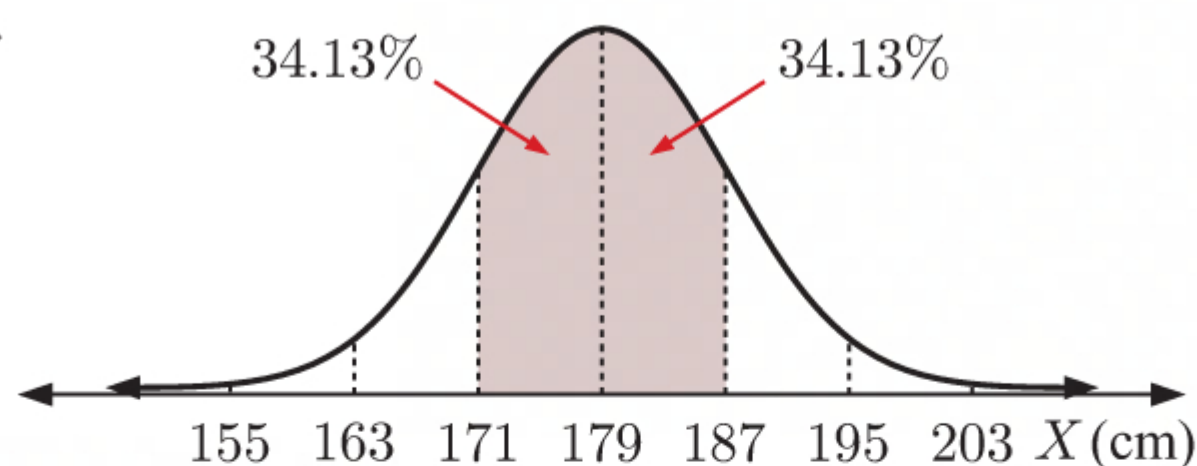
- 4 Let X be the height of a 17 year old boy.

$$X \sim N(179, 8^2)$$

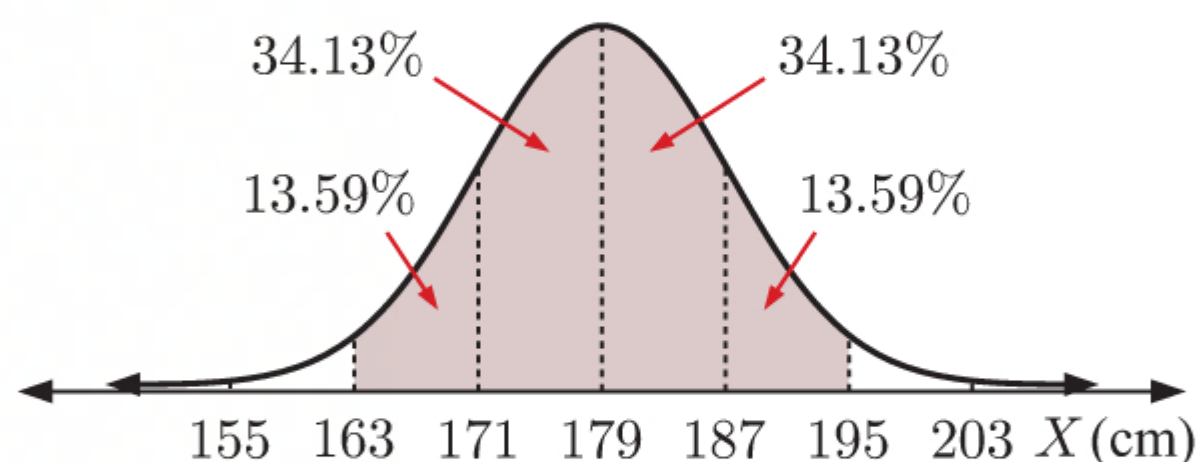
- a About $2.15\% + 0.13\% = 2.28\%$ of 17 year old boys have a height more than 195 cm.



- b** About $34.13\% + 34.13\% = 68.26\%$ of 17 year old boys have a height between 171 cm and 187 cm.

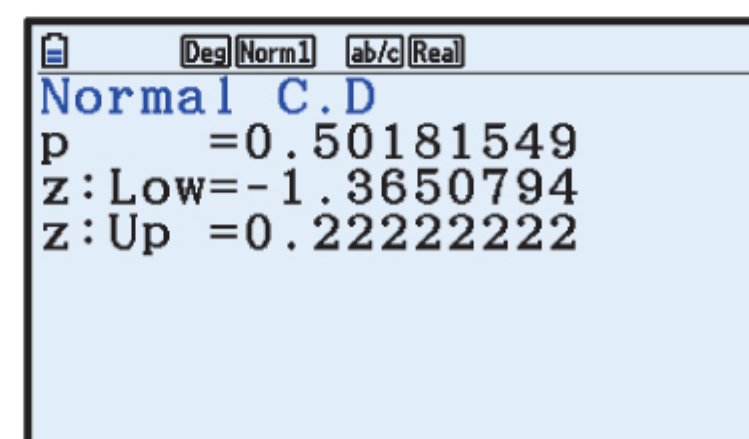
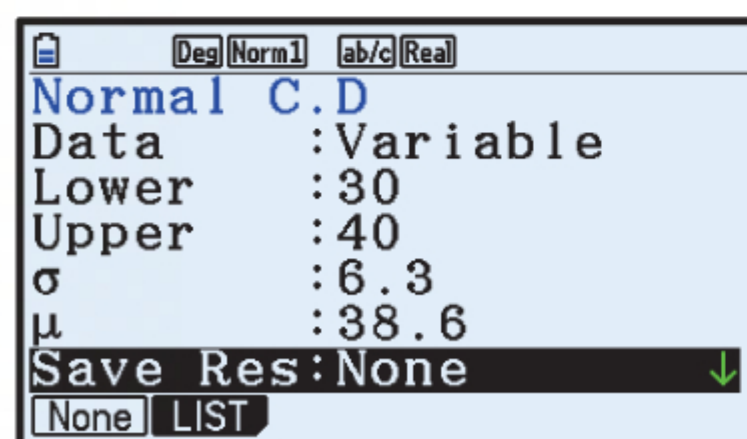
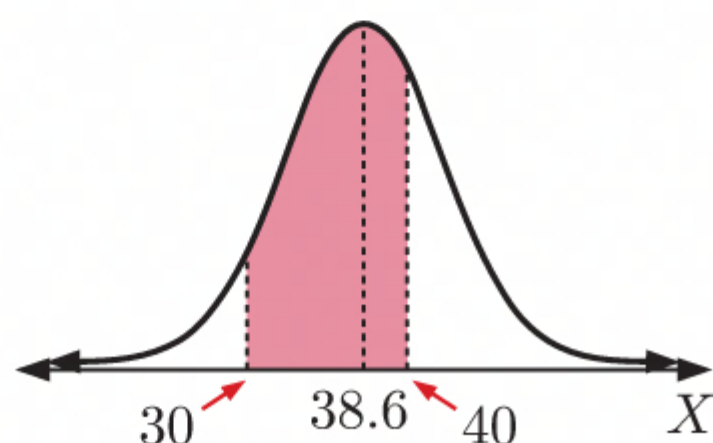


- c** About $13.59\% + 34.13\% + 34.13\% + 13.59\% = 95.44\%$ of 17 year old boys have a height between 163 cm and 195 cm.



- 5** Let X grams be the weight of the edible part of a randomly selected Coffin Bay oyster.

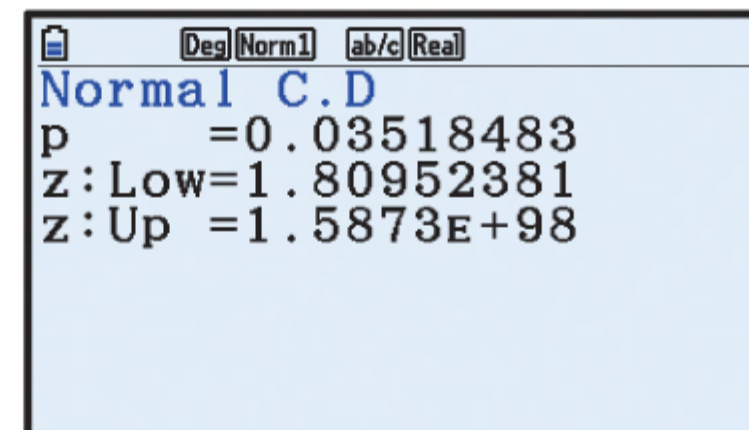
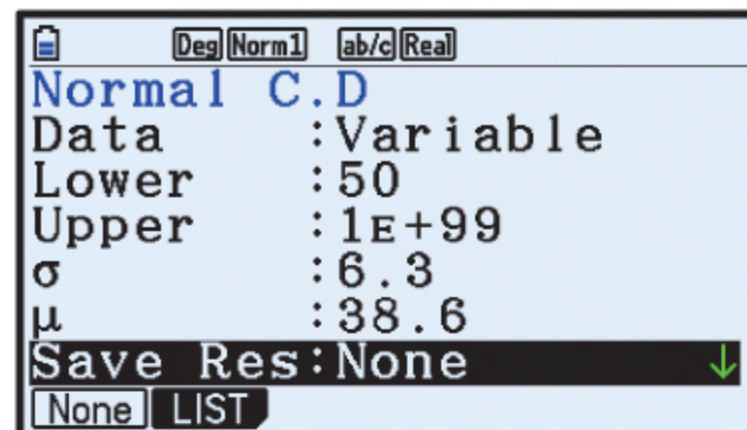
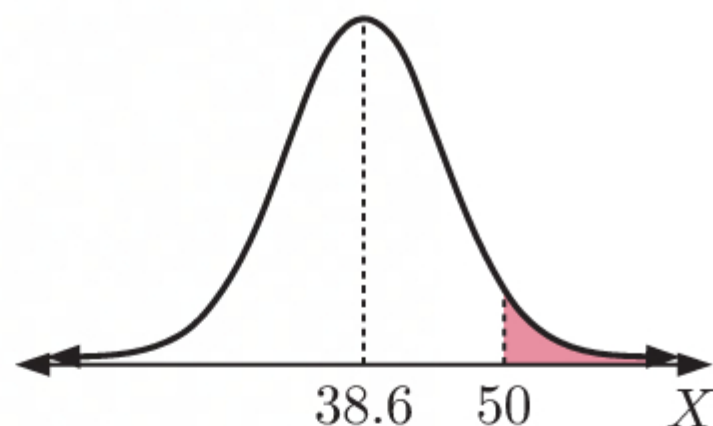
a



$$P(30 < X < 40) \approx 0.502 \approx 50.2\%$$

About 50.2% of oysters have an edible part that weighs between 30 g and 40 g.

b

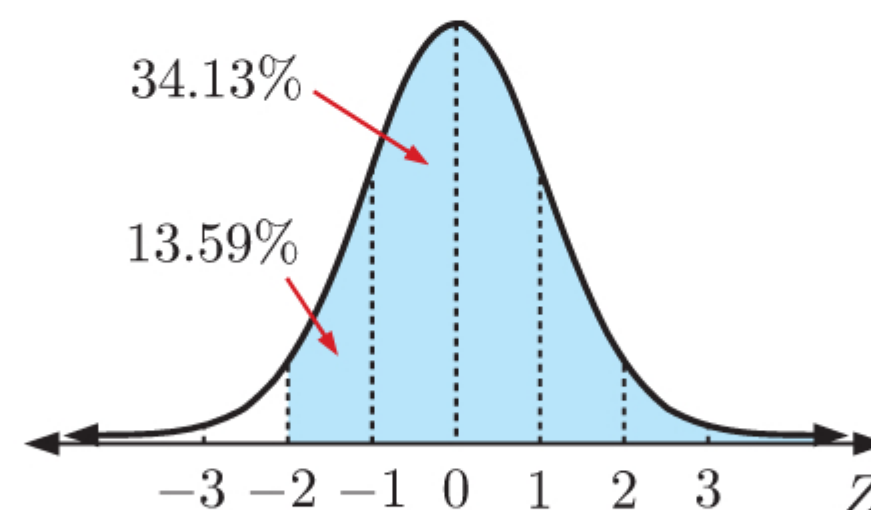


$$P(X > 50) \approx 0.0352$$

\therefore we would expect about $0.0352 \times 200 \approx 7$ oysters to have an edible part that weighs more than 50 g.

- 6 a** A z -score of -2 indicates that Harri's score is 2 standard deviations below the mean.

- b** About $13.59\% + 34.13\% + 50\% = 97.72\%$ of students obtained a better score than Harri.

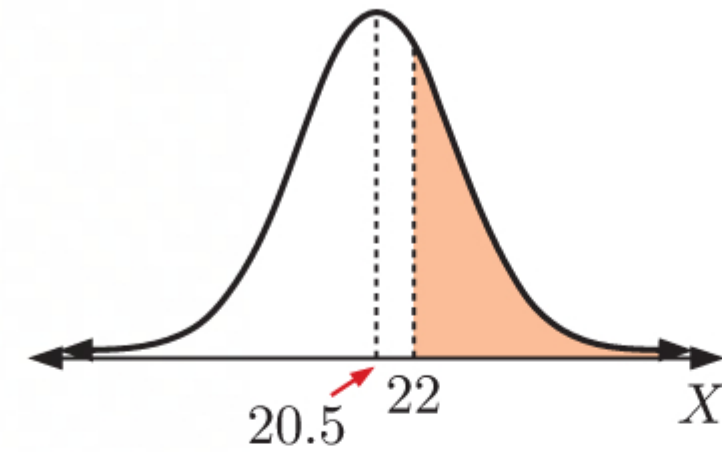
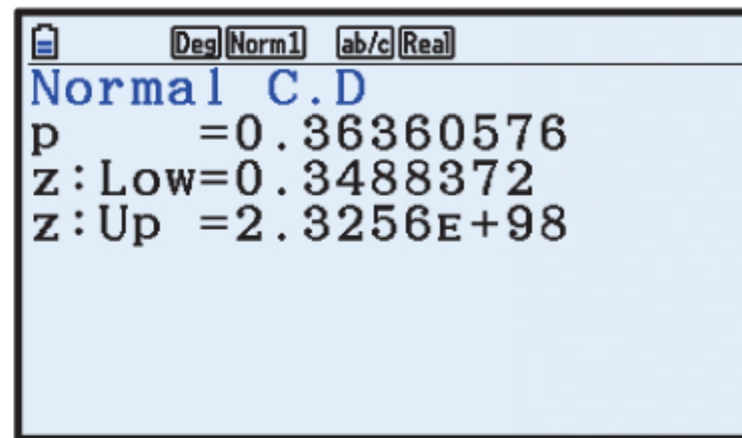
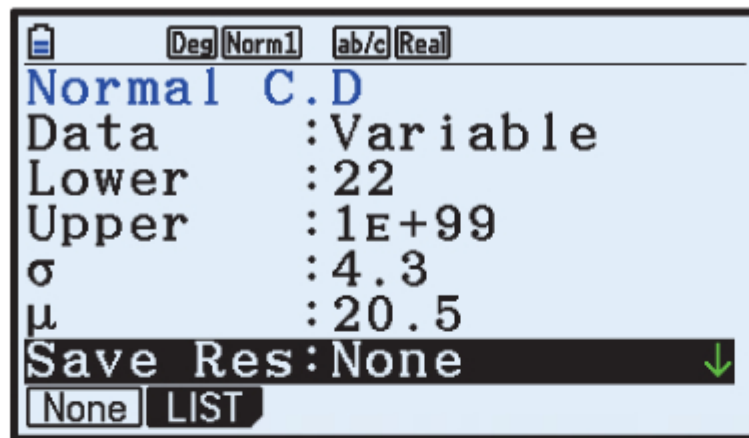


- c** $\mu = 61$ and $\mu - 2\sigma = 47$
 $\therefore 61 - 2\sigma = 47$
 $\therefore 2\sigma = 14$
 $\therefore \sigma = 7$

The standard deviation of the test scores was 7.

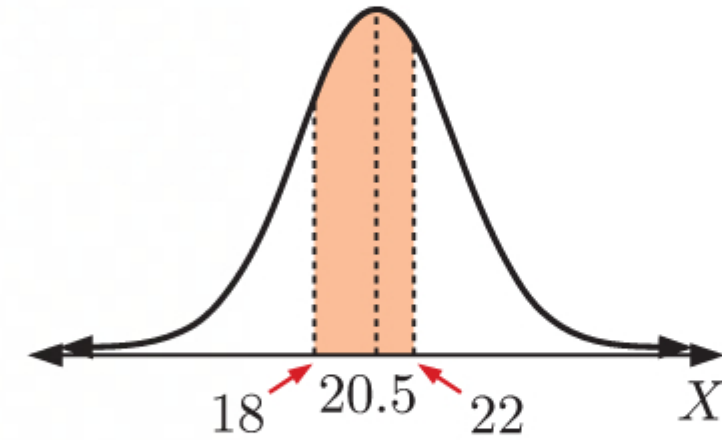
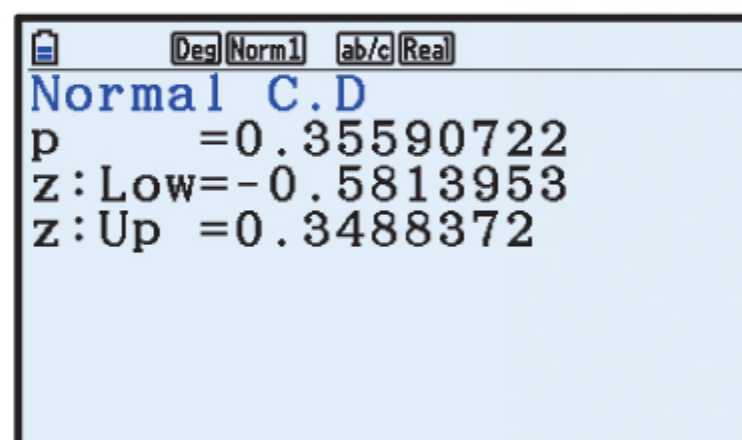
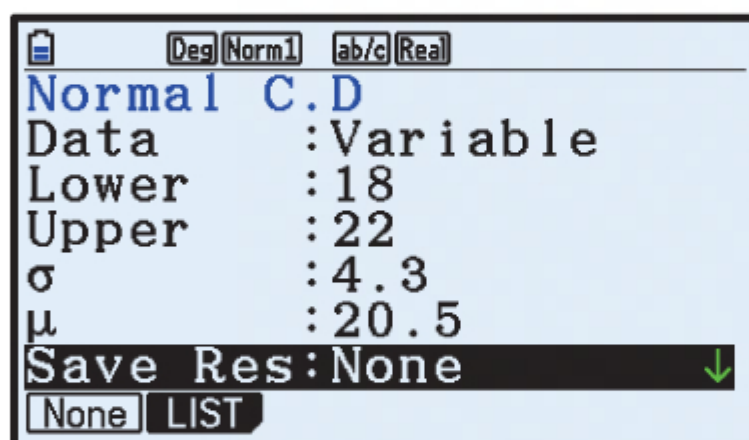
7 $X \sim N(20.5, 4.3^2)$

a



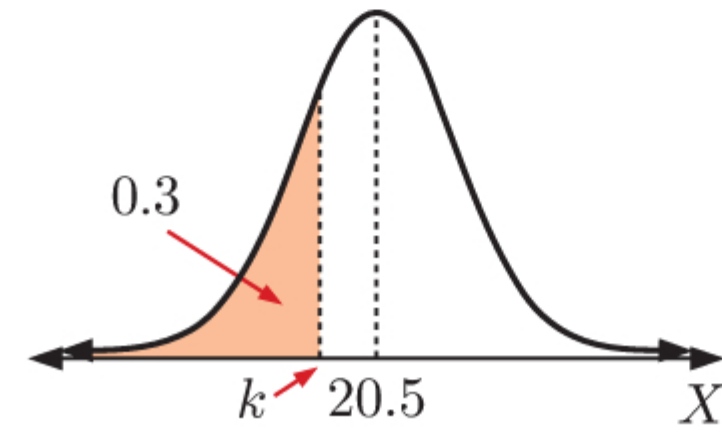
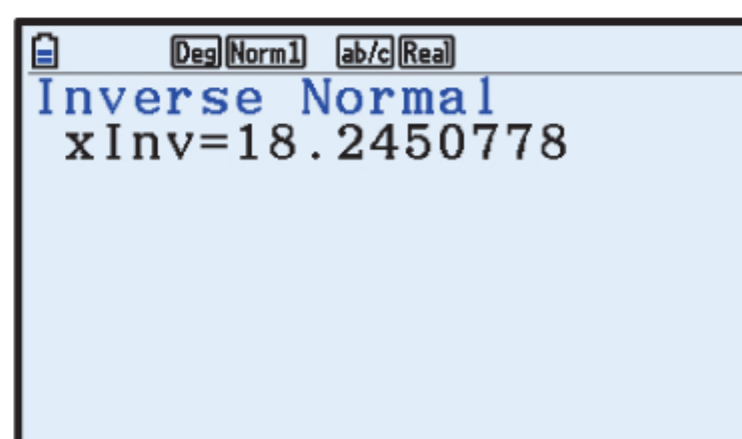
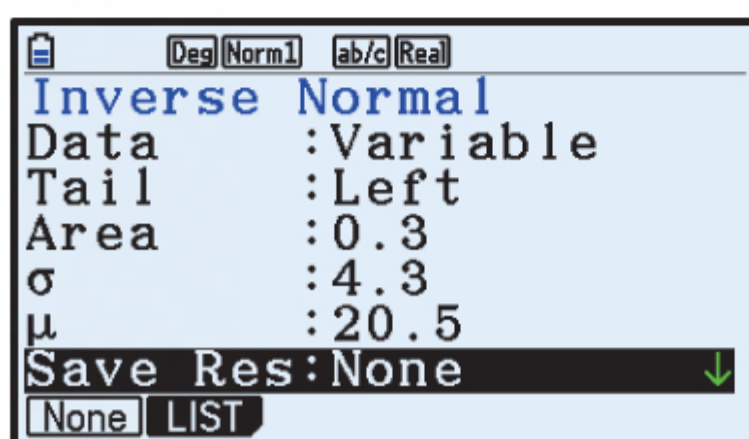
$P(X \geq 22) \approx 0.364$

b



$P(18 \leq X \leq 22) \approx 0.356$

c

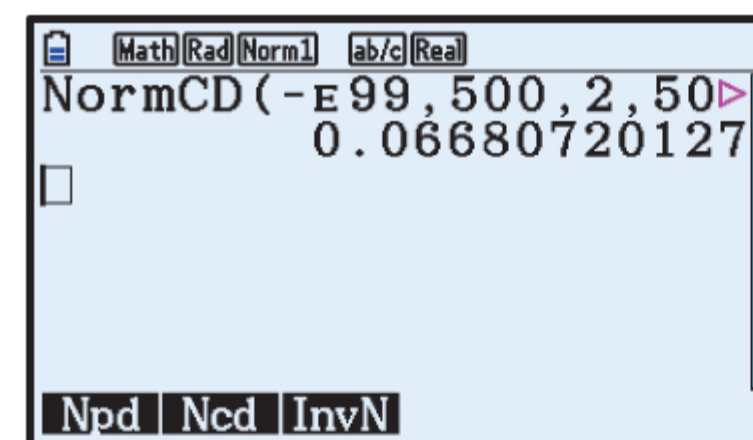


If $P(X \leq k) = 0.3$
then $k \approx 18.2$

8 a $X \sim N(503, 2^2)$

$P(X < 500) \approx 0.066\ 807$
 ≈ 0.0668

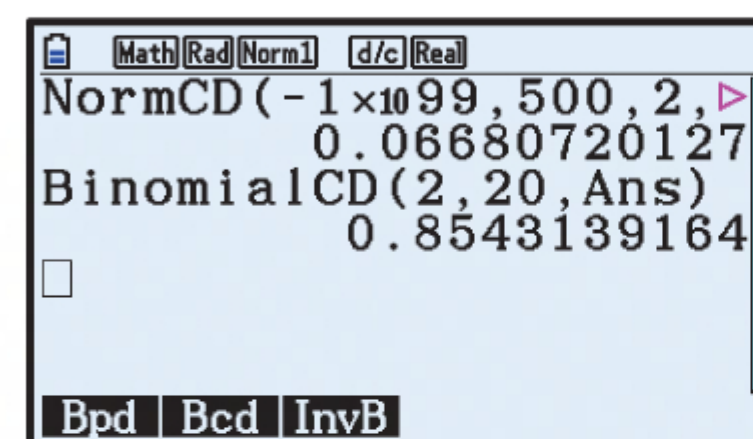
\therefore approximately 6.68% of the bags are underweight.



b Let Y be the number of bags which are underweight.

$Y \sim B(20, 0.066\ 807)$

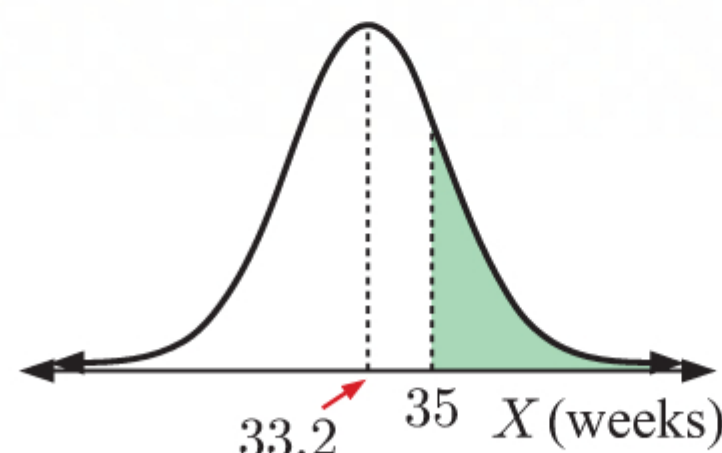
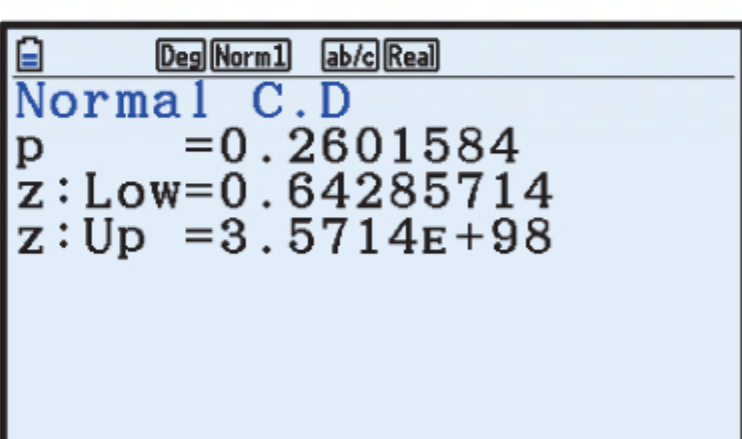
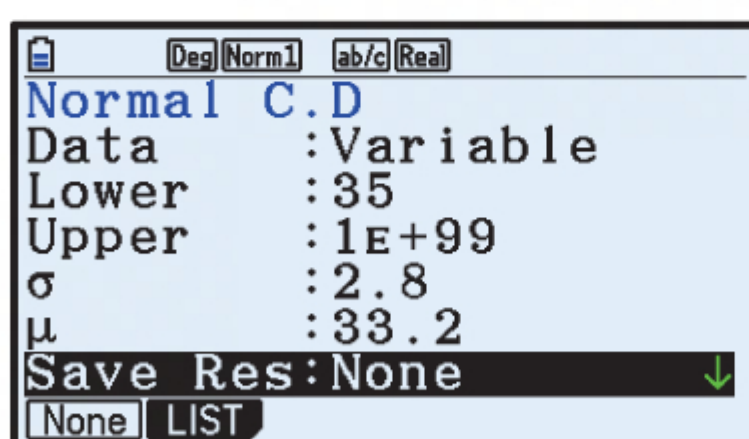
$\therefore P(Y \leq 2) \approx 0.854$



9 Let X be the life of a battery in weeks.

$X \sim N(33.2, 2.8^2)$

a



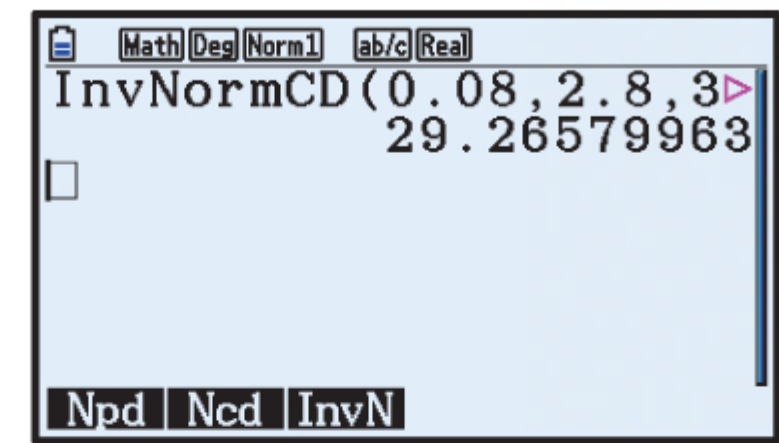
$P(X \geq 35) \approx 0.260$

- b** We need to find k such that

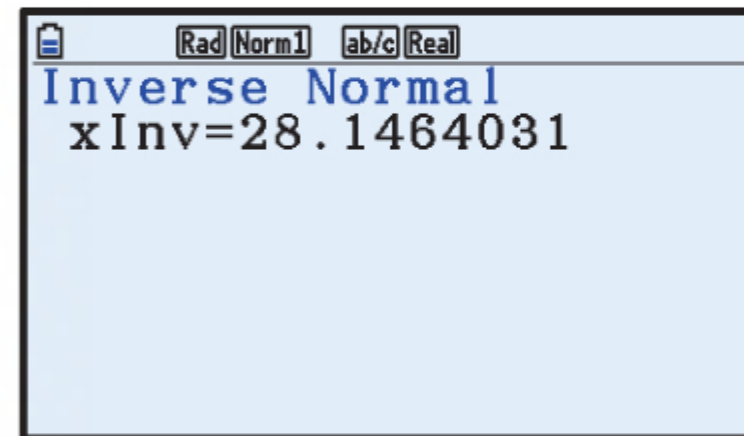
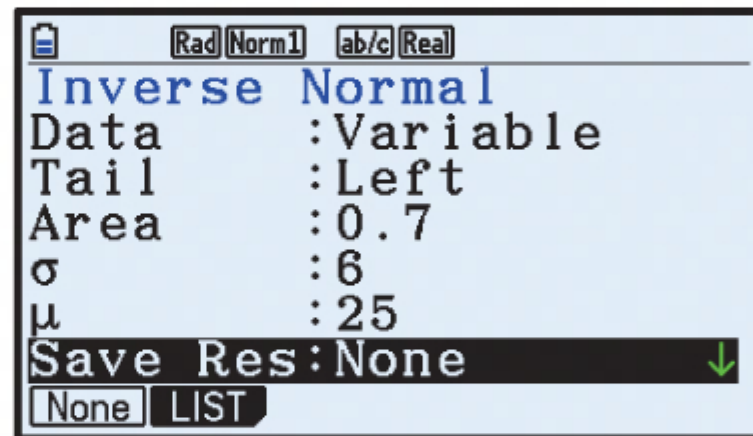
$$P(X \leq k) = 0.08$$

$$\therefore k \approx 29.3$$

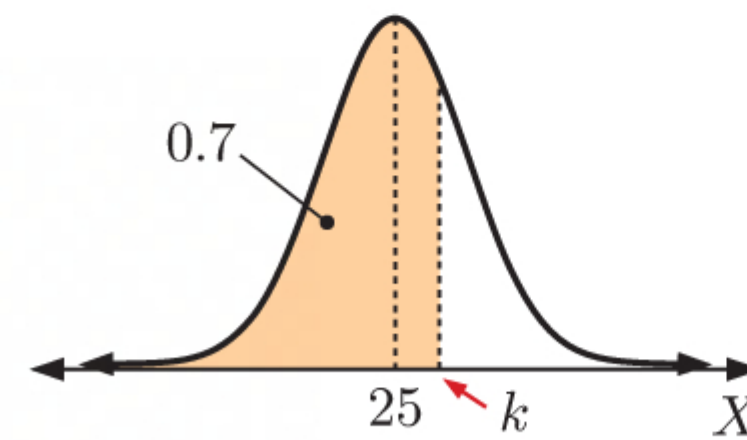
So, the manufacturer can expect the batteries to last about 29.3 weeks before 8% of them fail.



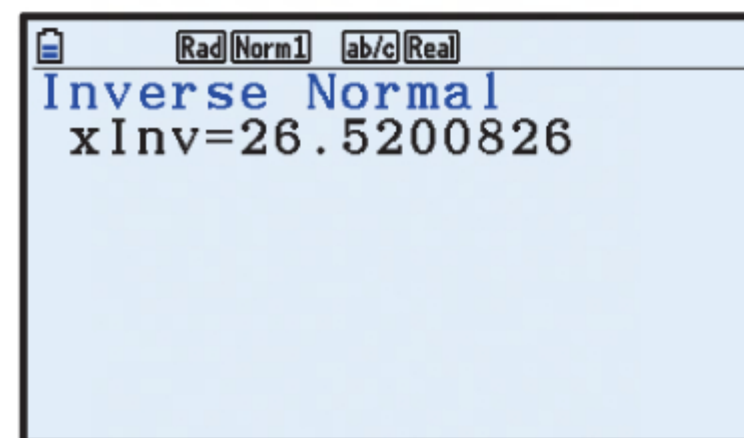
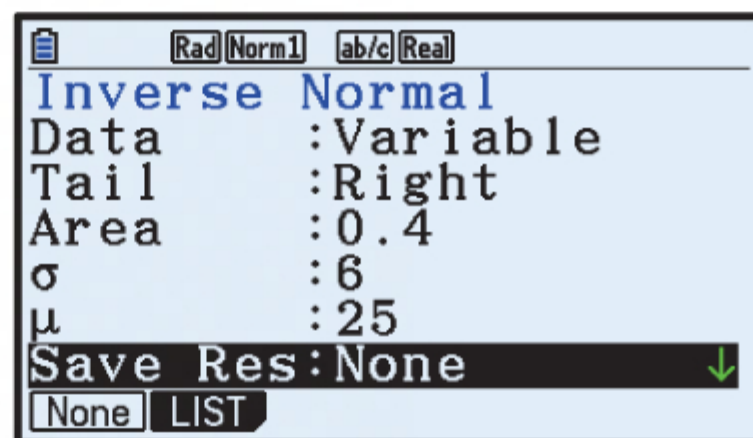
10 a



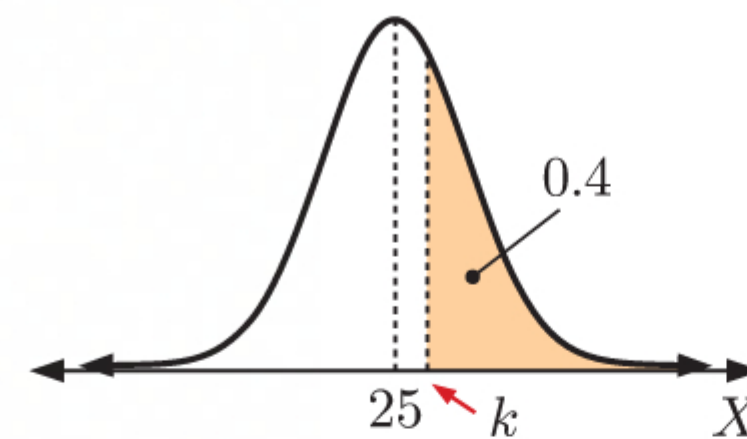
If $P(X \leq k) = 0.7$
then $k \approx 28.1$



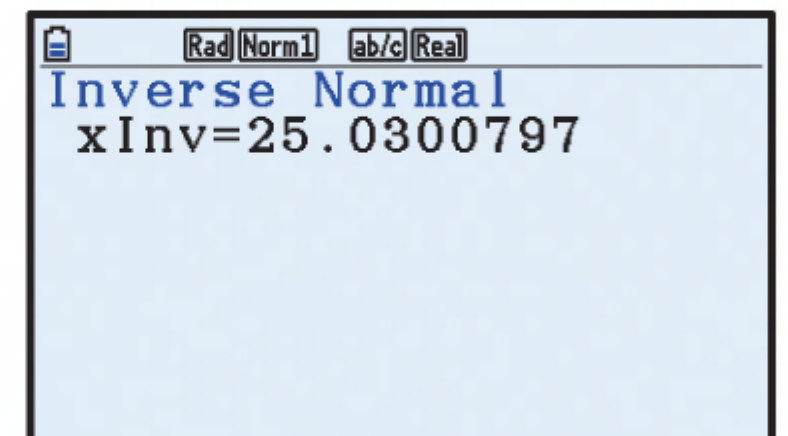
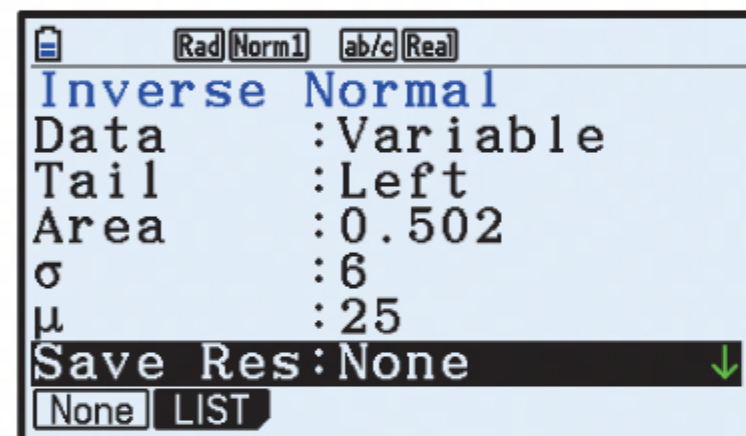
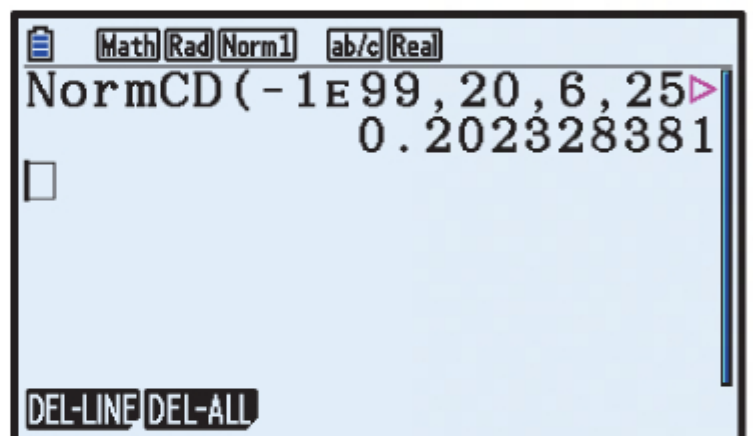
b



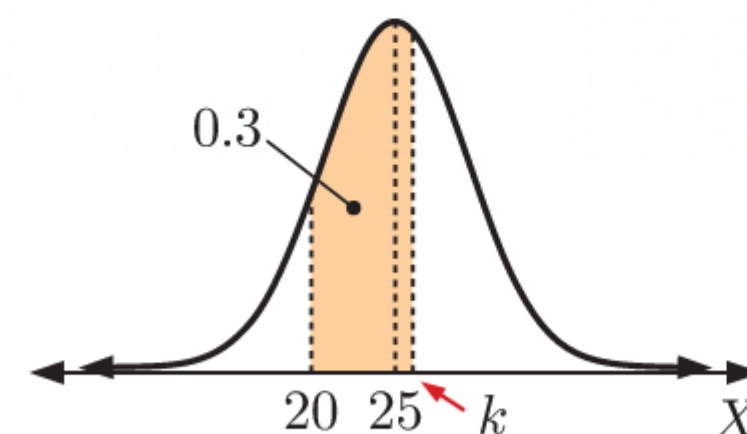
If $P(X \geq k) = 0.4$
then $k \approx 26.5$



c

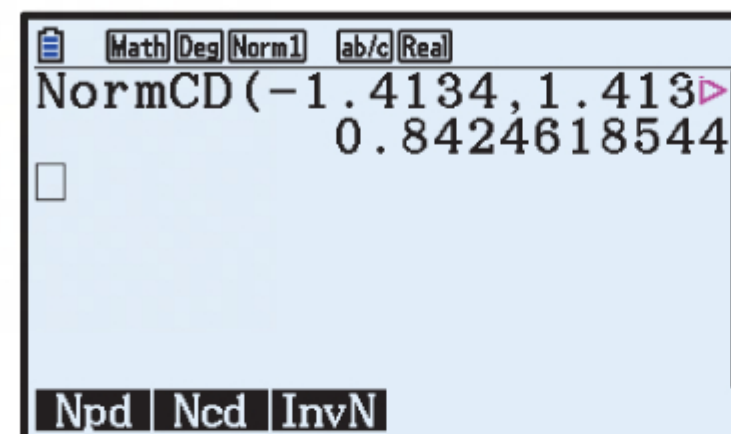


$$\begin{aligned} P(20 \leq X \leq k) &= 0.3 \\ \therefore P(X \leq k) - P(X \leq 20) &= 0.3 \\ \therefore P(X \leq k) - 0.202 &\approx 0.3 \\ \therefore P(X \leq k) &\approx 0.502 \\ \therefore k &\approx 25.0 \end{aligned}$$



11 $X \sim N(\mu, 2.83^2)$

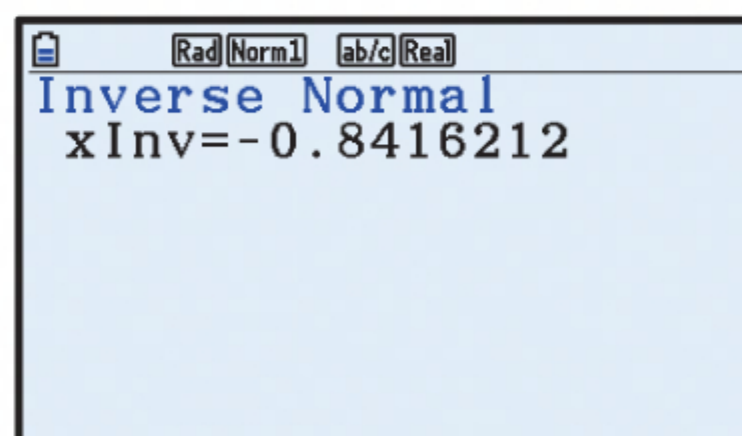
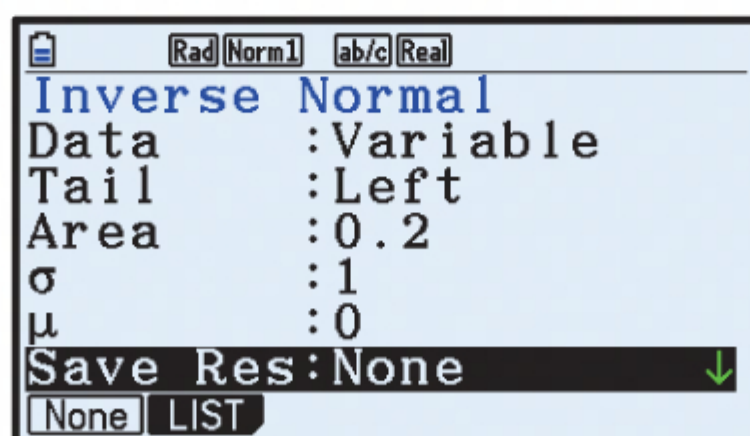
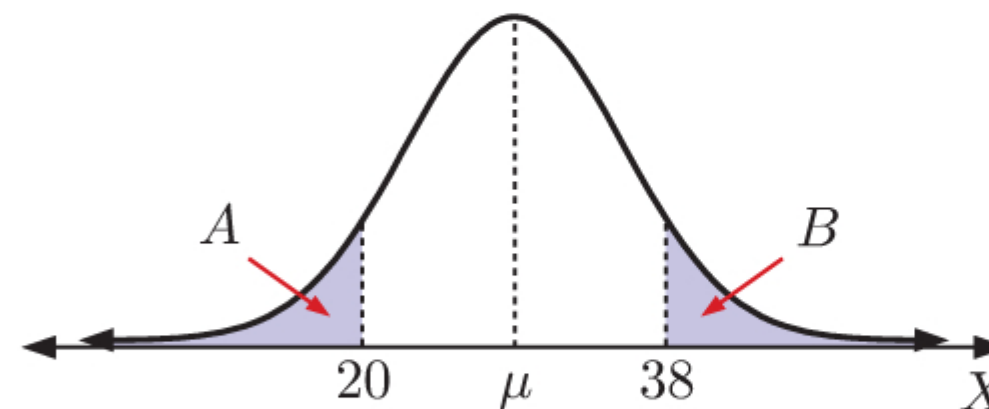
$$\begin{aligned}\therefore P(-4 < X - \mu < 4) &= P\left(\frac{-4}{2.83} < \frac{X - \mu}{2.83} < \frac{4}{2.83}\right) \\ &\approx P(-1.4134 < Z < 1.4134) \\ &\approx 0.842\end{aligned}$$



- 12 a** Since Area $A = \text{Area } B$, 20 and 38 must be equal distances away from the mean μ , because of the symmetry of the normal distribution.

$\therefore \mu$ is halfway between 20 and 38, so

$$\mu = \frac{20 + 38}{2} = 29$$



Now $P(X \leq 20) = 0.2$

$$\therefore P\left(Z \leq \frac{20 - 29}{\sigma}\right) = 0.2$$

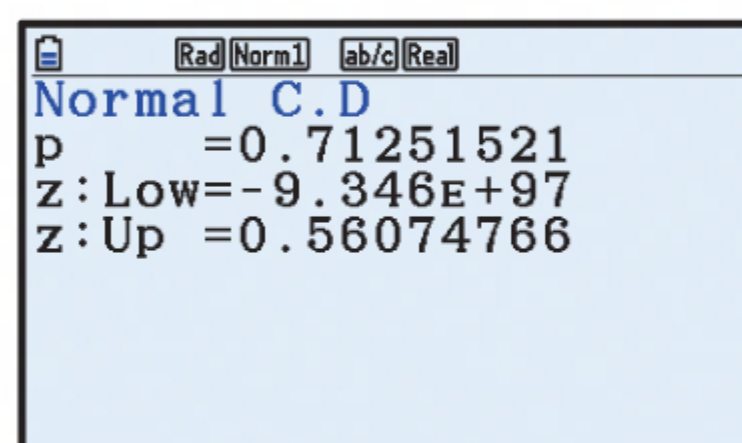
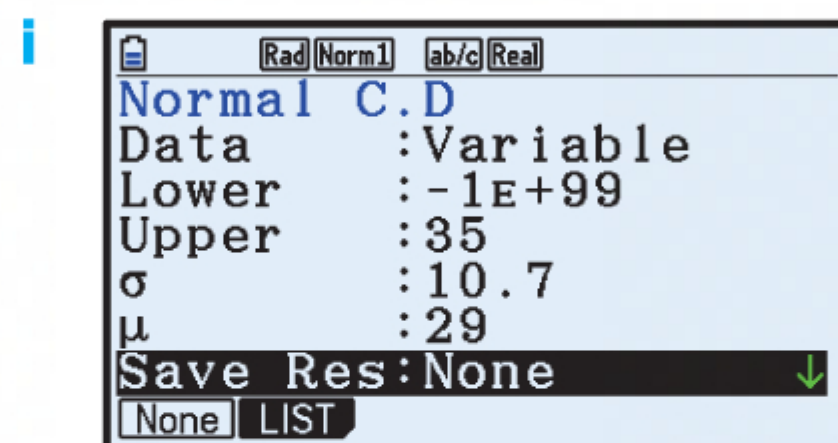
$$\therefore P\left(Z \leq -\frac{9}{\sigma}\right) = 0.2$$

$$\therefore -\frac{9}{\sigma} \approx -0.8416$$

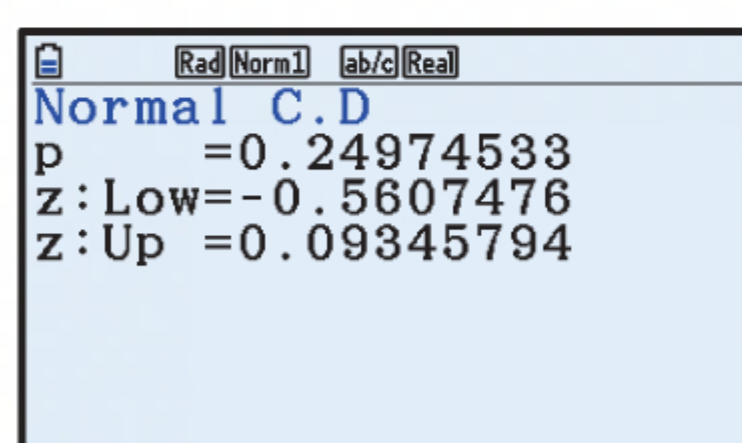
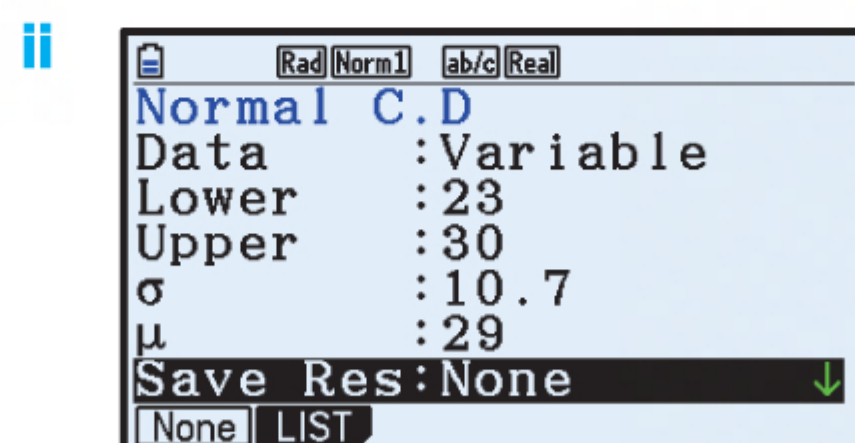
$$\therefore \sigma \approx 10.69$$

$$\therefore \mu = 29, \sigma \approx 10.7$$

- b** Using the values obtained for μ and σ in **a**:



$$P(X \leq 35) \approx 0.713$$

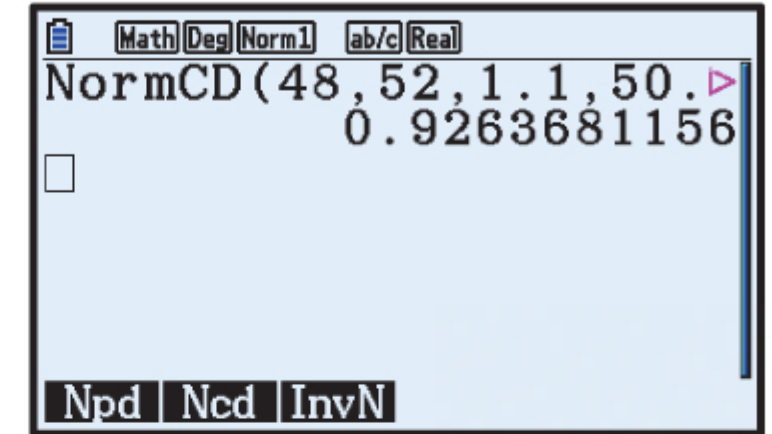


$$P(23 \leq X \leq 30) \approx 0.250$$

13 a i $X_A \sim N(50.2, 1.1^2)$

$$P(48 \leq X_A \leq 52) \approx 0.9264$$

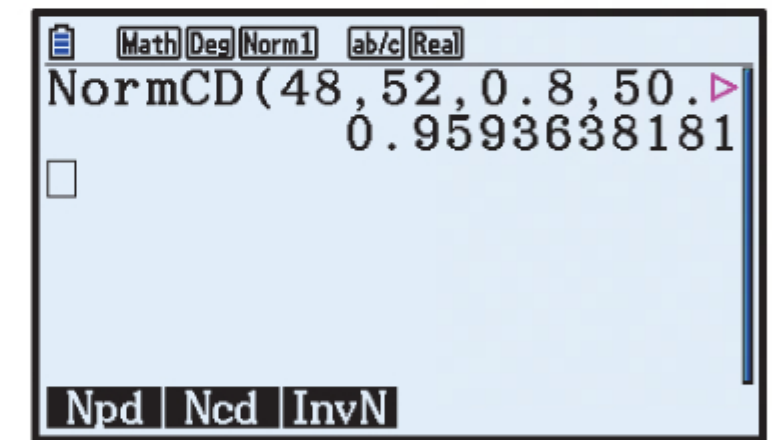
\therefore the probability that a nail from machine A needs to be rejected is about $1 - 0.9264 \approx 0.0736$.



ii $X_B \sim N(50.6, 0.8^2)$

$$P(48 \leq X_B \leq 52) \approx 0.9594$$

\therefore the probability that a nail from machine B needs to be rejected is about $1 - 0.9594 \approx 0.0406$.



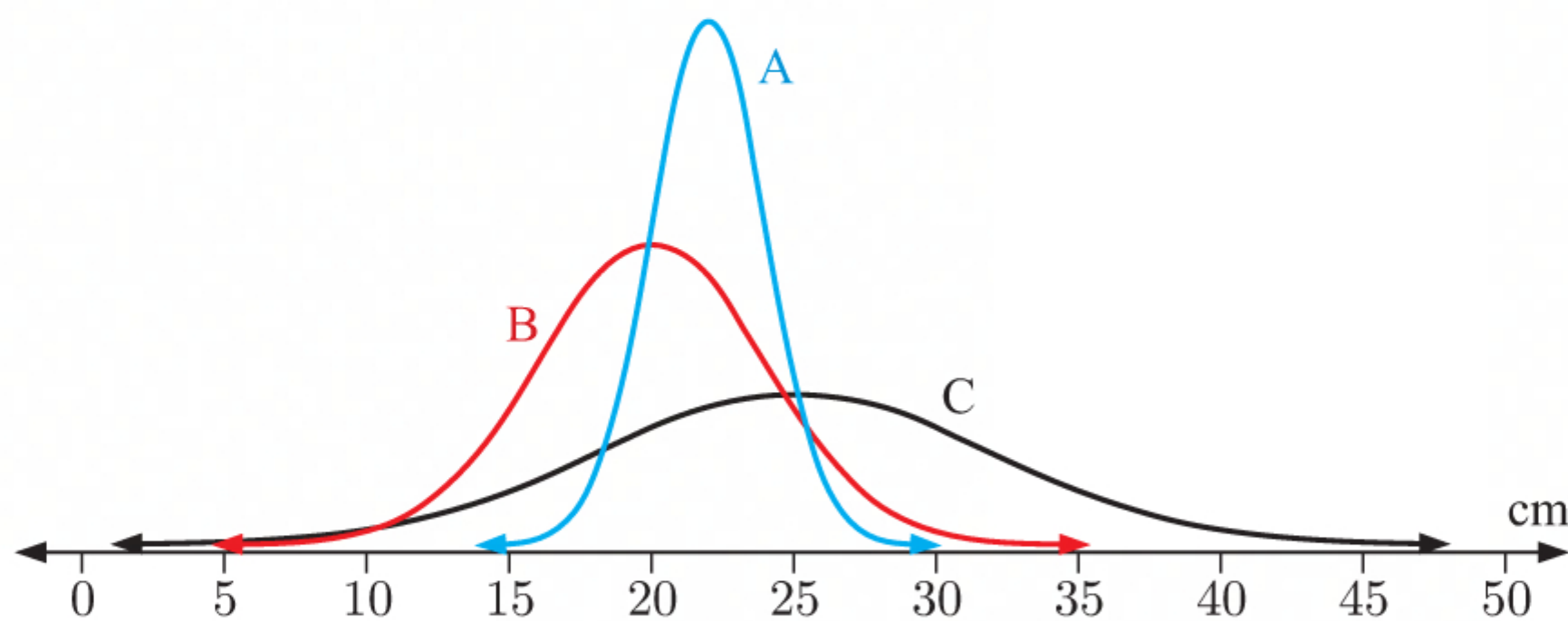
$$\begin{aligned} \text{b } P(\text{made by machine A} \mid \text{rejected}) &= \frac{P(\text{made by machine A} \cap \text{rejected})}{P(\text{rejected})} \\ &\approx \frac{0.5 \times 0.0736}{0.5 \times 0.0736 + 0.5 \times 0.0406} \\ &\approx 0.644 \end{aligned}$$

The probability that the nail was made by machine A *given* that it should be rejected is approximately 0.644.

REVIEW SET 21B

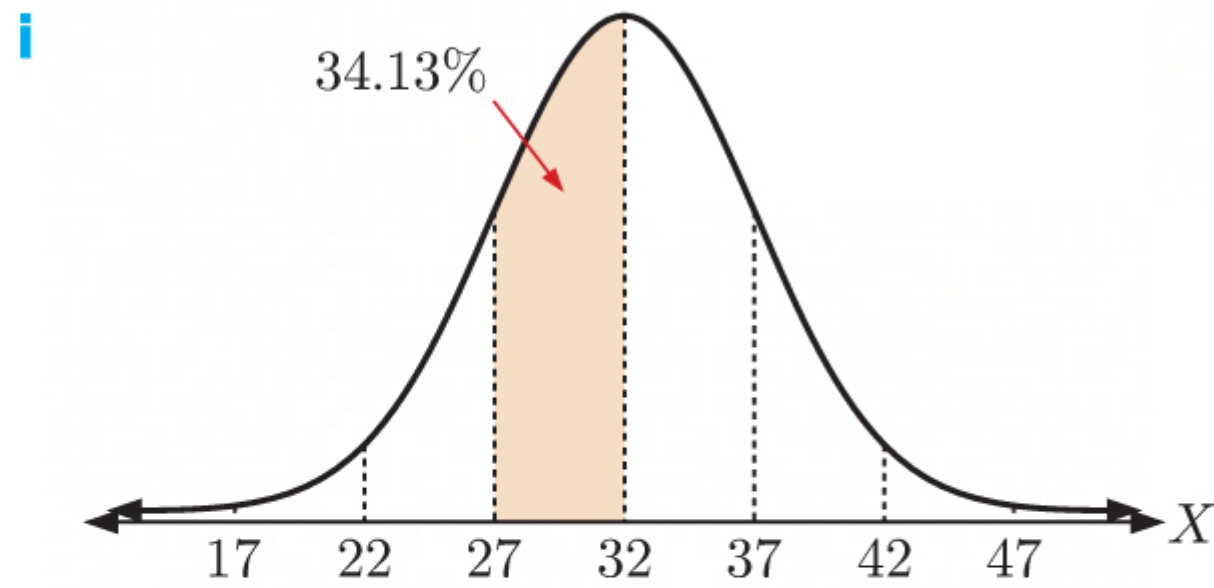
1

Distribution	Mean (cm)	Standard deviation (cm)
A	22	2
B	20	4
C	25	7

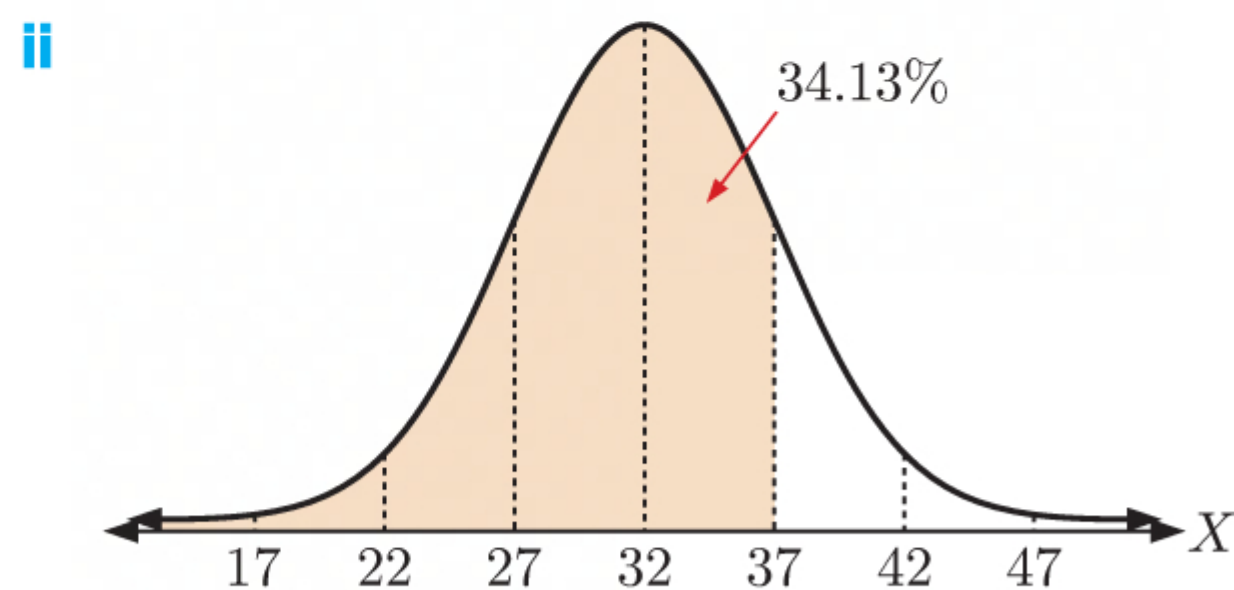


2 a The mean is $\mu = 32$, and the standard deviation is $\sigma = 5$.

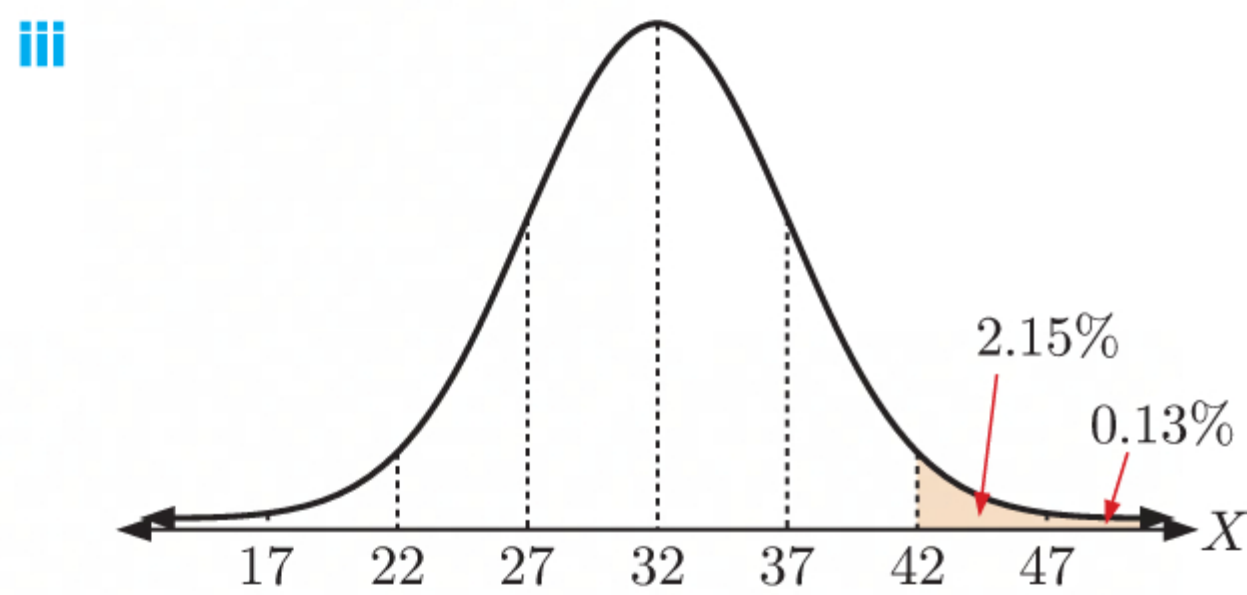
b $X \sim N(32, 5^2)$



About 34.13% of values of X are between 27 and 32.



About $50\% + 34.13\% = 84.13\%$ of values of X are less than 37.

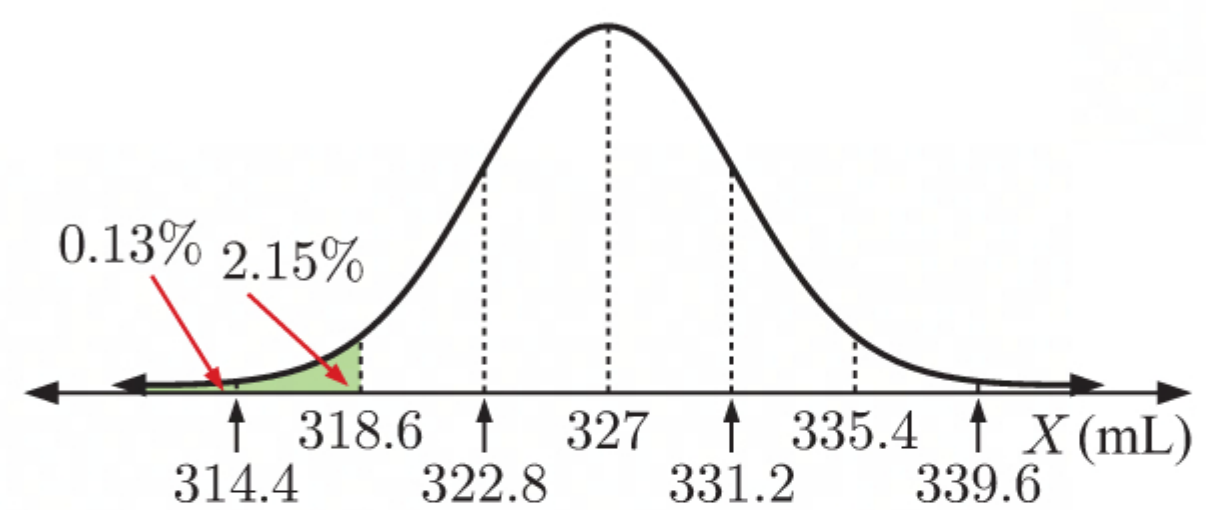


About $2.15\% + 0.13\% = 2.28\%$ of values of X are greater than 42.

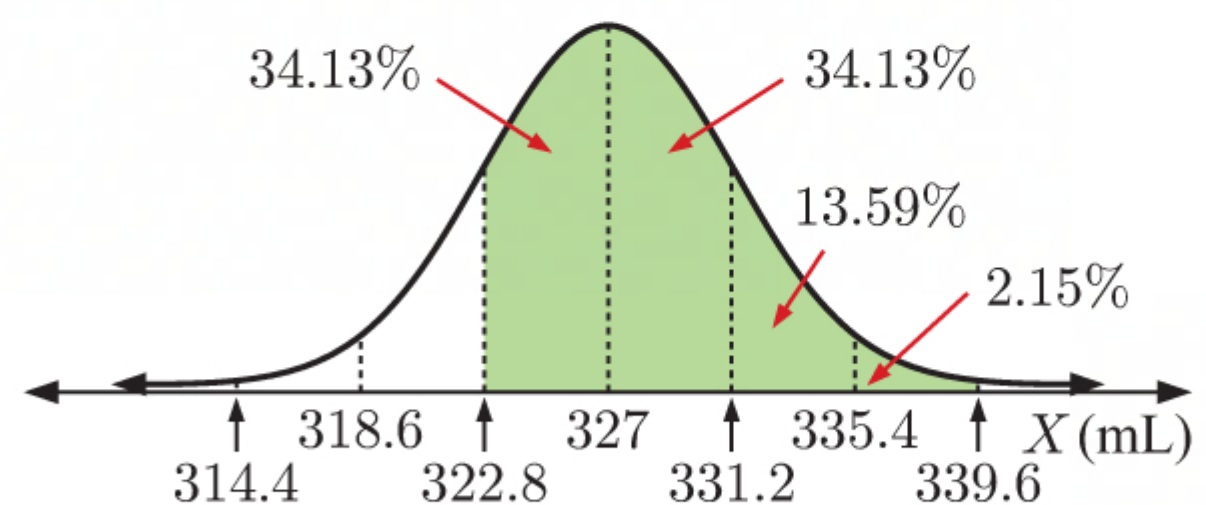
3 Let X mL be the contents of the container.

$X \sim N(327, 4.2^2)$

a i About $0.13\% + 2.15\% = 2.28\%$ of cans have contents less than 318.6 mL.

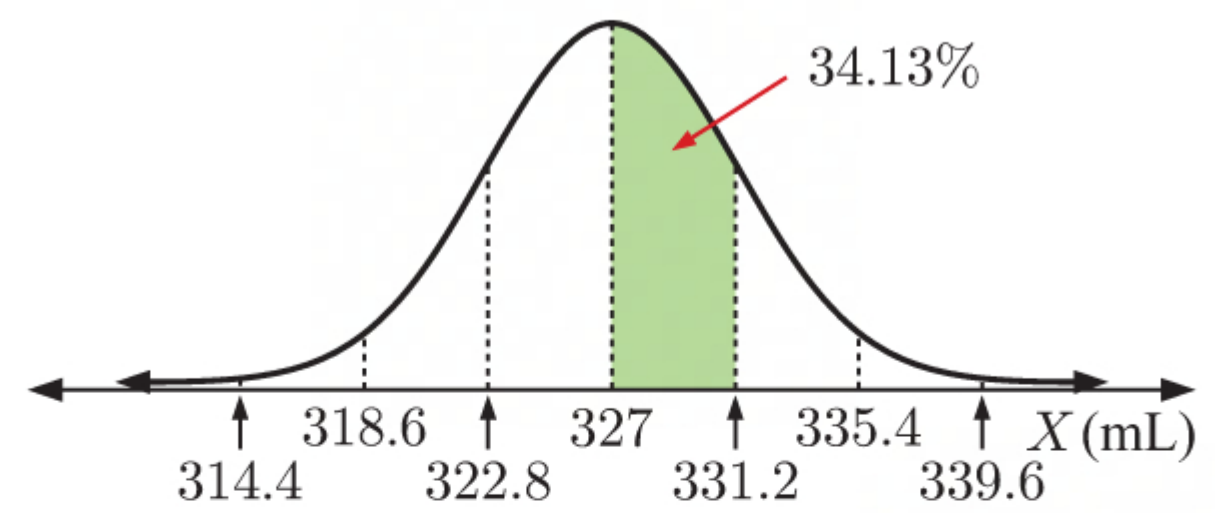


ii About
 $34.13\% + 34.13\% + 13.59\% + 2.15\%$
 $= 84.0\%$ of cans have contents
 between 322.8 mL and 339.6 mL.



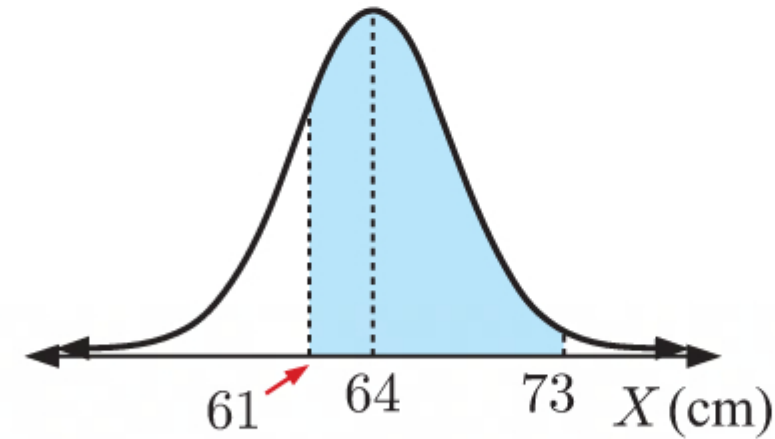
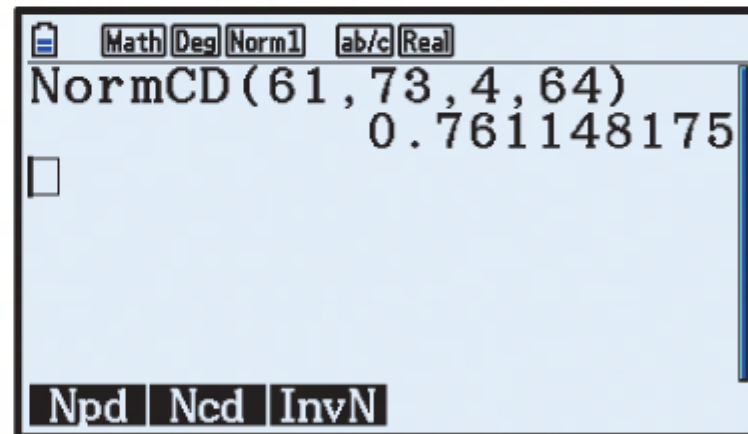
- b** About 34.13% of cans have contents between 327 mL and 331.2 mL.

$$P(\text{contents between 327 mL and 331.2 mL}) \\ \approx 0.3413$$



- 4** Let X cm be the arm length of a randomly selected 18 year old female.
 $X \sim N(64, 4^2)$

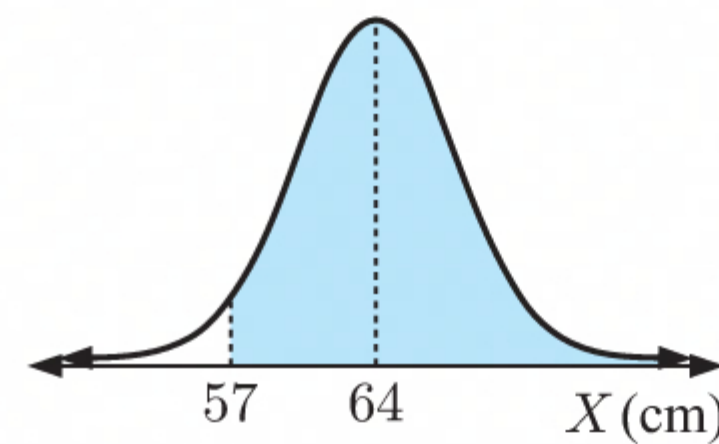
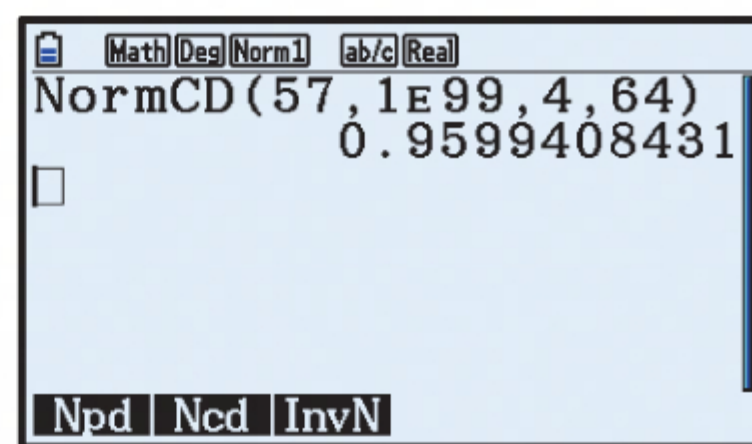
a i



$$P(61 < X < 73) \approx 0.761$$

\therefore approximately 76.1% of 18 year old females have an arm length between 61 cm and 73 cm.

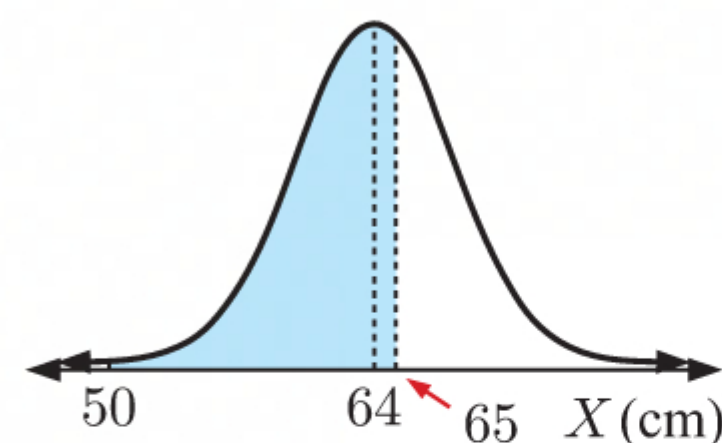
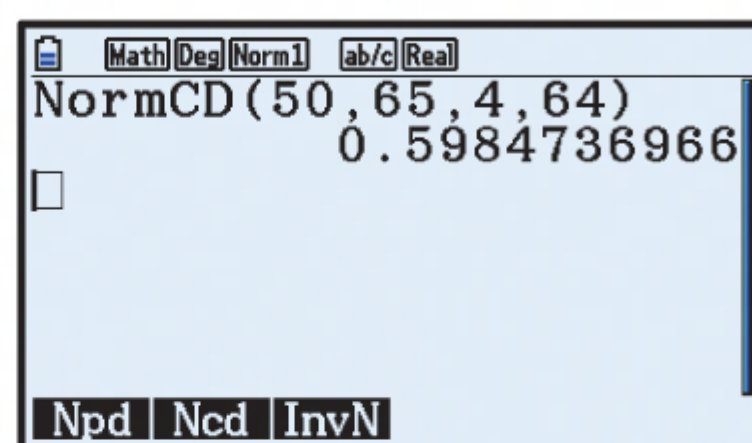
ii



$$P(X > 57) \approx 0.960$$

\therefore approximately 96.0% of 18 year old females have an arm length greater than 57 cm.

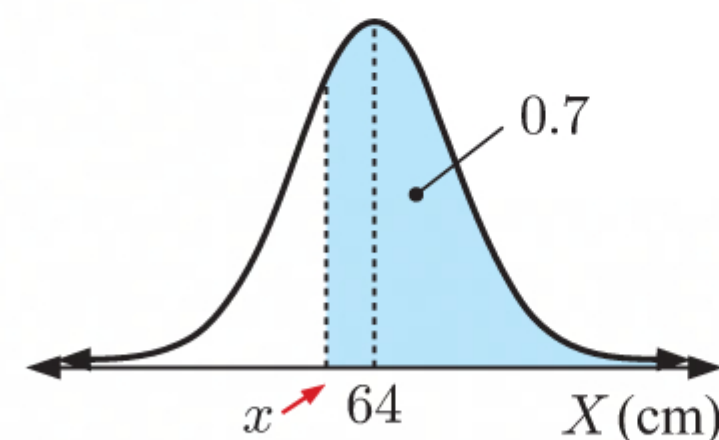
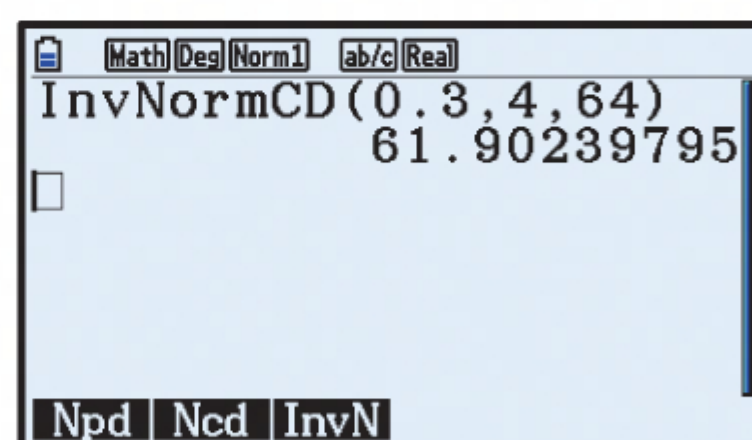
b



$$P(50 < X < 65) \approx 0.598$$

\therefore the probability that an 18 year old female has an arm length in the range 50 cm to 65 cm is approximately 0.598.

c

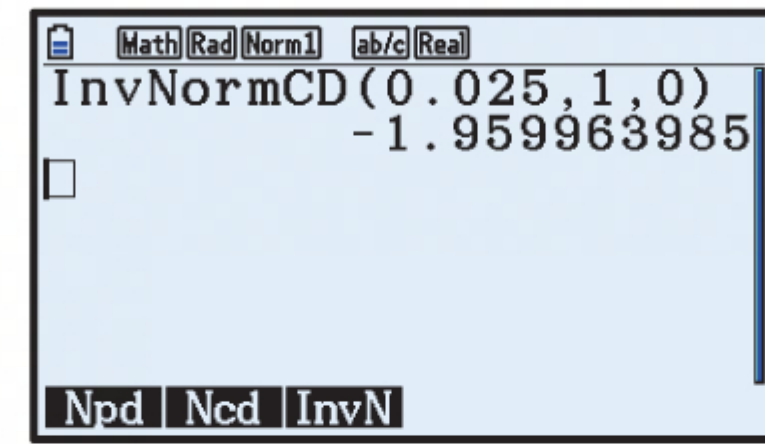


$$P(X > x) = 0.7$$

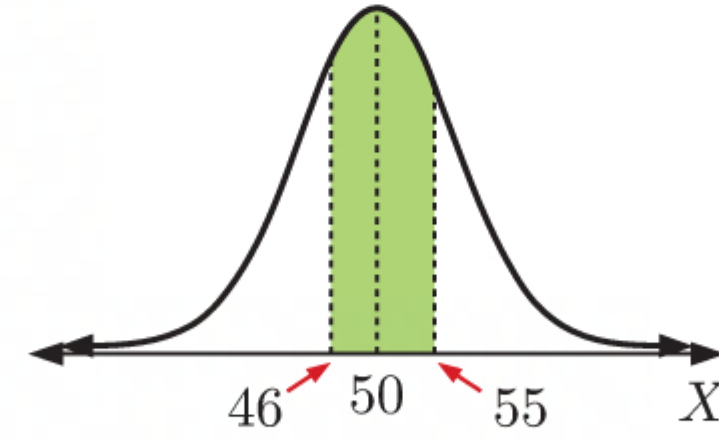
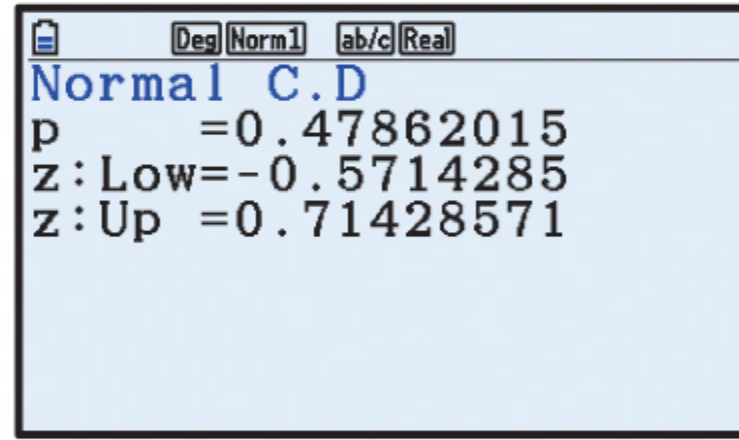
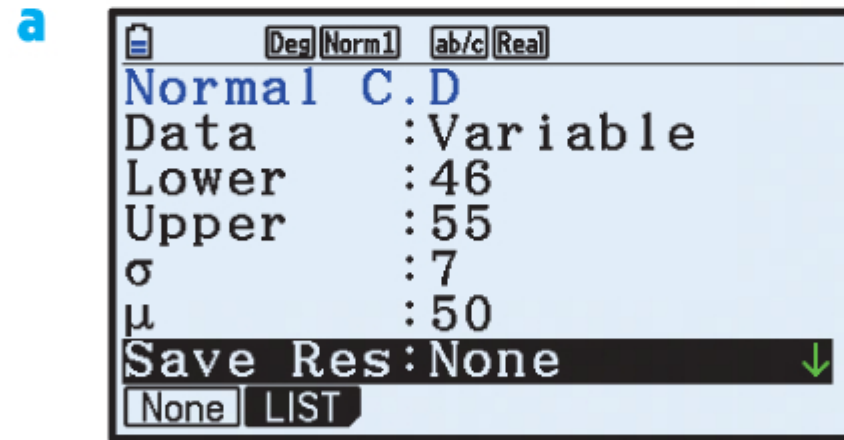
$$\therefore P(X < x) = 0.3$$

$$\therefore x \approx 61.9$$

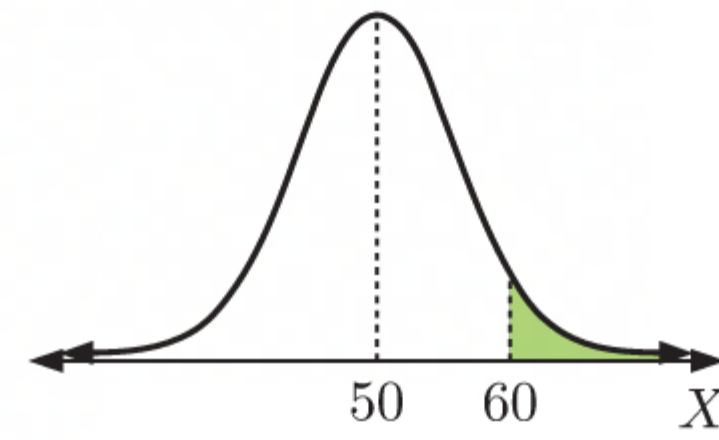
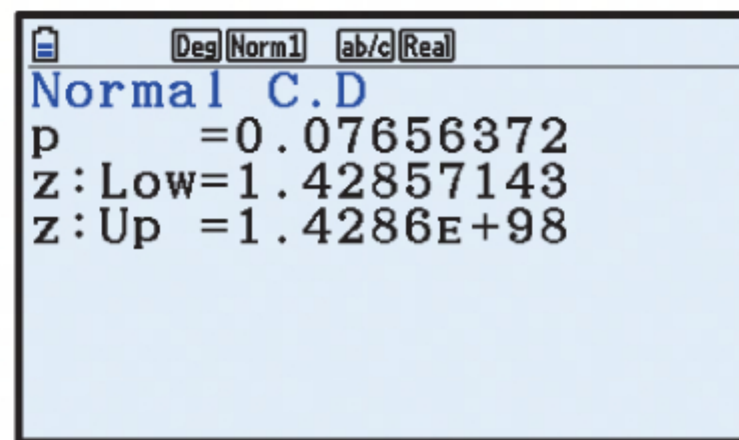
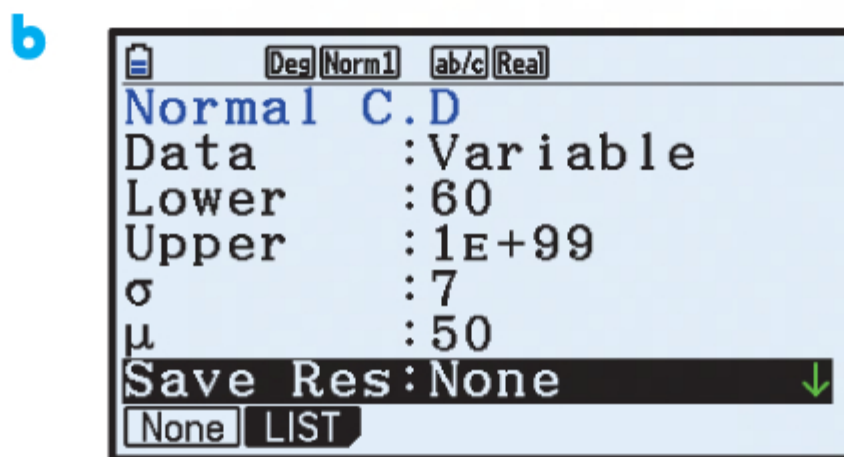
5 If $P(-k \leq Z \leq k) = 0.95$
 $\therefore 1 - P(Z \leq -k) - P(Z \geq k) = 0.95$
 $\therefore 1 - 2P(Z \leq -k) = 0.95$ {symmetry of the normal distribution}
 $\therefore 2P(Z \leq -k) = 0.05$
 $\therefore P(Z \leq -k) = 0.025$
 $\therefore k \approx 1.96$



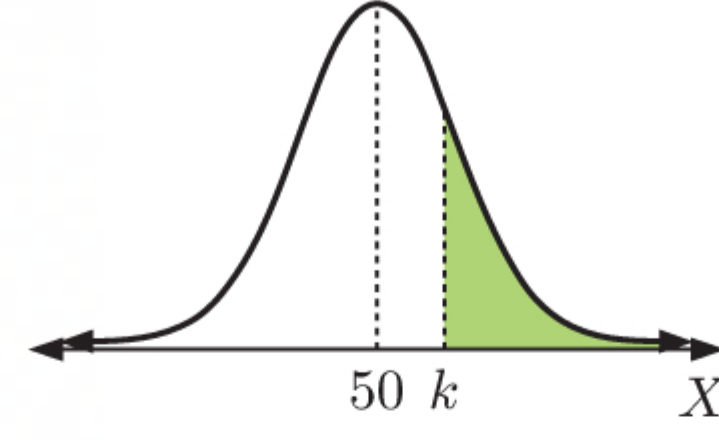
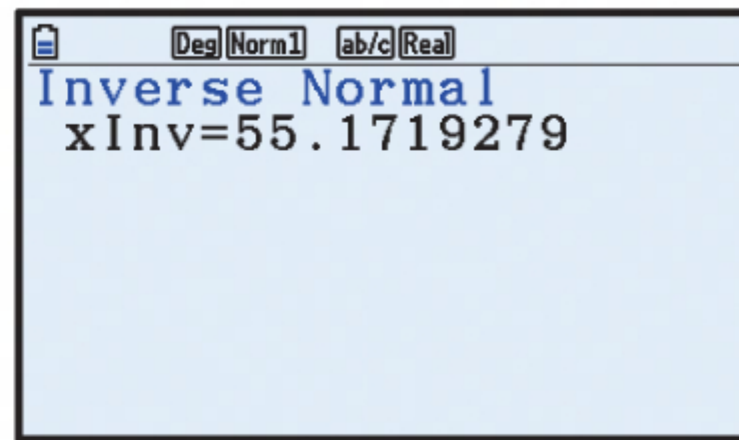
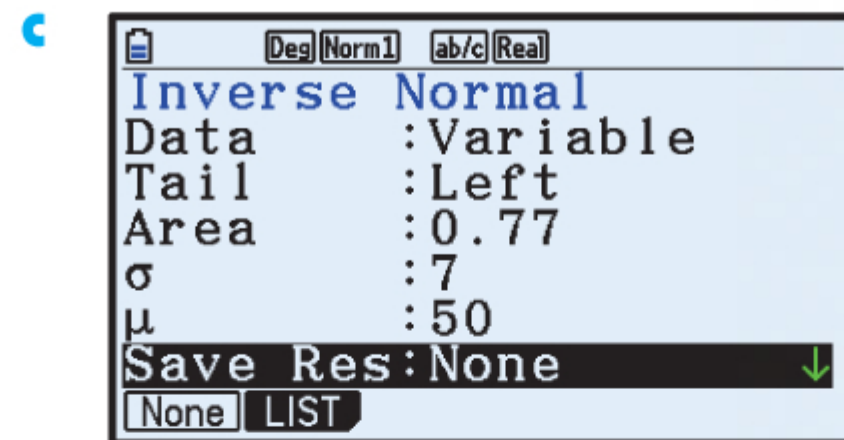
6 $X \sim N(50, 7^2)$



$P(46 \leq X \leq 55) \approx 0.479$



$P(X \geq 60) \approx 0.0766$



If $P(X > k) = 0.23$
 $\therefore P(X < k) = 0.77$
 $\therefore k \approx 55.2$

- 7 Let X seconds be the time a contestant holds their breath.

$$X \sim N(150, 12^2)$$

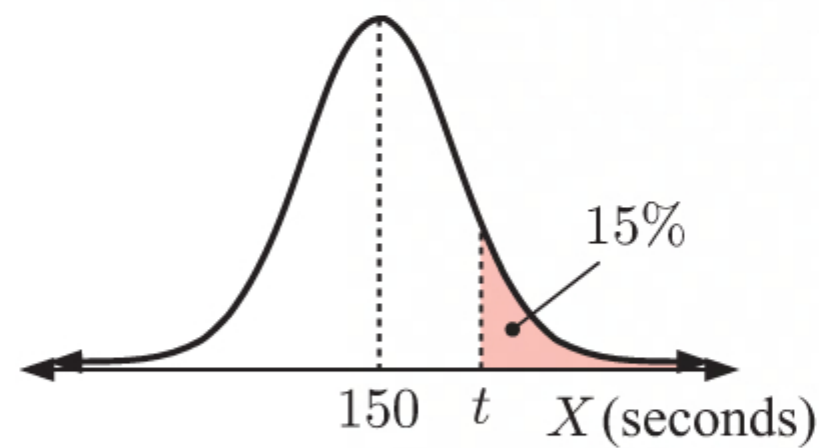
Deg Norm1 ab/c Real	
Inverse Normal	
Data	:Variable
Tail	:Left
Area	:0.85
σ	:12
μ	:150
Save Res:None	
None	LIST

Deg Norm1 ab/c Real	
Inverse Normal	
xInv=162.437201	

$$P(X > t) = 0.15$$

$$\therefore P(X < t) = 0.85$$

$$\therefore t \approx 162.4$$



To advance to the final round, a contestant would need to hold their breath for about 162 seconds.

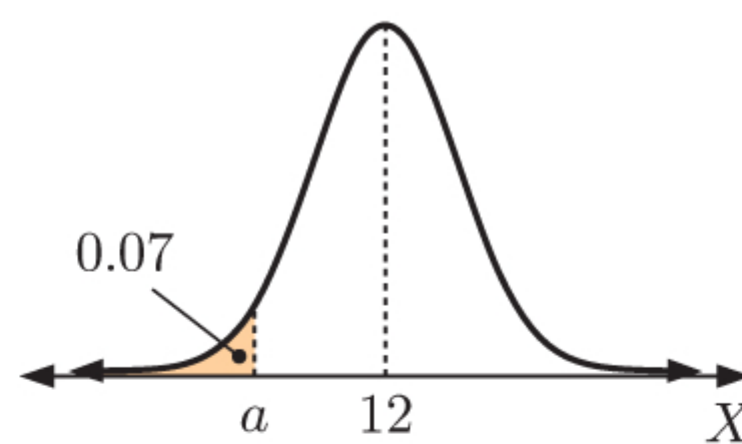
- 8 $X \sim N(12, 2^2)$

a

Rad Norm1 ab/c Real	
Inverse Normal	
Data	:Variable
Tail	:Left
Area	:0.07
σ	:2
μ	:12
Save Res:None	
None	LIST

Rad Norm1 ab/c Real	
Inverse Normal	
xInv=9.04841794	

If $P(X < a) = 0.07$
then $a \approx 9.05$

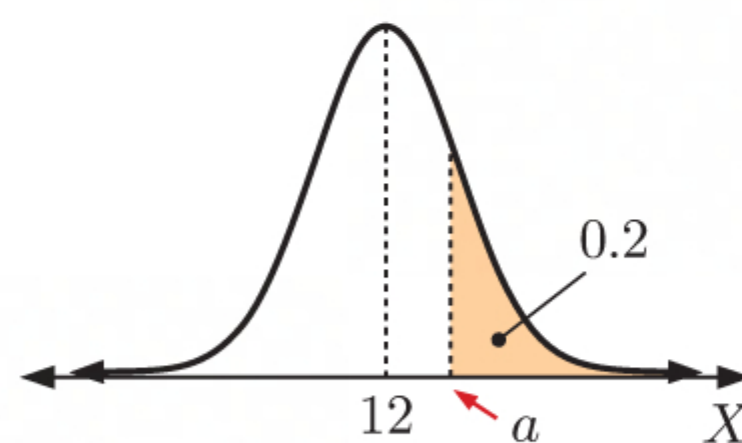


b

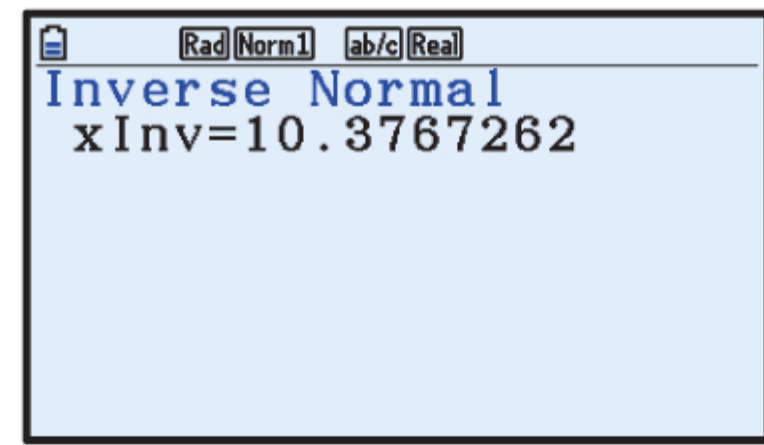
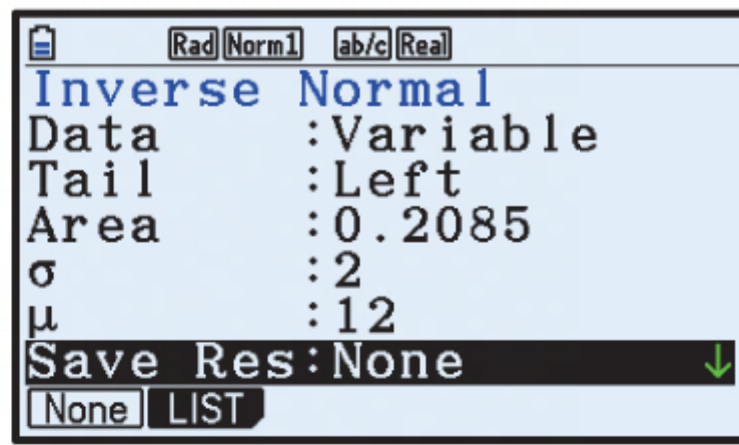
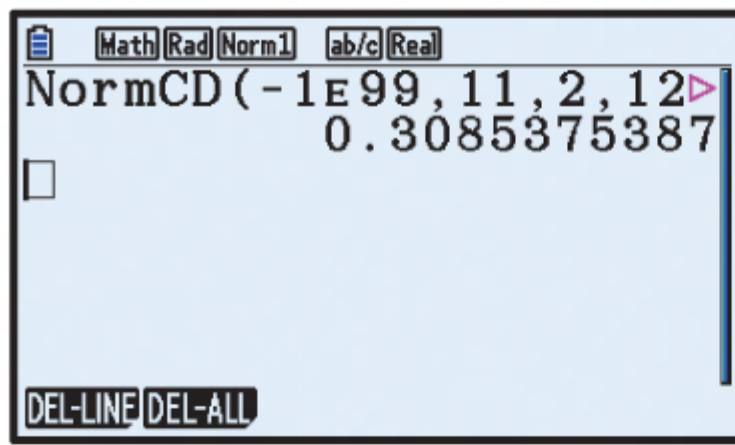
Rad Norm1 ab/c Real	
Inverse Normal	
Data	:Variable
Tail	:Left
Area	:0.8
σ	:2
μ	:12
Save Res:None	
None	LIST

Rad Norm1 ab/c Real	
Inverse Normal	
xInv=13.6832425	

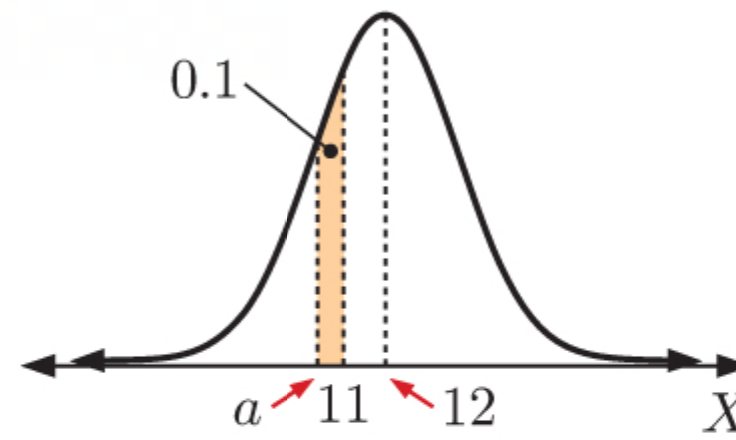
If $P(X > a) = 0.2$
 $\therefore P(X < a) = 0.8$
 $\therefore a \approx 13.7$



C



$$\begin{aligned} \text{If } P(a \leq X \leq 11) &= 0.1 \\ \therefore P(X \leq 11) - P(X \leq a) &= 0.1 \\ \therefore 0.3085 - P(X \leq a) &\approx 0.1 \\ \therefore P(X \leq a) &\approx 0.2085 \\ \therefore a &\approx 10.4 \end{aligned}$$

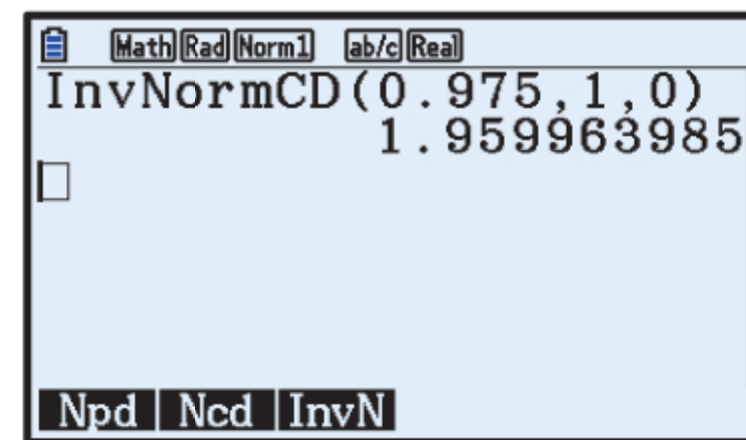


9 $X \sim N(\mu, 2.1^2), \quad Z \sim N(0, 1^2)$

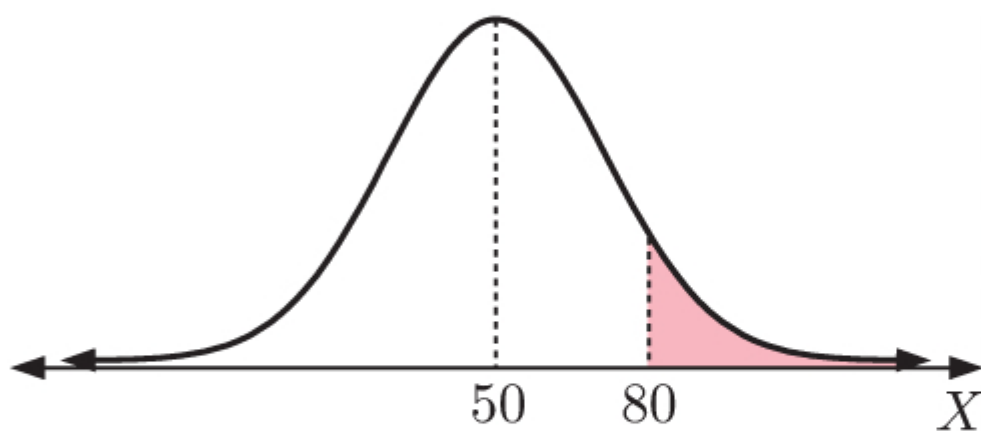
$$\begin{aligned} P(Z > -1.7) &= P(X > 5.4) \\ &= P\left(\frac{X - \mu}{2.1} > \frac{5.4 - \mu}{2.1}\right) \\ &= P\left(Z > \frac{5.4 - \mu}{2.1}\right) \\ \therefore \frac{5.4 - \mu}{2.1} &= -1.7 \\ \therefore 5.4 - \mu &= -3.57 \\ \therefore \mu &= 8.97 \end{aligned}$$

10 $P(X < 90) \approx 0.975$

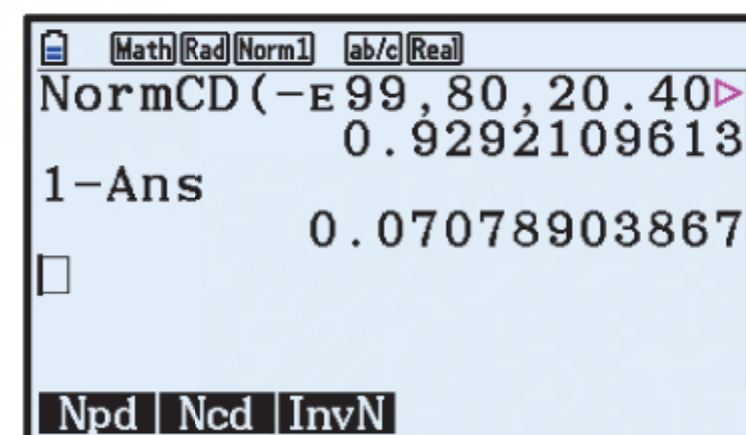
$$\begin{aligned} \therefore P\left(Z < \frac{90 - 50}{\sigma}\right) &\approx 0.975 \\ \therefore P\left(Z < \frac{40}{\sigma}\right) &\approx 0.975 \\ \therefore \frac{40}{\sigma} &\approx 1.95996 \\ \therefore \sigma &\approx 20.409 \end{aligned}$$



So, $X \sim N(50, 20.409^2)$



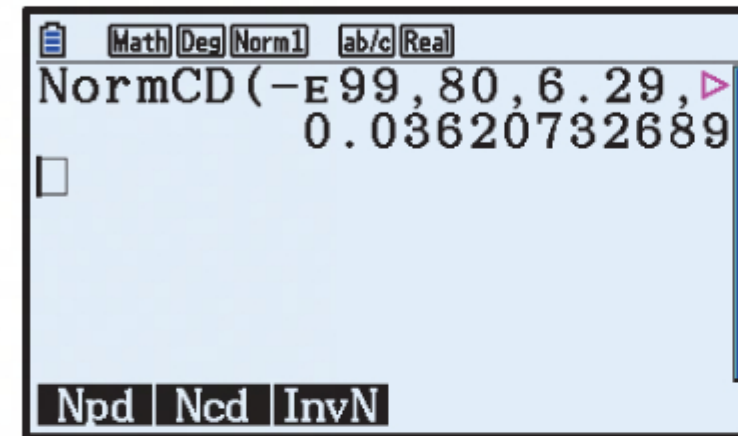
$$\begin{aligned} \text{Now, the shaded area} &= P(X \geq 80) \\ &= 1 - P(X < 80) \\ &\approx 1 - 0.9292 \\ &\approx 0.0708 \text{ units}^2 \end{aligned}$$



- 11 a** Let X_F be the weight in kilograms of a female ostrich and X_M be the weight in kilograms of a male ostrich.

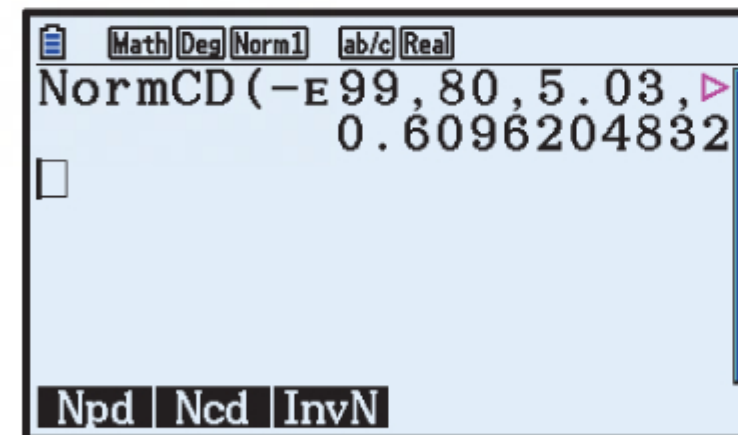
$$X_F \sim N(78.6, 5.03^2) \quad \text{and} \quad X_M \sim N(91.3, 6.29^2)$$

i $P(X_M < 80) \approx 0.0362$



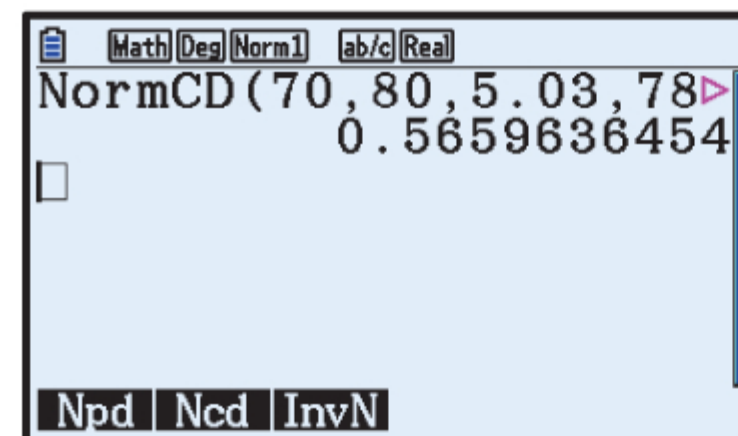
The probability that a randomly selected male ostrich will weigh less than 80 kg is about 0.0362.

ii $P(X_F < 80) \approx 0.610$



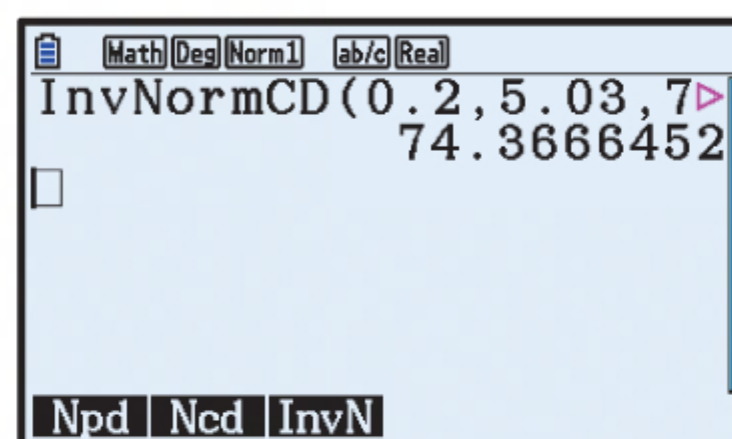
The probability that a randomly selected female ostrich will weigh less than 80 kg is about 0.610.

iii $P(70 < X_F < 80) \approx 0.566$



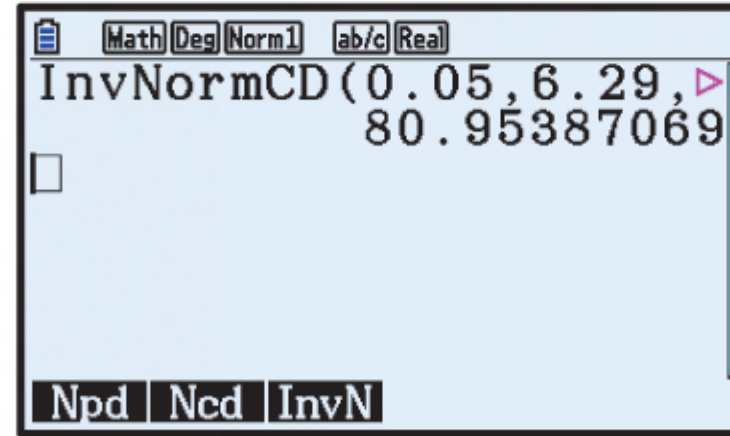
The probability that a randomly selected female ostrich will weigh between 70 and 80 kg is about 0.566.

b $P(X_F < k) = 0.2$
then $k \approx 74.4$

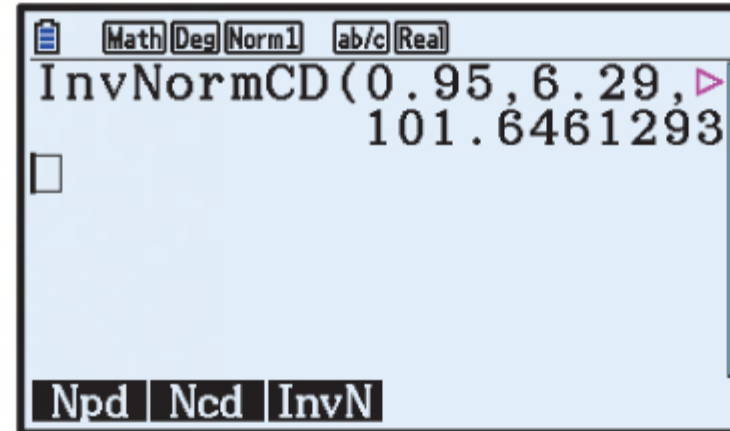


- c We need to find a and b such that $P(X_M < a) = 0.05$ and $P(X_M > b) = 0.05$.

If $P(X_M < a) = 0.05$
then $a \approx 81.0$

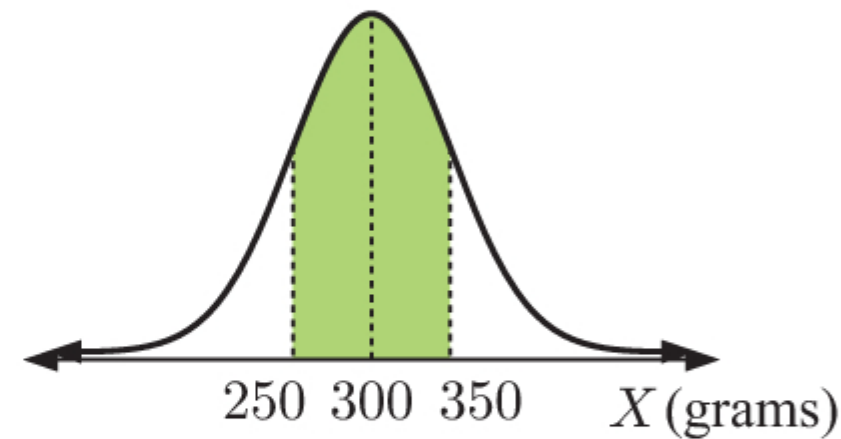
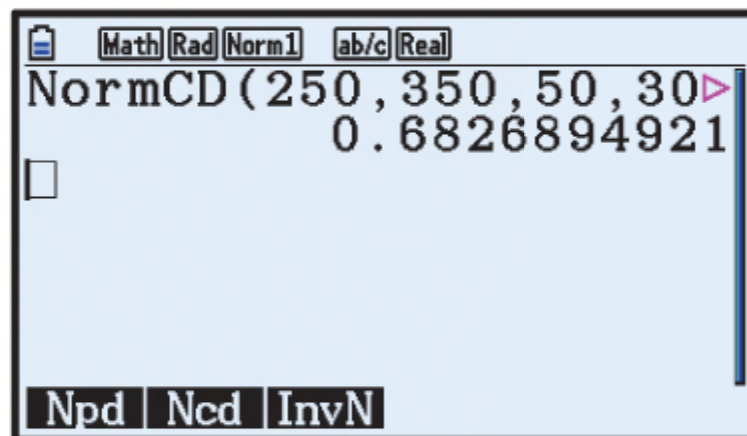


If $P(X_M > b) = 0.05$
then $P(X_M < b) = 0.95$
 $\therefore b \approx 102$



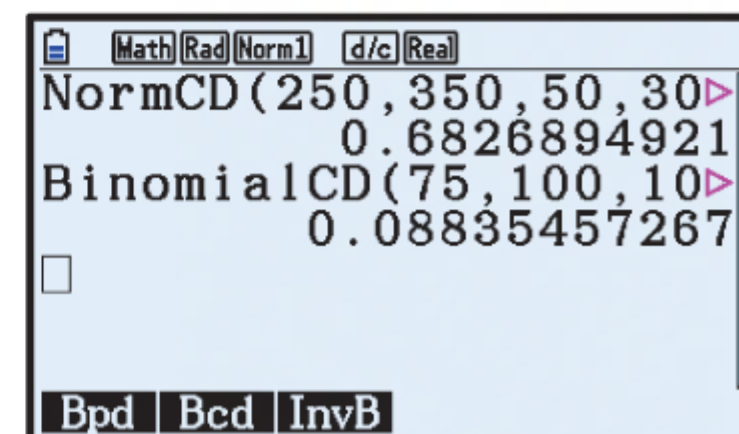
- d Probability that the ostrich weighs less than 80 kg
 $= P(\text{ostrich is female} \cap \text{less than 80 kg}) + P(\text{ostrich is male} \cap \text{less than 80 kg})$
 $\approx 0.82 \times 0.610 + 0.18 \times 0.0362$
 ≈ 0.506

- 12 a Let X grams be the weight of an apple.
 $X \sim N(300, 50^2)$



$P(250 \leq X \leq 350) \approx 0.682689$
 So, approximately 68.3% of apples are fit for sale.

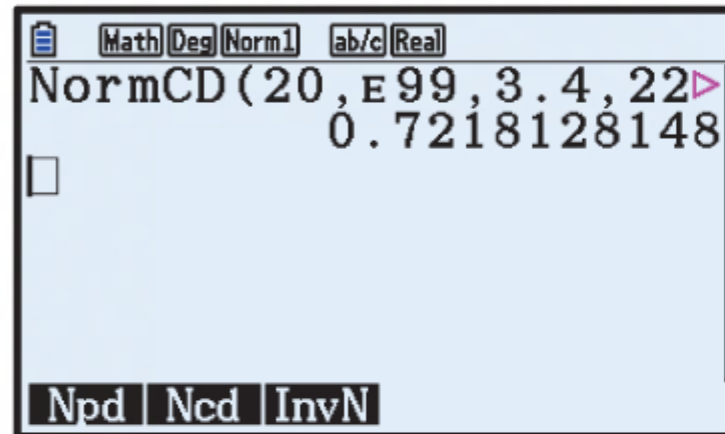
- b Let Y be the number of apples that are fit for sale.
 $Y \sim B(100, 0.682689)$
 $P(Y \geq 75) \approx 0.0884$



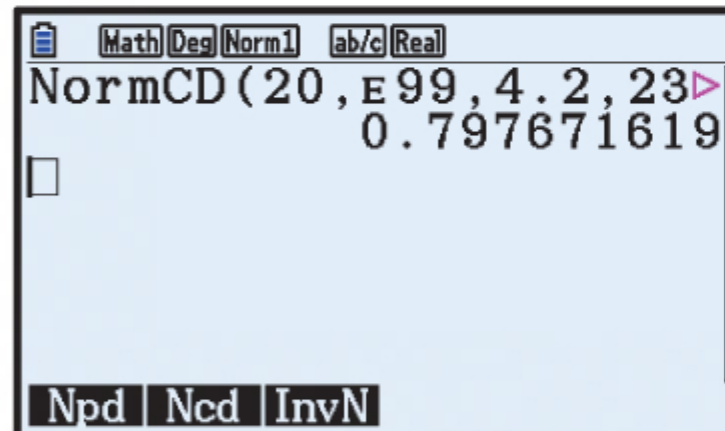
- 13 a** Let X_G be the length in cm of a carrot from Giovanni's farm, and X_B be the length in cm of a carrot from Beppe's farm.

$$X_G \sim N(22, 3.4^2) \quad \text{and} \quad X_B \sim N(23.5, 4.2^2)$$

i $P(X_G > 20) \approx 0.722$



ii $P(X_B > 20) \approx 0.798$



- b** Probability that neither carrot is longer than 20 cm = $P(X_G < 20) \times P(X_B < 20)$
 $\approx (1 - 0.722) \times (1 - 0.798)$
 ≈ 0.0563